14

UNIVERSALITY AND LIMITS

14.1 UNIVERSAL ARROWS

A **universal property** is commonly given in mathematics by some conditions of existence and uniqueness on morphisms, representing some sort of natural isomorphism. They can be used to define certain constructions up to isomorphism and to operate with them in an abstract setting. We will formally introduce universal properties using *universal arrows* from an object c to a functor S; the property of these arrows is that every arrow of the form $c \to Sd$ will factor uniquely through the universal arrow.

Definition 60 (Universal arrow)**.** A **universal arrow** from c to S is a morphism $u\colon c \to Sr$ such that for every $g\colon c \to Sd$ exists a unique morphism $f\colon r \to d$ making this diagram commute

$$\begin{tikzcd}
 & Sd & d \\
c \arrow[ur, "g"] \arrow[r, "u"'] & Sr \arrow[u, dashed, "Sf"'] & r \arrow[u, dashed, "\exists! f"']
\end{tikzcd}.$$

Note how an universal arrow is, equivalently, the initial object of the comma category $(c \downarrow S)$. Thus, universal arrows must be unique up to isomorphism.

Proposition 9 (Universality in terms of hom-sets)**.** *The arrow $u\colon c \to Sr$ is universal if and only if $f \mapsto Sf \circ u$ is a bijection* $\hom(r,d) \cong \hom(c,Sd)$ *natural in d. Any natural bijection of this kind is determined by a unique universal arrow.*

Proof. On the one hand, given an universal arrow, bijectivity follows from the definition of universal arrow; and naturality follows from the fact that $S(gf) \circ u = Sg \circ Sf \circ u$.

On the other hand, given a bijection φ, we define $u = \varphi(\mathrm{id}_r)$. By naturality, we have the bijection $\varphi(f) = Sf \circ u$, and every arrow is written in this way. $\square$

The categorical dual of an universal arrow from an object to a functor is the notion of universal arrow from a functor to an object. Note how, particularly in this case, we avoid the name *couniversal arrow*; as both arrows are representing what we usually call a *universal property*.

Definition 61 (Dual universal arrow)**.** A **universal arrow** from S to c is a morphism $v\colon Sr \to c$ such that for every $g\colon Sd \to c$ exists a unique morphism $f\colon d \to r$ making this diagram commute

$$\begin{tikzcd}
d \arrow[d, dashed, "\exists ! f"'] & Sd \arrow[d, dashed, "Sf"'] \arrow[dr, "g"] & \\
r & Sr \arrow[r, "v"'] & c
\end{tikzcd}$$

14.2 REPRESENTABILITY

Definition 62 (Representation of a functor)**.** A **representation** of a functor from an arbitrary category to the category of sets, $K\colon D \to \mathsf{Set}$, is a natural isomorphism making it isomorphic to a partially applied hom-functor,

$$\psi\colon \ \hom_D(r, -) \cong K.$$

A functor is *representable* if it has a representation. The object r is called a *representing object*. Note that, for this definition to work, D must have small hom-sets.

Proposition 10 (Representations in terms of universal arrows)**.** *If $u\colon * \to Kr$ is a universal arrow for a functor $K\colon D \to \mathsf{Set}$, then $f \mapsto K(f)(u*)$ is a representation. Every representation is obtained in this way.*

Proof. We know that $\hom(*, X) \xrightarrow{.} X$ is a natural isomorphism in X; in particular $\hom(*, K-) \xrightarrow{.} K-$. Every representation is built then as

$$\hom_D(r, -) \cong \hom(*, K-) \cong K,$$

for every natural isomorphism $D(r, -) \cong \mathsf{Set}(*, K-)$. But every natural isomorphism of this kind is universal arrow. $\square$

14.3 YONEDA LEMMA

Lemma 7 (Yoneda Lemma)**.** *For any $K\colon D \to \mathsf{Set}$ and $r \in D$, there is a bijection*

$$y\colon \mathrm{Nat}(\hom_D(r, -), K) \cong Kr$$

sending the natural transformation $\alpha\colon\ \hom_D(r, -) \xrightarrow{.} K$ to the image of the identity, $\alpha_r(\mathrm{id}_r)$.

