



# Geometric Control with Applications to Robotics

**Tutorial Session – ICIAfS 2021**

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**University of Peradeniya**

# Mechanical Systems on $\mathbb{R}^n$

→ Applied force,  $f^u$

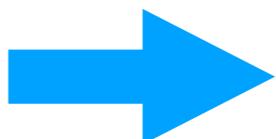


Configuration Space →  $\mathbb{R}^n$

Kinetic Energy →  $\frac{1}{2}\langle M\dot{x}, \dot{x} \rangle = \frac{M}{2}\dot{x}^T \dot{x}$

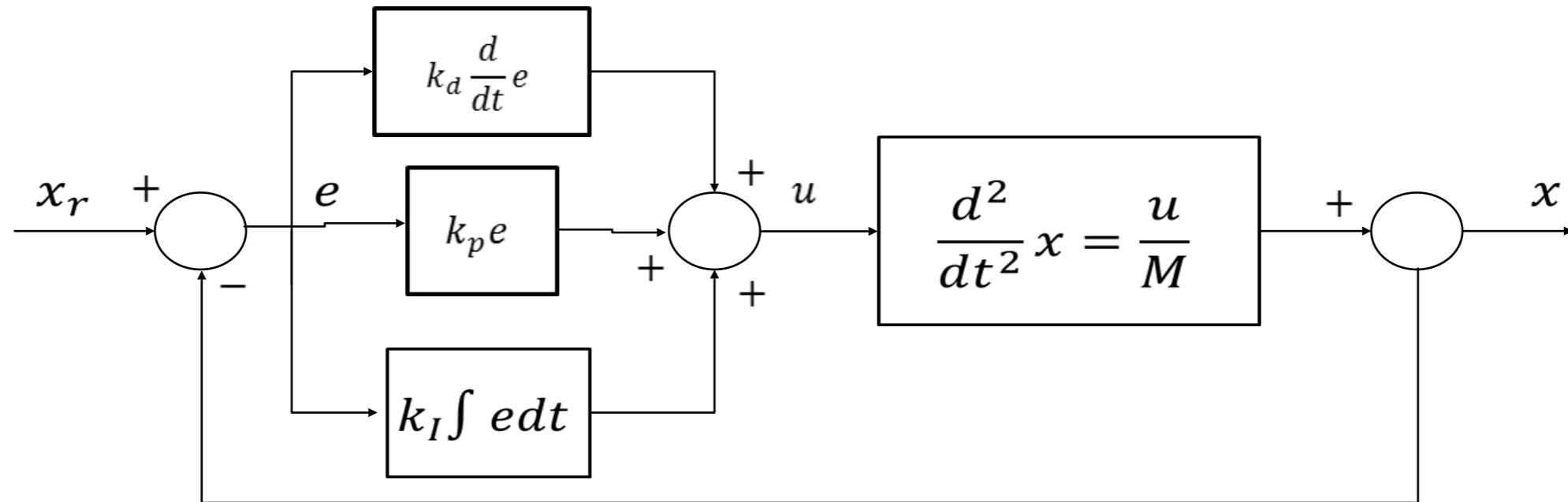
Newton's Equations

$$M\ddot{x} = f^d + f^u$$



$$\begin{aligned}\dot{x} &= v \\ M\dot{v} &= f^d + f^u\end{aligned}$$

# PID Control on $\mathbb{R}^n$



## PID Without Feedforward Control

$$e \triangleq (x - x_r)$$

$$\dot{e}_I = e$$

$$u_{PID} = -M(k_p e + k_d \dot{e} + k_I e_I) + M\ddot{x}_r$$

$$\lim_{t \rightarrow \infty} x(t) = x_r(t)$$

