16

LAWVERE THEORIES

This section follows [AB17].

16.1 MOTIVATION FOR ALGEBRAIC THEORIES

We will develop an unified approach to the study of algebraic structures based on constants, operations and equations; such as groups, modules or rings. Our **algebraic theories** are usually given by

- a *signature*, a family of sets $\{\Sigma_k\}_{k\in\mathbb{N}}$ whose elements are called *k-ary operations*. The *terms* of a signature are defined inductively, being variables or k-ary operations applied to k-tuples of terms;
- and a set of *axioms*, which are equations between terms.

For example, the theory of groups is given by a signature conaining

- a *binary operation* called $\cdot$,
- a *unary operation* written as $^{-1}$, and
- a *nullary operation* or a *constant* called e.

Satisfiying the following axioms

- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
- $x \cdot e = x$,
- $e \cdot x = x$,
- $x \cdot x^{-1} = e$, and
- $x^{-1} \cdot x = e$.

Note how quantifiers are not needed here, as we are interpreting each x, y, z as free variables and the universal quantification is therefore implicit. Theories in which the operations are not defined for every possible term, cannot be expressed in this way. Fields, in which the inverse of 0 is not defined, are not expressable in this form; this fact will be proved formally in Example 24.

A theory can be **interpreted** on a suitable category $\mathcal{C}$ as

- an object of the category, $A \in \mathcal{C}$;
- with morphisms $If \colon A^k \to A$ for every k-ary operation f.

Any interpretation of the theory induces an interpretation for every term on a context. That is, a term t can be given in the variable context $x_1, \dots, x_n$ if all variables that appear in t appear in $x_1, \dots, x_n$. We write that as

$$x_1, x_2, \dots, x_n \mid t;$$

and the **interpretation of the term** $x_1, \dots, x_n \mid t$ **on that context** is a morphism $I(x_1, \dots, x_n \mid t) \colon A^n \to A$ defined inductively knowing that

- the interpretation of the i-th variable is the i-th projection
$$I(x_1, \dots, x_n \mid x_i) = \pi_i : A^n \to A,$$
- the interpretation of an operation over a term is the interpretation of the morphism componsed with the componentwise interpretation of subterms
$$I(f\langle t_1, \dots, t_k \rangle) = If \circ \langle It_1, \dots, It_k \rangle \colon A^n \to A.$$
where we implicitely assume the context to be $x_1, \dots, x_n$.

The interpretation of a particular variable depends therefore on the context. We say that an interpretation **satisfies** an equation $\Gamma \mid u = v$ in a particular given context if the interpretation of both terms of the equation is the same on that context, $I(\Gamma \mid u) = I(\Gamma \mid v)$.

We usually would like to find interpretations where all the axioms of the theory were satisfied. These are called **models of the algebraic theory**. The problem with this notion of algebraic theory is that it is not representation-free; it is not independent of the choice of constants, operations or axioms. There may be multiple formulations of the same theory, with different but equivalent axioms. For instance, [Mcc91] discusses many single-equation axiomatizations of groups, such as

$$x \;/\; \big(((x/x)/y)/z\big) \;/\; \big((x/x)/x)/z\big) = y$$

with the binary operation $/$, related to the usual multiplication as $x/y = x \cdot y^{-1}$.

Our solution to this problem will be to capture all the algebraic information of a theory – all operations, constants and axioms – into a category. Differently presented but equivalent theories will give rise to the same category. This category will have *contexts* $[x_1, \dots, x_n]$ as objects. A morphism from $[x_1, \dots, x_n]$ to $[x_1, \dots, x_m]$ will be a tuple of terms

$$\langle t_1, \dots, t_k \rangle \colon [x_1, \dots, x_n] \to [x_1, \dots, x_m]$$

such that every t_k is given in the context $[x_1, \dots, x_n]$. Composition is defined componentwise as substitution of the terms of the first morphism into the variables of the second one, that is,

$$\langle s_1, \dots, s_n \rangle = \langle u_1, \dots, u_n \rangle \circ \langle t_1, \dots, t_m \rangle,$$

where

$$s_i = u_i[t_1, \dots, t_m / x_1, \dots, x_m].$$

Two morphisms in this category $\langle t_1, \dots, t_n \rangle$ and $\langle s_1, \dots, s_n \rangle$ are equal if the axioms of the theory imply the componentwise equality of its terms, that is, $t_i = s_i$.

