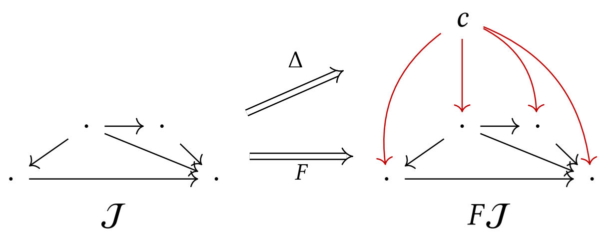
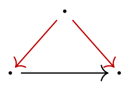
- the constantly-$c$ function for objects, $\Delta(j) = c$;
- and the constantly-$\mathrm{id}_c$ function for morphisms, $\Delta(f) = \mathrm{id}_c$.

The constant functor sends a morphism $g : c \longrightarrow c'$ to a natural transformation $\Delta g : \Delta c \longrightarrow \Delta c'$ whose components are all $g$.

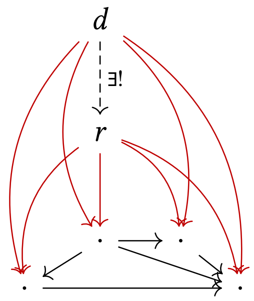
We could say that $\Delta c$ squeezes the whole category $\mathcal{J}$ into $c$. A natural transformation from this functor to some other $F : \mathcal{J} \longrightarrow C$ should be regarded as a ***cone*** from the object $c$ to a copy of $\mathcal{J}$ inside the category $C$; as the following diagram exemplifies




The components of the natural transformation appear highlighted in the diagram. The naturality of the transformation implies that each triangle



on that cone must be commutative. Thus, natural transformations are a way to recover all the information of an arbitrary ***index category*** $\mathcal{J}$ that was encoded in $c$ by the constant functor. As we did with products, we want to find the cone that best encodes that information; a universal cone, such that every other cone factorizes through it. Diagramatically an $r$ such that, for each $d$,




That factorization will be represented in the formal definition of limit by a universal natural transformation between the two constant functors.

**Definition 65** (Limit)**.** The **limit** of a functor $F : \mathcal{J} \longrightarrow C$ is an object $r \in C$ such that there exists a universal arrow $v : \Delta r \overset{\cdot}{\longrightarrow} F$ from $\Delta$ to $F$. It is usually written as $r = \varprojlim F$.

That is, for every natural transformation $w : \Delta d \overset{\cdot}{\longrightarrow} F$, there is a unique morphism $f : d \longrightarrow r$ such that

$$\begin{tikzcd} d \arrow[d, dashed, "\exists! f"'] \\ r \end{tikzcd} \qquad \begin{tikzcd} \Delta d \arrow[d, dashed, "\Delta f"'] \arrow[dr, "w"] & \\ \Delta r \arrow[r, "v"'] & F \end{tikzcd}$$

commutes. This reflects directly on the universality of the cone we described earlier and proves that limits are unique up to isomorphism.

By choosing different index categories, we will be able to define multiple different constructions on categories as limits.

## 14.5 EXAMPLES OF LIMITS

For our first example, we will take the following category, called $\downdownarrows$ as index category,

$$\begin{tikzcd} \cdot \arrow[r, shift left] \arrow[r, shift right] & \cdot \end{tikzcd}$$

A functor $F : \downdownarrows \longrightarrow \mathcal{C}$ is a pair of parallel arrows in $\mathcal{C}$. Limits of functors from this category are called **equalizers**. With this definition, the **equalizer** of two parallel arrows $f, g : a \longrightarrow b$ is an object $\mathrm{eq}(f, g)$ with a morphism $e : \mathrm{eq}(f, g) \longrightarrow a$ such that $f \circ e = g \circ e$; and such that any other object with a similar morphism factorizes uniquely through it

$$\begin{tikzcd} & d \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "e'"'] \arrow[ddr, bend left] & \\ & \mathrm{eq}(f,g) \arrow[dl, "e"'] \arrow[dr] & \\ a \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & b \end{tikzcd}$$

note how the right part of the cone is completely determined as $f \circ e$. Because of this, equalizers can be written without specifying it, and the diagram can be simplified to

$$\begin{tikzcd} \mathrm{eq}(f,g) \arrow[r, "e"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\ d \arrow[u, dashed, "\exists!"] \arrow[ur, "e'"'] & & \end{tikzcd}$$

.

