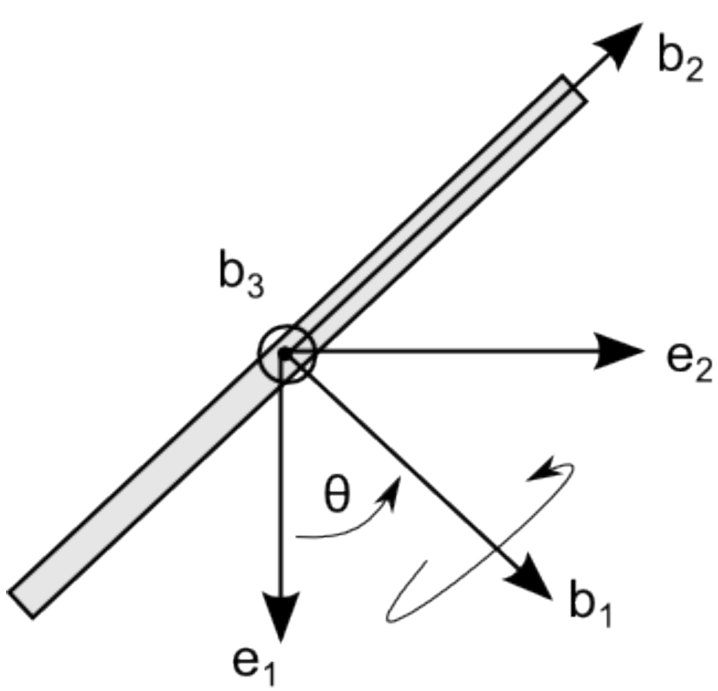
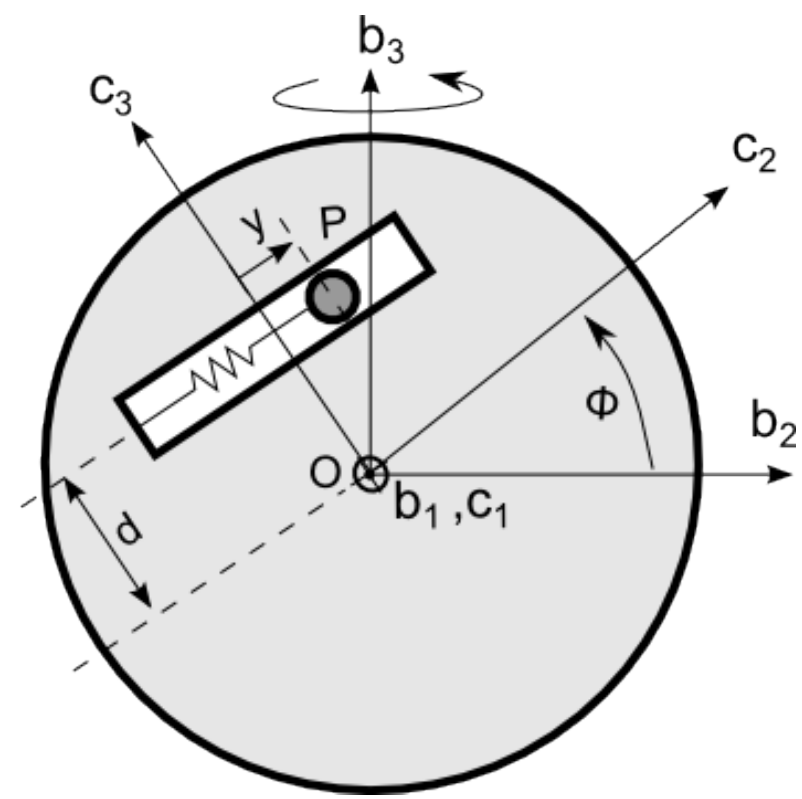
# Example: Description of Motion in Moving Frames

