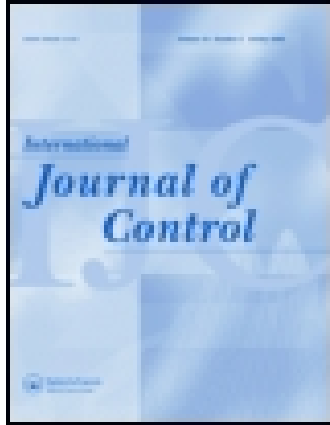


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DAVID S. BAYARD<sup>a</sup> & JOHN T. WEN<sup>a</sup>

<sup>a</sup> Jet Propulsion Laboratory California Institute of Technology, Pasadena, CA 91109, U.S.A.

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## New class of control laws for robotic manipulators

### Part 2. Adaptive case

DAVID S. BAYARD<sup>†</sup> and JOHN T. WEN<sup>†</sup>

A new class of asymptotically stable adaptive control laws is introduced for application to the robotic manipulator. Unlike most applications of adaptive control theory to robotic manipulators, this analysis addresses the non-linear dynamics directly without approximation, linearization, or *ad hoc* assumptions, and utilizes a parameterization based on physical (time-invariant) quantities. This approach is made possible by using energy-like Lyapunov functions that retain the non-linear character and structure of the dynamics, rather than simple quadratic forms, ubiquitous in adaptive control literature, which have bound the theory tightly to linear systems with unknown parameters. It is a unique feature of these results that the adaptive forms arise by straightforward certainty equivalence adaptation of their non-adaptive counterparts found in Wen and Bayard (1988)—i.e. by replacing unknown quantities by their estimates—and that this simple approach leads to asymptotically stable closed-loop adaptive systems. Furthermore, it is emphasized that this approach does not require convergence of the parameter estimates (i.e. via persistent excitation), invertibility of the mass matrix estimate, or measurement of joint accelerations.

#### 1. Introduction

In the past, many papers have appeared on the application of adaptive control theory to robotic manipulators (cf. Dubowsky and Desforges 1979, Takegaki and Arimoto 1981, Koivo and Guo 1983, Tomizuka and Horowitz 1983, Lee and Chung 1984, Lim and Eslami 1985, Hsia 1986 for an overview). It is a general property of adaptive designs based on Lyapunov's direct method that the Lyapunov function is chosen as a simple quadratic type, well known and well studied in the standard adaptive control literature (Landau 1974, Narendra and Monopoli 1980). However, this particular Lyapunov function was originally motivated for applications to the standard adaptive control problems (i.e. linear systems with unknown parameters), and not for non-linear dynamical systems. Hence, applications of standard adaptive control techniques to robotic manipulators invariably require the dynamics to be considered as linear. This, in turn, requires the use of *ad hoc* assumptions and/or analysis techniques including

- (i) the treatment of position dependent quantities as unknown constants, for which they must be assumed to vary slowly with time;
- (ii) the linearization of the system about some local operating point, valid only for small excursions from the nominal;
- (iii) the use of linear decoupled models for the links, which neglects non-linearities and cross-coupling the effects; and

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<sup>†</sup> Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109, U.S.A.

- (iv) the complete neglect of the non-linear and time-varying dynamics by assuming the plant to be linear.

Hence, stability results based on these assumptions are questionable, and a rigorous proof of adaptive control stability for robotic manipulators remains unresolved.

A recent exception to the above criticism is due to the work of Craig *et al.* (1986). Here, a useful 'linear-in-the-parameters' formulation is exploited to simplify the analysis, and to demonstrate global convergence of an adaptive version of the computed-torque control law, without approximation to the non-linear dynamics. However, the resulting adaptive controller requires the invertibility of the mass matrix estimate (which is not guaranteed *a priori*), and measurement of the joint accelerations (which is generally unavailable). It is suggested by Craig *et al.* (1986) that the former can be handled by projecting parameter estimates into known regions of parameter space for which the mass matrix inverse exists, and in which the true parameters are required to lie. However, knowledge and calculation of such regions is not straightforward and appears to be a weakness of the method.

In this paper, the 'linear-in-parameters' formulation of Craig *et al.* (1986) is used in conjunction with a different Lyapunov function. Here, the choice of Lyapunov function is more closely related to the energy of the system, and better retains the non-linear structure and character of the dynamics. In addition, many problems associated with adapting the computed-torque control law directly are avoided by making use of the new class of exponentially stabilizing controllers introduced by Wen and Bayard (1988). Although these controllers are very similar in form to the computed torque method, they have many advantages in the non-adaptive case and have the unique property that they can be made adaptive by using a straightforward certainty equivalence approach (i.e. by replacing unknown quantities by their on-line estimates). Furthermore, the class of adaptive systems defined in this manner can be shown to be asymptotically stable without approximation to the non-linear manipulator dynamics. This approach does not require convergence of parameter estimates (i.e. via persistent excitation), invertibility of the mass matrix estimate, or measurement of joint accelerations.

In the most recent literature (i.e. preprints, conference papers, etc.) there appears to be other work currently taking place that combines the linear-in-parameters formulation with a new Lyapunov function (Paden 1986, Slotine and Li 1986). Although this work is very new and is evolving very rapidly, we will try to contrast our results where possible, and provide an overall perspective.

The format of the paper is as follows. In § 2 the results of Wen and Bayard (1988) are reviewed and summarized as required for treatment of the adaptive control case. In § 3, asymptotic stability is proved for the class of systems arising from certainty equivalence adaptation of the control laws in Wen and Bayard (1988). In § 4, several remarks are made pertinent to the new adaptive designs, and conclusions are given in § 5.

## 2. Background and notation

### 2.1. Manipulator dynamics

The well-known Lagrange–Euler equations of motion for the  $n$ -joint manipulator are given as follows

$$\dot{q}_1 = q_2 \quad (2.1)$$

$$M(q_1)\dot{q}_2 = -C(q_1, q_2) - k(q_1) + u \quad (2.2)$$

where

$$C(q_1, q_2) = \sum_{i=1}^n \{[e_i q_2^T M_i(q_1)]^T - \frac{1}{2}[e_i q_2^T M_i(q_1)]\} \quad (2.3)$$

$e_i \triangleq$   $i$ th unit vector

$$M_i(q_1) = \frac{\partial M(q_1)}{\partial q_{1i}}; \quad q_{1i} \triangleq \text{ith component of } q_1$$

$k(q_1) \triangleq$  gravity load

Here,  $u \in R^n$  is a generalized torque vector,  $q_1, q_2, \dot{q}_2 \in R^n$  are generalized joint position, velocity and acceleration vectors (e.g.  $q_1$  is an angle or a distance for a revolute or prismatic joint, respectively),  $M(q_1) \in R^{n \times n}$  is the symmetric positive-definite mass inertia matrix;  $C(q_1, q_2) \in R^n$  is the coriolis and centrifugal force vector; and  $k(q_1) \in R^n$  is the gravitational load vector.

## 2.2. Some useful identities

The following notation is defined:

$$M_D(q_1, z) = \sum_{i=1}^n M_i(q_1) z e_i^T$$

$$\dot{M}(q_1, q_2) = \frac{d}{dt} M(q_1) = \sum_{i=1}^n M_i(q_1) e_i^T q_2$$

$$J(q_1, z) = \sum_{i=1}^n \{[e_i z^T M_i(q_1)] - [e_i z^T M_i(q_1)]^T\}$$

$$\Delta q_1 \triangleq q_1 - q_{1d}, \quad \Delta q_2 \triangleq q_2 - q_{2d}$$

$q_{1d}, q_{2d} \triangleq$  desired joint position and velocities, respectively ( $q_{2d} = \dot{q}_{1d}$ )

$$r(q_1, q_2, q_{2d}) = \Delta q_2^T [\frac{1}{2} M(q_1, q_2) \Delta q_2 - C(q_1, q_2) q_2]$$

Using the above notation, the following identities are quoted from Wen and Bayard (1988) without proof. In these identities,  $x$ ,  $y$  and  $z$  are used to denote arbitrary vectors of appropriate dimension.

