General Stability Criteria

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Introduction

This paper will examine general criteria for the stability and instability of inviscid parallel flows. The first section will explore unstratified flows, deriving Rayliegh's and Fjortoft's necessary (but not sufficient) criteria for instability. The second section will analyze stratified flows and derive stability criteria related to the local Richardson number of the flow.

1. Unstratified Flows

1.1 Introduction

Throughout this paper parallel flows of the following form will be considered:

$\mathbf{U}_* = U_*(z_*)\mathbf{i}$ for $z_{1*} \leq z_* \leq z_{2*}$

The following normalized quantities will be used throughout:

$$\mathbf{U} = \frac{\mathbf{U}_*}{max(|\mathbf{U}_*|)}$$

$$z = \frac{z_*}{(z_{2*} - z_{1*})}$$

$$p = \frac{p_*}{\rho_m}$$

Where ρ_m is the mean density of the flow.

The stability of these flows to small pertubations of the form $\mathbf{u}'(\mathbf{x}, t)$ will be examined with the aim of deriving general criteria for stability. As $\mathbf{U}$ arleady satisfies the boundary conditions, the pertubations are subject to the boundary conditions: $\mathbf{u}'(z = z_1) = \mathbf{u}'(z = z_2) = 0$.

1.2 Equations of Motion

The inviscid Navier-Stoke's equation with constant density and no external forces is:

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \cdot \mathbf{u})\mathbf{u} = -\nabla p$$

Let: $\mathbf{u} = \mathbf{U} + \mathbf{u}'$

Then:

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{u'}}{\partial t}$$

And:

$$(\nabla \cdot (\mathbf{U} + \mathbf{u'}))(\mathbf{U} + \mathbf{u'}) = (\nabla \cdot (\mathbf{U} + \mathbf{u'}))(\mathbf{U} + \mathbf{u'}) = \mathbf{U}\frac{\partial}{\partial x}(\mathbf{U} + \mathbf{u'}) + (\nabla \cdot (\mathbf{u'})\mathbf{U} + (\nabla \cdot (\mathbf{u'})\mathbf{u'}$$

Noting that $\frac{\partial}{\partial x}\mathbf{U} = 0$ and linearising ($(\nabla \cdot (\mathbf{u'})\mathbf{u'} \approx 0$):

$$\left(\nabla \cdot \left(\mathbf{U} + \mathbf{u'}\right)\right)\left(\mathbf{U} + \mathbf{u'}\right) \approx \mathbf{U}\frac{\partial}{\partial x}\mathbf{u'} + \left(\nabla \cdot \mathbf{u'}\right)\mathbf{U} = \mathbf{U}\frac{\partial}{\partial x}\mathbf{u'} + w'\frac{\partial}{\partial z}\mathbf{U}$$

Thus the equation of motion simplifies to:

$$\frac{\partial \mathbf{u'}}{\partial t} + U\frac{\partial}{\partial x}\mathbf{u'} + w'\frac{\partial}{\partial z}\mathbf{U} = -\nabla p$$

1.3 Squires Theorem

This section will examine a proof Squire's theorem: for any 3-dimensional pertubation, there is a 2 dimensional pertubation with the same wavenumber that is more unstable.

Intuitively, this is done by taking a 2-dimensional and rotating it align with the mean flow; this allows for more energy to be trasnferred from the mean flow to the disturbance causing it to be more unstable.

Noting that coefficients of the above equation of motion only depend on z, solutions of the following form will be searched for:

$$\mathbf{u}' = \hat{\mathbf{u}}(z)e^{i(ax+by+act)}$$

$$p' = \hat{p}(z)e^{i(ax+by+act)}$$

Let:

$$\mathbf{u}' = \hat{u}(z)e^{i(ax+by-act)}\hat{\mathbf{i}} + \hat{v}(z)e^{i(ax+by+act)}\hat{\mathbf{j}} + \hat{w}(z)e^{i(ax+by-act)}\hat{\mathbf{k}}$$

In order for the initial disturbances to have finite energy a and b must be real.

The equation of motion can be written as:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\hat{u} + w'\frac{\partial U}{\partial z} = -\frac{\partial p}{\partial x}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\hat{v} = -\frac{\partial p}{\partial y}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\hat{w} = -\frac{\partial p}{\partial z}$$

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{w}}{\partial z} = 0$$

Substituting the disturbances into these above equations and eliminating exponentials yields:

$$ia\left(U-c\right)\hat{u} + \hat{w}U' = -ia\hat{p}$$

$$ia\left(U-c\right)\hat{v} = -ib\hat{p}$$

$$ia\left(U-c\right)\hat{w} = \frac{d\hat{p}}{dz}$$

$$ia\hat{u} + ib\hat{v} + \frac{\partial \hat{w}}{\partial z} = 0$$

Summing a times the first equation and b times the second equations yields:

$$ia\left(U-c\right)\left(a\hat{u} + b\hat{v}\right) + a\hat{w}U' = -i\left(a^2+b^2\right)\hat{p}$$

Dividing through by a:

$$i\left(U-c\right)\left(a\hat{u}+b\hat{v}\right)+\hat{w}U' = -i\frac{\left(a^2+b^2\right)}{a}\hat{p}$$

Let $k = \sqrt{\left(a^2+b^2\right)}$

Let:

$$\tilde{u} = \frac{\left(a\hat{u}+b\hat{v}\right)}{k}$$
$$\tilde{v} = 0$$
$$\tilde{w} = \hat{w}$$

Let $\tilde{p} = \dfrac{k\hat{p}}{a}$

Then the above equations can be re-written as:

$$ik\left(U-c\right)\tilde{u}+\tilde{w}U' = -ik\hat{p}$$

$$ik\left(U-c\right)\tilde{w} = -ik\frac{d\hat{p}}{dz}$$

$$k\tilde{u}+\frac{\partial\tilde{w}}{\partial z} = 0$$

Thus, if a 3-dimensional disturbance of the form satisfies the equations of motion:

$$\mathbf{u}' = \hat{u}(z)e^{i(ax+by-act)}\hat{\mathbf{i}}+\hat{v}(z)e^{i(ax+by+act)}\hat{\mathbf{j}}+\hat{w}(z)e^{i(ax+by-act)}\hat{\mathbf{k}}$$

Then following 2-dimensional disturbance with the same wave number also satisfies the equations of motion:

$$\tilde{\mathbf{u}} = \tilde{u}(z)e^{i(kx-kct)}\hat{\mathbf{i}}+\hat{w}(z)e^{i(k-kct)}\hat{\mathbf{k}}$$

As $\left|k\right| \geq \left|a\right|$, $\left|kc\right| \geq \left|ac\right|$ thus this 2-dimensional distrubance is at least as unstable as the 3-dimensional disturbance.

