Quantum Information Processing Assignment 1

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Problem 1. Distinguishing between pairs of qubit states.
(a) $|0\rangle$ and $|+\rangle$
(b) $|0\rangle$ and $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$
(c) $\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ and $\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$

Solution. (a) To best ditinguish between these two states we will apply R_θ with $\theta = \frac{\pi}{8}$. Under this transformation the two states get transformed as follows.

$$R_\theta|0\rangle = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \qquad R_\theta|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\theta - \sin\theta \\ \cos\theta + \sin\theta\end{bmatrix}$$

Under this transformation, we have the following probability table.

	Probability of measuring state $\|0\rangle$	$\|1\rangle$
$R_\theta\|0\rangle$	$\cos^2\theta \approx 0.85$	$\sin^2\theta \approx 0.15$
$R_\theta\|+\rangle$	$\frac{1}{2}(\cos\theta - \sin\theta)^2 \approx 0.15$	$\frac{1}{2}(\cos\theta + \sin\theta)^2 \approx 0.85$

This leaves us with an overall success probability of 85%.

(b) Here we will make a clockwise rotation of 15° which we will do with R_θ with $\theta = -\frac{\pi}{12}$. Under this rotation the states are transformed as follows (using $|\psi\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ for notational convenience).

$$R_\theta|0\rangle = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \qquad R_\theta|\psi\rangle = \frac{-1}{2}\begin{bmatrix}\cos\theta + \sqrt{3}\sin\theta \\ \sin\theta - \sqrt{3}\cos\theta\end{bmatrix}$$

With these transformed states we can now evaluate the success probabilities given each state as above.

	Probability of measuring state $\|0\rangle$	$\|1\rangle$
$R_\theta\|0\rangle$	$\cos^2\theta \approx 0.93$	$\sin^2\theta \approx 0.07$
$R_\theta\|\psi\rangle$	$\left\lvert\frac{-1}{4}\left(\cos\theta + \sqrt{3}\sin\theta\right)^2\right\rvert \approx 0.07$	$\left\lvert\frac{-1}{4}\left(\sin\theta - \sqrt{3}\cos\theta\right)^2\right\rvert \approx 0.93$

So here our overall success probability would be 93%. Pretty solid I do say.

(c) These points are anti-podal on the Bloch sphere, and hence orthogonal, which means they are perfectly distinguishable. To find the unitary transformation U that allows us to measure the state in the computational basis, we need it to satisfy the following two equations.

$$U\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{\mathrm{i}}{\sqrt{2}}|1\rangle\right) = |0\rangle$$
$$U\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{\mathrm{i}}{\sqrt{2}}|1\rangle\right) = |1\rangle$$

These two equations are equivalent to the following two matrix equations.

$$\frac{1}{\sqrt{2}}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\frac{1}{\sqrt{2}}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

With the matrix written this way we have 4 equations with 4 unknowns. Solving for U gives us

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{bmatrix}.$$

This transformation maps our two states to the computational basis which allows us to measure and distinguish perfectly between the two states and hence have 100% success probability.