15

ADJOINTS, MONADS AND ALGEBRAS

15.1 ADJUNCTIONS

Definition 67 (Adjunction)**.** An **adjunction** from categories $\mathcal{X}$ to $\mathcal{Y}$ is a pair of functors $F\colon \mathcal{X} \to \mathcal{Y}$, $G\colon \mathcal{Y} \to \mathcal{X}$ with a bijection

$$\varphi\colon \hom(Fx, y) \cong \hom(x, Gy),$$

natural in both $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We say that F is *left-adjoint* to G and that G is *right-adjoint* to F. We write this as $F \dashv G$.

Naturality of φ means that both

$$\begin{tikzcd} \hom(Fx,y) \arrow[r, "\varphi_{x,y}"] \arrow[d, "-\circ Fh"'] & \hom(x,Gy) \arrow[d, "-\circ h"] \\ \hom(Fx',y) \arrow[r, "\varphi_{x,y}"'] & \hom(x',Gy) \end{tikzcd} \qquad \begin{tikzcd} \hom(Fx,y) \arrow[r, "\varphi_{x,y}"] \arrow[d, "k\circ -"'] & \hom(x,Gy) \arrow[d, "Gk\circ -"] \\ \hom(Fx,y') \arrow[r, "\varphi_{x,y}"'] & \hom(x,Gy') \end{tikzcd}$$

commute for every $h\colon x \to x'$ and $k\colon y \to y'$. That is, for every $f\colon Fx \to y$, $\varphi(f) \circ h = \varphi(f \circ Fh)$ and $Gk \circ \varphi(f) = \varphi(k \circ f)$. Equivalently, φ^{-1} is natural and that means that, for every $g\colon x \to Gy$, $k \circ \varphi^{-1}(g) = \varphi^{-1}(Gk \circ g)$ and $\varphi^{-1}(g) \circ Fh = \varphi^{-1}(g \circ h)$. A different and more intuitive way to write adjunctions is used by William Lawvere on his notes on logical operators (see [Law]). An adjunction $F \dashv G$ can be written as

$$\frac{Fx \xrightarrow{f} y}{\overline{}\; x \xrightarrow{\varphi(f)} Gy}$$

to emphasize that, for each morphism $f : Fx \to y$, there exists a unique morphism $\varphi(f) : x \to Gy$; written in a way that resembles bidirectional logical inference rules.

Naturality, in this setting, means that the precomposition and the postcomposition of arrows are preserved by the *inference rule*. Given morphisms $h\colon x' \to x$ and $k\colon y \to y'$, we have that the composition arrows of the following diagrams are adjoint to one another

$$
\begin{array}{c}
\begin{tikzcd} Fx' \arrow[r, "Fh"] \arrow[rr, bend left, "f\circ Fh"] & Fx \arrow[r, "f"] & y \end{tikzcd} \\
\hline\hline
\begin{tikzcd} x' \arrow[r, "h"'] \arrow[rr, bend right, "\varphi(f)\circ h"'] & x \arrow[r, "\varphi(f)"'] & Gy \end{tikzcd}
\end{array}
\qquad
\begin{array}{c}
\begin{tikzcd} Fx \arrow[r, "f"] \arrow[rr, bend left, "k\circ f"] & y \arrow[r, "k"] & y' \end{tikzcd} \\
\hline\hline
\begin{tikzcd} x \arrow[r, "\varphi(f)"'] \arrow[rr, bend right, "Gk\circ\varphi(f)"'] & Gy \arrow[r, "Gk"'] & Gy' \end{tikzcd}
\end{array}
$$

In other words, that $\varphi(f) \circ h = \varphi(f \circ Fh)$ and $Gk \circ \varphi(f) = \varphi(k \circ f)$, as we wrote earlier. In the following two propositions, we will characterize all this information in terms of natural transformations made up of universal arrows.

Definition 68 (Unit and counit of an adjunction)**.** Given an adjunction $F \dashv G$, we define

- the **unit** η as the family of morphisms $\eta_x = \varphi(\mathrm{id}_{Fx})\colon x \to GFx$, for each x;
- the **counit** ε as the family of morphisms $\varepsilon_y = \varphi^{-1}(\mathrm{id}_{Gy})\colon FGy \to y$, for each y.

