TWO PHASE STEFAN PROBLEM MATHEMATICA CODE

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ABSTRACT

TWO PHASE STEFAN PROBLEM WAS SOLVED USING EIGEN FUNCTION EXPANSION METHOD. TRANSCENDENTAL EQUATION WAS SOLVED AND EIGEN VALUES OBTAINED WERE GRAPHICALLY VALIDATED. IN ORDER TO FIND THE COEFFICIENTS NUMERICAL INTEGRATION SCHEME WAS APPLIED AND INTEGRATED FROM 0 TO 1. NUMERICAL DIFFERENTIATION SCHEME WAS APPLIED TO OBTAIN THE INTERFACE POSITION OF THE MELTED MATERIAL. IT WAS FOUND NUMERICAL SCHEME REDUCED THE COMPUTATIONAL TIME SIGNIFICANTLY AND MADE THE PROBLEM LESS TIME CONSUMING AND TRACEABLE.

1. MATHEMATICA CODE

1.1 Transcendental equation solution for Phase two

Transcendental equation was solved by assuming normalized initial interface location and and outer cylindrical boundary at 2.

```
eqn1:=BesselJ[0,β]BesselY[1,β*2]-BesselJ[1,β*2]BesselY[0,β]
Plot[Evaluate[eqn1],{β,0,30},Frame→False, FrameLabel→{Style["β",16],Style[ "Transdental Equation", 16]},PlotStyle→{Thick,Red} ]
```

1.2 Roots of the equation

```
Clear[root, guess, rootlist1];
guess = 1; rootlist1 = {}; n = 20;
Do[{root = FindRoot[Evaluate[eqn1 == 0], {β, guess}];
  AppendTo[rootlist1, root[[1, 2]]];
  guess = 3 + root[[1, 2]]}, {i, 1, n}];
rootlist1
```

```
{2.40483,5.52008,8.65373,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346,33.7758,36.9171,40.0584,43.1998,46.3412,49.4826,52.6241,55.7655,58.907,62.0485}
```

1.3 Transcendental equation for Phase One

```
eqn2:=BesselJ[0,g]
Plot[Evaluate[eqn2],{g,0,30},Frame→False,FrameLabel→{Style["g",16],Style[ "Transdental Equation", 16]},PlotStyle→{Thick,Green} ]
```

1.4 Roots of the equation

```
Clear[root,guess,rootlist2];
guess=1;rootlist2={};n=20;
Do[{root=FindRoot[Evaluate[eqn2==0],{g,guess}];
AppendTo[rootlist2,root[[1,2]]];
guess=3+root[[1,2]]},{i,1,n}];
rootlist2
```

{2.40483,5.52008,8.65373,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346,33.7758,36.9171,40.0584,43.1998,46.3412,49.4826,52.6241,55.7655,58.907,62.0485}

1.5 Coefficients for Phase One and Two

```
β[m_]:=rootlist1[[m]]
g[n_]:=rootlist2[[n]]
b1[n_]:=NIntegrate[R*BesselJ[0,g[n]R],{R,0,1}]
b2[n_]:=NIntegrate[BesselJ[0,g[n]R]^2,{R,0,1}]
B[n_]:=B[n]=(b1[n]/b2[n])
Table[B[n],{n,1,10}]
```

{0.455947,-0.244353,0.176975,-0.142462,0.121027,-0.106228,0.0952977,-0.0868404,0.0800687,-0.0745027}

```
c1[m_]:=BesselY[0,β[m]]NIntegrate[R*BesselJ[0,β[m]R],{R,0,1}]
c2[m_]:=BesselJ[0,β[m]]NIntegrate[R*BesselY[0,β[m]R],{R,0,1}]
b3[m_]:=BesselY[0,β[m]]^2 NIntegrate[R*BesselJ[0,β[m]R]^2,{R,0,1}]
b4[m_]:=BesselJ[0,β[m]]^2 NIntegrate[R*BesselY[0,β[m]R]^2,{R,0,1}]
b5[m_]:=2BesselJ[0,β[m]]BesselY[0,β[m]]NIntegrate[R*BesselJ[0,β[m]R]BesselY[0,β[m]R],{R,0,1}]

A[m_]:=A[m]=((c1[m]-c2[m])/(b3[m]+b4[m]-b5[m]))

Table[A[m],{m,1,20}]
```

{1.97837,4.3884,2.59092,3.81685,2.72071,3.64947,2.78627,3.56347,2.8278,3.50932,2.8572,3.47138,2.87946,3.44294,2.89709,3.42064,2.91151,3.40256,2.9236,3.38752}

