Equal temperament

An **equal temperament** is a musical temperament, or a system of tuning, in which every pair of adjacent notes has an identical frequency ratio. As pitch is perceived roughly as the logarithm of frequency, this means that the perceived "distance" from every note to its nearest neighbor is the same for every note in the system.

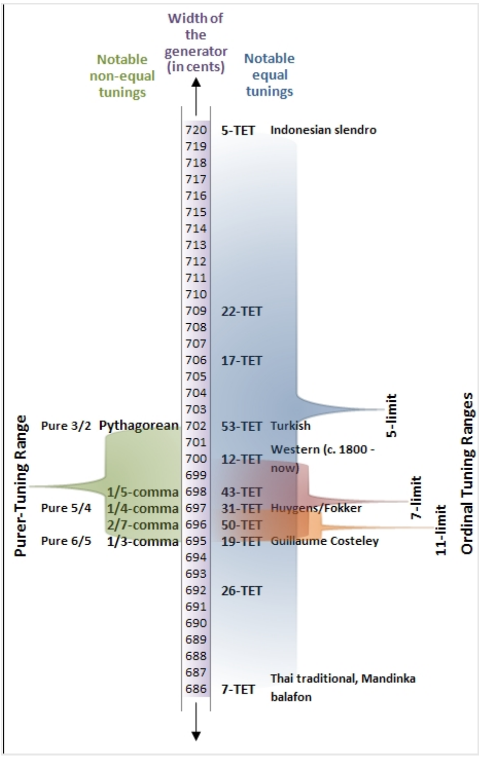


Figure 1: The syntonic tuning continuum, which includes many notable "equal temperament" tunings (Milne 2007).[1]

In equal temperament tunings, an interval – usually the octave – is divided into a series of equal steps (equal frequency ratios between successive notes). For classical music, the most common tuning system is **twelve-tone equal temperament** (also known as **12 equal temperament**), inconsistently abbreviated as **12-TET**, **12TET**, **12tET**, **12tet**, **12-ET**, **12ET**, or **12et**, which divides the octave into 12 parts, all of which are equal on a logarithmic scale. It is usually tuned relative to a standard pitch of 440 Hz, called A440.

Other equal temperaments exist (some music has been written in 19-TET and 31-TET for example, and 24-TET is used in Arabic music), but in Western countries when people use the term *equal temperament* without qualification, they usually mean 12-TET.

Equal temperaments may also divide some interval other than the octave, a pseudo-octave, into a whole number of equal steps. An example is an equal-tempered Bohlen–Pierce scale. To avoid ambiguity, the term **equal division of the octave**, or **EDO** is sometimes preferred. According to this naming system, *12-TET* is called *12-EDO*, *31-TET* is called *31-EDO*, and so on.

String ensembles and vocal groups, who have no mechanical tuning limitations, often use a tuning much closer to just intonation, as it is naturally more consonant. Other instruments, such as some wind, keyboard, and fretted instruments, often only approximate equal temperament, where technical limitations prevent exact tunings. Some wind instruments that can easily and spontaneously bend their tone, most notably double-reeds, use tuning similar to string ensembles and vocal groups.

The tuning continuum of the syntonic temperament, shown in Figure 1, includes a number of notable "equal temperament" tunings, including those that divide the octave equally into 5, 7, 12, 17, 19, 22, 26, 31, 43, 50, and 53 parts. On an isomorphic keyboard, the fingering of music written in any of these syntonic tunings is precisely the same as it is in any other syntonic tuning, so long as the notes are spelled properly—that is, with no assumption of enharmonicity. This consistency of fingering makes it possible to smoothly vary the tuning (and hence the pitches of all notes, systematically) all along the syntonic tuning continuum—a polyphonic tuning bend. The use of dynamic timbres lets consonance be maintained (or otherwise manipulated) across such tuning bends.

§History

The two figures frequently credited with the achievement of exact calculation of equal temperament are Zhu Zaiyu (also romanized as Chu-Tsaiyu. Chinese: 朱載堉) in 1584 and Simon Stevin in 1585. According to Fritz A. Kuttner, a critic of the theory,[2] it is known that "Chu-Tsaiyu presented a highly precise, simple and ingenious method for arithmetic calculation of equal temperament mono-chords in 1584" and that "Simon Stevin offered a mathematical definition of equal temperament plus a somewhat less precise computation of the corresponding numerical values in 1585 or later." The developments occurred independently.[3]

Kenneth Robinson attributes the invention of equal temperament to Zhu Zaiyu[4] and provides textual quotations as evidence.[5] Zhu Zaiyu is quoted as saying that, in a text dating from 1584, "I have founded a new system. I establish one foot as the number from which the others are to be extracted, and using proportions I extract them. Altogether one has to find the exact figures for the pitch-pipers in twelve operations."[5] Kuttner disagrees and remarks that his claim "cannot be considered correct without major qualifications."[2] Kuttner proposes that neither Zhu Zaiyu or Simon Stevin achieved equal temperament, and that neither of the two should be treated as inventors.[6]