Proof. The complete natural transformation α is determined by $\alpha_r(\mathrm{id}_r)$. By naturality, given any $f\colon r \to s$,

$$\begin{tikzcd}
\hom(r,r) \arrow[rrr, "\alpha_r"] \arrow[ddd, "f\circ -"'] & & & Kr \arrow[ddd, "Kf"] \\
& \mathrm{id}_r \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_r(\mathrm{id}_r) \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \alpha_s(f) & \\
\hom(r,s) \arrow[rrr, "\alpha_s"'] & & & Ks
\end{tikzcd}$$

it must be the case that $\alpha_s(f) = Kf(\alpha_r(\mathrm{id}_r))$. $\square$

Corollary 3 (Characterization of natural transformations between representable functors)**.** *Given $r, s \in D$, any natural transformation $\hom(r,-) \xrightarrow{\cdot} \hom(s,-)$ is of the form $- \circ h$ for a unique morphism $h\colon s \to r$.*

Proof. Using Yoneda Lemma (Lemma 7), we know that

$$\mathrm{Nat}(\hom_D(r,-), \hom_D(s,-)) \cong \hom_D(s,r),$$

sending the natural transformation to a morphism $\alpha(id_r) = h\colon s \to r$. The rest of the natural transformation is determined as $- \circ h$ by naturality. $\square$

Proposition 11 (Naturality of the Yoneda Lemma)**.** *The bijection on the Yoneda Lemma (Lemma 7),*

$$y\colon \mathrm{Nat}(\hom_D(r,-), K) \cong Kr,$$

is a natural isomorphism between two functors from $\mathsf{Set}^D \times D$ *to* Set.

Proof. We define $N\colon \mathsf{Set}^D \times D \to \mathsf{Set}$ on objects as $N\langle r, K\rangle = \mathrm{Nat}(\hom(r,-), K)$. Given $f\colon r \to r'$ and $F\colon K \xrightarrow{\cdot} K'$, the functor is defined on morphisms as

$$N\langle f, F\rangle(\alpha) = F \circ \alpha \circ (- \circ f) \in \mathrm{Nat}(\hom(r',-), K),$$

where $\alpha \in \mathrm{Nat}(\hom(r,-), K)$. We define $E\colon \mathsf{Set}^D \times D \to \mathsf{Set}$ on objects as $E\langle r, K\rangle = Kr$. Given $f\colon r \to r'$ and $F\colon K \xrightarrow{\cdot} K'$, the functor is defined on morphisms as

$$E\langle f, F\rangle(a) = F(Kf(a)) = K'f(Fa) \in K'r',$$

where $a \in Kr$, and the equality holds because of the naturality of F. The naturality of y is equivalent to the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{Nat}(\hom(r,-), K) \arrow[r, "y"] \arrow[d, "N\langle f,F\rangle"'] & Kr \arrow[d, "E\langle f,F\rangle"] \\
\mathrm{Nat}(\hom(r',-), K') \arrow[r, "y"] & K'r'
\end{tikzcd}$$

where, given any $\alpha \in \mathrm{Nat}(\hom(r,-),K)$, it follows from naturality of α that

$$\begin{aligned} y\big(N\langle f,F\rangle(\alpha)\big) &= y\big(F\circ\alpha\circ(-\circ f)\big) = F\circ\alpha\circ(-\circ f)(\mathrm{id}_{r'}) = F(\alpha(f)) \\ &= F(\alpha(\mathrm{id}_{r'}\circ f)) = F(Kf(\alpha_r(\mathrm{id}_r))) = E\langle f,F\rangle\alpha_r(\mathrm{id}_r) \\ &= E\langle f,F\rangle\big(y(\alpha)\big). \end{aligned}$$

□

Definition 63. In the conditions of the Yoneda Lemma (Lemma 7) the **Yoneda functor**, $Y\colon D^{op}\to \mathsf{Set}^D$, is defined with the arrow function

$$(f\colon s\to r)\mapsto \Big(\hom_D(f,-)\colon\ \hom_D(r,-)\to\hom_D(s,-)\Big).$$

It can be also written as $Y'\colon D\to\mathsf{Set}^{D^{op}}$.