*Example* 19 (Equalizers in Sets). The equalizer of two parallel functions $f, g : A \longrightarrow B$ in **Set** is $\{x \in A \mid f(a) = g(a)\}$ with the inclusion morphism. Given any other function $h : D \longrightarrow A$ such that $f \circ h = g \circ h$, we know that $f(h(d)) = g(h(d))$ for any $d \in D$. Thus, $h$ can be factorized through the equalizer.

$$\begin{tikzcd} \{x \in A \mid f(a) = g(a)\} \arrow[r, hook, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ D \arrow[u, dashed, "\exists!"] \arrow[ur, "h"'] & & \end{tikzcd}$$

*Example* 20 (Kernels). In the category of abelian groups, the kernel of a function $f$, $\ker(f)$, is the equalizer of $f : G \longrightarrow H$ and a function sending each element to the zero element of $H$. The same notion of kernel can be defined in the category of $R$Modules, for any ring $R$.

**Pullbacks** are defined as limits whose index category is $\cdot \longrightarrow \cdot \longleftarrow \cdot$. Any functor from that category is a pair of arrows with a common codomain; and the pullback is the universal cone over them.

$$\begin{tikzcd}
& d \arrow[ddl, bend right, "p'"'] \arrow[d, dashed, "\exists!"] \arrow[dd, bend right] \arrow[ddr, bend left, "q'"] & \\
& a \arrow[dl, bend right, "p"'] \arrow[d] \arrow[dr, bend left, "q"] & \\
x \arrow[r, "f"'] & z & y \arrow[l, "g"]
\end{tikzcd}$$

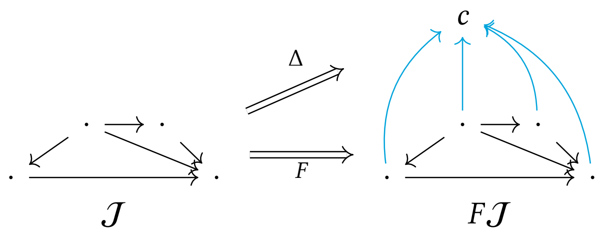
Again, the central arrow of the diagram is determined as $f \circ q = g \circ p$; so it can be ommited in the diagram. The usual definition of a pullback for two morphisms $f : x \longrightarrow z$ and $g : y \longrightarrow z$ is pair of morphisms $p : a \longrightarrow x$ and $q : a \longrightarrow y$ such that $f \circ q = g \circ p$ which are also universal, that is, given any pair of morphisms $p' : d \longrightarrow x$ and $q' : d \longrightarrow y$, there exists a unique $u : d \longrightarrow a$ making the diagram commute. Usually we write the pullback object as $x \times_z y$ and we write this property diagramatically as

$$\begin{tikzcd}
d \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "p'"] \arrow[ddr, bend right, "q'"'] & & \\
& x \times_z y \arrow[r, "p"] \arrow[d, "q"'] & x \arrow[d, "f"] \\
& y \arrow[r, "g"'] & z
\end{tikzcd}$$

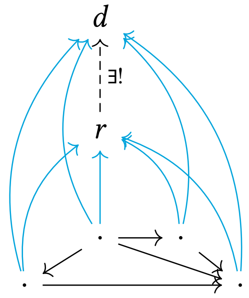
The square in this diagram is usually called a *pullback square*, and the pullback object is usually called a *fibered product*.

## 14.6 COLIMITS

A colimit is the dual notion of a limit. We could consider cocones to be the dual of cones and pick the universal one. Once an index category $\mathcal{J}$ and a base category $C$ are fixed, a ***cocone*** is a natural transformation from a functor on the base category to a constant functor. Diagramatically,



is an example of a cocone, and the universal one would be the $r$, such that, for each cone $d$,




and naturality implies that each triangle

$$\begin{tikzcd} & \cdot & \\ \cdot \arrow[ur] \arrow[rr] & & \cdot \arrow[ul] \end{tikzcd}$$

commutes.

