Theorem 8. *In intuitionistic logic, the double negation of the Law of Excluded Middle holds for every proposition, that is,*

$$\forall A : \neg\neg(A \vee \neg A)$$

Proof. Suppose $\neg(A \vee \neg A)$. We are going to prove first that, under this specific assumption, $\neg A$ holds. If A were true, $A \vee \neg A$ would be true and we would arrive to a contradition, so $\neg A$. But then, if we have $\neg A$ we also have $A \vee \neg A$ and we arrive to a contradiction with the assumption. We should conclude that $\neg\neg(A \vee \neg A)$. □

Note that this is, in fact, an intuitionistic proof. Although it seems to use the intuitionistically forbidden technique of proving by contradiction, it is actually only proving a negation. There is a difference between assuming A to prove $\neg A$ and assuming $\neg A$ to prove A: the first one is simply a proof of a negation, the second one uses implicitly the law of excluded middle.

This can be translated to the Mikrokosmos implementation of simply typed λ-calculus as the term

```
notnotlem = \f.absurd (f (inr (\a.f (inl a))))
notnotlem
--- [1]: λa.(ABSURD (a (INR λb.(a (INL b))))) :: ((A + (A → ⊥)) → ⊥) → ⊥
```

whose type is precisely $((\mathtt{A} + (\mathtt{A} \longrightarrow \bot)) \longrightarrow \bot) \longrightarrow \bot$. The derivation tree can be seen directly on the interpreter as Figure 1 shows.

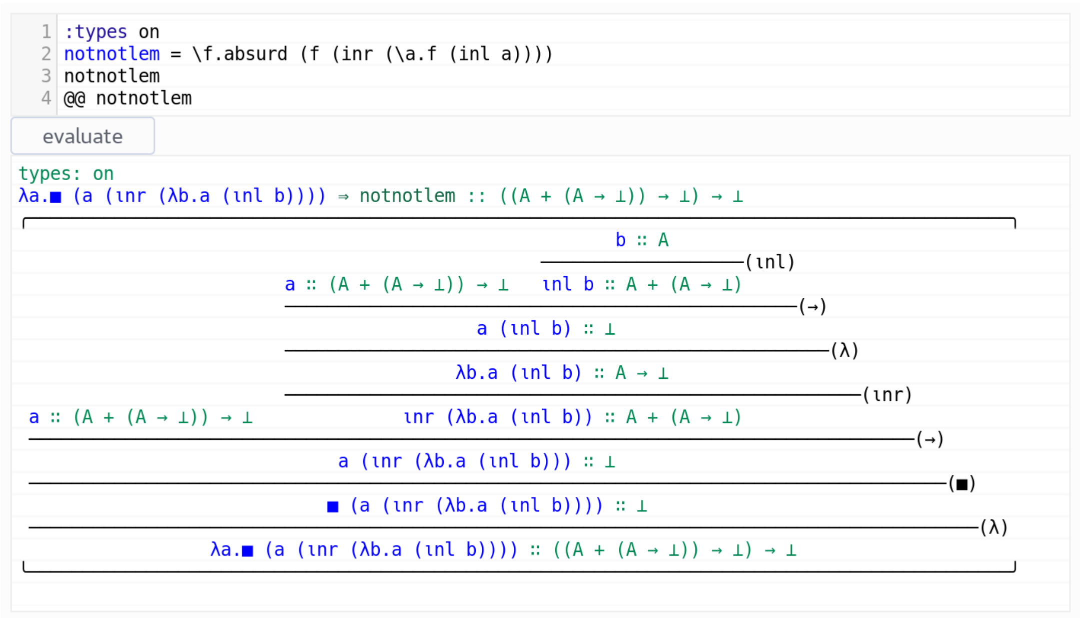

Figure 5: Proof of the double negation of LEM.

Part III

CATEGORY THEORY

11

CATEGORIES

We will think of a category as the algebraic structure that captures the notion of composition. A category will be built from some sort of objects linked by composable arrows; to which associativity and identity laws will apply.

Thus, a category has to rely in some notion of ***collection***. When interpreted inside set-theory, it is common to use this term to denote some unspecified formal notion of compilation of entities that could be given by sets or proper classes. We will want to define categories whose objects are all the possible sets and we will need the objects to form a proper class in order to avoid inconsistent results such as the Russell's paradox. This is why we will consider, from this approach, a particular class of categories of small set-theoretical size to be specially well-behaved.

Definition 29 (Small and locally small categories)**.** A category will be said to be **small** if the collection of its objects can be given by a set (instead of a proper class). It will be said to be **locally small** if the collection of arrows between any two objects can be given by a set.

A different approach, however, would be to simply take the *objects* and the *arrows* as fundamental concepts of our theory. These foundational concerns will not cause any explicit problem in this presentation of category theory, so we will keep them deliberately open to both interpretations.