D. H. S. Maithripala, PhD
Dept. of Mechanical Engineering
University of Peradeniya

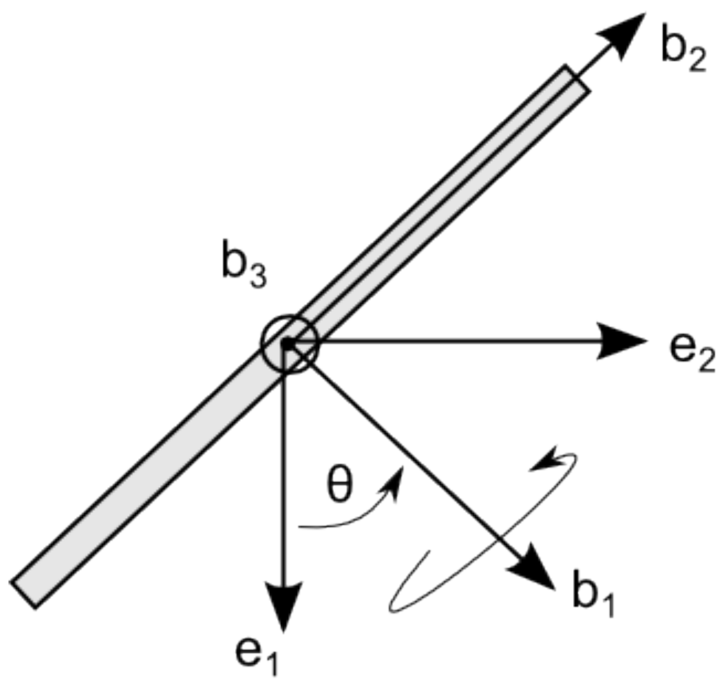
# Find

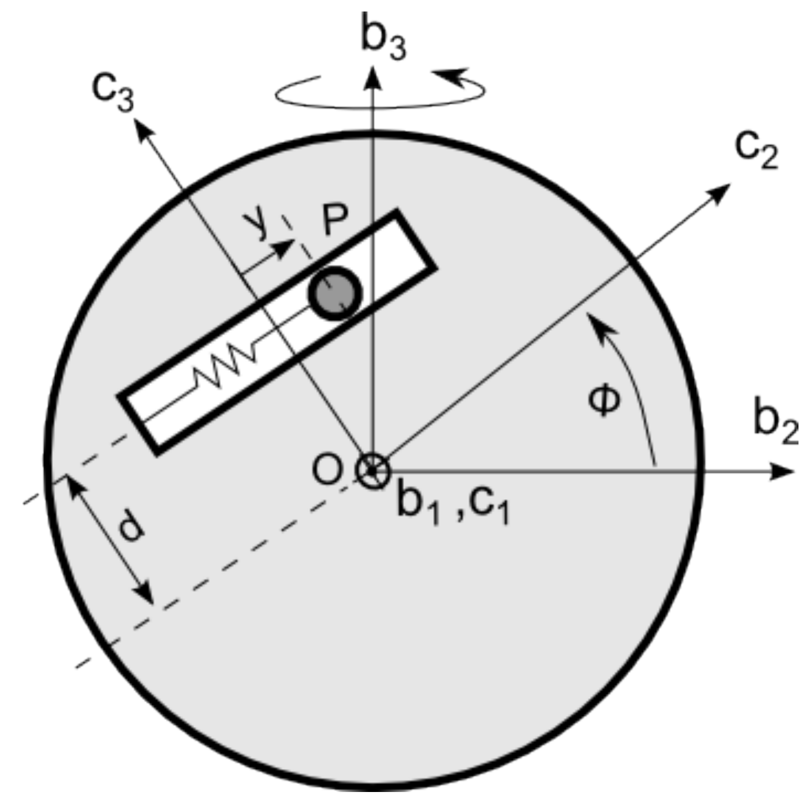



- **Angular velocity of frame c w.r.t e**
- **Angular Momentum**
- **Kinetic Energy**

