16

CARTESIAN CLOSED CATEGORIES AND LAMBDA CALCULUS

16.1 LAWVERE THEORIES

Definition 74 (Lawvere algebraic theory)**.** An **algebraic theory** [AB17] is a category $\mathbb{A}$ with all finite products whose objects form a sequence $A^0, A^1, A^2, \dots$ such that $A^m \times A^n = A^{m+n}$ for any m, n.

The usual notion of algebraic theory is given by a set of *k-ary operations* for each $k \in \mathbb{N}$ and certain axioms between the terms that can be constructed inductively from free veriables and our operations. For instance, the theory of groups is given by a binary operation $(\cdot)$, a unary operation $({}^{-1})$, and a constant or 0-ary operation e; satisfying the following axioms

$$x \cdot x^{-1} = e, \quad x^{-1} \cdot x = e, \quad (x \cdot y) \cdot z = x \cdot (y \cdot z), \quad x \cdot e = x, \quad e \cdot x = x,$$

for any free variables x, y, z. The problem with this notion of algebraic theory is that it is not independent from its representation: there may be multiple formulations for the same theory, with different but equivalent axioms. For example, [Mcc91] discusses many single-equation axiomatizations of groups, such as

$$x \;/\; \big(((x/x)/y)/z) \;/\; ((x/x)/x)/z\big) = y$$

with the binary operation /, related to the usual multiplication as $x/y = x \cdot y^{-1}$. Our solution to this problem will be to capture all the algebraic information of a theory – all operations, constants and axioms – into a category. Differently presented but equivalent signatures and axioms will give rise to the same category.

An algebraic theory can be built from a signature as follows: objects represent natural numbers, $A^0, A^1, A^2, \dots$, and morphisms from A^n to A^m are given by a tuple of m terms $t_1, \dots, t_m$ depending on n free variables $x_1, \dots, x_n$, written as

$$(x_1 \dots x_n \vdash \langle t_1, \dots, t_k \rangle) : A^n \longrightarrow A^m.$$

Composition is defined componentwise as the substitution of the terms of the first morphism into the variables of the second one; that is, given $(x_1 \dots x_k \vdash \langle t_1, \dots, t_m \rangle) : A^k \longrightarrow A^m$ and $(x_1 \dots x_m \vdash \langle u_1, \dots, u_n \rangle) : A^m \longrightarrow A^n$, composition is defined as $(x_1 \dots x_k \vdash \langle s_1, \dots, s_n \rangle)$, where

$s_i = u_i[t_1, \dots, t_m / x_1, \dots, x_m]$. Two morphisms are considered equal, $(x_1 \dots x_n \vdash \langle t_1, \dots, t_k \rangle) = (x_1 \dots x_n \vdash \langle t'_1, \dots, t'_k \rangle)$ if componentwise equality of terms $t_i = t'_i$ follows from the axioms of the theory. Identity is the morphism $(x_1 \dots x_n \vdash \langle x_1, \dots, x_n \rangle)$. The kth-projection from A^n is the term $(x_1 \dots x_n \vdash x_k)$, and it is easy to check that these projections make it the n-fold product of A.

Definition 75 (Model). A **model** of an algebraic theory $\mathbb{A}$ in a category C is a functor $M : \mathbb{A} \longrightarrow C$ preserving all finite products.

The **category of models**, $\mathrm{Mod}_C(\mathbb{A})$, is the subcategory of the functor category $C^{\mathbb{A}}$ given by the functors preserving all finite products, with natural transformations between them. We say that a category is **algebraic** if it is equivalent to a category of the form $\mathrm{Mod}_C(\mathbb{A})$.