The significance of this result is that when studying general criterea for modal instability only 2-dimensional disturbances need be considered. A flow is considered unstable if there exists any disturbance that will grow exponentially with time, thus if a 2-dimensional disturbance is found to grow exponentially with time then flow is unstable. Conversly, a flow is considered stable only if all disturbances do not grow exponentially with time; if a flow is found to be stable in response to all 2-dimensional disturbances then using squires theorem it must also be stable in response to all 3-dimensional disturbances and thus the flow is considered stable.

1.4 Rayleigh's Equation

This section will derive Rayleigh's equation for the stream function of 2-dimensional disturbances. It is convenient when studying 2-dimensional flows to use the stream function: $\psi(\mathbf{x}, t)$.

$$u' = \frac{\partial \psi}{\partial z}$$

$$w' = -\frac{\partial \psi}{\partial x}$$

Substituting these into the equations of motion:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x}\frac{\partial U}{\partial z} = -\frac{\partial p}{\partial x}$$

$$-\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial \psi}{\partial x} = -\frac{\partial p}{\partial z}$$

Once again, the coefficients only depend on z, thus solutions of the form $\psi = \phi e^{ik(x-ct)}$ will be searched for. Substituting this into the first equation yields:

$$ik\left(U - c\right)\phi' - ik\phi U' = -ik\tilde{p}$$

$$\tilde{p} = \phi U' - \left(U - c\right)\phi'$$

Differentiating $\tilde{p}$ with the purpose of substituting into the z-equation of motion:

$$\tilde{p}' = \phi U'' + \phi' U' - \left(U - c\right)\phi'' - U'\phi'$$

$$\tilde{p}' = \phi U'' - \left(U - c\right)\phi''$$

Substituing into the z equation of motion:

$$ik(U - c)(-ik\phi) = -\phi U'' + \left(U - c\right)\phi''$$

$$(U - c)\left(\phi'' - k^2\phi\right) - \phi U'' = 0$$

This last equation is known as Rayleigh's equation. Notice that if k satisfies the above equation then so does $-k$. Thus it can be taken without loss of generality that $k \geq 0$. Taking the complex conjugate of the entire equation yields:

$$(U - c^*)\left(\phi''^* - k^2\phi^*\right) - \phi^* U'' = 0$$

Thus if c satisfies Rayleigh's equation so does c^*. The condition for instability then becomes: $c_i \neq 0$.

1.5 Rayleigh's Necessary Condition for Instability

As discussed above, a pertubations is unstable iff c is complex. Let $c = c_r + ic_i$, then this condition is equivalent to $c_i \neq 0$.

If the pertubation is unstable, $c_i \neq 0$ and thus $\left(U - c\right) \neq 0$ as U is a real valued function. Therefore the equation below is non-singular:

$$\left(\phi'' - k^2\phi\right) - \frac{\phi U''}{(U - c)} = 0$$

Multiplying the above equation by ϕ^* and integrating yields the following:

$$\int_{z_1}^{z_2} \left[\left(\phi''\phi^* - k^2 \mid \phi \mid^2 \right) - \frac{|\phi \mid^2 U''}{(U - c)}\right] dz = 0$$

Integrating the first term by parts, noting that $\phi(z_1) = \phi(z_2) = 0$ by the boundary conditions and thus $\phi^*(z_1) = \phi^*(z_2) = 0$:

$$\int_{z_1}^{z_2} \phi''\phi^* dz = \left[\phi'\phi^*\right]_{z1}^{z2} - \int_{z_1}^{z_2} \phi'\phi'^* dz = -\int_{z_1}^{z_2} \mid \phi' \mid^2 dz$$

Substituting this into the equation yields:

$$\int_{z_1}^{z_2} \left[\left(\mid \phi' \mid^2 + k^2 \mid \phi \mid^2 \right) + \frac{|\phi \mid^2 U''}{(U - c)}\right] dz = 0$$

The first bracketed term is proportional the energy of one wavelength othe disturbance at time 0, this equation desribes the relationship between the energy of the disturbance and the mean flow. Noting that $\left(\mid \phi' \mid^2 + k^2 \mid \phi \mid^2 \right)$ is real the complex part of this integral is:

$$c_i \int_{z_1}^{z_2} \frac{|\phi \mid^2 U''}{|U - c|^2} = 0$$

Assuming $c_i \neq 0$ then:

$$\int_{z_1}^{z_2} \frac{|\phi \mid^2 U''}{|U - c|^2} = 0$$

$|\phi \mid^2 \geq 0, \mid U - c \mid^2 \geq 0$ therefore U'' changes sign at come point, as $U(z)''$ is continuos this implies that $U''(z_s) = 0$ for some $z_s, z_1 \leq z_s \leq z_2$. Therefore if U is an unstable flow ($c_i \neq 0$) then U has an inflexion point. In other words, $U'' = 0$ at some point within the range is a necessary condition for instability; it is not however, sufficient.

1.6 Fjortoft's Stronger Critereon

A stronger criterea for instability can be obtained from the real part of the above integrand:

$$\int_{z_1}^{z_2} \left[\left(\mid \phi' \mid^2 + k^2 \mid \phi \mid^2 \right) + \frac{|\phi\mid^2 \; U''(U - c_r)}{|U-c|^2} \right] dz = 0$$

Assuming instability:

$$\int_{z_1}^{z_2} \frac{|\phi\mid^2 \; U''}{|U-c|^2} = 0$$

Let $U_s = U(z_s)$, $(U''(z_s) = 0)$.

Adding the follwing to the real part of integral:

$$(c_r - U_s) \int_{z_1}^{z_2} \frac{|\phi\mid^2 \; U''}{|U-c|^2}$$

Yields:

$$\int_{z_1}^{z_2} \left[\left(\mid \phi' \mid^2 + k^2 \mid \phi \mid^2 \right) + \frac{|\phi\mid^2 \; U''(U - U_s)}{|U-c|^2} \right] dz = 0$$

As:

$$\int_{z_1}^{z_2} \left[\left(\mid \phi' \mid^2 + k^2 \mid \phi \mid^2 \right) \right] > 0$$

This implies:

$$\int_{z_1}^{z_2} \left[\frac{|\phi\mid^2 \; U''(U - U_s)}{|U-c|^2} \right] dz < 0$$

Thus $U''(U - U_s) < 0$ at some point on the range. This is Fjortoft's criterea, necessary condition for stability is that $U''(U - U_s) < 0$ at some point on the domain. An even stronger condition can be obtained if U is a monotonic function with only one inflexion point. To see this, without loss of generality that U(z) is monotonically increasing thus $(U - U_s)$ only changes sign at z_s. As U only has one inflexion point U'' only changes sign at z_s. As $(U - U_s)$ and U'' both change sign at the same point, the sign of $U''(U - U_s)$ does not change, therefore if $U''(U - U_s) < 0$ at some point on the domain then $U''(U - U_s) < 0$ for $z \neq z_s, z_1 \leq z \leq z_2$. This is stronger criterea, for a monotonically increasing function with only one inflexion point a necessary condition for instability is that $U''(U - U_s) \leq 0$ everwhere.