This interpretation will lead us to our definition of **algebraic theory** as a category with finite products.

Every model M in the previous sense could be seen as a functor from this category to a given category $\mathcal{C}$ preserving finite products. Once the image of $M[x_1] = A$ is chosen, the functor is determined on objects by

$$M[x_1, \dots, x_n] = A^k$$

and once it is defined for the basic operations, it is inductively determined on morphisms as

- $M\langle x_i \rangle = \pi_i \colon A^k \to A$, for any morphism $\langle x_i \rangle$;
- $M\langle t_1, \dots, t_m \rangle = \langle Mt_1, \dots, Mt_m \rangle \colon A^m \to A$, the componentwise interpretation of subterms;
- $M\langle f\langle t_1, \dots, t_m \rangle \rangle = Mf \circ \langle Mt_1, \dots, Mt_m \rangle \colon (M\mathbb{A})^m \to M\mathbb{A}$.

The fact that M is a well-defined functor follows from the assumption that it is a model.

16.2 ALGEBRAIC THEORIES AS CATEGORIES

Definition 74 (Lawvere algebraic theory)**.** An **algebraic theory** is a category $\mathbb{A}$ with finite products and objects forming a sequence $A^0, A^1, A^2, \dots$ such that $A^m \times A^n = A^{m+n}$ for any m, n.

From this definition, it follows that A^0 must be the terminal object.

Definition 75 (Model)**.** A **model** of an algebraic theory $\mathbb{A}$ in a category $\mathcal{C}$ is a functor $M \colon \mathbb{A} \to \mathcal{C}$ preserving all finite products.

Definition 76 (Category of models of a theory)**.** The **category of models** $\mathtt{Mod}_{\mathcal{C}}(\mathbb{A})$ is the full subcategory of functor category $\mathcal{C}^{\mathbb{A}}$ given by the functors preserving all finite products. Morphisms between models of a theory in a category are natural transformations.

Definition 77 (Algebraic category)**.** An **algebraic category** is category equivalent to a category of the form $\mathtt{Mod}_{\mathcal{C}}(\mathbb{A})$, where $\mathbb{A}$ is an algebraic theory.

Example 24 (Fields have no algebraic theory). The category `Fields` is not an algebraic category. Any algebraic category $\mathtt{Mod}_{\mathcal{C}}(\mathbb{A})$ has a terminal object given by the constant functor $\Delta_1 : \mathbb{A} \to \mathcal{C}$ to 1, the terminal object of $\mathcal{C}$. Note that $\mathcal{C}$ must have a terminal object for a model to it to exist, as models must preserve all finite products. We know that Δ_1 is a terminal object because, in general, it is the terminal object of the category of functors $\mathcal{C}^{\mathbb{A}}$. However, `Fields` has no terminal object.

16.3 COMPLETENESS FOR ALGEBRAIC THEORIES

When defining interpretation of algebraic theories as categories we should ensure the property of ***semantic completeness***. We already know that, if an equation can be proved from the axioms, it will be valid in all models; but we will also like to prove that, if every model of the theory satisfies a particular equation, it can actually be proved from the axioms of the theory.

Theorem 11 (Completeness for algebraic theories)**.** *Given $\mathbb{A}$ an algebraic theory, there exists a category $\mathcal{A}$ with a model $U \in \mathtt{Mod}_{\mathcal{A}}(\mathbb{A})$ such that, for every terms u, v,*

$$U \text{ satisfies } u = v \iff \mathbb{A} \text{ proves } u = v.$$

*This is called the **universal model** for $\mathbb{A}$. This theorem asserts that categorical semantics of algebraic theories are complete.*

Proof. Simply taking $\mathbb{A}$ with the identity functor, we have an universal model for $\mathbb{A}$. □

Note that this universal model needs not to be set-theoretic; but, even in this situation, we can always find a universal model in a presheaf category via the Yoneda embedding.

Proposition 18 (Yoneda embedding as a universal model)**.** *The Yoneda embedding $y\colon \mathbb{A} \to \widehat{\mathbb{A}}$ is a universal model for $\mathbb{A}$.*

Proof. It preserves finite products because it preserves all limits, hence it is a model. As it is a faithful functor, we know that any equation proved in the model is an eqation proved by the theory. □