Diagramatically, they can be obtained by taking y to be Fx and x to be Gy, respectively in the definition of adjunction

$$
\begin{array}{c}
Fx \xrightarrow{\mathrm{id}} Fx \\
\hline\hline
x \xrightarrow{\eta_x} GFx
\end{array}
\qquad
\begin{array}{c}
FGy \xrightarrow{\varepsilon_y} y \\
\hline\hline
Gy \xrightarrow{\mathrm{id}} Gy
\end{array}
$$

Proposition 13 (Units and counits are natural transformations)**.** *The* ***unit*** *and the* ***counit*** *are natural transformations such that*

1. *for each* $f\colon Fx \to y$, $\varphi(f) = Gf \circ \eta_x$;
2. *for each* $g\colon x \to Gy$, $\varphi^{-1}(g) = \varepsilon_y \circ Fg$;

that follow the ***triangle identities****,* $G\varepsilon \circ \eta G = \mathrm{id}_G$ *and* $\varepsilon F \circ F\eta = \mathrm{id}_F$.

$$
\begin{tikzcd}
G \arrow[r, "\eta G"] \arrow[dr, equal] & GFG \arrow[d, "G\varepsilon"] \\
& G
\end{tikzcd}
\qquad
\begin{tikzcd}
FGF \arrow[d, "\varepsilon F"'] & F \arrow[l, "F\eta"'] \arrow[dl, equal] \\
F &
\end{tikzcd}
$$

Proof. The right and left adjunct formulas are particular instances of the naturality equations we gave in the definition of φ;

- $Gf \circ \eta = Gf \circ \varphi(\mathrm{id}) = \varphi(f \circ \mathrm{id}) = \varphi(f)$;
- $\varepsilon_y \circ Fg = \varphi^{-1}(\mathrm{id}) \circ Fg = \varphi^{-1}(\mathrm{id} \circ g) = \varphi^{-1}(g)$;

diagramatically,

$$\begin{array}{c}
\begin{tikzcd} Fx \arrow[r, "Fg"] \arrow[rr, bend left, "\varepsilon\circ Fg"] & FGy \arrow[r, "\varepsilon_y"] & y \end{tikzcd} \\ \hline\hline
\begin{tikzcd} x \arrow[r, "g"'] \arrow[rr, bend right, "g"'] & Gy \arrow[r, "\mathrm{id}"'] & Gy \end{tikzcd}
\end{array}
\qquad
\begin{array}{c}
\begin{tikzcd} Fx \arrow[r, "\mathrm{id}"] \arrow[rr, bend left, "f"] & Fx \arrow[r, "f"] & y \end{tikzcd} \\ \hline\hline
\begin{tikzcd} x \arrow[r, "\eta_x"'] \arrow[rr, bend right, "Gf\circ\eta_x"'] & GFx \arrow[r, "Gf"'] & Gy \end{tikzcd}
\end{array}$$

The naturality of η and ε can be deduced again from the naturality of φ; given any two functions $h\colon x \to y$ and $k\colon x \to y$,

- $GFh \circ \eta_x = GFh \circ \varphi(\mathrm{id}_{Fx}) = \varphi(Fh) = \varphi(\mathrm{id}_{Fx}) \circ h = \eta_x \circ h$;
- $\varepsilon_x \circ FGk = \varphi^{-1}(\mathrm{id}_{Fx}) \circ FGk = \varphi^{-1}(Gk) = k \circ \varphi^{-1}(\mathrm{id}_{Gx}) = k \circ \varepsilon_x$;

diagramatically, we can prove that the adjunct of Fh is $GFh \circ \eta_x$ and $\eta_x \circ h$ at the same time; while the adjunct of Gk is $k \circ \varepsilon_x$ and $\varepsilon_x \circ FGk$,

$$\begin{array}{c}
x \xrightarrow{h} y \xrightarrow{\eta_y} GFy \\ \hline\hline
Fx \xrightarrow{\mathrm{id}} Fx \xrightarrow{Fh} Fy \xrightarrow{\mathrm{id}} Fy \\ \hline\hline
x \xrightarrow[\eta_x]{} GFx \xrightarrow[GFh]{} GFy
\end{array}
\qquad
\begin{array}{c}
FGx \xrightarrow{\varepsilon_x} x \xrightarrow{k} y \\ \hline\hline
Gx \xrightarrow{\mathrm{id}} Gx \xrightarrow{Gk} Gy \xrightarrow{\mathrm{id}} Gy \\ \hline\hline
FGx \xrightarrow[FGk]{} FGy \xrightarrow[\varepsilon_x]{} y
\end{array}$$