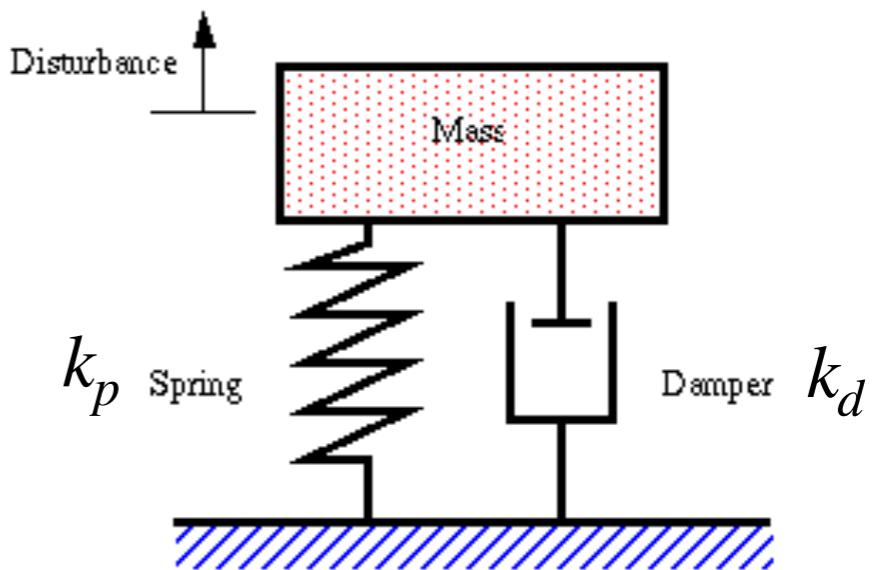
**Globally Exponentially**

- Global stability
- Robust
- Can track constant acceleration references
- Can reject constant input disturbances

# PD Control on $\mathbb{R}^n$

$$\begin{aligned} e &\triangleq (x - x_r) \\ f^u &= -M(k_p e + k_d \dot{e}) + M\ddot{x}_r \\ \ddot{e} + k_p e + k_d \dot{e} &= 0 \end{aligned}$$

**Error Dynamics**



**Momentum error**  $\rightarrow p_e \triangleq M\dot{e}$

$$W(e, \dot{e}) = \frac{k_p}{2} \langle M\dot{e}, \dot{e} \rangle + \frac{1}{2} \langle p_e, \dot{e} \rangle \geq 0$$

$$f^u = -M(k_p e + k_d \dot{e}) + M\ddot{x}_r \quad \rightarrow \quad \frac{dW}{dt} = k_p \langle M\dot{e}, \dot{e} \rangle + \langle M\ddot{\dot{e}}, \dot{e} \rangle = -k_d \langle M\dot{e}, \dot{e} \rangle \leq 0$$

