Advanced Quantum Theory Homework 1

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Exercise 2.1. Prove $\{c, f\} = 0$.

Solution. The antisymmetry of the Poisson bracket allows us to equivalently prove $\{f, c\} = 0$.

$$\begin{aligned}
\{f, c\} &= \{f, \sqrt{c}\}\sqrt{c} + \sqrt{c}\{f, \sqrt{c}\} \\
&= 2\sqrt{c}\{f, \sqrt{c}\} \\
&= -2\sqrt{c}\{\sqrt{c}, f\} \\
&= -2\{c, f\} \\
&= 2\{f, c\}
\end{aligned}$$

Since we now have $\{f, c\} = 2\{f, c\}$ we can subtract $\{f, c\}$ from both sides to obtain $\{f, c\} = 0$.

Exercise 2.2. Show that $\{f, f\} = 0$ for any f.

Solution. Again using the antisymmetry axiom we can see $\{f, f\} = -\{f, f\}$ and hence adding $\{f, f\}$ to both sides we obtain $\{f, f\} = 0$.

Exercise 2.3. Assume that n is a positive integer.
(a) Evaluate $\left\{x_1, p_1^n\right\}$
(b) Evaluate $\left\{x_1^n, p_1\right\}$

Solution. (a) I will use a proof by induction to prove $\left\{x_1, p_1^n\right\} = (2p)^{n-1}$. Starting with the base case of $n = 1$. Then we have $\{x_1, p_1\} = (2p)^0 = 1$ which agrees with the definition of the Poisson bracket for position and momentum. Assuming the formula is true for n we will show it's true for $n+1$.

$$\begin{aligned}\left\{x_1, p_1^{n+1}\right\} &= \{x_1, p_1^n\}p_1 + p_1\{x_1, p_1^n\} \\ &= 2p_1\{x_1, p_1^n\} \\ &= 2p_1(2p_1)^{n-1} \\ &= (2p_1)^n = (2p_1)^{(n+1)-1}\end{aligned}$$

(b) I know this is no longer part of the homework, but I did it before the changes were made, so I figured I would leave it in.

Here we will use the fact that $\{fg, h\} = \{f, h\}g + f\{g, h\}$ which can be derived from the product rule as follows.

$$\begin{aligned}\{fg, h\} &= -\{h, fg\} \\ &= -\{h, f\}g - f\{h, g\} \\ &= \{f, h\}g + f\{g, h\}\end{aligned}$$

Again I will use proof by induction to prove $\{x_2^n, p_2\} = nx_2^{n-1}$. Starting with the base case of $n = 1$. Then we have $\{x_2, p_2\} = 1 \cdot x_2^0 = 1$ which agrees with the definition. Assuming the formula is true for n we will show it's true for $n+1$.

$$\begin{aligned}\left\{x_2^{n+1}, p_2\right\} &= \{x_2^n, p_2\}x_2 + x_2^n\{x_2, p_2\} \\ &= \left(nx_2^{n-1}\right)x_2 + x_2^n \\ &= nx_2^n + x_2^n \\ &= (n+1)x_2^n = (n+1)x_2^{(n+1)-1}\end{aligned}$$

Exercise 2.4. Verify $\{-2p_1, 3x_1^2 + 7p_3^4 - 2x_2^2p_1^3 + 6\} = 12x_1$.

Solution.

$$\begin{aligned}
\text{LHS} &= -2\Big\{p_1, 3x_1^2 + 7p_3^4 - 2x_2^2p_1^3 + 6\Big\} \\
&= -2\left(3\Big\{p_1, x_1^2\Big\} + 7\cancelto{0}{\Big\{p_1, p_3^4\Big\}} - 2\Big\{p_1, x_2^2p_1^3\Big\} + \cancelto{0}{\{p_1, 6\}}\right) \\
&= -2\left(-6x_1 - 2\left(\cancelto{0}{\Big\{p_1, x_2^2\Big\}}p_1^3 + x_2^2\cancelto{0}{\Big\{p_1, p_1^3\Big\}}\right)\right) \\
&= 12x_1
\end{aligned}$$

Exercise 2.5. Show that the Poisson bracket is not associative by giving a counterexample.

