Lecture Note: Quantitative Methods in Finance

GENERAL EDUCATION SEMINAR

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Aims Of This Course

1. Learn basic concepts in finance such as continuous compound rate, present value and internal rate of return.
2. Learn basic principles of fixed income investing.
3. Learn the mean-variance approach in portfolio analysis.
4. Learn basic principles of corporate finance.
5. Hands-on practice of Python.

Reading List i

1. Comprehensive introduction to finance
   - Luenberger, D.G. (2013). *Investment Science*, 2nd ed., Oxford University Press.
2. Portfolio analysis
   - Elton, E.J, Gruber, M.J., Brown, S.J. and Goetzmann, W.N. (2013). *Modern Portfolio Theory and Investment Analysis*, 9th ed., Wiley.
3. Corporate finance
   - Brealey, R., Myers, S. and Allen, F. (2019). *Principles of Corporate Finance*, 13th ed., McGraw-Hill Education.

Reading List ii

- Koller, T., Goedhart, M. and Wessels, D. (2020). *Valuation: Measuring and Managing the Value of Companies*, 7th ed., Wiley.

4. Python for finance
   - Yves Hilpisch, Y. (2018). *Python for Finance: Mastering Data-Driven Finance*, 2nd ed., O'Reilly Media.

Python

- Python is a high-level programming language.
- Designed by Guido van Rossum
- Released in 1991
- Python is popular.
    - IEEE SPECTRUM
    - TIOBE

Why Python?

- It is free.
- It is slow in execution but highly manageable.
- Python codes are arguably more readable than other languages such as C/C++.
- Numerous packages have been developed for Python.
- Most of them are free and written in faster programming languages such as C/C++.

How To Obtain Python

- The official Python website
- Unfortunately, the plain Python does not include any useful tools for statistics / data science.
- Python distributions for scientific computing
    - Anaconda (we use this in the class)
    - ActivePython
    - Enthought Deployment Manager

Tools For Python Programming

- REPL (Read-Eval-Print-Loop)
  - Terminal-based REPL — **IPython**
  - Browser-based REPL — **Jupyter Notebook**
- An **integrated development environment (IDE)** is an application that consists of integrates an editor, a debugger, a profiler and other tools for developers.
  - **Spyder**
  - **PyCharm**
  - **Visual Studio Code**

Basic Packages

- **NumPy** — n-dimensional arrays and matrices
- **SciPy** — functions for scientific computing
- **Matplotlib** — 2D/3D plotting
- **Pandas** — data structure

Suppose $W(t)$ is the wealth at time t ($t \geqq 0$) and r is an interest rate.

- Simple interest rate

$$W(t) = (1 + rt)W(0).$$

- One-year compound interest rate

$$W(t) = (1 + r)^t W(0).$$

- $\frac{1}{M}$-year compound interest rate

$$W(t) = \left(1 + \frac{r}{M}\right)^{Mt} W(0).$$

Napier's constant is defined as

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}.$$

Interest Rate iii

Then

$$
\begin{aligned}
&\lim_{M\to\infty}\left(1+\frac{r}{M}\right)^{M} W(0)\\
&=\left\{\lim_{M\to\infty}\left(1+\frac{r}{M}\right)^{\frac{M}{r}}\right\}^{r} W(0)\\
&=\left\{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right\}^{r} W(0)\\
&=e^{r}W(0), \quad x=\frac{M}{r}.
\end{aligned}
$$

Interest Rate iv

Therefore

$$\begin{aligned} W(t) &= \left\{ \lim_{M \to \infty} \left(1 + \frac{r}{M}\right)^{M} \right\}^{t} W(0) \\ &= \left\{ \lim_{M \to \infty} \left(1 + \frac{r}{M}\right)^{\frac{M}{r}} \right\}^{rt} W(0) \\ &= e^{rt} W(0), \end{aligned}$$

which is called the continuous compound interest rate.

Present value

$$PV(0) = B(t)W(t),$$

where

$$B(t) = \begin{cases} \dfrac{1}{(1+r)^t}, & \text{(one-year compound)}, \\ \dfrac{1}{\left(1+\frac{r}{M}\right)^{Mt}}, & \left(\frac{1}{M}\text{-year compound}\right), \\ e^{-rt}, & \text{(continuous compound)}. \end{cases}$$

Present Value ii

The present value $PV(0)$ is interpreted as the amount of money you need to put in your bank account now ($t = 0$) to receive $W(t)$ at time t. In this context, the interest rate r is called the discount rate and $B(t)$ is called the discount factor.

Conversely,

$$W(t) = \frac{PV(0)}{B(t)},$$

is called the future value. The future value is simply regarded as the amount of money you expect to receive at time t.

Cash Flow i

A cash flow is, as its name suggests, a flow of cash payments. Suppose a project or enterprise (e.g., gold mine, oil well, power plant, factory, shopping mall) will produce cash payments for T years.

Let $C(t)$ denote the amount of payment at time t ($0 \leqq t \leqq T$). If $C(t) > 0$, it is treated as gain or profit (inflow); otherwise, it is loss (outflow). Suppose the number of payments is N but intervals between them is not necessarily regular. Let t_n $(n = 1, \ldots, N)$ denote the time of the n-th payment. As a convention, we suppose $t_0 = 0$ and $t_N = T$.

Cash Flow ii

Then the cash flow is represented as

$$\{C(t_1), \ldots, C(t_n), \ldots, C(t_N)\}.$$

Furthermore, let $B(t_n)$ denote the discount factor for $C(t_n)$.

Table 1: Cash flow, discount factor and present value

Time	t_1	$\cdots$	t_n	$\cdots$	t_N
Cash Flow	$C(t_1)$	$\cdots$	$C(t_n)$	$\cdots$	$C(t_N)$
Discount Factor	$B(t_1)$	$\cdots$	$B(t_n)$	$\cdots$	$B(t_N)$
Present Value	$B(t_1)C(t_1)$	$\cdots$	$B(t_n)C(t_n)$	$\cdots$	$B(t_N)C(t_N)$

Cash Flow iii

The present value of the cash flow is defined as the sum of all present values, i.e.,

$$
\begin{aligned}
PV(0) &= B(t_1)C(t_1) + \cdots + B(t_n)C(t_n) + \cdots + B(t_N)C(t_N) \\
&= \sum_{n=1}^{N} B(t_n)C(t_n).
\end{aligned}
$$

Net Present Value i

In practice, we need an initial investment to start the new project that will produce the cash flow $\{C(t_1), \ldots, C(t_N)\}$.

Let $C(0)$ denote the amount of the initial investment. In case of investment in a new factory, for example, $C(0)$ represents the amount of fund necessary for the purchase of equipments and machineries. Thus $C(0)$ must be negative.

A typical cash flow is a sequence of cash payments such that

$$C(0) < 0,\ C(t_1) \geqq 0, \ldots, C(t_n) \geqq 0, \ldots, C(t_N) \geqq 0.$$

Net Present Value ii

Recall $t_0 = 0$ and $B(t_0) = B(0) = 1$. The net present value (NPV) of a cash flow is defined as

$$
\begin{aligned}
NPV(0) &= C(0) + PV(0) \\
&= C(0) + \sum_{n=1}^{N} B(t_n)C(t_n) = \sum_{n=0}^{N} B(t_n)C(t_n).
\end{aligned}
$$

Suppose intervals between payments are equal, that is,

$$
t_n = \frac{n}{N}T = n\boldsymbol{\Delta}, \quad \boldsymbol{\Delta} = \frac{T}{N}, \quad (n = 0, 1, \dots, N).
$$

Net Present Value iii

Then $B(t_n) = \beta(r)^n$ where

$$\beta(r) = \begin{cases} \dfrac{1}{(1+r)^{\Delta}}, & \text{(one-year compound)} \\ \dfrac{1}{\left(1+\frac{r}{M}\right)^{M\Delta}}, & \left(\frac{1}{M}\text{-year compound}\right) \\ e^{-r\Delta}, & \text{(continuous compound).} \end{cases}$$

Therefore the NPV is expressed as a polynomial function:

$$NPV(0) = c_0 + c_1 x + \cdots + c_n x^n + \cdots + c_N x^N,$$

where $x = \beta(r)$ and $c_n = C(t_n)$ $(n = 0, 1, \ldots, N)$.

Internal Rate Of Return i

The internal rate of return (IRR) is the discount rate r^* that makes the NPV of the cash flow equal to zero:

$$0 = c_0 + c_1\beta(r^*) + \cdots + c_n\beta(r^*)^n + \cdots + c_N\beta(r^*)^N.$$

The solution r^* is obtained by solving the polynomial equation:

$$0 = c_0 + c_1x + \cdots + c_nx^n + \cdots + c_Nx^N,$$

with respect to x, and transform the solution x^* with

$$r^* = \begin{cases} \left(\frac{1}{x^*}\right)^{\frac{1}{\Delta}} - 1, & \text{(one-year compound)} \\ M\left\{\left(\frac{1}{x^*}\right)^{\frac{1}{M\Delta}} - 1\right\}, & \text{(}\tfrac{1}{M}\text{-year compound)} \\ -\frac{\log x^*}{\Delta}, & \text{(continuous compound)} \end{cases}$$

To simplify mathematical expressions, we suppose $\Delta = 1$ and $M = 1$, i.e., each payment occurs at the end of each year.