**Definition 66** (Colimits)**.** The **colimit** of a functor $F : J \longrightarrow C$ is an object $r \in C$ such that there exists a universal arrow $u : F \stackrel{.}{\longrightarrow} \Delta r$ from $F$ to $\Delta$. It is usually written as $r = \varinjlim F$.

That is, for every natural transformation $w : F \stackrel{.}{\longrightarrow} \Delta d$, there is a unique morphism $f : r \longrightarrow d$ such that

$$\begin{tikzcd} d & \Delta d & \\ r \arrow[u, dashed, "\exists! f"] & \Delta r \arrow[u, dashed, "\Delta f"] & F \arrow[l, "v"] \arrow[ul, "w"'] \end{tikzcd}$$

commutes. This reflects directly on the universality of the cocone we described earlier and proves that colimits are unique up to isomorphism.

## 14.7 EXAMPLES OF COLIMITS

**Coequalizers** are the dual of *equalizers*; colimits of functors from $\downarrow\downarrow$. The coequalizer of two parallel arrows is an object $\mathrm{coeq}(f, g)$ with a morphism $e : b \longrightarrow \mathrm{coeq}(f, g)$ such that $e \circ f = e \circ g$; and such that any other object with a similar morphism factorizes uniquely through it

$$\begin{tikzcd} & d & \\ & \mathrm{coeq}(f,g) \arrow[u, dashed, "\exists!"] & \\ a \arrow[ur] \arrow[uur, bend left] \arrow[rr, shift left, bend left, "f"] \arrow[rr, shift right, bend right, "g"'] & & b \arrow[ul, "e"'] \arrow[uul, bend right, "e'"'] \end{tikzcd}$$

as the right part of the cocone is completely determined by the left one, the diagram can be written as

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "e"] \arrow[dr, "e'"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists!"] \\ & & d \end{tikzcd}$$

*Example* 21 (Coequalizers in Sets). The coequalizer of two parallel functions $f, g : A \longrightarrow B$ in Set is $B/(\sim_{f=g})$, where $\sim_{f=g}$ is the minimal equivalence relation in which we have $f(a) \sim g(a)$ for each $a \in A$. Given any other function $h : B \longrightarrow D$ such that $h(f(a)) = h(g(a))$, it can be factorized in a unique way by $h' : B/\sim_{f=g} \longrightarrow D$.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] \arrow[dr, "e'"'] & B/(\sim_{f=g}) \arrow[d, dashed, "\exists!"] \\ & & D \end{tikzcd}$$

**Pushouts** are the dual of pullbacks; colimits whose index category is $\cdot \longleftarrow \cdot \longrightarrow \cdot$, that is, the dual of the index category for pullbacks. Diagramatically,

$$\begin{tikzcd} & d & \\ & a \arrow[u, dashed, "\exists!"] & \\ x \arrow[ur, bend left, "p"] \arrow[uur, bend left, "p'"] & z \arrow[l, "f"] \arrow[r, "g"'] \arrow[u] \arrow[uu, bend left] & y \arrow[ul, bend right, "q"'] \arrow[uul, bend right, "q'"'] \end{tikzcd}$$

and we can define the pushout of two morphisms $f : z \longrightarrow x$ and $g : z \longrightarrow y$ as a pair of morphisms $p : x \longrightarrow a$ and $q : y \longrightarrow a$ such that $p \circ f = q \circ g$ which are also universal, that is, given any pair of morphisms $p' : x \longrightarrow d$ and $q' : y \longrightarrow d$, there exists a unique $u : a \longrightarrow d$ making the diagram commute.

$$\begin{tikzcd} z \arrow[r, "g"] \arrow[d, "f"'] & y \arrow[d, "q"] \arrow[ddr, bend left, "q'"] & \\ x \arrow[r, "p"] \arrow[drr, bend right, "p'"'] & x \neq_z y \arrow[dr, dashed, "\exists! u"'] & \\ & & d \end{tikzcd}$$

The square in this diagram is usually called a *pushout square*, and the pullback object is usually called a *fibered coproduct*.