Example 24 (The algebraic theory of groups). Let $\mathbb{G}$ be the algebraic theory of groups built from its signature; we have all the tuples of terms that can be inductively built with

$$(x, y \vdash x \cdot y) : G^2 \longrightarrow G, \qquad (x \vdash x^{-1}) : G \longrightarrow G, \qquad (\vdash e) : G,$$

and the projections $(x_1 \dots x_n \vdash x_k) : G^n \longrightarrow G$, where the usual group axioms hold. A model $H : \mathbb{G} \longrightarrow C$ is be determined by an object of C and *morphisms of the catgeory* with the above signature for which the axioms hold; for instance,

- a model of $\mathbb{G}$ in Set is a classical **group**, a set with multiplication and inverse functions for which the axioms hold;
- a model of $\mathbb{G}$ in Top is a **topological group**, a topological space with continuous multiplication and inverse functions;
- a model of $\mathbb{G}$ in Mfd, the category of differentiable manifolds with smooth functions between them, is a **Lie group**;
- a model of $\mathbb{G}$ in Grp is an **abelian group**;
- a model of $\mathbb{G}$ in CRing, the category of commutative rings with homomorphisms, is a **Hopf algebra**.

The category of models $\mathrm{Mod}_{\mathsf{Set}}(\mathbb{A})$ is the usual category of groups, Grp; note that the natural transformations are precisely the group homomorphisms, as they have to preserve the unit, product and inverse of the group in order to be natural.

By construction we know that, if an equation can be proved from the axioms, it will be valid in all models (our semantics are *sound*); but we will also like to prove that, if every model of the theory satisfies a particular equation, it can actually be proved from the axioms of the theory (our semantics are *complete*). In general, we can actually prove a stronger result.

Theorem 11 (Universal model). *Given $\mathbb{A}$ an algebraic theory, there exists a category $\mathcal{A}$ with a model $U \in \mathtt{Mod}_{\mathcal{A}}(\mathbb{A})$ such that, for every terms u, v,*

$$u = v \text{ is satisfied under } U \iff \mathbb{A} \text{ proves } u = v.$$

*A category with this property is called a **universal model**.*

Proof. Indeed, taking $\mathcal{A} = \mathbb{A}$ as a model of itself with the identity functor $U = \mathrm{Id}$, $u = v$ is satisfied under the identity functor if and only if it is satisfied in the original category. □

This proof feels rather disappointing because this model is not even set-theoretic in general; but we can go further and assert the existence of a universal model in a presheaf category via the Yoneda embedding.

Corollary 4 (Completeness on presheaves)**.** *The Yoneda embedding* $y \colon \mathbb{A} \longrightarrow \mathsf{Set}^{\mathbb{A}^{op}}$ *is a universal model for* $\mathbb{A}$.

Proof. It preserves finite products because it preserves all limits, hence it is a model. As it is a faithful functor, we know that any equation proved in the model is an equation proved by the theory. □

Example 25 (Universal group). For instance, a universal model of the group would be the Yoneda embedding of $\mathbb{G}$ in $\mathsf{Set}^{\mathbb{G}^{op}}$. The group object would be the functor

$$U = \hom_{\mathbb{G}}(-, A^1);$$

which can be thought as a family of sets parametrized over the naturals: for each n we have $U_n = \hom_{\mathbb{G}}(A^n, A^1)$, which is the set of terms on n variables under the axioms of a group. In other words, the universal model for the theory of groups would be the free group on n generators, parametrized over n.

16.2 CARTESIAN CLOSED CATEGORIES

Definition 76 (Cartesian closed category)**.** A **cartesian closed category** is a category $\mathcal{C}$ in which the terminal, diagonal and product functors have right adjoints

$$! \colon \mathcal{C} \longrightarrow 1, \qquad \Delta \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \qquad (- \times A) \colon \mathcal{C} \longrightarrow \mathcal{C}.$$

These adjoints are given by terminal, product and exponential objects, written as

$$\frac{* \longrightarrow *}{C \xrightarrow{\;!\;} 1} \qquad \frac{C, C \xrightarrow{f,g} A, B}{C \xrightarrow{\langle f,g\rangle} A \times B} \qquad \frac{C \times A \xrightarrow{f} B}{C \xrightarrow{\tilde{f}} B^A}$$

