$\boxed{\text{Intermediate Goods Lagrangian (good } x_3\text{)}}$

$$\max_{x_3, x_4, n, \ell, e} \; p_3 x_3 - p_4 x_4 - p_n n - p_\ell \ell - S e^k \quad \text{s.t. } f(x_4, n, \ell, e) \geq x_3$$

$$\left(\text{where } f = A x_4^\beta n^\gamma \ell^\delta e^\varepsilon\right)$$

$$\mathcal{L}: \; p_3 x_3 - p_4 x_4 - p_n n - p_\ell \ell - S e^k + \lambda\left(A x_4^\beta n^\gamma \ell^\delta e^\varepsilon - x_3\right)$$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = A x_4^\beta n^\gamma \ell^\delta e^\varepsilon - x_3 = 0 \implies x_3 = A x_4^\beta n^\gamma \ell^\delta e^\varepsilon$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = p_3 - \lambda = 0 \implies p_3 = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial x_4} = -p_4 + \lambda A \beta x_4^{\beta-1} n^\gamma \ell^\delta e^\varepsilon = 0 \implies p_4 = \lambda A \beta x_4^{\beta-1} n^\gamma \ell^\delta e^\varepsilon$$

$$\frac{\partial \mathcal{L}}{\partial n} = -p_n + \lambda A \gamma x_4^\beta n^{\gamma-1} \ell^\delta e^\varepsilon = 0 \implies p_n = \lambda A \gamma x_4^\beta n^{\gamma-1} \ell^\delta e^\varepsilon$$

$$\frac{\partial \mathcal{L}}{\partial \ell} = -p_\ell + \lambda A \delta x_4^\beta n^\gamma \ell^{\delta-1} e^\varepsilon = 0 \implies p_\ell = \lambda A \delta x_4^\beta n^\gamma \ell^{\delta-1} e^\varepsilon$$

$$\frac{\partial \mathcal{L}}{\partial e} = -kSe^{k-1} + \lambda A \varepsilon x_4^\beta n^\gamma \ell^\delta e^{\varepsilon-1} \implies kSe^{k-1} = \lambda A \varepsilon x_4^\beta n^\gamma \ell^\delta e^{\varepsilon-1}$$

$$\Downarrow$$

$$p_3 = \frac{p_4}{A\beta x_4^{\beta-1} n^\gamma \ell^\delta e^\varepsilon} = \frac{p_n}{A\gamma x_4^\beta n^{\gamma-1} \ell^\delta e^\varepsilon} = \frac{p_\ell}{A\delta x_4^\beta n^\gamma \ell^{\delta-1} e^\varepsilon} = \frac{kSe^{k-1}}{A\varepsilon x_4^\beta n^\gamma \ell^\delta e^{\varepsilon-1}}$$

If $p_3 = p_4 = p_n = p_\ell = 1$, then $\dfrac{x_4}{\beta} = \dfrac{n}{\gamma} = \dfrac{\ell}{\delta} = \dfrac{kSe^k}{\varepsilon}$

$$x_4 = \sqrt[\beta-1]{\frac{p_4}{A\beta p_3 n^\gamma \ell^\delta e^\varepsilon}} = \sqrt[\beta]{\frac{p_n}{A\gamma p_3 n^{\gamma-1} \ell^\delta e^\varepsilon}} = \sqrt[\beta]{\frac{p_\ell}{A\delta p_3 n^\gamma \ell^{\delta-1} e^\varepsilon}} = \sqrt[\beta]{\frac{kSe^{k-1}}{A\varepsilon p_3 n^\gamma \ell^\delta e^{\varepsilon-1}}}$$

$$n = \sqrt[\gamma]{\frac{p_4}{A\beta x_4^{\beta-1} p_3 \ell^\delta e^\varepsilon}} = \sqrt[\gamma-1]{\frac{p_n}{A\cancel{\gamma} x_4^\beta p_3 \ell^\delta e^\varepsilon}} = \sqrt[\gamma]{\frac{p_\ell}{A\delta x_4^\beta p_3 \ell^{\delta-1} e^\varepsilon}} = \sqrt[\gamma]{\frac{kSe^{k-1}}{A\varepsilon x_4^\beta p_3 \ell^\delta e^{\varepsilon-1}}}$$

$$\ell = \sqrt[\delta]{\frac{p_4}{A\beta x_4^{\beta-1} n^\gamma p_3 e^\varepsilon}} = \sqrt[\delta]{\frac{p_n}{A\gamma x_4^\beta n^{\gamma-1} p_3 e^\varepsilon}} = \sqrt[\delta-1]{\frac{p_\ell}{A\delta x_4^\beta n^\gamma p_3 e^\varepsilon}} = \sqrt[\delta]{\frac{kSe^{k-1}}{A\varepsilon x_4^\beta n^\gamma p_3 e^\varepsilon}}$$

$$e = \sqrt[\varepsilon]{\frac{p_4}{A\beta x_4^{\beta-1} n^\gamma \ell^\delta p_3}} = \sqrt[\varepsilon]{\frac{p_n}{A\gamma x_4^\beta n^\gamma \ell^\delta p_3}} = \sqrt[\varepsilon]{\frac{p_\ell}{A\delta x_4^\beta n^\gamma \ell^\delta p_3}} = \sqrt[\varepsilon-1]{\frac{kSe^{k-1}}{A\varepsilon x_4^\beta n^\gamma \ell^\delta p_3}}$$