That is, if we start at an initial state $s \in S$ and the transition function is given by $\alpha : S \longrightarrow S$, the evolution of the system will be given by

$$s,\ \alpha(s),\ \alpha(\alpha(s)),\ \dots$$

and we could say that it evolves discretely over time, being $\alpha^t(s)$ the state of the system at the instant t.

This structure can be described as a functor from the monoid of natural numbers under addition. Note that any functor $D : \mathbb{N} \longrightarrow \mathsf{Set}$ has to choose a set, and an image for the $1 : \mathbb{N} \longrightarrow \mathbb{N}$, the only generator of the monoid. The image of any natural number n is determined by the image of 1; if $D(1) = \alpha$, it follows that $D(t) = \alpha^t$, where $\alpha^0 = \mathrm{id}$.

Once the structure has been described as a functor, the homomorphisms preserving this kind of structure can be described as natural transformations. A natural transformation between two functors $D, D' : \mathbb{N} \longrightarrow \mathsf{Set}$ describing two dynamic systems $(S, \alpha), (T, \beta)$ is given by a function $f : S \longrightarrow T$ such that the following diagram commutes

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\alpha^n"'] & T \arrow[d, "\beta^n"] \\
S \arrow[r, "f"] & T
\end{tikzcd}$$

that is, $f \circ \alpha = \beta \circ f$. A natural notion of homomorphism has arisen from the categorical interpretation of the structure.

A further generalization is now possible, if we want to consider continuously-evolving dynamical systems, we can define functors from the monoid of real numbers under adition instead of naturals, that is, considering functors $\mathbb{R} \longrightarrow \mathsf{Set}$. Note that these functors are given by a set S and a family of morphisms $\{\alpha_r\}_{r \in \mathbb{R}}$ such that

$$\alpha_r \circ \alpha_s = \alpha_{r+s} \qquad \forall r, s \in \mathbb{R}.$$

This example is described in [LS09].

13.4 COMMA CATEGORIES

The idea of functor categories leads us to think about categories whose objects are themselves diagrams on a category. The most relevant examples, which will be useful in our development of categorical logic, are the **comma categories**, and specially the particular case of a **slice category**.

Definition 56 (Comma category). Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories with functors $T : \mathcal{E} \longrightarrow \mathcal{C}$ and $S : \mathcal{D} \longrightarrow \mathcal{C}$. The **comma category** $(T \downarrow S)$ has

- morphisms of the form $f : Te \longrightarrow Sd$ as objects, for $e \in \mathcal{E}, d \in \mathcal{D}$;

- and pairs $\langle k, h\rangle : f \longrightarrow f'$, where $k : e \longrightarrow e'$ and $h : d \longrightarrow d'$ such that $f' \circ Tk = Sh \circ f$, as arrows.

Diagramatically, a morphism in this category is a commutative diagram

$$\begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"] & Sd' \end{tikzcd}$$

where the objects of the category are drawn in grey.

Definition 57 (Slice category)**.** A **slice category** is a particular case of a comma category $(T \downarrow S)$ in which $T = \mathrm{Id}$ is the identity functor and S is a functor from the terminal category, a category with only one object and its identity morphism.

A functor from the terminal category simply chooses an object of the category. If we call $a = S(*)$, objects of this category are morphisms $f : c \longrightarrow a$, where $c \in C$; and morphisms are $\langle k\rangle : f \longrightarrow f'$, where $k : c \longrightarrow c'$ such that $f' \circ k = f$. Diagramatically a morphism is drawn as

$$\begin{tikzcd} c \arrow[rr, "Tk"] \arrow[dr, "f"'] & & c' \arrow[dl, "f'"] \\ & a & \end{tikzcd}$$

This slice category is conventionally written as $(C \downarrow a)$. In general, we write $(T \downarrow a)$ when S is a functor from the terminal category picking an object a; and we write $(C \downarrow S)$ when T is the identity functor.

Definition 58 (Coslice category)**.** **Coslice categories** are the categorical dual of slice categories. It is the particular case of a comma category $(T \downarrow S)$ in which $S = \mathrm{Id}$ is the identity functor and T is a functor from the terminal category, a category with only one object and its identity morphism.

If we call $a = T(*)$, objects of this category are morphisms $f : c \longrightarrow a$, where $c \in C$; and morphisms are $\langle k\rangle : f \longrightarrow f'$, where $k : c \longrightarrow c'$ such that $k \circ f' = f$. Diagramatically a morphism is drawn as

$$\begin{tikzcd} & a \arrow[dl, "f'"'] \arrow[dr, "f"] & \\ c \arrow[rr, "Sk"] & & c' \end{tikzcd}$$

This slice category is conventionally written as $(a \downarrow C)$. In general, we write $(a \downarrow S)$ when T is a functor from the terminal category picking an object a; and we write $(T \downarrow C)$ when S is the identity functor.

Definition 59 (Arrow category)**. Arrow categories** are a particular case of comma categories $(T \downarrow S)$ in which both functors are the identity. They are usually written as $\mathcal{C}^{\rightarrow}$.