Internal Rate Of Return iii

Special cases

Case 1: When $N = 1$,

$$0 = c_0 + c_1 x^* \quad \Rightarrow \quad r^* = \frac{c_1 - (-c_0)}{-c_0}.$$

Case 2: When $N \to \infty$ and $c_n = c$ for all $n = 1, 2, 3, 4$,

$$0 = c_0 + \sum_{n=1}^{\infty} c(x^*)^n = c_0 + c\frac{x^*}{1 - x^*} \Rightarrow r^* = \frac{c}{-c_0}.$$

In general a higher-order polynomial equation could have multiple solutions and many of them are complex numbers. To ensure that the polynomial equation has the unique real-valued solution, the following two conditions must be satisfied:

(A1) $c_0 < 0$.

(A2) $c_n \geqq 0$ $(n = 1, \ldots, N)$ and $c_n > 0$ for some n.

To ensure that the IRR is positive, we need the extra condition:

(A3) $\sum_{n=0}^{N} c_n > 0$.

Bond

A bond is an instrument of borrowing. A typical bond promises to pay the predetermined amount of cash, the principal or the face value, to the bond holder on the predetermined future date, the maturity date. The length of time until the maturity date is called the time to maturity. A coupon-bearing bond promises to pay the predetermined amount of cash periodically to the bond holder. This payment is called the coupon. A bond without coupon payment is called a zero-coupon bond.

Issuers of bonds

- **Sovereign bond:** issued by national governments
- **Municipal bond:** issued by local governments
- **Corporate bond:** issued by corporations

Bond Vs. Bank Loan

As instruments to raise capital, bonds seem similar to bank loans (both are categorized as debts in the balance sheet), though there are notable differences between them:

1. Bonds are often issued at auctions (primary market).
2. Bonds can be transferred from one holder to others.
3. Bonds are tradable in the financial market (secondary market) until the maturity date.
4. When the bond issuer fails to pay either coupon or principal, it is deemed a default. Credit ratings are indicators for the default risk of bond issuers, which are assigned by credit rating agencies (e.g., S&P, Moody's, Fitch).

Cash Flows Of Bonds

Zero-coupon bond

t	0	1	2	$\cdots$	$T-1$	T
$C(t)$	$-V$	0	0	$\cdots$	0	F

Coupon-bearing bond

t	0	1	2	$\cdots$	$T-1$	T
$C(t)$	$-P$	C	C	$\cdots$	C	$C+F$

C — coupon

F — face value

T — time to maturity

V — price of the zero-coupon bond

P — price of the coupon-bearing bond

Yield To Maturity Of A Zero-Coupon Bond i

Let $V(t)$ denote the price of a zero-coupon bond that will mature at time t $(0 < t \leq T)$. The yield to maturity (YTM) or simply yield, denoted by $y(t)$, is defined as

Zero-coupon bond yield

$$y(t) = \begin{cases} \left(\frac{F}{V(t)}\right)^{\frac{1}{t}} - 1, & \text{(one-year compound)} \\ \frac{1}{t} \log \frac{F}{V(t)}, & \text{(continuous compound)} \end{cases}$$

Yield To Maturity Of A Zero-Coupon Bond ii

Basically, the yield $y(t)$ is equivalent to the internal rate of return on the cash flow of the zero-coupon bond. $y(t)$ as a function of time to maturity is called the zero yield curve or simply yield curve. Conversely, once the yield curve $y(t)$ is known, the price of a zero-coupon bond is

$$V(t) = \begin{cases} \dfrac{F}{\{1 + y(t)\}^t}, & \text{(one-year compound)} \\ \\ e^{-y(t)t}F, & \text{(continuous compound)} \end{cases}$$

The price of a zero-coupon bond is equal to the present value of the face value F discounted by the yield $y(t)$.

Yield To Maturity Of A Coupon-Bearing Bond

The yield to maturity of a coupon-bearing bond is the internal rate of return on its cash flow. In this context, $\boldsymbol{y(t)}$ must be constant for any t, though $\boldsymbol{y}$ is time-varying in reality. Therefore the yield $\boldsymbol{y(t)}$ is the real-valued solution of the following polynomial equation.

Coupon-bearing bond yield

$$P(0) = C\sum_{t=1}^{T-1} B(t,y) + (C+F)B(T,y),$$

$$B(t,y) = \begin{cases} \dfrac{1}{(1+y)^t}, & \text{(one-year compound)} \\ e^{-yt}, & \text{(continuous compound)} \end{cases}$$

Relationship Between Bond Price And Bond Yield

Bond price and bond yield

$$V(y) = B(t, y)F, \quad (t = 1, \ldots, T)$$

$$P(y) = C \sum_{t=1}^{T-1} B(t, y) + (C + F)B(T, y).$$

1. Both $V(y)$ and $P(y)$ are decreasing functions of y.
2. The yield will go up when the price goes down.
3. The yield is uniquely determined since (A1) and (A2) are always satisfied.
4. The yield is negative if

$$\sum_{t=0}^{T} C(t) = TC + F - P < 0.$$

Price Sensitivity To A Yield Curve Shift i

The yield curve frequently shifts due to business cycles, market sentiments, interventions by central banks and other numerous factors. In this lecture, we concentrate on a parallel shift in the yield curve: $y(t) + \lambda$ for all t. We consider the following measurement of sensitivity to a yield curve shift:

$$\text{sensitivity to yield curve shift} = \frac{\lim_{\lambda \to 0} \frac{P(y+\lambda) - P(y)}{\lambda}}{P(y)},$$

where y may be time-varying and "$\lim_{\lambda \to 0}$" means that the shift λ is infinitesimally small.

Price Sensitivity To A Yield Curve Shift ii

To obtain the exact formula, we need the differential of the bond price with respect to the yield. The differential of $f(x)$ is

$$\nabla_x f(x) \triangleq \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}.$$

Price Sensitivity To A Yield Curve Shift iii

Using the above notation, we have

$$\lim_{\lambda \to 0} \frac{P(y+\lambda) - P(y)}{\lambda} = \nabla_y P(y) = C \sum_{t=1}^{T} \nabla_y B(t,y) + F \nabla_y B(T,y)$$

$$= \begin{cases} -C \sum_{t=1}^{T} \frac{tB(t,y)}{1+y} - F \frac{TB(T,y)}{1+y}, & \text{(one-year compound)} \\ -C \sum_{t=1}^{T} tB(t,y) - FTB(T,y), & \text{(continuous compound)} \end{cases}$$

since

$$\nabla_y B(t,y) = \begin{cases} -\frac{t}{(1+y)^{t+1}}, & \text{(one-year compound)} \\ -te^{-yt}, & \text{(continuous compound)} \end{cases}$$

Price Sensitivity To A Yield Curve Shift iv

Hence the price sensitivity to a infinitesimally small shift in the yield curve is given by

$$\frac{\nabla_y P(y)}{P(y)} = \begin{cases} -\frac{D(y)}{1+y}, & \text{(one-year compound)} \\ -D(y), & \text{(continuous compound)} \end{cases}$$

$$D(y) = \frac{C\sum_{t=1}^{T} tB(t,y) + FTB(T,y)}{P(y)}.$$

$D(y)$ is called the duration.

Remarks On Duration

1. $D(y) > 0$. Thus the price will increase if $\lambda < 0$, and vice versa.
2. $D(y) = t$ for a zero-coupon bond with time to maturity t.
3. Therefore, a coupon-bearing bond with duration $D(y)$ is as sensitive to the yield curve shift as a zero-coupon bond with time to maturity $D(y)$.

Convexity

The convexity is defined as

$$C(y) = \frac{\nabla_y^2 P(y)}{P(y)} = \begin{cases} \frac{S(y)+(1+D(y))D(y)}{(1+y)^2}, & \text{(one-year compound)}, \\ S(y) + D(y)^2, & \text{(continuous compound)}, \end{cases}$$

$$S(y) = \frac{C\sum_{t=1}^{T}(t - D(y))^2 B(t,y) + F(T - D(y))^2 B(T,y)}{P(y)}.$$

1. $S(y)$ is called the dispersion.
2. $C(y) > 0$. Thus the rate of change in the bond price is a convex function of the shift λ.
3. Suppose there are two bonds with the same duration. Then the bond with large convexity will suffer less from the upward shift $(\lambda > 0)$ while it will gain more from the downward shift $(\lambda < 0)$.

Yield Curve Estimation i

Since long-term bonds are coupon-bearing ones, we have to estimate the yield curve with price data of coupon-bearing bonds. Suppose N bonds with various maturities $T_1, \ldots, T_N$ are traded in the market. They must be issued by the same agent. The price of bond n is P_n and its cash flow is $\{C_n(1), \ldots, C_n(\bar{T})\}$ where $\bar{T}$ is the longest maturity in the market. Since bond n will mature at T_n, $C_n(t) = 0$ for $t > T_n$. Then the bond price is given by

$$P_n = \sum_{t=1}^{T_n} B(t)C_n(t) = \sum_{t=1}^{\bar{T}} B(t)C_n(t), \quad (n = 1, \ldots, N)$$

Yield Curve Estimation ii

where $B(t)$ is the discount factor.

