Bayesian Estimation on Lie Groups

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Abstract

I. Introduction

[1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]

II. The Kalman Filter on $\mathbb{R}^n$

Let $\{x_k\}$ and $\{y_k\}$ be two Markovian Gaussian processes. We let $\{y_{m1}, y_{m2}, \cdots, y_{mk}\}$ be a certain realization of $\{y_k\}$ and let $\mathcal{F}_k$ denote the sigma algebra generated by $\{y_1, y_2, \cdots, y_k\}$. Then one sees that $\mathcal{F}_{k-1} \subset \mathcal{F}_k$. Denote by $x_k^- = E(x_k|\mathcal{F}_{k-1})$ and $y_k^- = E(y_k|\mathcal{F}_{k-1})$ estimates of x_k and y_k given the measurements upto $k-1$. We assume that

$$x_k^- = A_k\, x_{k-1}^+ + B_k\, u_k + w_k, \tag{1}$$
$$y_k^- = H_k\, x_k^- + z_k, \tag{2}$$

where $B_k\, u_k$ is assumed to be exactly known, $p(w_k) = \mathcal{N}(0, \Sigma_p)$, and $p(z_k) = \mathcal{N}(0, \Sigma_m)$ where $\mathcal{N}(\mu, \Sigma)$ denotes a multivariate gaussian with mean μ and covariance Σ. Please refer to the appendix for formulas for constructing the conditional probabilities of Gaussian distributions.

Denote by $p\left(E(x_k \mid \mathcal{F}_k)\right) = \mathcal{N}(m_k, P_k)$ and $p\left(E(x_k \mid \mathcal{F}_{k-1})\right) = \mathcal{N}(m_k^-, P_k^-)$. Then from (1) we see that

$$p\left(E(x_k \mid \mathcal{F}_{k-1})\right) = \mathcal{N}(A_k m_{k-1} + B_k u_k, A_k P_{k-1} A_k^T + B_k u_k u_k^T B_k^T + \Sigma_p),$$

and hence the predicted mean and the covariance satisfy

$$m_k^- = A_k\, m_{k-1} + B\, u_{k-1}, \tag{3}$$
$$P_k^- = A_k P_{k-1} A_k^T + B_k u_k u_k^T B_k^T + \Sigma_p. \tag{4}$$

From (2) we can write the joint distribution

$$p\begin{pmatrix} E(x_k \mid \mathcal{F}_{k-1}) \\ E(y_k \mid \mathcal{F}_{k-1}) \end{pmatrix} = \mathcal{N}\left(\begin{bmatrix} m_k^- \\ H_k m_k^- \end{bmatrix}, \begin{bmatrix} P_k^- & P_k^- H_k^T \\ H_k P_k^- & H_k P_k^- H_k^T + \Sigma_m \end{bmatrix}\right).$$

Thus from the properties of normal distributions we find that the conditional distribution is given by

$$p(E(x_k \mid \mathcal{F}_k)) = \mathcal{N}\left(m_k^- + P_k^- H_k^T (H_k P_k^- H_k^T + \Sigma_m)^{-1}(y_k - H_k m_k^-), P_k^- - P_k^- H_k^T (H_k P_k^- H_k^T + \Sigma_m)^{-1} H_k P_k^-\right)$$

Thus we have that the updated mean and covariances are given by

$$K_k \triangleq P_k^- H^T (H P_k^- H^T + \Sigma_m)^{-1}, \tag{5}$$
$$m_k = m_k^- + K_k (y_k - H m_k^-), \tag{6}$$
$$P_k = (I - K_k H_k) P_k^-. \tag{7}$$

where $\Sigma_p = E(w_k w_k^T)$ and $\Sigma_m = E(z_k z_k^T)$.

