# 11

# CATEGORIES

We will think of a category as the algebraic structure that captures the notion of composition. A category will be built from some sort of objects linked by composable arrows; to which associativity and identity laws will apply.

Thus, a category has to rely in some notion of ***collection***. When interpreted inside set-theory, it is common to use this term to denote some unspecified formal notion of compilation of entities that could be given by sets or proper classes. We will want to define categories whose objects are all the possible sets and we will need the objects to form a proper class in order to avoid inconsistent results such as the Russell's paradox. This is why we will consider, from this approach, a particular class of categories of small set-theoretical size to be specially well-behaved.

**Definition 29** (Small and locally small categories)**.** A category will be said to be **small** if the collection of its objects can be given by a set (instead of a proper class). It will be said to be **locally small** if the collection of arrows between any two objects can be given by a set.

A different approach, however, would be to simply take the *objects* and the *arrows* as fundamental concepts of our theory. These foundational concerns will not cause any explicit problem in this presentation of category theory, so we will keep them deliberately open to both interpretations.

## 11.1 DEFINITION OF CATEGORY

**Definition 30** (Category)**.** A **category** $\mathcal{C}$, as defined in [Lan78], is given by

- $\mathcal{C}_0$ (sometimes denoted $\operatorname{obj}(\mathcal{C})$ or simply $\mathcal{C}$), a *collection* whose elements are called **objects**, and
- $\mathcal{C}_1$, a *collection* whose elements are called **morphisms**.

Every morphism $f \in \mathcal{C}_1$ is assigned two objects: a **domain**, written as $\operatorname{dom}(f) \in \mathcal{C}_0$, and a **codomain**, written as $\operatorname{cod}(f) \in \mathcal{C}_0$; a common notation for such morphism is

$$f\colon \operatorname{dom}(f) \to \operatorname{cod}(f).$$

Given two morphisms $f\colon A \to B$ and $g\colon B \to C$, there exists a **composition morphism**, written as $g \circ f\colon A \to C$; or simply by yuxtaposition, as $gf$. Morphism composition is a binary associative operation with an identity element $\mathrm{id}_A\colon A \to A$ for every object $A$, that is,

$$h \circ (g \circ f) = (h \circ g) \circ f \quad \text{and} \quad f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f,$$

for any $f, g, h$, composable morphisms.

**Definition 31** (Hom-sets)**.** The **hom-set** of two objects $A, B$ on a category is the collection of morphisms between them. It is written as $\hom(A, B)$. The set of **endomorphisms** of an object $A$ is defined as $\mathrm{end}(A) = \hom(A, A)$.

We can use a subscript, as in $\hom_{\mathcal{C}}(A, B)$ to explicitly specify the category we are working in when necessary.

## 11.2 MORPHISMS

Objects in category theory are an atomic concept and can be only studied by their morphisms; that is, by how they are related to all the objects of the category. Thus, the essence of a category is given not by its objects, but by the morphisms between them and how composition is defined.

It is so much so, that we will consider two objects essentially equivalent (and we will call them ***isomorphic***) whenever they relate to other objects in the exact same way; that is, whenever an invertible morphism between them exists. This will constitute an equivalence relation on the category.

In a certain sense, morphisms are an abstraction of the notion of the structure-preserving homomorphisms that are defined between algebraic structures. From this perspective, ***monomorphisms*** and ***epimorphisms*** can be thought as abstractions of the usual injective and surjective homomorphisms. We will see, however, how some properties that we take for granted, such as "isomorphism" meaning exactly the same as "both injective and surjective", are not true in general.

**Definition 32** (Isomorphisms)**.** A morphism $f : A \to B$ is an **isomorphism** if there exist a morphism $f^{-1} : B \to A$ such that

- $f^{-1} \circ f = \mathrm{id}_A$,
- $f \circ f^{-1} = \mathrm{id}_B$.

This morphism is called an *inverse morphism*.

We call **automorphisms** to the morphisms which are both endomorphisms and isomorphisms.

**Proposition 2** (Unicity of inverses)**.** *If the inverse of a morphism exists, it is unique. In fact, if a morphism has a left-side inverse and a right-side inverse, they are both-sided inverses and they are equal.*

*Proof.* Given $f : A \to B$ with inverses $g_1, g_2 : B \to A$; we have that

$$g_1 = g_1 \circ \mathrm{id}_A = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \mathrm{id} \circ g_2 = g_2.$$

We have used associativity of composition, neutrality of the identity and the fact that $g_1$ is a left-side inverse and $g_2$ is a right-side inverse. □

**Definition 33.** Two objects are **isomorphic** if an isomorphism between them exists. We write $A \cong B$ when $A$ and $B$ are isomorphic.

