

Some claims concerning a tetrahedron ABCP based on an acute triangle ABC

Scale and position a tetrahedron ABCP in Cartesian space so that the coordinates of A, B and C are $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ and $(x_3, y_3, 0)$, respectively, with $x_j = \cos \phi_j, y_j = \sin \phi_j$ ($j = 1, 2, 3$) and $\phi_1 + \phi_2 + \phi_3 = 0$.

Define the following:

$$\angle A = \angle CAB, \angle B = \angle ABC, \angle C = \angle BCA, \alpha = \angle BPC, \beta = \angle CPA, \gamma = \angle APB,$$

$$C_0 = \cos \alpha \cos \beta \cos \gamma, C_1 = \cos^2 \alpha, C_2 = \cos^2 \beta, C_3 = \cos^2 \gamma,$$

$$L = 2[(y_1 + y_2 + y_3)(1 - C_0) + (1 - x_1)y_1(C_1 - 1) + (1 - x_2)y_2(C_2 - 1) + (1 - x_3)y_3(C_3 - 1)],$$

$$R = 2[(1 + x_1 + x_2 + x_3)(1 - C_0) + (1 + x_1)x_1(C_1 - 1) + (1 + x_2)x_2(C_2 - 1) + (1 + x_3)x_3(C_3 - 1)],$$

$$H = 1 + 2C_0 - C_1 - C_2 - C_3, E = L^2 + (R + H)^2,$$

$$D = E^2 + 18H^2E + 8H(R + H)[(R + H)^2 - 3L^2] - 27H^4.$$

The following (proven) constraints hold:

$$1. H > 0 \text{ (which is equivalent to } \alpha + \beta + \gamma < 2\pi, \alpha < \beta + \gamma, \beta < \gamma + \alpha \text{ and } \gamma < \alpha + \beta);$$

$$2. \angle A + \beta + \gamma < 2\pi, \alpha + \angle B + \gamma < 2\pi, \alpha + \beta + \angle C < 2\pi;$$

$$3. \beta + \gamma - \alpha < 2(\angle B + \angle C), \gamma + \alpha - \beta < 2(\angle C + \angle A), \alpha + \beta - \gamma < 2(\angle A + \angle B);$$

$$4. \alpha < \angle A \rightarrow [\beta < \max\{\angle B, \angle C + \alpha\} \wedge \gamma < \max\{\angle C, \angle B + \alpha\}] ,$$

$$\beta < \angle B \rightarrow [\gamma < \max\{\angle C, \angle A + \beta\} \wedge \alpha < \max\{\angle A, \angle C + \beta\}] ,$$

$$\gamma < \angle C \rightarrow [\alpha < \max\{\angle A, \angle B + \gamma\} \wedge \beta < \max\{\angle B, \angle A + \gamma\}] ;$$

$$5. \alpha < \angle A \rightarrow \cos \angle C \cos \beta + \cos \angle B \cos \gamma > 0 ,$$

$$\beta < \angle B \rightarrow \cos \angle A \cos \gamma + \cos \angle C \cos \alpha > 0 ,$$

$$\gamma < \angle C \rightarrow \cos \angle B \cos \alpha + \cos \angle A \cos \beta > 0 ;$$

$$6. (D > 0 \wedge \alpha < \angle A) \rightarrow (\beta < \angle B \vee \gamma < \angle C) ,$$

$$(D > 0 \wedge \beta < \angle B) \rightarrow (\gamma < \angle C \vee \alpha < \angle A) ,$$

$$(D > 0 \wedge \gamma < \angle C) \rightarrow (\alpha < \angle A \vee \beta < \angle B) ;$$

$$7. D > 0 \rightarrow [\alpha < \angle A \vee \alpha > \pi - \angle A \vee \beta < \angle B \vee \beta > \pi - \angle B \vee \gamma < \angle C \vee \gamma > \pi - \angle C].$$

Also, experimental evidence suggests that this collection of constraints is necessary and sufficient for a suitable tetrahedron ABCP to exist for a given acute triangle ABC. At present though, the sufficiency claim is only a conjecture. (Note: the arrows mean logical implication here.)