

Some claims concerning a tetrahedron based on an acute triangle ABC

Consider a tetrahedron ABCP such that the triangle ABC is acute. Without altering its shape, scale and position this tetrahedron in Cartesian space so that the coordinates of A, B and C are $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ and $(x_3, y_3, 0)$, respectively, with $x_j = \cos \phi_j, y_j = \sin \phi_j$ ($j = 1, 2, 3$) and $\phi_1 + \phi_2 + \phi_3 = \pi$ for angles ϕ_1, ϕ_2 and ϕ_3 . (This is straightforward to achieve.)

Define the following:

$$\begin{aligned}\angle A &= \angle CAB, \angle B = \angle ABC, \angle C = \angle BCA, \alpha = \angle BPC, \beta = \angle CPA, \gamma = \angle APB, \\ C_0 &= \cos \alpha \cos \beta \cos \gamma, C_1 = \cos^2 \alpha, C_2 = \cos^2 \beta, C_3 = \cos^2 \gamma, \\ L &= 2[(y_1 + y_2 + y_3)(1 - C_0) + (1 - x_1)y_1(C_1 - 1) + (1 - x_2)y_2(C_2 - 1) + (1 - x_3)y_3(C_3 - 1)], \\ R &= 2[(1 + x_1 + x_2 + x_3)(1 - C_0) + (1 + x_1)x_1(C_1 - 1) + (1 + x_2)x_2(C_2 - 1) + (1 + x_3)x_3(C_3 - 1)], \\ H &= 1 + 2C_0 - C_1 - C_2 - C_3, \quad E = L^2 + (R + H)^2, \\ D &= E^2 + 18H^2E + 8H(R + H)[(R + H)^2 - 3L^2] - 27H^4.\end{aligned}$$

The following (proven) constraints hold:

1. $H > 0$ (which is equivalent to $\alpha + \beta + \gamma < 2\pi$, $\alpha < \beta + \gamma$, $\beta < \gamma + \alpha$ and $\gamma < \alpha + \beta$);
2. $\angle A + \beta + \gamma < 2\pi$, $\alpha + \angle B + \gamma < 2\pi$, $\alpha + \beta + \angle C < 2\pi$;
3. $\beta + \gamma - \alpha < 2(\angle B + \angle C)$, $\gamma + \alpha - \beta < 2(\angle C + \angle A)$, $\alpha + \beta - \gamma < 2(\angle A + \angle B)$;
4. $\alpha < \angle A \rightarrow [\beta < \max\{\angle B, \angle C + \alpha\} \wedge \gamma < \max\{\angle C, \angle B + \alpha\}]$,
 $\beta < \angle B \rightarrow [\gamma < \max\{\angle C, \angle A + \beta\} \wedge \alpha < \max\{\angle A, \angle C + \beta\}]$,
 $\gamma < \angle C \rightarrow [\alpha < \max\{\angle A, \angle B + \gamma\} \wedge \beta < \max\{\angle B, \angle A + \gamma\}]$;
5. $\alpha < \angle A \rightarrow \cos \angle C \cos \beta + \cos \angle B \cos \gamma > 0$,
 $\beta < \angle B \rightarrow \cos \angle A \cos \gamma + \cos \angle C \cos \alpha > 0$,
 $\gamma < \angle C \rightarrow \cos \angle B \cos \alpha + \cos \angle A \cos \beta > 0$;
6. $(D > 0 \wedge \alpha < \angle A) \rightarrow (\beta < \angle B \vee \gamma < \angle C)$,
 $(D > 0 \wedge \beta < \angle B) \rightarrow (\gamma < \angle C \vee \alpha < \angle A)$,
 $(D > 0 \wedge \gamma < \angle C) \rightarrow (\alpha < \angle A \vee \beta < \angle B)$;
7. $D > 0 \rightarrow [\alpha < \angle A \vee \alpha > \pi - \angle A \vee \beta < \angle B \vee \beta > \pi - \angle B \vee \gamma < \angle C \vee \gamma > \pi - \angle C]$.

The first three items in this list are rather easily proved. The remaining four items require a significantly more technical analysis. Also, experimental evidence suggests that this collection of constraints is necessary and sufficient for a suitable tetrahedron ABCP to exist for a given acute triangle ABC. At present though, the sufficiency claim is only a conjecture. (Note: the arrows mean logical implication here.)