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New class of control laws for robotic manipulators Part 1. Non-adaptive case

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New class of control laws for robotic manipulators

Part 1. Non-adaptive case

JOHN T. WEN† and DAVID S. BAYARD†

A new class of exponentially stabilizing control laws for joint level control of robot arms is introduced. It has recently been recognized that the non-linear dynamics associated with robotic manipulators have certain inherent passivity properties. More specifically, the derivation of the robotic dynamic equations from Hamilton's principle gives rise to natural Lyapunov functions for control design based on total energy considerations. Through a slight modification of the energy Lyapunov function and the use of a convenient lemma to handle third-order terms in the Lyapunov function derivatives, closed-loop exponential stability for both the set point and tracking control problem is demonstrated. In one new design, the non-linear terms are decoupled from real-time measurements which completely removes the requirement for on-line computation of non-linear terms in the controller implementation. In general, the new class of control laws offers alternatives to the more conventional computed torque method, providing trade-offs between computation and convergence properties. Furthermore, these control laws have the unique feature that they can be adapted in a very simple fashion to achieve asymptotically stable adaptive control.

1. Introduction

The problem of joint level control for the multi-link rigid articulated robot arm is addressed in this paper. Accurate measurements of the joint variables, either angular or displacement, and the joint velocities are assumed available. Traditionally, this problem has been treated by the PID algorithm. Since the justification of using PID control is based on either linearization or some local stability argument (Arimoto and Miyazaki 1983), its application is limited to small angle manoeuvres. Large excursions usually require partitioning a desired trajectory into intermediate points and PID control is used to drive the arm between adjacent points. This approach is less than satisfactory since global stability and adequate performance are not guaranteed. This then motivates the computed torque method (Bejczy 1974) which compensates for non-linear terms in the robot dynamics. Assuming that the robot dynamics are known exactly, the compensated system appears like a decoupled system of double integrators and the closed-loop dynamics can be shaped into desirable forms.

A different approach has been advanced in the past few years. It is based on the recognition that robot arms belong to the class of natural systems, which means time-invariant, unconstrained and lying in a conservative force field (Meirovitch 1970). It is natural to investigate if the structure specific to this class of systems can be exploited in controller design. It has long been known (Pringle 1966, Meirovitch 1970, and earlier) that negative proportional (generalized position) and derivative (generalized

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velocity), or equivalently PD, feedback globally asymptotically stabilizes natural systems. The stability analysis is based on a Lyapunov function motivated by total energy considerations. Application of this result to robot arms has been relatively recent. In particular, global asymptotic stability under joint level PD feedback with gravity compensation has been shown (Golla 1979, Takegaki and Arimoto 1981, Koditschek 1984, Miyazaki and Arimoto 1985). Application to the tracking problem is more difficult owing to the time-varying nature of the problem. More specifically, the stability analysis requires a generalization of the invariance principle to time-varying systems; this issue has been partially addressed by Koditschek (1987) and Paden (1986).

In this paper, we introduce a new class of exponentially stabilizing control laws for both the set point and tracking control problems. The stability proof is achieved by making use of a particular class of energy-like Lyapunov functions in conjunction with a useful lemma for addressing the higher order terms in the Lyapunov function derivatives. In the set point control case, Lyapunov functions based on various artificial potential fields are used to derive control laws possessing desired properties. These include set point controllers having simple PD or PD + bias structures and the ability to handle joint stop constraints. In the tracking control case, this new class of Lyapunov functions avoids the need for a generalized invariance principle, which, as mentioned above, has been the major source of difficulty in existing approaches. This leads to a new class of exponentially stabilizing tracking control laws. In one design among this new class, the non-linear terms are decoupled from real-time measurements, which completely removes the requirement for on-line computation on non-linear terms in the controller implementation. This result is believed to have no counterpart in present day literature. In general, the new class of control laws offers alternatives to the more conventional computed torque method, providing trade-offs between computation and convergence properties. Furthermore, these control laws have the unique property that they can be adapted in a very simple fashion to achieve asymptotically stable adaptive control. This last property is elaborated on in Part 2 of this work (Bayard and Wen 1988).

This paper is organized as follows. Some background derivations, identities, notations, lemmas and relevant results in the literature are covered in § 2. Several useful set point controllers based on different artificial potential energies are presented in § 3. A new Lyapunov function is also introduced to demonstrate exponential convergence. In § 4, a new family of exponentially stabilizing tracking control laws are derived. We discuss the trade-off between the ease of implementation and the strength of assumptions for these controllers. Finally, conclusions are drawn in § 5 together with a table summarizing all of the controllers presented in this paper and the conditions for stability.