1.6 Solution to the Stefan Problem

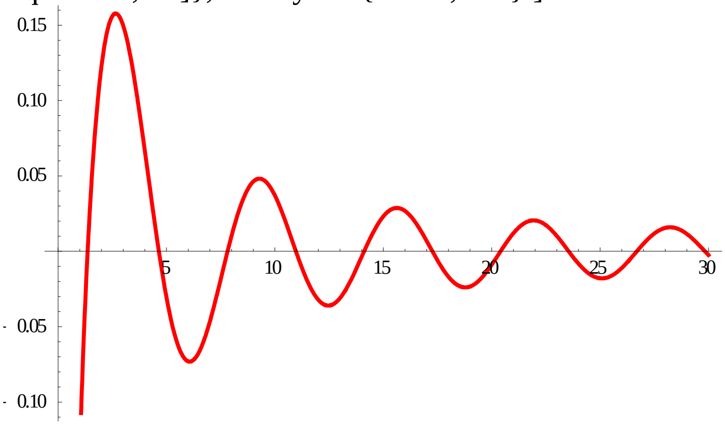
```
sol1=S/.NDSolve[{S'[t]==-Sum[A[m]E^(-t*β[m]^2)(-BesselJ[1,β[m]S[t]] BesselY[0,β[m]S[t]]+BesselJ[0,β[m]S[t]]BesselY[1,β[m]S[t]])0.5,{m,1,10}],S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

p1=Plot[Evaluate[Table[sol1[t], {S, 0, 2}]], {t, 0.01, 1}, FrameLabel -> {Style["Dimensionless time", 16], Style[ "Interface Position", 16]}, Frame -> True, PlotStyle -> {Black, Thick}]
```

Using this general code we can get graphs by changing dimensionless parameters and values.

2. Effect of Biot Number

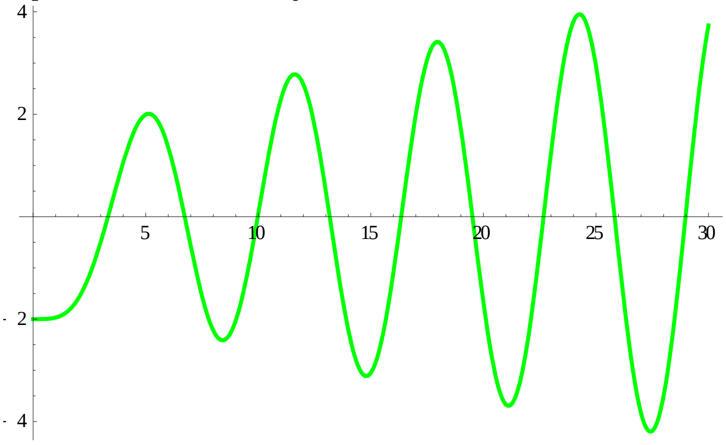
```
eqn1:=BesselJ[0,β]BesselY[1,β*2]-BesselJ[1,β*2]BesselY[0,β]
Plot[Evaluate[eqn1],{β,0,30},Frame->False,FrameLabel->{Style["β",16],Style[ "Transdental Equation", 16]},PlotStyle->{Thick,Red} ]
```



```
Clear[root,guess,rootlist1];
guess=1;rootlist1={};n=20;
Do[{root=FindRoot[Evaluate[eqn1==0],{β,guess}];
  AppendTo[rootlist1,root[[1,2]]];
  guess=3+root[[1,2]]},{i,1,n}];
rootlist1
```

{1.36078,4.6459,7.81416,10.9671,14.1151,17.2607,20.405,23.5487,26.6918,29.8347,32.9772,36.1197,39.262,42.4041,45.5462,48.6883,51.8303,54.9722,58.1141,61.256}

```
eqn2:=-g*BesselJ[1,g]-2*BesselJ[0,g]
Plot[Evaluate[eqn2],{g,0,30},Frame->False, FrameLabel->{Style["g",16],Style[ "Transdental Equation", 16]},PlotStyle->{Thick,Green} ]
```



```
Clear[root,guess,rootlist2];
guess=1;rootlist2={};n=20;
Do[{root=FindRoot[Evaluate[eqn2==0],{g,guess}];
  AppendTo[rootlist2,root[[1,2]]];
  guess=3+root[[1,2]]},{i,1,n}];
rootlist2
```

{9.97754,13.1739,16.3494,19.514,22.6723,25.8265,28.978,32.1276,35.2757,38.4228,41.569,44.7146,47.8597,51.0044,54.1486,57.2926,60.4364,63.5799,66.7233,69.8665}