Proposition 12. *The Yoneda functor is full and faithful.*

Proof. By Yoneda Lemma, we know that

$$y\colon\ \mathrm{Nat}(\hom(r,-),\hom(s,-))\cong\hom(s,r)$$

is a bijection, where $y(\hom(f,-)) = f$. □

14.4 LIMITS

In the definition of product, we chose two objects of the category, we considered all possible ***cones*** over two objects and we picked the universal one. Diagramtically,

$$\begin{tikzcd} & c \arrow[dl] \arrow[dr] & \\ a & & b \end{tikzcd} \qquad \begin{tikzcd} & d \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & a\times b \arrow[dl] \arrow[dr] & \\ a & & b \end{tikzcd}$$

c is a cone and $a\times b$ is the universal one: every cone factorizes through it. In this particular case, the base of each cone is given by two objects; or, in other words, by the image of a functor from the discrete category with only two objects, called the *index category*.

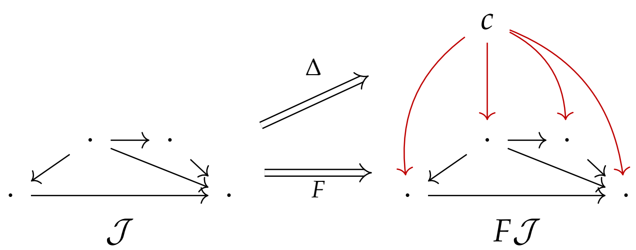
We will be able to create new constructions on categories by formalizing the notion of cone and generalizing to arbitrary bases, given as functors from arbitrariliy complex index categories. Constant functors are the first step into formalizing the notion of cone.

Definition 64 (Constant functor)**.** The **constant functor** $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ sends each object $c \in \mathcal{C}$ to a constant functor $\Delta c\colon \mathcal{J} \to \mathcal{C}$ defined as

- the constantly-c function for objects, $\Delta(j) = c$;
- and the constantly-id_c function for morphisms, $\Delta(f) = \mathrm{id}_c$.

The constant functor sends a morphism $g\colon c \to c'$ to a natural transformation $\Delta g\colon \Delta c \to \Delta c'$ whose components are all g.

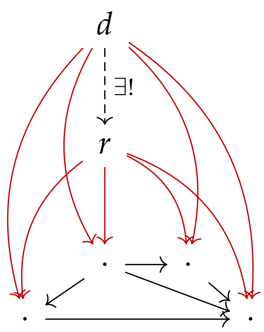
We could say that Δc squeezes the whole category $\mathcal{J}$ into c. A natural transformation from this functor to some other $F\colon \mathcal{J} \to \mathcal{C}$ should be regarded as a ***cone*** from the object c to a copy of $\mathcal{J}$ inside the category $\mathcal{C}$; as the following diagram exemplifies



The components of the natural transformation appear highlighted in the diagram. The naturality of the transformation implies that each triangle

$$\begin{tikzcd} & \cdot \arrow[dl] \arrow[dr] & \\ \cdot \arrow[rr] & & \cdot \end{tikzcd}$$

on that cone must be commutative. Thus, natural transformations are a way to recover all the information of an arbitrary ***index category*** $\mathcal{J}$ that was encoded in c by the constant functor. As we did with products, we want to find the cone that best encodes that information; a universal cone, such that every other cone factorizes through it. Diagramatically an r such that, for each d,



That factorization will be represented in the formal definition of limit by a universal natural transformation between the two constant functors.

Definition 65 (Limit)**.** The **limit** of a functor $F\colon \mathcal{J} \to \mathcal{C}$ is an object $r \in \mathcal{C}$ such that there exists a universal arrow $v\colon \Delta r \xrightarrow{\cdot} F$ from Δ to F. It is usually written as $r = \varprojlim F$.

That is, for every natural transformation $w \colon \Delta d \xrightarrow{\cdot} F$, there is a unique morphism $f \colon d \to r$ such that

$$\begin{tikzcd} d \arrow[d, dashed, "\exists! f"'] \\ r \end{tikzcd} \qquad \begin{tikzcd} \Delta d \arrow[d, dashed, "\Delta f"'] \arrow[dr, "w"] & \\ \Delta r \arrow[r, "v"'] & F \end{tikzcd}$$

commutes. This reflects directly on the universality of the cone we described earlier and proves that limits are unique up to isomorphism.

By choosing different index categories, we will be able to define multiple different constructions on categories as limits.