2. Background

2.1. *Robot dynamic equation*

In this section, the dynamic equation of a robot manipulator is derived. At first glance, it appears as a complex, tightly coupled set of non-linear equations. However, based on derivation from Hamilton's principle, the non-linearity actually contains a great deal of structure. As a result, some important identities are developed in the next section on which the rest of the paper is based.

An n-link rigid robot arm belongs to the class of so-called natural systems with kinetic and potential energies given by

$$\left.\begin{aligned} T &= \tfrac{1}{2} q_2^{\mathrm{T}} M(q_1) q_2 \\ U &= -q_1^{\mathrm{T}} u + g(q_1) \end{aligned}\right\} \tag{2.1}$$

where

T	is kinetic energy;
U	is potential energy;
q_1	is the joint angle or position vector $\in R^n$;
q_2	is the joint velocity vector $\in R^n$;
$M(q_1)$	is the mass inertia matrix $\in R^{n \times n}$;
$g(q_1)$	is the gravitational potential energy;
u	is the joint torque force vector $\in R^n$.

Note that since all the analysis is done at joint level, the arm can be redundant (more than six joints). To derive the differential form of the robot dynamics, first set up the lagrangian

$$L = T - U$$

Then apply Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_2}\right) - \frac{\partial L}{\partial q_1} = 0$$

This gives the dynamic equation of robot motion

$$\left.\begin{aligned} \dot{q}_1 &= q_2 \\ M(q_1)\dot{q}_2 &= -C(q_1, q_2) q_2 - k(q_1) + u \end{aligned}\right\} \tag{2.2}$$

where

$$C(q_1, q_2) = \sum_{i=1}^{n} \{[e_i q_2^{\mathrm{T}} M_i(q_1)]^{\mathrm{T}} - \tfrac{1}{2}[e_i q_2^{\mathrm{T}} M_i(q_1)]\} \tag{2.3}$$

$$e_i \triangleq i\text{th unit vector}$$

$$M_i(q_1) = \frac{\partial M(q_1)}{\partial q_{1i}}$$

$$k_i(q_1) = \frac{\partial g(q_1)}{\partial q_{1i}}$$

$$q_{1i} \triangleq i\text{th component of } q_1$$

The term $C(q_1, q_2)q_2$ represents the coriolis and centrifugal forces and $k(q_1)$ represents the gravity load. Note that $C(q_1, q_2)$ is determined entirely from the mass-inertia matrix. Many desirable properties, e.g. inherent passivity, well-posedness of the solution (no finite escape under any bounded control), the existence of a solution to the optimal control problem (Wen 1986) are consequences of this additional structure. Other important properties of (2.2) include that $M(q_1)$ and $M_i(q_1)$ are symmetric and $M(q_1)$ is positive-definite for all $q_1 \in R^n$. For later use, the matrices $C(q_1, q_2)$ and $M_i(q_1)$ are interpreted as $R^{n \times n}$ valued functions of two n-vectors (q_1 and q_2) and one n-vector (q_1), respectively.

2.2. *Some useful identities*

Some key identities that will be used throughout this paper and the companion paper (Bayard and Wen 1988) are derived in this section. First, some notation is defined.

$$M_D(q_1, z) = \sum_{i=1}^{n} M_i(q_1) z e_i^{\mathrm{T}} \tag{2.4}$$

$$\dot{M}(q_1, q_2) = \frac{d}{dt} M(q_1) \tag{2.5}$$

$$J(q_1, z) = \sum_{i=1}^{n} [(e_i z^{\mathrm{T}} M_i(q_1)) - (e_i z^{\mathrm{T}} M_i(q_1))^{\mathrm{T}}] \tag{2.6}$$

$$r(q_1, q_2, q_{2d}) = (q_2 - q_{2d})^{\mathrm{T}} [\tfrac{1}{2} \dot{M}(q_1, q_2)(q_2 - q_{2d}) - C(q_1, q_2) q_2] \tag{2.7}$$

where q_{1d}, q_{2d} are the desired joint position and joint velocities.

$$\left.\begin{aligned} \Delta q_1 &= q_1 - q_{1d} \\ \Delta q_2 &= q_2 - q_{2d} \end{aligned}\right\} \tag{2.8}$$

The first identity relates M_D to $\dot{M}$. Again, M_D and J are regarded as $R^{n \times n}$ valued functions of two n-vector arguments. Note that J is a skew systematic matrix, i.e. $J + J^{\mathrm{T}} = 0$.