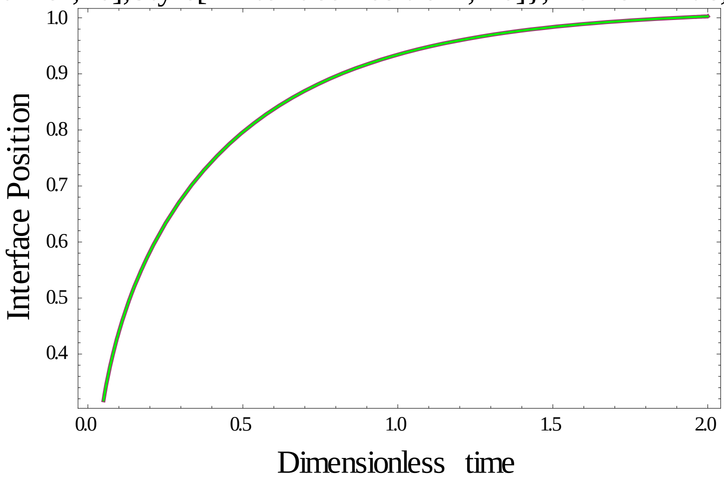
```
β [m_]:=rootlist1[[m]]
g[n_]:=rootlist2[[n]]
```

b1[n_]:=-g[n]NIntegrate[R*BesselJ[1,g[n]R],{R,0,1}]
b2[n_]:=2*NIntegrate[R*BesselJ[0,g[n]R],{R,0,1}]
c3[n_]:=g[n]^2 *NIntegrate[BesselJ[1,g[n]R]2 R,{R,0,1}]
c4[n_]:=4*NIntegrate[BesselJ[0,g[n]R]2 R,{R,0,1}]
c5[n_]:=2*NIntegrate[R*BesselJ[1,g[n]R]2*BesselJ[0,g[n]R],{R,0,1}]
B[n_]:=B[n]=((b1[n]-b2[n])/(c3[n]+c4[n]+c5[n]))

Table[B[n],{n,1,10}]
{-0.108782,0.0342083,-0.0493024,0.0207861,-0.0292216,0.0143691,-0.0197661,0.0107113,-0.0144673,0.00839091}

c1[m_]:=BesselY[0,β [m]]NIntegrate[R*BesselJ[0,β [m]R],{R,0,1}]
c2[m_]:=BesselJ[0,β [m]]NIntegrate[R*BesselY[0,β [m]R],{R,0,1}]
b3[m_]:=BesselY[0,β [m]]2 NIntegrate[BesselJ[0,β [m]R]2 R,{R,0,1}]
b4[m_]:=BesselJ[0,β [m]]2 NIntegrate[BesselY[0,β [m]R]2 R,{R,0,1}]
b5[m_]:=2BesselJ[0,β[m]]BesselY[0,β[m]]NIntegrate[R*BesselJ[0,β[m]R]BesselY[0,β[m]R],{R,0,1}]
A[m_]:=A[m]=((c1[m]-c2[m])/(b3[m]+b4[m]-b5[m]))

Table[A[m],{m,1,20}]
{1.97837,4.3884,2.59092,3.81685,2.72071,3.64947,2.78627,3.56347,2.8278,3.50932,2.8572,3.47138,2.87946,3.44294,2.89709,3.42064,2.91151,3.40256,2.9236,3.38752}
sol2=S/.NDSolve[{S'[t]β
$\sum_{n 1}^{10}$ B n Exp t g n ^2 g n BesselJ 2, g n S t 2 g n BesselJ 1, g n S t
-Sum[A[m]E^(-t*β[m]^2)(-BesselJ[1,β[m]S[t]]BesselY[0,β[m]S[t]]+BesselJ[0,β[m]S[t]]BesselY[1,β[m]S[t]]),{m,1,10}],S[0.01]=0.01},S,{t,0.01,3},MaxSteps-->10^8][[1,1]];
p2=Plot[Evaluate[Table[sol2[t],{S,0,2}]],{t,0.01,2},FrameLabel->{Style["Dimensionless time",16],Style["Interface Position", 16]},Frame->True,PlotStyle->{Green,Thick}]