The discount factors $B(1), \ldots, B(\bar{T})$ are obtained as the solution of the following system of equations:

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix} = \begin{bmatrix} C_1(1) & C_1(2) & \cdots & C_1(\bar{T}) \\ C_2(1) & C_2(2) & \cdots & C_2(\bar{T}) \\ \vdots & & \ddots & \vdots \\ C_N(1) & C_N(2) & \cdots & C_N(\bar{T}) \end{bmatrix} \begin{bmatrix} B(1) \\ B(2) \\ \vdots \\ B(\bar{T})) \end{bmatrix}.$$

Yield Curve Estimation iii

Finally, the yield curve $\boldsymbol{y(t)}$ is given by

$$y(t) = \begin{cases} \left(\dfrac{1}{B(t)}\right)^{\frac{1}{t}} - 1, & \textbf{(}\text{one-year compound}\textbf{)} \\[2ex] \dfrac{1}{t}\log\dfrac{1}{B(t)}, & \textbf{(}\text{continuous compound}\textbf{)} \end{cases}$$

Defaultable Bond And Credit Spread i

Suppose the probability of default is π and the recovery rate is ρ ($0 \leqq \pi, \rho < 1$). Then the payment of a defaultable zero-coupon bond is

$$\begin{cases} \rho F, & \text{(the issuer defaults with } \pi) \\ F, & \text{(the issuer does not default with } 1 - \pi) \end{cases}$$

The expected payment is

$$(1 - \pi)F + \pi\rho F = (1 - (1 - \rho)\pi)F.$$

Defaultable Bond And Credit Spread ii

The "fair price" of the zero-coupon bond is defined as

$$\tilde{V} = \begin{cases} \dfrac{(1-(1-\rho)\pi)F}{(1+\tilde{y})^T}, & \text{(one-year compound)} \\ e^{-\tilde{y}T}(1-(1-\rho)\pi)F, & \text{(continuous compound)} \end{cases}$$

The yield on the defaultable bond is given by

$$\tilde{y} = \begin{cases} \dfrac{1+y}{(1-(1-\rho)\pi)^{\frac{1}{T}}} - 1, & \text{(one-year compound)} \\ y - \dfrac{1}{T}\log(1-(1-\rho)\pi), & \text{(continous compound)} \end{cases}$$

Defaultable Bond And Credit Spread iii

where y is the yield on riskless bond with the same maturity.

Since $1 - (1-\rho)\pi < 1$, $\tilde{y} - y$ is positive. The difference between $\tilde{y}$ and y is called the credit spread.

1. The credit spread is increased when the probability of default is increased.
2. The credit spread is increased when the recovery rate is decreased.

Portfolio Analysis

Portfolio = a collection of assets that the investor holds

Types of assets: stocks, bonds, currencies, real estate

Questions to be answered

- How much we must invest our money in each asset?
- What is the best way to form a suitable portfolio?

Return On A Portfolio Of Assets

The return on a portfolio of assets is a weighted average of the return on the individual assets.

Return of a portfolio

$$R_P = \sum_{n=1}^{N} w_n R_n.$$

R_P: the return on the portfolio
R_n: the return on the n-th asset
w_n: the fraction of the funds invested in the n-th asset

Measure Of Average Outcome

Expected return

$$\mu_P = \mathbf{E}(R_P) = \mathbf{E}\left(\sum_{n=1}^{N} w_n R_n\right)$$
$$= \sum_{n=1}^{N} w_n \mathbf{E}(R_n) = \sum_{n=1}^{N} w_n \mu_n.$$

The expected return on a portfolio is the weighted average of the expected return on the individual assets.

Measure Of Dispersion

Variance

$$
\begin{aligned}
\sigma_P^2 = \mathbf{E}[(R_P - \mu_P)^2] = \mathbf{E}\left[\left\{\sum_{n=1}^{N} w_n(R_n - \mu_n)\right\}^2\right] \\
= \sum_{n=1}^{N} w_n^2 \mathbf{E}[(R_n - \mu_n)^2] \\
+ \sum_{n=1}^{N} \sum_{m \neq n}^{N} w_n w_m \mathbf{E}[(R_n - \mu_n)(R_m - \mu_m)] \\
= \sum_{n=1}^{N} w_n^2 \sigma_n^2 + \sum_{n=1}^{N} \sum_{m \neq n}^{N} w_n w_m \sigma_{nm}.
\end{aligned}
$$

Example: Portfolio Of Two Assets

The expected return on a portfolio of two assets is

$$\mu_P = w_1\mu_1 + w_2\mu_2,$$

and the variance is

$$\begin{aligned}\sigma_P^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2,\end{aligned}$$

where $\sigma_{12} = \rho_{12}\sigma_1\sigma_2$ and ρ_{12} is the correlation coefficient between R_1 and R_2.

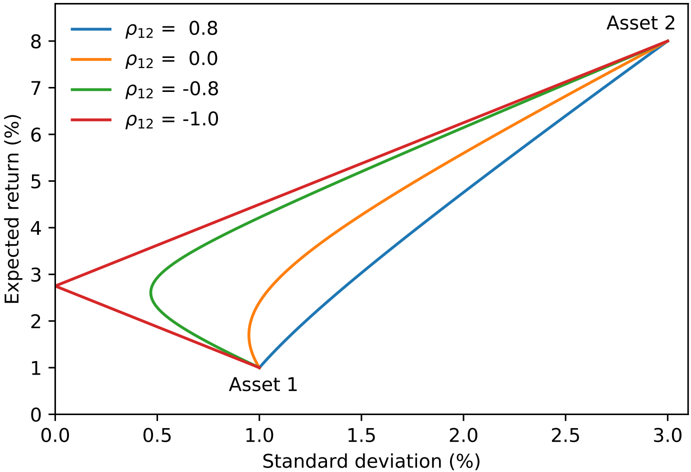

Figure 1: Trade-off between Risk and Return

Example: Portfolio Of Three Assets

The expected return on a portfolio of three assets is

$$\mu_P = w_1\mu_1 + w_2\mu_2 + w_3\mu_3,$$

and the variance is

$$\begin{aligned}
\sigma_P^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 \\
&\quad + 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_2w_3\sigma_{23} \\
&= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 \\
&\quad + 2w_1w_2\rho_{12}\sigma_1\sigma_2 + 2w_1w_3\rho_{13}\sigma_1\sigma_3 \\
&\quad + 2w_2w_3\rho_{23}\sigma_2\sigma_3,
\end{aligned}$$

Limit Of Diversification i

If $w_n = \frac{1}{N}$ for all assets, the variance of the portfolio is

$$
\begin{aligned}
\sigma_P^2 &= \sum_{n=1}^{N} \frac{\sigma_n^2}{N^2} + \sum_{n=1}^{N} \sum_{m \neq n}^{N} \frac{\sigma_{nm}}{N^2} \\
&= \frac{1}{N} \underbrace{\sum_{n=1}^{N} \frac{\sigma_n^2}{N}}_{\bar{\sigma}_n^2} + \frac{N-1}{N} \underbrace{\sum_{n=1}^{N} \sum_{m \neq n}^{N} \frac{\sigma_{nm}}{N(N-1)}}_{\bar{\sigma}_{nm}} \\
&= \frac{1}{N}(\bar{\sigma}_n^2 - \bar{\sigma}_{nm}) + \bar{\sigma}_{nm}.
\end{aligned}
$$

Limit Of Diversification ii

Thus σ_P^2 will converge to $\bar{\sigma}_{nm} \neq 0$ as N goes to infinity. Thus even well-diversified portfolios are not necessarily riskless.

In particular, when the variance is constant across assets ($\sigma_n^2 = \sigma^2$) and the correlation coefficient between any pair of asset is the same ($\sigma_{nm} = \rho\sigma^2$), we have

$$
\begin{aligned}
\sigma_P^2 &= \frac{1}{N}(\bar{\sigma}_n^2 - \bar{\sigma}_{nm}) + \bar{\sigma}_{nm} \\
&= \frac{1}{N}(\sigma^2 - \rho\sigma^2) + \rho\sigma^2 \to \rho\sigma^2,
\end{aligned}
$$

as n goes to infinity.

Basic Idea Of Portfolio Selection

- **Expected return as the performance measure**
  The investors love a higher return from their investment.
- **Variance as the risk measure**
  The investors want to avoid a higher variation in the value of their funds.

But we cannot have both. We need to find a “sweet spot” in the trade-off between risk and return. This trade-off is expressed as a utility maximization problem:

$$\max_{w_1,\ldots,w_N} \mu_P - \gamma\sigma_P^2, \tag{1}$$

where $\gamma \geqq 0$ is the degree of risk averse. Note that μ_P and σ_P^2 are functions of $w_1, \ldots, w_N$.

