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Global Convergence of the Adaptive PD Controller with Computed Feedforward for Robot Manipulators *

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Abstract

One of the simplest and naturally appealing adaptive tracking controllers for robot manipulators is the so-called adaptive PD controller with computed feedforward. Since its introduction, in the middle of eighties, to the best of author knowledge, only local convergence analyses have been presented. In this paper, based on some properties of the robot dynamics, which deal with continuously differentiable increasing saturating functions, we present the global convergence analysis of this adaptive tracking controller.

Keywords: *Tracking, Robot control, Stability analysis, Lyapunov function, Adaptive control.*

1 Introduction

In this paper we address the tracking control problem in the joint space of robot manipulators, which consists of following a given time–varying trajectory and its derivative. We consider that the robot dynamic parameters are constant but some of them are uncertain. By this, we mean that their exact values are unknown but upper bounds on them are available. To overcome this problem we resort to the adaptive PD control with computed feedforward.

The problem of designing adaptive control laws for rigid robot manipulators that ensure asymptotic trajectory tracking with boundedness of all internal signals is achieved by the so-called adaptive globally convergent controllers [13].

Several schemes for achieving these objectives have been reported in the literature, among them, we can find those proposed by: Craig *et al.* [4], Middleton and Goodwin [12], Slotine and Li [15], Spong and Ortega [16], Bayard and Wen [3], Sadegh and Horowitz [14], Kelly *et al* [7], Whitcomb *et al* [21], a large class of adaptive tracking controllers, presented by Bayard and Wen [3], and recently a nonlinear SP–D adaptive controller with computed feedforward, introduced by Arimoto [2].

Another adaptive tracking controller: the adaptive PD controller with computed feedforward, introduced by Bayard and Wen [3], has very interesting features: It is based on a linear PD feedback, plus an off–line adaptive feedforward computation of the nominal robot dynamics along the desired joint trajectory, there is no need of introducing a so–called sliding surface, as proposed Slotine and Li [15], and there is no need of real–time computation for the regressor matrix. This features make of this controller one of the simplest and most useful adaptive tracking controllers for robot manipulators. Furthermore, this controller has the nice property that it is reduced to adaptive PD control with desired gravity compensation in the particular case of set–point control. To the best of authors knowledge, until now, for this controller only local convergence analyses have been reported [3].

In this paper, our main contribution is to present the global convergence analysis of the adaptive PD controller with computed feedforward for robot manipulators. To this end, and inspired in the residual robot dynamic introduced by Arimoto [1], we present some interesting robot dynamics properties based on continuously differentiable increasing saturation functions. Besides, we give explicit lower bounds for tun-

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ing the proportional and derivative gains of the controller.

Throughout this paper, we use the notation $\lambda_{\rm m}\{A\}$ and $\lambda_{\rm M}\{A\}$ to indicate the smallest and largest eigenvalues, respectively, of a symmetric positive definite bounded matrix $A(\boldsymbol{x})$, for any $\boldsymbol{x} \in \mathbb{R}^n$. The norm of vector $\boldsymbol{x}$ is defined as $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$ and that of matrix A is defined as the corresponding induced norm $\|A\| = \sqrt{\lambda_{\rm M}\{A^TA\}}$.

2 Robot dynamics

In the absence of friction and other disturbances, the dynamics of a serial n-link rigid robot can be written as [17]:

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \qquad (1)$$

where $\boldsymbol{q}$ is the $n \times 1$ vector of joint displacements, $\dot{\boldsymbol{q}}$ is the $n \times 1$ vector of joint velocities, $\boldsymbol{\tau}$ is the $n \times 1$ vector of applied torques, $M(\boldsymbol{q})$ is the $n \times n$ symmetric positive definite manipulator inertia matrix, $C(\boldsymbol{q},\dot{\boldsymbol{q}})$ is the $n \times n$ matrix of centripetal and Coriolis torques, and $\boldsymbol{g}(\boldsymbol{q})$ is the $n \times 1$ vector of gravitational torques obtained as the gradient of the robot potential energy $\mathcal{U}(\boldsymbol{q})$, i.e., $\boldsymbol{g}(\boldsymbol{q}) = \nabla_{\boldsymbol{q}}\mathcal{U}(\boldsymbol{q})$.

We assume that the links are jointed together with revolute joints. Some important properties of dynamics (1) are the following:

Property 1. [11] The matrix $C(\boldsymbol{q},\dot{\boldsymbol{q}})$ and the time derivative $\dot{M}(\boldsymbol{q})$ of the inertia matrix satisfy:

1. $\dot{\boldsymbol{q}}^T\left[\frac{1}{2}\dot{M}(\boldsymbol{q}) - C(\boldsymbol{q},\dot{\boldsymbol{q}})\right]\dot{\boldsymbol{q}} = \boldsymbol{0} \qquad \forall\ \boldsymbol{q},\dot{\boldsymbol{q}} \in \mathbb{R}^n$.
2. $\dot{M}(\boldsymbol{q}) = C(\boldsymbol{q},\dot{\boldsymbol{q}}) + C(\boldsymbol{q},\dot{\boldsymbol{q}})^T \qquad \forall\ \boldsymbol{q},\dot{\boldsymbol{q}} \in \mathbb{R}^n$.