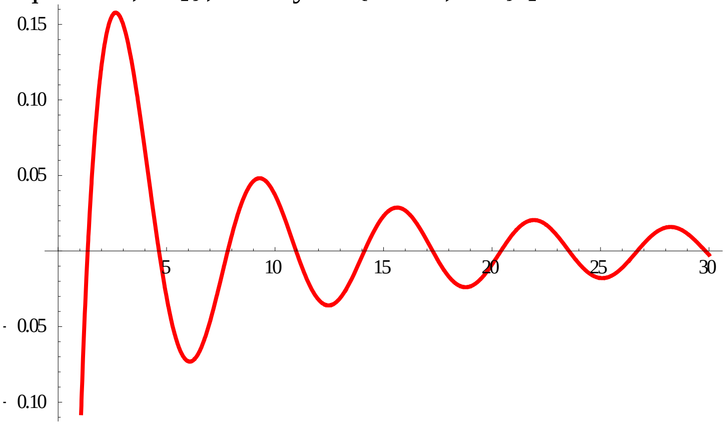
To export data in table form to Excel:

$$q = \mathbf{Table}[\mathbf{sol1}[t], \{t, 0, 3, 1/40\}]$$
$$q//\mathbf{TableForm}$$

Export["qfile.exe", *q*, "XLS"]

3. Effect of Natural Convection by Varying Rayleigh number

eqn1:=BesselJ[0,β]BesselY[1,β*2]-BesselJ[1,β*2]BesselY[0,β]
Plot[Evaluate[eqn1],{β,0,30},Frame→False, FrameLabel→{Style["β",16],Style["Transdental Equation", 16]},PlotStyle→{Thick,Red}]

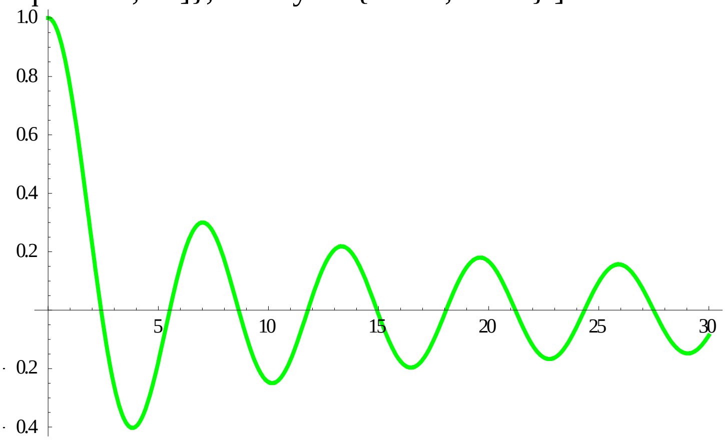


Clear[root,guess,rootlist1];
guess=1;rootlist1={};n=20;
Do[{root=FindRoot[Evaluate[eqn1==0],{β,guess}];
  AppendTo[rootlist1,root[[1,2]]];
  guess=3+root[[1,2]]},{i,1,n}];
rootlist1

{1.36078,4.6459,7.81416,10.9671,14.1151,17.2607,20.405,23.5487,26.6918,29.8347,32.9772,36.1197,39.262,42.4041,45.5462,48.6883,51.8303,54.9722,58.1141,61.256}

eqn2:=BesselJ[0,g]

Plot[Evaluate[eqn2],{g,0,30},Frame→False, FrameLabel→{Style["g",16],Style["Transdental Equation", 16]},PlotStyle→{Thick,Green}]



Clear[root,guess,rootlist2];
guess=1;rootlist2={};n=20;
Do[{root=FindRoot[Evaluate[eqn2==0],{g,guess}];
  AppendTo[rootlist2,root[[1,2]]];
  guess=3+root[[1,2]]},{i,1,n}];

rootlist2

{2.40483,5.52008,8.65373,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346,33.7758,36.9171,40.0584,43.1998,46.3412,49.4826,52.6241,55.7655,58.907,62.0485}

β[m_]:=rootlist1[[m]]
g[n_]:=rootlist2[[n]]
b1[n_]:=NIntegrate[R*BesselJ[0,g[n]R],{R,0,1}]
b2[n_]:=NIntegrate[BesselJ[0,g[n]R]^2,{R,0,1}]
B[n_]:=B[n]=(b1[n]/b2[n])
Table[B[n],{n,1,10}]

{0.455947,-0.244353,0.176975,-0.142462,0.121027,-0.106228,0.0952977,-0.0868404,0.0800687,-0.0745027}

c1[m_]:=BesselY[0,β[m]]NIntegrate[R*BesselJ[0,β[m]R],{R,0,1}]
c2[m_]:=BesselJ[0,β[m]]NIntegrate[R*BesselY[0,β[m]R],{R,0,1}]
b3[m_]:=BesselY[0,β[m]]^2 NIntegrate[BesselJ[0,β[m]R]^2 R,{R,0,1}]
b4[m_]:=BesselJ[0,β[m]]^2 NIntegrate[BesselY[0,β[m]R]^2 R,{R,0,1}]
b5[m_]:=2BesselJ[0,β[m]]BesselY[0,β[m]]NIntegrate[R*BesselJ[0,β[m]R]BesselY[0,β[m]R],{R,0,1}]