# 15

## ADJOINTS, MONADS AND ALGEBRAS

### 15.1 ADJUNCTIONS

**Definition 67** (Adjunction)**.** An **adjunction** from categories $\mathcal{X}$ to $\mathcal{Y}$ is a pair of functors $F : \mathcal{X} \longrightarrow \mathcal{Y}$, $G : \mathcal{Y} \longrightarrow \mathcal{X}$ with a bijection

$$\varphi : \ \hom(Fx, y) \cong \hom(x, Gy),$$

natural in both $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We say that $F$ is *left-adjoint* to $G$ and that $G$ is *right-adjoint* to $F$. We write this as $F \dashv G$.

Naturality of $\varphi$ means that both

$$\begin{tikzcd} \hom(Fx, y) \arrow[r, "\varphi_{x,y}"] \arrow[d, "- \circ Fh"'] & \hom(x, Gy) \arrow[d, "- \circ h"] \\ \hom(Fx', y) \arrow[r, "\varphi_{x,y}"'] & \hom(x', Gy) \end{tikzcd} \qquad \begin{tikzcd} \hom(Fx, y) \arrow[r, "\varphi_{x,y}"] \arrow[d, "k \circ -"'] & \hom(x, Gy) \arrow[d, "Gk \circ -"] \\ \hom(Fx, y') \arrow[r, "\varphi_{x,y}"'] & \hom(x, Gy') \end{tikzcd}$$

commute for every $h : x \longrightarrow x'$ and $k : y \longrightarrow y'$. That is, for every $f : Fx \longrightarrow y$, $\varphi(f) \circ h = \varphi(f \circ Fh)$ and $Gk \circ \varphi(f) = \varphi(k \circ f)$. Equivalently, $\varphi^{-1}$ is natural and that means that, for every $g : x \longrightarrow Gy$, $k \circ \varphi^{-1}(g) = \varphi^{-1}(Gk \circ g)$ and $\varphi^{-1}(g) \circ Fh = \varphi^{-1}(g \circ h)$. A different and more intuitive way to write adjunctions is used by William Lawvere on his notes on logical operators (see [Lawb]). An adjunction $F \dashv G$ can be written as

$$\frac{Fx \xrightarrow{\ f\ } y}{\overline{x \xrightarrow{\varphi(f)} Gy}}$$

to emphasize that, for each morphism $f : Fx \longrightarrow y$, there exists a unique morphism $\varphi(f) : x \longrightarrow Gy$; written in a way that resembles bidirectional logical inference rules.

Naturality, in this setting, means that the precomposition and the postcomposition of arrows are preserved by the *inference rule*. Given morphisms $h : x' \longrightarrow x$ and $k : y \longrightarrow y'$, we have that the composition arrows of the following diagrams are adjoint to one another

$$
\begin{array}{ccc}
\begin{tikzcd} Fx' \arrow[r, "Fh"] \arrow[rr, bend left, "f\circ Fh"] & Fx \arrow[r, "f"] & y \end{tikzcd}
&
\begin{tikzcd} Fx \arrow[r, "f"] \arrow[rr, bend left, "k\circ f"] & y \arrow[r, "k"] & y' \end{tikzcd}
\\ \hline\hline
\begin{tikzcd} x' \arrow[r, "h"'] \arrow[rr, bend right, "\varphi(f)\circ h"'] & x \arrow[r, "\varphi(f)"'] & Gy \end{tikzcd}
&
\begin{tikzcd} x \arrow[r, "\varphi(f)"'] \arrow[rr, bend right, "Gk\circ\varphi(f)"'] & Gy \arrow[r, "Gk"'] & Gy' \end{tikzcd}
\end{array}
$$

In other words, that $\varphi(f)\circ h = \varphi(f\circ Fh)$ and $Gk\circ\varphi(f) = \varphi(k\circ f)$, as we wrote earlier. In the following two propositions, we will characterize all this information in terms of natural transformations made up of universal arrows.

**Definition 68** (Unit and counit of an adjunction)**.** Given an adjunction $F \dashv G$, we define

- the **unit** $\eta$ as the family of morphisms $\eta_x = \varphi(\mathrm{id}_{Fx}) : x \longrightarrow GFx$, for each $x$;
- the **counit** $\varepsilon$ as the family of morphisms $\varepsilon_y = \varphi^{-1}(\mathrm{id}_{Gy}) : FGy \longrightarrow y$, for each $y$.