$\diamondsuit$

Property 2. [18], [6] There exist positive constants k_g and k_1 such that

1. $\|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})\| \le k_g\|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$
2. $\|\boldsymbol{g}(\boldsymbol{x})\| \le k_1$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

$\diamondsuit$

Property 3. [8] There exist positive constants k_{C1} and k_{C2} such that for all $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} \in \mathbb{R}^n$ we have:

1. $\|C(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z}\| \le k_{C1}\|\boldsymbol{y}\|\ \|\boldsymbol{z}\|$.
2. $\|C(\boldsymbol{x},\boldsymbol{z})\boldsymbol{w} - C(\boldsymbol{y},\boldsymbol{v})\boldsymbol{w}\| \le k_{C1}\|\boldsymbol{z} - \boldsymbol{v}\|\ \|\boldsymbol{w}\| + k_{C2}\|\boldsymbol{z}\|\ \|\boldsymbol{x} - \boldsymbol{y}\|\ \|\boldsymbol{w}\|$.

$\diamondsuit$

Property 4. [8] There exist positive constant k_M and k_2 such that for all $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} \in \mathbb{R}^n$ we have:

1. $\|M(\boldsymbol{x})\boldsymbol{z} - M(\boldsymbol{y})\boldsymbol{z}\| \le k_M\|\boldsymbol{x} - \boldsymbol{y}\|\ \|\boldsymbol{z}\|$.
2. $\lambda_{\rm M}\{M(\boldsymbol{x})\} \le k_2$.

$\diamondsuit$

Property 5. [10]. For all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^n$ we have:

$$M(\boldsymbol{q})\boldsymbol{u}+C(\boldsymbol{q},\boldsymbol{w})\boldsymbol{v}+\boldsymbol{g}(\boldsymbol{q}) = \Phi(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\boldsymbol{\theta}+\mathcal{K}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$$

where $\mathcal{K}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ is a $n \times 1$ vector, $\Phi(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ is a $n \times p$ matrix and $\boldsymbol{\theta} \in \mathbb{R}^p$ is a parameter vector containing only the interest robot and payload dynamic parameters.

$\diamondsuit$

In this paper we will denote the desired joint trajectory by $\boldsymbol{q}_d(t)$ which is chosen twice continuously differentiable with both of these derivatives bounded by $\|\dot{\boldsymbol{q}}_d\|_{\rm M}$ and $\|\ddot{\boldsymbol{q}}_d\|_{\rm M}$. The position and velocity errors will be denoted by $\tilde{\boldsymbol{q}} = \boldsymbol{q}_d - \boldsymbol{q}$ and $\dot{\tilde{\boldsymbol{q}}}_d = \dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}$.

For the purpose of this paper, it is convenient to state the following definition [9]:

Definition 1. $\mathcal{F}(m,p,\varepsilon,\boldsymbol{x})$ with $p > m > 0$, $\varepsilon > 0$ and $\boldsymbol{x} \in \mathbb{R}^n$ denotes the set of all continuous differentiable increasing functions $\boldsymbol{f}(\boldsymbol{x}) = [f_1(x_1)\ \ f_2(x_2)\ \ \cdots\ \ f_n(x_n)]^T$ such that:

- $p\,|x| \ge |\,f(x)| \qquad \forall\ x \in \mathbb{R}$
- $p\varepsilon \ge |\,f(x)| \qquad \forall\ x \in \mathbb{R}$
- $p\,|x| \ge |\,f(x)| \ge m\,|x| \qquad \forall\ x \in \mathbb{R} : |x| < \varepsilon$
- $p\varepsilon \ge |\,f(x)| \ge m\varepsilon \qquad \forall\ x \in \mathbb{R} : |x| \ge \varepsilon$
- $p \ge \frac{d}{dx}f(x) > 0 \qquad \forall\ x \in \mathbb{R}$

where $|\cdot|$ stands for the absolute value.

$\diamondsuit$

Some examples of this class of functions are: the hyperbolic tangent function $\tanh(x)$, the normalized function $\frac{x}{1+|x|}$ and Arimoto's Sine function [1].

An important property of functions $\boldsymbol{f}(\boldsymbol{x})$ belonging to $\mathcal{F}(m,p,\varepsilon,\boldsymbol{x})$ is now established.

Property 6. The Euclidean norm of $\boldsymbol{f}(\boldsymbol{x})$ satisfies for all $\boldsymbol{x} \in \mathbb{R}^n$:

$$\|\,\boldsymbol{f}(\boldsymbol{x})\| \ge \begin{cases} m\,\|\boldsymbol{x}\| & \text{if } \|\boldsymbol{x}\| < \varepsilon \\ m\varepsilon & \text{if } \|\boldsymbol{x}\| \ge \varepsilon \end{cases}$$

and

$$\|\,\boldsymbol{f}(\boldsymbol{x})\| \le \begin{cases} p\,\|\boldsymbol{x}\| & \forall\ \boldsymbol{x} \in \mathbb{R}^n \\ \sqrt{n}\,p\varepsilon & \forall\ \boldsymbol{x} \in \mathbb{R}^n \end{cases}.$$

$\diamondsuit$

Right away we give some useful lemmas, which we will use to present some additional properties of the robot dynamics. By reasons of paper length we omit their proofs.