§China

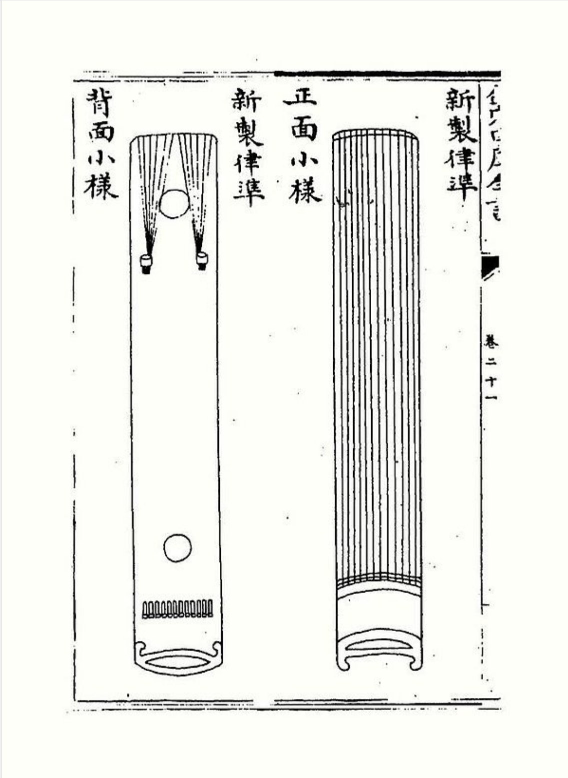
§Early history

The origin of the Chinese pentatonic scale is traditionally ascribed to the mythical Ling Lun. Allegedly his writings discussed the equal division of the scale in the 27th century BC.[7] However, evidence of the origins of writing in this period (the early Longshan) in China is limited to rudimentary inscriptions on oracle bones and pottery.[8]

A complete set of bronze chime bells, among many musical instruments found in the tomb of the Marquis Yi of Zeng (early Warring States, c. 5th century BCE in the Chinese Bronze Age), covers 5 full 7 note octaves in the key of C Major, including 12 note semi-tones in the middle of the range.[9]

An approximation for equal temperament was described by He Chengtian, a mathematician of Southern and Northern Dynasties around 400 AD.[10]

Historically, there was a seven-equal temperament or hepta-equal temperament practice in Chinese tradition.[11][12]



Prince Zhu Zaiyu constructed 12 string equal temperament tuning instrument, front and back view

Zhu Zaiyu (朱載堉), a prince of the Ming court, spent thirty years on research based on the equal temperament idea originally postulated by his father. He described his new pitch theory in his *Fusion of Music and Calendar 乐律融通* published in 1580. This was followed by the publication of a detailed account of the new theory of the equal temperament with a precise numerical specification for 12-TET in his 5,000-page work *Complete Compendium of Music and Pitch* (Yuelü quan shu 乐律全书) in 1584.[13] An extended account is also given by Joseph Needham.[14] Zhu obtained his result mathematically by dividing the length of string and pipe successively by

$$\sqrt[12]{2} = 1.059463094359295264561825$$

, and for pipe diameter by

$\sqrt[24]{2}$[15]

$$(1.059463094359295264561825)^{12} \approx 1.99999999999988$$

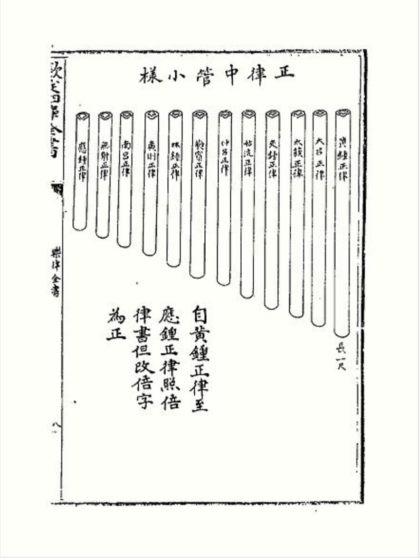
;

$$(1.059463094359295264561825)^{84} \approx 127.999999999946$$

(still in tune after 84/12 = 7 octaves)

§Zhu Zaiyu

According to Gene Cho, Zhu Zaiyu was the first person to solve the equal temperament problem mathematically.[16] Matteo Ricci, a Jesuit in China, was at Chinese trade fair in Canton the year Zhu published his solution, and very likely brought it back to the West.[17] Murray Barbour said, "The first known appearance in print of the correct figures for equal temperament was in China, where Prince Tsaiyü's brilliant solution remains an enigma."[18] The 19th-century German physicist Hermann von Helmholtz wrote in *On the Sensations of Tone* that a Chinese prince (see below) introduced a scale of seven notes, and that the division of the octave into twelve semitones was discovered in China.[19]

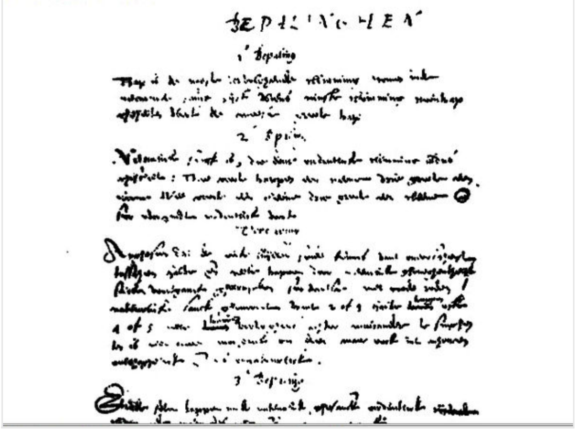


Zhu Zaiyu's equal temperament pitch pipes

Zhu Zaiyu illustrated his equal temperament theory by construction of a set of 36 bamboo tuning pipes ranging in 3 octaves, with instructions of the type of bamboo, color of paint, and detailed specification on their length and inner and outer diameters. He also constructed a 12-string tuning instrument, with a set of tuning pitch pipes hidden inside its bottom cavity. In 1890, Victor-Charles Mahillon, curator of the Conservatoire museum in Brussels, duplicated a set of pitch pipes according to Zhu Zaiyu's specification. He said that the Chinese theory of tones knew more about the diameter of pitch pipes than its Western counterpart, and that the set of pipes duplicated according to the Zaiyu data proved the accuracy of this theory.[20]