III. Intrinsic Extended Kalman Filter on Lie Groups

A. Continuous Model

1) Option I: The system evolves according to

$$\begin{aligned}
\dot{g} &= g \cdot (\zeta + \zeta_b + n_\zeta),\\
\dot{V} &= -\mathrm{ad}_{(\zeta+\zeta_b+n_\zeta)} V + a + a_b + n_a\\
y_{1k} &= \mathrm{Ad}_{g^{-1}} e_1 + n_1,\\
y_{2k} &= \mathrm{Ad}_{g^{-1}} e_2 + n_2,\\
y_{vk} &= V + n_V.
\end{aligned}$$

Consider the model

$$\begin{aligned}
\dot{m}_g &= m_g \cdot (\zeta + \zeta_b),\\
\dot{m}_V &= -\mathrm{ad}_{(\zeta+\zeta_b)} m_V + a + m_{a_b}\\
\dot{m}_{\zeta_b} &= 0,\\
\dot{m}_{a_b} &= 0,\\
m_{y_{1k}} &= \mathrm{Ad}_{m_g^{-1}} e_1,\\
m_{y_{2k}} &= \mathrm{Ad}_{m_g^{-1}} e_2,\\
m_{y_{vk}} &= m_V
\end{aligned}$$

Consider the estimation error $e_g \triangleq m_g^{-1} g$, $e_V = V - m_V$, $e_{a_b} = a_b - m_{a_b}$, and $e_{\zeta_b} = a_b - m_{\zeta_b}$. Then the estimated error dynamics evolve according to

$$\dot{e}_g = e_g \cdot \left((I - \mathrm{Ad}_{e_g^{-1}})(\zeta + \zeta_b) + e_{\zeta_b} + n_\zeta \right), \tag{8}$$

$$\dot{e}_V = -\mathrm{ad}_{(\zeta+\zeta_b)} e_V + \mathrm{ad}_{(m_V+e_V)} n_\zeta + e_{a_b} + n_a, \tag{9}$$

$$\dot{e}_{\zeta_b} = 0, \tag{10}$$

$$\dot{e}_{a_b} = 0, \tag{11}$$

$$e_{y_{1k}} = \mathrm{Ad}_{m_g^{-1}} (\mathrm{Ad}_{e_g^{-1}} - I) e_1 + n_1, \tag{12}$$

$$e_{y_{2k}} = \mathrm{Ad}_{m_g^{-1}} (\mathrm{Ad}_{e_g^{-1}} - I) e_2 + n_2, \tag{13}$$

$$e_{y_{vk}} = m_V + n_V \tag{14}$$

2) Option II: The system evolves according to

$$\begin{aligned}
\dot{g} &= g \cdot (\zeta + \zeta_b + n_\zeta),\\
\dot{v} &= \mathrm{Ad}_g (a + a_b + n_a)\\
y_{1k} &= \mathrm{Ad}_{g^{-1}} e_1 + n_1,\\
y_{2k} &= \mathrm{Ad}_{g^{-1}} e_2 + n_2,\\
y_{vk} &= v + n_v.
\end{aligned}$$

Consider the model

$$\dot{m}_g = m_g \cdot (\zeta + m_{\zeta_b}), \tag{15}$$

$$\dot{m}_v = \mathrm{Ad}_{m_g} (a + m_{a_b}) \tag{16}$$

$$\dot{m}_{\zeta_b} = 0, \tag{17}$$

$$\dot{m}_{a_b} = 0, \tag{18}$$

$$m_{y_{1k}} = \mathrm{Ad}_{m_g^{-1}} e_1, \tag{19}$$

$$m_{y_{2k}} = \mathrm{Ad}_{m_g^{-1}} e_2, \tag{20}$$
$$m_{y_{vk}} = m_v \tag{21}$$

Consider the estimation error $e_g \triangleq m_g^{-1} g$, $e_V = V - m_V$, $e_{a_b} = a_b - m_{a_b}$, and $e_{\zeta_b} = \zeta_b - m_{\zeta_b}$. Then the estimated error dynamics evolve according to

