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Some Claims Concerning a Tetrahedron with an Acute Triangle Face

Consider a tetrahedron *ABCP* with the triangle *ABC* acute. Without altering its shape, scale and position the tetrahedron in Cartesian space so that the coordinates of *A*, *B* and *C* are $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ and $(x_3, y_3, 0)$, respectively, with $x_j = \cos\phi_j$, $y_j = \sin\phi_j$ ($j = 1,2,3$) and $\phi_1 + \phi_2 + \phi_3 = 0$ for angles ϕ_1, ϕ_2 and ϕ_3. (This is straight-forward to achieve, and is done in my C++ programs at `github/mqrieck/tetrahedron_test.cpp`.)

Now define the following quantities:

$$\angle A = \angle CAB\,,\ \angle B = \angle ABC\,,\ \angle C = \angle BCA\,,\ \alpha = \angle BPC\,,\ \beta = \angle CPA\,,\ \gamma = \angle APB\,,$$

$$C_0 = \cos\alpha\cos\beta\cos\gamma,\ C_1 = \cos^2\alpha\,,\ C_2 = \cos^2\beta\,,\ C_3 = \cos^2\gamma\,,$$

$$L = 2\,[\,(y_1 + y_2 + y_3)(1 - C_0) + (1 - x_1)\,y_1\,(C_1 - 1) + (1 - x_2)\,y_2\,(C_2 - 1) + (1 - x_3)\,y_3\,(C_3 - 1)\,]\,,$$

$$R = 2\,[\,(1 + x_1 + x_2 + x_3)(1 - C_0) + (1 + x_1)\,x_1\,(C_1 - 1) + (1 + x_2)\,x_2\,(C_2 - 1) + (1 + x_3)\,x_3\,(C_3 - 1\,]\,,$$

$$H = 1 + 2C_0 - C_1 - C_2 - C_3\,, \quad K = -H - R\,,$$

$$D = (K^2 + L^2 + 12\,HK + 9\,H^2)^2 - 4\,H\,(2\,K + 3\,H)^3\,.$$

The following (proven) constraints hold:

1. $H > 0$ (equivalently, $\alpha + \beta + \gamma < 2\pi \ \wedge\ \alpha < \beta + \gamma \ \wedge\ \beta < \gamma + \alpha \ \wedge\ \gamma < \alpha + \beta$);

2. $\angle A + \beta + \gamma < 2\pi\,,\ \alpha + \angle B + \gamma < 2\pi\,,\ \alpha + \beta + \angle C < 2\pi$;

3. $\beta + \gamma - \alpha < 2(\angle B + \angle C)\,,\ \gamma + \alpha - \beta < 2(\angle C + \angle A)\,,\ \alpha + \beta - \gamma < 2(\angle \mathrm{A} + \angle B)$;

4. $\alpha < \angle A \ \rightarrow\ [\,\beta < \max\{\angle B,\ \angle C + \alpha\} \ \wedge\ \gamma < \max\{\angle C,\ \angle B + \alpha\}\,]$,
   $\beta < \angle B \ \rightarrow\ [\,\gamma < \max\{\angle C,\ \angle A + \beta\} \ \wedge\ \alpha < \max\{\angle A,\ \angle C + \beta\}\,]$,
   $\gamma < \angle C \ \rightarrow\ [\,\alpha < \max\{\angle A,\ \angle B + \gamma\} \ \wedge\ \beta < \max\{\angle B,\ \angle A + \gamma\}\,]$;

5. $\alpha < \angle A \ \rightarrow\ \cos\angle C\cos\beta + \cos\angle B\cos\gamma > 0$,
   $\beta < \angle B \ \rightarrow\ \cos\angle A\cos\gamma + \cos\angle C\cos\alpha > 0$,
   $\gamma < \angle C \ \rightarrow\ \cos\angle B\cos\alpha + \cos\angle A\cos\beta > 0$;

6. $[\,\alpha < \angle A \ \wedge\ \beta > \angle B \ \wedge\ \gamma > \angle C\,] \ \rightarrow\ D < 0$,
   $[\,\alpha > \angle A \ \wedge\ \beta < \angle B \ \wedge\ \gamma > \angle C\,] \ \rightarrow\ D < 0$,
   $[\,\alpha > \angle A \ \wedge\ \beta > \angle B \ \wedge\ \gamma < \angle C\,] \ \rightarrow\ D < 0$;

7. $[\,\angle A < \alpha < \pi - \angle A \ \wedge\ \angle B < \beta < \pi - \angle B \ \wedge\ \angle C < \gamma < \pi - \angle C\,] \ \rightarrow\ D < 0$.

The first three items in this list are relatively easy to prove. The remaining four items require significantly more technical analysis. Also, experimental evidence suggests that this collection of constraints is both necessary and sufficient for a suitable tetrahedron *ABCP* to exist for a given acute triangle *ABC*. At present though, the sufficiency claim is only a conjecture. (Note: the arrows mean logical implication.)