12

FUNCTORS AND NATURAL TRANSFORMATIONS

> "Category" has been defined in order to define "functor" and "functor" has been defined in order to define "natural transformation".
>
> – **Saunders MacLane**, *Categories for the working mathematician*, [EM42].

Functors and natural transformations were defined for the first time by Eilenberg and MacLane in [EM42] while studying Čech cohomology. While initially they were devised mainly as a language for studying homology, they have proven its foundational value with the passage of time. The notion of naturality will be a key element of our presentation of algebraic theories and categorical logic.

12.1 FUNCTORS

A ***functor*** will be interpreted as a homomorphism of categories preserving their structure. As we discussed in the previous section, the structure of a category is given by the composition of morphisms.

Definition 41 (Functor)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, a **functor** between them, $F : \mathcal{C} \to \mathcal{D}$, is given by

- an **object function**, $F : \mathrm{obj}(\mathcal{C}) \to \mathrm{obj}(\mathcal{D})$;
- and an **arrow function**, $F : \hom(A, B) \to \hom(FA, FB)$ for any two objects A, B of the category;

such that

- $F(\mathrm{id}_A) = \mathrm{id}_{FA}$, identities are preserved; and
- $F(f \circ g) = Ff \circ Fg$, the functor respects composition.

Functors can be composed as we did with morphisms. In fact, a category of categories can be defined, having functors as morphisms. In order to avoid paradoxes, we will only define the category of all small categories as a non-small category so it will not contain itself.

Definition 42 (The category of categories)**.** The category Cat is defined as the category of (small) categories as objects and functors as morphisms.

- Given two functors $F \colon \mathcal{C} \to \mathcal{B}$ and $G \colon \mathcal{B} \to \mathcal{A}$, their composite functor $G \circ F : \mathcal{C} \to \mathcal{A}$ is given by the composition of the object and arrow functions of the functors. This composition is trivially associative.
- The identity functor on a category $I_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is given by identity object and arrow functions. It is trivially neutral with respect to composition.

Definition 43 (Full functor)**.** A functor F is **full** if the arrow map between any pair of objects is surjective. That is, if every $g : FA \to FB$ is of the form Ff for some morphism $f \colon A \to B$.

Definition 44 (Faithful functor)**.** A functor F is **faithful** if the arrow map between any pair of objects is injective. That is, if, for every two arrows $f_1, f_2 : A \to B$, $Ff_1 = Ff_2$ implies $f_1 = f_2$.

It is easy to notice that the composition of faithful (respectively, full) functors is again a faithful functor (respectively, full).

Note that a faithful functor needs not to be injective on objects nor on morphisms. In particular, if A, A', B, B' are four different objects, it could be the case that $FA = FA'$ and $FB = FB'$; and, if $f : A \to B$ and $f' : A' \to B'$ were two morphisms, it could be the case that $Ff = Ff'$.

Definition 45 (Isomorphism of categories)**.** An **isomorphism of categories** is a functor T whose object and arrow functions are bijections. Equivalently, it is a functor T such that there exists an *inverse* functor S such that $T \circ S$ and $S \circ T$ are identity functors.

However, the notion of isomorphism of categories may be too strict. Sometimes, it will suffice if the two compositions $T \circ S$ and $S \circ T$ are not exactly the identity functor, but isomorphic in some sense to it. We will develop these weaker notions in the next section.

12.2 NATURAL TRANSFORMATIONS

Definition 46 (Natural transformation)**.** A **natural transformation** between two functors with the same domain and codomain, $\alpha \colon F \xrightarrow{\cdot} G$, is a family of morphisms parameterized by the objects of the domain category, $\alpha_C \colon FC \to GC$ such that the following diagram commutes

$$
\begin{tikzcd}
C \arrow[d, "f"] & & SC \arrow[r, "\tau_C"] \arrow[d, "Sf"'] & TC \arrow[d, "Tf"] \\
C' & & SC' \arrow[r, "\tau_{C'}"] & TC'
\end{tikzcd}
$$

for every arrow $f : C \to C'$.

Sometimes, it is also said that the family of morphisms τ is *natural* in its argument. This naturality property is what allows us to "translate" a commutative diagram from a functor to another.

$$
\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dd, "h"'] & \\
& B \arrow[dl, "g"] \\
C &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
FA \arrow[rr, "\tau A"] \arrow[dr, "Ff"] \arrow[dd, "Fh"'] & & GA \arrow[dr, "Gf"] \arrow[dd] & \\
& FB \arrow[rr, "\tau B"] \arrow[dl, "Fg"] & & GB \arrow[dl, "Gg"] \\
FC \arrow[rr, "\tau C"] & & GC &
\end{tikzcd}
$$

Definition 47 (Natural isomorphism)**.** A **natural isomorphism** is a natural transformation in which every component, every morphism of the parameterized family, is invertible.

The inverses of a natural transformation form another natural transformation, whose naturality follows from the naturality of the original transformation. We say that two functors T, S are **naturally isomorphic**, and we write this as $T \cong S$, if there is a natural isomorphism between them. The notion of a natural isomorphism between functors allows us to weaken the condition of strict equality that we imposed when talking about isomorphisms of categories. The generally more useful notion of *equivalence of categories* only needs the composition of the two functors to be naturally isomorphic to the identity.