§Europe

§Early history



Simon Stevin's *Van de Spiegheling der singconst* c. 1605.

One of the earliest discussions of equal temperament occurs in the writing of Aristoxenus in the 4th century BC.

Vincenzo Galilei (father of Galileo Galilei) was one of the first practical advocates of twelve-tone equal temperament. He composed a set of dance suites on each of the 12 notes of the chromatic scale in all the "transposition keys", and published also, in his 1584 "Fronimo", 24 +1 ricercars.[21] He used the 18:17 ratio for fretting the lute (although some adjustment was necessary for pure octaves).[22]

Galilei's countryman and fellow lutenist Giacomo Gorzanis had written music based on equal temperament by 1567.[23][24] Gorzanis was not the only lutenist to explore all modes or keys: Francesco Spinacino wrote a "Recercare de tutti li Toni" (Ricercar in all the Tones) as early as 1507.[25] In the 17th century lutenist-composer John Wilson wrote a set of 30 preludes including 24 in all the major/minor keys.[26][27]

Henricus Grammateus drew a close approximation to equal temperament in 1518. The first tuning rules in equal temperament were given by Giovani Maria Lanfranco in his "Scintille de musica".[28] Zarlino in his polemic with Galilei initially opposed equal temperament but eventually conceded to it in relation to the lute in his *Sopplimenti musicali* in 1588.

§Simon Stevin

The first mention of equal temperament related to Twelfth root of two in the West appeared in Simon Stevin's manuscript *Van De Spiegheling der singconst* (ca 1605) published posthumously nearly three centuries later in 1884.[29] However, due to insufficient accuracy of his calculation, many of the chord length numbers he obtained were off by one or two units from the correct values.[30] As a result, the frequency ratios of Simon Stevin's chords has no unified ratio, but one ratio per tone, which is claimed by Gene Cho as incorrect.[31]

The following were Simon Stevin's chord length from *Vande Spiegheling der singconst*:[32]

TONE	CHORD 10000 from Simon Stevin	RATIO	CORRECTED CHORD
semitone	9438	1.0595465	9438.7
whole tone	8909	1.0593781	
1.5 tone	8404	1.0600904	8409
ditone	7936	1.0594758	7937
ditone and a half	7491	1.0594046	7491.5
tritone	7071	1.0593975	7071.1
tritone and a half	6674	1.0594845	6674.2
four-tone	6298	1.0597014	6299
four-tone-and-half	5944	1.0595558	5946
five-tone	5611	1.0593477	5612.3
five-tone-and-half	5296	1.0594788	5297.2
full tone		1.0592000	

A generation later, French mathematician Marin Mersenne presented several equal tempered chord lengths obtained by Jean Beaugrand, Ismael Bouillaud and Jean Galle.[33]

In 1630 Johann Faulhaber published a 100 cent monochord table, with the exception of several errors due to his use of logarithmic tables . He did not explain how he obtained his results.[34]

§Baroque era

From 1450 to about 1800, plucked instrument players (lutenists and guitarists) generally favored equal temperament,[35] and the Brossard lute Manuscript compiled in the last quarter of the 17th century contains a series of 18 preludes attributed to Bocquet written in all keys, including the last prelude, entitled *Prelude sur tous les tons*, which enharmonically modulates through all keys.[36] Angelo Michele Bartolotti published a series of passacaglias in all keys, with connecting enharmonically modulating passages. Among the 17th-century keyboard composers Girolamo Frescobaldi advocated equal temperament. Some theorists, such as Giuseppe Tartini, were opposed to the adoption of equal temperament; they felt that degrading the purity of each chord degraded the aesthetic appeal of music, although Andreas Werckmeister emphatically advocated equal temperament in his 1707 treatise published posthumously.[37]

J. S. Bach wrote *The Well-Tempered Clavier* to demonstrate the musical possibilities of well temperament, where in some keys the consonances are even more degraded than in equal temperament. It is reasonable to believe[*weasel words*] that when composers and theoreticians of earlier times wrote of the moods and "colors" of the keys, they each described the subtly different dissonances made available within a particular tuning method. However, it is difficult to determine with any exactness the actual tunings used in different places at different times by any composer. (Correspondingly, there is a great deal of variety in the particular opinions of composers about the moods and colors of particular keys.)[*citation needed*]

Twelve tone equal temperament took hold for a variety of reasons. It conveniently fit the existing keyboard design, and permitted total harmonic freedom at the expense of just a little impurity in every interval. This allowed greater expression through enharmonic modulation, which became extremely important in the 18th century in music of such composers as Francesco Geminiani, Wilhelm Friedemann Bach, Carl Philipp Emmanuel Bach and Johann Gottfried Müthel.[*citation needed*]