1.8 Physical Intepretation

This section will aim to describe Fjortoft and Rayleihgh's criterea in terms of vorticity. The vorticity of the steady state is given by:

$$\nabla \times U\hat{\mathbf{i}} = -U'\hat{\mathbf{j}}$$

We will show that the absolutely value of the vorticity will attain a maximum for monotonic flows that satisfy Fjortoft's condition. Consider a monotonically increasing flow that satisfies Fjortoft's criteria with one inflexion point in a small region around this inflexion point z_s.

For $\delta z > 0$:

U is monotoically increasing and thus:

$$U(z + \delta z) > U_s$$

$$U(z + \delta z) - U_s > 0$$

But, if a monotonic flow satisfies Fjortoft's criteria somewhere then it satisfies Fjortoft's criteria everwhere. Therefore:

$$U(z + \delta z)''\left(U(z + \delta z) - U_s\right) < 0$$

And therefore:

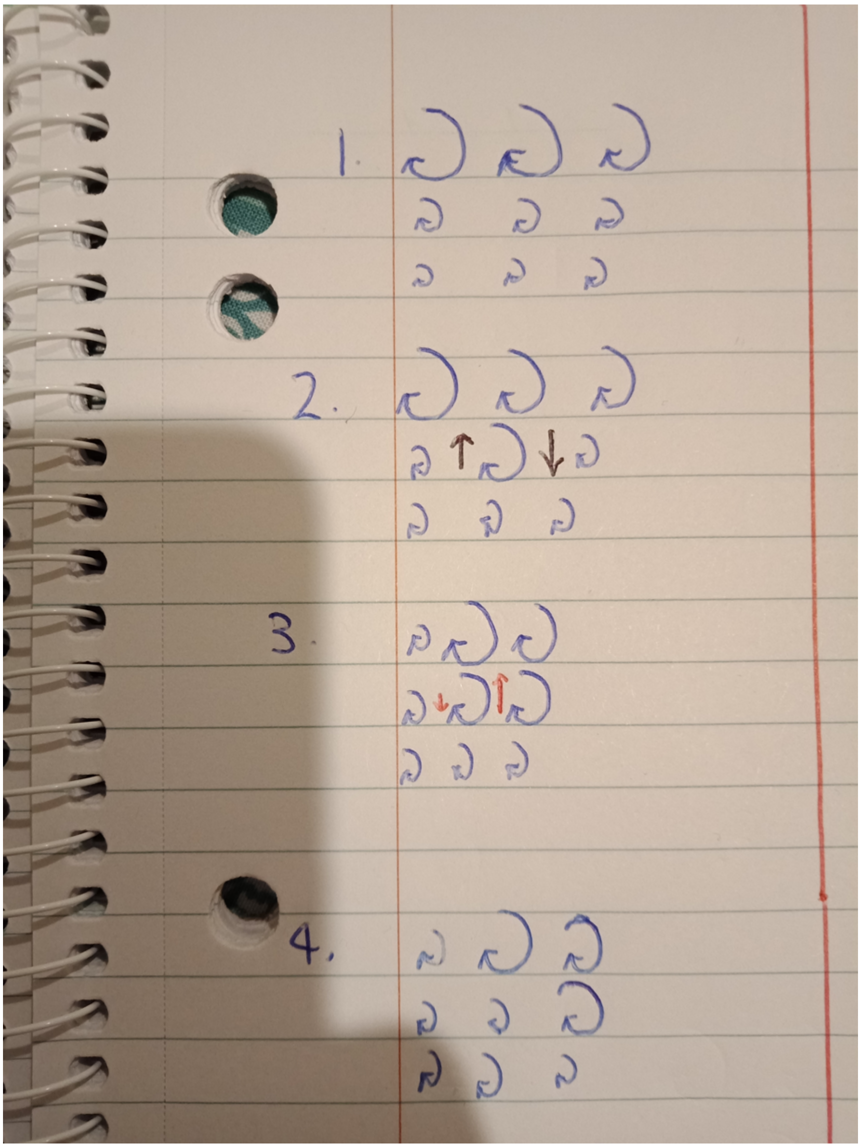
$$U(z + \delta z)'' < 0$$

Repeating this argument for $z - \delta z$ shows that:

