is denoted by $\tau' \circ \tau : S'S \longrightarrow T'T$ and is defined as the family of morphisms given by $\tau' \circ \tau = T'\tau \circ \tau' = \tau' \circ S'\tau$, that is, by the diagonal of the following commutative square

$$\begin{tikzcd}
S'Sc \arrow[r, "\tau'_{Sc}"] \arrow[d, "S'\tau_c"'] \arrow[dr, "(\tau' \circ \tau)_c"] & T'Sc \arrow[d, "T'\tau_c"] \\
S'Tc \arrow[r, "\tau'_{Tc}"] & T'Tc
\end{tikzcd}$$

Proposition 6 (Horizontal composition is a natural transformation)**.** *The horizontal composition of two natural transformations is in fact a natural transformation.*

Proof. It is natural as the following diagram is the composition of two naturality squares

$$\begin{tikzcd}
S'Sc \arrow[r, "S'\tau"] \arrow[d, "S'Sf"] & S'Tc \arrow[r, "\tau'"] \arrow[d, "S'Tf"] & T'Tc \arrow[d, "T'Tf"] \\
S'Sb \arrow[r, "S'\tau"] & S'Tb \arrow[r, "\tau'"] & T'Tb
\end{tikzcd}$$

defined respectively by the naturality of $S'\tau$ and τ'. □

13

CONSTRUCTIONS ON CATEGORIES

13.1 PRODUCT CATEGORIES

Definition 51 (Product category)**.** The **product category** of two categories $\mathcal{C}$ and $\mathcal{D}$, denoted by $\mathcal{C} \times \mathcal{D}$ is a category having

- pairs $\langle c, d \rangle$ as objects, where $c \in \mathcal{C}$ and $d \in \mathcal{D}$;
- and pairs $\langle f, g \rangle : \langle c, d \rangle \longrightarrow \langle c', d' \rangle$ as morphisms, where $f : c \longrightarrow c'$ and $g : d \longrightarrow d'$ are morphisms in their respective categories.

The identity morphism of any object $\langle c, d \rangle$ is $\langle \mathrm{id}_c, \mathrm{id}_d \rangle$, and composition is defined componentwise as

$$\langle f', g' \rangle \circ \langle f, g \rangle = \langle f' \circ f, g' \circ g \rangle.$$

We also define **projection functors** $P : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{C}$ and $Q : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{D}$ on arrows as $P\langle f, g \rangle = f$ and $Q\langle f, g \rangle = g$. Note that this definition of product, using these projections, would be the product of two categories on a category of categories with functors as morphisms.

Definition 52 (Product of functors)**.** The **product functor** of two functors $F : \mathcal{C} \longrightarrow \mathcal{C}'$ and $G : \mathcal{D} \longrightarrow \mathcal{D}'$ is a functor $F \times G : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{C}' \times \mathcal{D}'$ which can be defined

- on objects as $(F \times G)\langle c, d \rangle = \langle Fc, Gd \rangle$;
- and on arrows as $(F \times G)\langle f, g \rangle = \langle Ff, Gg \rangle$.

It can be seen as the unique functor making the following diagram commute

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "F"] & \mathcal{C} \times \mathcal{D} \arrow[l, "P"'] \arrow[r, "Q"] \arrow[d, dashed, "F \times G"] & \mathcal{D} \arrow[d, "G"] \\
\mathcal{C}' & \mathcal{C}' \times \mathcal{D}' \arrow[l, "P'"] \arrow[r, "Q'"'] & \mathcal{D}'
\end{tikzcd}$$

In this sense, the $\times$ operation is itself a functor acting on objects and morphisms of the `Cat` category of all categories. A **bifunctor** is a functor from a product category; and it also can be seen as a functor on two variables. As we will show in the following proposition, it is completely determined by the two families of functors that we obtain when we fix any of the elements.