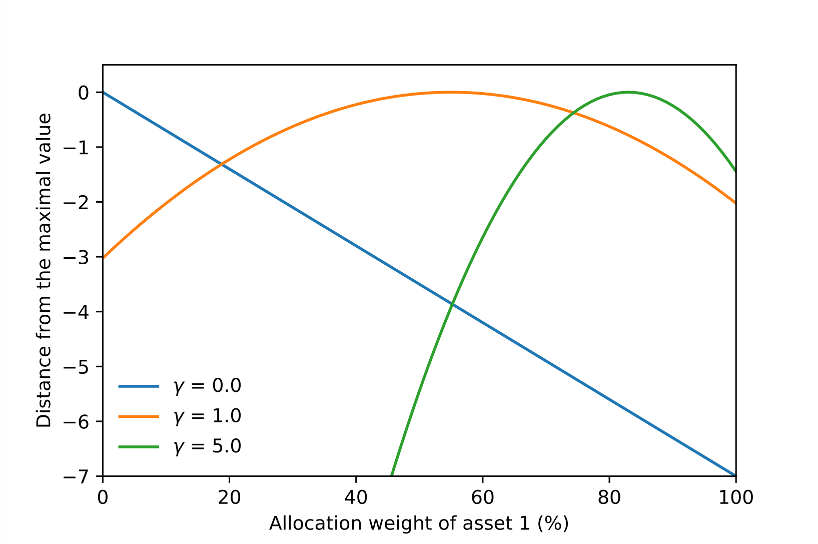

Figure 2: Investor's Utility Function

$$\begin{aligned}
\min_{w_1,\ldots,w_N} \quad & \sigma_P^2 = \sum_{n=1}^{N} w_n^2\sigma_n^2 + \sum_{n=1}^{N}\sum_{m\neq n}^{N} w_n w_n \sigma_{nm}, \\
\text{subject to} \quad & \text{(a)} \ \sum_{n=1}^{N} w_n\mu_n = \mu_P, \\
& \text{(b)} \ \sum_{n=1}^{N} w_n = 1, \\
& \text{(c)} \ w_n \geqq 0, \quad (n = 1, \ldots, N).
\end{aligned} \tag{2}$$

Minimum Variance Portfolio Problem ii

- The constraint (a) sets the target return of the portfolio.
- The constraint (b) is required due to the definition of $w_1, \ldots, w_n$.
- The constraint (c) prohibits short selling.

The solution of the above problem (2) gives us the trade-off relationship between the risk $\boldsymbol{\sigma_P}$ and the return $\boldsymbol{\mu_P}$.

The framework for portfolio selection in (2) is called the mean-variance approach and was proposed by Markowitz (1952). Thus it is also known as the Markowitz model.

Matrix Form

$$\begin{aligned}
\min_{w} \quad & \sigma_P^2 = w^\intercal \boldsymbol{\Sigma} w, \\
\text{subject to} \quad & \text{(a) } w^\intercal \boldsymbol{\mu} = \mu_P, \\
& \text{(b) } w^\intercal \boldsymbol{\iota} = 1, \\
& \text{(c) } w \geqq \mathbf{0},
\end{aligned} \tag{3}$$

where

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \ \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}, \ \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_N^2 \end{bmatrix},$$

and $\boldsymbol{\iota}$ is the $N \times 1$ vector whose elements are equal to 1.

Remarks On Minimum Variance Portfolio Selection

1. The minimum variance portfolio problem without short selling (3) has no closed-form solution.
2. Since the minimum variance portfolio problem (3) is a type of quadratic programming problem, it can be solved by a quadratic programming solver.
3. We may incorporate additional conditions (trading fees, taxes, upper and lower bounds of weights, etc.) into the minimum variance portfolio problem (3).

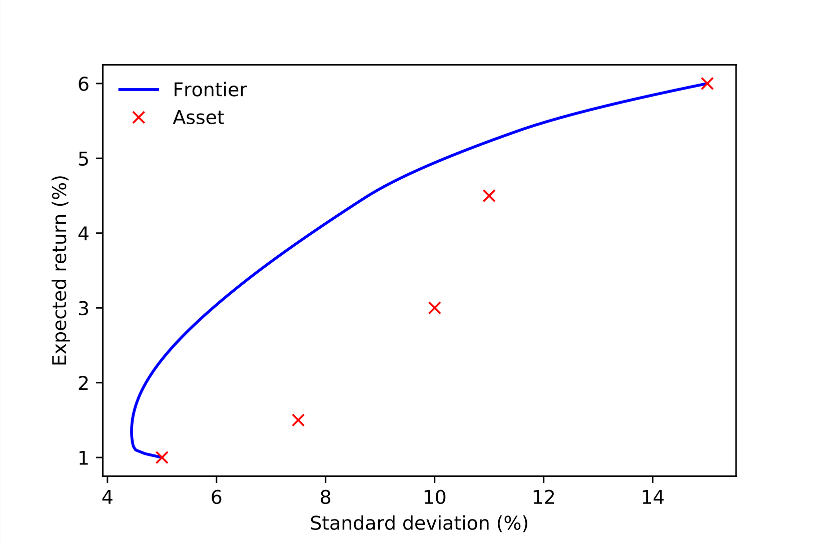

Figure 3: Minimum Variance Frontier without Short Selling

Optimal Portfolio Selection Procedure i

- For each given μ_P, the minimum variance σ_P^2 is obtained by solving the minimum variance portfolio problem (3). The graph of (σ_P, μ_P) is called the minimum variance frontier.
- In particular, the upper half of the minimum variance frontier is called the efficient frontier because the investor prefers a portfolio with a higher rate of return when the risk is the same.

Optimal Portfolio Selection Procedure ii

The efficient frontier shows us the trade-off between risk and return. The investor must choose the optimal portfolio among those on the efficient frontier.

Step 1. Construct the efficient frontier.

Step 2. Set the tolerable level of risk or the target expected return.

Step 3. Pick the corresponding portfolio on the efficient frontier.

Closed-form Solution

If we drop the constraint (c) which prohibits short selling, we can derive the solution of the minimum variance portfolio problem in a closed form:

$$w = \frac{C\mu_P - A}{D}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{B - A\mu_P}{D}\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}$$

$$A = \boldsymbol{\mu}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}, \quad B = \boldsymbol{\mu}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu},$$
$$C = \boldsymbol{\iota}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}, \quad D = BC - A^2.$$

The relationship between $\boldsymbol{\mu}_P$ and $\boldsymbol{\sigma}_P$ is given by

$$\boldsymbol{\sigma}_P = \sqrt{\frac{C\boldsymbol{\mu}_P^2 - 2A\boldsymbol{\mu}_P + B}{D}} = \sqrt{\frac{C}{D}\left(\boldsymbol{\mu}_P - \frac{A}{C}\right)^2 + \frac{1}{C}}.$$

This is the minimum variance frontier.

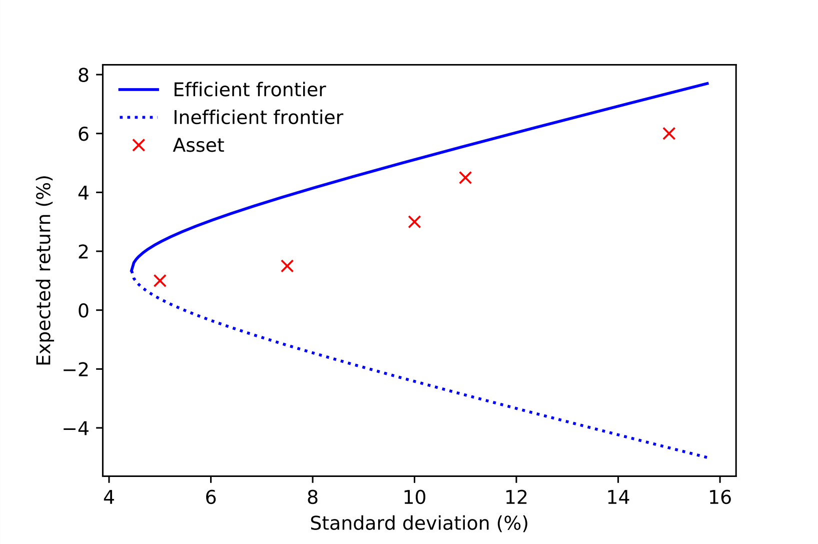

Figure 4: Minimum Variance Frontier with Short Selling

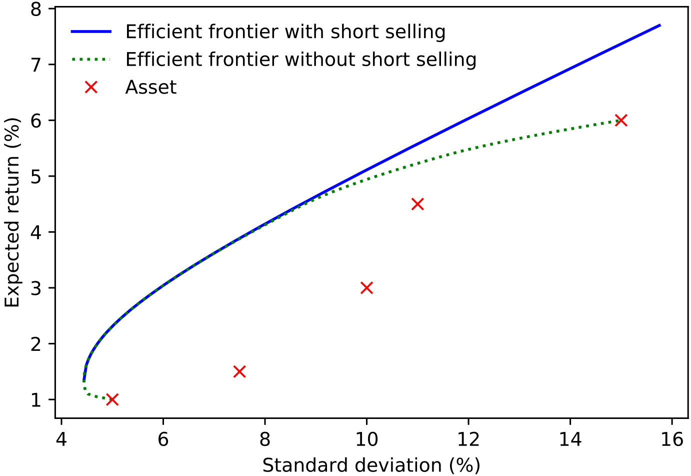

Figure 5: Minimum Variance Frontier with/without Short Selling

Risk-free Asset And Excess Return

Suppose there exists an asset on which the internal rate of return is constant under any circumstances. Such an asset is called the risk-free asset (riskless asset or safe asset). For example, a bank account is supposed to be that kind of asset.

Let R_f denote the return on the risk-free asset. We call it the risk-free rate.

Let us define

$$
\begin{aligned}
\tilde{R}_n &= R_n - R_f, \quad (n = 1\ldots, N), \\
\tilde{R}_{\mathcal{M}} &= R_{\mathcal{M}} - R_f.
\end{aligned}
$$

$\tilde{R}_n$ and $\tilde{R}_{\mathcal{M}}$ are called the excess return of asset n and the market portfolio respectively.