The progress of equal temperament from the mid-18th century on is described with detail in quite a few modern scholarly publications: it was already the temperament of choice during the

Classical era (second half of the 18th century),[citation needed] and it became standard during the Early Romantic era (first decade of the 19th century),[citation needed] except for organs that switched to it more gradually, completing only in the second decade of the 19th century. (In England, some cathedral organists and choirmasters held out against it even after that date; Samuel Sebastian Wesley, for instance, opposed it all along. He died in 1876.)[citation needed]

A precise equal temperament is possible using the 17th-century Sabbatini method of splitting the octave first into three tempered major thirds.[38] This was also proposed by several writers during the Classical era. Tuning without beat rates but employing several checks, achieving virtually modern accuracy, was already done in the first decades of the 19th century.[39] Using beat rates, first proposed in 1749, became common after their diffusion by Helmholtz and Ellis in the second half of the 19th century.[40] The ultimate precision was available with 2-decimal tables published by White in 1917.[41]

It is in the environment of equal temperament that the new styles of symmetrical tonality and polytonality, atonal music such as that written with the twelve tone technique or serialism, and jazz (at least its piano component) developed and flourished.

§General properties

In an equal temperament, the distance between each step of the scale is the same interval. Because the perceived identity of an interval depends on its ratio, this scale in even steps is a geometric sequence of multiplications. (An arithmetic sequence of intervals would not sound evenly spaced, and would not permit transposition to different keys.) Specifically, the smallest interval in an equal-tempered scale is the ratio:

$$r^n = p$$
$$r = \sqrt[n]{p}$$

where the ratio r divides the ratio p (typically the octave, which is 2/1) into n equal parts. (*See Twelve-tone equal temperament below.*)

Scales are often measured in cents, which divide the octave into 1200 equal intervals (each called a cent). This logarithmic scale makes comparison of different tuning systems easier than comparing ratios, and has considerable use in Ethnomusicology. The basic step in cents for any equal temperament can be found by taking the width of p above in cents (usually the octave, which is 1200 cents wide), called below w, and dividing it into n parts:

$$c = \frac{w}{n}$$

In musical analysis, material belonging to an equal temperament is often given an integer notation, meaning a single integer is used to represent each pitch. This simplifies and generalizes discussion of pitch material within the temperament in the same way that taking the logarithm of a multiplication reduces it to addition. Furthermore, by applying the modular arithmetic where the modulus is the number of divisions of the octave (usually 12), these integers can be reduced to pitch classes, which removes the distinction (or acknowledges the similarity) between pitches of the same name, e.g. 'C' is 0 regardless of octave register. The MIDI encoding standard uses integer note designations.

§Twelve-tone equal temperament

In twelve-tone equal temperament, which divides the octave into 12 equal parts, the width of a semitone, i.e. the frequency ratio of the interval between two adjacent notes, is the twelfth root of two:

$$\sqrt[12]{2} = 2^{\frac{1}{12}} \approx 1.059463$$

This interval is divided into 100 cents.

§Calculating absolute frequencies

See also: Piano key frequencies

To find the frequency, P_n, of a note in 12-TET, the following definition may be used:

$$P_n = P_a(\sqrt[12]{2})^{(n-a)}$$

In this formula P_n refers to the pitch, or frequency (usually in hertz), you are trying to find. P_a refers to the frequency of a reference pitch (usually 440Hz). n and a refer to numbers assigned to the desired pitch and the reference pitch, respectively. These two numbers are from a list of consecutive integers assigned to consecutive semitones. For example, A4 (the reference pitch) is the 49th key from the left end of a piano (tuned to 440 Hz), and C4 (middle C) is the 40th key. These numbers can be used to find the frequency of C4:

$$P_{40} = 440(\sqrt[12]{2})^{(40-49)} \approx 261.626 \text{ Hz}$$

§Historical comparison

[42]

YEAR	NAME	RATIO	CENTS
400	He Chengtian	1.060070671	101.0
1580	Vincenzo Galilei	18:17 [1.058823529]	99.0
1581	Zhu Zaiyu	1.059463094	100.0
1585	Simon Stevin	1.059546514	100.1
1630	Marin Mersenne	1.059322034	99.8
1630	Johann Faulhaber	1.059490385	100.0

§Comparison to just intonation

The intervals of 12-TET closely approximate some intervals in just intonation. The fifths and fourths are almost indistinguishably close to just.

In the following table the sizes of various just intervals are compared against their equal-tempered counterparts, given as a ratio as well as cents.

§Seven-tone equal division of the fifth

Violins, violas and cellos are tuned in perfect fifths (G – D – A – E, for violins, and C – G – D – A, for violas and cellos), which suggests that their semi-tone ratio is slightly higher than in the conventional twelve-tone equal temperament. Because a perfect fifth is in 3:2 relation with its base tone, and this interval is covered in 7 steps, each tone is in the ratio of

$$\sqrt[7]{3/2}$$

to the next (100.28 cents),[*citation needed*] which provides for a perfect fifth with ratio of 3:2 but a slightly widened octave with ratio of ≈ 517:258 or ≈ 2.00388:1 rather than the usual 2:1 ratio, because twelve perfect fifths do not equal seven octaves.[43] During actual play, however, the violinist chooses pitches by ear, and only the four unstopped pitches of the strings are guaranteed to exhibit this 3:2 ratio.