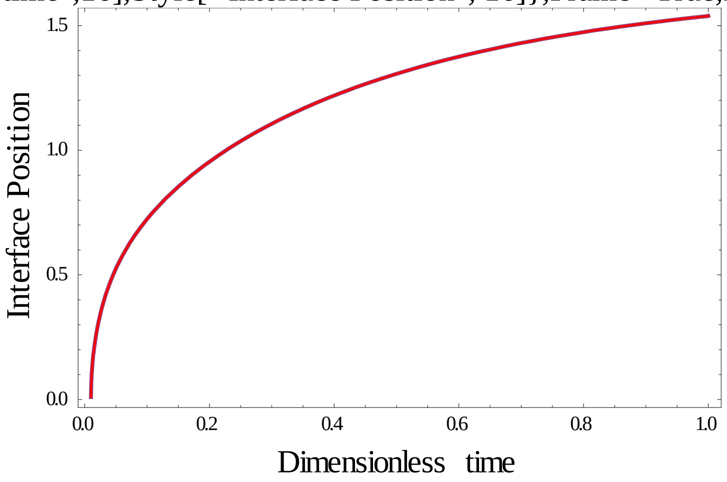
A[m_]:=A[m]=((c1[m]-c2[m])/(b3[m]+b4[m]-b5[m]))

Table[A[m],{m,1,20}]

{1.97837,4.3884,2.59092,3.81685,2.72071,3.64947,2.78627,3.56347,2.8278,3.50932,2.8572,3.47138,2.87946,3.44294,2.89709,3.42064,2.91151,3.40256,2.9236,3.38752}

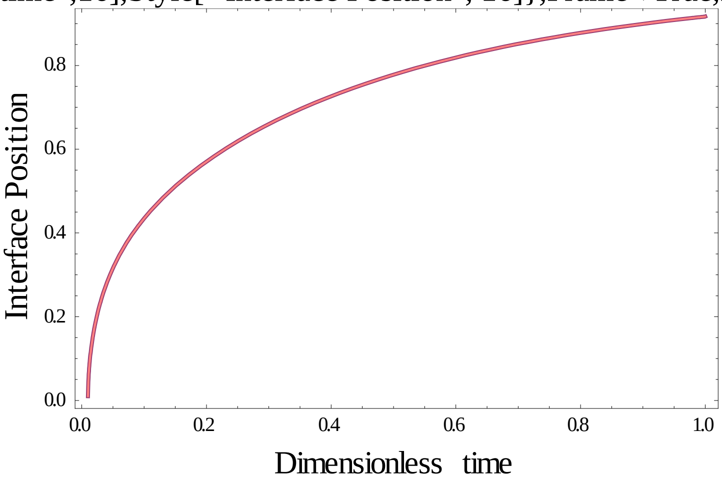
sol1=S/.NDSolve[{S'[t]== -Sum[A[m]E^(-t*β[m]^2) (-BesselJ[1,β[m]S[t]] BesselY[0,β[m]S[t]]+BesselJ[0,β[m]S[t]] BesselY[1,β[m] S[t]])2.77,{m,1,10}],S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

p1=Plot[Evaluate[Table[sol1[t],{S,0,2}]],{t,0.01,1},FrameLabel→{Style["Dimensionless time",16],Style["Interface Position", 16]},Frame→True,PlotStyle→{Red,Thick}]



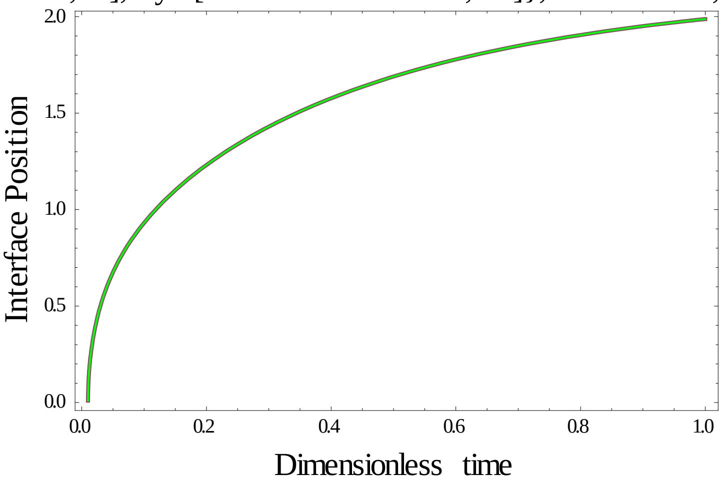
sol2=S/.NDSolve[{S'[t]== -Sum[A[m]E^(-t*β[m]^2) (-BesselJ[1,β[m]S[t]] BesselY[0,β[m]S[t]]+BesselJ[0,β[m]S[t]] BesselY[1,β[m] S[t]]),{m,1,10}],S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

```
p2=Plot[Evaluate[Table[sol2[t],{S,0,2}]],{t,0.01,1},FrameLabel→{Style["Dimensionless time",16],Style[ "Interface Position", 16]},Frame→True,PlotStyle→{Pink,Thick}]
```



