Definition 39 (Product object)**.** An object C is the **product** of two objects A, B on a category if there are two morphisms

$$A \xleftarrow{\pi_A} C \xrightarrow{\pi_B} B$$

such that, for any other object D with two morphisms $f_1 : D \longrightarrow A$ and $f_2 : D \longrightarrow B$, an unique morphism $h : D \longrightarrow C$, such that $f_1 = \pi_A \circ h$ and $f_2 = \pi_B \circ h$. Diagramatically,

$$\begin{tikzcd}
 & D \arrow[d, dashed, "\exists! h"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\
 & C \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
A & & B
\end{tikzcd}$$

Note that the product of two objects does not have to exist on a category; but when it exists, it is essentially unique. In fact, we will be able later to construct a category in which the product object is the final object of the category and Proposition 4 can be applied. We will write *the* product object of A, B as $A \times B$.

Definition 40 (Coproduct object)**.** An object C is the **coproduct** of two objects A, B on a category if there are two morphisms

$$A \xrightarrow{i_A} C \xleftarrow{i_B} B$$

such that, for any other object D with two morphisms $f_1 : D \longrightarrow A$ and $f_2 : D \longrightarrow B$, an unique morphism $h : D \longrightarrow C$, such that $f_1 = i_A \circ h$ and $f_2 = i_B \circ h$. Diagramatically,

$$\begin{tikzcd}
 & D & \\
 & C \arrow[u, dashed, "\exists! h"] & \\
A \arrow[uur, bend left, "f_1"] \arrow[ur, "i_A"] & & B \arrow[uul, bend right, "f_2"'] \arrow[ul, "i_B"']
\end{tikzcd} .$$

The same discussion we had earlier for the product can be rewritten here for the coproduct only reversing the direction of the arrows. We will write *the* coproduct of A, B as $A \neq B$. As we will see later, the notion of a coproduct is dual to the notion of product; and the same proofs can be applied on both cases, only by reversing the arrows.

11.4 EXAMPLES OF CATEGORIES

Example 8 (Discrete categories)**.** A category is **discrete** if it has no other morphisms than the identities. A discrete category is uniquely defined by the class of its objects and every class of objects defines a discrete category. Thus, discrete categories are classes or sets, without any additional categorical structure.

Example 9 (Monoids, groups). A single-object category is a **monoid**. A monoid in which every morphism is an isomorphism is a **group**.

This definition is equivalent to the usual definition of monoid if we take the morphisms as elements of the monoid and composition of morphisms as the monoid operation. Groupoids are also a particular case of categories.

Example 10 (Groupoids). A category in which every morphism is an isomorphism is a **groupoid**.

Example 11 (Partially ordered sets). Every partial ordering defines a category in which the elements are the objects and an only morphism between two objects $\rho_{a,b} : a \longrightarrow b$ exists

In particular, every ordinal can be seen as a partially ordered set and defines a category.

For example, if we take the finite ordinal $[n] = (0 < \cdots < n)$, it could be interpreted as the category given by the following diagram

$$\begin{tikzcd}
0 \arrow[loop above] \arrow[r] \arrow[rr, bend right] \arrow[rrrr, bend right] & 1 \arrow[loop above] \arrow[r] \arrow[rrr, bend right] & 2 \arrow[loop above] \arrow[r] \arrow[rr, bend right] & \dots \arrow[r] & n \arrow[loop above]
\end{tikzcd}$$

in which every object p has an identity arrow and a unique arrow to every q such that $p \leq q$. Note how the composition of arrows can be only defined in a single way.

In a partially ordered set, the product of two objects would be its join, the coproduct would be its meet and the initial and terminal objects would be the greatest and the least element, respectively.

Example 12 (The category of sets). The category Set is defined as the category with all the possible sets as objects and functions between them as morphisms. It is trivial to check associativity of composition and the existence of the identity function for any set.