### Identity 1

$$\dot{M}(q_1, q_2) z = M_D(q_1, z) q_2$$

### Identity 2

$$C(q_1, z) z = \frac{1}{2} [M_D(q_1, z) - J(q_1, z)] z$$

### Identity 3

$$J(q_1, z) = M_D^T(q_1, z) - M_D(q_1, z)$$

## 2.3. Important lemma

In this section, a useful lemma is reviewed, quoted directly without proof from Wen and Bayard (1988). For convenience, this result will be alternatively referred to as the  $\beta$ -Ball Lemma, owing to the method used to prove it.

**Lemma 2.1:  $\beta$ -Ball lemma**

Given a dynamical system

$$\dot{x}_i = f_i(x_1, \dots, x_N, t), \quad x_i \in \mathbb{R}^{n_i}, \quad t \geq 0, \quad i = 1, \dots, N$$

let  $f_i$  values be locally Lipschitz with respect to  $x_1, \dots, x_N$  uniformly in  $t$  on bounded intervals and continuous in  $t$  for  $t \geq 0$ . Suppose a function  $V: \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is given such that

$$(a) \quad V(x_1, \dots, x_N, t) = \sum_{i,j=1}^N x_i^T P_{ij}(x_1, \dots, x_N, t) x_j$$

where for each  $i = 1, \dots, N$  there exists  $\xi_i > 0$  such that

$$\xi_i \|x_i\|^2 \leq V(x_1, \dots, x_N, t) \quad (2.4)$$

$$(b) \quad \dot{V}(x_1, \dots, x_N, t) \leq - \sum_{i \in I_1} \left( \alpha_i - \sum_{j \in I_{2i}} \gamma_{ij} \|x_j(t)\|^{k_{ij}} \right) \|x_i(t)\|^2 \quad (2.5)$$

where  $\alpha_i, \gamma_{ij}, k_{ij} > 0$ ,  $I_{2i} \subset I_1 \subset \{1, \dots, N\}$ .

Let  $V_0 \triangleq V(x_1(0), \dots, x_N(0), 0)$ . If for all  $i \in I_1$ ,

$$\alpha_i > \sum_{j \in I_{2i}} \gamma_{ij} \left( \frac{V_0}{\xi_j} \right)^{k_{ij}/2} \quad (2.6)$$

then for all

$$\lambda_i \in \left( 0, \alpha_i - \sum_{j \in I_{2i}} \gamma_{ij} \left( \frac{V_0}{\xi_j} \right)^{k_{ij}/2} \right)$$

the following inequality holds

$$\dot{V}(x_1, \dots, x_N, t) \leq - \sum_{i \in I_1} \lambda_i \|x_i\|^2, \quad \forall t \geq 0$$

Wen and Bayard (1988) introduce various new exponentially stabilizing compensators for both the set-point and tracking control problems. For the purposes of adaptive control, it is of interest to consider the subset of this class summarized in Table 1. In addition, the well-known computed torque control has also been included in Table 1 for comparison purposes. It is noted that the desired potential field  $U^*(\Delta q_1)$  used in Wen and Bayard (1988) has been chosen here simply as

$$U^*(\Delta q_1) = \frac{1}{2} \Delta q_1^T K_p \Delta q_1 \quad (2.7)$$

so as not to obscure the presentation with additional obstacle-avoidance objectives. Nevertheless, many of the adaptive control results presented herein are easily extended to the more general case:

It is useful to observe that all the Control Laws (1)–(7) differ from the computed torque method in that the mass matrix  $M(q_1)$  does not premultiply the position and velocity feedback gains  $K_p$  and  $K_v$ , respectively. This property is critical since it renders this entire class of control laws amenable to simple adaptation schemes (i.e. certainty equivalence adaptation) that can be shown to lead to desired asymptotic stability properties. The presence of the mass matrix premultiplier otherwise prevents simple cancellations in the Lyapunov function derivative, hindering most attempts to

	Conditions for stability†	Cross-reference Wen and Bayard (1988)‡
Computed torque control law $u = -M(q_1)(K_p \Delta q_1 + K_v \Delta \dot{q}_2) + k(q_1) + M(q_1)\dot{q}_{2d} + C(q_1, q_2)q_2$	None required	(2.34)
New exponentially stable control laws		
(1) $u = -K_p \Delta q_1 - K_v \Delta \dot{q}_2 + k(q_1) + M(q_1)\dot{q}_{2d} - \frac{1}{2}J(q_1, q_2)q_{2d} + \frac{1}{2}M_D(q_1, q_{2d})q_2$	None required	(4.2 a)
(2) $u = -K_p \Delta q_1 - K_v \Delta \dot{q}_2 + k(q_1) + M(q_1)\dot{q}_{2d} - \frac{1}{2}J(q_1, q_{2d})q_2 + \frac{1}{2}M_D(q_1, q_2)q_{2d}$	None required	(4.2 b)
(3) $u = -K_p \Delta q_1 - K_v \Delta \dot{q}_2 + k(q_1) + M(q_1)\dot{q}_{2d} - \frac{1}{2}J(q_1, q_{2d})q_{2d} + \frac{1}{2}M_D(q_1, q_2)q_{2d}$	None required	(4.2 c)
(4) $u = -K_p \Delta q_1 - K_v \Delta \dot{q}_2 + k(q_1) + M(q_1)\dot{q}_{2d} - \frac{1}{2}J(q_1, q_2)q_2 + \frac{1}{2}M_D(q_1, q_{2d})q_2$	None required	(4.2 d)
(5) $u = -K_p \Delta q_1 - K_v \Delta \dot{q}_2 + k(q_1) + M(q_1)\dot{q}_{2d} + C(q_1, q_{2d})q_{2d}$	$\sigma_{\min}(K_v) > \frac{\eta_2}{2}$	(4.6)
(6) $u = -K_p \Delta q_1 - K_v \Delta \dot{q}_2 + k(q_1) + M(q_1)\dot{q}_{2d} + C(q_1, q_2)q_2$	$\sigma_{\min}(K_v)$ sufficiently large w.r.t. initial condition	(4.7)
(7) $u = -K_p \Delta q_1 - K_v \Delta \dot{q}_2 + k(q_{1d}) + M(q_{1d})\dot{q}_{2d} + C(q_{1d}, q_{2d})q_{2d}$	$\sigma_{\min}(K_v)$ sufficiently large w.r.t. initial condition	(4.8)

† General assumptions:  $\|q_{1d}\|, \|q_{2d}\|, \|\dot{q}_{2d}\|$  bounded;  $K_v = K_v^T > 0, K_p = K_p^T > 0$ .

‡ Let  $U^*(\Delta q_1) \triangleq \frac{1}{2}\Delta q_1^T K_p \Delta q_1$ .

Table 1.

apply adaptive control directly to the non-linear dynamic manipulator equations. A recent exception to this can be found in the work of Craig *et al.* (1986). However, the resulting adaptation law requires that the estimated mass matrix be invertible for all values of estimated parameters. This in turn requires on-line projections of parameter estimates into prespecified bounded regions of parameter space where  $\hat{M}(q_1)$  is not only invertible, but where the true parameters are certain to lie. This approach not only requires tight bounds on parameter uncertainty, but involves a very difficult (albeit off-line) determination of the proper parameter projection domains. This problem is further exacerbated by the fact that the adaptation law is not parameterized by physical parameters, and is of the form where the transformation back to physical parameters is neither straightforward nor unique. These problems are overcome in this paper by using the exponentially stabilizing control laws of Table 1, that do not involve a premultiplying mass matrix on the feedback gains.