Identity 1

$$\dot{M}(q_1, q_2) z = M_D(q_1, z) q_2 \tag{2.9}$$

Proof

$$\begin{aligned} \dot{M}(q_1, q_2) z &= \sum_{i=1}^{n} q_{2i} M_i(q_1) z = \sum_{i=1}^{n} M_i(q_1) z e_i^{\mathrm{T}} q_2 \\ &= M_D(q_1, z) q_2 \end{aligned}$$

□

The next identity relates C to M_D and J.

Identity 2

$$C(q_1, z) z = \tfrac{1}{2} [M_D(q_1, z) - J(q_1, z)] z \tag{2.10}$$

Proof

$$\begin{aligned} C(q_1, z) &= \sum_{i=1}^{n} \left\{ [e_i z^{\mathrm{T}} M_i(q_1)]^{\mathrm{T}} - \frac{1}{2} [e_i z^{\mathrm{T}} M_i(q_1)] \right\} z \\ &= \frac{1}{2} \sum_{i=1}^{n} \{ [e_i z^{\mathrm{T}} M_i(q_1)]^{\mathrm{T}} - [e_i z^{\mathrm{T}} M_i(q_1)] \} z + \frac{1}{2} \sum_{i=1}^{n} M_i(q_1) z e_i^{\mathrm{T}} z \\ &= -\tfrac{1}{2} J(q_1, z) z + \tfrac{1}{2} M_D(q_1, z) z \\ &= \tfrac{1}{2} [M_D(q_1, z) - J(q_1, z)] z \end{aligned}$$

□

The next identity relates J to M_D.

Identity 3

$$J(q_1, z) = M_D^{\mathrm{T}}(q_1, z) - M_D(q_1, z) \tag{2.11}$$

Proof

Direct substitution. □

The next identity involves M_D^{T}.

Identity 4

$$M_D^{\mathrm{T}}(q_1, x)y = M_D^{\mathrm{T}}(q_1, y)x \tag{2.12}$$

Proof

$$\begin{aligned} M_D^{\mathrm{T}}(q_1, x)y &= \sum_{i=1}^{n} e_i x^{\mathrm{T}} M_i(q_1) y \\ &= \sum_{i=1}^{n} e_i y^{\mathrm{T}} M_i(q_1) x = M_D^{\mathrm{T}}(q_1, y)x \end{aligned}$$ □

The next identity simplifies (2.7) in several different ways. It is used in the proof of the tracking problem.

Identity 5

$$r(q_1, q_2, q_{2d}) = \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[J(q_1, q_2)q_{2d} - M_D(q_1, q_{2d})q_2] \tag{2.13}$$

$$= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[J(q_1, q_{2d})q_2 - M_D(q_1, q_2)q_{2d}] \tag{2.14}$$

$$= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[J(q_1, q_{2d})q_{2d} - M_D(q_1, q_2)q_{2d}] \tag{2.15}$$

$$= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[J(q_1, q_2)q_2 - M_D(q_1, q_{2d})q_2] \tag{2.16}$$

Proof

From (2.7)

$$r(q_1, q_2, q_{2d}) = \Delta q_2^{\mathrm{T}}[\tfrac{1}{2}M_D(q_1, \Delta q_2)q_2 - C(q_1, q_2)q_2]$$

Apply Identities 2 and 3

$$\begin{aligned} r(q_1, q_2, q_{2d}) &= \Delta q_2^{\mathrm{T}}[\tfrac{1}{2}M_D(q_1, \Delta q_2)q_2 - M_D(q_1, q_2)q_2 + \tfrac{1}{2}M_D^{\mathrm{T}}(q_1, q_2)q_2] \\ &= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[M_D^{\mathrm{T}}(q_1, q_2)q_2 - M_D(q_1, q_2)q_2 - M_D(q_1, q_{2d})q_2] \\ &= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}\{[M_D^{\mathrm{T}}(q_1, q_2) - M_D(q_1, q_2)]\Delta q_2 + M_D^{\mathrm{T}}(q_1, q_2)q_{2d} \\ &\qquad - M_D(q_1, q_2)q_{2d} - M_D(q_1, q_{2d})q_2\} \\ &= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[J(q_1, q_2)q_{2d} - M_D(q_1, q_{2d})q_2] \end{aligned}$$