Finally, the triangle identities follow directly from the previous ones,

- $\mathrm{id} = \varphi(\varepsilon) = G\varepsilon \circ \eta$;
- $\mathrm{id} = \varphi^{-1}(\eta) = \varepsilon \circ F\eta$. □

Proposition 14 (Characterization of adjunctions). *Each adjunction is $F \dashv G$ between categories $\mathcal{X}$ and $\mathcal{Y}$ is completely determined by any of the following data,*

1. *functors F, G and $\eta\colon 1 \dot{\to} GF$ where $\eta_x\colon x \to GFx$ is universal to G.*
2. *functor G and universals $\eta_x\colon x \to GF_0x$, where $F_0x \in \mathcal{Y}$, creating a functor F.*
3. *functors F, G and $\varepsilon\colon FG \dot{\to} 1$ where $\varepsilon_a\colon FGa \to a$ is universal from F.*
4. *functor F and universals $\varepsilon_a\colon FG_0a \to a$, where $G_0x \in \mathcal{X}$, creating a functor G.*
5. *functors F, G, with natural transformations satisfiying the triangle identities $G\varepsilon \circ \eta G = \mathrm{id}$ and $\varepsilon F \circ F\eta = \mathrm{id}$.*

Proof. 1. Universality of η_x gives a isomorphism $\varphi\colon \hom(Fx, y) \cong \hom(x, Gy)$ between the arrows in the following diagram

$$\begin{tikzcd} & Gy & & y \\ x \arrow[ur, "f"] \arrow[r, "\eta_x"'] & GFx \arrow[u, dashed, "Gg"'] & & Fx \arrow[u, dashed, "\exists! g"'] \end{tikzcd}$$

defined as $\varphi(g) = Gg \circ \eta_x$. This isomorphism is natural in x; for every $h\colon x' \to x$ we know by naturality of η that $Gg \circ \eta \circ h = G(g \circ Fh) \circ \eta$. The isomorphism is also natural in y; for every $k\colon y \to y'$ we know by functoriality of G that $Gh \circ Gg \circ \eta = G(h \circ g) \circ \eta$.

2. We can define a functor F on objects as $Fx = F_0 x$. Given any $h\colon x \to x'$, we can use the universality of η to define Fh as the unique arrow making this diagram commute

$$\begin{tikzcd}
& GFx' & Fx' \\
x \arrow[ur, "\eta_{x'}\circ h"] \arrow[r, "\eta_x"'] & GFx \arrow[u, dashed, "GFh"'] & Fx \arrow[u, dashed, "\exists!Fh"']
\end{tikzcd}$$

and this choice makes F a functor and η a natural transformation, as it can be checked in the following diagrams using the existence and uniqueness given by the universality of η in both cases

$$\begin{tikzcd}
& & & x'' \arrow[r, "\eta_{x''}"] & GFx'' & Fx'' \\
& GFx & Fx & x' \arrow[u, "h'"] \arrow[r, "\eta_{x'}"] & GFx' \arrow[u, "GFh'"'] & Fx' \arrow[u, dashed, "\exists!Fh'"] \\
x \arrow[ur, "\eta_x"] \arrow[r, "\eta_x"'] & GFx \arrow[u, dashed, "\mathrm{id}"'] & Fx \arrow[u, dashed, "\mathrm{id}"'] & x \arrow[u, "h"] \arrow[r, "\eta_x"] & GFx \arrow[u, "GFh"'] & Fx \arrow[u, dashed, "\exists!Fh'"] \arrow[uu, dashed, bend right, "\exists!F(h'\circ h)"']
\end{tikzcd}$$