In the non-adaptive case, comparisons between the new control laws of Table 1 and the computed torque method can be found in Wen and Bayard (1988). Nevertheless, a brief account is in order here. In particular, Control Laws (1), (2), (3), (4) are roughly 'on a par' with the computed torque method in the non-adaptive case, guaranteeing exponential stability with no conditions on  $K_p$  or  $K_v$ . Unlike the computed torque method, however, they are not in a form suitable for application of the recursive Newton–Euler computation technique. This presently appears to be their major disadvantage. In order to overcome this difficulty, Control Laws (5), (6) and (7) were developed in a form suitable for recursive Newton–Euler computation. Relative to the computed torque method, Control Law (5) utilizes the desired velocity signal  $q_{2d}$  in place of the measured velocity  $q_2$  in the non-linear terms of the controller. This 'cleans up' the feedback signal in the sense that non-idealities due to sensor dynamics and measurement noise in  $q_2$  are avoided in the non-linear feedback terms. Control Law (7) further replaces  $q_1$  in  $K$ ,  $M$  and  $C$  by  $q_{1d}$ . This decouples the non-linear terms from real-time measurements, which completely removes the requirement for on-line computation of non-linear terms in the controller implementation. Control Law (6) is exactly the computed torque method without the premultiplying mass matrix term described earlier. The advantages of these controllers are off-set slightly by the conditions imposed on  $K_p$  and  $K_v$  for guaranteeing asymptotic stability, i.e. that  $K_v$  be chosen sufficiently large for Control Laws (1), (2), (3), (4), (5), (6) and that both  $K_v$  and  $K_p$  be chosen sufficiently large for Control Law (7). It will be seen in the adaptive case that these requirements can be removed by adapting these feedback gains appropriately.

The use of  $q_{2d}$  rather than  $q_2$  in many of the new control laws offers additional advantages. In particular, in the set-point control application  $q_{2d} = \dot{q}_{2d} = 0$ . Hence, there is considerable simplification in the control laws relative to the computed torque method, i.e. the non-linear terms vanish from the control law. This simplification carries over directly to the adaptive case, and provides substantial simplification in set-point control relative to the recent adaptive control laws of Slotine and Li (1986) and Paden (1986).

### 3. New class of asymptotically stable adaptive control laws

All of the new exponentially stabilizing control laws summarized in Table 1 have the unique property that they can be adapted in real-time so as to yield asymptotically stable adaptive control systems. Furthermore, the adaptation is done in a certainty equivalence fashion, i.e. simply by relacing unknown quantities in the control laws by

their estimates—as generated by an appropriate parameter adaptation algorithm. In this section, asymptotic stability for the various control laws will be proved, and the proper mechanisms for parameter adaptation will be derived. At this point, the reader may wish to glance at Table 2, for an overview of the results which will follow. In particular, in § 3.2 a detailed stability proof will be given for the adaptation of Control Law (1). This analysis then extends easily to Control Laws (2), (3) and (4). In § 3.3 the discussion will focus on the adaptation of Control Laws (5) and (6), and in § 3.4 the emphasis will be on the adaptation of Control Law (7).

The simplicity in structure of the adaptive control schemes presented here is largely due to a 'linear-in-parameters' formulation of the problem. This particular parameterization has become increasingly popular in recent literature (cf. Paden 1986, Atkinson *et al.* 1985, Khosia and Kanade 1985, Craig *et al.* 1986) and will be discussed in more detail below.

### 3.1. Linear in the parameters formulation

A useful parameterization of the non-linear dynamical equations arises by noting the following relations ( $x$ ,  $y$  and  $z$  arbitrary vectors)

$$C(x, y)y = H_C(x, y)\theta_C$$

$$M(x)z = H_M(x, z)\theta_M$$

$$k(x) = H_k(x)\theta_k$$

$$M_D(x, y)y = H_D(x, y)\theta_D$$

where  $H_C$ ,  $H_M$ ,  $H_k$  and  $H_D$  are known matrix functions of  $x$ ,  $y$  and  $z$ , and where  $\theta_C$ ,  $\theta_M$ ,  $\theta_k$  and  $\theta_D$  are vectors of constant parameters related directly to true physical parameters (masses, inertias, link lengths, centres of gravity, etc.). It is emphasized that this parameterization does not contain any hidden 'slowly varying' states in the parameter vector definition and does not require any linearization of the dynamical equations of motion.

### 3.2. Global asymptotic stability for adaptation of control laws (1), (2), (3), (4)

In this section, global asymptotic stability is proved for adaptation of Control Laws (1), (2), (3) and (4). In order to avoid redundant analysis, the details of the proof will be considered only for Control Law (1), and the extension to the other control laws will follow immediately by taking advantage of the unified treatment of these control laws given by Wen and Bayard (1988).

#### 3.2.1. Asymptotic stability. Consider Control Law (1)

$$u^0 = -K_p \Delta q_1 - K_v \Delta \dot{q}_2 + k(q_1) + M(q_1) \dot{q}_{2d} - \frac{1}{2} J(q_1, q_2) q_{2d} + \frac{1}{2} M_D(q_1, q_{2d}) q_2 \quad (3.1)$$

Here, superscript 'o' is used to denote the ideal non-adaptive control law, i.e. the completely 'tuned' control law which would be used if the parameters were known exactly. Using the linear-in-parameters formulation discussed in § 3.1 there exists a matrix  $H_1(q_1, q_2, q_{2d}, \dot{q}_{2d})$  and a vector of parameters  $\theta$  such that

$$u^0 = -K_p \Delta q_1 - K_v \Delta \dot{q}_2 + H_1 \theta \quad (3.2)$$



where

$$H_1 \theta \triangleq M(q_1) \dot{q}_{2d} - \frac{1}{2} J(q_1, q_2) q_{2d} + \frac{1}{2} M_D(q_1, q_{2d}) q_2 \quad (3.3)$$

Here, the parameters in  $\theta$  are constant with time and are related directly to physical link and payload parameters. When these parameters are unknown, the parameter vector  $\theta$  is replaced by its estimate  $\hat{\theta}(t)$  in real-time to give the following adaptive control law

$$u = -K_p \Delta q_1 - K_v \Delta q_2 + H_1 \hat{\theta} \quad (3.4)$$

Subtracting (3.2) from (3.4) and rearranging gives

$$u = u^o + H_1 (\hat{\theta} - \theta) \triangleq u^o + H_1 \phi \quad (3.5)$$

This is an important relation since it shows that the adaptive control is equal to the non-adaptive control plus an expression that is linear in the parameter error  $\phi \triangleq \hat{\theta} - \theta$ .

The proof of stability then follows by choosing the following Lyapunov function

$$V = V^o + \frac{1}{2} \phi^T \Gamma \phi, \quad \Gamma = \Gamma^T > 0 \quad (3.6 a)$$

where  $V^o$  is the Lyapunov function for the non-adaptive control law used by Wen and Bayard (1988), and where  $\phi^T \Gamma \phi$  is a positive-definite function in the parameter error  $\phi$ . For completeness,  $V^o$  is rewritten here (cf. (4.4) in Wen and Bayard 1988) where  $U^*(\Delta q_1) \triangleq \frac{1}{2} \Delta q_1^T K_p \Delta q_1$ :

$$V^o = \frac{1}{2} \Delta q_2^T M(q_1) \Delta q_2 + \frac{1}{2} \Delta q_1^T (K_p + cK_v) \Delta q_1 + c \Delta q_1^T M(q_1) \Delta q_2 \quad (3.6 b)$$

Taking the derivative of  $V$  along system trajectories and substituting control law (3.5) gives upon rearranging

$$\dot{V} = \dot{V}^o + (\Delta q_2 + c \Delta q_1)^T H_1 \phi + \dot{\phi}^T \Gamma \phi \quad (3.7)$$

where  $\dot{V}^o$  is the Lyapunov function derivative for the non-adaptive case, and where the additional terms involving  $\phi$  on the right-hand side of (3.7) arise directly from the additional terms involving  $\phi$  in the control law (3.5) and the Lyapunov function (3.6), respectively.