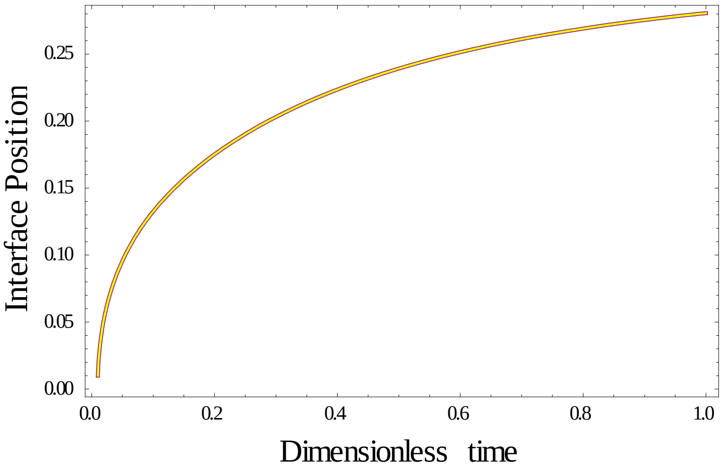

```
sol3=S/.NDSolve[{S'[t]== -Sum[A[m]E^(-t*β[m]^2 ) (-BesselJ[1,β[m]S[t]] BesselY[0,β[m]S[t]]+BesselJ[0,β[m]S[t]] BesselY[1,β[m] S[t]])4.57,{m,1,10}],S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

p3=Plot[Evaluate[Table[sol3[t],{S,0,2}]],{t,0.01,1},FrameLabel→{Style["Dimensionless time",16],Style[ "Interface Position", 16]},Frame→True,PlotStyle→{Green,Thick}]
```

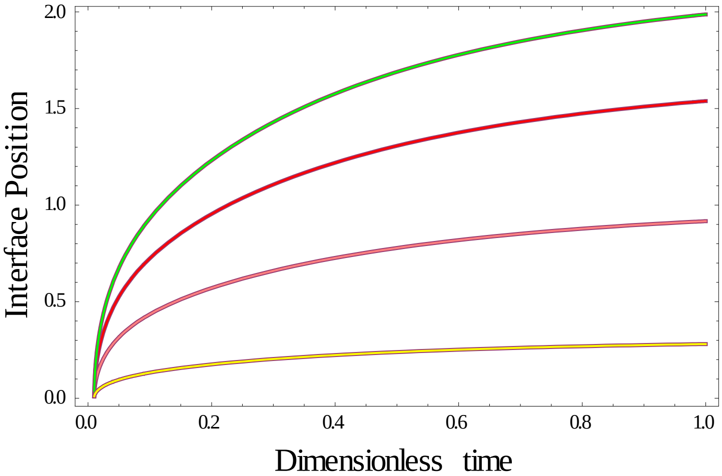



```
sol4=S/.NDSolve[{S'[t]== -Sum[A[m]E^(-t*β[m]^2 ) (-BesselJ[1,β[m]S[t]] BesselY[0,β[m]S[t]]+BesselJ[0,β[m]S[t]] BesselY[1,β[m] S[t]])0.09,{m,1,10}],S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

p4=Plot[Evaluate[Table[sol4[t],{S,0,2}]],{t,0.01,1},FrameLabel→{Style["Dimensionless time",16],Style[ "Interface Position", 16]},Frame→True,PlotStyle→{Yellow,Thick}]
```



Show[p1,p2,p3,p4,PlotRange→All]



4. Comparison with Exact Solution

sol1=S/.NDSolve[{S'[t]==-Sum[A[m]E^(-t*β[m]^2)(-BesselJ[1,β[m]S[t]] BesselY[0,β[m]S[t]]+BesselJ[0,β[m]S[t]]BesselY[1,β[m] S[t]])0.2,{m,1,10}],S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

w=Table[sol1[t],{t,0,1,1/40}]