11.1 DEFINITION OF CATEGORY

Definition 30 (Category)**.** A **category** $\mathcal{C}$, as defined in [Lan78], is given by

- $\mathcal{C}_0$ (sometimes denoted $\mathrm{obj}(\mathcal{C})$ or simply $\mathcal{C}$), a *collection* whose elements are called **objects**, and
- $\mathcal{C}_1$, a *collection* whose elements are called **morphisms**.

Every morphism $f \in \mathcal{C}_1$ is assigned two objects: a **domain**, written as $\mathrm{dom}(f) \in \mathcal{C}_0$, and a **codomain**, written as $\mathrm{cod}(f) \in \mathcal{C}_0$; a common notation for such morphism is

$$f : \mathrm{dom}(f) \longrightarrow \mathrm{cod}(f).$$

Given two morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$, there exists a **composition morphism**, written as $g \circ f : A \longrightarrow C$; or simply by yuxtaposition, as gf. Morphism composition is a binary associative operation with an identity element $\mathrm{id}_A : A \longrightarrow A$ for every object A, that is,

$$h \circ (g \circ f) = (h \circ g) \circ f \quad \text{and} \quad f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f,$$

for any f, g, h, composable morphisms.

Definition 31 (Hom-sets)**.** The **hom-set** of two objects A, B on a category is the collection of morphisms between them. It is written as $\hom(A, B)$. The set of **endomorphisms** of an object A is defined as $\mathrm{end}(A) = \hom(A, A)$.

We can use a subscript, as in $\hom_C(A, B)$ to explicitly specify the category we are working in when necessary.

11.2 MORPHISMS

Objects in category theory are an atomic concept and can be only studied by their morphisms; that is, by how they are related to all the objects of the category. Thus, the essence of a category is given not by its objects, but by the morphisms between them and how composition is defined.

It is so much so, that we will consider two objects essentially equivalent (and we will call them ***isomorphic***) whenever they relate to other objects in the exact same way; that is, whenever an invertible morphism between them exists. This will constitute an equivalence relation on the category.

In a certain sense, morphisms are an abstraction of the notion of the structure-preserving homomorphisms that are defined between algebraic structures. From this perspective, ***monomorphisms*** and ***epimorphisms*** can be thought as abstractions of the usual injective and surjective homomorphisms. We will see, however, how some properties that we take for granted, such as "isomorphism" meaning exactly the same as "both injective and surjective", are not true in general.

Definition 32 (Isomorphisms)**.** A morphism $f : A \longrightarrow B$ is an **isomorphism** if there exist a morphism $f^{-1} : B \longrightarrow A$ such that

- $f^{-1} \circ f = \mathrm{id}_A$,
- $f \circ f^{-1} = \mathrm{id}_B$.

This morphism is called an *inverse morphism.*

We call **automorphisms** to the morphisms which are both endomorphisms and isomorphisms.

Proposition 2 (Unicity of inverses)**.** *If the inverse of a morphism exists, it is unique. In fact, if a morphism has a left-side inverse and a right-side inverse, they are both-sided inverses and they are equal.*

Proof. Given $f : A \longrightarrow B$ with inverses $g_1, g_2 : B \longrightarrow A$; we have that

$$g_1 = g_1 \circ \mathrm{id}_A = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \mathrm{id} \circ g_2 = g_2.$$

We have used associativity of composition, neutrality of the identity and the fact that g_1 is a left-side inverse and g_2 is a right-side inverse. □

Definition 33. Two objects are **isomorphic** if an isomorphism between them exists. We write $A \cong B$ when A and B are isomorphic.

Proposition 3 (Isomorphy is an equivalence relation)**.** *The relation of being isomorphic is an equivalence relation. In particular,*

- *the identity,* $\mathrm{id} = \mathrm{id}^{-1}$*;*
- *the inverse of an isomorphism,* $(f^{-1})^{-1} = f$*;*
- *and the composition of isomorphisms,* $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$*;*

are all isomorphisms.

Proof. We can check that those are in fact inverses. From their existance follows

- reflexivity, $A \cong A$;
- symmetry, $A \cong B$ implies $B \cong A$;
- transitivity, $A \cong B$ and $B \cong C$ imply $A \cong C$.

□

Definition 34 (Monomorphisms and epimorphisms)**.** A **monomorphism** is a left-cancellable morphism, that is, $f : A \longrightarrow B$ is a monomorphism if, for every $g, h : B \longrightarrow A$,

$$f \circ g = f \circ h \implies g = h.$$

An **epimorphism** is a right-cancellable morphism, that is, $f : A \longrightarrow B$ is an epimorphism if, for every $g, h : B \longrightarrow A$,

$$g \circ f = h \circ f \implies g = h.$$

A morphism that is a monomorphism and an epimorphism at the same time is called a **bimorphism**.

Remark 1. A morphism can be a bimorphism without being an isomorphism. We will cover examples of this fact later.

Definition 35 (Retractions and sections)**.** A **retraction** is a left inverse, that is, a morphism that has a right inverse; conversely, a **section** is a right inverse, a morphism that has a left inverse.

By virtue of Proposition 2, a morphism that is both a retraction and a section is an isomorphism. Thus, not every epimorphism is a section and not every monomorphism is a retraction.