The second and third terms of (3.7) are cancelled exactly by the choice of adaptation law

$$\dot{\phi} = \hat{\dot{\theta}} = -\Gamma^{-1} H_1^T (\Delta q_2 + c \Delta q_1) \quad (3.8)$$

The expression for the remaining term  $\dot{V}^o$  is simply taken from Wen and Bayard (1988) as (see (4.5) where  $v \triangleq \sigma_{\min}(K_p)$ ); also note that Control Law (1) corresponds to case (4.2 b) for which  $a = 3/2$ )

$$\dot{V} = \dot{V}^o = -\alpha_1 \|\Delta q_1\|^2 - \alpha_2 \|\Delta q_2\|^2 + \gamma_{21} \|\Delta q_1\| \|\Delta q_2\|^2 \quad (3.9)$$

where

$$\left. \begin{aligned} \alpha_1 &= c(\sigma_{\min}(K_p) - \frac{3}{4} \eta_2 \rho^2) \\ \alpha_2 &= \sigma_{\min}(K_v) - c \left( \mu + \frac{3\eta_2}{4\rho^2} \right) \\ \gamma_{21} &= \frac{c}{2} \eta_1 \end{aligned} \right\} 0 < c < l^2 \text{ arbitrary} \quad (3.10 a)$$

$$(3.10 b)$$

$$(3.10 c)$$

$$\eta_2 = \max_{q_{2d}} \|q_{2d}\| \eta_1 \quad (3.11 a)$$

$$\eta_1 = \max_{q_1} \left( \sum_i \|M_i(q_1)\| \right) \quad (3.11 b)$$

$$\mu = \max_{q_1} \|M(q_1)\| \quad (3.11 c)$$

Applying the  $\beta$ -ball argument of Lemma 2.1 to (3.9) using the values of  $\alpha_1, \alpha_2$  and  $\gamma_{21}$  given in (3.10), it follows that if

$$\sigma_{\min}(K_p) > \frac{3}{4} \eta_2 \rho^2 \quad (3.12 a)$$

$$\sigma_{\min}(K_v) > c \left( \mu + \frac{3}{4} \frac{\eta_2}{\rho^2} + \frac{\eta_1}{2} \left( \frac{V_0}{\xi_1} \right)^{1/2} \right) \quad (3.12 b)$$

then,

$$\dot{V} \leq -\lambda_1 \|\Delta q_1\|^2 - \lambda_2 \|\Delta q_2\|^2 \quad (3.13)$$

for any  $\lambda_1$  and  $\lambda_2$  such that

$$\lambda_1 \in (0, c(\sigma_{\min}(K_p) - \frac{3}{4} \eta_2 \rho^2)) \quad (3.14 a)$$

$$\lambda_2 \in \left( 0, \sigma_{\min}(K_v) - c \left[ \mu + \frac{3}{4} \frac{\eta_2}{\rho^2} + \frac{\eta_1}{2} \left( \frac{V_0}{\xi_1} \right)^{1/2} \right] \right) \quad (3.14 b)$$

where

$$V_0 = V|_{t=0} \quad (3.15)$$

$$\xi_1 = \frac{1}{2} [\sigma_{\min}(K_p + cK_v) - l^2 c \sigma(M)] \quad (3.16 a)$$

$$\xi_2 = \frac{1}{2} \left( l - \frac{c}{l^2} \right) \sigma(M) \quad (3.16 b)$$

$$\sigma(M) \triangleq \min_{q_1} \sigma_{\min}(M(q_1))$$

Since  $\rho^2$  is arbitrary, it can be chosen sufficiently small so that (3.12 a) is satisfied. With this choice of  $\rho^2$ , the value of  $c$  in (3.12 b) can be chosen sufficiently small so that inequality (3.12 b) is satisfied. Hence (3.13) follows. This is essentially the same proof of stability as in the non-adaptive case (cf. Theorem 4.1 in Wen and Bayard 1988) with the following exceptions.

- (i) The value of  $V_0$  in (3.12 b) and (3.14 b) now includes the initial parameter error  $\frac{1}{2} \phi^T(0) \Gamma \phi(0)$ .
- (ii) The value of  $c$  is now required for implementation of the parameter adaptation law (3.8).
- (iii)  $\dot{V}$  in (3.13) is now only negative semidefinite in the state since the full state vector in the adaptive case is augmented by  $\phi$ .

It is noted that property (iii) destroys the simple exponential stability argument used earlier in the non-adaptive case (cf. Theorem 4.1 in Wen and Bayard 1988) to insure asymptotic convergence of  $\|\Delta q_1\|$  and  $\|\Delta q_2\|$ . In addition since the error system in  $(\Delta q_1, \Delta q_2, \phi)$  is non-autonomous (and in general, non-periodic), standard in-

Case no.	Adaptive control	Linear-in-parameters expression	Parameter adaptation	Stability condition†
(1 a)	$u = -K_p \Delta q_1 - K_v \Delta q_2 + H_1 \dot{\theta}$	$H_1 \theta = k(q_1) + M(q_1) \ddot{q}_{2d} - \frac{1}{2} J(q_1, q_2) \ddot{q}_{2d} + \frac{1}{2} M_D(q_1, q_{2d}) \ddot{q}_2$	$\dot{\hat{\theta}} = -\Gamma^{-1} H_1^T (\Delta q_2 + c \Delta q_1)$	$\sigma_{\min}(K_v)$ sufficiently large or $c$ sufficiently small (see (3.12)) Asymptotically stable‡
(1 b)	$u = -K_p \Delta q_1 - \hat{K}_v \Delta q_2 + H_1 \dot{\hat{\theta}}$	$H_1 \theta = \text{same as (1 a) above}$	$\dot{\hat{\theta}} = -\Gamma^{-1} H_1^T (\Delta q_2 + c \Delta q_1)$ $\dot{\hat{K}}_v = \frac{1}{\delta} (\Delta q_2 + c \Delta q_1) \Delta q_2^T$	None required Globally asymptotically stable‡
(2 a)	$u = -K_p \Delta q_1 - K_v \Delta q_2 + H_2 \dot{\theta}$	$H_2 \theta = k(q_1) + M(q_1) \ddot{q}_{2d} - \frac{1}{2} J(q_1, q_{2d}) \ddot{q}_2 + \frac{1}{2} M_D(q_1, q_2) \ddot{q}_{2d}$	$\dot{\hat{\theta}} = -\Gamma^{-1} H_2^T (\Delta q_2 + c \Delta q_1)$	$\sigma_{\min}(K_v)$ sufficiently large or $c$ sufficiently small Asymptotically stable‡
(2 b)	$u = -K_p \Delta q_1 - \hat{K}_v \Delta q_2 + H_2 \dot{\hat{\theta}}$	$H_2 \theta = \text{same as (2 a) above}$	$\dot{\hat{\theta}} = -\Gamma^{-1} H_2^T (\Delta q_2 + c \Delta q_1)$ $\dot{\hat{K}}_v = \frac{1}{\delta} (\Delta q_2 + c \Delta q_1) \Delta q_2^T$	None required Globally asymptotically stable‡
(3 a)	$u = -K_p \Delta q_1 - K_v \Delta q_2 + H_3 \dot{\theta}$	$H_3 \theta = k(q_1) + M(q_1) \ddot{q}_{2d} - \frac{1}{2} J(q_1, q_{2d}) \ddot{q}_{2d} + \frac{1}{2} M_D(q_1, q_2) \ddot{q}_{2d}$	$\dot{\hat{\theta}} = -\Gamma^{-1} H_3^T (\Delta q_2 + c \Delta q_1)$	$\sigma_{\min}(K_v)$ sufficiently large or $c$ sufficiently small Asymptotically stable‡
(3 b)	$u = -K_p \Delta q_1 - \hat{K}_v \Delta q_2 + H_3 \dot{\hat{\theta}}$	$H_3 \theta = \text{same as (3 a) above}$	$\dot{\hat{\theta}} = -\Gamma^{-1} H_3^T (\Delta q_2 + c \Delta q_1)$ $\dot{\hat{K}}_v = \frac{1}{\delta} (\Delta q_2 + c \Delta q_1) \Delta q_2^T$	None required Globally asymptotically stable‡
(4 a)	$u = -K_p \Delta q_1 - K_v \Delta q_2 + H_4 \dot{\theta}$	$H_4 \theta = k(q_1) + M(q_1) \ddot{q}_{2d} - \frac{1}{2} J(q_1, q_2) \ddot{q}_2 + \frac{1}{2} M_D(q_1, q_{2d}) \ddot{q}_2$	$\dot{\hat{\theta}} = -\Gamma^{-1} H_4^T (\Delta q_2 + c \Delta q_1)$	$\sigma_{\min}(K_v)$ sufficiently large or $c$ sufficiently small Asymptotically stable
(4 b)	$u = -K_p \Delta q_1 - \hat{K}_v \Delta q_2 + H_4 \dot{\hat{\theta}}$	$H_4 \theta = \text{same as (4 a) above}$	$\dot{\hat{\theta}} = -\Gamma^{-1} H_4^T (\Delta q_2 + c \Delta q_1)$ $\dot{\hat{K}}_v = \frac{1}{\delta} (\Delta q_2 + c \Delta q_1) \Delta q_2^T$	None required Globally asymptotically stable