Exponentials are characterized by the **evaluation morphism** $\varepsilon \colon B^A \times A \longrightarrow B$ which is the counit of the adjunction

$$\begin{array}{c}
\begin{tikzcd} C \times A \arrow[r, "\tilde{f}\times \mathrm{id}"] \arrow[rr, bend left, "f"] & B^A \times A \arrow[r, "\varepsilon"] & B \end{tikzcd} \\
\hline\hline
\begin{tikzcd} C \arrow[r, "\tilde{f}"'] \arrow[rr, bend right, "\tilde{f}"'] & B^A \arrow[r, "\mathrm{id}"'] & B^A \end{tikzcd}
\end{array}$$

Example 26 (Presheaf categories are cartesian closed). Set is cartesian closed, with exponentials given by function sets. In general, any presheaf category $\mathsf{Set}^{C^{op}}$ from a small C is cartesian closed. Given any two presheaves Q, P, if the exponential exists, its component in any A should be

$$Q^P A \cong \mathrm{Nat}(\hom(-, A), Q^P) \cong \mathrm{Nat}(\hom(-, A) \times P, Q)$$

by the Yoneda lemma, which is a set when C is small. A family of evaluation functions $\varepsilon_A : \mathrm{Nat}(\hom(-, A) \times P, Q) \times PA \longrightarrow QA$ can be defined as $\varepsilon_A(\eta, p) = \eta(\mathrm{id}_A, p)$. We show that each is a universal arrow: given any $n : R \times P \overset{\cdot}{\longrightarrow} Q$, we can show that there exists a unique ϕ making the diagram

$$\begin{tikzcd}
R \times P \arrow[d, dashed, "\phi\times\mathrm{Id}"'] \arrow[dr, "n"] & \\
\mathrm{Nat}(\hom(-, A) \times P, Q) \times P \arrow[r, "\varepsilon"'] & Q
\end{tikzcd}$$

commute. In fact, we know that, by commutativity $\phi_r(\mathrm{id}, p) = \varepsilon_A(\phi_r, p) = n(r, p)$ must hold; and then by naturality, for every $f : B \longrightarrow A$,

$$\begin{tikzcd}
RA \arrow[r, "\phi_A"] \arrow[d, "Rf"'] & \mathrm{Nat}(\hom(-, A) \times P, Q) \arrow[d, "-\circ((f\circ -)\times \mathrm{id})"] \\
RB \arrow[r, "\phi_B"] & \mathrm{Nat}(\hom(-, B) \times P, Q)
\end{tikzcd}$$

Thus, the complete natural transformation is completely determined for any $r \in RA$, $p \in PA$ as

$$\phi(r)(f, p) = \phi(Rf(r))(\mathrm{id}, p) = n(Rf\, r, p).$$

The existence of a universal family of evaluations characterizes the adjunction by Proposition 14.

In general, cartesian-closed categories with functors preserving products and exponentials form a category called Ccc.

16.3 SIMPLY-TYPED λ-THEORIES

If we read $\Gamma \vdash a : A$ as a morphism from the context Γ to the output type A, the rules of simply-typed lambda calculus with product and unit types match the adjoints that determine a cartesian closed category

$$\frac{}{\Gamma \vdash * : 1} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \langle a, b \rangle : A \times B} \qquad \frac{\Gamma, a : A \vdash b : B}{\Gamma \vdash (\lambda a.b) : A \longrightarrow B}$$

A λ-**theory** $\mathbb{T}$ is the analog of a Lawvere theory for cartesian closed categories. It is given by a set of basic types and constants over the simply-typed lambda calculus and a set of equality axioms, determining a **definitional equality** $\equiv$, an equivalence relation preserving the structure of the simply-typed lambda calculus; that is

