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Some claims concerning a tetrahedron ABCP based on an acute triangle ABC

Scale and position the tetrahedron so that the coordinates of A, B and C are $(x_1\,,\ y_1\,, 0)$, $(x_2\,,\ y_2\,, 0)$ and $(x_3\,,\ y_3\,, 0)$, respectively, with $x_j = \cos\phi_j\,, y_j = \sin\phi_j$ (j = 1,2,3) and $\phi_1 + \phi_2 + \phi_3 = 0$.

Define the following:

$$\angle A = \angle \mathrm{CAB}\,,\ \angle B = \angle \mathrm{ABC}\,,\ \angle C = \angle \mathrm{BCA}\,,\ \alpha = \angle \mathrm{BPC}\,,\ \beta = \angle \mathrm{CPA}\,,\ \gamma = \angle \mathrm{APB}\,,$$

$$C_0 = \cos\alpha\cos\beta\cos\gamma,\ C_1 = \cos^2\alpha\,,\ C_2 = \cos^2\beta\,,\ C_3 = \cos^2\gamma\,,$$

$$L = 2\,[\,(y_1 + y_2 + y_3)(1 - C_0) + (1 - x_1)\,y_1\,(C_1 - 1) + (1 - x_2)\,y_2\,(C_2 - 1) + (1 - x_3)\,y_3\,(C_3 - 1)\,]\,,$$

$$R = 2\,[\,(1 + x_1 + x_2 + x_3)(1 - C_0) + (1 + x_1)\,x_1\,(C_1 - 1) + (1 + x_2)\,x_2\,(C_2 - 1) + (1 + x_3)\,x_3\,(C_3 - 1)\,]\,,$$

$$H = 1 + 2C_0 - C_1 - C_2 - C_3\,, \qquad E = L^2 + (R{+}H)^2\,,$$

$$D = E^2 + 18\,H^2 E + 8H\,(R{+}H)\,[(R{+}H)^2 - 3L^2] - 27H^4\,.$$

The following (proven) constraints hold:

1. $H > 0$ (which is equivalent to $\alpha + \beta + \gamma < 2\pi$, $\alpha < \beta + \gamma$, $\beta < \gamma + \alpha$ and $\gamma < \alpha + \beta$) ;

2. $\angle A + \beta + \gamma < 2\pi$, $\alpha + \angle B + \gamma < 2\pi$, $\alpha + \beta + \angle C < 2\pi$;

3. $\beta + \gamma - \alpha < 2(\angle B + \angle C)$, $\gamma + \alpha - \beta < 2(\angle C + \angle A)$, $\alpha + \beta - \gamma < 2(\angle \mathrm{A} + \angle B)$;

4. $\alpha < \angle A \rightarrow [\ \beta < \max\{\ \angle B, \angle C + \alpha\ \} \ \wedge\ \gamma < \max\{\ \angle C, \angle B{+}\alpha\ \}\ ]$,
   $\beta < \angle B \rightarrow [\ \gamma < \max\{\ \angle C, \angle A + \beta\ \} \ \wedge\ \alpha < \max\{\ \angle A, \angle C{+}\beta\ \}\ ]$,
   $\gamma < \angle C \rightarrow [\ \alpha < \max\{\ \angle A, \angle B + \gamma\ \} \ \wedge\ \beta < \max\{\ \angle B, \angle A{+}\gamma\ \}\ ]$;

5. $\alpha < \angle A \rightarrow \cos\angle C\cos\beta + \cos\angle B\cos\gamma > 0$,
   $\beta < \angle B \rightarrow \cos\angle A\cos\gamma + \cos\angle C\cos\alpha > 0$,
   $\gamma < \angle C \rightarrow \cos\angle B\cos\alpha + \cos\angle A\cos\beta > 0$;

6. $(D > 0 \ \wedge\ \alpha < \angle A) \rightarrow (\beta < \angle B \ \vee\ \gamma < \angle C)$,
   $(D > 0 \ \wedge\ \beta < \angle B) \rightarrow (\gamma < \angle C \ \vee\ \alpha < \angle A)$,
   $(D > 0 \ \wedge\ \gamma < \angle C) \rightarrow (\alpha < \angle A \ \vee\ \beta < \angle B)$;

7. $D > 0 \rightarrow [\ \alpha < \angle A \ \vee\ \alpha > \pi - \angle A \ \vee\ \beta < \angle B \ \vee\ \beta > \pi - \angle B \ \vee\ \gamma < \angle C \ \vee\ \gamma > \pi - \angle C\ ]$.

Also, experimental evidence suggests that this collection of constraints is necessary and sufficient for a suitable tetrahedron ABCP to exist for a given acute triangle ABC. At present though, the sufficiency claim is only a conjecture. (Note: the arrows mean logical implication here.)