(5 a) $u = -K_p \Delta q_1 - K_v \Delta q_2 + H_5 \dot{\theta}$	$H_5 \theta = k(q_1) + M(q_1) \dot{q}_{2d} + C(q_1, q_{2d}) q_{2d}$	$\dot{\theta} = -\Gamma^{-1} H_5^T (\Delta q_2 + c \Delta q_1)$	$\sigma_{\min}(K_v)$ sufficiently large (see (3.36), (3.37)) Asymptotically stable <sup>†,§</sup>
(5 b) $u = -K_p \Delta q_1 - \hat{K}_v \Delta q_2 + H_5 \dot{\theta}$	$H_5 \theta = \text{same as (3 a) above}$	$\dot{\theta} = -\Gamma^{-1} H_5^T (\Delta q_2 + c \Delta q_1)$ $\dot{\hat{K}}_v = \frac{1}{\delta} (\Delta q_2 + c \Delta q_1) \Delta q_2^T$	None required Globally asymptotically stable <sup>†,§</sup>
(6 a) $u = -K_p \Delta q_1 - K_v \Delta q_2 + H_6 \dot{\theta}$	$H_6 \theta = k(q_1) + M(q_1) \dot{q}_{2d} + C(q_1, q_2) q_2$	$\dot{\theta} = -\Gamma^{-1} H_6^T (\Delta q_2 + c \Delta q_1)$	$\sigma_{\min}(K_v)$ sufficiently large (see (3.45), (3.46)) Asymptotically stable <sup>§</sup>
(6 b) $u = -K_p \Delta q - \hat{K}_v \Delta q_2 + H_6 \dot{\theta}$	$H_6 \theta = \text{same as (6 a) above}$	$\dot{\theta} = -\Gamma^{-1} H_6^T (\Delta q_2 + c \Delta q_1)$ $\dot{\hat{K}}_v = \frac{1}{\delta} (\Delta q_2 + c \Delta q_1) \Delta q_2^T$	$\delta$ sufficiently small (see (3.55), (3.56)) Globally asymptotically stable <sup>§</sup>
(7 a) $u = -K_p \Delta q_1 - K_v \Delta q_2 + H_7 \dot{\theta}$	$H_7 \theta = k(q_{1d}) + M(q_{1d}) \dot{q}_{2d} + C(q_{1d}, q_{2d}) q_{2d}$	$\dot{\theta} = -\Gamma^{-1} H_7^T (\Delta q_2 + c \Delta q_1)$	$\sigma_{\min}(K_p)$ and $\sigma_{\min}(K_v)$ sufficiently large w.r.t. initial condition (see (3.55), (3.56)) Asymptotically stable <sup>†,§</sup>
(7 b) $u = -\hat{K}_p \Delta q_1 - \hat{K}_v \Delta q_2 + H_7 \dot{\theta}$	$H_7 \theta = \text{same as (5 a) above}$	$\dot{\theta} = -\Gamma^{-1} H_7^T (\Delta q_2 + c \Delta q_1)$ $\dot{\hat{K}}_v = \frac{1}{\delta_v} (\Delta q_2 + c \Delta q_1) \Delta q_2^T$ $\delta_v > 0$ $\dot{\hat{K}}_p = \frac{1}{\delta_p} (\Delta q_2 + c \Delta q_1) \Delta q_1^T$ $\delta_p > 0$	None required Globally asymptotically stable <sup>†,§</sup>

<sup>†</sup> General assumptions:  $K_v = K_v^T > 0$ ,  $K_p = K_p^T > 0$ ,  $\Gamma = \Gamma^T > 0$ ,  $c > 0$ ,  $\delta > 0$ ,  $\delta_p > 0$ ,  $\delta_v > 0$ .

<sup>‡</sup> Significant simplification for set-point control  $\dot{q}_{2d} = q_{2d} = 0$ .

<sup>§</sup> Recursive Newton–Euler applicable in non-adaptive case.

Table 2. Summary of asymptotically stable adaptive control laws.

variance principles are not applicable. Alternatively, we make use of a lemma due originally to Barbalat, quoted without proof from Popov (1973).

**Lemma 3.1** (Barbalat)

If  $W$  is a real function of the real variable  $t$ , defined and uniformly continuous for  $t \geq 0$  and if the limit of the integral

$$\lim_{t \rightarrow \infty} \int_0^t W(t') dt'$$

exists and is a finite number, then  $\lim_{t \rightarrow \infty} W(t) = 0$ .

For our purposes let

$$W(t) \triangleq \lambda_1 \|\Delta q_1(t)\|^2 + \lambda_2 \|\Delta q_2(t)\|^2$$

so that

$$\dot{V} \leq -W \quad (3.17)$$

Integrating both sides of (3.17) from 0 to  $t$ , yields upon rearranging

$$\int_0^t W dt' \leq V_0 - V(t) \quad (3.18)$$

Since  $V_0$  is bounded, and  $V(t)$  is non-increasing and bounded below, it follows that

$$\lim_{t \rightarrow \infty} \int_0^t W dt' < \infty$$

Also, since  $\dot{W}$  is bounded,  $W(t)$  is uniformly continuous. Hence, application of Barbalat's Lemma gives,

$$\lim_{t \rightarrow \infty} W = 0 \quad (3.19)$$

or equivalently  $\|\Delta q_1\| \rightarrow 0$  and  $\|\Delta q_2\| \rightarrow 0$ .