Proposition 7 (Conditions for the existence of bifunctors)**.** *Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ categories with two families of functors*

$$\{L_c : \mathcal{B} \longrightarrow \mathcal{D}\}_{c \in \mathcal{C}} \quad \textit{and} \quad \{M_b : \mathcal{C} \longrightarrow \mathcal{D}\}_{b \in \mathcal{B}},$$

such that $M_b(c) = L_c(b)$ for all b, c. A bifunctor $S : \mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{D}$ such that $S(-, c) = L_c$ and $S(b, -) = M_b$ for all b, c exists if and only if for every $f : b \longrightarrow b'$ and $g : c \longrightarrow c'$,

$$M_{b'} g \circ L_c f = L_{c'} f \circ M_b g.$$

Proof. If the equality holds, the bifunctor can be defined as $S(b, c) = M_b(c) = L_c(b)$ in objects and as $S(f, g) = M_{b'} g \circ L_c f = L_{c'} f \circ M_b g$ on morphisms. This bifunctor preserves identities, as

$$S(\mathrm{id}_b, \mathrm{id}_c) = M_b(\mathrm{id}_c) \circ L_c(\mathrm{id}_b) = \mathrm{id}_{M_b(c)} \circ \mathrm{id}_{L_c(b)} = \mathrm{id},$$

and it preserves composition, as

$$S(f', g') \circ S(f, g) = Mg' \circ Lf' \circ Mg \circ Lf = Mg' \circ Mg \circ Lf' \circ Lf = S(f' \circ f, g' \circ g)$$

for any composable morphisms f, f', g, g'. On the other hand, if a bifunctor exists,

$$\begin{aligned} M_{b'}(g) \circ L_c(f) &= S(\mathrm{id}_{b'}, g) \circ S(f, \mathrm{id}_c) = S(\mathrm{id}_{b'} \circ f, g \circ \mathrm{id}_c) \\ &= S(f \circ \mathrm{id}_b, \mathrm{id}_{c'} \circ g) = S(f, \mathrm{id}_{c'}) \circ S(\mathrm{id}_b, g) \\ &= L_{c'}(f) \circ M_b(f). \end{aligned}$$

□

Proposition 8 (Naturality for bifunctors)**.** *Given S, S' bifunctors, $\alpha_{b,c} : S(b, c) \longrightarrow S'(b, c)$ is a natural transformation if and only if $\alpha(b, c)$ is natural in b for each c and natural in c for each b.*

Proof. If α is natural, in particular, we can use the identities to prove that it must be natural in its two components

$$\begin{tikzcd} S(b,c) \arrow[r, "\alpha_{b,c}"] \arrow[d, "S\langle f,\mathrm{id}_c\rangle"'] & S'(b,c) \arrow[d, "S'\langle f,\mathrm{id}_c\rangle"] \\ S(b',c) \arrow[r, "\alpha_{b',c}"] & S'(b',c) \end{tikzcd} \qquad \begin{tikzcd} S(b,c) \arrow[r, "\alpha_{b,c}"] \arrow[d, "S\langle \mathrm{id}_b,g\rangle"'] & S'(b,c) \arrow[d, "S'\langle \mathrm{id}_b,g\rangle"] \\ S(b,c') \arrow[r, "\alpha_{b,c'}"] & S'(b,c') \end{tikzcd}$$

If both components of α are natural, the naturality of the natural transformation follows from the composition of these two squares

$$\begin{tikzcd}
S(b,c) \arrow[r, "\alpha_{b,c}"] \arrow[d, "S\langle f,\mathrm{id}_c\rangle"'] & S'(b,c) \arrow[d, "S'\langle f,\mathrm{id}_c\rangle"] \\
S(b',c) \arrow[r, "\alpha_{b',c}"] \arrow[d, "S\langle \mathrm{id}_{b'},g\rangle"'] & S'(b',c) \arrow[d, "S'\langle \mathrm{id}_{b'},g\rangle"] \\
S(b',c') \arrow[r, "\alpha_{b',c'}"] & S'(b',c')
\end{tikzcd}$$

where each square is commutative by the naturality of each component of α. $\square$

13.2 OPPOSITE CATEGORIES AND CONTRAVARIANT FUNCTORS

Definition 53 (Opposite category)**.** The **opposite category** $\mathcal{C}^{op}$ of a category $\mathcal{C}$ is a category with the same objects as $\mathcal{C}$ but with all its arrows reversed. That is, for each morphism $f : A \longrightarrow B$, there exists a morphism $f^{op} : B \longrightarrow A$ in $\mathcal{C}^{op}$. Composition is defined as

$$f^{op} \circ g^{op} = (g \circ f)^{op},$$

exactly when the composite $g \circ f$ is defined in $\mathcal{C}$.