§Rational semitone

For any semitone that is a proper fraction of a whole tone, exactly one equal division of the octave lets the circle of fifths generate all the notes of the equal division while preserving the order of the notes. (That is, C is lower than D, D is lower than E, etc., and F♯ is indeed sharper than F.) The number of divisions needed for the octave is seven times the number of divisions of a whole tone minus twice the number of divisions of the semitone. The corresponding fifth spans a number of divisions equal to four whole tones minus one semitone. Hence, for a semitone of one-half of a whole tone, the corresponding equal temperament scheme is 12-EDO with a fifth of seven divisions. A semitone of one-third of a whole tone corresponds to 19-EDO with a fifth of eleven divisions.

12-EDO is the equal temperament with the smallest number of divisions that allows for a rational semitone to preserve the desired properties concerning note order and the circle of fifths. It also has the desirable property of making the semitone exactly one-half of a whole tone. These are additional reasons why 12-EDO became the predominant form of equal temperament.

While each rational semitone corresponds to only one equal temperament, the reverse is not the case. For example, both a semitone of one-seventh, and a semitone of eight-ninths both use 47-EDO, which is the smallest number of divisions that has two different semitones. However, they have different values for the fifth, as a semitone of one-seventh uses a fifth of twenty-seven divisions while a semitone of eight ninths uses a fifth of twenty-eight divisions.

§Other equal temperaments

See also: Sonido 13

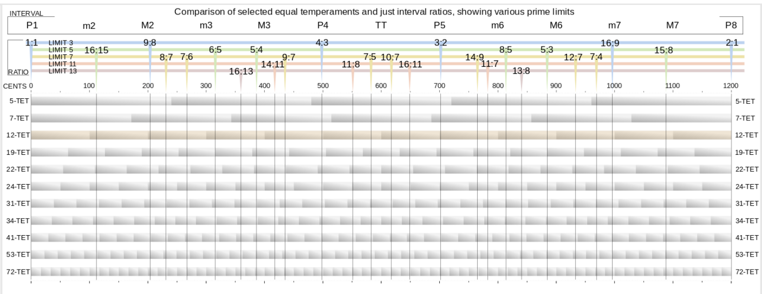
§5 and 7 tone temperaments in ethnomusicology

Five and seven tone equal temperament (**5-TET** 🔊 Play (help·info) and **7-TET**🔊 Play (help·info)), with 240 🔊 Play (help·info) and 171 🔊 Play (help·info) cent steps respectively, are fairly common. A Thai xylophone measured by Morton (1974) "varied only plus or minus 5 cents," from 7-TET. A Ugandan Chopi xylophone measured by Haddon (1952) was also tuned to this system. According to Morton, "Thai instruments of fixed pitch are tuned to an equidistant system of seven pitches per octave ... As in Western traditional music, however, all pitches of the tuning system are not used in one mode (often referred to as 'scale'); in the Thai system five of the seven are used in principal pitches in any mode, thus establishing a pattern of nonequidistant intervals for the mode."[44] 🔊 Play (help·info) Indonesian gamelans are tuned to 5-TET according to Kunst (1949), but according to Hood (1966) and McPhee (1966) their tuning varies widely, and according to Tenzer (2000) they contain stretched octaves. It is now well-accepted that of the two primary tuning systems in gamelan music, slendro and pelog, only slendro somewhat resembles five-tone equal temperament while pelog is highly unequal; however, Surjodiningrat et al. (1972) has analyzed pelog as a seven-note subset of nine-tone equal temperament (133 cent steps 🔊 Play (help·info)). A South American Indian scale from a preinstrumental culture measured by Boiles (1969) featured 175 cent seven tone equal temperament, which stretches the octave slightly as with instrumental gamelan music.

5-TET and 7-TET mark the endpoints of the syntonic temperament's valid tuning range, as shown in Figure 1.

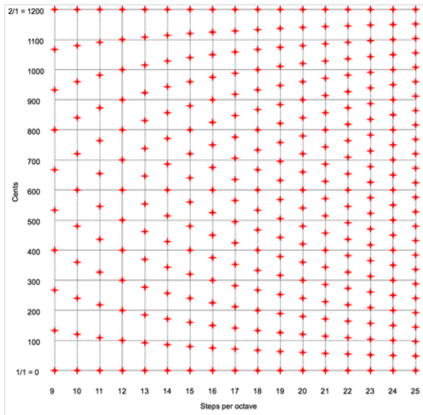
- In 5-TET the tempered perfect fifth is 720 cents wide (at the top of the tuning continuum), and marks the endpoint on the tuning continuum at which the width of the minor second shrinks to a width of 0 cents.
- In 7-TET the tempered perfect fifth is 686 cents wide (at the bottom of the tuning continuum), and marks the endpoint on the tuning continuum, at which the minor second expands to be as wide as the major second (at 171 cents each).

§Various Western equal temperaments



A comparison of some equal temperament scales.[46] The graph spans one octave horizontally, and each shaded rectangle is the width of one step in a scale. The just interval ratios are separated in rows by their prime limits.

31 tone equal temperament was advocated by Christiaan Huygens and Adriaan Fokker. 31-TET has a slightly less accurate fifth than 12-TET, but provides near-just major thirds, and provides decent matches for harmonics up to at least 13, of which the seventh harmonic is particularly accurate.