Definition 48 (Equivalence of categories)**.** An **equivalence of categories** is given by two functors T and S such that its two compositions are naturally isomorphic to the identity functor, $T \circ S \cong \mathrm{id}$ and $S \circ T \cong \mathrm{id}$.

12.3 COMPOSITION OF NATURAL TRANSFORMATIONS

There is an obvious way in which two natural transformations $\sigma : R \to S$ and $\tau : S \to T$ can be composed into a new natural transformation $R \to T$; this will be used later to define categories whose objects are functors and whose morphisms are natural transformations. But there is also a different notion of composition of natural transformations, which applies two natural transformations, in parallel, to the composition of two functors.

That is, if we draw a natural transformation between two functors as a double arrow

$$
\begin{tikzcd}
A \arrow[r, bend left, "S"{name=U}] \arrow[r, bend right, "T"'{name=D}] & B
\arrow[from=U, to=D, Rightarrow, "\tau"]
\end{tikzcd}
$$

- we have a *vertical* composition of natural transformations, which, diagramatically, composes the two natural transformations of the left diagram into a transformation like in the right one

$$
\begin{tikzcd}
A \arrow[r, bend left=60, "R"] \arrow[r, "T"] \arrow[r, bend right=60, "S"'] \arrow[r, phantom, shift left=3, "\Downarrow\tau"] \arrow[r, phantom, shift right=3, "\Downarrow\sigma"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, bend left, "R"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\tau\cdot\sigma"] & B
\end{tikzcd}
$$

- and we have a *horizontal* composition of natural transformations, which composes the two natural transformations of the first diagram into the second one

$$
\begin{tikzcd}
A \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\tau"] & B \arrow[r, bend left, "S'"] \arrow[r, bend right, "T'"'] \arrow[r, phantom, "\Downarrow\sigma"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, bend left, "S\circ S'"] \arrow[r, bend right, "T\circ T'"'] \arrow[r, phantom, "\Downarrow\tau\circ\sigma"] & C
\end{tikzcd}
$$

Definition 49 (Vertical composition of natural transformations)**.** The **vertical composition** of two natural transformations $\tau : S \to T$ and $\sigma : R \to S$, denoted by $\tau \cdot \sigma$ is the family of morphisms defined by the objectwise composition of the components of the two natural transformations, that is

$$
\begin{tikzcd}
Rc \arrow[r, "Rf"] \arrow[d, "\sigma_c"] \arrow[dd, bend right=60, "(\tau\circ\sigma)_c"'] & Rc' \arrow[d, "\sigma_{c'}"] \arrow[dd, bend left=60, "(\tau\circ\sigma)_{c'}"] \\
Sc \arrow[r, "Sf"] \arrow[d, "\tau_c"] & Sc' \arrow[d, "\tau_{c'}"] \\
Tc \arrow[r, "Tf"] & Tc'
\end{tikzcd}
$$

Proposition 5 (Vertical composition is a natural transformation)**.** *The vertical composition of two natural transformations is in fact a natural transformation.*

Proof. Naturality of the composition follows from the naturality of its two factors. In other words, the commutativity of the external square on the above diagram follows from the commutativity of the two internal squares. □

Definition 50 (Horizontal composition of natural transformations)**.** The **horizontal composition** of two natural transformations $\tau\colon S \to T$ and $\tau'\colon S' \to T'$, with domains and codomains as in the following diagram

$$
\begin{tikzcd}
A \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\tau"] & B \arrow[r, bend left, "S'"] \arrow[r, bend right, "T'"'] \arrow[r, phantom, "\Downarrow\sigma"] & C
\end{tikzcd}
$$

is denoted by $\tau' \circ \tau \colon S'S \to T'T$ and is defined as the family of morphisms given by $\tau' \circ \tau = T'\tau \circ \tau' = \tau' \circ S'\tau$, that is, by the diagonal of the following commutative square

$$\begin{tikzcd} S'Sc \arrow[r, "\tau'_{Sc}"] \arrow[d, "S'\tau_c"'] \arrow[dr, "(\tau'\circ\tau)_c"] & T'Sc \arrow[d, "T'\tau_c"] \\ S'Tc \arrow[r, "\tau'_{Tc}"] & T'Tc \end{tikzcd}$$

Proposition 6 (Horizontal composition is a natural transformation)**.** *The horizontal composition of two natural transformations is in fact a natural transformation.*

Proof. It is natural as the following diagram is the composition of two naturality squares

$$\begin{tikzcd} S'Sc \arrow[r, "S'\tau"] \arrow[d, "S'Sf"] & S'Tc \arrow[r, "\tau'"] \arrow[d, "S'Tf"] & T'Tc \arrow[d, "T'Tf"] \\ S'Sb \arrow[r, "S'\tau"] & S'Tb \arrow[r, "\tau'"] & T'Tb \end{tikzcd}$$

defined respectively by the naturality of $S'\tau$ and τ'. $\square$