This completes the proof of asymptotic stability. The proof, however, is not a global one owing to property (ii), i.e. the value of  $c$  which was not required in the non-adaptive case now appears in the parameter adaptation law (3.8). Hence, one is committed to choosing a particular value of  $c$  in the adaptive implementation. Of course,  $c$  can always be chosen sufficiently small to satisfy the requirement, however, the position tracking performance determined by the magnitude of  $\lambda_1$  in (3.14 a) must be compromised as a result. Hence in practice, the initial choice of  $c$  can be made using whatever bounds on  $\eta_1$ ,  $\eta_2$ ,  $\mu$ ,  $\sigma(M)$  and  $V_0$  are available *a priori*, and the value of  $c$  can be improved (increased) on-line as more information becomes available. It is noted that (3.16 a) and (3.16 b) impose additional constraints on how large  $c$  can become, since it is required that  $\xi_1 > 0$  and  $\xi_2 > 0$  for a positive-definite  $V$  (these conditions can be shown to be sufficient).

The asymptotic stability proof presented above for the adaptation of Control Law (1) is easily extended to the adaptation of Control Laws (2), (3) and (4), since the corresponding non-adaptive Lyapunov function derivatives for these control laws are of exactly the same form as  $V^\circ$  in (3.25) (see Theorem 4.1 in Wen and Bayard 1988 for

details). For convenience, all asymptotically stable adaptive control laws discussed thus far and their appropriate parameter adaptation laws are summarized in Table 2, corresponding to cases (1 a), (2 a), (3 a) and (4 a), respectively.

An alternative to choosing  $c$  sufficiently small in the above asymptotic stability argument is to choose  $K_v$  sufficiently large. In this case, the condition on  $c$  above can be removed completely by adapting  $K_v$  on-line. This modification insures global asymptotic stability of the adaptive control system (i.e. choice of  $c$  independent of the initial condition  $V_0$ ) and is discussed in more detail below.

**3.2.2. Global asymptotic stability-adapting  $K_v$ .** Since the velocity gain  $K_v$  enters linearly in the control law, it can be adapted as if it were an unknown parameter using the same formulation as § 3.2.1. It will be shown that this approach removes the dependence of the choice of  $c$  on the initial condition  $V_0$  and this completes the proof of global asymptotic stability for the adaptive case.

Consider Control Law (1) written in adaptive form, where both  $\theta$  and  $K_v$  are adapted in real time, i.e.

$$u = -K_p \Delta q_1 - \hat{K}_v \Delta q_2 + H_1 \hat{\theta} \quad (3.20)$$

Here,  $\hat{K}_v$  is a time-varying quantity that remains to be specified, and  $H_1$  is as defined earlier in (3.3). The non-adaptive control law  $u^o$  in (3.1) is subtracted from (3.20) to give the following expression

$$u = u^o - \Delta K_v \Delta q_2 + H_1 \phi \quad (3.21)$$

where  $\Delta K_v \triangleq \hat{K}_v - K_v$  and  $\phi = \hat{\theta} - \theta$ .

The Lyapunov function for the stability analysis is given as

$$V = V^o + \frac{1}{2} \phi^T \Gamma \phi + \frac{1}{2} \delta \text{tr} \{ \Delta K_v^T \Delta K_v \}, \quad \delta > 0, \quad \Gamma = \Gamma^T > 0 \quad (3.22)$$

where a new term has been added relative to (3.6 a), quadratic in the error  $\Delta K_v$ . Taking the derivative of  $V$  along system trajectories and substituting control law (3.21) gives upon rearranging

$$\begin{aligned} \dot{V} = \dot{V}^o + (\Delta q_2 + c \Delta q_1)^T H_1 \phi + \dot{\phi}^T \Gamma \phi \\ + \text{tr} \{ [\delta \Delta \dot{K}_v^T - \Delta q_2 (\Delta q_2 + c \Delta q_1)^T] \Delta K_v \} \end{aligned} \quad (3.23)$$

The latter terms are cancelled exactly by the choice of parameter adaptation laws

$$\dot{\phi} = \dot{\hat{\theta}} = -\Gamma^{-1} H_1^T (\Delta q_2 + c \Delta q_1) \quad (3.24 a)$$

$$\Delta \dot{K}_v = \dot{\hat{K}}_v = \frac{1}{\delta} (\Delta q_2 + c \Delta q_1) \Delta q_2^T \quad (3.24 b)$$

This choice leaves  $\dot{V}$  exactly of the form (3.9), i.e. applying the  $\beta$ -ball Lemma 2.1

$$\dot{V} = \dot{V}^o \leq -\lambda_1 \|\Delta q_1\|^2 - \lambda_2 \|\Delta q_2\|^2 \quad (3.25)$$

if

$$\sigma_{\min}(K_p) > \frac{3}{4} \eta_2 \rho^2 \quad (3.26)$$

$$\sigma_{\min}(K_v) > c \left( \mu + \frac{3}{4} \frac{\eta_2}{\rho^2} + \frac{\eta_1}{2} \left( \frac{V_0}{\xi_1} \right)^{1/2} \right) \quad (3.27)$$

In (3.26) and (3.27), all quantities are defined exactly as in (3.12 a) and (3.12 b),

respectively, except for  $V_0$  which is presently the initial value of  $V$  in (3.22). Furthermore, the values of  $\xi_1$  and  $\xi_2$  are once again given as

$$\xi_1 = \frac{1}{2}[\sigma_{\min}(K_p + cK_v) - l^2 c \sigma(M)] \quad (3.28)$$

$$\xi_2 = \frac{1}{2} \left( 1 - \frac{c}{l^2} \right) \sigma(M) \quad (3.29)$$

An important observation is that

$$V_0 \sim \mathcal{O}(\|K_v\|^2), \quad \|K_v\| \rightarrow \infty \quad (3.30a)$$

$$\xi_1 \sim \mathcal{O}(\|K_v\|), \quad \|K_v\| \rightarrow \infty \quad (3.30b)$$

Hence, for any choice of  $\delta > 0$ ,  $c > 0$  and  $K_p = K_p^T > 0$ , there exist values of  $\rho^2$ ,  $l^2$  and  $K_v = K_v^T > 0$  (with  $\sigma_{\min}(K_v)$  sufficiently large) such that inequalities (3.26) and (3.27) are satisfied, and  $\xi_1 > 0$ ,  $\xi_2 > 0$  in (3.28) and (3.29), respectively. Global asymptotic stability of this adaptive control scheme then follows immediately by application of Barbalat's Lemma to the Lyapunov function derivative (3.25), as was done earlier in (3.17)–(3.19).

The global asymptotic stability of adaptive controllers based on Control Laws (2), (3) and (4) (where  $K_v$  is adapted on-line) follows from an identical argument, since  $\dot{V}^\circ$  corresponding to the non-adaptive Lyapunov function derivatives for these control laws are of exactly the same form as  $\dot{V}^\circ$  in this analysis (see Theorem 4.1 in Wen and Bayard 1988 for details). For convenience, these adaptive control laws involving adaptation of  $K_v$  are summarized in Table 2, corresponding to cases (1 b), (2 b), (3 b) and (4 b), respectively.

### 3.3. Global asymptotic stability for adaptation of Control Laws (5), (6) and (7)

Global asymptotic stability for adaptation of Control Laws (5), (6) and (7) can be proved using exactly the same techniques as applied in § 3.2. The only difference lies in slight variations in the non-adaptive Lyapunov function derivative  $\dot{V}^\circ$  that arise in each adaptive control analysis. These differences have been detailed in the non-adaptive treatment found by Wen and Bayard (1988), and will be applied directly in this derivation.

