Conics and Quadratic Forms

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Abstract—**Solved problems from JEE mains papers related to conic sections in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.**

1 Two parabolas with a common vertex and with axes along x-axis and y-axis, respectively, intersect each other in the first quadrant. If the length of the latus rectum of each parabola is 3, find the equation of the common tangent to the two parabolas.

2 A hyperbola passes through the point

$$\mathbf{P} = \begin{pmatrix} \sqrt{2} \\ \sqrt{3} \end{pmatrix} \tag{2.1}$$

and has foci at $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$. Find the equation of the tangent to this hyperbola at $\mathbf{P}$.

3 Find the product of the perpendiculars drawn from the foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 225 \tag{3.1}$$

upon the tangent to it at the point

$$\frac{1}{2} \begin{pmatrix} 3 \\ 5\sqrt{3} \end{pmatrix} \tag{3.2}$$

4 Let P be the point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{4.1}$$

which is at a minimum distance from the centre C of the circle

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 0 & 12 \end{pmatrix} \mathbf{x} = 1 \tag{4.2}$$

Find the equation of the circle passing through C and having its centre at $\left(P\right)$.

5 Consider an ellipse, whose centre is at the origin and its major axis is along the x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then find the area of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse.

6 Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation

$$9e^2 - 18e + 5 = 0 \tag{6.1}$$

If

$$\mathbf{S} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{6.2}$$

is a focus and

$$\begin{pmatrix} 5 & 0 \end{pmatrix} \mathbf{x} = 9 \tag{6.3}$$

is the corresponding directrix of this hyperbola, then find $a^2 - b^2$.

7 A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{7.1}$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 12 \tag{7.2}$$

meets the coordinate axes at $\mathbf{A}$ and $\mathbf{B}$, then find the locus of the midpoint of AB.

Solution: The intersection of the lines in (7.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{7.3}$$

by forming the augmented matrix and row

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reduction as

$$\begin{pmatrix}4 & 3 & 12\\3 & 4 & 12\end{pmatrix} \leftrightarrow \begin{pmatrix}4 & 3 & 12\\0 & 7 & 12\end{pmatrix} \leftrightarrow \begin{pmatrix}28 & 0 & 48\\0 & 7 & 12\end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix}7 & 0 & 12\\0 & 7 & 12\end{pmatrix} \tag{7.4}$$

resulting in

$$\mathbf{C} = \frac{1}{7}\begin{pmatrix}12\\12\end{pmatrix} \tag{7.5}$$

Let the $\mathbf{R}$ be the mid point of AB. Then,

$$\mathbf{A} = 2\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{R} \tag{7.6}$$

$$\mathbf{B} = 2\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{R} \tag{7.7}$$

Let the equation of AB be

$$\mathbf{n}^T(\mathbf{x} - \mathbf{C}) = 0 \tag{7.8}$$

Since $\mathbf{R}$ lies on AB,

$$\mathbf{n}^T(\mathbf{R} - \mathbf{C}) = 0 \tag{7.9}$$

Also,

$$\mathbf{n}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{7.10}$$

Substituting from (7.6) in (7.10),

$$\mathbf{n}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{R} = 0 \tag{7.11}$$

From (7.9) and (7.11),

$$(\mathbf{R} - \mathbf{C}) = k\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{R} \tag{7.12}$$

for some constant k. Multiplying both sides of (7.12) by

$$\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \tag{7.13}$$

$$\begin{aligned}\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}(\mathbf{R} - \mathbf{C}) &= k\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{R}\\ &= k\mathbf{R}^T\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{R} = 0\end{aligned} \tag{7.14}$$

$$\because \mathbf{R}^T\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{R} = 0 \tag{7.15}$$

which can be easily verified for any $\mathbf{R}$. from (7.14),

$$\begin{aligned}&\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}(\mathbf{R} - \mathbf{C}) = 0\\ \implies &\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{R} - \mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{C} = 0\\ \implies &\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{R} - \mathbf{C}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{R} = 0\end{aligned} \tag{7.16}$$

which is the desired locus.