14.5 EXAMPLES OF LIMITS

For our first example, we will take the following category, called $\downarrow\downarrow$ as index category,

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] & \cdot \end{tikzcd}$$

A functor $F \colon \downarrow\downarrow \to \mathcal{C}$ is a pair of parallel arrows in $\mathcal{C}$. Limits of functors from this category are called **equalizers**. With this definition, the **equalizer** of two parallel arrows $f, g \colon a \to b$ is an object $\mathrm{eq}(f,g)$ with a morphism $e \colon \mathrm{eq}(f,g) \to a$ such that $f \circ e = g \circ e$; and such that any other object with a similar morphism factorizes uniquely through it

$$\begin{tikzcd} & d \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "e'"'] \arrow[ddr, bend left] & \\ & \mathrm{eq}(f,g) \arrow[dl, "e"'] \arrow[dr] & \\ a \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & & b \end{tikzcd}$$

note how the right part of the cone is completely determined as $f \circ e$. Because of this, equalizers can be written without specifying it, and the diagram can be simplified to

$$\begin{tikzcd} \mathrm{eq}(f,g) \arrow[r, "e"] & a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & b \\ d \arrow[u, dashed, "\exists!"] \arrow[ur, "e'"'] & & \end{tikzcd}$$

.

Example 19 (Equalizers in Sets). The equalizer of two parallel functions $f, g \colon A \to B$ in Set is $\{x \in A \mid f(a) = g(a)\}$ with the inclusion morphism. Given any other function

$h\colon D \to A$ such that $f \circ h = g \circ h$, we know that $f(h(d)) = g(h(d))$ for any $d \in D$. Thus, h can be factorized through the equalizer.

$$\begin{tikzcd}
\{x \in A \mid f(a) = g(a)\} \arrow[r, hook, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
D \arrow[u, dashed, "\exists!"] \arrow[ur, "h"'] & &
\end{tikzcd}$$

Example 20 (Kernels). In the category of abelian groups, the kernel of a function f, $\ker(f)$, is the equalizer of $f\colon G \to H$ and a function sending each element to the zero element of H. The same notion of kernel can be defined in the category of R-Modules, for any ring R.

Pullbacks are defined as limits whose index category is $\cdot \to \cdot \leftarrow \cdot$. Any functor from that category is a pair of arrows with a common codomain; and the pullback is the universal cone over them.

$$\begin{tikzcd}
& d \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "p'"'] \arrow[ddr, bend left, "q'"] \arrow[dd, bend right] & \\
& a \arrow[dl, bend right, "p"'] \arrow[dr, bend left, "q"] \arrow[d] & \\
x \arrow[r, "f"'] & z & y \arrow[l, "g"]
\end{tikzcd}$$

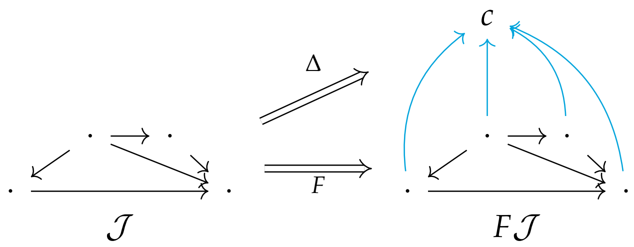
Again, the central arrow of the diagram is determined as $f \circ q = g \circ p$; so it can be ommited in the diagram. The usual definition of a pullback for two morphisms $f\colon x \to z$ and $g\colon y \to z$ is pair of morphisms $p\colon a \to x$ and $q\colon a \to y$ such that $f \circ q = g \circ p$ which are also universal, that is, given any pair of morphisms $p'\colon d \to x$ and $q'\colon d \to y$, there exists a unique $u\colon d \to a$ making the diagram commute. Usually we write the pullback object as $x \times_z y$ and we write this property diagramatically as

$$\begin{tikzcd}
d \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "p'"] \arrow[ddr, bend right, "q'"'] & & \\
& x \times_z y \arrow[r, "p"] \arrow[d, "q"'] & x \arrow[d, "f"] \\
& y \arrow[r, "g"'] & z
\end{tikzcd}$$

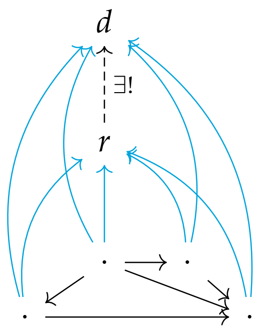
The square in this diagram is usually called a *pullback square*, and the pullback object is usually called a *fibered product*.