Objects in this category are morphisms in $\mathcal{C}$, and morphisms in this category are commutative squares in $\mathcal{C}$. Diagramatically,

$$\begin{tikzcd} a \arrow[r, "k"] \arrow[d, "f"'] & b \arrow[d, "f'"] \\ a' \arrow[r, "h"] & b' \end{tikzcd}$$

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14

UNIVERSALITY AND LIMITS

14.1 UNIVERSAL ARROWS

A **universal property** is commonly given in mathematics by some conditions of existence and uniqueness on morphisms, representing some sort of natural isomorphism. They can be used to define certain constructions up to isomorphism and to operate with them in an abstract setting. We will formally introduce universal properties using *universal arrows* from an object c to a functor S; the property of these arrows is that every arrow of the form $c \longrightarrow Sd$ will factor uniquely through the universal arrow.

Definition 60 (Universal arrow)**.** A **universal arrow** from c to S is a morphism $u : c \longrightarrow Sr$ such that for every $g : c \longrightarrow Sd$ exists a unique morphism $f : r \longrightarrow d$ making this diagram commute

$$\begin{tikzcd} & Sd & & d \\ c \arrow[ur, "g"] \arrow[r, "u"'] & Sr \arrow[u, dashed, "Sf"'] & & r \arrow[u, dashed, "\exists! f"'] \end{tikzcd}.$$

Note how an universal arrow is, equivalently, the initial object of the comma category $(c \downarrow S)$. Thus, universal arrows must be unique up to isomorphism.

Proposition 9 (Universality in terms of hom-sets)**.** *The arrow $u : c \longrightarrow Sr$ is universal if and only if $f \mapsto Sf \circ u$ is a bijection* $\hom(r, d) \cong \hom(c, Sd)$ *natural in d. Any natural bijection of this kind is determined by a unique universal arrow.*

Proof. On the one hand, given an universal arrow, bijectivity follows from the definition of universal arrow; and naturality follows from the fact that $S(gf) \circ u = Sg \circ Sf \circ u$.

On the other hand, given a bijection φ, we define $u = \varphi(\mathrm{id}_r)$. By naturality, we have the bijection $\varphi(f) = Sf \circ u$, and every arrow is written in this way. $\square$

The categorical dual of an universal arrow from an object to a functor is the notion of universal arrow from a functor to an object. Note how, particularly in this case, we avoid the name *couniversal arrow*; as both arrows are representing what we usually call a *universal property.*

Definition 61 (Dual universal arrow)**.** A **universal arrow** from S to c is a morphism $v : Sr \longrightarrow c$ such that for every $g : Sd \longrightarrow c$ exists a unique morphism $f : d \longrightarrow r$ making this diagram commute

$$\begin{tikzcd}
d \arrow[d, dashed, "\exists! f"'] & Sd \arrow[d, dashed, "Sf"'] \arrow[dr, "g"] & \\
r & Sr \arrow[r, "v"'] & c
\end{tikzcd}$$

14.2 REPRESENTABILITY

Definition 62 (Representation of a functor)**.** A **representation** of a functor from an arbitrary category to the category of sets, $K : D \longrightarrow \mathsf{Set}$, is a natural isomorphism making it isomorphic to a partially applied hom-functor,

$$\psi : \hom_D(r, -) \cong K.$$

A functor is *representable* if it has a representation. The object r is called a *representing object.* Note that, for this definition to work, D must have small hom-sets.

Proposition 10 (Representations in terms of universal arrows)**.** *If $u : * \longrightarrow Kr$ is a universal arrow for a functor $K : D \longrightarrow \mathsf{Set}$, then $f \mapsto K(f)(u\, *)$ is a representation. Every representation is obtained in this way.*

Proof. We know that $\hom(*, X) \overset{\cdot}{\longrightarrow} X$ is a natural isomorphism in X; in particular $\hom(*, K-) \overset{\cdot}{\longrightarrow} K-$. Every representation is built then as

$$\hom_D(r, -) \cong \hom(*, K-) \cong K,$$

for every natural isomorphism $D(r, -) \cong \mathsf{Set}(*, K-)$. But every natural isomorphism of this kind is universal arrow. $\square$

14.3 YONEDA LEMMA

Lemma 7 (Yoneda Lemma)**.** *For any $K : D \longrightarrow \mathsf{Set}$ and $r \in D$, there is a bijection*

$$y : \mathrm{Nat}(\hom_D(r, -), K) \cong Kr$$

sending any natural transformation $\alpha : \hom_D(r, -) \overset{\cdot}{\longrightarrow} K$ to its image on the identity, $\alpha_r(\mathrm{id}_r)$.