Portfolio Selection With The Risk-free Asset

The minimum variance portfolio problem with the risk-free asset but without the short-selling constraint is formulated as

$$\begin{aligned} \min_{w} \quad & \sigma_P^2 = w^\intercal \boldsymbol{\Sigma} w, \\ \text{subject to} \quad & (1 - w^\intercal \boldsymbol{\iota}) R_f + w^\intercal \boldsymbol{\mu} = \mu_P, \end{aligned} \tag{4}$$

where the definitions of μ_P, σ_P^2, $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, w and $\boldsymbol{\iota}$ are the same as before.

Note: The allocation weight for the risk-free asset is $1 - w^\intercal \boldsymbol{\iota}$. Since $1 - w^\intercal \boldsymbol{\iota} + w^\intercal \boldsymbol{\iota} = 1$, the sum of all allocation weights is always equal to 1. Thus we drop the constraint $w^\intercal \boldsymbol{\iota} = 1$ from the portfolio selection problem (4).

Closed-form Solution

The solution of the minimum variance portfolio problem (4) is

$$w = \frac{\mu_P - R_f}{CR_f^2 - 2AR_f + B}\boldsymbol{\Sigma}^{-1}(\mu_P - R_f\,\iota),$$

where

$$A = \mu^\top \boldsymbol{\Sigma}^{-1}\iota, \quad B = \mu^\top \boldsymbol{\Sigma}^{-1}\mu,$$
$$C = \iota^\top \boldsymbol{\Sigma}^{-1}\iota, \quad D = BC - A^2.$$

Then the relationship between μ_P and σ_P is given by

$$\mu_P = R_f \pm \sigma_P\sqrt{CR_f^2 - 2AR_f + B}. \tag{5}$$

This is also called the minimum variance frontier.

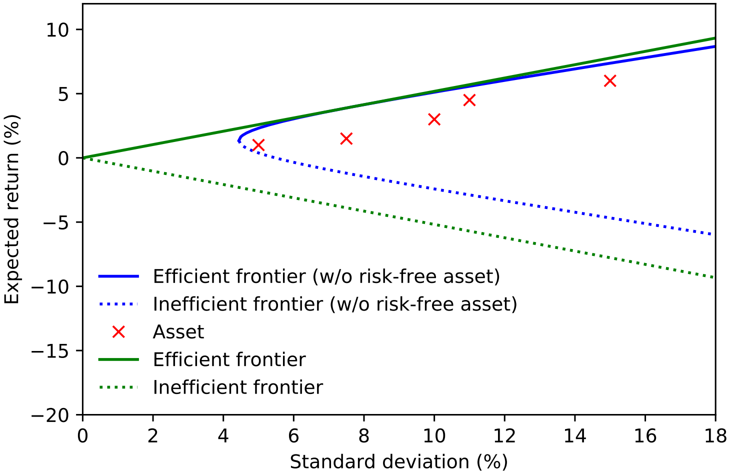

Figure 6: Frontier with/without the Risk-free Asset

One Fund Theorem

- Investors choose their optimal portfolios on the efficient frontier in (5) that is the straight line connecting $(0, R_f)$ and the tangent point on the efficient frontier of the minimum variance portfolio with no risk-free asset.
- The minimum variance portfolio corresponding the tangent point is called the tangent portfolio.
- Every portfolio on this line is a combination of the risk-free asset and the tangent portfolio. This property is called the one fund theorem or mutual fund theorem.
- Therefore the portfolio selection problem (4) is boiled down to the question: "How much of money should we invest in the tangent portfolio?"

Market Portfolio

Suppose N assets are traded in the financial market and M_n is the market capitalization of asset n $(n = 1, \ldots, N)$.

We consider a well-diversified portfolio that consists of all traded assets in the financial market:

Market portfolio

$$R_{\mathcal{M}} = \sum_{n=1}^{N} \bar{w}_n R_n, \quad \bar{w}_n = \frac{M_n}{\sum_{n=1}^{N} M_n}. \tag{6}$$

Under certain conditions (we will discuss later), in the market equilibrium, the market portfolio is the tangent portfolio for all investors.

Capital Asset Pricing Model (CAPM)

- The capital asset pricing model or CAPM (pronounced *Cap-M*) was independently discovered by Sharpe (1964), Lintner (1965) and Mossin (1966).
- As its name indicates, CAPM is used for pricing assets such as stocks.
- Although CAPM is out of fashion in the academic field of finance, it is still widely applied for estimating the cost of capital (e.g., WACC) in corporate valuation.

Key Assumptions For CAPM

1. Investors select their portfolios in the mean-variance approach (4).
2. Investors are able to invest in all traded assets.
3. $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ of all traded assets are common among investors.
4. Investors can short-sell assets without limits.
5. Investors can lend and borrow money at the same interest rate R_f without limits.
6. Investors can trade assets without paying any transaction costs or taxes.

Formula Of CAPM

Under the above assumptions, the expected return on asset n, $\mu_n = \mathbf{E}[R_n]$, is expressed as

CAPM

$$\mu_n = R_f + \beta_n(\mu_{\mathcal{M}} - R_f), \quad n = 1, \dots, N, \tag{7}$$

where $\mu_{\mathcal{M}} = \mathbf{E}[R_{\mathcal{M}}]$ and

$$\beta_n = \frac{\mathbf{Cov}[\tilde{R}_n, \tilde{R}_{\mathcal{M}}]}{\mathbf{Var}[\tilde{R}_{\mathcal{M}}]}. \tag{8}$$

This β_n is called the beta.

Note: The beta of a portfolio with return $R_P = \sum_{n=1}^{N} w_n R_n$ is $\beta_P = \sum_{n=1}^{N} w_n \beta_n$.

SML And CML

1. **Security Market Line (SML)**
   Note that $\mu_{\mathcal{M}} - R_f$ is fixed for all assets in (7). Thus it is regarded as a relationship between expected returns and betas.

$$\mu = R_f + (\mu_{\mathcal{M}} - R_f)\beta. \tag{9}$$

   This relationship is called the security market line.

2. **Capital Market Line (CML)**

$$\mu = R_f + \frac{\mu_{\mathcal{M}} - R_f}{\sigma_{\mathcal{M}}}\sigma, \tag{10}$$

   where $\frac{\mu_{\mathcal{M}} - R_f}{\sigma_{\mathcal{M}}}$ is called the Sharpe ratio of the market portfolio.

Estimation Of Beta

Suppose we have the following data sets:

- Returns on asset n at period t : R_{nt}, $(n = 1, \ldots, N,\ t = 1, \ldots, T)$
- Returns on the market portfolio at period t: $R_{\mathcal{M}t}$, $(t = 1, \ldots, T)$
- Risk-free rate at period t: R_{ft}, $(t = 1, \ldots, T)$,

and consider the following regression model:

$$\tilde{R}_{nt} = \alpha_n + \beta_n \tilde{R}_{\mathcal{M}t} + u_{nt}, \tag{11}$$

where u_{nt} is the error term.

We can obtain the estimate of β_n by estimating (α_n, β_n) with the ordinary least squares (OLS) estimation.

Cost Of Capital

- The cost of capital is a measure of opportunity cost investors must factor in when they make a decision on investment in a firm (e.g, M & A).
- The opportunity cost in this context is the expected return on an alternative instrument that is supposed to produce cash flows on par with the target firm.
- Investors receive cash payments from a firm in two forms: interest payments as debt-holders, and dividends and retained earnings as share-holders. So investors can regard the firm as an asset that produces cash flows.
- The cost of capital, if it is properly designed, can be used in discounting the cash flows of the firm.

Enterprise Value And Modigliani-Miller Theorem

- The enterprise value (EV) of a firm is defined as

$$\begin{aligned} EV = &\text{ Value of debt } (D) + \text{Value of equity } (E) \\ &- \text{Cash and cash equivalents}. \end{aligned}$$

- Note that the fair value of E is unknown in general.
- For simplicity, we ignore cash and cash equivalents in EV.
- In corporate valuation, the fair EV of a firm is defined as the present value of forecasted cash flows that the firm will produce in the future.
- Profitability of a firm has nothing to do with how it raised funds via debt financing or equity financing.
- Thus EV should be independent of the leverage D/E.
- This is a conclusion of the Modigliani-Miller (MM)

Cost Of Capital

As an alternative instrument, we consider a portfolio of debt (supposed risk-free) and equity that mimics the capital structure (composition of debt and equity on the B/S). Let R_A denote the return on the alternative instrument. Then we have

Cost of capital

$$\mathbb{E}[R_A] = \frac{D}{D+E}R_f + \frac{E}{D+E}\mathbb{E}[R_E], \tag{12}$$

where $\mathbb{E}[R_E]$ is the expected return on equity (cost of equity), D is the amount of debt on the firm's B/S and E is the amount of equity on the B/S (book value) or the market capitalization of the firm (market value)[1].