**LaSalle's Invariance Principle**  $\rightarrow \lim_{t \rightarrow \infty} e(t) = 0$

**Integral Control**  $u_I = -k_I e_I$  **Adds Robustness**

# Can We Always Use PID ?



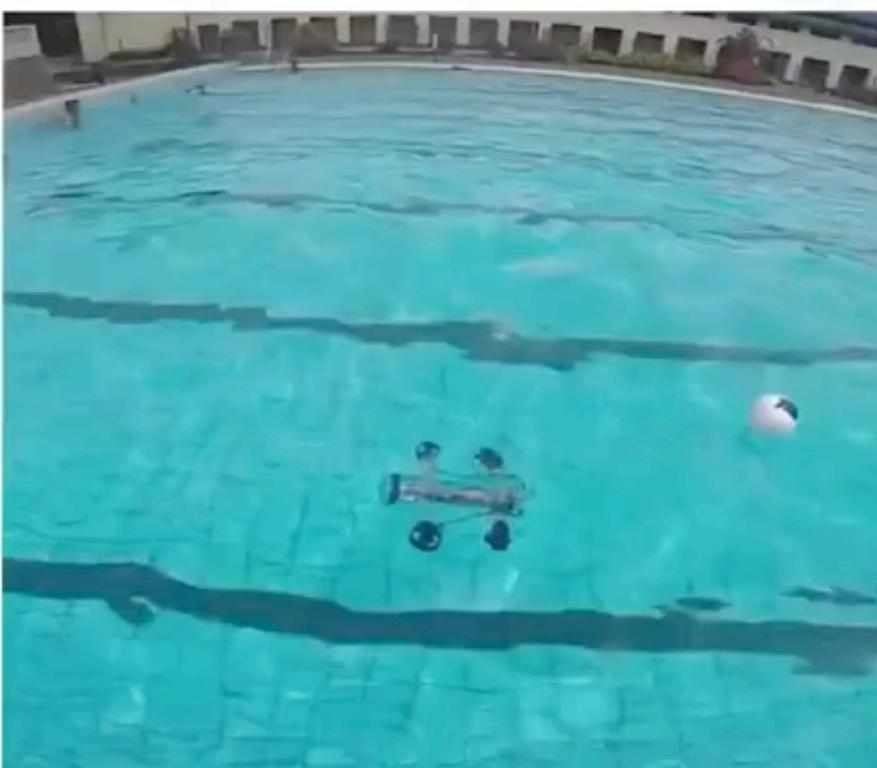
# Nonlinear Systems are Tricky



<https://youtu.be/ZNpi-IHnAh0>



# What We Have Developed



<https://youtu.be/6E9WDQNVSYA>



# Almost Globally Stable PID



[https://youtu.be/zq05N8m\\_9SA](https://youtu.be/zq05N8m_9SA)



# Fully Autonomous Control and Navigation



<https://youtu.be/SQLYFBChrAk>



# Fully Onboard Autonomous Obstacle Avoidance



## Development of an Autonomous Robot Control and Navigation Platform

Faculty of Engineering University of Peradeniya

&

CodeGen International (Pvt) Ltd

<https://youtu.be/9dfmOx7TPgs>



# **Underwater Vehicle Control and Navigation**

**Mathematical Modelling, Simulation and Testing  
of Underwater Vehicle Platform.**

Department of Mechanical Engineering  
University of Peradeniya  
Collaborate with Codegen International