# Angular Velocity



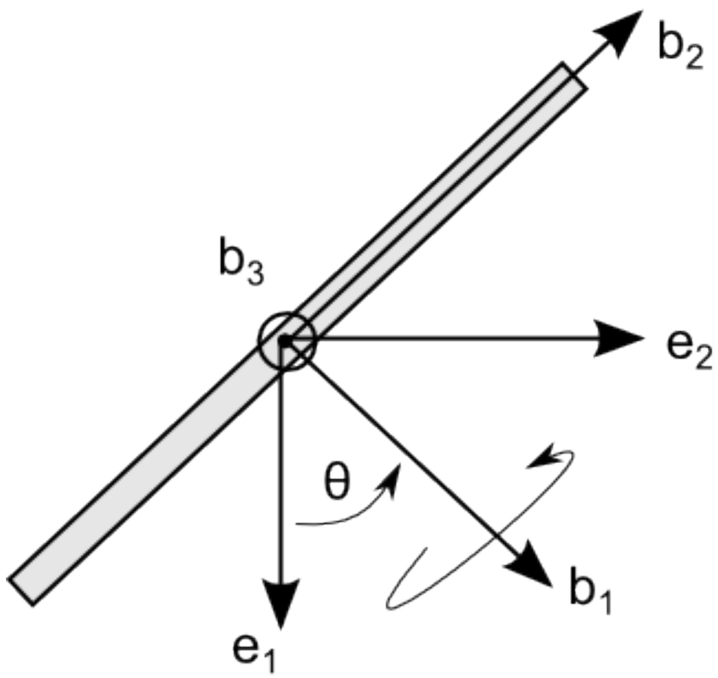

$$\mathbf{c} = \mathbf{e}\,R$$

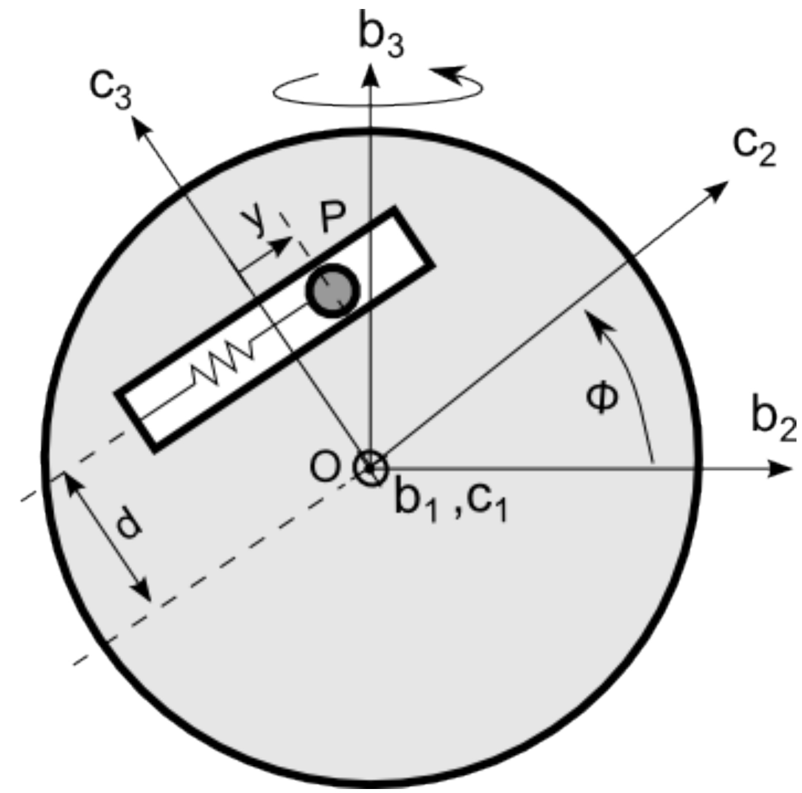
$$\widehat{\Omega} = R^T \dot{R}$$

**Angular velocity of c w.r.t e in c is** $\Omega$

**Angular velocity of c w.r.t e in e is** $\omega = R\Omega$

# Angular Velocity



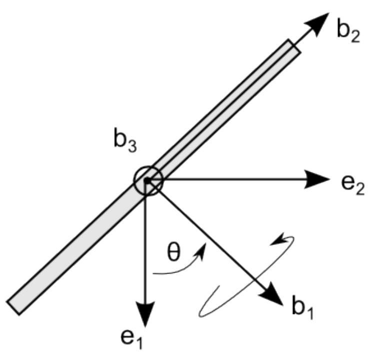



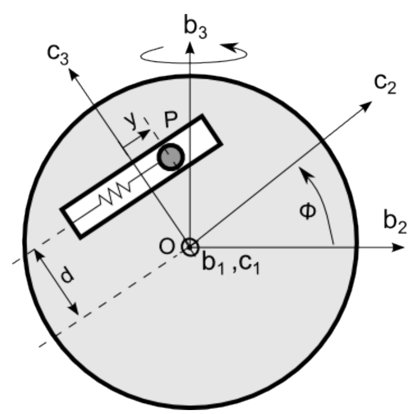

$$\mathbf{b} = \mathbf{e}\, R_3(\theta) \qquad\qquad \mathbf{c} = \mathbf{b}\, R_1(\phi)$$

$$\mathbf{c} = \mathbf{b}\, R_1(\phi) = \mathbf{e}\ \underbrace{R_3(\theta) R_1(\phi)}_{R}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