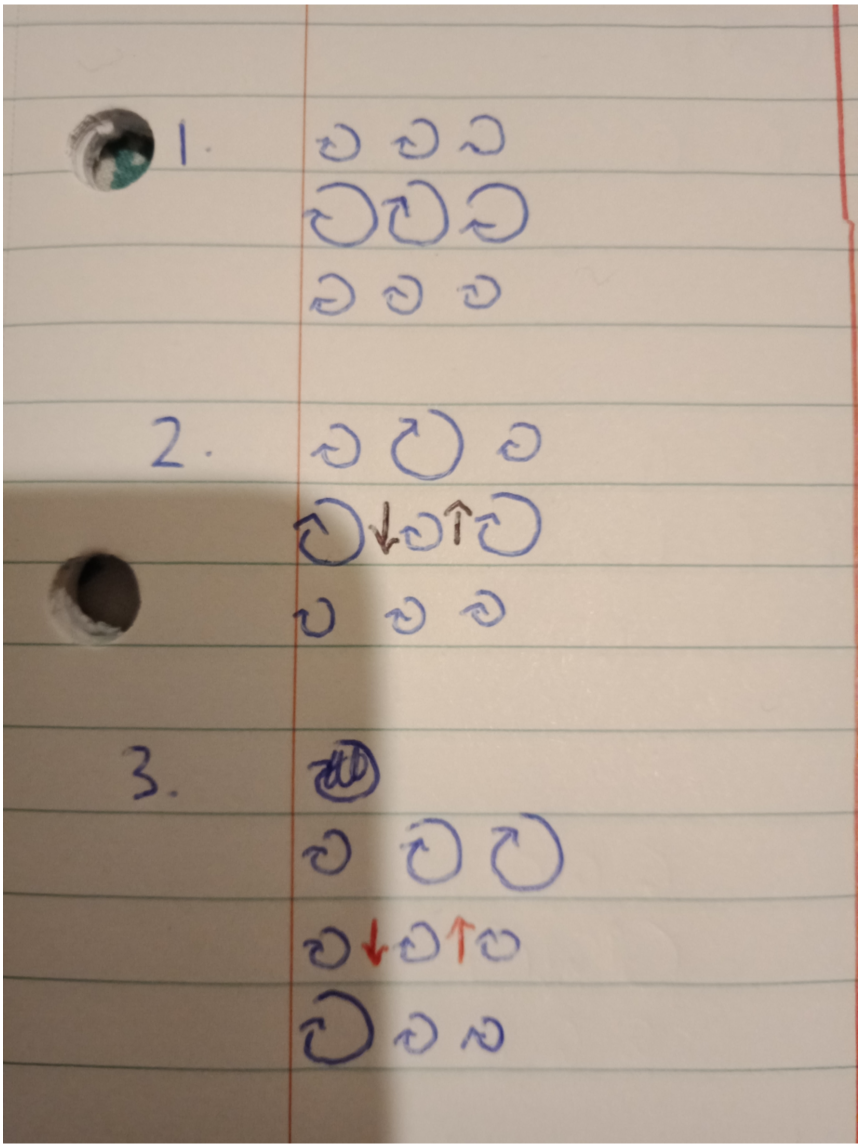
$$U(z - \delta z)'' < 0$$

Therefore U'' decreases fron $z - \delta z$ to $z + \delta z$ and hence $U''' < 0$. As $U'' = 0$ and $U(z_s)''' < 0$ U' attains a local maximum at z_s. At U is monotonic, $U' > 0$ and thus $\left|U'\right|$ attains a local maximum at z_s. The same argument can be repeated for monotonically decreasing functions yielding a local minimum and thus for monotonic function Fjortoft's criteria implies the $\left|U'\right|$ attains a local maximum somewhere. In other words the absolute value of the vorticity attains a local maximum somewhere.

To undestand this intuitively consider the following drawing of a small section of the vorticity for a parrallel flow where there is not maximima of the magnitude of the vorticity. State 1 is the steady state. In state 2, a fluid element of the top layer has been perturbed to the middle layer bringing with it excess vorticity. This excess vorticity induces flow in the z-direction as shown by the black arrows. This z-flow then causes fluid from the top and bottom layers to move resulting in state 3. In state 3, there is more vorticity to the right of centre element in the middle row than to the left, causing the unbalances forces shown in red to be exerted on this element forcing it back to its original position. Notice, in state 4 there is now excess vorticity in the right element of the middle layer suggesting that this excess vorticity is transported in a wave-like manner. This illustrates how stable modes can exist where there exists no maximum of the magnitude of the vorticity.



Now consider the below drawing of the vorticity field about a local maximum of the absolute value of vorticity. The logic followed is the same except in state 3 the forces are balanced and there is no restoring force on the originally perturbed fluid element and therefore the flow is not stable to small pertubations.



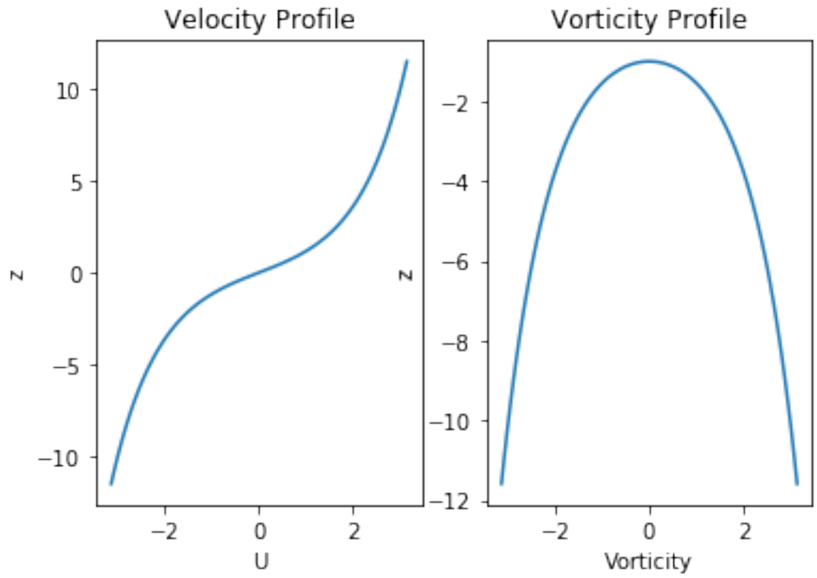
1.7 Examples

Consider the flow: $U(z) = sinh(z)$ on the range $[-\pi, \pi]$.

$U''(0) = 0$ is an inflexion point and thus the flow satisfies rayleighs criteria.

$U(0) = 0$, therefore $U''(U - U_s) = sinh(z)(sinh(z) - 0) = \left(sinh(z)\right)^2 \geq 0$and hence Fjortoft's criterea is not fulfilled and the flow is stable, this flow is a counterexample to the sufficiency of Rayleigh's criteria. Visually inspecing the flow it can be seen that $\left|U'\right|$ attains a local minimima, not a maxima, at z_s.

In [7]:
```
import numpy as np
from matplotlib import pyplot as plt

zs = np.linspace(-np.pi, np.pi, num=100)
us = np.sinh(zs)
dus = -np.cosh(zs)


plt.subplot(1, 2, 1)
plt.plot(zs,us)
plt.title('Velocity Profile')
plt.xlabel('U')
plt.ylabel('z')

plt.subplot(1, 2, 2)
plt.plot(zs,dus)
plt.title("Vorticity Profile")
plt.xlabel('Vorticity')
plt.ylabel('z')
plt.show()
```



Consider the flow $U(z) = sinz$.