[1] The latter is more commonly used than the former in practice

Cost Of Capital And Modigliani-Miller Theorem

- Although the cost of capital in (12) seems to depend on the leverage D/E , this is not the case.
- Under the conditions in Modigliani and Miller (1958), the cost of capital must be independent of the leverage. This can be proved with the same reasoning behind the leverage-independence of EV.
- To make the cost of capital constant, the cost of equity must be a function of the leverage, i.e.,

$$\mathbf{E}[R_E] = \mathbf{E}[R_A] + \frac{D}{E}(\mathbf{E}[R_A] - R_f). \tag{13}$$

- Note that $\mathbf{E}[R_E] = \mathbf{E}[R_A]$ if $D = 0$.
- This is another conclusion of the MM theorem.

Weighted Average Cost Of Capital (WACC)

Interest payments are deducted from earnings before taxation, though investors receive them as debt-holders. As a result, interest payments are "tax-free" while dividends and retained earnings are not. This phenomenon is called tax shield. The cost of capital that factors the tax shield in, which is called the weighted average cost of capital (WACC), is given by

WACC

$$
\begin{aligned}
WACC &= \frac{D}{D+E}(1-T_x)R_f + \frac{E}{D+E}\mathbb{E}[R_E] \qquad (14)\\
&= \mathbb{E}[R_A] - \frac{D}{D+E}T_x R_f,
\end{aligned}
$$

where T_x is the effective rate of corporate tax.

DCF Approach

The discounted cash flow (DCF) approach is widely used among practitioners for corporate valuation. In the DCF approach, EV is evaluated as

DCF approach

$$EV = \sum_{t=1}^{T} \frac{FCF(t)}{(1+WACC)^t} + \frac{TV(T)}{(1+WACC)^T}, \tag{15}$$

where T is the end of the forecast period (typically 5 years), $FCF(t)$ is the free cash flow[2] at time t and $TV(T)$ is the terminal value at time T.

[2] Roughly speaking, the free cash flow is the fraction of earnings that can be distributable to investors.

Terminal Value i

In practice, the terminal value $TV(t)$ is evaluated in the following ways.

1. **Perpetual growth approach**
   In this approach, the terminal value is defined as the present value of the free cash flow that will grow at the same rate g forever, namely

$$TV(T) = \sum_{t=1}^{\infty} \frac{FCF(T)(1+g)^t}{(1+WACC)^t} = \frac{FCF(T)(1+g)}{WACC-g}.$$

Terminal Value ii

2. **Exit multiple approach**

   A multiple in corporate valuation is defined as

   $$\text{Multiple} = \frac{\text{Market-based EV}}{\text{Performance metric}},$$

   where "market-based EV" is calculated with the market value of equity. A popular performance metric is earnings before interest, taxes, depreciation and amortization (EBITDA, pronounced E-Bit-Da). Then

   $$\begin{aligned} TV(T) = {} & \text{Average multiple among comparable firms} \\ & \times \text{Forecasted performance metric at time } T. \end{aligned}$$

Levered And Unlevered Beta

With the formula of CAPM (7), $\mathbf{E}[R_A]$ is expressed as

$$\mathbf{E}[R_A] = R_f + \beta_A(\mu_{\mathcal{M}} - R_f),$$

where β_A is called the unlevered beta or asset beta. According to the MM theorem, the cost of capital is independent of the leverage and so is β_A. In general, however, most firms are in debt. Thus $\mathbf{E}[R_E]$ is not equal to $\mathbf{E}[R_A]$. As a result, the beta in

$$\mathbf{E}[R_E] = R_f + \beta_E(\mu_{\mathcal{M}} - R_f),$$

does depend on the leverage. This β_E is called the levered beta or equity beta.

Case 1: Constant Leverage

Suppose the firm continuously rebalances the capital structure so that the leverage D/E should be constant. In this scenario, the cost of equity in (13) is also constant. Applying CAPM (7), we have

$$\begin{aligned}\mathbb{E}[R_E] &= R_f + \beta_A(\mu_{\mathcal{M}} - R_f) + \frac{D}{E}\beta_A(\mu_{\mathcal{M}} - R_f)\\ &= R_f + \frac{D+E}{E}\beta_A(\mu_{\mathcal{M}} - R_f).\end{aligned}$$

Thus

$$\beta_A = \frac{E}{D+E}\beta_E. \tag{16}$$

Case 2: General Cost Of Debt

In general, the expected return on debt $\mathbf{E}[R_D]$ is not equal to the risk-free rate R_f. Then we have

$$\mathbf{E}[R_D] = R_f + \beta_D(\mu_{\mathcal{M}} - R_f).$$

Suppose the firm keeps the leverage constant. Then

$$\begin{aligned}
\mathbf{E}[R_E] &= \mathbf{E}[R_A] + \frac{D}{E}(\mathbf{E}[R_A] - \mathbf{E}[R_D]) \\
&= R_f + \beta_A(\mu_{\mathcal{M}} - R_f) + \frac{D}{E}(\beta_A - \beta_D)(\mu_{\mathcal{M}} - R_f) \\
&= R_f + \left(\frac{D+E}{E}\beta_A - \frac{D}{E}\beta_D\right)(\mu_{\mathcal{M}} - R_f). \\
&\Rightarrow \beta_A = \frac{D}{D+E}\beta_D + \frac{E}{D+E}\beta_E. \qquad (17)
\end{aligned}$$

Note: $\beta_D = 0$ if $\mathbf{E}[R_D] = R_f$.

Case 3: Constant Debt

Suppose D is fixed. Modigliani and Miller (1963) showed that the cost of equity in this scenario was expressed as

$$\mathbb{E}[R_E] = \mathbb{E}[R_A] + \frac{D}{E}(1 - T_x)(\mathbb{E}[R_A] - R_f). \tag{18}$$

Apply CAPM (7), we have

$$\begin{aligned}
\mathbb{E}[R_E] &= R_f + \beta_A(\mu_{\mathcal{M}} - R_f) + \frac{D}{E}(1 - T_x)\beta_A(\mu_{\mathcal{M}} - R_f) \\
&= R_f + \left\{1 + \frac{D}{E}(1 - T_x)\right\}\beta_A(\mu_{\mathcal{M}} - R_f). \\
&\Rightarrow \beta_A = \frac{\beta_E}{1 + \frac{D}{E}(1 - T_x)}. \qquad (19)
\end{aligned}$$

In all cases, once we obtain the estimate of β_E, we can transform it into the estimate of β_A with (16), (17) or (19).

Levered Or Unlevered?

- Since data used in estimation of betas mostly come from firms in debt, estimated betas are levered ones.
- This may not necessarily cause serious problems as long as there is no sizable change in the capital structure of the target firm between the sample period of estimation and the forecast period of the DCF model (15).
- However, it is problematic for transactions that will drastically alter the capital structure (e.g., LBO).
- Another example is an IPO. Because the firm is not listed yet, no stock returns data are available. We need to estimate its beta by using stock returns data of comparable listed firms instead.

Death Of CAPM

- Although CAPM is still ubiquitous among practitioners, it is not supported by empirical studies (the “death” of CAPM).
- Key assumptions are too restrictive.
- The market portfolio must include ALL traded assets. In practice, however, a broad market index of the stock market (e.g., S&P 500, TOPIX) is used as the market portfolio.
- In terms of econometrics/statistics, (11) is merely a simple regression model. Thus its explanatory power is fairly limited by construction.
- Researchers have proposed numerous alternatives models. Most of them are categorized as factor models.

Factor Models

1. Arbitrage pricing theory (APT) [Ross (1976)]
2. Fama-French-type factor models
    - 3-factor model [Fama and French (1993)]
        - market $R_{\mathcal{M}} - R_f$
        - firm size (small-minus-big, SMB)
        - book-to-market ratio (high-minus-low, HML)
    - momentum factor [Carhart (1997)]
    - 5-factor model [Fama and French (2015)]
        - 3 factors in Fama and French (1993)
        - profitability (robust-minus-weak, RMW)
        - investment (conservative-minus-aggressive, CMA)

Harvey, Liu and Zhou (2016) tested 316 factors as of 2012.

So far we assume we know $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ in (3). In reality, however, we need to estimate them with data. Let r_{nt} denote realized return on asset n at period t $(t = 1, \ldots, T)$. The sample mean $\bar{r}_n$ and the sample covariance s_{nm} (or the sample variance s_n^2 when $n = m$) are defined as

$$\bar{r}_n = \frac{1}{T}\sum_{t=1}^{T} r_{it}, \quad s_{nm} = \frac{1}{T}\sum_{t=1}^{T}(r_{nt} - \bar{r}_n)(r_{mt} - \bar{r}_m),$$
$$(n, m = 1, \ldots, N).$$

When $\boldsymbol{\mu}$ And $\boldsymbol{\Sigma}$ Are Unknown ii

A straightforward way is to replace $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in (3) with

$$\bar{r} = \begin{bmatrix} \bar{r}_1 \\ \vdots \\ \bar{r}_N \end{bmatrix}, \quad S = \begin{bmatrix} s_1^2 & \cdots & s_{1N} \\ \vdots & \ddots & \vdots \\ s_{N1} & \cdots & s_N^2 \end{bmatrix},$$

respectively, but a more elegant method is known in the literature.