In this category, the product is given by the usual cartesian product

$$A \times B = \left\{ (a, b) \mid a \in A,\ b \in B \right\},$$

with the projections $\pi_A(a, b) = a$ and $\pi_B(a, b) = b$. We can easily check that, if we have $f : C \longrightarrow A$ and $g : C \longrightarrow B$, there is a unique function given by $h(c) = (f(c), g(c))$ such that $\pi_A \circ h = f$ and $\pi_B \circ h = g$.

The initial object in Set is given by the empty set $\emptyset$: given any set A, the only function of the form $f : \emptyset \longrightarrow A$ is the empty one. The final object, however, is only defined up to isomorphism: given any set with a single object $\{*\}$, there exists a unique function of the form $f : A \longrightarrow \emptyset$ for any set A; namely, the one defined as $\forall a \in A : f(a) = *$. Every two sets with exactly one object are terminal objects and they are trivially isomorphic.

Similarly, the coproduct is defined only to isomorphism. The coproduct of two sets A, B is given by its disjoint union $A \sqcup B$; but this union can be defined in many different (but equivalent) ways. For instance, we can add a label to the elements of each sets before joining them in order to ensure that this will be a disjoint union; that is,

$$A \sqcup B = \{(a, 0) \mid a \in A\} \cup \{(b, 1) \mid b \in B\}$$

with the inclusions $i_A(a) = (a, 0)$ and $i_B(b) = (b, 1)$, is a possible coproduct. Given any two functions $f : A \longrightarrow C$ and $g : A \longrightarrow C$, there exists a unique function $h : A \sqcup B \longrightarrow C$, given by

$$h(x, n) = \begin{cases} f(x) & \text{if } n = 0, \\ g(x) & \text{if } n = 1, \end{cases}$$

such that $f = h \circ i_A$ and $g = h \circ i_B$.

The category of sets is a very special category, whose properties we will study in detail later.

Example 13 (The category of groups). The category Grp is defined as the category with groups as objects and group homomorphisms between them as morphisms.

Example 14 (The category of R-modules). The category R-Mod is defined as the category with R-modules as objects and module homomorphisms between them as morphisms. We know that the composition of module homomorphisms and the identity are also module homomorphisms.

In particular, abelian groups form a category as $\mathbb{Z}$-modules.

Example 15 (The category of topological spaces). The category Top is defined as the category with topological spaces as objects and continuous functions between them as morphisms.

12

FUNCTORS AND NATURAL TRANSFORMATIONS

> "Category" has been defined in order to define "functor" and "functor" has been defined in order to define "natural transformation".
>
> – **Saunders MacLane**, *Categories for the working mathematician*, [EM42].

Functors and natural transformations were defined for the first time by Eilenberg and MacLane in [EM42] while studying Čech cohomology. While initially they were devised mainly as a language for studying homology, they have proven its foundational value with the passage of time. The notion of naturality will be a key element of our presentation of algebraic theories and categorical logic.

12.1 FUNCTORS

A ***functor*** will be interpreted as a homomorphism of categories preserving their structure. As we discussed in the previous section, the structure of a category is given by the composition of morphisms.

Definition 41 (Functor)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, a **functor** between them, $F : \mathcal{C} \longrightarrow \mathcal{D}$, is given by

- an **object function**, $F : \mathrm{obj}(\mathcal{C}) \longrightarrow \mathrm{obj}(\mathcal{D})$;
- and an **arrow function**, $F : \hom(A,B) \longrightarrow \hom(FA,FB)$ for any two objects A, B of the category;

such that

- $F(\mathrm{id}_A) = \mathrm{id}_{FA}$, identities are preserved; and
- $F(f \circ g) = Ff \circ Fg$, the functor respects composition.

Functors can be composed as we did with morphisms. In fact, a category of categories can be defined, having functors as morphisms. In order to avoid paradoxes, we will only define the category of all small categories as a non-small category so it will not contain itself.

