Numerical Analysis Assignment 1

Name: Nate Stemen (20906566)
Email: `nate.stemen@uwaterloo.ca`

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Problem 1. Eigenvalues and eigenvectors of the 1D Laplacian.

(a) Show that the n eigenvectors are given by the vectors $\mathbf{x}^{(p)}$ with components

$$x_j^{(p)} = \sin(jp\pi h)$$

and with eigenvalues

$$\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1).$$

(b) Verify the functions $u^{(p)}(x) = \sin(p\pi x)$ with $p \in \mathbb{N}$ are eigenfunctions of the continuous differential operator $\mathrm{d}^2/\mathrm{d}x^2$ on domain $[0, 1]$ with boundary conditions $u(0) = 0 = u(1)$.

(c) Compare the eigenvectors and the eigenvalues for the discrete and continuous operators and comment. Are the discrete and continuous eigenvalues similar for small values of $h \cdot p$?

Solution. (a) We start by verifying the the eigenvectors and eigenvalues given are correct.

$$\begin{aligned}
A\mathbf{x}^{(p)} &= \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} \sin(p\pi h) \\ \sin(2p\pi h) \\ \vdots \\ \sin((n-1)p\pi h) \\ \sin(np\pi h) \end{bmatrix} \\
&= \frac{1}{h^2}\begin{bmatrix} -2\sin(p\pi h) + \sin(2p\pi h) \\ \sin(p\pi h) - 2\sin(2p\pi h) + \sin(3p\pi h) \\ \vdots \\ \sin((n-1)p\pi h) - 2\sin(np\pi h) \end{bmatrix}
\end{aligned}$$

We can compute a an elemente $(A\mathbf{x}^{(p)})_j$ as follows. We use $\varphi = p\pi h$ to make the trig identity easier to see.

$$\begin{aligned}
(A\mathbf{x}^{(p)})_j &= \frac{1}{h^2}(\sin((j-1)\varphi) - 2\sin(j\varphi) + \sin((j+1)\varphi)) \\
&= \frac{1}{h^2}(-2\sin(j\varphi) + \sin(j\varphi + \varphi) + \sin(j\varphi - \varphi)) \\
&= \frac{1}{h^2}(-2\sin(j\varphi) + 2\sin(j\varphi)\cos(\varphi)) && \text{By product to sum identity} \\
&= \frac{2}{h^2}(\cos(p\pi h) - 1)\sin(jp\pi h) \\
&= \lambda_p \sin(jp\pi h) = \lambda_p(\mathbf{x}^{(p)})_j
\end{aligned}$$

It's worth noting that the first and last elements of $\mathbf{x}^{(p)}$ are slightly different because they don't get 3 terms, but the above calculation still works. For the first element $(A\mathbf{x}^{(p)})_1$ the first sin term disappears because $\sin 0 = 0$, and for $(A\mathbf{x}^{(p)})_n$ the last sin term vanishes because $(n+1)h = 1$ and $\sin(n\pi) = 0$.

(b) First it's simple to verify the boundary conditions because $\sin 0 = 0$ and $\sin(p\pi) = 0$ for $p \in \mathbb{N}$. Now to show it's an eigenvector of the second derivative operator.

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u^{(p)}(x) = p\pi\frac{\mathrm{d}}{\mathrm{d}x}\cos(p\pi x) = \overbrace{-p^2\pi^2}^{\lambda_p}\sin(p\pi x) = \lambda_p u^{(p)}(x)$$

So the eigenvalues here are $\lambda_p = -p^2\pi^2$.

(c) At first glance the eigenvectors look very similar for these two problems, but the eigenvalues look quite different. However if we make n very large (make the numerical grid much finer) then we can use the Taylor series for cos get get the follow approximation.

$$\begin{aligned}
\frac{2}{h^2}(\cos(p\pi h) - 1) &\approx \frac{2}{h^2}\left(1 - \frac{p^2\pi^2h^2}{2} + \mathcal{O}\left(h^4\right) - 1\right) \\
&= -p^2\pi^2 + \mathcal{O}\left(h^2\right)
\end{aligned}$$

So in the limit $n \to \infty$ we do recover the continuous eigenvalues which is a sign we are doing something right.