# Angular Velocity




$$\widehat{\Omega}_1 = R_1^T \dot{R}_1 \qquad \widehat{\Omega}_3 = R_3^T \dot{R}_3$$

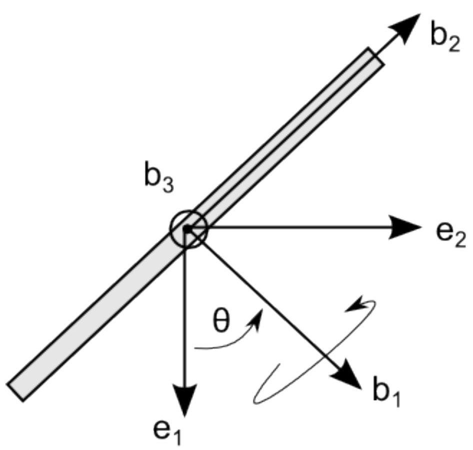
$$\Omega_1 = [\dot{\phi}\ \ 0\ \ 0]^T \qquad \Omega_3 = [0\ \ 0\ \ \dot{\theta}]^T$$

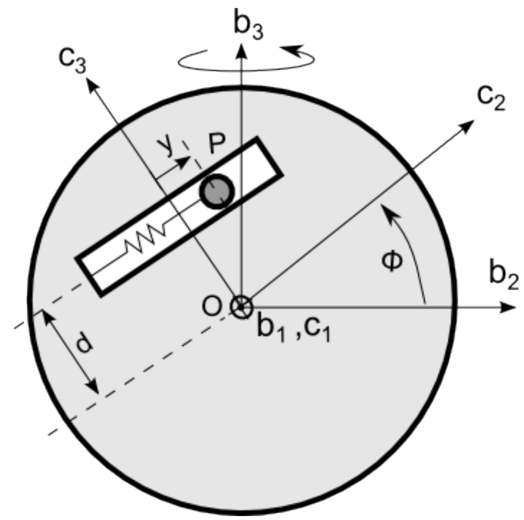
$$\widehat{\Omega} = R^T \dot{R} = R_1^T \widehat{\Omega}_3 R_1 + \widehat{\Omega}_1 = \widehat{R_1^T \Omega_3} + \widehat{\Omega}_1 = \widehat{R_1^T \Omega_3 + \Omega_1}$$

$$\Omega = R_1^T \Omega_3 + \Omega_1 = \begin{bmatrix} \dot{\phi} \\ \dot{\theta}\sin\phi \\ \dot{\theta}\cos\phi \end{bmatrix}$$

$$\omega = R\Omega = \begin{bmatrix} \cos\theta & -\sin\theta & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta}\sin\phi \\ \dot{\theta}\cos\phi \end{bmatrix}$$

# Angular Momentum



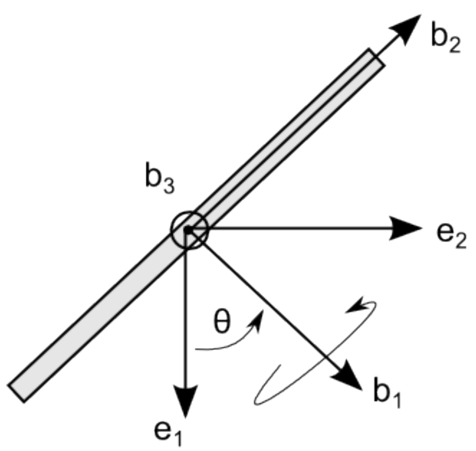

$$\pi = x \times m\dot{x} = mRX \times (R(\Omega \times X + \dot{X}))$$
$$= R\,\underbrace{m\left(X \times (\Omega \times X + \dot{X})\right)}_{\Pi} = R\,\Pi\,.$$