Alternative Expression Of The Portfolio Variance

$$
\begin{aligned}
w^{\mathsf{T}} S w &= \sum_{n=1}^{N} \sum_{m=1}^{N} w_n w_m s_{nm} \\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} w_n w_m \left\{ \frac{1}{T} \sum_{t=1}^{T} (r_{nt} - \bar{r}_n)(r_{mt} - \bar{r}_m) \right\} \\
&= \frac{1}{T} \sum_{t=1}^{T} \sum_{n=1}^{N} w_n (r_{nt} - \bar{r}_n) \sum_{m=1}^{N} w_n (r_{nt} - \bar{r}_n) \\
&= \frac{1}{T} \sum_{t=1}^{T} \left\{ \sum_{n=1}^{N} w_n (r_{nt} - \bar{r}_n) \right\}^2 = \frac{1}{T} \sum_{t=1}^{T} (r_{Pt} - \bar{r}_P)^2 , \\
r_{Pt} &= \sum_{n=1}^{N} w_n r_{nt}, \quad \bar{r}_P = \sum_{n=1}^{N} w_n \bar{r}_n = \frac{1}{T} \sum_{t=1}^{T} r_{Pt},
\end{aligned}
$$

Remarks

- r_{Pt} is a *realized return* of the portfolio with allocation weights w at period t
- $\bar{r}_P$ is the average of realized returns of the portfolio with allocation weights w from period 1 to period T.
- $\frac{1}{T}\sum_{t=1}^{T}(r_{Pt} - \bar{r}_P)^2$ is the sample variance of realized returns of the portfolio with allocation weights w from period 1 to period T.
- Using $w^\intercal S w$ implies that we use the realized variance of a portfolio in a hypothetical situation; What if we invest our money on a portfolio with allocation weights w from period 1 to period T?

Alternative Form Of The Minimization Problem

Define $v_t = r_{Pt} - \bar{r}_P$. Then the minimum variance problem with unknown $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ is given by

Minimum variance portfolio problem

$$\begin{aligned} \min_{w,v} \quad & \widehat{\mathbf{Var}}[R_P] = \frac{1}{T} v^\top v, \\ \text{s.t.} \quad & Dw = v, \quad w^\top \bar{r} = \mu_P, \quad w^\top \boldsymbol{\iota} = 1, \\ & w_1 \geqq 0, \ \cdots, \ w_N \geqq 0, \end{aligned} \tag{20}$$

where

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_T \end{bmatrix}, \quad D = \begin{bmatrix} r_{11} - \bar{r}_1 & \cdots & r_{N1} - \bar{r}_N \\ \vdots & \ddots & \vdots \\ r_{1T} - \bar{r}_1 & \cdots & r_{NT} - \bar{r}_N \end{bmatrix}.$$

Alternative Risk Criterion

Other than the variance $\mathbf{E}[(R_P - \mu_P)^2]$, many alternative risk criteria have been proposed in the literature.

In this lecture, we study the following three alternatives:

- Mean absolute deviation: $\mathbf{E}[|R_P - \mu_P|]$
- Semivariance: $\mathbf{E}[(R_P - \mu_P)^2 | R_{Pt} \leqq \mu_P]$
- Expected shortfall: $\mathbf{E}[-R_P | R_P \leqq \mathrm{VaR}_\alpha]$ ($\mathbf{Pr}\{R_P \leqq \mathrm{VaR}_\alpha\} = \alpha$)

Mean Absolute Deviation Optimization

The sample mean absolute deviation is

$$\varrho^{AD}(w) = \frac{1}{T}\sum_{t=1}^{T} |r_{Pt} - \bar{r}_P|. \tag{21}$$

Define $v_t = r_{Pt} - \bar{r}_P$ and use the same notations as in (20).

Minimum mean absolute deviation portfolio problem

$$\begin{aligned} \min_{w,v} \quad & \varrho^{AD}(w) = \frac{1}{T}\sum_{t=1}^{T} |v_t|, \\ \text{subject to} \quad & Dw = v, \\ & w^\top \bar{r} = \mu_P, \quad w^\top \iota = 1, \\ & w_1 \geqq 0, \cdots, w_N \geqq 0, \end{aligned} \tag{22}$$

Semivariance Optimization

The sample semivariance is

$$\varrho^{SV}(w) = \frac{1}{T}\sum_{t=1}^{T}\{[r_{Pt} - \bar{r}_P]^-\}^2, \tag{23}$$

where $[r_{Pt} - \bar{r}_P]^- = \max\{-(r_{Pt} - \bar{r}_P), 0\}$ is call the negative part. Defining $v_t = [r_{Pt} - \bar{r}_P]^-$, we have

Minimum semivariance portfolio problem

$$\begin{aligned} \min_{w,v} \quad & \varrho^{SV}(w) = \frac{1}{T}\sum_{t=1}^{T} v_t^2, \\ \text{subject to} \quad & w^\top \bar{r} = \mu_P, \quad w^\top \iota = 1, \\ & w_1 \geqq 0, \cdots, w_N \geqq 0, \; v_1 \geqq 0, \cdots, v_T \geqq 0, \\ & r_{P1} - \bar{r}_P + v_1 \geqq 0, \cdots, r_{PT} - \bar{r}_P + v_T \geqq 0, \end{aligned} \tag{24}$$

Value-At-Risk And Expected Shortfall

The value-at-risk (VaR) of a portfolio is defined as

VaR

$$\mathbf{Pr}\{R_P \leqq \mathrm{VaR}_{\alpha}\} = \alpha. \tag{25}$$

The expected shortfall (ES) is defined as

Expected shortfall

$$\mathrm{ES}_{\alpha} = \mathbf{E}[-R_P | R_P \leqq \mathrm{VaR}_{\alpha}] \tag{26}$$

ES is interpreted as the conditional expected loss under a severe market condition in which our portfolio suffers from a rare but huge loss ($R_P \leqq \mathrm{VaR}_{\alpha}$).

Coherent Risk Measure

Suppose $\mathcal{X}$ is a set of random variables. We regard each $X \in \mathcal{X}$ as the return (or value) of a portfolio and let $\varrho(X)$ denote a risk measure of X. $\varrho(\cdot)$ is said to be coherent if it satisfies the following conditions [Artzner et al. (1999)]:

Monotonicity: For any $X, Y \in \mathcal{X}$,

$$\mathbf{Pr}\{X \leqq Y\} = 1 \text{ implies } \varrho(X) \geqq \varrho(Y).$$

Cash invariance: For $M \in \mathcal{X}$ such that $\mathbf{Pr}\{M = R\} = 1$,

$$\varrho(X + M) = \varrho(X) - R.$$

Sub-additivity: For any $X, Y \in \mathcal{X}$, $\varrho(X + Y) \leqq \varrho(X) + \varrho(Y)$.

Positive homogeneity: For any $\lambda \geqq 0$, $\varrho(\lambda X) = \lambda \varrho(X)$.

Interpretations Of Conditions

Monotonicity: A surely profitable portfolio should be less risky.

Cash invariance: Adding the riskless asset should reduce the risk.

Sub-additivity: Diversification should not make the portfolio risker.

Positive homogeneity: The risk should be proportional to the position.

Remark: ES is coherent but VaR is not.

Expected Shortfall Optimization

If the probability density function of R_P, say $p(R_P)$, is known, the ES is given by

$$\mathrm{ES}_\alpha = \frac{1}{\alpha}\int_{-\infty}^{\infty}[R_P - \mathrm{VaR}_\alpha]^- p(R_P)dR_P - \mathrm{VaR}_\alpha. \tag{27}$$

The portfolio selection problem with the ES is given by

Minimum expected shortfall portfolio problem

$$\begin{aligned} \min_{w} \quad & \frac{1}{\alpha}\int_{-\infty}^{\infty}[R_P - \mathrm{VaR}_\alpha]^- p(R_P)dR_P - \mathrm{VaR}_\alpha, \\ \text{subject to} \quad & w^\intercal \bar{r} = \mu_P, \quad w^\intercal \iota = 1, \quad w_1 \geqq 0, \dots, w_N \geqq 0. \end{aligned} \tag{28}$$

Unfortunately, (28) is a non-linear optimization problem which is difficult to solve unless we introduce additional assumptions such as the normality of asset returns.

Approximation Method

To circumvent this obstacle, Rockafellar and Uryasev (2000) proposed an approximation method. If T is large enough, the ES (27) is approximately equivalent to

$$\varrho^{ES}(w, c) = \frac{1}{\alpha T} \sum_{t=1}^{T} [r_{Pt} - c]^{-} - c. \tag{29}$$

Thus, by defining $v_t = [r_{Pt} - c]^{-}$, we have

Minimum expected shortfall portfolio problem

$$\begin{aligned} \min_{w,v,c} \quad & \varrho^{ES}(w, c) = \frac{1}{\alpha T} \sum_{t=1}^{T} v_t - c, \\ \text{subject to} \quad & w^{\intercal} \bar{r} = \mu_P, \quad w^{\intercal} \iota = 1, \\ & w_1 \geqq 0, \ \cdots, \ w_N \geqq 0, \ v_1 \geqq 0, \ \cdots, \ v_T \geqq 0, \\ & r_{P1} - c + v_1 \geqq 0, \ \cdots, \ r_{PT} - c + v_T \geqq 0. \end{aligned} \tag{30}$$

Risk Parity Approach

A risk parity approach of portfolio management focuses on balancing "risk allocation" among assets, instead of balancing the target return and the risk of the portfolio.

1. The mean-variance approach tends to produce an extremely skewed portfolio.
2. It is hard to obtain a reliable estimate of the expected return on any asset.
3. Many fund managers feel increasingly uncomfortable with the traditional asset allocation techniques since they were useless during such financial turmoil as the Global Financial Crisis.