**Proposition 3** (Isomorphy is an equivalence relation)**.** *The relation of being isomorphic is an equivalence relation. In particular,*

- *the identity,* $\mathrm{id} = \mathrm{id}^{-1}$*;*
- *the inverse of an isomorphism,* $(f^{-1})^{-1} = f$*;*
- *and the composition of isomorphisms,* $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$*;*

*are all isomorphisms.*

*Proof.* We can check that those are in fact inverses. From their existance follows

- reflexivity, $A \cong A$;
- symmetry, $A \cong B$ implies $B \cong A$;
- transitivity, $A \cong B$ and $B \cong C$ imply $A \cong C$.

□

**Definition 34** (Monomorphisms and epimorphisms)**.** A **monomorphism** is a left-cancellable morphism, that is, $f : A \to B$ is a monomorphism if, for every $g, h : B \to A$,

$$f \circ g = f \circ h \implies g = h.$$

An **epimorphism** is a right-cancellable morphism, that is, $f : A \to B$ is an epimorphism if, for every $g, h : B \to A$,

$$g \circ f = h \circ f \implies g = h.$$

A morphism that is a monomorphism and an epimorphism at the same time is called a **bimorphism**.

*Remark* 1. A morphism can be a bimorphism without being an isomorphism. We will cover examples of this fact later.

**Definition 35** (Retractions and sections)**.** A **retraction** is a left inverse, that is, a morphism that has a right inverse; conversely, a **section** is a right inverse, a morphism that has a left inverse.

By virtue of Proposition 2, a morphism that is both a retraction and a section is an isomorphism. Thus, not every epimorphism is a section and not every monomorphism is a retraction.

## 11.3 TERMINAL OBJECTS, PRODUCTS AND COPRODUCTS

***Products and coproducts*** are very widespread notions in mathematics. Whenever a new structure is defined, it is common to wonder what the product or sum of two of these structures would be. Examples of products are the cartesian product of sets, the product topology or the product of abelian groups; examples of coproducts are the disjoint union of sets, topological sum or the free product of groups.

We will abstract categorically these notions in terms of *universal properties*. This viewpoint, however, is an important shift with respect to how these properties are usually defined. We will not define the product of two objects in terms of its internal structure (categorically, objects are atomic and do not have any); but in terms of all the other objects, that is, in terms of the complete structure of the category. This turns inside-out the focus of definitions. Moreover, objects defined in terms of universal properties are usually not uniquely determined, but only determined up to isomorphism. This reinforces our previous idea of considering two isomorphic objects in a category as *essentialy* the same object.

Initial and terminal objects will be a first example of this viewpoint based on universal properties.

**Definition 36** (Initial object)**.** An object $I$ is an **initial object** if every object is the domain of exactly one morphism from it. That is, for every object $A$ exists an unique morphism $i_A \colon I \to A$.

**Definition 37** (Terminal object)**.** An object $T$ is a **terminal object** (also called *final object*) if every object is the codomain of exactly one morphism to it. That is, for every object $A$ exists an unique $t_A \colon A \to T$.

**Definition 38** (Zero object)**.** A **zero object** is an object which is both initial and terminal at the same time.

**Proposition 4** (Initial and final objects are essentially unique)**.** *Initial and final objects in a category are essentially unique; that is, any two initial objects are isomorphic and any two final objects are isomorphic.*

*Proof.* If $A, B$ were initial objects, by definition, there would be only one morphism $f : A \to B$ and only one morphism $g : B \to A$. Moreover, there would be only an endomorphism in $\mathrm{End}(A)$ and $\mathrm{End}(B)$ which should be the identity. That implies,

- $f \circ g = \mathrm{id}$,
- $g \circ f = \mathrm{id}$.

As a consequence, $A \cong B$. A similar proof can be written for the terminal object. □

Note, however, that these objects may not exist in any given category.

**Definition 39** (Product object)**.** An object $C$ is the **product** of two objects $A, B$ on a category if there are two morphisms

$$A \xleftarrow{\pi_A} C \xrightarrow{\pi_B} B$$

such that, for any other object $D$ with two morphisms $f_1 : D \to A$ and $f_2 : D \to B$, an unique morphism $h : D \to C$, such that $f_1 = \pi_A \circ h$ and $f_2 = \pi_B \circ h$. Diagramatically,

$$\begin{tikzcd}
 & D \arrow[d, dashed, "\exists! h"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\
 & C \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
A & & B
\end{tikzcd}$$

Note that the product of two objects does not have to exist on a category; but when it exists, it is essentially unique. In fact, we will be able later to construct a category in which the product object is the final object of the category and Proposition 4 can be applied. We will write *the* product object of $A, B$ as $A \times B$.