University of  
**PERADENIYA**  
Sri Lanka

Powered by  
**CODEGEN**  
www\_codegen\_co\_uk

<https://youtu.be/AZrR0rp7-p0>



# Balancing Vehicle Control and Navigation

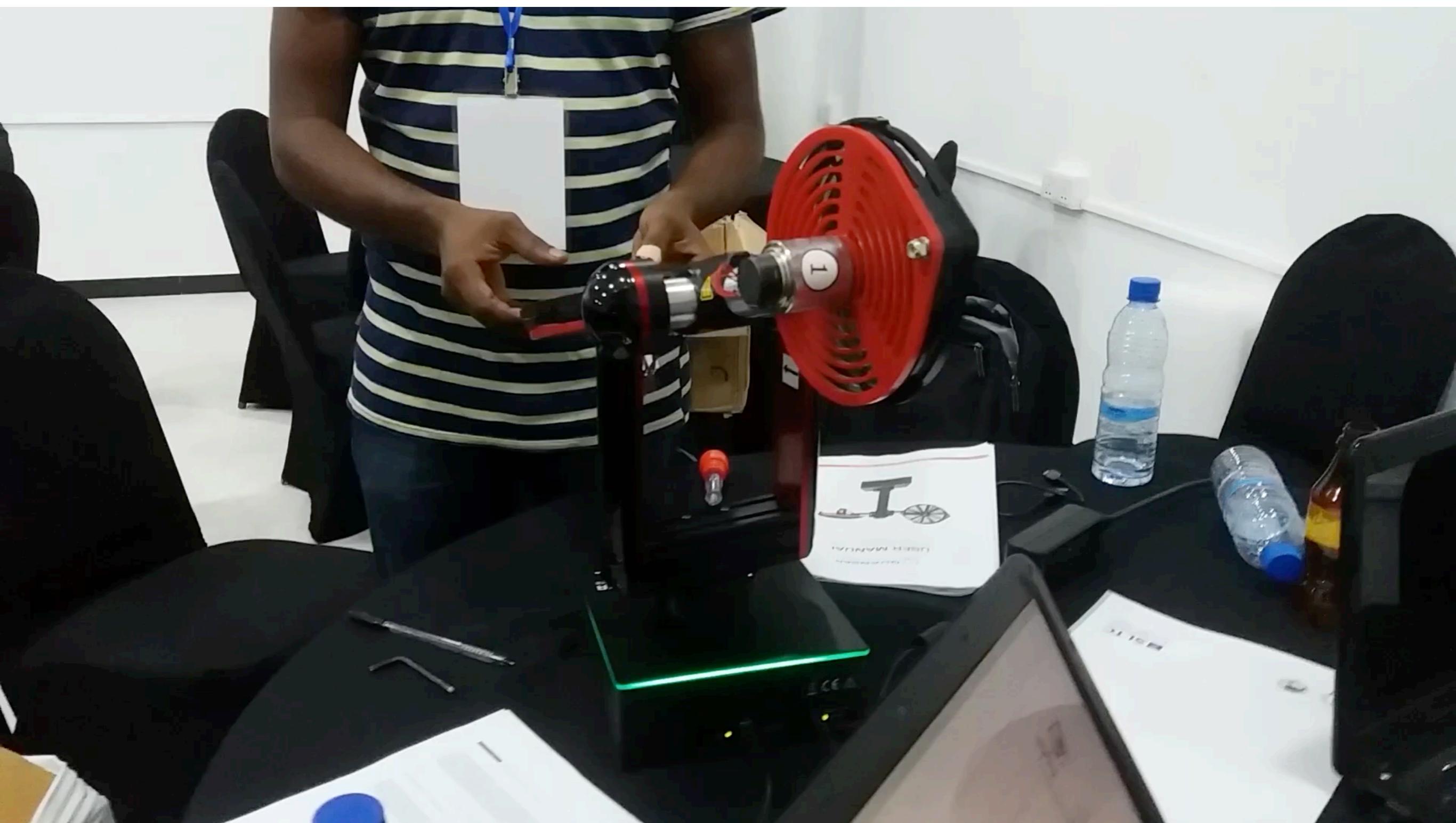


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<https://youtu.be/uUKxXImRMOA>



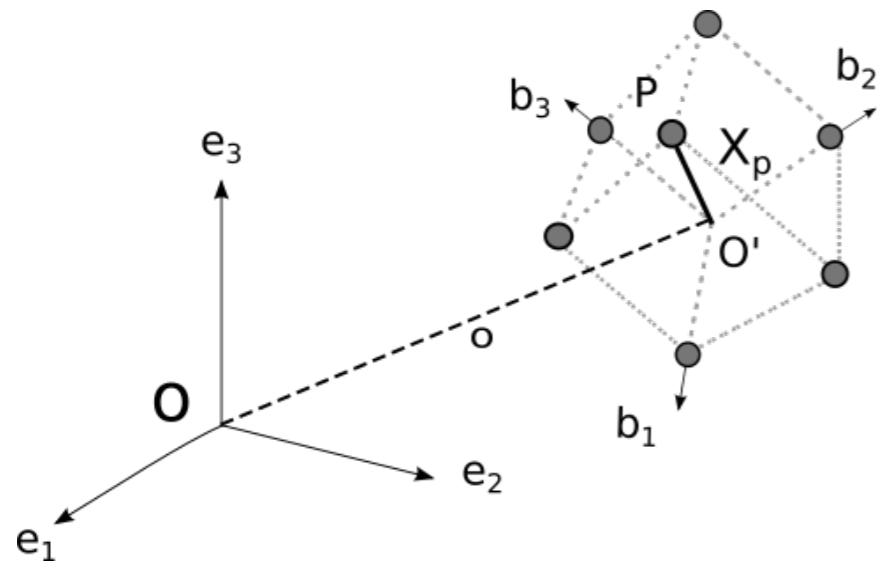
# Control of the QUANSER AEORO



<https://youtu.be/J5MMp6Be3tU>



# Rigid Body Control



$$\begin{aligned}\pi &= \mathbb{I}_R \omega \\ \dot{R} &= \widehat{\omega} R \\ \dot{\pi} &= \tau^u\end{aligned}$$