**3.3.1. Adaptation of Control Law (5).** Consider Control Law (5) of Table 1 written as follows

$$u = u^\circ + H_5 \phi \quad (3.31)$$

where

$$u^\circ = -K_p \Delta q_1 - K_v \Delta q_2 + H_5 \theta$$

$$\phi = \hat{\theta} - \theta$$

$$H_5 \theta = k(q_1) + M(q_1) \dot{q}_{2d} + C(q_1, q_{2d}) q_{2d} \quad (3.32)$$

The Lyapunov function for the analysis is chosen as earlier, i.e.

$$V = V^\circ + \frac{1}{2} \phi^T \Gamma \phi \quad (3.33)$$

where  $V^\circ$  is given by (3.6 b). Taking the derivative of  $V$  along system trajectories induced by the control law (3.31) and the parameter adaptation law,

$$\dot{\phi} = \dot{\hat{\theta}} = -\Gamma^{-1} H_5^T (\Delta q_2 + c \Delta q_1) \quad (3.34)$$

yields upon application of Lemma 2.1 (see Corollary 4.1 in Wen and Bayard 1988)

$$\dot{V} = \dot{V}^o \leq -\lambda_1 \|\Delta q_1\|^2 - \lambda_2 \|\Delta q_2\|^2 \quad (3.35)$$

if

$$\sigma_{\min}(K_p) > \frac{3\eta_2 \rho^2}{2} \quad (3.36)$$

$$\sigma_{\min}(K_v) > \frac{1}{2}\eta_2 + c \left[ \mu + \frac{3\eta_2}{2\rho^2} + \frac{\eta_2}{2} \left( \frac{V_0}{\xi_1} \right)^{1/2} \right] \quad (3.37)$$

Owing to the presence of  $\frac{1}{2}\eta_2$  in (3.37),  $\sigma_{\min}(K_v)$  must now be chosen sufficiently large, regardless of how small  $c$  is chosen, in order to guarantee asymptotic stability. This adaptive control law is summarized in Table 2, case (5a).

The condition on  $\sigma_{\min}(K_v)$  above can once again be removed by adapting  $K_v$  on-line according to

$$\dot{K}_v = \frac{1}{\delta} (\Delta q_2 + c \Delta q_1) \Delta q_2^T \quad (3.38)$$

This follows from the fact that  $V_0$  grows as  $\mathcal{O}(\|K_v\|^2)$  for the required Lyapunov function of the form (3.22), while  $\xi_1$  grows as  $\mathcal{O}(\|K_v\|)$ . This implies that

$$\left( \frac{V_0}{\xi_1} \right)^{1/2} \sim \mathcal{O}(\|K_v\|^{1/2}) \quad (3.39)$$

Hence, a  $K_v$  always exists (not needed for implementation) such that (3.36) and (3.37) are satisfied for any choice of  $c > 0$  and  $\delta > 0$ , and global asymptotic stability follows. This controller with  $K_v$  adaptation is summarized in Table 2, case (5 b).

3.3.2. *Adaptation of Control Law (6).* Consider Control Law (6) written as follows

$$u = u^o + H_6 \phi \quad (3.40)$$

where

$$u^o = -K_p \Delta q_1 - K_v \Delta q_2 + H_6 \theta \quad (3.41)$$

$$H_6 \theta = k(q_1) + M(q_1) \dot{q}_{2d} + C(q_1, q_2) q_2 \quad (3.42)$$

The Lyapunov function for the analysis is chosen as in (3.33). Taking the derivative along system trajectories induced by (3.40) and parameter adaptation law

$$\dot{\phi} = \dot{\theta} = -\Gamma^{-1} H_6^T (\Delta q_2 + c \Delta q_1) \quad (3.43)$$

yields upon application of Lemma 2.1 (see Corollary 4.2 in Wen and Bayard 1988)

$$\dot{V} = \dot{V}^o \leq -\lambda_1 \|\Delta q_1\|^2 - \lambda_2 \|\Delta q_2\|^2 \quad (3.44)$$

if

$$\sigma_{\min}(K_p) > \frac{\eta_2 \rho^2}{2} \quad (3.45)$$

$$\sigma_{\min}(K_v) > \frac{\eta_2}{2} + c \left[ \mu + \frac{\eta_2}{2\rho^2} + \eta_1 \left( \frac{V_0}{\xi_1} \right)^{1/2} \right] + \frac{\eta_1}{2} \left( \frac{V_0}{\xi_2} \right)^{1/2} \quad (3.46)$$



Since

$$\left(\frac{V_0}{\xi_2}\right)^{1/2} \sim \mathcal{O}(\|K_v\|^{1/2}), \quad \text{and} \quad \left(\frac{V_0}{\xi_1}\right)^{1/2}$$

is bounded as  $\|K_v\| \rightarrow \infty$ , it follows that  $\sigma_{\min}(K_v)$  can and must be chosen sufficiently large (required for implementation) such that inequalities (3.45) and (3.46) are satisfied. This asymptotically stable adaptive controller is summarized in Table 2, case (6 a).

Once again the condition on  $\sigma_{\min}(K_v)$  above can be removed by adapting  $K_v$  on-line, thus insuring global asymptotic stability. However, a new condition arises in this case on the adaptation gain  $\delta$ . This is shown below.

Let  $K_v$  be adapted on-line according to

$$\dot{K}_v = \frac{1}{\delta} (\Delta q_2 + c \Delta q_1) \Delta q_2^T \quad (3.47)$$

Let the Lyapunov function for the analysis be given by

$$V = V^o + \frac{1}{2} \phi^T \Gamma \phi + \frac{\delta}{2} \text{tr} \{ \Delta K_v^T \Delta K_v \} \quad (3.48)$$

Then  $\dot{V}$  is given by (3.44) if conditions (3.45) and (3.46) are satisfied. Remembering that  $V_0$  in (3.46) now corresponds to the initial condition of  $V$  given in (3.48), it follows that

$$\left(\frac{V_0}{\xi_1}\right)^{1/2} \sim \mathcal{O}(\|K_v\|^{1/2}), \quad \|K_v\| \rightarrow \infty \quad (3.49)$$

$$\left(\frac{V_0}{\xi_2}\right)^{1/2} \sim \mathcal{O}(\|K_v\|), \quad \|K_v\| \rightarrow \infty \quad (3.50)$$

Since the quantity in (3.50) grows at the same rate as  $\sigma_{\min}(K_v)$ , inequality (3.46) can only be satisfied for choice of  $\delta$  sufficiently small. With this condition, the adaptive scheme is globally asymptotically stable and is summarized in Table 2, case (6 b).

3.3.3. *Adaptation of Control Law (7).* Consider Control Law (7) written as follows

$$u = u^o + H_7 \phi \quad (3.51)$$

where

$$\begin{aligned} u^o &= -K_p \Delta q_1 - K_v \Delta q_2 + H_7 \theta \\ H_7 \theta &= k(q_{1d}) + M(q_{1d}) \dot{q}_{2d} + C(q_{1d}, q_{2d}) q_{2d} \end{aligned} \quad (3.52)$$

The Lyapunov function for the analysis is chosen as (3.33). Taking the derivative along system trajectories induced by (3.51) and the parameter adaptation law,

$$\dot{\phi} = \dot{\theta} = -\Gamma^{-1} H_7^T (\Delta q_2 + c \Delta q_1) \quad (3.53)$$

yields upon application of Lemma 2.1 (see Theorem 4.2 in Wen and Bayard 1988)

$$\dot{V} = \dot{V}^o \leq -\lambda_1 \|\Delta q_1\|^2 - \lambda_2 \|\Delta q_2\|^2 \quad (3.54)$$

if

$$c \sigma_{\min}(K_p) > c(\eta_5 + \eta_4 \eta_7 + \frac{3}{2} \eta_6^2 \eta_3) + \rho^2 \eta_8 \quad (3.55)$$

$$\sigma_{\min}(K_v) > \frac{1}{2}\eta_2 + \frac{\eta_8}{\rho^2} + c \left[ \mu + \frac{\eta_1}{2} \left( \frac{V_0}{\xi_1} \right)^{1/2} \right] \quad (3.56)$$

Here,  $\eta_i$ ,  $i = 2, 3, 4, 5, 6, 7, 8$ , are bounded scalars defined by Wen and Bayard (1988). It is seen in this case that both  $\sigma_{\min}(K_v)$  and  $\sigma_{\min}(K_p)$  must be chosen sufficiently large for satisfaction of inequalities (3.55), (3.56)—thus ensuring asymptotic stability of the adaptive algorithm. This adaptive law is summarized in Table 2, case (7 a).

