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Some claims concerning a tetrahedron ABCP based on an acute triangle ABC

Scale and position the tetrahedron so that the coordinates of A, B and C are $(x_1, y_1, 0)$, $(x_2, y_2, 0)$, $(x_3, y_3, 0)$, respectively, with $x_j = \cos\phi_j$, $y_j = \sin\phi_j$ $(j = 1,2,3)$, and $\phi_1 + \phi_2 + \phi_3 = 0$.

Define the following:

$$\angle A = \angle \mathrm{CAB},\ \angle B = \angle \mathrm{ABC},\ \angle C = \angle \mathrm{BCA},\ \alpha = \angle \mathrm{BPC},\ \beta = \angle \mathrm{CPA},\ \gamma = \angle \mathrm{APB},$$

$$C_0 = \cos\alpha\cos\beta\cos\gamma,\ C_1 = \cos^2\alpha,\ C_2 = \cos^2\beta,\ C_3 = \cos^2\gamma,$$

$$L = 2\,[\,(y_1 + y_2 + y_3)(1 - C_0) + (1 - x_1)\,y_1\,(C_1 - 1) + (1 - x_2)\,y_2\,(C_2 - 1) + (1 - x_3)\,y_3\,(C_3 - 1)\,],$$

$$R = 2\,[\,(1 + x_1 + x_2 + x_3)(1 - C_0) + (1 + x_1)\,x_1\,(C_1 - 1) + (1 + x_2)\,x_2\,(C_2 - 1) + (1 + x_3)\,y_3\,(C_3 - 1)\,],$$

$$H = 1 + 2\,C_0 - C_1 - C_2 - C_3, \qquad E = L^2 + (R+H)^2,$$

$$D = E^2 + 18\,EH^2 + 8\,(R+H)\,H\,[\,(R+H)^2 - 3L^2\,] - 27\,H^4.$$

The following (proven) constraints hold:

1. $H > 0$ (which is equivalent to $\alpha+\beta+\gamma < 2\pi$, $\alpha < \beta+\gamma$, $\beta < \alpha+\gamma$, $\gamma < \alpha+\beta$);
2. $\angle A + \beta + \gamma < 2\pi$, $\alpha + \angle B + \gamma < 2\pi$, $\alpha + \beta + \angle C < 2\pi$;
3. $\beta + \gamma - \alpha < 2\,(\angle B + \angle C)$, $\gamma + \alpha - \beta < 2\,(\angle C + \angle A)$, $\alpha + \beta - \gamma < 2\,(\angle C + \angle A)$;
4. $\alpha < \angle A \rightarrow [\,\beta < \max\{\angle B, \angle C + \alpha\} \wedge \gamma < \max\{\angle C, \angle B + \alpha\}\,]$,
   $\beta < \angle B \rightarrow [\,\gamma < \max\{\angle C, \angle A + \beta\} \wedge \alpha < \max\{\angle A, \angle C + \beta\}\,]$,
   $\gamma < \angle C \rightarrow [\,\alpha < \max\{\angle A, \angle B + \gamma\} \wedge \beta < \max\{\angle B, \angle A + \gamma\}\,]$;
5. $\alpha < \angle A \rightarrow \cos\angle C\cos\beta + \cos\angle B\cos\gamma > 0$,
   $\beta < \angle B \rightarrow \cos\angle A\cos\gamma + \cos\angle C\cos\alpha > 0$,
   $\gamma < \angle C \rightarrow \cos\angle B\cos\alpha + \cos\angle A\cos\beta > 0$;
6. $(D > 0 \wedge \alpha < \angle A) \rightarrow (\beta < \angle B \vee \gamma < \angle C)$,
   $(D > 0 \wedge \beta < \angle B) \rightarrow (\gamma < \angle C \vee \alpha < \angle A)$,
   $(D > 0 \wedge \gamma < \angle C) \rightarrow (\alpha < \angle A \vee \beta < \angle B)$.

Also, experimental evidence strongly suggests that this collection of constraints is also sufficient for a suitable tetrahedron ABCP to exist for a given acute triangle ABC. At present though, this is just a conjecture.