Lemma 1: The increasing function $\boldsymbol{f}(\boldsymbol{x}) \in \mathcal{F}(m_f, p_f, \varepsilon_f, \boldsymbol{x})$ satisfies: $\alpha \boldsymbol{f}(\boldsymbol{x}) \in \mathcal{F}(\alpha\, m_f, \alpha\, p_f, \varepsilon_f, \boldsymbol{x})$ with a constant $\alpha > 0$. □

Lemma 2: Consider the increasing functions $\boldsymbol{f}(\boldsymbol{x}) \in \mathcal{F}(m_f, p_f, \varepsilon_f, \boldsymbol{x})$, $\boldsymbol{g}(\boldsymbol{x}) \in \mathcal{F}(m_g, p_g, \varepsilon_g, \boldsymbol{x})$ and $\boldsymbol{h}(\boldsymbol{x}) \in \mathcal{F}(m_h, p_h, \varepsilon_h, \boldsymbol{x})$. Suppose that: $m_h = \sqrt{n}\,[p_f + p_g]$, $p_h = \sqrt{n}\,[p_f + p_g + \delta]$, $\varepsilon_h = \varepsilon_f + \varepsilon_g$, with $\delta > 0$, then the following expressions are satisfied: $\|\boldsymbol{f}(\boldsymbol{x})\| \leq \|\boldsymbol{h}(\boldsymbol{x})\| \quad \forall \quad \boldsymbol{x} \in \mathbb{R}^n$ and $\|\boldsymbol{g}(\boldsymbol{x})\| \leq \|\boldsymbol{h}(\boldsymbol{x})\| \quad \forall \quad \boldsymbol{x} \in \mathbb{R}^n$. □

Now, it is possible to demonstrate, by using lemmas 1 and 2, some additional properties of the robot dynamics (1), which will be useful to show our main stability result.

Property 7. There exist continuous differentiable increasing functions; $\boldsymbol{f}_g(\tilde{\boldsymbol{q}}) \in \mathcal{F}\left(k_g, p_g, \frac{2k_1}{k_g}, \tilde{\boldsymbol{q}}\right)$, $\boldsymbol{f}_M(\tilde{\boldsymbol{q}}) \in \mathcal{F}\left(k_M\|\ddot{\boldsymbol{q}}_d\|_M, p_M, \frac{2k_2}{k_M}, \tilde{\boldsymbol{q}}\right)$, and $\boldsymbol{f}_C(\tilde{\boldsymbol{q}}) \in \mathcal{F}\left(k_{C2}\|\dot{\boldsymbol{q}}_d\|_M^2, p_C, \frac{2k_{C1}}{k_{C2}}, \tilde{\boldsymbol{q}}\right)$, such that for all $\tilde{\boldsymbol{q}} \in \mathbb{R}^n$ we have:

$$\|\boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q})\| \leq \|\boldsymbol{f}_g(\tilde{\boldsymbol{q}})\| \quad \forall \quad \tilde{\boldsymbol{q}} \in \mathbb{R}^n. \tag{2}$$

$$\|[M(\boldsymbol{q}_d) - M(\boldsymbol{q})]\ddot{\boldsymbol{q}}_d\| \leq \|\boldsymbol{f}_M(\tilde{\boldsymbol{q}})\| \quad \forall \quad \tilde{\boldsymbol{q}} \in \mathbb{R}^n. \tag{3}$$

$$\begin{aligned} \|[C(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d) - C(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}}_d\| &\leq k_{C1}\|\dot{\boldsymbol{q}}_d\|_M\|\dot{\tilde{\boldsymbol{q}}}\| \\ &+ \|\boldsymbol{f}_C(\tilde{\boldsymbol{q}})\| \quad \forall \quad \tilde{\boldsymbol{q}} \in \mathbb{R}^n. \end{aligned} \tag{4}$$

◇

In order to be ready to show our main stability result we select the class of continuous differentiable increasing functions

$$\mathbf{sat}(\tilde{\boldsymbol{q}}) = [\mathrm{sat}(q_1)\ \mathrm{sat}(q_2)\ \cdots\ \mathrm{sat}(q_n)]^T \in \mathcal{F}(m, p, \varepsilon, \tilde{\boldsymbol{q}}) \tag{5}$$

where

- $m = \sqrt{n}\,[k_g + k_M\|\ddot{\boldsymbol{q}}_d\|_M + k_{C2}\|\dot{\boldsymbol{q}}_d\|_M^2]$
- $p = m + \delta$ with $\delta > 0$
- $\varepsilon = \frac{2k_1}{k_g} + \frac{2k_2}{k_M} + \frac{2k_{C1}}{k_{C2}}$.