$$\dot{e}_g = e_g \cdot \Big((I - \mathrm{Ad}_{e_g^{-1}})(\zeta + m_{\zeta_b}) + e_{\zeta_b} + n_\zeta \Big), \tag{22}$$
$$\dot{e}_v = \mathrm{Ad}_{m_g} \left((\mathrm{Ad}_{e_g} - I)(a + m_{a_b}) + \mathrm{Ad}_{e_g} e_{a_b} + \mathrm{Ad}_{e_g} n_a \right), \tag{23}$$
$$\dot{e}_{\zeta_b} = 0, \tag{24}$$
$$\dot{e}_{a_b} = 0, \tag{25}$$
$$e_{y_{1k}} = \mathrm{Ad}_{m_g^{-1}} (\mathrm{Ad}_{e_g^{-1}} - I) e_1 + n_1, \tag{26}$$
$$e_{y_{2k}} = \mathrm{Ad}_{m_g^{-1}} (\mathrm{Ad}_{e_g^{-1}} - I) e_2 + n_2, \tag{27}$$
$$e_{y_{vk}} = e_v + n_v \tag{28}$$

B. Discretized Model

1) Option II:

$$m_{g_k} = m_{g_{k-1}} \exp\left(\Delta T (\zeta_{k-1} + m_{\zeta_{b_{k-1}}})\right), \tag{29}$$
$$m_{v_k} = m_{v_{k-1}} + \Delta T \, \mathrm{Ad}_{m_{g_{k-1}}} (a_{k-1} + m_{a_{b_{k-1}}}), \tag{30}$$
$$m_{\zeta_{b_k}} = m_{\zeta_{b_{k-1}}}, \tag{31}$$
$$m_{a_{b_k}} = m_{a_{b_{k-1}}}, \tag{32}$$
$$m_{y_{1\,k}} = \mathrm{Ad}_{m_{g_k}{}^{-1}}(e_1), \tag{33}$$
$$m_{y_{2\,k}} = \mathrm{Ad}_{m_{g_k}{}^{-1}}(e_2), \tag{34}$$
$$m_{y_{v\,k}} = m_{v_k} \tag{35}$$

$$\begin{aligned} e_{g_k} &= e_{g_{k-1}} \exp\Big(\Delta T (I - \mathrm{Ad}_{e^{-1}_{g_{k-1}}})(\zeta_{k-1} + m_{\zeta_{b_{k-1}}}) + \Delta T e_{\zeta_{b_{k-1}}} + \Delta T n_{\zeta_{k-1}} \Big), \\ &= \exp\left(\eta_{e\,k-1}\right) \exp\Big(\Delta T (I - \exp\left(-\mathrm{ad}_{\eta_{e\,k-1}}\right))(\zeta_{k-1} + m_{\zeta_{b_{k-1}}}) + \Delta T e_{\zeta_{b_{k-1}}} + \Delta T n_{\zeta_{k-1}} \Big). \end{aligned}$$

where we denote by η_{ek} the exponential coordinate of e_{g_k}. That is $e_{g_k} = e^{\eta_{e\,k}}$. Then we have that if the noise is small

$$\begin{aligned} \eta_{ek} &\approx \eta_{e\,k-1} + \Delta T \exp\left(\mathrm{ad}_{\eta_{e\,k-1}}\right) \Big(\left(I - \exp\left(-\mathrm{ad}_{\eta_{e\,k-1}}\right)\right) (\zeta_{k-1} + m_{\zeta_{b_{k-1}}}) + e_{\zeta_{b_{k-1}}} + n_{\zeta_{k-1}} \Big), \\ &\approx \eta_{e\,k-1} + \Delta T \exp\left(\mathrm{ad}_{\eta_{e\,k-1}}\right) \Big(\mathrm{ad}_{\eta_{e\,k-1}} (\zeta_{k-1} + m_{\zeta_{b_{k-1}}}) + e_{\zeta_{b_{k-1}}} + n_{\zeta_{k-1}} \Big), \\ &\approx \eta_{e\,k-1} + \Delta T \big(I + \mathrm{ad}_{\eta_{e\,k-1}}\big) \Big(\mathrm{ad}_{\eta_{e\,k-1}} (\zeta_{k-1} + m_{\zeta_{b_{k-1}}}) + n_{\zeta_{k-1}} \Big) + \Delta T \exp\left(\mathrm{ad}_{\eta_{e\,k-1}}\right) e_{\zeta_{b_{k-1}}}, \\ &\approx \Big(I - \Delta T \mathrm{ad}_{(\zeta_{k-1} + m_{\zeta_{b_{k-1}}})} \Big) \eta_{e\,k-1} + \Delta T e_{\zeta_{b_{k-1}}} + \Delta T n_{\zeta_{k-1}}. \end{aligned} \tag{36}$$