3. The proof is dual to that of 1.
4. The proof is dual to that of 2.
5. We can define two functions $\varphi(f) = Gf \circ \eta_x$ and $\theta(g) = \varepsilon_y \circ Fg$. We checked in 1 (and 3) that these functions are natural in both arguments; now we will see that they are inverses of each other using naturality and the triangle identities
   - $\varphi(\theta(g)) = G\varepsilon_a \circ GFg \circ \eta_x = G\varepsilon_a \circ \eta_x \circ g = g$;
   - $\theta(\varphi(f)) = \varepsilon \circ FGf \circ F\eta = f \circ \varepsilon \circ F\eta = f$.

□

Proposition 15 (Essential uniqueness of adjoints)**.** *Two adjoints to the same functor $F, F' \dashv G$ are naturally isomorphic.*

Proof. Note that the two different adjunctions give two units η, η', and for each x both $\eta_x\colon x \to GFx$ and $\eta'_x\colon x \to GF'x$ are universal arrows from x to G. As universal arrows are unique up to isomorphism, we have a unique isomorphism $\theta_x\colon Fx \to F'x$ such that $G\theta_x \circ \eta_x = \eta'_x$.

We know that θ is natural because there are two arrows, $\theta \circ Ff$ and $F'f \circ \theta$, making this universal diagram commute

$$\begin{tikzcd}
y \arrow[r, "\eta'"] & GF'y & F'y \\
x \arrow[u, "f"] \arrow[r, "\eta"] & GFx \arrow[u, dashed] & Fx \arrow[u, dashed, "\exists!"]
\end{tikzcd}$$

because

- $G(\theta \circ Ff) \circ \eta = G\theta \circ GFf \circ \eta = G\theta \circ \eta \circ f = \eta' \circ f;$
- $G(F'f \circ \theta) \circ \eta = GF'f \circ G\theta \circ \eta = GF'f \circ \eta' = \eta' \circ f;$

thus, they must be equal, $\theta \circ Ff = F'f \circ \theta$. □

Theorem 9 (Composition of adjunctions)**.** *Given two adjunctions between categories $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Y}, \mathcal{Z}$ respectively,*

$$\varphi\colon \hom(Fx, y) \cong \hom(x, Gy) \qquad \theta\colon \hom(F'y, z) \cong \hom(y, G'z)$$

the composite functors yield a composite adjunction

$$\varphi \cdot \theta\colon \hom(F'Fx, z) \cong \hom(x, GG'z).$$

If the unit and counit of φ are $\langle \eta, \varepsilon \rangle$ and the unit and counit of θ are $\langle \eta', \varepsilon' \rangle$; the unit and counit of the composite adjunction are $\langle G\eta' F \circ \eta,\ \varepsilon' \circ F'\varepsilon G' \rangle$.

Proof. We saw previously that the componentwise composition of two natural isomorphisms is itself an natural isomorphism. Diagramatically, we compose

$$\begin{array}{c}
F'Fx \xrightarrow{\ f\ } y \\
\hline\hline
Fx \xrightarrow{\theta(f)} G'y \\
\hline\hline
x \xrightarrow{\varphi\theta(f)} GG'y
\end{array}$$

If we apply the two natural isomorphisms to the identity, we find the unit and the counit of the adjunction.

$$\begin{array}{c}
F'Fx \xrightarrow{\mathrm{id}} F'Fx \\
\hline\hline
Fx \xrightarrow{\mathrm{id}} Fx \xrightarrow{\eta'_{Fx}} G'F'Fx \\
\hline\hline
x \xrightarrow[\eta]{} GFx \xrightarrow[G\eta'_{Fx}]{} GG'F'Fx
\end{array}
\qquad
\begin{array}{c}
GG'z \xrightarrow{\mathrm{id}} GG'z \\
\hline\hline
FGG'z \xrightarrow{\varepsilon_{G'z}} G'z \xrightarrow{\mathrm{id}} G'z \\
\hline\hline
F'FGG'z \xrightarrow[F'\varepsilon_{G'z}]{} F'G'z \xrightarrow[\varepsilon']{} z
\end{array}$$

□

15.2 EXAMPLES OF ADJOINTS

Example 22 (Product and coproduct as adjoints). Given any category $\mathcal{C}$, we define a **diagonal functor** to a product category $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, sending every object x to a pair (x, x), and each morphism $f : x \to y$ to the pair $\langle f, f\rangle\colon (x, x) \to (y, y)$.