Comparison of equal temperaments from 9 to 25 (after Sethares (2005), p.58).

In the 20th century, standardized Western pitch and notation practices having been placed on a 12-TET foundation made the quarter tone scale (or 24-TET) a popular microtonal tuning.

29-TET is the lowest number of equal divisions of the octave which produces a better perfect fifth than 12-TET. Its major third is roughly as inaccurate as 12-TET, however it is tuned 14 cents flat rather than 14 cents sharp.

41-TET is the second lowest number of equal divisions that produces a better perfect fifth than

12-TET. Its major third is more accurate than 12-ET and 29-ET, about 6 cents flat.

53-TET is better at approximating the traditional just consonances than 12, 19 or 31-TET, but has had only occasional use. Its extremely good perfect fifths make it interchangeable with an extended Pythagorean tuning, but it also accommodates schismatic temperament, and is sometimes used in Turkish music theory. It does not, however, fit the requirements of meantone temperaments, which put good thirds within easy reach via the cycle of fifths. In 53-TET the very consonant thirds would be reached instead by strange enharmonic relationships. A consequence of this is that chord progressions like I-vi-ii-V-I won't land you back where you started in 53-TET, but rather one 53-tone step flat (unless the motion by I-vi wasn't by the 5-limit minor third).

Another extension of 12-TET is 72-TET (dividing the semitone into 6 equal parts), which though not a meantone tuning, approximates well most just intonation intervals, even less traditional ones such as 7/4, 9/7, 11/5, 11/6 and 11/7. 72-TET has been taught, written and performed in practice by Joe Maneri and his students (whose atonal inclinations interestingly typically avoid any reference to just intonation whatsoever).

Other equal divisions of the octave that have found occasional use include 14-TET, 15-TET, 16-TET, 17-TET, 19-TET, 22-TET, 34-TET, 46-TET, 48-TET, 99-TET, and 171-TET.

2, 5, 12, 41, 53, 306, 665 and 15601 are denominators of first convergents of

$$\log_2(3)$$

, so 2, 5, 12, 41, 53, 306, 665 and 15601 twelfths (and fifths), being in correspondent equal temperaments equal to en integer number of octaves, are better approximation of 2, 5, 12, 41, 53, 306, 665 and 15601 just twelfths/fifths than for any equal temperaments with less tones.[47][48]

1, 2, 3, 5, 7, 12, 29, 41, 53, 200... (sequence A060528 in OEIS) is the sequence of divisions of octave that provide better and better approximations of the perfect fifth. Related sequences contain divisions approximating other just intervals.[49] It is noteworthy that many elements of this sequences are sums of previous elements. This application: http://equal.meteor.com/ calculates the frequencies, amounts of cents and Pitch Bend values for any systems of Equal Division of the Octave. Note, that both 'rounded' and 'floored' notations are equivalent, produces the same MIDI value.

§Equal temperaments of non-octave intervals

The equal-tempered version of the Bohlen–Pierce scale consists of the ratio 3:1, 1902 cents, conventionally a perfect fifth and an octave, called in this theory a tritave (🔊 play (help·info)), and split into a thirteen equal parts. This provides a very close match to justly tuned ratios consisting only of odd numbers. Each step is 146.3 cents (🔊 play (help·info)), or $\sqrt[13]{3}$.

Wendy Carlos created three unusual equal temperaments after a thorough study of the properties of possible temperaments having a step size between 30 and 120 cents. These were called *alpha*, *beta*, and *gamma*. They can be considered as equal divisions of the perfect fifth.[*citation needed*] Each of them provides a very good approximation of several just intervals.[50] Their step sizes:

- *alpha*:

  $$\sqrt[9]{3/2}$$

  (78.0 cents)
- *beta*:

  $$\sqrt[11]{3/2}$$

  (63.8 cents)
- *gamma*:

  $$\sqrt[20]{3/2}$$

  (35.1 cents)

Alpha and Beta may be heard on the title track of her 1986 album *Beauty in the Beast*.

§See also

- Musical acoustics (the physics of music)
- Music and mathematics
- Microtuner
- Microtonal music
- Piano tuning
- List of meantone intervals
- Diatonic and chromatic
- Electronic tuner