$$\begin{array}{ll}
t \equiv *, & \text{for each } t : 1; \\
\langle a, b \rangle \equiv \langle a', b' \rangle, & \text{for each } a \equiv a',\ b \equiv b' \\
\mathtt{fst}\langle a, b \rangle \equiv a,\ \mathtt{snd}\langle a, b \rangle \equiv b, & \text{for each } a : A, b : B; \\
\mathtt{fst}\ m \equiv \mathtt{fst}\ m',\ \mathtt{snd}\ m \equiv \mathtt{snd}\ m', & \text{for each } m \equiv m'; \\
m \equiv \langle \mathtt{fst}\ m, \mathtt{snd}\ m \rangle, & \text{for each } m : A \times B; \\
f\ x \equiv f'\ x', & \text{for each } f \equiv f', x \equiv x'; \\
(\lambda x.f\ x) \equiv f, & \text{for each } f : A \longrightarrow B; \\
(\lambda x.m) \equiv (\lambda x.m'), & \text{for each } m \equiv m'; \\
(\lambda x.m)\ n \equiv m[n/x] & \text{for each } m : B, n : A.
\end{array}$$

Two types are *isomorphic*, $A \cong A'$ if there exist terms $f : A \longrightarrow A'$ and $g : A' \longrightarrow A$ such that $f\ (g\ a) \equiv a$ for each $a : A$, and $g\ (f\ a') \equiv a'$ for each $a' : A'$.

Example 27 (System T). Gödel's **System T** [GTL89] is defined as a λ-theory with the basic types `nat` and `bool`; the constants $0 : \mathtt{nat}$, $S : \mathtt{nat} \longrightarrow \mathtt{nat}$, $\mathrm{true} : \mathtt{bool}$, $\mathrm{false} : \mathtt{bool}$, $\mathrm{ifelse} : \mathtt{bool} \longrightarrow C \longrightarrow C \longrightarrow C$ and $\mathrm{rec} : C \longrightarrow (\mathtt{nat} \longrightarrow C \longrightarrow C) \longrightarrow \mathtt{nat} \longrightarrow C$; and the axioms

$$\begin{array}{ll}
\mathrm{ifelse}\ \mathrm{true}\ a\ b \equiv a, & \mathrm{rec}\ c_0\ c_s\ 0 \equiv c_0, \\
\mathrm{ifelse}\ \mathrm{false}\ a\ b \equiv b, & \mathrm{rec}\ c_0\ c_s\ (Sn) \equiv c_s\ n\ (\mathrm{rec}\ c_0\ c_s\ n).
\end{array}$$

Example 28 (Untyped $\lambda - calculus$). Untyped λ calculus can be recovered as a λ theory with a single basic type $\mathtt{D}$ and a type isomorphism $\mathtt{D} \cong \mathtt{D} \longrightarrow \mathtt{D}$ given by two constants $r : \mathtt{D} \longrightarrow (\mathtt{D} \longrightarrow \mathtt{D})$ and $s : (\mathtt{D} \longrightarrow \mathtt{D}) \longrightarrow \mathtt{D}$ such that $r\ (s\ f) \equiv f$ for each $f : \mathtt{D} \longrightarrow \mathtt{D}$ and $s\ (r\ x) \equiv x$ for each $x : \mathtt{D}$. We assume that each term of the untyped calculus is of type $\mathtt{D}$ and apply r, s as needed to construct well-typed terms.

Definition 77 (Translation)**.** The reasonable notion of homomorphism between lambda theories is called a **translation** between λ-theories $\tau : \mathbb{T} \longrightarrow \mathbb{U}$, and it given by a function on types and terms

1. preserving type constructors

$$\tau 1 = 1, \qquad \tau(A \times B) = \tau A \times \tau B, \qquad \tau(A \longrightarrow B) = \tau A \longrightarrow \tau B;$$

2. preserving the term structure

$$\begin{array}{lll}
\tau(\mathtt{fst}\ m) \equiv \mathtt{fst}\ (\tau m), & \tau(\mathtt{snd}\ m) \equiv \mathtt{snd}\ (\tau m), & \tau\langle a, b \rangle \equiv \langle \tau a, \tau b \rangle, \\
\tau(f\ x) \equiv (\tau f)\ (\tau x), & \tau(\lambda x.m) \equiv \lambda x.(\tau m);
\end{array}$$

3. and preserving all equations, $t \equiv u$ implies $\tau t \equiv \tau u$.