$$\begin{aligned} e_{v_k} &= e_{v_{k-1}} + \Delta T \, \mathrm{Ad}_{m_{g_{k-1}}} \Big(\left(\mathrm{Ad}_{e^{\eta_{e\,k-1}}} - I\right) (a_{k-1} + m_{a_{b_{k-1}}}) + \mathrm{Ad}_{e^{\eta_{e\,k-1}}} e_{a_{b_{k-1}}} + \mathrm{Ad}_{e^{\eta_{e\,k-1}}} n_{a_{k-1}} \Big) \\ &= e_{v_{k-1}} + \Delta T \, \mathrm{Ad}_{m_{g_{k-1}}} \Big(\left(\exp\left(\mathrm{ad}_{\eta_{e\,k-1}}\right) - I\right) (a_{k-1} + m_{a_{b_{k-1}}}) + \exp\left(\mathrm{ad}_{\eta_{e\,k-1}}\right) e_{a_{b_{k-1}}} + \exp\left(\mathrm{ad}_{\eta_{e\,k-1}}\right) n_{a_{k-1}} \Big) \\ &\approx e_{v_{k-1}} - \Delta T \, \mathrm{Ad}_{m_{g_{k-1}}} \, \mathrm{ad}_{(a_{k-1} + m_{a_{b_{k-1}}})} \eta_{e\,k-1} + \Delta T \, \mathrm{Ad}_{m_{g_{k-1}}} \exp\left(\mathrm{ad}_{\eta_{e\,k-1}}\right) \left(e_{a_{b_{k-1}}} + n_{a_{k-1}}\right) \end{aligned}$$

$$\approx e_{v_{k-1}} - \Delta T \operatorname{Ad}_{m_{g_{k-1}}} \operatorname{ad}_{(a_{k-1}+m_{a_{b_{k-1}}})}\eta_{ek-1} + \Delta T \operatorname{Ad}_{m_{g_{k-1}}} e_{a_{b_{k-1}}} + \Delta T \operatorname{Ad}_{m_{g_{k-1}}} n_{a_{k-1}}. \tag{37}$$

We also see that

$$\begin{aligned} e_{y_{ik}} &\triangleq (y_{1k} - m_{y_{1k}}) = \operatorname{Ad}_{g_k^{-1}}(e_i) + n_{ik} - \operatorname{Ad}_{m_{g_k}{}^{-1}}(e_i) \\ &= \operatorname{Ad}_{m_{g_k}^{-1}}\left(\operatorname{Ad}_{e^{-\eta_{ek}}} - I\right)(e_i) + n_{ik} \approx \operatorname{Ad}_{m_{g_k}^{-1}} \operatorname{ad}_{e_i}\eta_{ek} + n_{ik} \end{aligned} \tag{38}$$

$$e_{y_{vk}} \triangleq (y_{2k} - m_{y_{2k}}) = e_{v_k} + n_{v_k}. \tag{39}$$

Thus when ζ_{k-1} is considered to be a known input we have that the linearized estimation error dynamics evolve according to

$$e_k = A_{k-1}e_{k-1} + G_{k-1}q_{a_{k-1}}, \tag{40}$$

$$y_{e_k} = He_k + n_k \tag{41}$$

where

$$e_k = \begin{bmatrix} \eta_{ek} \\ e_{v_k} \\ e_{\zeta_{b_k}} \\ e_{a_{b_k}} \end{bmatrix}, \quad y_{e_k} = \begin{bmatrix} e_{y_{1k}} \\ e_{y_{2k}} \\ e_{y_{vk}} \end{bmatrix}, \quad n_k = \begin{bmatrix} n_{1k} \\ n_{2k} \\ n_{v_k} \end{bmatrix}, \quad q_k = \begin{bmatrix} n_{\zeta_k} \\ n_{a_k} \end{bmatrix}$$