The right adjoint to this functor will be the categorical product $\times\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, sending each pair of objects to their product and each pair of morphisms to their unique product. The left adjoint to this functor will be the categorical sum, $+\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, sending each pair of objects to their sum and each pair of morphisms to their unique sum. That is, we have the following chain of adjoints,

$$+ \dashv \Delta \dashv \times.$$

More precisely, knowing that a morphism $(x, x') \to (y, z)$ is actually pair of morphisms $x \to y$ and $x' \to z$, the adjoint properties for the diagonal functor

$$\frac{\Delta x \longrightarrow (y,z)}{x \longrightarrow \times(y,z)} \qquad \frac{+(x,y) \longrightarrow z}{(x,y) \longrightarrow \Delta z}$$

can be rewritten as bidirectional inference rules with two premises

$$\frac{x \to y \qquad x \to z}{x \longrightarrow y \times z} \qquad \frac{x + y \longrightarrow z}{x \to z \qquad y \to z}$$

which are exactly the universal properties of the product and the sum. The necessary natural isomorphism is given by the existence and uniqueness provided by the inference rule.

Example 23 (Free and forgetful functors). Let Mon be the category of monoids with monoid homomorphisms. A functor $U\colon \mathsf{Mon} \to \mathsf{Set}$ can be defined by sending each morphism to its underlying set and each monoid homomorphism to its underlying function between sets. Funtors of this kind are called **forgetful functors**, as they simply *forget* part of the algebraic structure.

Left adjoints to forgetful functors are called **free functors**. In this case, the functor $F\colon \mathsf{Set} \to \mathsf{Mon}$ taking each set to the free monoid over it and each function to its unique extension to the free monoid. Note how it preserves identities and composition. The adjunction can be seen diagramatically as

$$\frac{FA \xrightarrow{\overline{f}} M}{A \xrightarrow{f} UM}$$

where each monoid homormorphism from the free monoid, $FA \to M$ can be seen as the unique possible extension of a function from the set of generators $f\colon A \to UM$ to a full monoid homomorphism, $\overline{f}$.

Note how, while this characterizes the notion of free monoid, it does not provide an explicit construction of it. Indeed, given $f\colon A \to UM$, if we take the free monoid FA to consist on words over the elements of A endowed with the concatenation operator; the only way to extend f to an homomorphism is to define

$$\overline{f}(a_1 a_2 \dots a_n) = f(a_1) f(a_2) \dots f(a_n);$$

and note how every homomorphism from the free monoid is determined by how it acts on the generator set.

The notion of forgetful and free functors can be generalized to algebraic structures other than monoids.

15.3 MONADS

The notion of **monads** is pervasive in category theory. A monad is a certain type of endofunctor that arises naturally when considering adjoints. They will be useful to model algebraic notions inside category theory and to model a variety of effects and contextual computations in functional programming languages.

Definition 69 (Monad)**.** A **monad** is a functor $T\colon X \to X$ with natural transformations

- $\eta\colon I \dot{\to} T$, called *unit*; and
- $\mu\colon T^2 \dot{\to} T$, called *multiplication*;

such that the following two diagrams commute

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} IT \arrow[r, "\eta T"] \arrow[dr, "\cong"'] & T^2 \arrow[d, "\mu"] & TI \arrow[l, "T\eta"'] \arrow[dl, "\cong"] \\ & T & \end{tikzcd} .$$

The first diagram is encoding some form of associativity of the multiplication, while the second one is encoding the fact that η creates a neutral element with respect to this multiplication. These statements will be made precise when we talk about algebraic theories.