We consider the category $\lambda\mathsf{Thr}$ of λ-theories with translations, note that the identity and composition of translations are translations. Our goal is to prove that this category is equivalent to that of cartesian closed categories with functors preserving products and exponentials.

Apart from the natural definition of isomorphism, we consider the weaker notion of **equivalence of theories**. Two theories with translations $\tau : \mathbb{T} \longrightarrow \mathbb{U}$ and $\sigma : \mathbb{U} \longrightarrow \mathbb{T}$ are equivalent $\mathbb{T} \simeq \mathbb{U}$ if there exist two families of *type isomorphisms* $\tau\sigma A \cong A$ and $\sigma\tau B \cong B$.

Proposition 18 (Syntactic category). *Given a λ-theory $\mathbb{T}$, its **syntactic category** $\mathcal{S}(\mathbb{T})$, has an object for each type of the theory and a morphism $A \longrightarrow B$ for each term $a : A \vdash b : B$. The composition of two morphisms $a : A \vdash b : B$ and $b' : B \vdash c : C$ is given by $a{:}A \vdash c[b/b']{:}C$; and any two morphisms $\Gamma \vdash b : B$ and $\Gamma \vdash b' : A$ are equal if $b \equiv b'$.*

The syntactic category is cartesian closed and this induces a functor $\mathcal{S} : \lambda\mathsf{Thr} \longrightarrow \mathsf{Ccc}$.

Proof. The type 1 is terminal because every morphism $\Gamma \vdash t{:}1$ is $t \equiv *$. The type $A \times B$ is the product of A and B; projections from a pair morphism are given by $\mathtt{fst}\ \langle a, b\rangle \equiv a$ and $\mathtt{snd}\ \langle a, b\rangle \equiv b$. Any other morphism under the same conditions must be again the pair morphism because $d \equiv \langle \mathtt{fst}\ d, \mathtt{snd}\ d\rangle \equiv \langle a, b\rangle$.

Finally, given two types A, B, its exponential is $A \longrightarrow B$ with the evaluation morphism

$$m : (A \longrightarrow B) \times A \vdash (\mathtt{fst}\ m)\,(\mathtt{snd}\ m) : B.$$

It is universal: for any $p{:}C \times A \vdash q{:}B$, there exists a morphism $z{:}C \vdash \lambda x.q[\langle z, x\rangle/p] : A \longrightarrow B$ such that

$$(\lambda x.q[\langle \mathtt{fst}\ p, x\rangle/p])(\mathtt{snd}\ p) \equiv q[\langle \mathtt{fst}\ p, \mathtt{snd}\ p\rangle/p] \equiv q[p/p] \equiv q;$$

and if any other morphism $z{:}C \vdash d{:}A \longrightarrow B$ satisfies $(d[\mathtt{fst}\ p/z](\mathtt{snd}\ p)) \equiv q$ then

$$\lambda x.q[\langle z, x\rangle/p] \equiv \lambda x.(d[\mathtt{fst}\ p/z]\,(\mathtt{snd}\ p))[\langle z, x\rangle/p] \equiv \lambda x.(d[z/z]\ x) \equiv d.$$

Given a translation $\tau : \mathbb{T} \longrightarrow \mathbb{U}$, we define a functor $\mathcal{S}(\tau) : \mathcal{S}(\mathbb{T}) \longrightarrow \mathcal{S}(\mathbb{U})$ mapping the object $A \in \mathcal{S}(\mathbb{T})$ to $\tau A \in \mathcal{S}(\mathbb{U})$ and any morphism $\vdash b : B$ to $\vdash \tau b : \tau B$. The complete structure of the functor is then determined because it must preserve products and exponentials. $\square$

