Quantum Information Processing Assignment 1

Name: Nate Stemen (20906566) **Due:** Thur, Sep 17, 2020 11:59 PM
Email: nate.stemen@uwaterloo.ca **Course:** QIC 710

Problem 1. Distinguishing between pairs of qubit states.
(a) $|0\rangle$ and $|+\rangle$
(b) $|0\rangle$ and $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$
(c) $\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ and $\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$

Solution. (a) To best ditinguish between these two states we will apply R_θ with $\theta = \frac{\pi}{8}$. Under this transformation the two states get transformed as follows.

$$R_\theta|0\rangle = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \qquad R_\theta|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\theta - \sin\theta \\ \cos\theta + \sin\theta\end{bmatrix}$$

Under this transformation, we have the following probability table.

	Probability of measuring state $\|0\rangle$	Probability of measuring state $\|1\rangle$
$R_\theta\|0\rangle$	$\cos^2\theta \approx 0.85$	$\sin^2\theta \approx 0.15$
$R_\theta\|+\rangle$	$\frac{1}{2}(\cos\theta - \sin\theta)^2 \approx 0.15$	$\frac{1}{2}(\cos\theta + \sin\theta)^2 \approx 0.85$

This leaves us with an overall success probability of 85%.

(b) Here we will make a clockwise rotation of 15° which we will do with R_θ with $\theta = -\frac{\pi}{12}$. Under this rotation the states are transformed as follows (using $|\psi\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ for notational convenience).

$$R_\theta|0\rangle = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \qquad R_\theta|\psi\rangle = \frac{-1}{2}\begin{bmatrix}\cos\theta + \sqrt{3}\sin\theta \\ \sin\theta - \sqrt{3}\cos\theta\end{bmatrix}$$

With these transformed states we can now evaluate the success probabilities given each state as above.

	Probability of measuring state $\|0\rangle$	Probability of measuring state $\|1\rangle$
$R_\theta\|0\rangle$	$\cos^2\theta \approx 0.93$	$\sin^2\theta \approx 0.07$
$R_\theta\|\psi\rangle$	$\left\lvert\frac{-1}{4}\left(\cos\theta + \sqrt{3}\sin\theta\right)^2\right\rvert \approx 0.07$	$\left\lvert\frac{-1}{4}\left(\sin\theta - \sqrt{3}\cos\theta\right)^2\right\rvert \approx 0.93$

So here our overall success probability would be 93%. Pretty solid I do say.

(c) These points are anti-podal on the Bloch sphere, and hence orthogonal, which means they are perfectly distinguishable. To find the unitary transformation U that allows us to measure the state in the computational basis, we need it to satisfy the following two equations.

$$U\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{\mathrm{i}}{\sqrt{2}}|1\rangle\right) = |0\rangle$$
$$U\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{\mathrm{i}}{\sqrt{2}}|1\rangle\right) = |1\rangle$$

These two equations are equivalent to the following two matrix equations.

$$\frac{1}{\sqrt{2}}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\frac{1}{\sqrt{2}}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

With the matrix written this way we have 4 equations with 4 unknowns. Solving for U gives us

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{bmatrix}.$$

This transformation maps our two states to the computational basis which allows us to measure and distinguish perfectly between the two states and hence have 100% success probability.

Problem 2. Product states versus entangled states.

(a) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

(b) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

(c) $\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle$

Solution. For this problem we will use the fact that the tensor products of two general single qubit states is $\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$.

(a) With the above formula we have the following four equations.

$$\alpha_0\beta_0 = \frac{1}{2} \tag{2.1}$$

$$\alpha_0\beta_1 = \frac{1}{2} \tag{2.2}$$

$$\alpha_1\beta_0 = \frac{1}{2} \tag{2.3}$$

$$\alpha_1\beta_1 = -\frac{1}{2} \tag{2.4}$$

Now if we divide eq. (2.1) by eq. (2.2) we get $\beta_0 = \beta_1$ which we will call β. Dividing eq. (2.3) by eq. (2.4) yields $\frac{\alpha_1\beta}{\alpha_1\beta} = \frac{1/2}{-1/2} \implies 1 = -1$. With this we can conclude this state is not a possible product state and hence an **entangled state**.

(b) The following tensor prodduct gives rise to the desired state which tells us the state is a **product state**, not an entangled state.

$$\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

(c) Given the $|11\rangle$ term is nonexistent, we know $\alpha_1\beta_1 = 0$. This implies $\alpha_1 = 0$ or $\beta_1 = 0$. If this was the case then either the $\alpha_1\beta_0|10\rangle$ term would be 0 or the $\alpha_0\beta_1|01\rangle$ term would be zero. This is not the case so we conclude this is an **entangled state**.