§References

§Citations

1. ^ Milne, A., Sethares, W.A. and Plamondon, J.,"Isomorphic Controllers and Dynamic Tuning: Invariant Fingerings Across a Tuning Continuum", *Computer Music Journal*, Winter 2007, Vol. 31, No. 4, Pages 15-32.
2. ^ ***a b*** Fritz A. Kuttner. p. 163.
3. ^ Fritz A. Kuttner. "Prince Chu Tsai-Yü's Life and Work: A Re-Evaluation of His Contribution to Equal Temperament Theory", p.200, *Ethnomusicology*, Vol. 19, No. 2 (May, 1975), pp. 163–206.
4. ^ Kenneth Robinson: *A critical study of Chu Tsai-yü's contribution to the theory of equal temperament in Chinese music*. (Sinologica Coloniensia, Bd. 9.) x, 136 pp. Wiesbaden: Franz Steiner Verlag GmbH, 1980. DM 36. p.vii "Chu-Tsaiyu the first formulator of the mathematics of "equal temperament" anywhere in the world
5. ^ ***a b*** Robinson, Kenneth G., and Joseph Needham. 1962. "Physics and Physical Technology". In Science and Civilisation in China, vol. 4: "Physics and Physical Technology", Part 1: "Physics", edited by Joseph Needham. Cambridge: University Press. p. 221.
6. ^ Fritz A. Kuttner. p. 200.
7. ^ Owen Henry Jorgensen, Tuning (East Lansing: Michigan State University Press, 1991)
8. ^ The Formation of Chinese Civilization: An Archaeological Perspective, Zhang Zhongpei, Shao Wangping and others, Yale (New Haven Conn)and New World Press (Beijing),2005
9. ^ The Formation of Chinese Civilization, 2005 (opt cit), the Eastern Zhou and the Growth of Regionalism, Lu Liancheng, pp 140 ff
10. ^ J. Murray Barbour Tuning and Temperament p55-56, Michigan State University Press 1951
11. ^ 有关"七平均律"新文献著作的发现 [Findings of new literatures concerning the hepta – equal temperament] (in Chinese). Archived from the original on 27 October 2007. "'Hepta-equal temperament' in our folk music has always been a controversial issue."
12. ^ 七平均律"琐谈--兼及旧式均孔曲笛制作与转调 [abstract of *About "Seven- equal- tuning System"*] (in Chinese). "From the flute for two thousand years of the production process, and the Japanese shakuhachi remaining in the production of Sui and Tang Dynasties and the actual temperament, identification of people using the so-called 'Seven Laws' at least two thousand years of history; and decided that this law system associated with the flute law."
13. ^ "Quantifying Ritual: Political Cosmology, Courtly Music, and Precision Mathematics in Seventeenth-Century China Roger Hart Departments of History and Asian Studies, University of Texas, Austin". Uts.cc.utexas.edu. Retrieved 2012-03-20.
14. ^ Science and Civilisation in China, Vol IV:1 (Physics), Joseph Needham, Cambridge

University Press, 1962–2004, pp 220 ff

15. ^ The Shorter Science & Civilisation in China, An abridgement by Colin Ronan of Joseph Needham's original text, p385
16. ^ Gene J. Cho "The Significance of the Discovery of the Musical Equal Temperament In the Cultural History," http://en.cnki.com.cn/Article_en/CJFDTOTAL-XHYY201002002.htm
17. ^ "EQUAL TEMPERAMENT". University of Houston. Retrieved October 5, 2014.
18. ^ J. Murray Barbour, *Tuning and Temperament* Michigan University Press, 1951 p7
19. ^ [1] "The division of Octave into twelve Semitones, and the transposition of scales have also been discovered by this intelligent and skillful nation" Hermann von Helmholtz *On the Sensations of Tone as a Physiological basis for the theory of music*, p 258, 3rd edition, Longmans, Green and Co, London, 1895
20. ^ Lau Hanson, Abacus and Practical Mathematics p389 (in Chinese 劳汉生 《珠算与实用数学》 389页)
21. ^ Galilei, V. (1584). Il Fronimo... Dialogo sopra l'arte del bene intavolare. G. Scotto: Venice, ff. 80–89.
22. ^ J. Murray Barbour *Tuning and Temperament* p8 1951 Michigan State University Press
23. ^ "Resound – Corruption of Music". Philresound.co.uk. Retrieved 2012-03-20.
24. ^ Giacomo Gorzanis, c. 1525 - c. 1575 Intabolatura di liuto. Geneva, 1982
25. ^ "Spinacino 1507a: Thematic Index". Appalachian State University. Archived from the original on 2011-07-25. Retrieved 14 June 2012.
26. ^ John Wilson, 26 Preludes, Diapason press (DP49, Utrecht)
27. ^ in "English Song, 1600–1675, Facsimile of Twenty-six Manuscripts and an Edition of the Texts" vol. 7, Manuscripts at Oxford, Part II. introduction by Elise Bickfoer Jorgens, Garland Publishing Inc., New York and London, 1987
28. ^ "Scintille de musica", (Brescia, 1533), p. 132
29. ^ "Van de Spiegheling der singconst, ed by Rudolf Rasch, The Diapason Press". Diapason.xentonic.org. 2009-06-30. Retrieved 2012-03-20.
30. ^ Thomas S. Christensen, The Cambridge history of western music theory p205, Cambridge University Press
31. ^ Gene Cho, The discovery of musical equal temperament in China and Europe – Page 223, Mellen Press
32. ^ Gene Cho, The discovery of musical equal temperament in China and Europe – Page 222, Mellen Press
33. ^ Thomas S. Christensen, The Cambridge history of western music theory p207, Cambridge University Press
34. ^ Thomas S. Christensen, The Cambridge history of western music theory p78, Cambridge