**Definition 40** (Coproduct object)**.** An object $C$ is the **coproduct** of two objects $A, B$ on a category if there are two morphisms

$$A \xrightarrow{i_A} C \xleftarrow{i_B} B$$

such that, for any other object $D$ with two morphisms $f_1 : D \to A$ and $f_2 : D \to B$, an unique morphism $h : D \to C$, such that $f_1 = i_A \circ h$ and $f_2 = i_B \circ h$. Diagramatically,

$$\begin{tikzcd}
 & D & \\
 & C \arrow[u, dashed, "\exists! h"] & \\
A \arrow[ur, "i_A"] \arrow[uur, bend left, "f_1"] & & B \arrow[ul, "i_B"'] \arrow[uul, bend right, "f_2"']
\end{tikzcd} .$$

The same discussion we had earlier for the product can be rewritten here for the coproduct only reversing the direction of the arrows. We will write *the* coproduct of $A, B$ as $A \amalg B$. As we will see later, the notion of a coproduct is dual to the notion of product; and the same proofs can be applied on both cases, only by reversing the arrows.

## 11.4 examples of categories

*Example* 8 (Discrete categories). A category is **discrete** if it has no other morphisms than the identities. A discrete category is uniquely defined by the class of its objects and every class of objects defines a discrete category. Thus, discrete categories are classes or sets, without any additional categorical structure.

*Example* 9 (Monoids, groups). A single-object category is a **monoid**. A monoid in which every morphism is an isomorphism is a **group**.

This definition is equivalent to the usual definition of monoid if we take the morphisms as elements of the monoid and composition of morphisms as the monoid operation. Groupoids are also a particular case of categories.

*Example* 10 (Groupoids). A category in which every morphism is an isomorphism is a **groupoid**.

*Example* 11 (Partially ordered sets). Every partial ordering defines a category in which the elements are the objects and an only morphism between two objects $\rho_{a,b} : a \to b$ exists

In particular, every ordinal can be seen as a partially ordered set and defines a category.

For example, if we take the finite ordinal $[n] = (0 < \cdots < n)$, it could be interpreted as the category given by the following diagram

$$\begin{tikzcd}
0 \arrow[loop above] \arrow[r] \arrow[rr, bend right] \arrow[rrrr, bend right] & 1 \arrow[loop above] \arrow[r] \arrow[rrr, bend right] & 2 \arrow[loop above] \arrow[r] \arrow[rr, bend right] & \dots \arrow[r] & n \arrow[loop above]
\end{tikzcd}$$

in which every object $p$ has an identity arrow and a unique arrow to every $q$ such that $p \leq q$. Note how the composition of arrows can be only defined in a single way.

In a partially ordered set, the product of two objects would be its join, the coproduct would be its meet and the initial and terminal objects would be the greatest and the least element, respectively.

*Example* 12 (The category of sets). The category Set is defined as the category with all the possible sets as objects and functions between them as morphisms. It is trivial to check associativity of composition and the existence of the identity function for any set.

In this category, the product is given by the usual cartesian product

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\},$$

with the projections $\pi_A(a, b) = a$ and $\pi_B(a, b) = b$. We can easily check that, if we have $f : C \to A$ and $g : C \to B$, there is a unique function given by $h(c) = (f(c), g(c))$ such that $\pi_A \circ h = f$ and $\pi_B \circ h = g$.

The initial object in Set is given by the empty set $\varnothing$: given any set $A$, the only function of the form $f : \varnothing \to A$ is the empty one. The final object, however, is only defined up to isomorphism: given any set with a single object $\{*\}$, there exists a unique function of the form $f : A \to \varnothing$ for any set $A$; namely, the one defined as $\forall a \in A : f(a) = *$. Every two sets with exactly one object are terminal objects and they are trivially isomorphic.

Similarly, the coproduct is defined only to isomorphism. The coproduct of two sets $A, B$ is given by its disjoint union $A \sqcup B$; but this union can be defined in many different (but equivalent) ways. For instance, we can add a label to the elements of each sets before joining them in order to ensure that this will be a disjoint union; that is,

$$A \sqcup B = \{(a,0) \mid a \in A\} \cup \{(b,1) \mid b \in B\}$$

with the inclusions $i_A(a) = (a,0)$ and $i_B(b) = (b,1)$, is a possible coproduct. Given any two functions $f : A \to C$ and $g : A \to C$, there exists a unique function $h : A \sqcup B \to C$, given by

$$h(x,n) = \begin{cases} f(x) & \text{if } n = 0, \\ g(x) & \text{if } n = 1, \end{cases}$$

such that $f = h \circ i_A$ and $g = h \circ i_B$.

The category of sets is a very special category, whose properties we will study in detail later.

*Example* 13 (The category of groups). The category $\mathsf{Grp}$ is defined as the category with groups as objects and group homomorphisms between them as morphisms.

*Example* 14 (The category of R-modules). The category $R$-Mod is defined as the category with $R$-modules as objects and module homomorphisms between them as morphisms. We know that the composition of module homomorphisms and the identity are also module homomorphisms.

In particular, abelian groups form a category as $\mathbb{Z}$-modules.

*Example* 15 (The category of topological spaces). The category $\mathsf{Top}$ is defined as the category with topological spaces as objects and continuous functions between them as morphisms.