Proof. The complete natural transformation α is determined by $\alpha_r(\mathrm{id}_r)$. By naturality, given any $f : r \longrightarrow s$,

$$\begin{tikzcd}
\hom(r,r) \arrow[rrr, "\alpha_r"] \arrow[ddd, "f\circ-"'] & & & Kr \arrow[ddd, "Kf"] \\
& \mathrm{id}_r \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_r(\mathrm{id}_r) \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \alpha_s(f) & \\
\hom(r,s) \arrow[rrr, "\alpha_s"'] & & & Ks
\end{tikzcd}$$

it must be the case that $\alpha_s(f) = Kf(\alpha_r(\mathrm{id}_r))$. □

Corollary 3 (Characterization of natural transformations between representable functors)**.** *Given $r, s \in D$, any natural transformation $\hom(r,-) \overset{\cdot}{\longrightarrow} \hom(s,-)$ is of the form $-\circ h$ for a unique morphism $h : s \longrightarrow r$.*

Proof. Using Yoneda Lemma (Lemma 7), we know that

$$\mathrm{Nat}(\hom_D(r,-), \hom_D(s,-)) \cong \hom_D(s,r),$$

sending the natural transformation to a morphism $\alpha(id_r) = h : s \longrightarrow r$. The rest of the natural transformation is determined as $-\circ h$ by naturality. □

Proposition 11 (Naturality of the Yoneda Lemma)**.** *The bijection on the Yoneda Lemma (Lemma 7),*

$$y : \mathrm{Nat}(\hom_D(r,-), K) \cong Kr,$$

is a natural isomorphism between two functors from $\mathsf{Set}^D \times D$ to Set.

Proof. We define $N : \mathsf{Set}^D \times D \longrightarrow \mathsf{Set}$ on objects as $N\langle r, K\rangle = \mathrm{Nat}(\hom(r,-), K)$. Given $f : r \longrightarrow r'$ and $F : K \overset{\cdot}{\longrightarrow} K'$, the functor is defined on morphisms as

$$N\langle f, F\rangle(\alpha) = F\circ\alpha\circ(-\circ f) \in \mathrm{Nat}(\hom(r',-), K),$$

where $\alpha \in \mathrm{Nat}(\hom(r,-), K)$. We define $E : \mathsf{Set}^D \times D \longrightarrow \mathsf{Set}$ on objects as $E\langle r, K\rangle = Kr$. Given $f : r \longrightarrow r'$ and $F : K \overset{\cdot}{\longrightarrow} K'$, the functor is defined on morphisms as

$$E\langle f, F\rangle(a) = F(Kf(a)) = K'f(Fa) \in K'r',$$

where $a \in Kr$, and the equality holds because of the naturality of F. The naturality of y is equivalent to the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{Nat}(\hom(r,-), K) \arrow[r, "y"] \arrow[d, "N\langle f,F\rangle"'] & Kr \arrow[d, "E\langle f,F\rangle"] \\
\mathrm{Nat}(\hom(r',-), K') \arrow[r, "y"] & K'r'
\end{tikzcd}$$

where, given any $\alpha \in \mathrm{Nat}(\hom(r,-),K)$, it follows from naturality of α that

$$\begin{aligned} y\big(N\langle f,F\rangle(\alpha)\big) &= y\big(F\circ\alpha\circ(-\circ f)\big) = F\circ\alpha\circ(-\circ f)(\mathrm{id}_{r'}) = F(\alpha(f)) \\ &= F(\alpha(\mathrm{id}_{r'}\circ f)) = F(Kf(\alpha_r(\mathrm{id}_r))) = E\langle f,F\rangle\alpha_r(\mathrm{id}_r) \\ &= E\langle f,F\rangle\big(y(\alpha)\big). \end{aligned}$$

□

Definition 63. In the conditions of the Yoneda Lemma (Lemma 7) the **Yoneda functor**, $Y : D^{op} \longrightarrow \mathsf{Set}^D$, is defined with the arrow function

$$(f : s \to r) \mapsto \Big(\hom_D(f,-) : \hom_D(r,-) \longrightarrow \hom_D(s,-) \Big).$$

It can be also written as $Y' : D \longrightarrow \mathsf{Set}^{D^{op}}$.

Proposition 12. *The Yoneda functor is full and faithful.*

Proof. By Yoneda Lemma, we know that

$$y : \mathrm{Nat}(\hom(r,-),\hom(s,-)) \cong \hom(s,r)$$

is a bijection, where $y(\hom(f,-)) = f$. □

14.4 LIMITS

In the definition of product, we chose two objects of the category, we considered all possible ***cones*** over two objects and we picked the universal one. Diagramtically,

$$\begin{tikzcd} & c \arrow[dl] \arrow[dr] & \\ a & & b \end{tikzcd} \qquad \begin{tikzcd} & d \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & a \times b \arrow[dl] \arrow[dr] & \\ a & & b \end{tikzcd}$$

c is a cone and $a \times b$ is the universal one: every cone factorizes through it. In this particular case, the base of each cone is given by two objects; or, in other words, by the image of a functor from the discrete category with only two objects, called the *index category*.

We will be able to create new constructions on categories by formalizing the notion of cone and generalizing to arbitrary bases, given as functors from arbitrariliy complex index categories. Constant functors are the first step into formalizing the notion of *cone*.

Definition 64 (Constant functor)**.** The **constant functor** $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{J}}$ sends each object $c \in \mathcal{C}$ to a constant functor $\Delta c : \mathcal{J} \longrightarrow \mathcal{C}$ defined as