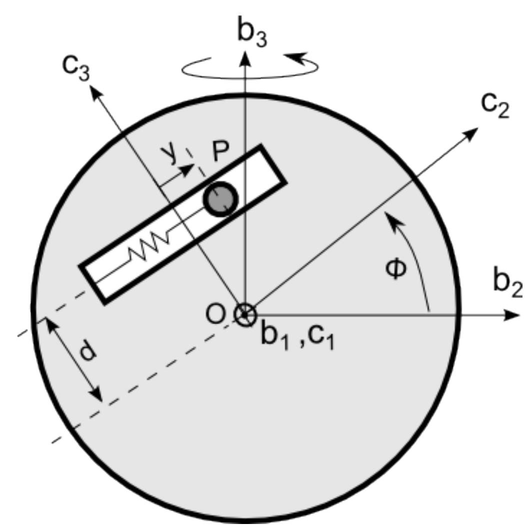
**Angular Momentum in the frame c**

$$\Pi = mX \times (\Omega \times X + \dot{X}) = mX \times (-X \times \Omega + \dot{X})$$
$$= -m\,\widehat{X}^{2}\Omega + m\,\widehat{X}\,\dot{X} = \mathbb{I}_p\Omega + m\,\widehat{X}\,\dot{X}$$

$$\boxed{\begin{aligned} \mathbb{I}_p &\triangleq -m\,\widehat{X}^{2} = m\left(||X||^2 I_{3\times 3} - XX^T\right) \\ &= m\begin{bmatrix} m(y^2+d^2) & 0 & 0 \\ 0 & md^2 & -myd \\ 0 & -myd & my^2 \end{bmatrix} \end{aligned}}$$

# Angular Momentum




**Angular Momentum in the frame c**

$$\Pi = mX \times (\Omega \times X + \dot{X}) = mX \times (-X \times \Omega + \dot{X})$$

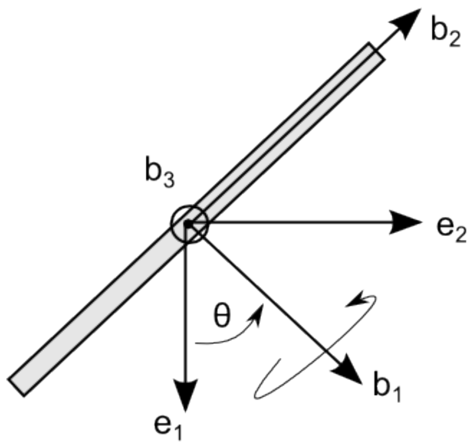
$$= -m\,\widehat{X}^2\Omega + m\,\widehat{X}\,\dot{X} = \mathbb{I}_p\Omega + m\,\widehat{X}\,\dot{X}$$

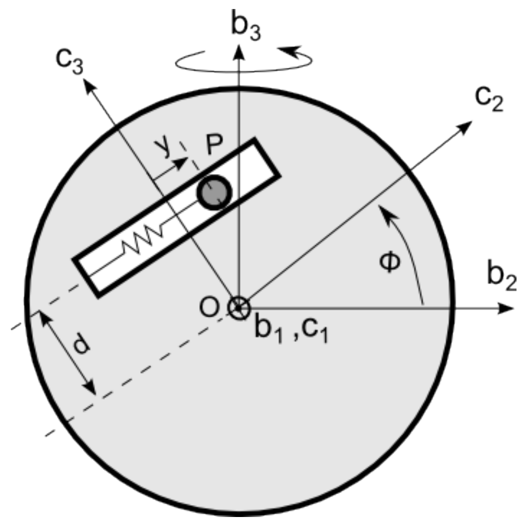
$$X \times \dot{X} = \widehat{X}\,\dot{X} = \begin{bmatrix} 0 & -d & y \\ d & 0 & 0 \\ -y & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -d\dot{y} \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} \mathbb{I}_p &\triangleq -m\,\widehat{X}^2 = m\left(||X||^2 I_{3\times 3} - XX^T\right) \\ &= m\begin{bmatrix} m(y^2+d^2) & 0 & 0 \\ 0 & md^2 & -myd \\ 0 & -myd & my^2 \end{bmatrix} \end{aligned}}$$

$$\mathbb{I}_p\Omega = m\begin{bmatrix} m(y^2+d^2) & 0 & 0 \\ 0 & md^2 & -myd \\ 0 & -myd & my^2 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta}\sin\phi \\ \dot{\theta}\cos\phi \end{bmatrix} = \begin{bmatrix} m(y^2+d^2)\dot{\phi} \\ md\dot{\theta}(d\sin\phi + y\cos\phi) \\ my\dot{\theta}(d\sin\phi + y\cos\phi) \end{bmatrix}$$

# Angular Momentum




**Angular Momentum in the frame c**

$$\Pi = mX \times (\Omega \times X + \dot{X}) = mX \times (-X \times \Omega + \dot{X})$$