This class of functions, in agreement with Lemma 2, satisfy:

$$\|\boldsymbol{f}_g(\tilde{\boldsymbol{q}})\| \leq \|\mathbf{sat}(\tilde{\boldsymbol{q}})\| \tag{6}$$

$$\|\boldsymbol{f}_M(\tilde{\boldsymbol{q}})\| \leq \|\mathbf{sat}(\tilde{\boldsymbol{q}})\| \tag{7}$$

$$\|\boldsymbol{f}_C(\tilde{\boldsymbol{q}})\| \leq \|\mathbf{sat}(\tilde{\boldsymbol{q}})\| \tag{8}$$

3 Adaptive PD Control with Computed Feedforward: Global Convergence Analysis

In this section we present our main contribution: the global convergence analysis of a robot manipulator system in closed loop with the adaptive PD controller with computed feedforward.

It is well known that the goal of the trajectory tracking control problem is to find $\boldsymbol{\tau}(t)$ such that $\lim_{t\to\infty} \tilde{\boldsymbol{q}}(t) = \boldsymbol{0}$.

Let us consider the PD controller with computed feedforward given by [20]

$$\boldsymbol{\tau}(t) = K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + M(\boldsymbol{q}_d)\ddot{\boldsymbol{q}}_d + C(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d)\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}_d). \tag{9}$$

where K_p is a $n \times n$ diagonal positive definite matrix and K_v is a $n \times n$ positive definite matrix.

In this note we consider that the robot dynamic parameters are constant but some of them are uncertain. By this, we mean that their exact values are unknown but upper bounds on them are available. With this information it is possible to obtain upper bounds on $\lambda_{\mathrm{M}}\{M\}$, k_{C1} and k_g required in the design of the adaptive PD controller with computed feedforward.

It is convenient to remember that the robot dynamics can be linear reparametrized (Property 3) into a parameter vector $\boldsymbol{\theta}^* \in \mathbb{R}^p$ containing only the unknown robot and payload dynamic parameters [10]. In particular, we have

$$\begin{aligned} &M(\boldsymbol{q}_d, \boldsymbol{\theta}^*)\boldsymbol{u} + C(\boldsymbol{q}_d, \boldsymbol{w}, \boldsymbol{\theta}^*)\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{q}_d, \boldsymbol{\theta}^*) = \\ &\Phi(\boldsymbol{q}_d, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\boldsymbol{\theta}^* + M_0(\boldsymbol{q}_d)\boldsymbol{u} + C_0(\boldsymbol{q}_d, \boldsymbol{w})\boldsymbol{v} + \boldsymbol{g}_0(\boldsymbol{q}_d) \end{aligned} \tag{10}$$

where the matrices $M_0(\boldsymbol{q}_d)$, $C_0(\boldsymbol{q}_d, \boldsymbol{w})$ and the vector $\boldsymbol{g}_0(\boldsymbol{q}_d)$ represent the known part of $M(\boldsymbol{q}_d; \boldsymbol{\theta}^*)$, $C(\boldsymbol{q}_d, \boldsymbol{w}; \boldsymbol{\theta}^*)$ and $\boldsymbol{g}(\boldsymbol{q}_d; \boldsymbol{\theta}^*)$ respectively, and $\Phi(\boldsymbol{q}_d, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \mathbb{R}^{n\times p}$ is a regressor matrix which contains known nonlinear functions of $\boldsymbol{q}_d, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

To denote the dependence of the control law (9) on some robot and and payload parameters, let us rewrite it as:

$$\begin{aligned} &\boldsymbol{\tau}(t) = \\ &K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + M(\boldsymbol{q}_d, \boldsymbol{\theta}^*)\ddot{\boldsymbol{q}}_d + C(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \boldsymbol{\theta}^*)\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}_d; \boldsymbol{\theta}^*) \end{aligned} \tag{11}$$

The adaptive version of the control law (11) can be expressed by

$$\begin{aligned} \boldsymbol{\tau}(t) &= K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + \Phi(\boldsymbol{q}_d, \ddot{\boldsymbol{q}}_d, \dot{\boldsymbol{q}}_d, \boldsymbol{q}_d)\hat{\boldsymbol{\theta}} \\ &+ M_0(\boldsymbol{q}_d)\ddot{\boldsymbol{q}}_d + C_0(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d)\dot{\boldsymbol{q}}_d + \boldsymbol{g}_0(\boldsymbol{q}_d) \end{aligned} \tag{12}$$

where $\hat{\theta} \in \mathbb{R}^p$ is the adaptive parameter vector given by the following update law:

$$\hat{\theta}(t) =$$

$$\Gamma \int_0^t \Phi(q_d, \ddot{q}_d, \dot{q}_d, q_d)^T \left[\mathbf{sat}(\tilde{q}(\sigma)) + \dot{\tilde{q}}(\sigma)\right] \, d\sigma + \hat{\theta}(0) \quad (13)$$

where $\Gamma \in \mathbb{R}^{p\times p}$ is the symmetric positive definite adaptation gain matrix, $\mathbf{sat}(\tilde{q})$ is defined in (5) and, $\hat{\theta}(0) \in \mathbb{R}^p$ is any vector, but usually selected in practice as the "best" *a priori* approximation available on the unknown parameter vector θ^*.