Problem 2. Find the LU decomposition of

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$$

and briefly explain the steps.

Solution.

$$\begin{aligned}
\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} &= \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}^{L} \overbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}^{U} \\
&= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{23}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}
\end{aligned}$$

With this we can immediately see $u_{11} = 1, u_{12} = 4, u_{13} = 7, l_{21} = 2$ and $l_{31} = 3$. We can then plug these numbers into the other 4 equations to work out the rest of the components. With that we obtain the following lower and upper matrices.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 6 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -1 & -6 \\ 0 & 0 & 25 \end{bmatrix}$$

Problem 3. Computational work for recursive determinant computation.

Solution. Using the following recursive definition of the determinant

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

we can calculate the work needed to compute the determinant of an $n \times n$ matrix as W_n.

$$W_n = \sum_{i=1}^{n} (1\text{M} + W_{n-1}) = n(1 + W_{n-1})$$

In order to solve this recursive recurrence relation it is helpful to expand it out a few times.

$$\begin{aligned} W_n &= n(1 + W_{n-1}) \\ &= n(1 + (n-1)(1 + (n-2)(1 + W_{n-3}))) \\ &= n + n(n-1) + n(n-1)(n-2) + n(n-1)(n-2)W_{n-3} \\ &= \frac{n!}{(n-1)!} + \frac{n!}{(n-2)!} + \frac{n!}{(n-3)!} W_{n-3} \end{aligned}$$

Writing the expression in the last form allows us to more easily see a pattern arising. We are summing progressively less "cut off" forms of the factorial which can be expressed as follows.

$$W_n = n! \sum_{k=1}^{n-1} \frac{1}{k!}$$

In the limit of large n this approaches $W_n = en!$. Nice.

Problem 4. Vector norm inequalities.
Show that $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty$ for $\mathbf{x} \in \mathbb{R}^n$.

Solution. First, let $|x_j| := \max_i |x_i| = \|\mathbf{x}\|_\infty$.

$$\begin{aligned}
\|\mathbf{x}\|_\infty &= \max_{1\leq i\leq n} |x_i| \\
&\leq |x_j| + \sum_{\substack{i=1\\i\neq j}}^{n} |x_i| && \text{(bc second term is positive)} \\
&= \sum_{i=1}^{n} |x_i| = \|\mathbf{x}\|_1 \\
&\leq \sum_{i=1}^{n} n|x_j| && \text{(bc } |x_j| \geq |x_i| \text{ for all } i\text{)} \\
&= n\sum_{i=1}^{n} |x_j| = n\|\mathbf{x}\|_\infty && \square
\end{aligned}$$

Problem 5. Matrix norm formula.
Let $A \in \mathbb{R}^{n\times n}$. Show that

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|.$$

Solution. We begin by showing the 1-norm of a matrix must be less or equal to the maximum absolute column sum. Once that is established we will find a vector that brings the matrix norm up to that bound, which shows the maximum can be attained and hence the equality true.

$$\begin{aligned}
\|A\mathbf{x}\|_1 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \\
&\le \sum_i \sum_j |a_{ij} x_j| \\
&\le \sum_j |x_j| \sum_i |a_{ij}| \\
&\le \left[\max_k \sum_i |a_{ik}| \right] \underbrace{\sum_j |x_j|}_{\|\mathbf{x}\|_1}
\end{aligned}$$

If we use the the following definition of the matrix norm $\|A\|_1 = \max_{\|\mathbf{x}_1\|=1} \|A\mathbf{x}\|_1$, then the last term in the above inequality vanishes (goes to 1) and hence we have established the 1-norm of this matrix is always less than or equal to the maximum absolute column sum.