$$A_k = \begin{bmatrix} \left(I - \Delta T \operatorname{ad}_{(\zeta_{k-1}+m_{\zeta_{b_{k-1}}})}\right) & 0 & \Delta T I & 0 \\ -\Delta T \operatorname{Ad}_{m_{g_k}} \operatorname{ad}_{(a_{k-1}+m_{a_{b_{k-1}}})} & I & 0 & \Delta T \operatorname{Ad}_{m_{g_k}} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad H_k = \begin{bmatrix} \operatorname{Ad}_{m_{g_k}^{-1}} \operatorname{ad}_{e_1} & 0 & 0 & 0 \\ \operatorname{Ad}_{m_{g_k}^{-1}} \operatorname{ad}_{e_2} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$

$$G_k = \Delta T \begin{bmatrix} I & 0 \\ 0 & \operatorname{Ad}_{m_{g_{k-1}}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus we have the prediction model

$$m_{g_k}^- = m_{g_{k-1}} \exp\left(\Delta T(\zeta_{k-1} + m_{\zeta_{b_{k-1}}})\right), \tag{42}$$

$$m_{v_k}^- = m_{v_{k-1}} + \Delta T \operatorname{Ad}_{m_{g_{k-1}}}(a_{k-1} + m_{a_{b_{k-1}}}), \tag{43}$$

$$m_{\zeta_{b_k}}^- = m_{\zeta_{b_{k-1}}}, \tag{44}$$

$$m_{a_{b_k}}^- = m_{a_{b_{k-1}}}, \tag{45}$$

$$m_{y_{1k}} = \operatorname{Ad}_{m_{g_k}^-{}^{-1}}(e_1), \tag{46}$$

$$m_{y_{2k}} = \operatorname{Ad}_{m_{g_k}^-{}^{-1}}(e_2), \tag{47}$$

$$m_{y_{vk}} = m_{v_k}^- \tag{48}$$

$$e_k^- = A_{k-1}\, e_{k-1}, \tag{49}$$

$$P_k^- = A_{k-1}P_{k-1}A_{k-1}^T + G_k\Sigma_q G_k^T, \tag{50}$$

$$\tag{51}$$

and the correction

$$K_k \triangleq P_k^- H_k^T (H_k P_k^- H_k^T + \Sigma_m)^{-1}, \tag{52}$$

$$e_k = e_k^- + K_k(y_{e_k} - H_k e_k^-), \tag{53}$$

$$P_k = (I - K_k H_k) P_k^- . \tag{54}$$

where $\Sigma_m = E(n_k n_k^T)$ and $\Sigma_p = E(q_k q_k^T)$.

Then the estimated attitude and translational velocity are

$$m_{g_k} = m_{g_k}^- \exp(\eta_{ek}), \tag{55}$$

$$m_{v_k} = m_{v_k}^- + e_{v_k}, \tag{56}$$

$$\zeta_{b_k} = m_{\zeta_{b_k}}^- + e_{\zeta_{b_k}}, \tag{57}$$

$$a_{b_k} = m_{a_{b_k}}^- + e_{a_{b_k}}, \tag{58}$$

For rigid body attitude estimation

$$A_k = \begin{bmatrix} \left(I - \Delta T(\widehat{\zeta}_k + \widehat{m}_{\zeta_{b_k}})\right) & 0 & \Delta T & 0 \\ -\Delta T\, R_k(\widehat{a}_k + \widehat{m}_{a_{b_k}})R_k^T & I & 0 & \Delta T R_k \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad H_k = \begin{bmatrix} R^{-T}_{k}\widehat{e}_1 R^{-}{}_k & 0 & 0 & 0 \\ R^{-T}_{k}\widehat{e}_2 R^{-}{}_k & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \quad G_k = \Delta T \begin{bmatrix} I & 0 \\ 0 & R_k \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Appendix

Let $\phi : G \mapsto H$ be a Lie-group homomorphism.