University Press

35. ^ "Lutes, Viols, Temperaments" Mark Lindley ISBN 978-0-521-28883-5
36. ^ Vm7 6214
37. ^ Andreas Werckmeister: Musicalische Paradoxal-Discourse, 1707
38. ^ Di Veroli, Claudio. Unequal Temperaments: Theory, History and Practice. 2nd edition, Bray Baroque, Bray, Ireland 2009, pp. 140, 142 and 256.
39. ^ Moody, Richard. "Early Equal Temperament, An Aural Perspective: Claude Montal 1836" in Piano Technicians Journal. Kansas City, USA, Feb. 2003.
40. ^ Helmholtz, Hermann L.F. Lehre von den Tonempfindungen... Heidelberg 1862. English edition: On the Sensations of Tone, Longmans, London, 1885, p.548.
41. ^ White, William Braid. Piano Tuning and Allied Arts. 1917, 5th enlarged edition, Tuners Supply Co., Boston 1946, p.68.
42. ^ Date,name,ratio,cents: from equal temperament monochord tables p55-p78; J. Murray Barbour *Tuning and Temperament*, Michigan State University Press 1951
43. ^ Cordier, Serge. "Le tempérament égal à quintes justes" (in French). Association pour la Recherche et le Développement de la Musique. Retrieved 2 June 2010.
44. ^ Morton, David (1980). "The Music of Thailand", *Musics of Many Cultures*, p.70. May, Elizabeth, ed. ISBN 0-520-04778-8.
45. ^ Myles Leigh Skinner (2007). *Toward a Quarter-tone Syntax: Analyses of Selected Works by Blackwood, Haba, Ives, and Wyschnegradsky*, p.55. ISBN 9780542998478.
46. ^ Sethares compares several equal temperament scales in a graph with axes reversed from the axes here. (fig. 4.6, p. 58)
47. ^ "665edo". xenoharmonic (microtonal wiki). Retrieved 2014-06-18.
48. ^ "convergents(log2(3), 10)". WolframAlpha. Retrieved 2014-06-18.
49. ^ 3/2 and 4/3, 5/4 and 8/5, 6/5 and 5/3 (A054540);
3/2 and 4/3, 5/4 and 8/5 (A060525);
3/2 and 4/3, 5/4 and 8/5, 7/4 and 8/7 (A060526);
3/2 and 4/3, 5/4 and 8/5, 7/4 and 8/7, 16/11 and 11/8 (A060527);
4/3 and 3/2, 5/4 and 8/5, 6/5 and 5/3, 7/4 and 8/7, 16/11 and 11/8, 16/13 and 13/8 (A060233);
3/2 and 4/3, 5/4 and 8/5, 6/5 and 5/3, 9/8 and 16/9, 10/9 and 9/5, 16/15 and 15/8, 45/32 and 64/45 (A061920);
3/2 and 4/3, 5/4 and 8/5, 6/5 and 5/3, 9/8 and 16/9, 10/9 and 9/5, 16/15 and 15/8, 45/32 and 64/45, 27/20 and 40/27, 32/27 and 27/16, 81/64 and 128/81, 256/243 and 243/128 (A061921);
5/4 and 8/5 (A061918);

6/5 and 5/3 (A061919);
6/5 and 5/3, 7/5 and 10/7, 7/6 and 12/7 (A060529);
11/8 and 16/11 (A061416)

50. ^ *Three Asymmetric Divisions of the Octave* by Wendy Carlos

§Bibliography

- Cho, Gene Jinsiong. (2003). *The Discovery of Musical Equal Temperament in China and Europe in the Sixteenth Century*. Lewiston, NY: Edwin Mellen Press.
- Duffin, Ross W. *How Equal Temperament Ruined Harmony (and Why You Should Care).* W.W.Norton & Company, 2007.
- Jorgensen, Owen. *Tuning*. Michigan State University Press, 1991. ISBN 0-87013-290-3
- Sethares, William A. (2005). *Tuning, Timbre, Spectrum, Scale* (2nd ed. ed.). London: Springer-Verlag. ISBN 1-85233-797-4.
- Surjodiningrat, W., Sudarjana, P.J., and Susanto, A. (1972) *Tone measurements of outstanding Javanese gamelans in Jogjakarta and Surakarta*, Gadjah Mada University Press, Jogjakarta 1972. Cited on http://web.telia.com/~u57011259/pelog_main.htm, accessed May 19, 2006.
- Stewart, P. J. (2006) "From Galaxy to Galaxy: Music of the Spheres" [2]
- Khramov, Mykhaylo. "Approximation of 5-limit just intonation. Computer MIDI Modeling in Negative Systems of Equal Divisions of the Octave", *Proceedings of the International Conference SIGMAP-2008*, 26–29 July 2008, Porto, pp. 181–184, ISBN 978-989-8111-60-9

§External links

- "Explaining the Equal Temperament" by Yuval Nov, *YuvalNov.org*.
- "An Introduction to Historical Tunings" by Kyle Gann, *KyleGann.com*.
- Huygens-Fokker Foundation Centre for Microtonal Music
- A.Orlandini: Music Acoustics
- "Temperament" from *A supplement to Mr. Chambers's cyclopædia* (1753)
- "12TET Frequency Table Maker" – Creates a frequency table (Hz.) for all 12TET pitches
- "HANDBOOK OF ACOUSTICS – online book" Brief description of Scheibler's tonometer
- Barbieri, Patrizio. Enharmonic instruments and music, 1470–1900. (2008) Latina, Il Levante Libreria Editrice
- "Fractal Microtonal Music", *Jim Kukula*.
- All existing 18th century quotes on J.S. Bach and temperament: https://www.academia.edu/5210832/18th_Century_Quotes_on_J.S._Bachs_Temperament

Dominic Eckersley: "Rosetta Revisited: Bach's Very Ordinary Temperament". https://www.academia.edu/3368760/Rosetta_Revisited_Bachs_Very_Ordinary_Temperament.