Now let ν be the index where the maximum absolute column sum lives ($\max_j \sum_i |a_{ij}| = \sum_i |a_{i\nu}|$). Choose $\mathbf{x} = \mathbf{e}_\nu$ where $\mathbf{e}_\nu$ is the unit normal vector with 1 in the νth position, and 0 everywhere else. Now we can evaluate the norm of A times this vector.

$$\begin{aligned}
\|A\mathbf{x}\|_1 = \|A\mathbf{e}_\nu\|_1 &= \sum_i \left| \sum_j a_{ij} e_j \right| \\
&= \sum_i |a_{i\nu}| \\
&= \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|
\end{aligned}$$

Clearly $\|\mathbf{e}_\nu\|_1 = 1$, so we've found a vector on the unit sphere that attains the maximum which shows the equality of the given statement.

Problem 6. Inverse update formula.

Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Show that if $A + \mathbf{u}\mathbf{v}^\intercal$ is nonsingular, then it's inverse can be expressed by the formula

$$\left(A + \mathbf{u}\mathbf{v}^\intercal\right)^{-1} = A^{-1} - \frac{1}{1 + \mathbf{v}^\intercal A^{-1}\mathbf{u}} A^{-1}\mathbf{u}\mathbf{v}^\intercal A^{-1}$$

Solution. We start by showing $1 + \mathbf{v}^\intercal A^{-1}\mathbf{u} \neq 0$ by contradiction. So assume $1 + \mathbf{v}^\intercal A^{-1}\mathbf{u} = 0$.

$$\begin{aligned}
1 + \mathbf{v}^\intercal A^{-1}\mathbf{u} &= 0 \\
\mathbf{u} + \mathbf{u}\mathbf{v}^\intercal A^{-1}\mathbf{u} &= \mathbf{0} \\
\left(\mathbb{1} + \mathbf{u}\mathbf{v}^\intercal A^{-1}\right)\mathbf{u} &= \mathbf{0} \\
\mathbb{1} + \mathbf{u}\mathbf{v}^\intercal A^{-1} &= 0^{n\times n} \\
A + \mathbf{u}\mathbf{v}^\intercal &= 0^{n\times n}
\end{aligned}$$

Where we've arrived at a contradiction on the last equation, because we took $A + \mathbf{u}\mathbf{v}^\intercal$ to be nonsingular (and hence not be the 0 matrix).

With this proved we can now show the formula is indeed an inverse. For notational convenience we use $\alpha = \frac{1}{1+\mathbf{v}^\intercal A^{-1}\mathbf{u}}$.

$$\begin{aligned}
&\left(A + \mathbf{u}\mathbf{v}^\intercal\right)\left(A^{-1} - \frac{1}{1 + \mathbf{v}^\intercal A^{-1}\mathbf{u}} A^{-1}\mathbf{u}\mathbf{v}^\intercal A^{-1}\right) \\
&\quad = \mathbb{1} - \alpha\mathbf{u}\mathbf{v}^\intercal A^{-1} + \mathbf{u}\mathbf{v}^\intercal A^{-1} - \alpha\mathbf{u}\mathbf{v}^\intercal A^{-1}\mathbf{u}\mathbf{v}^\intercal A^{-1} \\
&\quad = \mathbb{1} + \mathbf{u}\left(-\alpha + 1 - \alpha\mathbf{v}^\intercal A^{-1}\mathbf{u}\right)\mathbf{v}^\intercal A^{-1} \\
&\quad = \mathbb{1} + \mathbf{u}\left(1 + -\alpha\left[1 + \mathbf{v}^\intercal A^{-1}\mathbf{u}\right]\right)\mathbf{v}^\intercal A^{-1} \\
&\quad = \mathbb{1} + \mathbf{u}(1-1)\mathbf{v}^\intercal A^{-1} = \mathbb{1}
\end{aligned}$$

If a square matrix has a left (or right) inverse, then it also has a right (left) inverse and they are equal.[1] We can now conclude that the formula given is indeed an inverse for $A + \mathbf{u}\mathbf{v}^\intercal$.

[1] If $AB = \mathbb{1}$, then $1 = \det AB = \det A \det B$ so we know B is nonsingular. $BAB = B \implies (BA - \mathbb{1})B = 0 \implies BA = \mathbb{1}$