OR

$$\begin{aligned}\dot{R} &= R \widehat{\Omega} \\ \mathbb{I} \dot{\Omega} &= \mathbb{I} \Omega \times \Omega + T^u\end{aligned}$$

$$(R_r(t), \dot{R}_r = R_r \widehat{\Omega}_r(t))$$

Reference Trajectory

Control Objective



$$\begin{aligned}\omega &= R \Omega \\ \tau^u &= R T^u\end{aligned}$$

Fully actuated

find  $T^u$  such that  
 $\lim_{t \rightarrow \infty} R(t) = R_r(t)$

Almost Globally and  
Locally Exponentially

# Intrinsic PD on $SO(3)$

On  $\mathbb{R}^n$

$$\begin{aligned} e &\triangleq (x - x_r) \\ f^u &= -M(k_p e + k_d \dot{e}) + M\ddot{x}_r \\ \ddot{e} + k_p e + k_d \dot{e} &= 0 \end{aligned}$$

**Error Dynamics**

$$V: \mathbb{R}^n \rightarrow \mathbb{R} \quad \rightarrow \quad V(e) = \frac{k_p}{2} \langle M e, e \rangle$$

$$W(e, \dot{e}) = V(e) + \frac{1}{2} \langle M \dot{e}, \dot{e} \rangle \geq 0$$

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$$\frac{dW}{dt} = -k_d \langle M \dot{e}, \dot{e} \rangle \leq 0$$

On  $SO(3)$

$$\left( R_r^T \dot{R}_r = R_r \widehat{\Omega_r(t)} \right)$$

**Reference Trajectory**

$$\begin{aligned} E &= R_r^T R \\ \text{Tracking Error} \end{aligned}$$

$$\begin{array}{ccc} \textbf{Velocity Error} & \rightarrow & \dot{E} = E \cdot \underbrace{(\Omega - E^T \Omega_r)}_{\widehat{\Omega}_E} \end{array}$$

$$V: SO(3) \rightarrow \mathbb{R} \quad \rightarrow \quad V(E) = \frac{k_p}{2} \text{trace} \left( K(I - E) \right)$$

$$K = \text{diag}\{\mathbb{I}_2 + \mathbb{I}_3 - \mathbb{I}_1, \mathbb{I}_3 + \mathbb{I}_1 - \mathbb{I}_2, \mathbb{I}_1 + \mathbb{I}_2 - \mathbb{I}_3\}$$

$$E = I + \sin \theta_e \hat{n}_e + (1 - \cos \theta_e) \hat{n}_e^2$$

$$V(E) = \alpha_n \frac{k_p}{2} (1 - \cos \theta_e) \geq 0$$

$$\alpha_n \triangleq \mathbb{I}_1 n_{e_1}^2 + \mathbb{I}_2 n_{e_2}^2 + \mathbb{I}_3 n_{e_3}^2$$

# Intrinsic PD on $SO(3)$

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$$V(E) = \alpha_n \frac{k_p}{2} (1 - \cos \theta_e) \geq 0$$

**Unique minimum when**  $E = I_{3 \times 3}$

**Three maxima when**  $E = \text{diag}\{1, -1, -1\}$

$$E = \text{diag}\{-1, 1, -1\} \quad E = \text{diag}\{-1, -1, 1\}$$

$$\alpha_n \triangleq \mathbb{I}_1 n_{e_1}^2 + \mathbb{I}_2 n_{e_2}^2 + \mathbb{I}_3 n_{e_3}^2$$