Reversing all the arrows is a process that directly translates every property of the category into its ***dual*** property. A morphism f is a monomorphism if and only if f^{op} is an epimorphism; a terminal object in $\mathcal{C}$ is an initial object in $\mathcal{C}^{op}$ and a right inverse becomes a left inverse on the opposite category. This process is also an *involution*, where $(f^{op})^{op}$ can be seen as f and $(\mathcal{C}^{op})^{op}$ is trivially isomorphic to $\mathcal{C}$.

Definition 54 (Contravariant functor)**.** A **contravariant** functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor from the opposite category, that is, $F : \mathcal{C}^{op} \longrightarrow \mathcal{D}$. Non-contravariant functors are often called **covariant** functors, to emphasize the difference.

Example 16 (Hom functors). In a locally small category $\mathcal{C}$, the **Hom-functor** is the bifunctor

$$\hom : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathsf{Set},$$

defined as $\hom(a,b)$ for any two objects $a, b \in \mathcal{C}$. Given $f : a \longrightarrow a'$ and $g : b \longrightarrow b'$, this functor is defined on any $p \in \hom(a,b)$ as

$$\hom(f,g)(p) = f \circ p \circ g \in \hom(a',b').$$

Partial applications of the functors give rise to

- hom$(a, -)$, a covariant functor for any fixed $a \in \mathcal{C}$. Given $g : b \longrightarrow b'$,

$$\hom(a, f) : \hom(a, b) \longrightarrow \hom(a, b')$$

is defined as the postcomposition with g, that we write as $-\circ g$.
- hom$(-, b)$, a contravariant functor for any fixed $b \in \mathcal{C}$. Given $f : a \longrightarrow a'$,

$$\hom(f, b) : \hom(a', b) \longrightarrow \hom(a, b)$$

is defined as the precomposition with f, that we write as $f\circ-$.

This kind of functor, contravariant on the first variable and covariant on the second is usually called a **profunctor**.

13.3 FUNCTOR CATEGORIES

Definition 55 (Functor category)**.** Given two categories $\mathcal{B}, \mathcal{C}$, the **functor category** $\mathcal{B}^{\mathcal{C}}$ has all functors $\mathcal{C} \longrightarrow \mathcal{B}$ as objects and natural transformations between them as morphisms.

If we consider the category of small categories $\mathtt{Cat}$, there is a profunctor $-^{-} : \mathtt{Cat}^{op} \times \mathtt{Cat} \longrightarrow \mathtt{Cat}$ sending any two categories to their functor category.

In [LS09], multiple examples of usual mathematical constructions in terms of functor categories can be found. Graphs, for instance, can be seen as functors; and graphs homomorphisms as the natural transformations between them.

Example 17 (Graphs as functors)*.* We consider the category given by two objects and two non-identity morphisms,

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] & \cdot \end{tikzcd}$$

usually called $\downarrow\downarrow$. To define a functor from this category to Set amounts to choose two sets E, V (not necessarily different) called the set of *edges* and the set of *vertices*; and two functions $s, t : E \longrightarrow V$, called *source* and *target.* That is, our usual definition of directed multigraph,

$$\begin{tikzcd} E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & V \end{tikzcd}$$

can be seen as an object in the category $\mathsf{Set}^{\downarrow\downarrow}$. Note how a natural transformation between two graphs (E, V) and (E', V') is a pair of morphisms $\alpha_E : E \longrightarrow E'$ and $\alpha_V : V \longrightarrow V'$ such that $s\circ\alpha_E = \alpha_V\circ s$ and $t\circ\alpha_E = \alpha_V\circ t$. This provides a convenient notion of graph homomorphism: a pair of morphisms preserving the incidence of edges. We can call $\mathtt{Graph}$ to this functor category.

Example 18 (Dynamical systems as functors)*.* A set endowed with an endomorphism (S, α) can be regarded as a *dynamical system* in an informal way. Each state of the system is represented by an element of the set and and the transition function is represented by the endomorphism.