Problem 2. Product states versus entangled states.
(a) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$
(b) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$
(c) $\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle$

Solution. For this problem we will use the fact that the tensor products of two general single qubit states is $\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$.

(a) With the above formula we have the following four equations.

$$\alpha_0\beta_0 = \frac{1}{2} \tag{2.1}$$

$$\alpha_0\beta_1 = \frac{1}{2} \tag{2.2}$$

$$\alpha_1\beta_0 = \frac{1}{2} \tag{2.3}$$

$$\alpha_1\beta_1 = -\frac{1}{2} \tag{2.4}$$

Now if we divide eq. (2.1) by eq. (2.2) we get $\beta_0 = \beta_1$ which we will call β. Dividing eq. (2.3) by eq. (2.4) yields $\frac{\alpha_1\beta}{\alpha_1\beta} = \frac{1/2}{-1/2} \implies 1 = -1$. With this we can conclude this state is not a possible product state and hence an **entangled state**.

(b) The following tensor prodduct gives rise to the desired state which tells us the state is a **product state**, not an entangled state.

$$\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

(c) Given the $|11\rangle$ term is nonexistent, we know $\alpha_1\beta_1 = 0$. This implies $\alpha_1 = 0$ or $\beta_1 = 0$. If this was the case then either the $\alpha_1\beta_0|10\rangle$ term would be 0 or the $\alpha_0\beta_1|01\rangle$ term would be zero. This is not the case so we conclude this is an **entangled state**.