$$\phi(\exp(t\zeta)) = \exp(tT_e\phi \cdot \zeta). \tag{59}$$

In the special case where $\phi = \mathrm{I}_g : G \mapsto G$ for each g, we have that

$$h(t) = g\exp(\zeta t)g^{-1} = \exp(t\mathrm{Ad}_g\zeta) \tag{60}$$

In the special case where $\phi = \mathrm{Ad} : G \mapsto GL(\mathcal{G})$, we have that

$$h(t) = \mathrm{Ad}_{\exp(\zeta t)} = \exp(t\mathrm{ad}_\zeta) = I + t\mathrm{ad}_\zeta + \frac{t^2}{2!}\mathrm{ad}_\zeta^2 + \cdots \tag{61}$$

From which we get

$$\frac{d}{dt}e^{\zeta(t)} = e^{\zeta(t)}\left(\frac{I - e^{-\mathrm{ad}_\zeta}}{\mathrm{ad}_\zeta}\right)\frac{d\zeta}{dt} \tag{62}$$

Let $e^{\chi(t)} = e^{\zeta}e^{t\eta}$. Then

$$\eta = e^{-\chi(t)}\frac{d}{dt}e^{\chi(t)} = \left(\frac{I - e^{-\mathrm{ad}_\chi}}{\mathrm{ad}_\chi}\right)\frac{d\chi(t)}{dt}.$$

Thus formally

$$\frac{d\chi(t)}{dt} = \left(\frac{\mathrm{ad}_\chi}{I - e^{-\mathrm{ad}_\chi}}\right)\eta.$$

Hence integrating from 0 to 1 with respect to t we have

$$\log\left(e^{\zeta}e^{\eta}\right) = \chi(1) = \zeta + \left(\int_0^1 \frac{\mathrm{ad}_\chi}{I - e^{-\mathrm{ad}_\chi}}\,dt\right)\eta.$$

The formal series in the integral is defined as

$$\frac{\mathrm{ad}_\chi}{I - e^{-\mathrm{ad}_\chi}} = \psi(e^{\mathrm{ad}_\chi}) = \psi(\mathrm{Ad}_{e^\chi}) = \psi(\mathrm{Ad}_{e^\zeta e^{t\eta}}) = \psi(e^{\mathrm{ad}_\zeta}e^{t\mathrm{ad}_\eta})$$

where

$$\psi(w) \triangleq \frac{w \log w}{w-1} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m(m+1)}(w-1)^m, \quad ||w|| < 1.$$

Thus we have the Baker-Campbell-Hausdorff formula

$$\log\left(e^X e^Y\right) = X + \left(\int_0^1 \psi(e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y})\, dt\right) Y \tag{63}$$

$$= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) \tag{64}$$

$$- \frac{1}{24}[Y,[X,[X,Y]]] \tag{65}$$

$$- \frac{1}{720}\left([Y,[Y,[Y,[Y,X]]]] + [X,[X,[X,[X,Y]]]]\right) \tag{66}$$

$$+ \frac{1}{360}\left([X,[Y,[Y,[Y,X]]]] + [Y,[X,[X,[X,Y]]]]\right) \tag{67}$$

$$+ \frac{1}{120}\left([Y,[X,[Y,[X,Y]]]] + [X,[Y,[X,[Y,X]]]]\right) + \cdots . \tag{68}$$

Alternatively

$$\log(e^X e^Y) = X + \frac{\mathrm{ad}_X\ e^{\mathrm{ad}_X}}{e^{\mathrm{ad}_X} - I}\ Y + O(Y^2). \tag{69}$$

The quaternion representation of $\dot{R} = R\widehat{\Omega}$.

$$\begin{bmatrix} \dot{q}_0 \\ \dot{w} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\Omega \cdot w \\ q_0\Omega - \Omega \times w \end{bmatrix}$$

$$\dot{q} = \frac{1}{2}\begin{bmatrix} 0 & -\Omega^T \\ \Omega & -\widehat{\Omega} \end{bmatrix} q$$

$$\exp\left(\widehat{\Omega}\right) = I + \frac{\sin||\Omega||}{||\Omega||}\widehat{\Omega} + \frac{1}{2}\left(\frac{\sin\frac{||\Omega||}{2}}{\frac{||\Omega||}{2}}\right)^2 \widehat{\Omega}^2. \tag{70}$$

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