InterpolatingFunction::dmval: Input value _{0}_ lies outside the range of data in the interpolating function. Extrapolation will be used. ≫

{0,0.0929048,0.142154,0.17324,0.196771,0.216017,0.232458,0.24689,0.259787,0.271459,0.282114,0.291906,0.300951,0.309338,0.31714,0.324417,0.33122,0.337591,0.343568,0.349183,0.354464,0.359438,0.364126,0.368549,0.372726,0.376672,0.380404,0.383935,0.387277,0.390443,0.393443,0.396288,0.398985,0.401545,0.403974,0.406281,0.408471,0.410552,0.412529,0.414409,0.416196}

4.1 Exact Solution Code

```
sol2=S/.NDSolve[{(S'[t]*S[t])/2==1.1(Exp[-S[t]^2/(4t)])0.2+(Exp[-S[t]^2/(4t)]/ExpIntegralEi[-S[t]^2/(4t)])0.2,S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

e=Table[sol2[t],{t,0,1,1/40}]
```

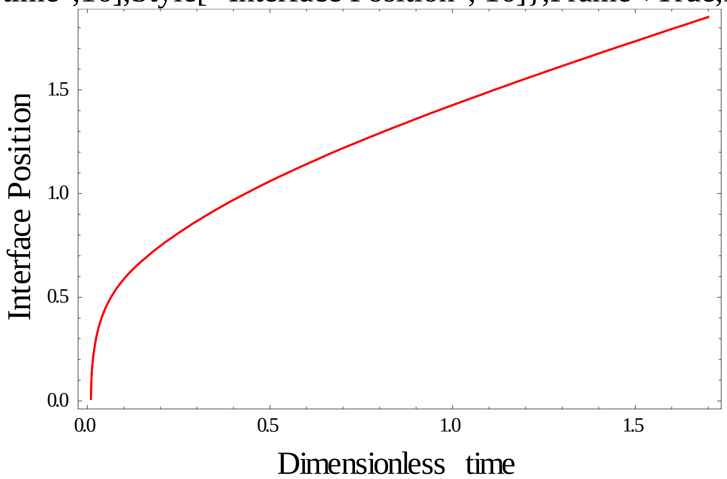
InterpolatingFunction::dmval: Input value _{0}_ lies outside the range of data in the interpolating function. Extrapolation will be used. ≫

{0,0.0886324,0.136277,0.169882,0.197535,0.221635,0.243295,0.263141,0.281568,0.298846,0.315167,0.330676,0.345484,0.359679,0.37333,0.386497,0.399228,0.411563,0.423537,0.435181,0.446521,0.457578,0.468374,0.478926,0.489249,0.499359,0.509267,0.518987,0.528526,0.537897,0.547107,0.556164,0.565075,0.573848,0.582489,0.591003,0.599396,0.607673,0.615839,0.623897,0.631853}

5. Effect of Stefan Number

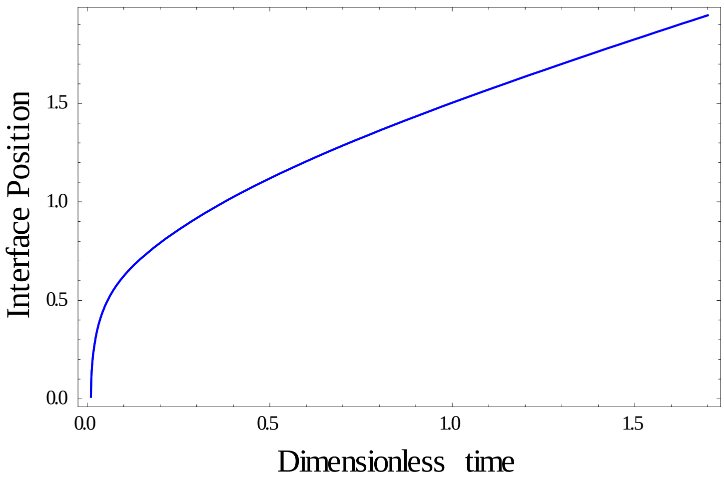
```
sol1=S/.NDSolve[{S'[t]==-(8/(4*3.14))(-Exp[-S[t]^2/(4t)]+ExpIntegralEi[-S[t]^2/(4t)])0.75-Sum[A[m]E^(-t*β[m]^2 ) (-β[m]BesselJ[1,β[m]S[t]] BesselY[0,β[m]S[t]]+β[m]BesselJ[0,β[m]S[t]] BesselY[1,β[m] S[t]])0.75,{m,1,10}],S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

p=Plot[Evaluate[Table[sol1[t],{S,0,2}]],{t,0.01,1.7},FrameLabel→{Style["Dimensionless time",16],Style[ "Interface Position", 16]},Frame→True,PlotStyle→{Red,Thin}]
```