Diagramatically, they can be obtained by taking $y$ to be $Fx$ and $x$ to be $Gy$, respectively in the definition of adjunction

$$
\begin{array}{cc}
Fx \xrightarrow{\mathrm{id}} Fx & FGy \xrightarrow{\varepsilon_y} y \\ \hline\hline
x \xrightarrow{\eta_x} GFx & Gy \xrightarrow{\mathrm{id}} Gy
\end{array}
$$

**Proposition 13** (Units and counits are natural transformations)**.** *The **unit** and the **counit** are natural transformations such that*

1. *for each $f : Fx \longrightarrow y$, $\varphi(f) = Gf\circ\eta_x$;*
2. *for each $g : x \longrightarrow Gy$, $\varphi^{-1}(g) = \varepsilon_y\circ Fg$;*

*that follow the **triangle identities**, $G\varepsilon\circ\eta G = \mathrm{id}_G$ and $\varepsilon F\circ F\eta = \mathrm{id}_F$.*

$$
\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, equal] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}
\qquad
\begin{tikzcd} FGF \arrow[d, "\varepsilon F"'] & F \arrow[l, "F\eta"'] \arrow[dl, equal] \\ F & \end{tikzcd}
$$

*Proof.* The right and left adjunct formulas are particular instances of the naturality equations we gave in the definition of $\varphi$;

- $Gf\circ\eta = Gf\circ\varphi(\mathrm{id}) = \varphi(f\circ\mathrm{id}) = \varphi(f)$;
- $\varepsilon_y\circ Fg = \varphi^{-1}(\mathrm{id})\circ Fg = \varphi^{-1}(\mathrm{id}\circ g) = \varphi^{-1}(g)$;

diagramatically,

$$
\begin{array}{c}
\begin{tikzcd} Fx \arrow[r, "Fg"] \arrow[rr, bend left, "\varepsilon\circ Fg"] & FGy \arrow[r, "\varepsilon_y"] & y \end{tikzcd} \\ \hline\hline
\begin{tikzcd} x \arrow[r, "g"'] \arrow[rr, bend right, "g"'] & Gy \arrow[r, "\mathrm{id}"'] & Gy \end{tikzcd}
\end{array}
\qquad
\begin{array}{c}
\begin{tikzcd} Fx \arrow[r, "\mathrm{id}"] \arrow[rr, bend left, "f"] & Fx \arrow[r, "f"] & y \end{tikzcd} \\ \hline\hline
\begin{tikzcd} x \arrow[r, "\eta_x"'] \arrow[rr, bend right, "Gf\circ\eta_x"'] & GFx \arrow[r, "Gf"'] & Gy \end{tikzcd}
\end{array}
$$

The naturality of $\eta$ and $\varepsilon$ can be deduced again from the naturality of $\varphi$; given any two functions $h : x \longrightarrow y$ and $k : x \longrightarrow y$,

- $GFh\circ\eta_x = GFh\circ\varphi(\mathrm{id}_{Fx}) = \varphi(Fh) = \varphi(\mathrm{id}_{Fx})\circ h = \eta_x\circ h$;
- $\varepsilon_x\circ FGk = \varphi^{-1}(\mathrm{id}_{Fx})\circ FGk = \varphi^{-1}(Gk) = k\circ\varphi^{-1}(\mathrm{id}_{Gx}) = k\circ\varepsilon_x$;

diagramatically, we can prove that the adjunct of $Fh$ is $GFh\circ\eta_x$ and $\eta_x\circ h$ at the same time; while the adjunct of $Gk$ is $k\circ\varepsilon_x$ and $\varepsilon_x\circ FGk$,

$$
\begin{array}{c}
x \xrightarrow{h} y \xrightarrow{\eta_y} GFy \\ \hline\hline
Fx \xrightarrow{\mathrm{id}} Fx \xrightarrow{Fh} Fy \xrightarrow{\mathrm{id}} Fy \\ \hline\hline
x \xrightarrow[\eta_x]{} GFx \xrightarrow[GFh]{} GFy
\end{array}
\qquad
\begin{array}{c}
FGx \xrightarrow{\varepsilon_x} x \xrightarrow{k} y \\ \hline\hline
Gx \xrightarrow{\mathrm{id}} Gx \xrightarrow{Gk} Gy \xrightarrow{\mathrm{id}} Gy \\ \hline\hline
FGx \xrightarrow[FGk]{} FGy \xrightarrow[\varepsilon_x]{} y
\end{array}
$$