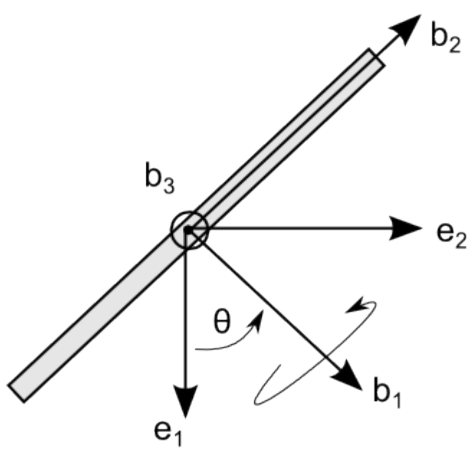
$$= -m\widehat{X}^2\Omega + m\widehat{X}\dot{X} = \mathbb{I}_p\Omega + m\widehat{X}\dot{X}$$

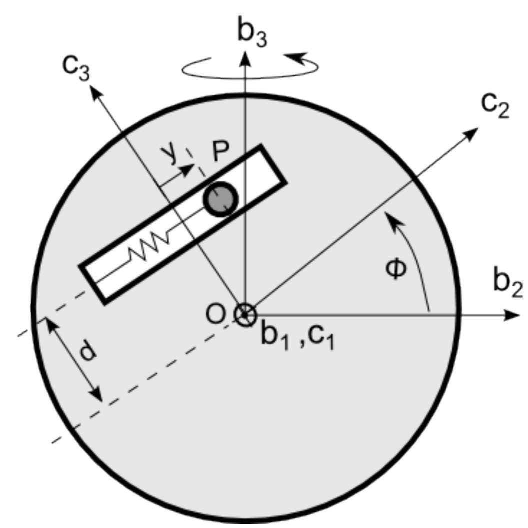
$$\boxed{\begin{aligned}\mathbb{I}_p \triangleq -m\widehat{X}^2 &= m\left(||X||^2 I_{3\times3} - XX^T\right)\\ &= m\begin{bmatrix} m(y^2+d^2) & 0 & 0 \\ 0 & md^2 & -myd \\ 0 & -myd & my^2 \end{bmatrix}\end{aligned}}$$

$$\Pi = \mathbb{I}_p\Omega + m\widehat{X}\dot{X} = \begin{bmatrix} m(y^2+d^2)\dot{\phi} - md\dot{y} \\ md\dot{\theta}(d\sin\phi + y\cos\phi) \\ my\dot{\theta}(d\sin\phi + y\cos\phi) \end{bmatrix}$$

$$\pi = R\Pi = R(\mathbb{I}_p\Omega + m\widehat{X}\dot{X}) = \begin{bmatrix} \cos\theta & -\sin\theta & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} m(y^2+d^2)\dot{\phi} - md\dot{y} \\ md\dot{\theta}(d\sin\phi + y\cos\phi) \\ my\dot{\theta}(d\sin\phi + y\cos\phi) \end{bmatrix}$$

# Kinetic Energy




$$\boxed{\begin{aligned} \mathbb{I}_p &\triangleq -m\widehat{X}^2 = m\left( ||X||^2 I_{3\times3} - XX^T \right) \\ &= m \begin{bmatrix} m(y^2+d^2) & 0 & 0 \\ 0 & md^2 & -myd \\ 0 & -myd & my^2 \end{bmatrix} \end{aligned}}$$

$$\mathrm{KE} = \frac{m}{2}||\dot{x}||^2 = \frac{m}{2}||R(\Omega\times X + \dot{X})||^2 = \frac{m}{2}||(\Omega\times X + \dot{X})||^2$$

$$\mathrm{KE} = \frac{m}{2}\left|\left| \begin{bmatrix} 0 & -\dot{\theta}\cos\phi & \dot{\theta}\sin\phi \\ \dot{\theta}\cos\phi & 0 & -\dot{\phi} \\ -\dot{\theta}\sin\phi & \dot{\phi} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y \\ d \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} \right|\right|^2 = \frac{m}{2}\left|\left| \begin{bmatrix} \dot{\theta}(d\sin\phi - y\cos\phi) \\ \dot{y} - d\dot{\phi} \\ y\dot{\phi} \end{bmatrix} \right|\right|^2$$

$$= \frac{m}{2}\left( \dot{\theta}^2(d\sin\phi - y\cos\phi)^2 + (\dot{y} - d\dot{\phi})^2 + y^2\dot{\phi}^2 \right)$$

# Thank You