Proposition 19 (Internal language). *Given a cartesian closed category $\mathcal{C}$, its **internal language** $\mathbb{L}(\mathcal{C})$ is a λ-theory with a type $\ulcorner A \urcorner$ for each object $A \in \mathcal{C}$, a constant $\ulcorner f \urcorner : \ulcorner A \urcorner \longrightarrow \ulcorner B \urcorner$ for each morphism $f : A \longrightarrow B$, axioms*

$$\ulcorner \mathrm{id} \urcorner\ x \equiv x, \qquad \ulcorner g \circ f \urcorner\ x \equiv \ulcorner g \urcorner\,(\ulcorner f \urcorner\ x),$$

and three families of constants

$$\mathtt{T} : 1 \longrightarrow \ulcorner 1 \urcorner,\quad \mathtt{P}_{AB} : \ulcorner A \urcorner \times \ulcorner B \urcorner \longrightarrow \ulcorner A \times B \urcorner,\quad \mathtt{E}_{AB} : (\ulcorner A \urcorner \longrightarrow \ulcorner B \urcorner) \longrightarrow \ulcorner B^A \urcorner,$$

that act as type isomorphisms*, which means that they create the following pairs of two-side inverses relating the categorical and type-theoretical structures*

$$
\begin{aligned}
&t \equiv \mathrm{T}\ * && \textit{for each } u\ :\ \ulcorner 1 \urcorner,\\
&m \equiv \mathrm{P}\ \langle \ulcorner \pi_1 \urcorner\ m, \ulcorner \pi_2 \urcorner\ m \rangle && \textit{for each } z\ :\ \ulcorner A \times B \urcorner,\\
&n \equiv \langle \ulcorner \pi_0 \urcorner\ (\mathrm{P}\ n), \ulcorner \pi_1 \urcorner\ (\mathrm{P}\ n) \rangle && \textit{for each } n\ :\ \ulcorner A \urcorner \times \ulcorner B \urcorner,\\
&f \equiv \mathrm{E}\ (\lambda x. \ulcorner e \urcorner\ (\mathrm{P}\ \langle f, x \rangle)) && \textit{for each } f\ :\ \ulcorner B^A \urcorner,\\
&g \equiv \lambda x. \ulcorner e \urcorner\ (\mathrm{P}\ \langle \mathrm{E}\ g, x \rangle) && \textit{for each } g\ :\ \ulcorner A \urcorner \longrightarrow \ulcorner B \urcorner.
\end{aligned}
$$

This extends to a functor $\mathbb{L} : \mathsf{Ccc} \longrightarrow \lambda\mathsf{Thr}$.

Proof. Given any functor preserving products and exponentials $F : \mathcal{C} \longrightarrow \mathcal{D}$, we define a translation $\mathbb{L}(F) : \mathbb{L}(\mathcal{C}) \longrightarrow \mathbb{L}(\mathcal{D})$ taking each basic type $\ulcorner A \urcorner$ to $\ulcorner FA \urcorner$ and each constant $\ulcorner f \urcorner$ to $\ulcorner Ff \urcorner$; equations are preserved because F is a functor and types are preserved up to isomorphism because F preserves products and exponentials. □

Theorem 12 (Equivalence between cartesian closed categories and lambda calculus)**.** *There exists a equivalence of categories* $\mathcal{C} \simeq \mathcal{S}(\mathbb{L}(\mathcal{C}))$ *for any* $\mathcal{C} \in \mathsf{Ccc}$ *and an equivalence of theories* $\mathbb{T} \simeq \mathbb{L}(\mathcal{S}(\mathbb{T}))$ *for any* $\mathbb{T} \in \lambda\mathsf{Thr}$.