Finally, the triangle identities follow directly from the previous ones,

- $\mathrm{id} = \varphi(\varepsilon) = G\varepsilon\circ\eta$;
- $\mathrm{id} = \varphi^{-1}(\eta) = \varepsilon\circ F\eta$. $\square$

**Proposition 14** (Characterization of adjunctions). *Each adjunction is $F \dashv G$ between categories $\mathcal{X}$ and $\mathcal{Y}$ is completely determined by any of the following data,*

1. *functors $F, G$ and $\eta : 1 \dot{\longrightarrow} GF$ where $\eta_x : x \longrightarrow GFx$ is universal to $G$.*
2. *functor $G$ and universals $\eta_x : x \longrightarrow GF_0x$, where $F_0x \in \mathcal{Y}$, creating a functor $F$.*
3. *functors $F, G$ and $\varepsilon : FG \dot{\longrightarrow} 1$ where $\varepsilon_a : FGa \longrightarrow a$ is universal from $F$.*
4. *functor $F$ and universals $\varepsilon_a : FG_0a \longrightarrow a$, where $G_0x \in \mathcal{X}$, creating a functor $G$.*
5. *functors $F, G$, with natural transformations satisfiying the triangle identities $G\varepsilon\circ\eta G = \mathrm{id}$ and $\varepsilon F\circ F\eta = \mathrm{id}$.*

*Proof.* 1. Universality of $\eta_x$ gives a isomorphism $\varphi : \hom(Fx, y) \cong \hom(x, Gy)$ between the arrows in the following diagram

$$
\begin{tikzcd}
 & Gy & y \\
x \arrow[r, "\eta_x"'] \arrow[ur, "f"] & GFx \arrow[u, dashed, "Gg"'] & Fx \arrow[u, dashed, "\exists!g"']
\end{tikzcd}
$$

defined as $\varphi(g) = Gg \circ \eta_x$. This isomorphism is natural in $x$; for every $h : x' \longrightarrow x$ we know by naturality of $\eta$ that $Gg \circ \eta \circ h = G(g \circ Fh) \circ \eta$. The isomorphism is also natural in $y$; for every $k : y \longrightarrow y'$ we know by functoriality of $G$ that $Gh \circ Gg \circ \eta = G(h \circ g) \circ \eta$.

2. We can define a functor $F$ on objects as $Fx = F_0 x$. Given any $h : x \longrightarrow x'$, we can use the universality of $\eta$ to define $Fh$ as the unique arrow making this diagram commute

$$\begin{tikzcd}
 & GFx' & & Fx' \\
x \arrow[ur, "\eta_{x'} \circ h"] \arrow[r, "\eta_x"'] & GFx \arrow[u, dashed, "GFh"'] & & Fx \arrow[u, dashed, "\exists! Fh"']
\end{tikzcd}$$

and this choice makes $F$ a functor and $\eta$ a natural transformation, as it can be checked in the following diagrams using the existence and uniqueness given by the universality of $\eta$ in both cases

$$\begin{tikzcd}
 & GFx & Fx \\
x \arrow[ur, "\eta_x"] \arrow[r, "\eta_x"'] & GFx \arrow[u, dashed, "\mathrm{id}"'] & Fx \arrow[u, dashed, "\mathrm{id}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x'' \arrow[r, "\eta_{x''}"] & GFx'' & Fx'' \\
x' \arrow[u, "h'"] \arrow[r, "\eta_{x'}"] & GFx' \arrow[u, "GFh'"'] & Fx' \arrow[u, dashed, "\exists! Fh'"] \\
x \arrow[u, "h"] \arrow[r, "\eta_x"] & GFx \arrow[u, "GFh"'] & Fx \arrow[u, dashed, "\exists! Fh'"] \arrow[uu, dashed, bend right, "\exists! F(h' \circ h)"']
\end{tikzcd}$$

3. The proof is dual to that of 1.
4. The proof is dual to that of 2.
5. We can define two functions $\varphi(f) = Gf \circ \eta_x$ and $\theta(g) = \varepsilon_y \circ Fg$. We checked in 1 (and 3) that these functions are natural in both arguments; now we will see that they are inverses of each other using naturality and the triangle identities
   - $\varphi(\theta(g)) = G\varepsilon_a \circ GFg \circ \eta_x = G\varepsilon_a \circ \eta_x \circ g = g$;
   - $\theta(\varphi(f)) = \varepsilon \circ FGf \circ F\eta = f \circ \varepsilon \circ F\eta = f$.

