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Some claims concerning a tetrahedron ABCP based on an acute triangle ABC

Scale and position the tetrahedron so that the coordinates of A, B and C are $(x_1, y_1, 0)$, $(x_2, y_2, 0)$, $(x_3, y_3, 0)$, respectively, with $x_j = \cos \phi_j$, $y_j = \sin \phi_j$ $(j = 1,2,3)$, and $\phi_1 + \phi_2 + \phi_3 = 0$.

Define the following:

$\angle A = \angle CAB$, $\angle B = \angle ABC$, $\angle C = \angle BCA$, $\alpha = \angle BPC$, $\beta = \angle CPA$, $\gamma = \angle APB$,

$C_0 = \cos \alpha \cos \beta \cos \gamma$, $C_1 = \cos^2 \alpha$, $C_2 = \cos^2 \beta$, $C_3 = \cos^2 \gamma$,

$L = 2 [(y_1 + y_2 + y_3)(1 - C_0) + (1 - x_1) y_1 (C_1 - 1) + (1 - x_2) y_2 (C_2 - 1) + (1 - x_3) y_3 (C_3 - 1)]$,

$R = 2 [(1 + x_1 + x_2 + x_3)(1 - C_0) + (1 + x_1) x_1 (C_1 - 1) + (1 + x_2) x_2 (C_2 - 1) + (1 + x_3) y_3 (C_3 - 1)]$,

$H = 1 + 2 C_0 - C_1 - C_2 - C_3$, $\qquad E = L^2 + (R+H)^2$,

$D = E^2 + 18\ E H^2 + 8 (R+H) H [(R+H)^2 - 3L^2] - 27 H^4$.

The following (proven) constraints hold:

1. $H > 0$ (which is equivalent to $\alpha+\beta+\gamma < 2\pi$, $\alpha < \beta+\gamma$, $\beta < \alpha+\gamma$, $\gamma < \alpha+\beta$);

2. $\angle A + \beta + \gamma < 2\pi$, $\alpha + \angle B + \gamma < 2\pi$, $\alpha + \beta + \angle C < 2\pi$;

3. $\beta + \gamma - \alpha < 2 (\angle B + \angle C)$, $\gamma + \alpha - \beta < 2 (\angle C + \angle A)$, $\alpha + \beta - \gamma < 2 (\angle A + \angle B)$;

4. $\alpha < \angle A \rightarrow [\beta < \max \{ \angle B, \angle C + \alpha \} \wedge \gamma < \max \{ \angle C, \angle B+\alpha \}]$,
    $\beta < \angle B \rightarrow [\gamma < \max \{ \angle C, \angle A + \beta \} \wedge \alpha < \max \{ \angle A, \angle C+\beta \}]$,
    $\gamma < \angle C \rightarrow [\alpha < \max \{ \angle A, \angle B + \gamma \} \wedge \beta < \max \{ \angle B, \angle A+\gamma \}]$;

5. $\alpha < \angle A \rightarrow \cos \angle C \cos \beta + \cos \angle B \cos \gamma > 0$,
    $\beta < \angle B \rightarrow \cos \angle A \cos \gamma + \cos \angle C \cos \alpha > 0$,
    $\gamma < \angle C \rightarrow \cos \angle B \cos \alpha + \cos \angle A \cos \beta > 0$;

6. $(D > 0 \wedge \alpha < \angle A) \rightarrow (\beta < \angle B \vee \gamma < \angle C)$,
    $(D > 0 \wedge \beta < \angle B) \rightarrow (\gamma < \angle C \vee \alpha < \angle A)$,
    $(D > 0 \wedge \gamma < \angle C) \rightarrow (\alpha < \angle A \vee \beta < \angle B)$;

7. $D > 0 \rightarrow [\alpha < \angle A \vee \alpha > \pi - \angle A \vee \beta < \angle B \vee \beta > \pi - \angle B \vee \gamma < \angle C \vee \gamma > \pi - \angle C]$.

Also, experimental evidence strongly suggests that this collection of constraints is necessary and sufficient for a suitable tetrahedron ABCP to exist for a given acute triangle ABC. At present though, the sufficiency claim is only a conjecture.