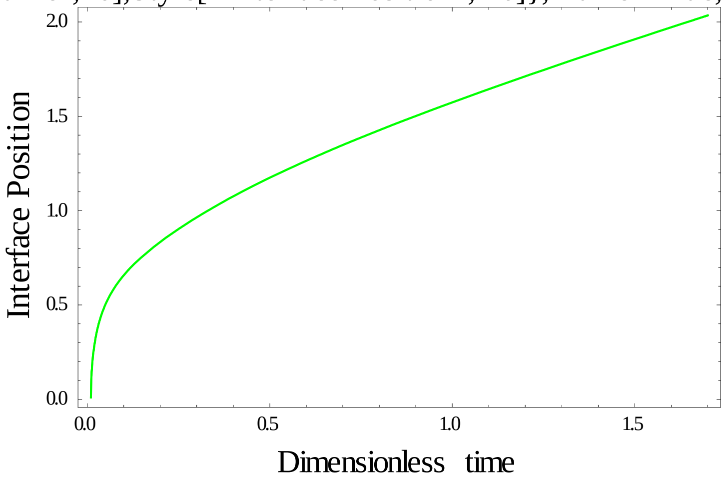

```
sol2=S/.NDSolve[{S'[t]==-(8/(4*3.14))(-Exp[-S[t]^2/(4t)]+ExpIntegralEi[-S[t]^2/(4t)])0.85-Sum[A[m]E^(-t*β[m]^2 ) (-β[m]BesselJ[1,β[m]S[t]] BesselY[0,β[m]S[t]]+β[m]BesselJ[0,β[m]S[t]] BesselY[1,β[m] S[t]])0.85,{m,1,10}],S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

p2=Plot[Evaluate[Table[sol2[t],{S,0,2}]],{t,0.01,1.7},FrameLabel→{Style["Dimensionless time",16],Style[ "Interface Position", 16]},Frame→True,PlotStyle→{Blue,Thin}]
```

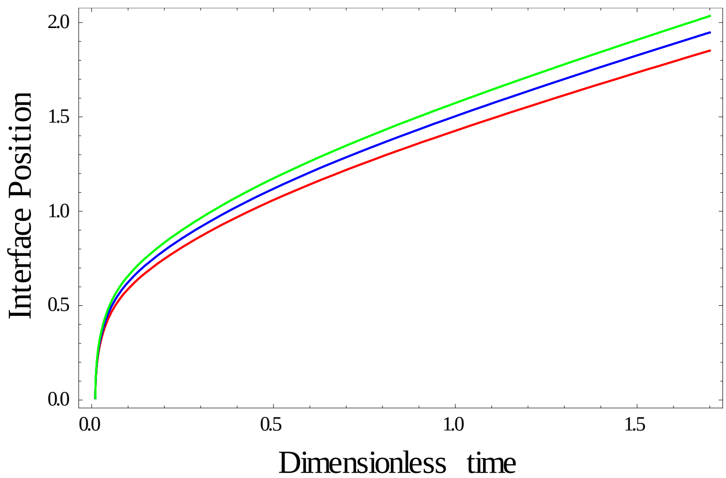



```
sol3=S/.NDSolve[{S'[t]==-(8/(4*3.14))(-Exp[-S[t]^2/(4t)]+ExpIntegralEi[-S[t]^2/(4t)])0.95-Sum[A[m]E^(-t*β[m]^2 ) (-β[m]BesselJ[1,β[m]S[t]] BesselY[0,β[m]S[t]]+β[m]BesselJ[0,β[m]S[t]] BesselY[1,β[m] S[t]])0.95,{m,1,10}],S[0.01]==0.01},S,{t,0.01,3},MaxSteps→10^8][[1,1]];

p3=Plot[Evaluate[Table[sol3[t],{S,0,2}]],{t,0.01,1.7},FrameLabel→{Style["Dimensionless time",16],Style[ "Interface Position", 16]},Frame→True,PlotStyle→{Green,Thin}]
```




```
Show[p,p2,p3,PlotRange→All]
```



Acknowledgments

I would like to acknowledge Prof B.G Higgens and Prof Andres Granados for their useful suggestions.

REFERENCES

[1] Wolfram Research, Inc., Mathematica, Version 5.2, Champaign, IL, 2005.
[2] [1] B.G Higgens, “Transient Heat Conduction in a Finite Length”, sites.google.com/site/chemengwithmathematica/home/ech51-material-balances/class-notes.