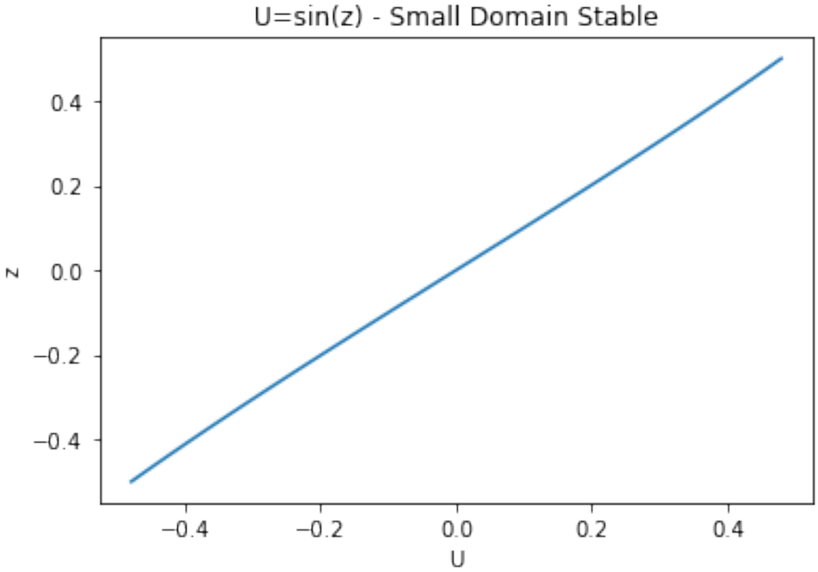
$$U''(0) = 0$$
$$U(0) = 0$$

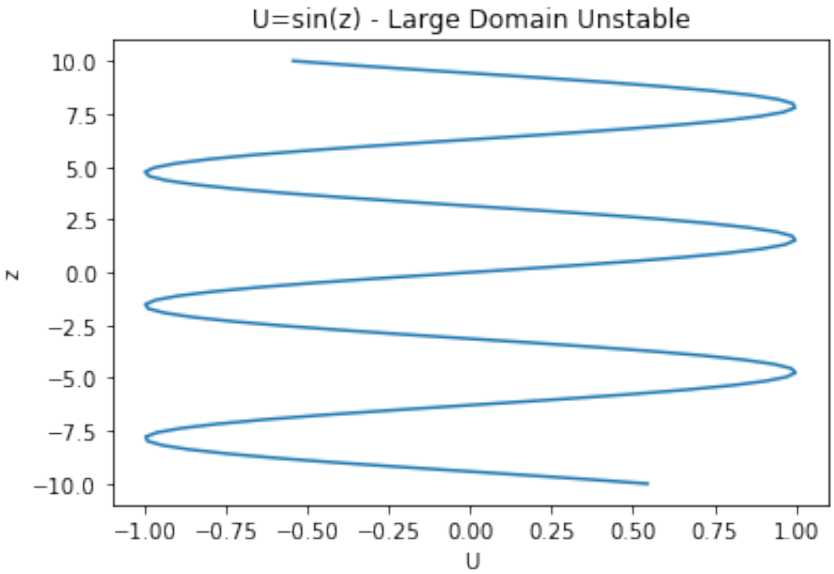
Therefore:

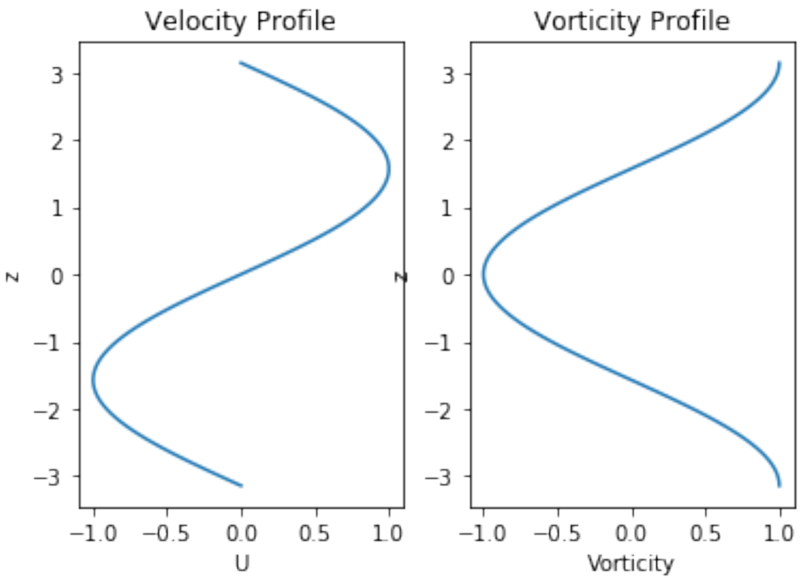
$$U''(U - U_s) = sinz(-sinz) = -(sinz)^2 \leq 0$$

It can be seen that both Rayleigh's and Fjortoft's criteria are satisfied. However, when the domain is restricted to $[-\delta z, \delta z]$ the flow approximates Couette flow which is stable. When the range is extended to mutiple wavelengths of $sin(z)$ the flow is clearly unstable, and thus Fjortoft's criteria is not sufficient for instability. There are further criteria related to the size of the domain and allowable wavelengths of the disturbances. If the domain is smaller than a critical value unstable wavelengths cannot exist, in fact at this point neutral modes cannot exist either. The last graphs show that the vorticity magnitude attains a maximium at $z = 0$.

In [6]:
```
zs = np.linspace(-0.5, 0.5, num=100)
us = np.sin(zs)

plt.plot(us,zs)
plt.title('U=sin(z) - Small Domain Stable')
plt.xlabel('U')
plt.ylabel('z')
plt.show()

zs = np.linspace(-10, 10, num=100)
us = np.sin(zs)

plt.plot(us,zs)
plt.title('U=sin(z) - Large Domain Unstable')
plt.xlabel('U')
plt.ylabel('z')
plt.show()

zs = np.linspace(-np.pi, np.pi, num=100)
us = np.sin(zs)
dus = -np.cos(zs)

plt.subplot(1, 2, 1)
plt.plot(us,zs)
plt.title('Velocity Profile')
plt.xlabel('U')
plt.ylabel('z')

plt.subplot(1, 2, 2)
plt.plot(dus,zs)
plt.title("Vorticity Profile")
plt.xlabel("Vorticity")
plt.ylabel('z')
plt.show()
```


U=sin(z) - Small Domain Stable
z
U


U=sin(z) - Large Domain Unstable
z
U


Velocity Profile
Vorticity Profile
z
U
Vorticity

1.9 Discontinuous Case

The motivation for this section is the study of the Rayliegh profile:

$$U(z) = \begin{cases} -U_0 & z \leq -a \\ \frac{U_0}{a} z & -a < z < a \\ U_0 & z \geq a \end{cases}$$

A non-smooth version of the Rayleigh criteria will presented and Fjortoft's criteria will be re-examined.

The integral that Rayleigh's criteria is derived from is:

$$\int_{z_1}^{z_2} \frac{|\phi\ |^2\ U''}{|U-c|^2} = 0$$

Which implies that U'' changes sign somewhere. If $U''(z)$ is continuous this in turn implies an inflexion point. However, if U'' is not continuous Rayliegh's criteria simply becomes: U'' changes sign somewhere.