14.6 COLIMITS

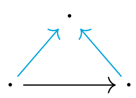
A colimit is the dual notion of a limit. We could consider cocones to be the dual of cones and pick the universal one. Once an index category $\mathcal{J}$ and a base category $\mathcal{C}$ are fixed, a ***cocone*** is a natural transformation from a functor on the base category to a constant functor. Diagramatically,



is an example of a cocone, and the universal one would be the r, such that, for each cone d,



and naturality implies that each triangle



commutes.

Definition 66 (Colimits)**.** The **colimit** of a functor $F\colon J \to \mathcal{C}$ is an object $r \in \mathcal{C}$ such that there exists a universal arrow $u\colon F \xrightarrow{\cdot} \Delta r$ from F to Δ. It is usually written as $r = \varinjlim F$.

That is, for every natural transformation $w\colon F \xrightarrow{\cdot} \Delta d$, there is a unique morphism $f\colon r \to d$ such that

$$\begin{tikzcd} d & & \Delta d & \\ r \arrow[u, dashed, "\exists! f"] & & \Delta r \arrow[u, dashed, "\Delta f"] & F \arrow[l, "v"] \arrow[ul, "w"'] \end{tikzcd}$$

commutes. This reflects directly on the universality of the cocone we described earlier and proves that colimits are unique up to isomorphism.

14.7 EXAMPLES OF COLIMITS

Coequalizers are the dual of *equalizers*; colimits of functors from $\downarrow\downarrow$. The coequalizer of two parallel arrows is an object $\mathrm{coeq}(f,g)$ with a morphism $e\colon b \to \mathrm{coeq}(f,g)$ such that $e \circ f = e \circ g$; and such that any other object with a similar morphism factorizes uniquely through it

$$\begin{tikzcd}
& d & \\
& \mathrm{coeq}(f,g) \arrow[u, dashed, "\exists!"] & \\
a \arrow[ur] \arrow[uur, bend left] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & b \arrow[ul, "e"'] \arrow[uul, bend right, "e'"']
\end{tikzcd}$$

as the right part of the cocone is completely determined by the left one, the diagram can be written as

$$\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "e"] \arrow[dr, "e'"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists!"] \\
& & d
\end{tikzcd}$$

Example 21 (Coequalizers in Sets). The coequalizer of two parallel functions $f,g\colon A \to B$ in Set is $B/(\sim_{f=g})$, where $\sim_{f=g}$ is the minimal equivalence relation in which we have $f(a) \sim g(a)$ for each $a \in A$. Given any other function $h\colon B \to D$ such that $h(f(a)) = h(g(a))$, it can be factorized in a unique way by $h'\colon B/\sim_{f=g} \to D$.

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] \arrow[dr, "e'"'] & B/(\sim_{f=g}) \arrow[d, dashed, "\exists!"] \\
& & D
\end{tikzcd}$$

Pushouts are the dual of pullbacks; colimits whose index category is $\cdot \leftarrow \cdot \rightarrow \cdot$, that is, the dual of the index category for pullbacks. Diagramatically,

$$\begin{tikzcd}
& d & \\
& a \arrow[u, dashed, "\exists!"] & \\
x \arrow[ur, bend left, "p"] \arrow[uur, bend left, "p'"] & z \arrow[l, "f"] \arrow[r, "g"'] \arrow[u] \arrow[uu, bend left] & y \arrow[ul, bend right, "q"'] \arrow[uul, bend right, "q'"']
\end{tikzcd}$$

and we can define the pushout of two morphisms $f\colon z \to x$ and $g\colon z \to y$ as a pair of morphisms $p\colon x \to a$ and $q\colon y \to a$ such that $p \circ f = q \circ g$ which are also universal, that is, given any pair of morphisms $p' \colon x \to d$ and $q' \colon y \to d$, there exists a unique $u \colon a \to d$ making the diagram commute.

$$
\begin{tikzcd}
z \arrow[r, "g"] \arrow[d, "f"'] & y \arrow[d, "q"] \arrow[ddr, bend left, "q'"] & \\
x \arrow[r, "p"] \arrow[drr, bend right, "p'"'] & x \amalg_z y \arrow[dr, dashed, "\exists! u"'] & \\
 & & d
\end{tikzcd}
$$

The square in this diagram is usually called a *pushout square*, and the pullback object is usually called a *fibered coproduct*.