Proof. On the one hand, we define $\eta : \mathcal{C} \longrightarrow \mathcal{S}(\mathbb{L}(\mathcal{C}))$ as $\eta A = \ulcorner A \urcorner$ in objects and $\eta f = (a : \ulcorner A \urcorner \vdash f\ a\ :\ \ulcorner B \urcorner)$ for any morphism $f : A \longrightarrow B$. It is a functor because $\ulcorner \mathrm{id} \urcorner\ a \equiv a$ and $\ulcorner g \circ f \urcorner\ a \equiv g\ (f\ a)$. We define $\theta : \mathcal{S}(\mathbb{L}(\mathcal{C})) \longrightarrow \mathcal{C}$ on types inductively as $\theta(1) = 1$, $\theta(\ulcorner A \urcorner) = A$, $\theta(B \times C) = \theta(A) \times \theta(C)$ and $\theta(B \longrightarrow C) = \theta(C)^{\theta(B)}$. Now there is a natural isomorphism $\eta\theta \stackrel{\cdot}{\longrightarrow} \mathrm{Id}$, using the isomorphisms induced by the constants $\mathrm{T}, \mathrm{P}, \mathrm{E}$,

$$
\begin{aligned}
&\eta(\theta \ulcorner A \urcorner) = \eta A = \ulcorner A \urcorner,\\
&\eta(\theta 1) = \eta 1 = \ulcorner 1 \urcorner \cong 1,\\
&\eta(\theta(A \times B)) = \ulcorner \theta(A) \times \theta(B) \urcorner \cong \ulcorner \theta A \urcorner \times \ulcorner \theta B \urcorner = \eta\theta A \times \eta\theta B \cong A \times B.\\
&\eta(\theta(A \longrightarrow B)) = \ulcorner \theta(A) \longrightarrow \theta(B) \urcorner = \ulcorner \theta B \urcorner^{\ulcorner \theta A \urcorner} = (\eta\theta B)^{(\eta\theta A)} = B^A;
\end{aligned}
$$

and a natural isomorphism $\mathrm{Id} \stackrel{\cdot}{\longrightarrow} \theta\eta$ which is in fact an identity, $A = \theta \ulcorner A \urcorner = \theta\eta(A)$.

On the other hand, we define a translation $\tau : \mathbb{T} \longrightarrow \mathbb{L}(\mathcal{S}(\mathbb{T}))$ as $\tau A = \ulcorner A \urcorner$ in types and $\tau(a) = \ulcorner (\vdash a\ :\ \tau A) \urcorner$ in constants. We define $\sigma : \mathbb{L}(\mathcal{S}(\mathbb{T})) \longrightarrow \mathbb{T}$ as $\sigma \ulcorner A \urcorner = A$ in types and as

$$
\sigma(\ulcorner a\ :\ A \vdash b\ :\ B \urcorner) = \lambda a.b, \quad \sigma\mathrm{T} = \lambda x.x, \quad \sigma\mathrm{P} = \lambda x.x, \quad \sigma\mathrm{E} = \lambda x.x,
$$

in the constants of the internal language. We have $\sigma(\tau(A)) = A$, so we will check that $\tau(\sigma(A)) \cong A$ by structural induction on the constructors of the type:

- if $A = \ulcorner B \urcorner$ is a basic type, we apply structural induction over the type B to get
  - if B is a basic type, $\tau\sigma(\ulcorner B \urcorner) = \ulcorner B \urcorner$;
  - if $B = 1$, then $\tau\sigma(\ulcorner 1 \urcorner) = 1$ and $\ulcorner 1 \urcorner \cong 1$ thanks to the constant T;
  - if $B = C \times D$, then $\tau\sigma(\ulcorner C \times D \urcorner) = \ulcorner C \urcorner \times \ulcorner D \urcorner$ and $\ulcorner C \times D \urcorner \cong \ulcorner C \urcorner \times \ulcorner D \urcorner$ thanks to the constant P;

  - if $B = D^C$, then $\tau\sigma(\ulcorner D^C \urcorner) = \ulcorner C \urcorner \longrightarrow \ulcorner D \urcorner$ and $\ulcorner D^C \urcorner \cong \ulcorner C \urcorner \longrightarrow \ulcorner D \urcorner$ thanks to the constant $\mathtt{E}$.
- if $A = 1$, then $\tau\sigma 1 = 1$;
- if $A = C \times D$, then $\tau\sigma(C \times D) = \tau\sigma(C) \times \tau\sigma(D) \cong C \times D$ by induction hypothesis;
- if $A = C \longrightarrow D$, then $\tau\sigma(C \longrightarrow D) = \tau\sigma(C) \longrightarrow \tau\sigma(D) \cong C \longrightarrow D$ by induction hypothesis.