Before obtaining the closed loop equation, let us define the parameter error vector as $\tilde{\theta} = \hat{\theta} - \theta^*$. Thus, using (10), with $u = \ddot{q}_d$, $v = \dot{q}_d$ and $w = \dot{q}_d$, the control law (12) can be written as

$$\tau(t) = K_p\tilde{q} + K_v\dot{\tilde{q}} + \Phi(q_d, \ddot{q}_d, \dot{q}_d, q_d)\tilde{\theta}$$

$$+ M(q_d;\theta^*)\ddot{q}_d + C(q_d, \dot{q}_d;\theta^*)\dot{q}_d + g(q_d;\theta^*). \quad (14)$$

In order to make easy the notation, henceforth, we will omit the argument of $\Phi(q_d, \ddot{q}_d, \dot{q}_d, q_d)$.

On the other hand, from definition of $\tilde{\theta}$ and since the parameter vector θ^* was assumed to be constant, hence we have $\dot{\tilde{\theta}} = \dot{\hat{\theta}}$. Using this together with update law (13) and substituting the control action (14) into the robot dynamics, we obtain the following closed loop equation

$$\frac{d}{dt}\begin{bmatrix}\tilde{q}\\ \dot{\tilde{q}}\\ \tilde{\theta}\end{bmatrix} =$$

$$\begin{bmatrix} \dot{\tilde{q}} \\ M(q)^{-1}\left[-K_p\tilde{q} - K_v\dot{\tilde{q}} - C(q,\dot{q})\dot{\tilde{q}} - h(\tilde{q},\dot{\tilde{q}}) - \Phi\tilde{\theta}\right] \\ \Gamma\Phi^T\left[\mathbf{sat}(\tilde{q}) + \dot{\tilde{q}}\right]\end{bmatrix} \quad (15)$$

where $h(\tilde{q},\dot{\tilde{q}})$ denotes the residual robot dynamics introduced by [1], and is given by

$$h(\tilde{q},\dot{\tilde{q}}) = [M(q_d) - M(q)]\ddot{q}_d + [C(q_d,\dot{q}_d) - C(q,\dot{q})]\dot{q}_d$$

$$+ g(q_d) - g(q) \quad (16)$$

Notice that the closed loop system (15) is a nonlinear nonautonomous differential equation whose origin $[\tilde{q}^T \;\; \dot{\tilde{q}}^T \;\; \tilde{\theta}]^T = 0 \in \mathbb{R}^{2n+p}$ is an equilibrium.

To carry out the stability analysis we propose the following Lyapunov function candidate:

$$V(\tilde{q},\dot{\tilde{q}},\tilde{\theta}) = \frac{1}{2}\dot{\tilde{q}}^T M(q)\dot{\tilde{q}}$$

$$+ \frac{1}{2}\tilde{q}^T K_p\tilde{q} + \mathbf{sat}(\tilde{q})^T M(q)\dot{\tilde{q}} + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \quad (17)$$

where $\mathbf{sat}(\tilde{q})$ was defined in (5).

To show that Lyapunov function candidate (17) is a radially unbounded and positive definite function we rewrite it as

$$V(\tilde{q},\dot{\tilde{q}},\tilde{\theta}) = \frac{1}{2}[\dot{\tilde{q}} + \mathbf{sat}(\tilde{q})]^T M(q)[\dot{\tilde{q}} + \mathbf{sat}(\tilde{q})]$$

$$+\frac{1}{2}\tilde{q}^T K_p\tilde{q} - \frac{1}{2}\mathbf{sat}^T(\tilde{q})M(q)\mathbf{sat}(\tilde{q}) + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \quad (18)$$

thus, it will be a radially unbounded positive definite function provided that

$$\frac{1}{2}\tilde{q}^T K_p\tilde{q} - \frac{1}{2}\mathbf{sat}^T(\tilde{q})M(q)\mathbf{sat}(\tilde{q}) \quad (19)$$

is also a radially unbounded positive definite function in $\tilde{q}$.

Since (19) vanishes at $\tilde{q} = 0$, then it only remains to show that it is radially unbounded and positive for all $\tilde{q} \neq 0 \in \mathbb{R}^n$. To this end we use the following expressions, which hold for all $\tilde{q} \in \mathbb{R}^n$:

$$\begin{aligned}\tilde{q}^T K_p\tilde{q} &\geq \lambda_m\{K_p\}\|\tilde{q}\|^2 \\ &\geq \frac{\lambda_m\{K_p\}}{p^2}\|\mathbf{sat}(\tilde{q})\|^2 \quad \text{and}\end{aligned} \quad (20)$$

$$-\frac{1}{2}\mathbf{sat}^T(\tilde{q})M(q)\mathbf{sat}(\tilde{q}) \geq -\lambda_M\{M(q)\}\,\|\mathbf{sat}(\tilde{q})\|^2 \quad (21)$$

where we have used Property 6.

Hence, from (20) and (21), we ensure that (19) is a positive definite function in $\tilde{q}$ provided that

$$\lambda_m\{K_p\} > p^2\lambda_M\{M(q)\} \quad (22)$$

where p was defined in (5).

In sum, we have the Lyapunov function candidate (17), under condition (22) is a radially unbounded and globally positive definite function.