Differentiating U:

$$U'(z) = \begin{cases} 0 & z \leq -a \\ \frac{U_0}{a} & -a < z < a \\ 0 & z \geq a \end{cases}$$

Differentiating again:

$$U'(z) = \frac{U_0}{a}\delta(z+a) - \frac{U_0}{a}\delta(z-a)$$

Loosely speaking:

$$U''(z) = \begin{cases} \infty & z = -a \\ -\infty & z = a \\ 0 & z \neq a, -a \end{cases}$$

Thus U'' changes sign and Rayleigh's criteria is fulfilled. As Rayleigh's criteria no longer ensures that $U'' = 0$, U_s may no longer exist for unstable flows and thus a more general Fjortoft's criteria must be considered. Notice that the choice of U_s is arbitrary in Fjortoft's criteria, any real number r will satisfy:

$$\int_{z_1}^{z_2} \left[\frac{|\phi\ |^2\ U''(U-r)}{|U-c|^2}\right] dz < 0$$

Assuming of course that the flow is unstable. Therefore, $U''(U-r) < 0$ somewhere for all values of r. To examine whether Rayleigh flow satisfies this condition consider 2 cases:

Case 1 $r < U_0$**:**

$$U_0 - r > 0$$

$$U''(a) = -\frac{U_0}{a}\delta(0) = -\infty < 0$$

Also, $U(a) = U_0$, therefore:

$$U''(a)\big(U(a) - r\big) < 0$$

Case 2 $r \geq U_0$**:**

$$-U_0 - r < 0$$

$$U''(-a) = \frac{U_0}{a}\delta(0) = \infty > 0$$

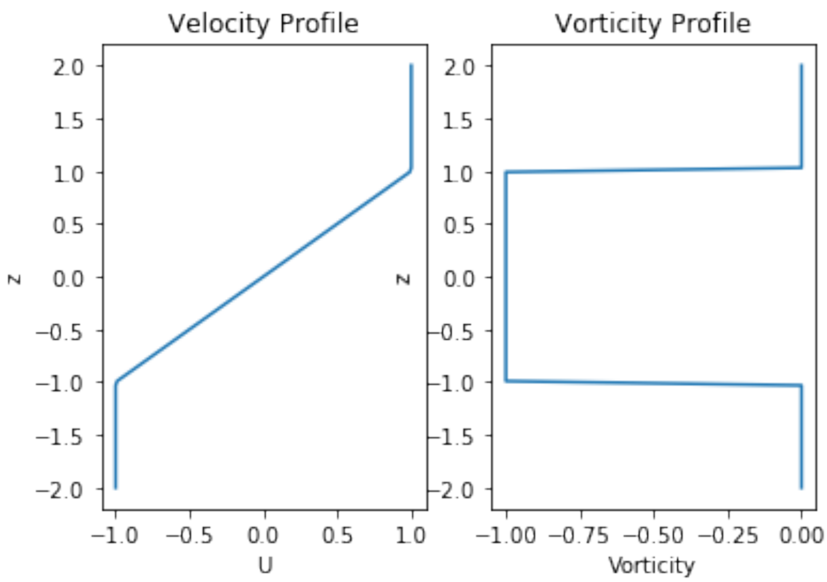
Also, $U(a) = -U_0$, therefore:

$$U''(-a)\big(U(-a) - r\big) < 0$$

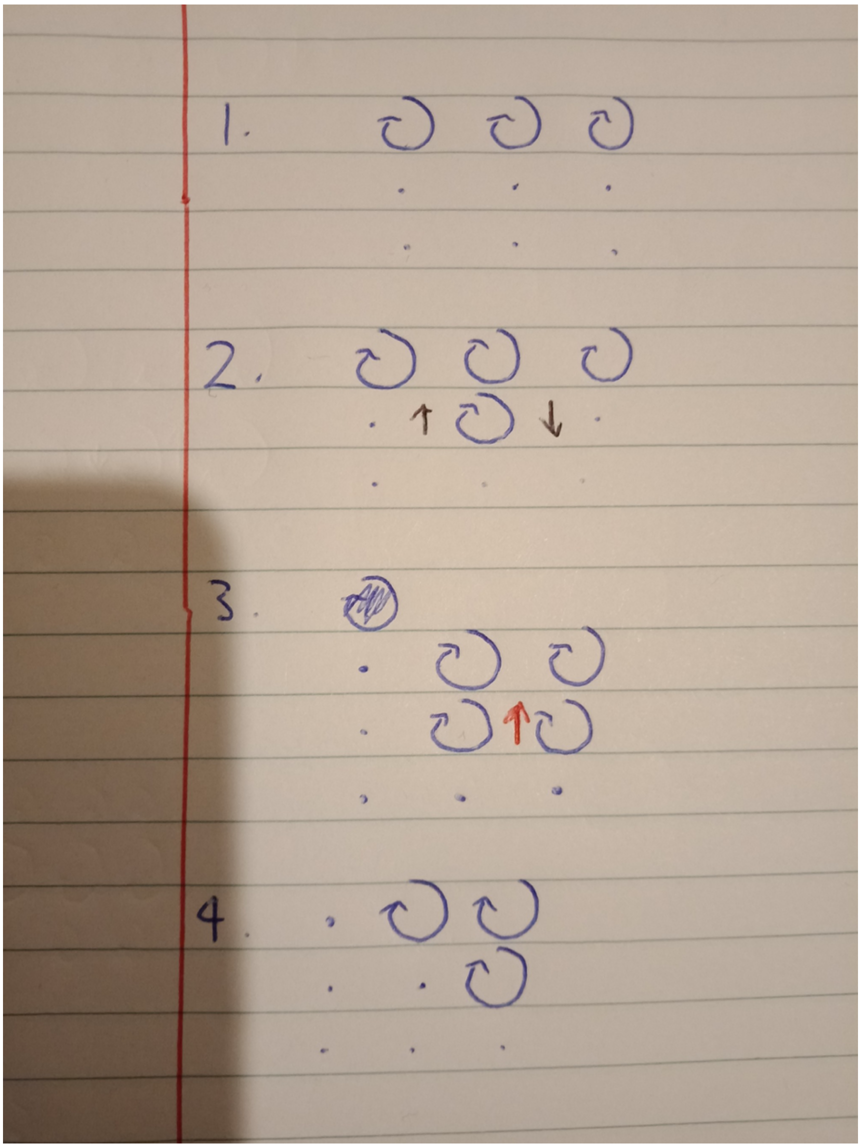
And therefore Fjortoft's criteria is satisfied. Notice, that to statisfy Rayleigh's and Fjortoft's requires both discontinuities in U'', without one of them there would be stability. The question remains of whether the previous physical explanation for instability, a maximum of the absolute value of the vorticity, still holds. Below the flow velocity and vorticity is plotted for Rayleigh flow. It can clearly be seen that the magnitude of the vorticity attains a maximum $\frac{U_0}{a}$ for $-a < z < a$.