Since both gains  $K_v$  and  $K_p$  enter Control Law (7) linearly, they can be adapted on-line. Adapting both  $K_v$  and  $K_p$  on-line removes the above conditions on  $\sigma_{\min}(K_v)$  and  $\sigma_{\min}(K_p)$  above, and hence guarantees global asymptotic stability. This is shown by considering the Lyapunov function,

$$V = V^0 + \frac{1}{2}\phi^T \Gamma \phi + \frac{\delta_v}{2} \text{tr} \{ \Delta K_v^T \Delta K_v \} + \frac{\delta_p}{2} \text{tr} \{ \Delta K_p^T \Delta K_p \} \quad (3.57)$$

and the adaptive control law

$$\begin{aligned} u &= -\hat{K}_p \Delta q_1 - \hat{K}_v \Delta q_2 + H_7 \hat{\theta} \\ &= u^0 - \Delta K_p \Delta q_1 - \Delta K_v \Delta q_2 + H \phi \end{aligned} \quad (3.58)$$

where  $\Delta K_p = \hat{K}_p - K_p$  and  $K_v = \hat{K}_v - K_v$ . Taking the derivative of (3.57) along system trajectories induced by (3.58) and parameter adaptation laws (3.57) augmented by

$$\begin{aligned} \dot{\hat{K}}_v &= \frac{1}{\delta_v} (\Delta q_2 + c \Delta q_1) \Delta q_2^T \\ \dot{\hat{K}}_p &= \frac{1}{\delta_p} (\Delta q_2 + c \Delta q_1) \Delta q_1^T \end{aligned}$$

yields upon application of Lemma 2.1 the same expression for  $\dot{V}$  earlier in (3.54), assuming that inequalities (3.55) and (3.56) are satisfied. Remembering that  $V_0$  in (3.56) now represents the initial condition of (3.57), it follows that

$$\begin{aligned} \left( \frac{V_0}{\xi_1} \right)^{1/2} &\sim \mathcal{O}(\|K_p\|^{1/2}), \quad \|K_p\| \rightarrow \infty \\ \left( \frac{V_0}{\xi_1} \right)^{1/2} &\sim \mathcal{O}(\|K_v\|^{1/2}), \quad \|K_v\| \rightarrow \infty \end{aligned}$$

Hence,  $K_p$  and  $K_v$  always exist (and are not needed for implementation) such that inequalities (3.55) and (3.56) are satisfied for any choice of  $c > 0$ ,  $\delta_v > 0$  and  $\delta_p > 0$ . This ensures global asymptotic stability of the adaptive algorithm. For a summary of this algorithm with both  $K_v$  and  $K_p$  adaptation, see Table 2, case (7 b).

### 3.4. Remarks on adaptive robustness

The adaptation law for  $\hat{K}_v$  in (3.47) can be written as follows

$$\frac{1}{2}(\hat{K}_v^T(t) + \hat{K}_v(t)) = \frac{1}{\delta} \int_0^t \Delta q_2 \Delta q_2^T dt' + \frac{c}{2\delta} [\Delta q_1(t) \Delta q_1^T(t) - \Delta q_1(0) \Delta q_1^T(0)] \quad (3.59)$$

where we have used the fact that

$$\Delta q_2 \Delta q_1^T + \Delta q_1 \Delta q_2^T = \frac{d}{dt} (\Delta q_1 \Delta q_1^T)$$

(All adaptation laws involving  $\hat{K}_v$  considered in this paper are of this form.) From (4.47), it is seen that  $K_v(t)$  is non-decreasing. Although from the Lyapunov analysis we know that  $\hat{K}_v$  must remain bounded, the form of (3.59) indicates that  $\hat{K}_v(t)$  will have a tendency to 'grow' with time over the course of many manoeuvres. Since global-asymptotic convergence is assured for any initial gain  $\hat{K}_v(0)$ , there is nothing lost by resetting this gain to some nominal value at the outset of each manoeuvre. Alternatively,  $K_v$  can be clamped to some sufficiently large value such that the algorithm reduces to the fixed  $K_v$  type, and asymptotic stability is assured.

Although the techniques described above provide a practical solution to the problem of drifting gains under ideal circumstances, it does not address the robustness problem associated with implementing adaptive loops of the form (3.44) and (3.45) in the presence of unmodelled dynamics and measurement noise. It presently appears that many techniques from the linear adaptive control literature can be applied to the problem (e.g.  $\sigma$ -modification (Ioannou and Kokotovic 1984),  $\sigma$ -switching (Ioannou 1986), branch filter (Bayard *et al.* 1987), etc.) in order to provide robustness in the non-linear adaptive control case. Such approaches are presently under investigation.

#### 4. Summary and remarks

The adaptive control laws derived herein, along with the sufficient conditions for stability and appropriate parameter adaptation laws, are summarized in Table 2. Several remarks are in order at this point in the discussion.

##### Remark 1

All adaptive control laws in this paper were derived for the general tracking control law. However, significant simplification occurs in many of these designs for the special case of set-point control (i.e.  $q_{2d} = \dot{q}_{2d} = 0$ ).

##### Remark 2

The adaptive robustness issue remains open. Certainly for parameter adaptation laws of the form given in Table 2, there will be sensitivities to noise disturbances and unmodelled dynamics directly analogous to those that arise in the linear adaptive control case. It presently appears that many of the robustness techniques developed in the linear adaptive control literature will carry over to the non-linear adaptive control application. This conjecture, however, remains to be investigated.

##### Remark 3

In the non-adaptive case, many of the control laws in Table 2 are in a form appropriate for application of the recursive Newton–Euler computational algorithm. However, the Newton–Euler algorithm requires knowledge of all physical parameters—more parameters than are actually needed to control the system adaptively and more than are actually adapted on-line in the vector  $\hat{\theta}$  of Table 2. Hence, transformation from  $\hat{\theta}$  back to physical parameters is required in order to salvage use of the Newton–Euler algorithm in the adaptive case. However, the transformation is generally non-linear and will not lead to a unique solution unless further constraints are imposed. One typical set of constraints arises when only the payload mass is unknown. In the more general adaptive case, it is emphasized that all linear-in-parameters expressions can be implemented directly, since representations of the form  $H\theta$  are assumed to be available in symbolic form.

##### Remark 4

The control laws of Table 1 were derived by Wen and Bayard (1988) for the

general desired potential energy function. This feature was dropped in the adaptive case in order to simplify the analysis. However, it appears that the adaptive control laws developed herein can be extended to the more general case and this line of research is presently under investigation.

#### Remark 5

A brief comparison with the recent results of Paden (1986) and Slotine and Li (1986) is useful. In these papers, adaptive control laws are derived by choosing  $u$  to cancel various terms in the Lyapunov function derivative, rather than overbounding them (via Lemma 2.1) as was done here. This approach has the advantage of providing global asymptotic convergence without adapting gains  $K_v$  and  $K_p$ . The control laws, however, are by necessity more complex than those designs considered here, and do not simplify in the set-point control case.

### 5. Conclusions

A new class of asymptotically stable adaptive control laws is defined by adapting the control laws of Wen and Bayard (1988) in a certainty equivalence fashion. These algorithms are proved to be asymptotically stable without approximations, linearizations or *ad hoc* assumptions concerning the non-linear manipulator dynamics. Furthermore, the asymptotic convergence properties can be made global by appropriate adaptation of feedback gains. Ongoing research efforts are directed at adaptive robustness, computation, and obstacle-avoidance problems.

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