The time derivative of the Lyapunov function candidate (17) along the trajectories of the closed loop equation (15) and using Property 1, can be written as

$$\dot{V}(\tilde{q},\dot{\tilde{q}},\tilde{\theta}) = -\dot{\tilde{q}}^T K_v\dot{\tilde{q}} + \dot{\mathbf{sat}}(\tilde{q})^T M(q)\dot{\tilde{q}}$$

$$-\mathbf{sat}(\tilde{q})^T K_p\tilde{q} - \mathbf{sat}(\tilde{q})^T K_v\dot{\tilde{q}} + \mathbf{sat}(\tilde{q})^T C(q,\dot{q})^T\dot{\tilde{q}}$$

$$-\dot{\tilde{q}}^T h(\tilde{q},\dot{\tilde{q}}) - \mathbf{sat}\tilde{q}^T h(\tilde{q},\dot{\tilde{q}})$$

$$-\dot{\tilde{q}}^T\Phi\tilde{\theta} - \mathbf{sat}^T(\tilde{q})\Phi\tilde{\theta} + \dot{\tilde{\theta}}^T\Gamma^{-1}\tilde{\theta} \quad (23)$$

Now we provide upper bounds on the following terms:

$$\begin{aligned}
\mathbf{sat}(\tilde{q})^T C(q,\dot{q})^T \dot{\tilde{q}} &\leq \sqrt{n}\, k_{C1} p\varepsilon ||\dot{\tilde{q}}||^2 \\
&\quad + k_{C1} ||\dot{q}_d||_{\mathrm{M}} \, ||\mathbf{sat}(\tilde{q})|| \, ||\dot{\tilde{q}}|| && (24)\\
\dot{\mathbf{sat}}(\tilde{q})^T M(q)\dot{\tilde{q}} &\leq p\lambda_{\mathrm{M}}\{M\} ||\dot{\tilde{q}}||^2, && (25)\\
-\dot{\tilde{q}}^T K_v \dot{\tilde{q}} &\leq -\lambda_{\mathrm{m}}\{K_v\} ||\dot{\tilde{q}}||^2, && (26)\\
-\mathbf{sat}(\tilde{q})^T K_v \dot{\tilde{q}} &\leq \lambda_{\mathrm{M}}\{K_v\} ||\mathbf{sat}(\tilde{q})|| \, ||\dot{\tilde{q}}|| && (27)\\
-\mathbf{sat}(\tilde{q})^T K_p \tilde{q} &\leq -\frac{\lambda_{\mathrm{m}}\{K_p\}}{p} ||\mathbf{sat}(\tilde{q})||^2 && (28)\\
-\dot{\tilde{q}}^T h(\tilde{q},\dot{\tilde{q}}) &\leq 3\, ||\dot{\tilde{q}}|| \, ||\mathbf{sat}(\tilde{q})|| \\
&\quad + k_{C1} ||\dot{q}_d||_{\mathrm{M}} ||\dot{\tilde{q}}||^2 && (29)\\
-\mathbf{sat}(\tilde{q})^T h(\tilde{q},\dot{\tilde{q}}) &\leq 3\, ||\mathbf{sat}(\tilde{q})||^2 \\
&\quad + k_{C1} ||\dot{q}_d||_{\mathrm{M}} \, ||\dot{\tilde{q}}|| \, ||\mathbf{sat}(\tilde{q})|| && (30)
\end{aligned}$$

where we have used; properties 3 and 6, for expression (24); equation $\dot{\mathbf{sat}} = F(\tilde{q})\dot{\tilde{q}}$ with $F(\tilde{q})$ being a diagonal matrix whose entries $\partial \mathrm{sat}(\tilde{q}_i)/\partial \tilde{q}_i$ are nonnegative and smaller than p given in (5), for expression (25); Property 6 for expression (28); and finally, property 7, inequalities (6), (7) and (8), for expressions (29) and (30).

From above bounds, and taking the update law (13) it now follows that the time derivative $\dot{V}(\tilde{q},\dot{\tilde{q}},\tilde{\theta})$ in (23) gives

$$\begin{aligned}
&\dot{V}(\tilde{q},\dot{\tilde{q}},\tilde{\theta}) \leq \\
&-\left[\frac{\lambda_{\mathrm{m}}\{K_v\}}{2} - \sqrt{n}\, k_{C1} p\varepsilon - p\lambda_{\mathrm{M}}\{M\} - k_{C1}||\dot{q}_d||_{\mathrm{M}}||\right] ||\dot{\tilde{q}}||^2 \\
&+ \left[2k_{C1}||\dot{q}_d||_{\mathrm{M}} + \lambda_{\mathrm{M}}\{K_v\} + 3\right] ||\dot{\tilde{q}}|| \, ||\mathbf{sat}(\tilde{q})|| \\
&- \frac{\lambda_{\mathrm{m}}\{K_v\}}{2} ||\dot{\tilde{q}}||^2 - \left[\frac{\lambda_{\mathrm{m}}\{K_p\}}{p} - 3\right] ||\mathbf{sat}(\tilde{q})||^2. \qquad (31)
\end{aligned}$$