Definition 42 (The category of categories)**.** The category Cat is defined as the category of (small) categories as objects and functors as morphisms.

- Given two functors $F : \mathcal{C} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$, their composite functor $G \circ F : \mathcal{C} \longrightarrow \mathcal{A}$ is given by the composition of the object and arrow functions of the functors. This composition is trivially associative.
- The identity functor on a category $I_C : C \longrightarrow C$ is given by identity object and arrow functions. It is trivially neutral with respect to composition.

Definition 43 (Full functor)**.** A functor F is **full** if the arrow map between any pair of objects is surjective. That is, if every $g : FA \longrightarrow FB$ is of the form Ff for some morphism $f : A \longrightarrow B$.

Definition 44 (Faithful functor)**.** A functor F is **faithful** if the arrow map between any pair of objects is injective. That is, if, for every two arrows $f_1, f_2 : A \longrightarrow B$, $Ff_1 = Ff_2$ implies $f_1 = f_2$.

It is easy to notice that the composition of faithful (respectively, full) functors is again a faithful functor (respectively, full).

Note that a faithful functor needs not to be injective on objects nor on morphisms. In particular, if A, A', B, B' are four different objects, it could be the case that $FA = FA'$ and $FB = FB'$; and, if $f : A \longrightarrow B$ and $f' : A' \longrightarrow B'$ were two morphisms, it could be the case that $Ff = Ff'$.

Definition 45 (Isomorphism of categories)**.** An **isomorphism of categories** is a functor T whose object and arrow functions are bijections. Equivalently, it is a functor T such that there exists an *inverse* functor S such that $T \circ S$ and $S \circ T$ are identity functors.

However, the notion of isomorphism of categories may be too strict. Sometimes, it will suffice if the two compositions $T \circ S$ and $S \circ T$ are not exactly the identity functor, but isomorphic in some sense to it. We will develop these weaker notions in the next section.

12.2 NATURAL TRANSFORMATIONS

Definition 46 (Natural transformation)**.** A **natural transformation** between two functors with the same domain and codomain, $\alpha : F \stackrel{\cdot}{\longrightarrow} G$, is a family of morphisms parameterized by the objects of the domain category, $\alpha_C : FC \longrightarrow GC$ such that the following diagram commutes

$$
\begin{tikzcd}
C \arrow[d, "f"] & SC \arrow[r, "\tau_C"] \arrow[d, "Sf"'] & TC \arrow[d, "Tf"] \\
C' & SC' \arrow[r, "\tau_{C'}"] & TC'
\end{tikzcd}
$$

for every arrow $f : C \longrightarrow C'$.

Sometimes, it is also said that the family of morphisms τ is *natural* in its argument. This naturality property is what allows us to "translate" a commutative diagram from a functor to another.

$$\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dd, "h"'] & \\
& B \arrow[dl, "g"] \\
C &
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[rr, "\tau A"] \arrow[dr, "Ff"] \arrow[dd, "Fh"'] & & GA \arrow[dr, "Gf"] \arrow[dd] & \\
& FB \arrow[rr, "\tau B"] \arrow[dl, "Fg"] & & GB \arrow[dl, "Gg"] \\
FC \arrow[rr, "\tau C"] & & GC &
\end{tikzcd}$$

Definition 47 (Natural isomorphism)**.** A **natural isomorphism** is a natural transformation in which every component, every morphism of the parameterized family, is invertible.

The inverses of a natural transformation form another natural transformation, whose naturality follows from the naturality of the original transformation. We say that two functors T, S are **naturally isomorphic**, and we write this as $T \cong S$, if there is a natural isomorphism between them. The notion of a natural isomorphism between functors allows us to weaken the condition of strict equality that we imposed when talking about isomorphisms of categories. The generally more useful notion of *equivalence of categories* only needs the composition of the two functors to be naturally isomorphic to the identity.