Solution. If the Poisson bracket was associative it would mean the following: $\{f, \{g, h\}\} = \{\{f, g\}, h\}$ for f, g, h polynomials in x_i, p_i. Take $f = p_1^2$, $g = x_1$, and $h = p_1$. We can then evaluate both sides to see this does not hold.

$$\begin{aligned} \text{LHS} &= \left\{p_1^2, \{x_1, p_1\}\right\} = \left\{p_1^2, 1\right\} = 0 \\ \text{RHS} &= \left\{\left\{p_1^2, x_1\right\}, p_1\right\} = \{-2p_1, p_1\} = 1 \end{aligned}$$

Hence the Poisson bracket is not associative.

Exercise 2.6. Look up and state the axioms of
(a) a Lie algebra
(b) an associative algebra
(c) a Poisson algebra

Solution. (a) A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with bilinear map (called a Lie Bracket) $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is anticommutative ($[x,y] = -[y,x]$) and satisfies the Jacobi Identity.

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \forall x,y,z \in \mathfrak{g}$$

(b) If R is a commutative ring, then an *associative algebra* is an R-module V together with a bilinear map $V \times V \to V$ that is associative and has an identity.

(c) A *Poisson algebra* is a vector space equipped with two bilinear products $- \cdot -$ and $\{\cdot,\cdot\}$ such that

- $- \cdot -$ forms an associative algebra,
- $\{\cdot,\cdot\}$ is antisymmetric, obeys the Jacobi identity and forms a Lie algebra
- The Poisson bracket $\{\cdot,\cdot\}$ acts as $\{x, y \cdot z\} = \{x,y\} \cdot z + y \cdot \{x,z\}$ for all x,y,z in the Poisson algebra.

Exercise 2.7. Prove that both methods of calculating $\dot{f}$ yield the same result.

Solution.

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \dot{g}h + g\dot{h} \\
&= \{g, H\}h + g\{h, H\} \\
&= -(\{H, g\}h + g\{H, h\}) \\
&= -\{H, gh\} \\
&= \{gh, H\} = \{f, H\}
\end{aligned}$$

Exercise 2.8. Use the Jacobi identity to prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\{f,g\} = \{\dot{f},g\} + \{f,\dot{g}\}$$

Solution.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\{f,g\} &= \{\{f,g\},H\} \\
&= -\{H,\{f,g\}\} \\
&= \{f,\{g,H\}\} + \{g,\{H,f\}\} && \text{(Jacobi identity)} \\
&= \{f,\dot{g}\} + \{g,-\dot{f}\} \\
&= \{\dot{f},g\} + \{f,\dot{g}\}
\end{aligned}$$

Exercise 2.9. Show that if H is a polynomial in the positions and momenta with arbitrary (and possibly time-dependent) coefficients, it is true that $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}$.

Solution.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H(x, p, t) &= \frac{\partial H}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial H}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial H}{\partial t} \\
&= \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t} \\
&= \frac{\partial H}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial x} + \frac{\partial H}{\partial t} \\
&= \frac{\partial H}{\partial t}
\end{aligned}$$

Exercise 2.10. Show that the total momentum is conserved.

Solution. To show $p_i^{(1)} + p_i^{(2)}$ is conserved for all i we will show it's time derivative is 0 by taking it's Poisson bracket with the Hamiltonian.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i^{(1)} + p_i^{(2)}\right) &= \left\{p_i^{(1)} + p_i^{(2)}, H\right\} \\
&= \left\{p_i^{(1)} + p_i^{(2)}, \frac{k}{2}\sum_j \left(x_j^{(1)} - x_j^{(2)}\right)^2\right\} && \text{no contribution from kinetic} \\
&= \frac{k}{2}\left\{p_i^{(1)} + p_i^{(2)}, \left(x_i^{(1)} - x_i^{(2)}\right)^2\right\} \\
&= \frac{k}{2}\left[\left\{p_i^{(1)}, {x_i^{(1)}}^2\right\} - 2\left\{p_i^{(1)}, x_i^{(1)}x_i^{(2)}\right\} + \left\{p_i^{(2)}, {x_i^{(2)}}^2\right\} - 2\left\{p_i^{(2)}, x_i^{(1)}x_i^{(2)}\right\}\right] \\
&= \frac{k}{2}\left[-2x_i^{(1)} + 2x_i^{(2)} - 2x_i^{(2)} + 2x_i^{(1)}\right] \\
&= 0
\end{aligned}
$$