□

**Proposition 15** (Essential uniqueness of adjoints). *Two adjoints to the same functor $F, F' \dashv G$ are naturally isomorphic.*

*Proof.* Note that the two different adjunctions give two units $\eta, \eta'$, and for each $x$ both $\eta_x : x \longrightarrow GFx$ and $\eta'_x : x \longrightarrow GF'x$ are universal arrows from $x$ to $G$. As universal arrows are unique up to isomorphism, we have a unique isomorphism $\theta_x : Fx \longrightarrow F'x$ such that $G\theta_x \circ \eta_x = \eta'_x$.

We know that $\theta$ is natural because there are two arrows, $\theta \circ Ff$ and $F'f \circ \theta$, making this universal diagram commute

$$\begin{tikzcd}
y \arrow[r, "\eta'"] & GF'y & F'y \\
x \arrow[u, "f"] \arrow[r, "\eta"] & GFx \arrow[u, dashed] & Fx \arrow[u, dashed, "\exists!"]
\end{tikzcd}$$

because

- $G(\theta \circ Ff) \circ \eta = G\theta \circ GFf \circ \eta = G\theta \circ \eta \circ f = \eta' \circ f$;
- $G(F'f \circ \theta) \circ \eta = GF'f \circ G\theta \circ \eta = GF'f \circ \eta' = \eta' \circ f$;

thus, they must be equal, $\theta \circ Ff = F'f \circ \theta$. □

**Theorem 9** (Composition of adjunctions)**.** *Given two adjunctions between categories $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Y}, \mathcal{Z}$ respectively,*

$$\varphi : \ \hom(Fx, y) \cong \hom(x, Gy) \qquad \theta : \ \hom(F'y, z) \cong \hom(y, G'z)$$

*the composite functors yield a composite adjunction*

$$\varphi \cdot \theta : \ \hom(F'Fx, z) \cong \hom(x, GG'z).$$

*If the unit and counit of $\varphi$ are $\langle \eta, \varepsilon \rangle$ and the unit and counit of $\theta$ are $\langle \eta', \varepsilon' \rangle$; the unit and counit of the composite adjunction are $\langle G\eta' F \circ \eta,\ \varepsilon' \circ F' \varepsilon G' \rangle$.*

*Proof.* We saw previously that the componentwise composition of two natural isomorphisms is itself an natural isomorphism. Diagramatically, we compose

$$\begin{array}{c} F'Fx \xrightarrow{f} y \\ \hline\hline Fx \xrightarrow{\theta(f)} G'y \\ \hline\hline x \xrightarrow{\varphi\theta(f)} GG'y \end{array}$$

If we apply the two natural isomorphisms to the identity, we find the unit and the counit of the adjunction.

$$\begin{array}{c} F'Fx \xrightarrow{\mathrm{id}} F'Fx \\ \hline\hline Fx \xrightarrow{\mathrm{id}} Fx \xrightarrow{\eta'_{Fx}} G'F'Fx \\ \hline\hline x \xrightarrow[\eta]{} GFx \xrightarrow[G\eta'_{Fx}]{} GG'F'Fx \end{array} \qquad \begin{array}{c} GG'z \xrightarrow{\mathrm{id}} GG'z \\ \hline\hline FGG'z \xrightarrow{\varepsilon_{G'z}} G'z \xrightarrow{\mathrm{id}} G'z \\ \hline\hline F'FGG'z \xrightarrow[F'\varepsilon_{G'z}]{} F'G'z \xrightarrow[\varepsilon']{} z \end{array}$$

□

## 15.2 EXAMPLES OF ADJOINTS

*Example* 22 (Product and coproduct as adjoints). Given any category $\mathcal{C}$, we define a **diagonal functor** to a product category $\Delta : \ \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$, sending every object $x$ to a pair $(x, x)$, and each morphism $f : x \longrightarrow y$ to the pair $\langle f, f \rangle : \ (x, x) \longrightarrow (y, y)$.