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$$W(E, \Omega_E) = V(E) + \frac{1}{2} \langle \mathbb{I} \Omega_E, \Omega_E \rangle \geq 0$$

$$\frac{d}{dt} W(E, \Omega_E) = \frac{d}{dt} V(E) + \frac{1}{2} \frac{d}{dt} \langle \mathbb{I} \Omega_E, \Omega_E \rangle$$

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$$\frac{d}{dt} V(E) = -\frac{k_p}{2} \text{trace}(K \dot{E}) = -\frac{k_p}{2} \text{trace}(K E \widehat{\Omega}_E)$$

$$= -\frac{k_p}{2} \text{trace} \left( \underbrace{\frac{(E - E^T)}{2}}_{\widehat{\zeta}_E} K \widehat{\Omega}_E \right) = k_p \langle \mathbb{I} \zeta_E, \Omega_E \rangle$$

$$K = \text{diag}\{\mathbb{I}_2 + \mathbb{I}_3 - \mathbb{I}_1, \mathbb{I}_3 + \mathbb{I}_1 - \mathbb{I}_2, \mathbb{I}_1 + \mathbb{I}_2 - \mathbb{I}_3\}$$

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$$\frac{d}{dt} \langle \pi_E, \omega_E \rangle = \frac{d}{dt} \langle \mathbb{I} \Omega_E, \Omega_E \rangle = 2 \langle \mathbb{I} \dot{\Omega}_E, \Omega_E \rangle$$

# Intrinsic PD on SO(3)

On  $\mathbb{R}^n$

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$$\text{Velocity Error} \quad \rightarrow \quad \dot{E} = E \cdot \underbrace{\left( \widehat{\Omega} - E^T \Omega_r \right)}_{\widehat{\Omega}_E}$$

$$\begin{aligned} \|\dot{\Omega}_E\| &= \|\dot{\Omega} - \|E^T \dot{\Omega}_r + \|\Omega_E \times (E^T \Omega_r)\| \\ &= \|\Omega \times \Omega + T^u - \|E^T \dot{\Omega}_r + \|\Omega_E \times (E^T \Omega_r)\| \end{aligned}$$

$$\begin{aligned} \|\Omega \times \Omega\| &= \|\Omega_E + E^T \Omega_r\| \times \|\Omega_E + E^T \Omega_r\| \\ &= \|\Omega_E \times \Omega_E + \|\Omega_E \times E^T \Omega_r + \|E^T \Omega_r \times \Omega_E + \|E^T \Omega_r \times E^T \Omega_r \end{aligned}$$

$$\|\dot{\Omega}_E\| = \|\Omega_E \times \Omega_E + T^u - F_r(E, \Omega_E, \Omega_r)\|$$

$$F_r = \|E^T \dot{\Omega}_r - 2\|\Omega_E \times (E^T \Omega_r) - \|E^T \Omega_r \times \Omega_E - \|E^T \Omega_r \times E^T \Omega_r\|$$

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$$\begin{array}{ccc} \textbf{Velocity Error} & \rightarrow & \dot{E} = E \cdot \underbrace{(\Omega - E^T \Omega_r)}_{\widehat{\Omega}_E} \\ \textbf{grad of } \mathbf{V} & \rightarrow & \nabla V \triangleq \zeta_E \end{array}$$

$$\begin{aligned} \|\dot{\Omega}_E\| &= \|\Omega_E \times \Omega_E + T^u - F_r(E, \Omega_E, \Omega_r)\| \\ \frac{1}{2} \frac{d}{dt} \langle \pi_E, \omega_E \rangle &= \langle \dot{\Omega}_E, \Omega_E \rangle = \langle (T^u - F_r), \Omega_E \rangle \end{aligned}$$

$$\frac{d}{dt} V(E) = k_p \langle \zeta_E, \Omega_E \rangle$$

$$W(E, \Omega_E) = V(E) + \frac{1}{2} \langle \|\Omega_E\|, \Omega_E \rangle \geq 0$$

$$\frac{d}{dt} W(E, \Omega_E) = \left\langle (k_p \zeta_E + T^u - F_r), \Omega_E \right\rangle$$