which is (2.13). Now apply Identity 4,

$$r(q_1, q_2, q_{2d}) = \tfrac{1}{2}\Delta q_2^{\mathrm{T}}\{[M_D^{\mathrm{T}}(q_1, q_{2d}) - M_D(q_1, q_{2d})]q_2 - M_D(q_1, q_2)q_{2d}\}$$

which is (2.14). Further manipulation yields

$$\begin{aligned} r(q_1, q_2, q_{2d}) &= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[M_D^{\mathrm{T}}(q_1, q_2 - q_{2d})q_{2d} + M_D^{\mathrm{T}}(q_1, q_{2d})q_{2d} - M_D(q_1, q_2)q_{2d} \\ &\quad - M_D(q_1, q_{2d})(q_2 - q_{2d}) - M_D(q_1, q_{2d})q_{2d}] \\ &= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[J(q_1, q_{2d})\Delta q_2 + J(q_1, q_{2d})q_{2d} - M_D(q_1, q_2)q_{2d}] \\ &= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[J(q_1, q_{2d})q_{2d} - M_D(q_1, q_2)q_{2d}] \end{aligned}$$

which is (2.15). From (2.14)

$$\begin{aligned} r(q_1, q_2, q_{2d}) &= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[-M_D^{\mathrm{T}}(q_1, q_2 - q_{2d})q_2 + M_D^{\mathrm{T}}(q_1, q_2)q_2 - M_D(q_1, q_{2d})q_2 \\ &\quad + M_D(q_1, q_2)(q_2 - q_{2d}) - M_D(q_1, q_2)q_2] \\ &= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[-J(q_1, q_2)\Delta q_2 + J(q_1, q_2)q_2 - M_D(q_1, q_{2d})q_2] \\ &= \tfrac{1}{2}\Delta q_2^{\mathrm{T}}[J(q_1, q_2)q_2 - M_D(q_1, q_{2d})q_2] \end{aligned}$$

which is (2.16). □

The last identity relates to the expressions in the brackets in (2.13)–(2.16).

Identity 6

$$\begin{aligned} &M_D(q_1, \Delta q_2)q_2 - C(q_1, q_2)q_2 - \tfrac{1}{2}(J(q_1, q_2)q_{2d} - M_D(q_1, q_{2d})q_2) \\ &\quad = M_D(q_1, \Delta q_2)q_2 - C(q_1, q_2)q_2 - \tfrac{1}{2}(J(q_1, q_{2d})q_2 - M_D(q_1, q_2)q_{2d}) \\ &\quad = \tfrac{1}{2}(M_D^{\mathrm{T}}(q_1, \Delta q_2)\Delta q_2 + M_D^{\mathrm{T}}(q_1, q_{2d})\Delta q_2 - M_D(q_1, q_{2d})\Delta q_2 + M_D(q_1, \Delta q_2)q_{2d}) \end{aligned} \tag{2.17}$$

$$\begin{aligned} &M_D(q_1, \Delta q_2)q_2 - C(q_1, q_2)q_2 - \tfrac{1}{2}(J(q_1, q_{2d})q_{2d} - M_D(q_1, q_2)q_{2d}) \\ &= (M_D^{\mathrm{T}}(q_1, q_{2d}) - M_D(q_1, q_{2d}))\Delta q_2 + \tfrac{1}{2}M_D^{\mathrm{T}}(q_1, \Delta q_2)\Delta q_2 + \tfrac{1}{2}M_D(q_1, \Delta q_2)q_{2d} \end{aligned} \tag{2.18}$$

$$\begin{aligned} &M_D(q_1, \Delta q_2)q_2 - C(q_1, q_2)q_2 - \tfrac{1}{2}(J(q_1, q_2)q_2 - M_D(q_1, q_{2d})q_2) \\ &\qquad = \tfrac{1}{2}(M_D(q_1, \Delta q_2)\Delta q_2 + M_D(q_1, \Delta q_2)q_{2d}) \end{aligned} \tag{2.19}$$

Proof

$$\begin{aligned} &M_D(q_1, \Delta q_2)q_2 - C(q_1, q_2)q_2 \\ &\qquad = M_D(q_1, q_2)q_2 - M_D(q_1, q_{2d})q_2 - M_D(q_1, q_2)q_2 + \tfrac{1}{2}M_D^{\mathrm{T}}(q_1, q_2)q_2 \\ &\qquad = -M_D(q_1, q_{2d})q_2 + \tfrac{1}{2}M_D^{\mathrm{T}}(q_1, q_2)q_2 \end{aligned}$$