With this, and the fact that has no explicit time dependence, we can conclude that momentum is conserved.

Exercise 2.11. Use the derivative definition of the Poisson bracket to evaluate $\{x^8p^6, x^3p^4\}$.

Solution.

$$\begin{aligned}
\left\{x^8p^6, x^3p^4\right\} &= \frac{\partial}{\partial x}\left(x^8p^6\right)\frac{\partial}{\partial p}\left(x^3p^4\right) - \frac{\partial}{\partial p}\left(x^8p^6\right)\frac{\partial}{\partial x}\left(x^3p^4\right) \\
&= \left(8x^7p^6\right)\left(4x^3p^3\right) - \left(6x^8p^5\right)\left(3x^2p^4\right) \\
&= 32x^{10}p^9 - 18x^{10}p^9 \\
&= 14x^{10}p^9
\end{aligned}$$

Exercise 2.12. Show that the derivative definition of the Poisson bracket is indeed a representation of the Poisson bracket defined by the axioms.

Solution. First, antisymmetry.

$$
\begin{aligned}
\{f,g\} &= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\frac{\partial g}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial g}{\partial x_i^{(r)}}\right) \\
&= -\sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial g}{\partial x_i^{(r)}}\frac{\partial f}{\partial p_i^{(r)}} - \frac{\partial g}{\partial p_i^{(r)}}\frac{\partial f}{\partial x_i^{(r)}}\right) \\
&= -\{g,f\}
\end{aligned}
$$

Second, linearity.

$$
\begin{aligned}
\{cf,g\} &= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial (cf)}{\partial x_i^{(r)}}\frac{\partial g}{\partial p_i^{(r)}} - \frac{\partial (cf)}{\partial p_i^{(r)}}\frac{\partial g}{\partial x_i^{(r)}}\right) \\
&= c\sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\frac{\partial g}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial g}{\partial x_i^{(r)}}\right) \\
&= c\{f,g\}
\end{aligned}
$$

Third, the addition rule.

$$
\begin{aligned}
\{f,g+h\} &= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\frac{\partial (g+h)}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial (g+h)}{\partial x_i^{(r)}}\right) \\
&= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\frac{\partial g}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial g}{\partial x_i^{(r)}} + \frac{\partial f}{\partial x_i^{(r)}}\frac{\partial h}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial g}{\partial x_i^{(r)}}\right) \\
&= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\frac{\partial g}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial g}{\partial x_i^{(r)}}\right) + \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\frac{\partial h}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial g}{\partial x_i^{(r)}}\right) \\
&= \{f,g\} + \{f,h\}
\end{aligned}
$$

Fourth, the product rule.

$$
\begin{aligned}
\{f, gh\} &= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\frac{\partial (gh)}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial (gh)}{\partial x_i^{(r)}}\right) \\
&= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\left[\frac{\partial g}{\partial p_i^{(r)}}h + g\frac{\partial h}{\partial p_i^{(r)}}\right] - \frac{\partial f}{\partial p_i^{(r)}}\left[\frac{\partial g}{\partial x_i^{(r)}}h + g\frac{\partial h}{\partial x_i^{(r)}}\right]\right) \\
&= h\sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\frac{\partial g}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial g}{\partial x_i^{(r)}}\right) \\
&\qquad + g\sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i^{(r)}}\frac{\partial h}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial h}{\partial x_i^{(r)}}\right) \\
&= \{f, g\}h + g\{f, h\}
\end{aligned}
$$