It only remains to obtain the conditions on K_p and K_v to ensure that $\dot{V}(\tilde{q},\dot{\tilde{q}},\tilde{\theta})$ is a negative semidefinite function. Toward this end, first notice that for all $x, y \in \mathbb{R}$ and strict positive constants a, b, and c, the function

$$a|x|^2 - b|x||y| + c|y|^2 \qquad (32)$$

is positive definite provided that $4ac - b^2 > 0$. Therefore by taking $a = \lambda_{\mathrm{m}}\{K_v\}/2$, $b = 2k_{C1}||\dot{q}_d||_{\mathrm{Max}} + \lambda_{\mathrm{M}}\{K_v\} + 3$ and $c = \frac{\lambda_{\mathrm{m}}\{K_p\}}{p} - 3$, above condition, together with the resulting conditions for the coefficients of (31) be positive, become

$$\lambda_{\mathrm{m}}\{K_v\} > 2\left[\sqrt{n}\, k_{C1} p\varepsilon + p\lambda_{\mathrm{M}}\{M\} + k_{C1}||\dot{q}_d||_{\mathrm{M}}||\right] \qquad (33)$$

$$\lambda_{\mathrm{m}}\{K_p\} > 3p + \frac{p\left[2k_{C1}||\dot{q}_d||_{\mathrm{M}} + \lambda_{\mathrm{M}}\{K_v\} + 3\right]^2}{2\lambda_{\mathrm{m}}\{K_v\}} \qquad (34)$$

where p and ε were defined in (5).

Thus we conclude that $\dot{V}(\tilde{q},\dot{\tilde{q}},\tilde{\theta})$ is a negative semidefinite function, provided that (33) and (34) are satisfied. It is not difficult to show that the Lyapunov function $V(\tilde{q},\dot{\tilde{q}},\tilde{\theta})$ can be upper bounded by a class $\mathcal{K}$ function of the norm of the state vector, and thus, by invoking the Lyapunov's direct method [19] we conclude uniform stability of the equilibrium point $[\tilde{q}^T \ \dot{\tilde{q}}^T \ \tilde{\theta}^T]^T = \mathbf{0} \in \mathbb{R}^{2n}$ for the closed loop equation (15).

Since (31) is a globally negative semidefinite function in the full state and the Lyapunov function (17) is a radially unbounded globally positive definite function, so we can guarantee that $\tilde{q}$, $\dot{q}$ and $\tilde{\theta}$ are bounded vector functions; that is,

$$0 \leq V(\tilde{q}(t), \dot{q}(t), \tilde{\theta}(t)) \leq V(\tilde{q}(0), \dot{q}(0), \tilde{\theta}(0)),$$

implies $\tilde{q}, \dot{q} \in L_\infty^n$, and $\tilde{\theta} \in L_\infty^p$.

By using the following properties: $p||\tilde{q}|| \geq ||\mathbf{sat}(\tilde{q})|| \ \forall\, \tilde{q} \in \mathbb{R}^n$ and $||\frac{d}{dt}\mathbf{sat}(\tilde{q})|| \leq p||\dot{q}|| \ \forall\, \dot{q} \in \mathbb{R}^n$ we can assure that $\mathrm{sat}(\tilde{q})$ and $\frac{d}{dt}\mathrm{sat}(\tilde{q}) \in L_\infty^n$.

Next, it is possible to rearrange (31) to demonstrate, by integrating both sides of the rearranged inequality, that $\dot{\tilde{q}}$ and $\mathbf{sat}(\tilde{q})$ are square integrable vector functions; that is, $\dot{\tilde{q}} \in L_2^n$ and $\mathbf{sat}(\tilde{q}) \in L_2^n$. From this and using the previous conclusion on $\frac{d}{dt}\mathbf{sat}(\tilde{q})$ and $\mathrm{sat}(\tilde{q})$ boundedness we have $\lim_{t\to\infty} \mathrm{sat}(\tilde{q}(t)) = \mathbf{0}$ (a bounded and square integrable function whose derivative is bounded must tend to zero [5]; $\boldsymbol{f} \in L_2^n, \boldsymbol{f}$ and $\dot{\boldsymbol{f}} \in L_\infty^n \Rightarrow \lim_{t\to\infty} \boldsymbol{f}(t) = \mathbf{0} \in \mathbb{R}^n$). Finally owing to $\mathbf{sat}(\tilde{q}) = \mathbf{0} \Leftrightarrow \tilde{q} = \mathbf{0}$ we have $\lim_{t\to\infty} \tilde{q}(t) = \mathbf{0}$, as desired.

In sum we have proven the following:

Proposition. Consider the adaptive tracking controller (12)–(13) with a K_p and K_v satisfying (22), (33) and (34). Then, the equilibrium $\left[\tilde{q}^T \ \dot{\tilde{q}}^T \ \tilde{\theta}^T\right]^T = \mathbf{0} \in \mathbb{R}^{2n+p}$ of the closed-loop system (15) is uniformly stable and $\lim_{t\to\infty} \tilde{q}(t) = \mathbf{0}$. A Lyapunov function to prove it, is given by

$$\begin{aligned}
V(\tilde{q},\dot{\tilde{q}}) &= \frac{1}{2}\dot{\tilde{q}}^T M(q)\dot{\tilde{q}} \\
&+ \frac{1}{2}\tilde{q}^T K_p \tilde{q} + \mathbf{sat}(\tilde{q})^T M(q)\dot{\tilde{q}} + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta}
\end{aligned}$$

where $\mathbf{sat}(\tilde{q})$ was defined in (5).