Definition 48 (Equivalence of categories)**.** An **equivalence of categories** is given by two functors T and S such that its two compositions are naturally isomorphic to the identity functor, $T \circ S \cong \mathrm{id}$ and $S \circ T \cong \mathrm{id}$.

12.3 COMPOSITION OF NATURAL TRANSFORMATIONS

There is an obvious way in which two natural transformations $\sigma : R \longrightarrow S$ and $\tau : S \longrightarrow T$ can be composed into a new natural transformation $R \longrightarrow T$; this will be used later to define categories whose objects are functors and whose morphisms are natural transformations. But there is also a different notion of composition of natural transformations, which applies two natural transformations, in parallel, to the composition of two functors.

That is, if we draw a natural transformation between two functors as a double arrow

$$\begin{tikzcd}
A \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\tau"] & B
\end{tikzcd}$$

- we have a *vertical* composition of natural transformations, which, diagramatically, composes the two natural transformations of the left diagram into a transformation like in the right one

$$
\begin{tikzcd}
A \arrow[r, bend left=60, "R"{name=U}] \arrow[r, "T"{name=M, swap}] \arrow[r, bend right=60, "S"{name=D, swap}] & B
\arrow[from=U, to=M, phantom, "\Downarrow\tau"]
\arrow[from=M, to=D, phantom, "\Downarrow\sigma"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, bend left=50, "R"] \arrow[r, bend right=50, "T"'] \arrow[r, phantom, "\Downarrow\tau\cdot\sigma"] & B
\end{tikzcd}
$$

- and we have a *horizontal* composition of natural transformations, which composes the two natural transformations of the first diagram into the second one

$$
\begin{tikzcd}
A \arrow[r, bend left=50, "S"] \arrow[r, bend right=50, "T"'] \arrow[r, phantom, "\Downarrow\tau"] & B \arrow[r, bend left=50, "S'"] \arrow[r, bend right=50, "T'"'] \arrow[r, phantom, "\Downarrow\sigma"] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, bend left=50, "S\circ S'"] \arrow[r, bend right=50, "T\circ T'"'] \arrow[r, phantom, "\Downarrow\tau\circ\sigma"] & C
\end{tikzcd}
$$

Definition 49 (Vertical composition of natural transformations)**.** The **vertical composition** of two natural transformations $\tau : S \longrightarrow T$ and $\sigma : R \longrightarrow S$, denoted by $\tau \cdot \sigma$ is the family of morphisms defined by the objectwise composition of the components of the two natural transformations, that is

$$
\begin{tikzcd}
Rc \arrow[r, "Rf"] \arrow[d, "\sigma_c"] \arrow[dd, bend right=60, "(\tau\circ\sigma)_c"'] & Rc' \arrow[d, "\sigma_{c'}"] \arrow[dd, bend left=60, "(\tau\circ\sigma)_{c'}"] \\
Sc \arrow[r, "Sf"] \arrow[d, "\tau_c"] & Sc' \arrow[d, "\tau_{c'}"] \\
Tc \arrow[r, "Tf"] & Tc'
\end{tikzcd}
$$

Proposition 5 (Vertical composition is a natural transformation)**.** *The vertical composition of two natural transformations is in fact a natural transformation.*

Proof. Naturality of the composition follows from the naturality of its two factors. In other words, the commutativity of the external square on the above diagram follows from the commutativity of the two internal squares. □

Definition 50 (Horizontal composition of natural transformations)**.** The **horizontal composition** of two natural transformations $\tau : S \longrightarrow T$ and $\tau' : S' \longrightarrow T'$, with domains and codomains as in the following diagram

$$
\begin{tikzcd}
A \arrow[r, bend left=50, "S"] \arrow[r, bend right=50, "T"'] \arrow[r, phantom, "\Downarrow\tau"] & B \arrow[r, bend left=50, "S'"] \arrow[r, bend right=50, "T'"'] \arrow[r, phantom, "\Downarrow\sigma"] & C
\end{tikzcd}
$$