Equal-weight (1/N) Portfolio

One trivial example of such portfolios that equalize the impact of each asset onto the total risk is

$1/N$ portfolio

$$w_n^{1/N} = \frac{1}{N}, \quad (n = 1, \ldots, N). \tag{31}$$

This is often refered to as the $1/N$ portfolio.

Global Minimum Variance Portfolio

The global minimum variance portfolio is the solution of the following optimization problem:

$$\begin{aligned} \min_{w} \quad & w^\intercal \boldsymbol{\Sigma} w \\ \text{subject to} \quad & w^\intercal \boldsymbol{\iota} = 1, \end{aligned} \tag{32}$$

that is,

Global minimum variance portfolio

$$w^{MV} = \frac{1}{\boldsymbol{\iota}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}. \tag{33}$$

Note that (33) does not depend on the expected return vector $\boldsymbol{\mu}$. Thus we do not need to estimate $\boldsymbol{\mu}$.

Properties i

1. Suppose the variance of the return on the n-th asset is σ_n^2 $(n = 1, \ldots, n)$. If there is no correlation among the asset returns, the allocation weight in the global minimum variance portfolio is given by

$$w_n^{MV} = \frac{\sigma_n^{-2}}{\sum_{n=1}^{N} \sigma_n^{-2}}, \quad (n = 1, \ldots, N).$$

2. Suppose $\sigma_1^2 = \cdots = \sigma_N^2$ and the correlation coefficient between any pair of assets is constant. Then the global minimum variance portfolio is equivalent to the $1/N$ portfolio.

Properties ii

3. Consider a Lagrangian for (32):

$$\mathcal{L} = \frac{1}{2} w^\intercal \boldsymbol{\Sigma} w + \lambda(1 - w^\intercal \boldsymbol{\iota}),$$

where λ is the Lagrange multiplier. The first-order condition is

$$\nabla_w \mathcal{L} = \boldsymbol{\Sigma} w - \lambda \boldsymbol{\iota} = 0 \quad \Rightarrow \quad \boldsymbol{\Sigma} w = \lambda \boldsymbol{\iota}, \tag{34}$$

Note that the first derivative of $\sigma(w) = \sqrt{w^\intercal \boldsymbol{\Sigma} w}$ is given by

$$\nabla \sigma(w) = \begin{bmatrix} \nabla_{w_1} \sigma(w) \\ \vdots \\ \nabla_{w_N} \sigma(w) \end{bmatrix} = \begin{bmatrix} \nabla_1 \sigma(w) \\ \vdots \\ \nabla_N \sigma(w) \end{bmatrix} = \frac{1}{\sigma(w)} \boldsymbol{\Sigma} w.$$

Properties iii

Therefore the first-order condition (34) implies

$$\nabla_1 \sigma(w) = \cdots = \nabla_N \sigma(w). \tag{35}$$

$\nabla_n \sigma(w)$ $(n = 1, \ldots, N)$ is called the MRC (Marginal Risk Contribution).

Risk Decomposition

The standard deviation $\sigma(w)$ is decomposed as

$$\begin{aligned}\sigma(w) &= \frac{1}{\sigma(w)} w^\top \Sigma w = w^\top \left(\frac{1}{\sigma(w)} \Sigma w \right) \\ &= w^\top \nabla \sigma(w) \\ &= w_1 \nabla_1 \sigma(w) + \cdots + w_N \nabla_N \sigma(w), \end{aligned} \tag{36}$$

$w_n \nabla_n \sigma(w)$ $(n = 1, \ldots, N)$ is interpreted as the contribution of the n-th asset on the total risk of the portfolio and is called the TRC (Total Risk Contribution).

A portfolio that satisfies

$$w_1 \nabla_1 \boldsymbol{\sigma}(w) = \cdots = w_N \nabla_N \boldsymbol{\sigma}(w), \tag{37}$$

is called the risk parity portfolio. Dividing both sides of (36) with $\boldsymbol{\sigma}(w)$, we have

$$1 = \frac{w_1 \nabla_1 \boldsymbol{\sigma}(w)}{\boldsymbol{\sigma}(w)} + \cdots + \frac{w_N \nabla_N \boldsymbol{\sigma}(w)}{\boldsymbol{\sigma}(w)}.$$

Under (37),

$$\frac{w_n \nabla_n \boldsymbol{\sigma}(w)}{\boldsymbol{\sigma}(w)} = \frac{1}{N}, \quad (n = 1, \ldots, N).$$

Risk Parity Portfolio ii

w^{RP} is the solution of the system of non-linear equations:

Risk parity portfolio

$$\boldsymbol{\Sigma} w^{RP} = \frac{\boldsymbol{\kappa}}{w^{RP}} = \begin{bmatrix} \frac{\boldsymbol{\kappa}}{w_1^{RP}} \\ \vdots \\ \frac{\boldsymbol{\kappa}}{w_N^{RP}} \end{bmatrix}, \quad \boldsymbol{\kappa} = \frac{\boldsymbol{\sigma}^2(w^{RP})}{N}, \tag{38}$$

$$\boldsymbol{\iota}^\top w^{RP} = 1,$$

(38) has no closed-form solution, but can be solved numerically.

Properties

1. If all correlation coefficients are equal,
$$w_n^{RP} = \frac{\boldsymbol{\sigma}_n^{-1}}{\sum_{n=1}^{N} \boldsymbol{\sigma}_n^{-1}}, \quad (n = 1, \dots, N),$$
   that is,
$$w^{RP} = \frac{\boldsymbol{\sigma}^{-1}}{\boldsymbol{\iota}^\top \boldsymbol{\sigma}^{-1}}, \quad \boldsymbol{\sigma}^{-1} = \begin{bmatrix} \boldsymbol{\sigma}_1^{-1} \\ \vdots \\ \boldsymbol{\sigma}_N^{-1} \end{bmatrix}.$$
2. Suppose $\boldsymbol{\sigma}_1^2 = \cdots = \boldsymbol{\sigma}_n^2$ and the correlation coefficient between any pair of assets is constant. Then the risk parity portfolio is equivalent to the $1/N$ portfolio.

Maximum Diversification Portfolio

The maximum diversification portfolio is the solution of

$$\max_{w} \quad \frac{\boldsymbol{\sigma}^\top w}{\sqrt{w^\top \boldsymbol{\Sigma} w}}. \tag{39}$$

The optimal w in (39) is given by

Maximum diversification portfolio

$$w^{MD} = \frac{1}{\boldsymbol{\iota}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\sigma}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\sigma}. \tag{40}$$

(40) equalizes the risk allocation as

$$\frac{\nabla_1 \sigma(w^{MD})}{\sigma_1} = \cdots = \frac{\nabla_N \sigma(w^{MD})}{\sigma_N}. \tag{41}$$

Properties

1. If all correlation coefficients are equal,

$$w_n^{MD} = \frac{\sigma_n^{-1}}{\sum_{n=1}^{N} \sigma_n^{-1}}, \quad (n = 1, \ldots, N).$$

2. Suppose $\sigma_1^2 = \cdots = \sigma_n^2$ and the correlation coefficient between any pair of assets is constant. Then the maximum diversification portfolio is equivalent to the $1/N$ portfolio.

Passive Management Vs. Active Management

So far we have reviewed how to manage our portfolio in terms of the balance between the expected return and the risk (the variance or the expected shortfall). This style of portfolio management is called active management. Active management also involves discretionary selection of assets.

Passive management of a portfolio, on the other hand, is a investment strategy in which an investor tries to mimic a benchmark index. Passive management funds that mimic indices are called index funds.

The goal in passive management is to minimize a discrepancy between a portfolio and the benchmark index.

Tracking Error

Let y_t denote the return on the benchmark index at time t, r_{nt} denote the return on asset n $(n = 1, \ldots, N)$ at time t $(t = 1, \ldots, T)$, and w_n denote the allocation weight for asset n. Then a discrepancy between a portfolio and the benchmark index at time t is given by

$$\begin{aligned} e_t &= y_t - \sum_{n=1}^{N} w_n r_{nt} = y_t - \begin{bmatrix} r_{1t} & \cdots & r_{Nt} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \\ &= y_t - r_t^\top w. \end{aligned} \tag{42}$$

This discrepancy is called a tracking error.

Tracking Error Minimization

A tracking error minimization problem is formulated as

Tracking error minimization

$$\begin{aligned} \min_{w} \quad & \frac{1}{T}\sum_{t=1}^{T}(y_t - r_t^{\top} w)^2, \\ \text{subject to} \quad & \sum_{n=1}^{N} w_n = 1, \quad w_1 \geqq 0, \ldots, w_N \geqq 0. \end{aligned} \tag{43}$$

Tracking Error Minimization (Matrix Form)

By defining

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix}, \quad R = \begin{bmatrix} r_1^\top \\ \vdots \\ r_T^\top \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ \vdots \\ e_T \end{bmatrix}.$$

Then the tracking error minimization problem is defined as

Tracking error minimization (matrix form)

$$\begin{aligned} \min_{w} \quad & \frac{1}{T} e^\top e, \\ \text{subject to} \quad & e = y - Rw, \quad w^\top \iota = 1, \quad w \geqq 0. \end{aligned} \tag{44}$$