Fifth, six, and seventh: the rules for positions and momenta.

$$
\begin{aligned}
\left\{x_i^{(r)}, p_j^{(s)}\right\} &= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\cancelto{1}{\frac{\partial x_i^{(r)}}{\partial x_i^{(r)}}}\frac{\partial p_j^{(s)}}{\partial p_i^{(r)}} - \frac{\partial x_i^{(r)}}{\partial x_i^{(r)}}\cancelto{0}{\frac{\partial p_j^{(s)}}{\partial p_i^{(r)}}}\frac{\partial x_i^{(r)}}{\partial x_i^{(r)}}\right) \\
&= \sum_{r=1}^{n}\sum_{i=1}^{3}\delta_{i,j}\delta_{s,r} \\
&= \delta_{i,j}\delta_{s,r}
\end{aligned}
$$

$$
\begin{aligned}
\left\{x_i^{(r)}, x_j^{(s)}\right\} &= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial x_i^{(r)}}{\partial x_i^{(r)}}\cancelto{0}{\frac{\partial x_j^{(s)}}{\partial p_i^{(r)}}} - \cancelto{0}{\frac{\partial x_i^{(r)}}{\partial p_i^{(r)}}}\frac{\partial x_j^{(s)}}{\partial x_i^{(r)}}\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\left\{p_i^{(r)}, p_j^{(s)}\right\} &= \sum_{r=1}^{n}\sum_{i=1}^{3}\left(\cancelto{0}{\frac{\partial p_i^{(r)}}{\partial x_i^{(r)}}}\frac{\partial p_j^{(s)}}{\partial p_i^{(r)}} - \frac{\partial p_i^{(r)}}{\partial p_i^{(r)}}\cancelto{0}{\frac{\partial p_j^{(s)}}{\partial x_i^{(r)}}}\right) \\
&= 0
\end{aligned}
$$

Exercise 2.13. Find the representation of the Hamilton equations

Solution. We start with eq. 2.21 and eq. 2.22 and then use those results for eq. 2.19.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x_i^{(r)} &= \left\{x_i^{(r)}, H\right\} \\
&= \sum_{s,j} \frac{\partial x_i^{(r)}}{\partial x_j^{(s)}}\frac{\partial H}{\partial p_j^{(s)}} - \cancelto{0}{\frac{\partial x_i^{(r)}}{\partial p_j^{(s)}}\frac{\partial H}{\partial x_j^{(s)}}} \\
&= \sum_{s,j} \delta_{i,j}\delta_{r,s}\frac{\partial H}{\partial p_j^{(s)}} \\
\dot{x}_i^{(r)} &= \frac{\partial H}{\partial p_i^{(r)}}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}p_i^{(r)} &= \left\{p_i^{(r)}, H\right\} \\
&= \sum_{s,j} \cancelto{0}{\frac{\partial p_i^{(r)}}{\partial x_j^{(s)}}\frac{\partial H}{\partial p_j^{(s)}}} - \frac{\partial p_i^{(r)}}{\partial p_j^{(s)}}\frac{\partial H}{\partial x_j^{(s)}} \\
&= -\sum_{s,j} \delta_{i,j}\delta_{r,s}\frac{\partial H}{\partial x_j^{(s)}} \\
\dot{p}_i^{(r)} &= -\frac{\partial H}{\partial x_i^{(r)}}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \{f, H\} + \frac{\partial f}{\partial t} \\
&= \sum_{r,i} \frac{\partial f}{\partial x_i^{(r)}}\frac{\partial H}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}}\frac{\partial H}{\partial x_i^{(r)}} + \frac{\partial f}{\partial t} \\
&= \sum_{r,i} \frac{\partial f}{\partial x_i^{(r)}}\dot{x}_i^{(r)} + \frac{\partial f}{\partial p_i^{(r)}}\dot{p}_i^{(r)} + \frac{\partial f}{\partial t}
\end{aligned}$$

Which is exactly the chain rule for a function $f(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}, \mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(n)}, t)$.