In [5]:

```
def ray_U(z):
    return min(1,abs(z))*z/(abs(z))
def ray_dU(z):
    dU = 1 if abs(z) < 1 else 0
    return dU

zs = np.linspace(-2, 2, num=100)
us = [ray_U(z) for z in zs]
dus = [-ray_dU(z) for z in zs]

plt.subplot(1, 2, 1)
plt.plot(us,zs)
plt.title('Velocity Profile')
plt.xlabel('U')
plt.ylabel('z')

plt.subplot(1, 2, 2)
plt.plot(dus,zs)
plt.title('Vorticity Profile')
plt.xlabel("Vorticity")
plt.ylabel('z')
plt.show()
```



Repeating the infintismal vorticity pertubation reasoning above it can be seen that pertubation across both discontinuities are stable. If a fluid element is perturbed over one of these discontinuities it is pushed back - how then does instability occur? One potential explanation is to consider the discontinuities are reflecting boundaries, if a fluid element is perturbed past one boundary it is pushed back pushing fluid across the other boundary. In this way perhaps standing waves can resonate between the boundaries, increasing in energy over time causing instability. Boundaries such as these don't require discontinuities, they only require that the gradient of the vorticity in 2 regions are of opposite sign - if the flow function is smooth this implies that $U'' = 0$ somewhere and possibly this phenomena is the physical explanation for instability.



2. Stratified Flows

2.1 Introduction

The dimensionless quantities above will be reused. The treatment of the stratified case is very similar to the procedure used above: the equations will be linearised, stream functions of the form $\psi = \phi e^{ik(x-ct)}$ will be searched for and this will results in an equation for ψ that will multiplied by a complex conjugate and integrated yielding stability criteria.

2.2 Equations of Motion

This section will use the Boussineque approximation of the Navier-Stokes equation. The Boussinesque equations of motion are as follows:

Let: $\rho(\mathbf{x}, t) = \rho_m + \bar{\rho}(z) + \rho'(\mathbf{x}, t)$

Where: $\rho_m \gg \bar{\rho} \gg \rho'$

$$\frac{\partial \mathbf{u}}{\partial t} + \left(\nabla \cdot \mathbf{u}\right)\mathbf{u} = -\nabla p - g\frac{\rho'}{\rho_m}\hat{\mathbf{k}}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \left(\nabla \rho\right) = 0$$

Writing $\mathbf{u} = U(z)\hat{\mathbf{i}} + \mathbf{u}'$ and linearising as before yields:

$$\frac{\partial \mathbf{u}'}{\partial t} + \left(\nabla \cdot \mathbf{u}'\right)U\hat{\mathbf{i}} + U\frac{\partial \mathbf{u}'}{\partial x} = -\nabla p' - g\frac{\rho'}{\rho_m}\hat{\mathbf{k}}$$

Seperating out the 2-dimensional version of this equation into components:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + w'\frac{\partial U}{\partial z} = -\frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} - g\frac{\rho'}{\rho_m}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \rho'}{\partial t} + U\frac{\partial \rho'}{\partial x} + w'\frac{\partial \bar{\rho}}{\partial z} = 0$$

2.3 The Search for Eigenfunctions

As before stream functions of the form $\psi = \phi(z)e^{ik(x-ct)}$ will be searched for.

$$u' = \frac{\partial \psi}{\partial z} = \phi' e^{ik(x-ct)}$$

$$w' = -\frac{\partial \psi}{\partial x} = -ik\phi e^{ik(x-ct)}$$

Subbing these into the x-momentum equation yields:

$$-ikc\phi' + Uik\phi' + -ik\phi U' = -ik\hat{p}$$

$$\left(U - c\right)\phi' - \phi U' = -\hat{p}$$

Therefore:

$$-\hat{p}' = \left(U - c\right)\phi'' + U'\phi' - \phi' U' - \phi U''$$

$$-\hat{p}' = \left(U - c\right)\phi'' - \phi U''$$

Subbing this into the z-momentum equation yields:

$$-ikc\left(-ik\right)\phi + ik\left(-ik\right)\phi U + g\frac{\rho'}{\rho_m} = \left(U - c\right)\phi'' - \phi U''$$

$$k^2\left(U - c\right)\phi + g\frac{\rho'}{\rho_m} = \left(U - c\right)\phi'' - \phi U''$$

$$\left(U - c\right)\left(\phi'' - k^2\phi\right) - \phi U'' - g\frac{\hat{\rho}}{\rho_m} = 0$$

The last equation yields:

$$\left(U - c\right)\hat{\rho} + \bar{\rho}' = 0$$

$$\hat{\rho} = -\frac{\bar{\rho}'}{\left(U - c\right)}$$

Substituting this into the above equation yields:

$$\left(U - c\right)\left(\phi'' - k^2\phi\right) - \phi U'' + g\frac{\bar{\rho}'\phi}{\rho_m\left(U - c\right)} = 0$$

This is known as the Taylor-Goldstein equation, the stratified equivalant of the Rayleigh equation.

2.4 Manipulation of the Taylor Goldstein Equation

The aim of this section is to multiply the Taylor-Goldstein equation by some complex conjugae and integrate to obtain stability criteria. In order to obtain this criteria we define a quantity H and re-write the Taylor-Goldstein equation in terms of H.

$$H = \frac{\phi}{\left(U-c\right)^{\frac{1}{2}}}$$

H can be thought of as the amplitude of the disturbance divided by the square root of the flow velocity as seen from the reference frame of a travelling wave.

Writing ϕ in terms of H:

$$\phi = H\left(U-c\right)^{\frac{1}{2}}$$

$$\phi' = H'\left(U-c\right)^{\frac{1}{2}} + \frac{HU'}{2\left(U-c\right)^{\frac{1}{2}}}$$

$$\phi'' = H''\left(U-c\right)^{\frac{1}{2}} + 2\frac{H'U'}{2\left(U-c\right)^{\frac{1}{2}}} + \frac{HU''}{2\left(U-c\right)^{\frac{1}{2}}} - \frac{H(U')^2}{4\left(U-c\right)^{\frac{3}{2}}}$$

Thus:

$$\left(U-c\right)\phi'' = H''\left(U-c\right)^{\frac{3}{2}} + H'U'\left(U-c\right)^{\frac{1}{2}} + \frac{1}{2}HU''\left(U-c\right)^{\frac{1}{2}} - \frac{H(U')^2}{4\left(U-c\right)^{\frac{1}{2}}}$$

