Quantum Information Processing Assignment 2

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Course: QIC 710

Problem 1. Simple operations on quantum states.

(a) Apply R_θ to the qubit in state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

(b) Apply R_θ to the *first* qubit of state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$

(c) Apply R_θ to *both* qubits of state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$

(d) Apply $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix}$ to *both* qubits of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Solution. (a)

$$\begin{aligned} R_\theta\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta - \sin\theta \\ \sin\theta + \cos\theta \end{pmatrix} \end{aligned}$$

This is the $|+\rangle$ state rotated about the origin by an angle of θ.

(b)

$$\begin{aligned} R_\theta \otimes \mathbb{1}\left(\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\right) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta \\ \sin\theta \\ \sin\theta \\ -\cos\theta \end{pmatrix} \end{aligned}$$

(c) First we need to calculate $R_\theta \otimes R_\theta$ in order to apply it to our system.

$$R_\theta \otimes R_\theta = \begin{pmatrix} \cos^2\theta & -\cos\theta\sin\theta & -\cos\theta\sin\theta & \sin^2\theta \\ \cos\theta\sin\theta & \cos^2\theta & -\sin^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ \sin^2\theta & \cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta \end{pmatrix}$$

With that we can now apply it to our state.

$$R_\theta \otimes R_\theta \left(\frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |11\rangle \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos^2\theta - \sin^2\theta \\ 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta \\ \sin^2\theta - \cos^2\theta \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \\ \sin 2\theta \\ -\cos 2\theta \end{pmatrix}$$

Which is interesting because it's the same state as the previous part, but with double the angle.

(d) Again we must first calculate the tensor product of the gate with itself. We call the gate A.

$$A \otimes A = \frac{1}{2} \begin{pmatrix} A & \mathrm{i}A \\ \mathrm{i}A & A \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i} & \mathrm{i} & -1 \\ \mathrm{i} & 1 & -1 & \mathrm{i} \\ \mathrm{i} & -1 & 1 & \mathrm{i} \\ -1 & \mathrm{i} & \mathrm{i} & 1 \end{pmatrix}$$

And now we can apply this to our state as follows.

$$A \otimes A \left(\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \right) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 \\ 2\mathrm{i} \\ 2\mathrm{i} \\ 0 \end{pmatrix}$$
$$= \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$
$$= \frac{\mathrm{i}}{\sqrt{2}} (|01\rangle + |10\rangle)$$

Problem 2. Simulating a controlled-rotation with CNOT and one-qubit gates.

Solution. To begin, as per the hint, we will take $U = R_\alpha$ and $V = R_\beta$. We can then construct the total LHS gate by multiplying the sequence of gates together making sure to reverse the order of multiplication.

$$\mathrm{CNOT} \cdot (\mathbb{1} \otimes V) \cdot \mathrm{CNOT} \cdot (\mathbb{1} \otimes U) = \mathrm{CNOT} \cdot \left(\mathbb{1} \otimes R_\beta\right) \cdot \mathrm{CNOT} \cdot (\mathbb{1} \otimes R_\alpha)$$

We can now expand that and set it equal to a controlled rotation. In the following calculation, we will treat each "element" as a 2×2 block matrix.

$$\begin{aligned}
\begin{bmatrix} \mathbb{1} & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} R_\beta & 0 \\ 0 & R_\beta \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} R_\alpha & 0 \\ 0 & R_\alpha \end{bmatrix} &= \begin{bmatrix} R_\beta & 0 \\ 0 & R_\beta X \end{bmatrix} \begin{bmatrix} R_\alpha & 0 \\ 0 & R_\alpha X \end{bmatrix} \\
&= \begin{bmatrix} R_\beta R_\alpha & 0 \\ 0 & X R_\beta X R_\alpha \end{bmatrix}
\end{aligned}$$

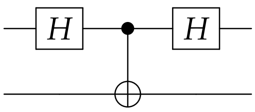
Now this combined gate is supposed to be equal to a controlled rotation, which has form $\left[\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & R_\theta \end{smallmatrix}\right]$, so we can conclude $R_\beta R_\alpha = \mathbb{1}$ and $X R_\beta X R_\alpha = R_\theta$. The first condition tells use that $\alpha + \beta = 2n\pi$ for some $n \in \mathbb{Z}$ and the second condition requires us to expand the matrices.

$$\begin{aligned}
R_\theta = X R_\beta X R_\alpha &= \begin{bmatrix} \sin\beta & \cos\beta \\ \cos\beta & -\sin\beta \end{bmatrix} \begin{bmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{bmatrix} \\
&= \begin{bmatrix} \sin\alpha\sin\beta + \cos\alpha\cos\beta & -\sin\alpha\cos\beta + \cos\alpha\sin\beta \\ -\cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\alpha - \beta) & -\sin(\alpha - \beta) \\ \sin(\alpha - \beta) & \cos(\alpha - \beta) \end{bmatrix} = R_{\alpha - \beta}
\end{aligned}$$

This implies that $\alpha - \beta = \theta$. This, together the the fact that we can choose $n = 0$ in the above criteria means that $\alpha = \beta = \theta/2$.

Problem 3. Circuit for constructing a state.

Solution. The following quantum circuit turns $|00\rangle$ into the desired state.



In matrix form we have the following.

$$\begin{aligned}
(H \otimes \mathbb{1}) \cdot \text{CNOT} \cdot (H \otimes \mathbb{1}) &= \frac{1}{2} \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \mathbb{1} + X & \mathbb{1} - X \\ \mathbb{1} - X & \mathbb{1} + X \end{bmatrix} \\
D &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

Sure enough $D|00\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$ as desired.