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Some claims concerning a tetrahedron ABCP based on an acute triangle ABC

Scale and position the tetrahedron so that the coordinates of A, B and C are $(x_1, y_1, 0)$, $(x_2, y_2, 0)$, $(x_3, y_3, 0)$, respectively, with $x_j = \cos \phi_j$, $y_j = \sin \phi_j$ ($j = 1,2,3$), and $\phi_1 + \phi_2 + \phi_3 = 0$.

Define the following:

$\angle A = \angle \mathrm{CAB}$, $\angle B = \angle \mathrm{ABC}$, $\angle C = \angle \mathrm{BCA}$, $\alpha = \angle \mathrm{BPC}$, $\beta = \angle \mathrm{CPA}$, $\gamma = \angle \mathrm{APB}$,

$C_0 = \cos\alpha \cos\beta \cos\gamma$, $C_1 = \cos^2\alpha$, $C_2 = \cos^2\beta$, $C_3 = \cos^2\gamma$,

$L = 2\,[\,(y_1 + y_2 + y_3)(1 - C_0) + (1 - x_1)\,y_1\,(C_1 - 1) + (1 - x_2)\,y_2\,(C_2 - 1) + (1 - x_3)\,y_3\,(C_3 - 1)\,]$,

$R = 2\,[\,(1 + x_1 + x_2 + x_3)(1 - C_0) + (1 + x_1)\,x_1\,(C_1 - 1) + (1 + x_2)\,x_2\,(C_2 - 1) + (1 + x_3)\,y_3\,(C_3 - 1)\,]$,

$H = 1 + 2\,C_0 - C_1 - C_2 - C_3$, $E = L^2 + (R{+}H)^2$,

$D = E^2 + 18\;EH^2 + 8\,(R{+}H)\,H\,[\,(R{+}H)^2 - 3L^2\,] - 27\,H^4$.

The following (proven) constraints hold:

1. $H > 0$ (which is equivalent to $\alpha{+}\beta{+}\gamma < 2\pi$, $\alpha < \beta{+}\gamma$, $\beta < \alpha{+}\gamma$, $\gamma < \alpha{+}\beta$);

2. $\angle A + \beta + \gamma < 2\pi$, $\alpha + \angle B + \gamma < 2\pi$, $\alpha + \beta + \angle C < 2\pi$;

3. $\beta + \gamma - \alpha < 2\,(\angle B + \angle C)$, $\gamma + \alpha - \beta < 2\,(\angle C + \angle A)$, $\alpha + \beta - \gamma < 2\,(\angle C + \angle A)$;

4. $\alpha < \angle A \;\rightarrow\; [\,\beta < \max\{\angle B, \angle C + \alpha\} \;\wedge\; \gamma < \max\{\angle C, \angle B{+}\alpha\}\,]$,
   $\beta < \angle B \;\rightarrow\; [\,\gamma < \max\{\angle C, \angle A + \beta\} \;\wedge\; \alpha < \max\{\angle A, \angle C{+}\beta\}\,]$,
   $\gamma < \angle C \;\rightarrow\; [\,\alpha < \max\{\angle A, \angle B + \gamma\} \;\wedge\; \beta < \max\{\angle B, \angle A{+}\gamma\}\,]$;

5. $\alpha < \angle A \;\rightarrow\; \cos\angle C \cos\beta + \cos\angle B \cos\gamma > 0$,
   $\beta < \angle B \;\rightarrow\; \cos\angle A \cos\gamma + \cos\angle C \cos\alpha > 0$,
   $\gamma < \angle C \;\rightarrow\; \cos\angle B \cos\alpha + \cos\angle A \cos\beta > 0$;

6. $(D > 0 \;\wedge\; \alpha < \angle A) \;\rightarrow\; (\beta < \angle B \;\vee\; \gamma < \angle C)$,
   $(D > 0 \;\wedge\; \beta < \angle B) \;\rightarrow\; (\gamma < \angle C \;\vee\; \alpha < \angle A)$,
   $(D > 0 \;\wedge\; \gamma < \angle C) \;\rightarrow\; (\alpha < \angle A \;\vee\; \beta < \angle B)$.

Also, experimental evidence strongly suggests that this collection of constraints is also sufficient for a suitable tetrahedron ABCP to exist for a given acute triangle ABC. At present though, this is just a conjecture.