Substituing this into the Taylor-Goldstein equations yields:

$$H''\left(U-c\right)^{\frac{3}{2}} + H'U'\left(U-c\right)^{\frac{1}{2}} + \frac{1}{2}HU''\left(U-c\right)^{\frac{1}{2}} - \frac{H(U')^2}{4\left(U-c\right)^{\frac{1}{2}}} - k^2H\left(U-c\right)^{\frac{3}{2}} - U''H\left(U -\right.$$

$$+ g\frac{\bar{\rho}'}{\rho_m}\frac{H}{\left(U-c\right)^{\frac{1}{2}}} = 0$$

Multiplying by $\left(U-c)\right)^{\frac{1}{2}}$:

$$H''\left(U-c\right)^2 + H'U'\left(U-c\right) + \frac{1}{2}HU''\left(U-c\right) - \frac{H(U')^2}{4} - k^2H\left(U-c\right)^2 - U''H\left(U-c\right) + g$$

$$\big(U-c\big)^2\big(H''-k^2H\big)+\big(U-c\big)\big(H'U'+\frac{1}{2}HU''-U''H\big)+\big(g\frac{\bar{\rho}'}{\rho_m}-\frac{H(U')^2}{4}\big)=0$$

$$\big(U-c\big)^2\big(H''-k^2H\big)+\big(U-c\big)\big(H'U'-\frac{1}{2}HU''\big)+H\left(g\frac{\bar{\rho}'}{\rho_m}-\frac{(U')^2}{4}\right)=0$$

Dividing by $\big(U-c\big)$:

$$\big(U-c\big)\big(H''-k^2H\big)+\big(H'U'-\frac{1}{2}HU''\big)+\frac{H}{\big(U-c\big)}\left(g\frac{\bar{\rho}'}{\rho_m}-\frac{(U')^2}{4}\right)=0$$

This is Taylor-Goldstein equation written in terms of H, as will be seen below the term $H'U'$ will make integration much more tractable.

2.5 Integration to Obtain Stability Criteria

Similar to the unstratified case, this equation will be multiplied by H^* and integrated to obtain stability criterea. These results will rely on use of integration by parts and exploitation of the boundary conditions:

$$\phi(z_1)=\phi(z_2)=\phi^*(z_1)=\phi^*(z_2)=0$$

Thus:

$$H(z_1)=H(z_2)=H^*(z_1)=H^*(z_2)=0$$

The resulting integral is:

$$\int_{z_1}^{z_2}H^*\left[\big(U-c\big)\big(H''-k^2H\big)+\big(H'U'-\frac{1}{2}HU''\big)+\frac{H}{\big(U-c\big)}\left(g\frac{\bar{\rho}'}{\rho_m}-\frac{(U')^2}{4}\right)\right]dz=0$$

Using integration by parts of $H''\big(U-c\big)$:

$$\int\big(U-c\big)H''dz=H'\big(U-c\big)-\int U'H'dz$$

Therefore:

$$\int_{z_1}^{z_2}\big(U-c\big)H''H^*dz=\left[H^*\left[H'\big(U-c\big)-\int U'H'dz\right]\right]_{z_1}^{z_2}-\int_{z_1}^{z_2}\left[H'^*\left[H'\big(U-c\big)-\int U'H'd\right.\right.$$

$$\int_{z_1}^{z_2}\big(U-c\big)H''H^*dz=-\int_{z_1}^{z_2}\left|H'\right|^2\big(U-c\big)dz+\int_{z_1}^{z_2}\left[H'^*\int\left[U'H'\right]dz\right]dz$$

Also:

$$\int_{z_1}^{z_2} H^* H' U' dz = \left[H^* \int \left(H'U'\right) dz\right]_{z1}^{z_2} - \int_{z_1}^{z_2} \left[H'^* \left[\int H'U'dz\right]\right] dz$$

$$\int_{z_1}^{z_2} H^* H' U' dz = -\int_{z_1}^{z_2} \left[H'^* \left[\int H'U'dz\right]\right] dz$$

Therefore the sum of these two integrals yields:

$$\int_{z_1}^{z_2} \left(H^*\left(U-c\right)H'' + H^* H' U'\right) dz = -\int_{z_1}^{z_2} \left|H'\right|^2 \left(U-c\right) dz$$

Substituing this result into the original integral yields:

$$\int_{z_1}^{z_2} \left[-\left(U-c\right)\left(\left|H'\right|^2 + k^2\left|H\right|^2\right) - \frac{1}{2}U''\left|H\right|^2 + \frac{\left|H\right|^2}{\left(U-c\right)}\left(g\frac{\bar{\rho}'}{\rho_m} - \frac{(U')^2}{4}\right)\right] dz = 0$$

Or:

$$\int_{z_1}^{z_2} \left[\left(U-c\right)\left(\left|H'\right|^2 + k^2\left|H\right|^2\right) + \frac{1}{2}U''\left|H\right|^2 + \frac{\left|H\right|^2}{\left(U-c\right)}\left(\frac{(U')^2}{4} - g\frac{\bar{\rho}'}{\rho_m}\right)\right] dz = 0$$

Assuming $c_i \neq 0$, the complex part of the above integral is:

$$-c_i \int_{z_1}^{z_2} \left[\left|H'\right|^2 + \frac{\left|H\right|^2}{\left|U-c\right|^2}\left(\frac{(U')^2}{4} - g\frac{\bar{\rho}'}{\rho_m}\right)\right] dz = 0$$

$\int_{z_1}^{z_2} \left|H'\right|^2 dz > 0$, therefore:

$$\int_{z_1}^{z_2} \frac{\left|H\right|^2}{\left|U-c\right|^2}\left(\frac{(U')^2}{4} - g\frac{\bar{\rho}'}{\rho_m}\right) < 0$$

Thus:

$$\left(\frac{(U')^2}{4} - g\frac{\bar{\rho}'}{\rho_m}\right) < 0$$

At some point on the domain. In other words:

$$g\frac{\bar{\rho}'}{\rho_m (U')^2} < \frac{1}{4}$$

This quantity is known as the local Richardson number. A necessary condition for instability is that the local Richardson number is somewhere less than $\frac{1}{4}$. Intuitively, this can be understood as the ratio between the buyouncy term: $g\frac{d\bar{p}}{dz}$ to the shear term: $p_m(U')^2$. Greater shear forces cause the flow to be more unstable, hence the stability of the flow is inversely proportional to the shear term. The more stably stratified a density density distribution the more stable the flow hence the stability of the flow is proportional to the buyouncy term. If the Richardson number is less than 0.25 the shear forces can supply enough energy to overcome the stable stratification, allowing small pertubations to grow.