□

Thus, we can say that the simply-typed lambda calculus is the language of cartesian closed categories; each theory is a model inside a cartesian closed category.

16.4 WORKING IN CARTESIAN CLOSED CATEGORIES

We can now talk internally about cartesian closed categories using lambda calculus. Note each closed λ term $\vdash a : A$ can also be seen as a morphism from the terminal object $1 \longrightarrow A$. We say that a morphism $f : A \longrightarrow B$ in a cartesian closed category is **point-surjective** if, for every $b : B$, there exists an $a : A$ such that $f\ a \equiv b$.

Theorem 13 (Lawvere's fixed point theorem)**.** *In any cartesian closed category, if there exists a point-surjective $d : A \longrightarrow B^A$, then each $f : B \longrightarrow B$ has a fixed point, a $b : B$ such that $f\ b \equiv b$. [lawa]*

Proof. As d is point-surjective, there exists $x : A$ such that $d\ x \equiv \lambda a.f\ (d\ a\ a)$, but then, $d\ x\ x \equiv (\lambda a.f\ (d\ a\ a))\ x \equiv f\ (d\ x\ x)$ is a fixed point. □

This theorem has nontrivial consequences when interpreted in different contexts:

- **Cantor's theorem** is a corollary in Set; as there exists a nontrivial permutation of the two-element set, a point-surjective $A \longrightarrow 2^A$ cannot exist;
- **Russell's paradox** follows because $\in :\ \text{Sets} \longrightarrow 2^{\text{Sets}}$ is point-surjective if we assume that for any property on the class of sets $P :\ \text{Sets} \longrightarrow 2$ there exists a comprehension set $\{y \in \text{Sets} \mid P(y)\}$;
- the **existence of fixed points** for any term of **in untyped λ-calculus** follows from the fact that there exists a type isomorphism (in particular, a point-surjection) $\mathtt{D} \longrightarrow (\mathtt{D} \longrightarrow \mathtt{D})$;
- **Gödel first incompleteness theorem** and **Tarski's theorem** follow a similar reasoning (see [Yan03] for details). Let our category be an (algebraic) theory $A^0, A^1, A^2, \dots$ with a supplementary object 2; we say that the theory is **consistent** if there exists a morphism $\text{not} :\ 2 \longrightarrow 2$ such that $\text{not} \circ \varphi \neq \varphi$ for every $\varphi :\ A \longrightarrow 2$; we say that **satisfiability** is definable if there exists a map $\text{sat} :\ A \times A \longrightarrow 2$ such that for every "predicate" $\varphi :\ A \longrightarrow 2$, there exists a "Gödel number" $c : A$ such that $\text{sat}(c, a) = \varphi(a)$. We get that, if satisfiability is definible in a theory, then it is inconsistent.

16.5 BICARTESIAN CLOSED CATEGORIES

Definition 78 (Bicartesian closed category)**.** A **bicartesian closed category** is a cartesian closed category in which the terminal and diagonal functors have left adjoints.

These adjoints are given by initial and coproduct objects, written as

$$\frac{* \longrightarrow *}{0 \xrightarrow{\;!\;} C} \qquad \frac{A, B \xrightarrow{f,g} C, C}{A + B \xrightarrow{f+g} C}$$

The rules of union and void types in simply-typed lambda calculus can be rewritten to match the structure of these adjoints

$$\frac{}{\Gamma, z : 0 \vdash \text{abort} : C} \qquad \frac{\Gamma, a : A \vdash c : C \qquad \Gamma, b : B \vdash c' : C}{\Gamma, u : A + B \vdash \text{case } u \text{ of } c; c'}$$

In a similar way to how our previous fragment of simply-typed lambda calculus was the internal language of cartesian closed categories, this extended version is the internal language of bicartesian closed categories.