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On  $\mathbb{R}^n$

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$$\text{grad of } V \quad \rightarrow \quad \nabla V \triangleq \zeta_E$$

$$\frac{d}{dt} W(E, \Omega_E) = \left\langle (k_p \mathbb{I} \zeta_E + T^u - F_r), \Omega_E \right\rangle$$

$$T^u = -\mathbb{I}(k_p \zeta_E + k_d \Omega_E) + F_r$$

$$\frac{d}{dt} W(E, \Omega_E) = -k_d \langle \mathbb{I} \Omega_E, \Omega_E \rangle \leq 0$$

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$$\left( R_r(t), \dot{R}_r = R_r \widehat{\Omega_r(t)} \right)$$

**Reference Trajectory**

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**Tracking Error**

$$\text{Velocity Error} \rightarrow \dot{E} = E \cdot (\widehat{\Omega} - \overbrace{E^T \Omega_r}^{\widehat{\Omega}_E})$$

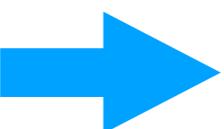
$$\text{grad of } V \rightarrow \nabla V \triangleq \zeta_E$$

$$V: SO(3) \rightarrow \mathbb{R} \rightarrow V(E) = \frac{k_p}{2} \text{trace} (K(I - E))$$

$$W(E, \Omega_E) = V(E) + \frac{1}{2} \langle \mathbb{I}\Omega_E, \Omega_E \rangle$$

$$T^u = -\mathbb{I}(k_p \zeta_E + k_d \Omega_E) + F_r$$

**Intrinsic PD control**



$$\frac{d}{dt} W(E, \Omega_E) = -k_d \langle \mathbb{I}\Omega_E, \Omega_E \rangle \leq 0$$

**LaSalle's Invariance Principle**

$$\mathbb{I}\dot{\Omega}_E = \mathbb{I}\Omega_E \times \Omega_E - (k_p \zeta_E + k_d \Omega_E)$$

$$\mathbb{I}\nabla_{\Omega_E} \Omega_E \triangleq \mathbb{I}\dot{\Omega}_E - \mathbb{I}\Omega_E \times \Omega_E$$

$$\mathbb{I}(\nabla_{\Omega_E} \Omega_E + k_p \zeta_E + k_d \Omega_E) = 0$$

**Error Dynamics on SO(3)**



$$\lim_{t \rightarrow \infty} R(t) = R_r(t)$$

**For all ICs except three !**

# Intrinsic PID

**Velocity Error**       $\rightarrow \dot{E} = E \cdot (\widehat{\Omega - E^T \Omega_r})$

**grad of V**       $\rightarrow \nabla V \triangleq \zeta_E \underbrace{\widehat{\Omega}_E}_{\widehat{\Omega}_E}$

$$\mathbb{I} \nabla_{\zeta_E} \Omega_I = \mathbb{I} \zeta_E,$$

$$T^u = - \mathbb{I} (k_p \zeta_E + k_d \Omega_E + k_I \Omega_I)$$

$$V : SO(3) \rightarrow \mathbb{R} \quad \rightarrow \quad V(E) = \frac{k_p}{2} \text{trace} (K(I - E))$$

$$W(E, \Omega_E, \Omega_I) = V(E) + \frac{1}{2} \langle \mathbb{I} \Omega_E, \Omega_E \rangle + \frac{\gamma}{2} \langle \mathbb{I} \Omega_I, \Omega_I \rangle + \text{cross terms}$$

**Works even for problems with singular Inertia Tensors !**

D.H.S. Maithripala and J.M. Berg, “An intrinsic robust PID on Lie groups”, IEEE 53rd Annual Conference on CDC, USA, 2014

D.H.S. Maithripala and J.M. Berg, “An intrinsic PID controller for mechanical systems on Lie groups”, Automatica, vol. 54, no. 0, pp. 89-200, 2015



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**Thank You**