$\nabla\nabla\nabla$

4 Conclusions

In this paper we have presented the global convergence analysis of one of the simplest controller that may be used for adaptive trajectory tracking control of robot manipulators: the so-called adaptive PD controller with computed feedforward. To this end, we have presented some particular properties of the robot dynamics dealing with continuously differentiable increasing saturating functions. Besides we have provided explicit lower bounds to compute proportional and derivative gains. Obviously the global uniform asymptotic stability of the nonadaptive version of this controller can be easily proved following similar steps to those given in this paper.

References

[1] Arimoto S., "Fundamental problems of robot control: Part I. Innovations in the realm of robot servo-loops". *Robotica*, **13**, pp, 19–27, 1995.

[2] S. Arimoto, T. Naniwa, V. Parra-Vega, and L. L. Whitcomb, " A class of quasi-natural potentials for robot servo-loops and its role in adaptive and learning controls". *Intelligent Automation and Soft Computing*, **1**, No. 1, pp. 85–98, 1995.

[3] D. Bayard and J. T. Wen, "New class of control laws for robotic manipulators. Part 2: Adaptive case". *International Journal of Control*, Vol. 47, No. 5, pp. 1387–1406, 1988.

[4] J. J. Craig, P. Hsu and S. Sastry, "Adaptive control of mechanical manipulators". *Proc. IEEE International Conference on Robotics and Automation*, San Francisco, CA, pp. 190–195, 1986.

[5] C. A. Desoer and M. Vidyasagar, *Feedback systems: Input–output properties*, Academic Press, 1975.

[6] R. Gunawardana and F. Ghorbel, "The class of robot manipulators with bounded Jacobian of the gravity vector". *Proceeding of the 1996 IEEE International Conference on Robotics and Automation*, Minneapolis, MN., April 1996, pp. 3677–3682.

[7] R. Kelly, R. Carelli and R. Ortega, "Adaptive motion control design of robot manipulators: an input–output approach". *International Journal of Control*, Vol. 50, No. 6, pp. 2563–2581, 1989.

[8] Kelly R., and R. Salgado, "PD control with computed feedforward of robot manipulators: A design procedure". *Trans. Robotics and Automn*, Vol. 10, No.4, pp. 566-571, 1994.

[9] R. Kelly, "Global positioning on robot manipulators via PD control plus a class of nonlinear integral actions". *IEEE Trans. on Aut. Contr.*, Vol. 43, No. 7, pp. 934–938, 1998.

[10] P. Khosla P. and T. Kanade, "Parameter identification of robot dynamics". *Proc. IEEE Conf. on Decision and Control*, F. Lauderdale, FL., 1985.

[11] D. Koditschek, "Natural motion for robot arms". In *Proc. of the 1984 IEEE Conf. on Decision and Control*, Las Vegas CA., pp. 733-735, 1984.

[12] R. H. Middleton and G. Goodwin, "Adaptive computed torque control for rigid links manipulators", *Proc. 25th Conference on Decision and Control*, Athens, Greece, pp. 68–73, 1986.

[13] R. Ortega and M.Spong. "Adaptive motion control of rigid robots: A tutorial". *Automatica*, Vol. 25, No.6 pp. 877–888, 1989.

[14] N. Sadegh and R. Horowitz, 'An exponential stable adaptive control law for robot manipulators', *IEEE Transactions on Robotics and Automation*, Vol. 6, No.4, pp. 491–496, 1990.

[15] J. J. Slotine and W. Li, "On the adaptive control of robot manipulators". *The Int. J. of Robotics Research*, Vol. 6, No. 3, pp. 49–59, 1987.

[16] M. Spong and R. Ortega, "On adaptive inverse dynamics control of rigid robots". *IEEE Transactions on Automatic Control* Vol. 35, pp. 92–95, 1990.

[17] Spong M. and M. Vidyasagar, *Robot Dynamics and Control.* John Wiley and Sons, New York, 1989.

[18] P. Tomei, "Adaptive PD controller for robot manipulators". *IEEE Trans. Robotics Automn*, Vol. 7, No.4, pp. 565-57, 1991.

[19] M. Vidyasagar, *Nonlinear systems analysis.* Prentice Hall, Englewood Cliffs, NJ., 1993.

[20] J. T. Wen and D. S. Bayard, "New class of control laws for robotic manipulators. Part 1: Non–adaptive case", *International Journal of Control*, Vol. 47, No. 5, 1988.

[21] L. L. Whitcomb, A. A. Rizzi and D.E. Koditschek, "Comparative experiments with a new adaptive controller for robot arms", *IEEE Trans. on Robotics and Automation*, Vol. 9, No. 1, pp. 59-70, 1993.