Proposition 16 (Each adjunction gives rise to a monad)**.** *Given $F \dashv G$, the composition GF is a monad.*

Proof. We take the unit of the adjunction as the monad unit. We define the product as $\mu = G\varepsilon F$. Associativity follows from these diagrams

$$\begin{tikzcd} FGFG \arrow[r, "FG\varepsilon"] \arrow[d, "\varepsilon FG"'] & FG \arrow[d, "\varepsilon"] \\ FG \arrow[r, "\varepsilon"] & I \end{tikzcd} \qquad \begin{tikzcd} GFGFGF \arrow[r, "GFG\varepsilon F"] \arrow[d, "G\varepsilon FGF"'] & GFGF \arrow[d, "G\varepsilon F"] \\ GFGF \arrow[r, "G\varepsilon F"] & GF \end{tikzcd},$$

where the first is commutative by Proposition [BROKEN LINK: prop-interchangelaw] and the second is obtained by applying functors G and F. Unit laws follow from the [BROKEN LINK: *Unit and counit] after applying F and G. □

Definition 70 (Comonad)**.** A **comonad** is a functor $L\colon X \to X$ with natural transformations

- $\varepsilon\colon L \to I$, called *counit*
- $\delta\colon L \to L^2$, called *comultiplication*

such that

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L^2 \arrow[d, "L\delta"] \\ L^2 \arrow[r, "\delta L"] & L^3 \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, "\cong"'] \arrow[d, "\delta"] \arrow[dr, "\cong"] & \\ IL & L^2 \arrow[l, "\varepsilon L"] \arrow[r, "L\varepsilon"'] & LI \end{tikzcd}.$$

15.4 ALGEBRAS FOR A MONAD

Definition 71 (T-algebra)**.** For a monad T, a T**-algebra** is an object x with an arrow $h\colon Tx \to x$ called *structure map* making these diagrams commute

$$\begin{tikzcd} T^2x \arrow[r, "Th"] \arrow[d, "\mu"'] & Tx \arrow[d, "h"] \\ Tx \arrow[r, "h"] & x \end{tikzcd}$$

Definition 72 (Morphism of T-algebras)**.** A **morphism of T-algebras** is an arrow $f\colon x \to x'$ making the following square commute

$$\begin{tikzcd} Tx \arrow[r, "h"] \arrow[d, "Tf"'] & Tx \arrow[d, "f"] \\ Tx' \arrow[r, "h'"'] & Tx' \end{tikzcd}$$

Proposition 17 (Category of T-algebras)**.** *The set of all T-algebras and their morphisms form a category X^T.*

Proof. Given $f\colon x \to x'$ and $g\colon x' \to x''$, T-algebra morphisms, their composition is also a T-algebra morphism, due to the fact that this diagram

$$\begin{tikzcd}
Tx \arrow[r, "h"] \arrow[d, "Tf"'] & x \arrow[d, "f"] \\
Tx' \arrow[r, "h'"] \arrow[d, "Tg"'] & x' \arrow[d, "g"] \\
Tx'' \arrow[r, "h''"] & x''
\end{tikzcd}$$

commutes. □

15.5 KLEISLI CATEGORIES

Definition 73 (Kleisli category)**.** The **Kleisli category** of a monad $T\colon \mathcal{X} \to \mathcal{X}$ is written as $\mathcal{X}_T$ and is given by

- an object x_T for every $x \in \mathcal{X}$; and
- an arrow $f^\flat\colon x_T \to y_T$ for every $f\colon x \to Ty$.

And the composite of two morphisms is defined as

$$g^\flat \circ f^\flat = (\mu \circ Tg \circ f)^\flat.$$

Theorem 10 (Adjunction on a Kleisli category)**.** *The functors $F_T\colon \mathcal{X} \to \mathcal{X}_T$ and $G\colon \mathcal{X}_T \to \mathcal{X}$ from and to the Kleisli category which are defined on objects as $F_T(x) = x_T$ and $G_T(x_T) = Tx$, and defined on morphisms as*

$$\begin{aligned}
F_T\colon (k\colon x \to y) \quad &\mapsto (\eta_y \circ k)^\flat\colon x_T \to y_T \\
G_T\colon (f^\flat\colon x_T \to y_T) \quad &\mapsto (\eta_y \circ Tf)\colon Tx \to Ty
\end{aligned}$$

form an adjunction $\flat\colon \hom(x, Ty) \to \hom(x_T, y_T)$ whose monad is precisely T.

Proof. □