We show the first equality of (2.17)

$$\begin{aligned} &-M_D(q_1, q_{2d})q_2 + \tfrac{1}{2}M_D^{\mathrm{T}}(q_1, q_2)q_2 - \tfrac{1}{2}[J(q_1, q_2)q_{2d} - M_D(q_1, q_{2d})q_2] \\ &= \tfrac{1}{2}M_D^{\mathrm{T}}(q_1, q_2)\Delta q_2 - \tfrac{1}{2}M_D(q_1, q_{2d})q_2 + \tfrac{1}{2}M_D(q_1, q_2)q_{2d} \\ &= \tfrac{1}{2}[M_D^{\mathrm{T}}(q_1, \Delta q_2)\Delta q_2 + M_D^{\mathrm{T}}(q_1, q_{2d})\Delta q_2 - M_D(q_1, q_{2d})\Delta q_2 + M_D(q_1, \Delta q_2)q_{2d}] \end{aligned}$$

The second equality of (2.17) follows by direct computation

$$\begin{aligned} &-M_D(q_1, q_{2d})q_2 + \tfrac{1}{2}M_D^{\mathrm{T}}(q_1, q_2)q_2 - \tfrac{1}{2}[J(q_1, q_{2d})q_2 - M_D(q_1, q_2)q_{2d}] \\ &\qquad = \tfrac{1}{2}M_D^{\mathrm{T}}(q_1, q_2)\Delta q_2 - \tfrac{1}{2}M_D(q_1, q_{2d})q_2 + \tfrac{1}{2}M_D(q_1, q_2)q_{2d} \end{aligned}$$

which is exactly the second equality above.

In (2.18), the left-hand side is the same as the second expression in (2.17) with the additional term $\frac{1}{2}[M_D^T(q_1, q_{2d})\Delta q_2 - M_D(q_1, q_{2d})\Delta q_2]$. Add it to the right-hand side of (2.17) and we obtain

$$\begin{aligned}\tfrac{1}{2}[&M_D^T(q_1, q_2)\Delta q_2 - M_D(q_1, q_{2d})\Delta q_2 + M_D(q_1, \Delta q_2)q_{2d} + M_D^T(q_1, q_{2d})\Delta q_2 \\ &- M_D(q_1, q_{2d})\Delta q_2] \\ &= \tfrac{1}{2}M_D^T(q_1, \Delta q_2)\Delta q_2 + \tfrac{1}{2}M_D(q_1, \Delta q_2)q_{2d} + [M_D^T(q_1, q_{2d}) - M_D(q_1, q_{2d})]\Delta q_2\end{aligned}$$

Similarly, the left-hand side of (2.19) is the same as the first expression in (2.17) with the additional term $\frac{1}{2}(-M_D^T(q_1, q_2)\Delta q_2 + M_D(q_1, q_2)\Delta q_2)$. Add it to the right-hand side of (2.17) and we obtain

$$\begin{aligned}\tfrac{1}{2}[&M_D^T(q_1, q_2)\Delta q_2 - M_D(q_1, q_{2d})\Delta q_2 + M_D(q_1, \Delta q_2)q_{2d} - M_D^T(q_1, q_2)\Delta q_2 \\ &+ M_D(q_1, q_2)\Delta q_2] = \tfrac{1}{2}(M_D(q_1, \Delta q_2)\Delta q_2 + M_D(q_1, \Delta q_2)q_{2d})\end{aligned}$$ □

2.3. *Important lemma*

In this section, an important stability lemma is presented that will play pivotal roles in later sections. The lemma is essentially a local stability theorem that establishes a region of convergence. It will be shown that if the initial state is within some ball of radius β, then the state never escapes the β-ball.

Lemma 2.1

Given a dynamical system

$$\dot{x}_i = f_i(x_i, \ldots, x_N, t), \quad x_i \in R^{n_i}, \quad t \geqslant 0, \quad i = 1, \ldots, N$$

Let f_i be locally Lipschitz with respect to $x_1, \ldots, x_N$, uniformly in t on bounded intervals and continuous in t for $t \geqslant 0$.

Suppose a function $V: R^{n_1 \times \cdots \times n_N} \times R_+ \to R_+$ is given such that

(i) $$V(x_1, \ldots, x_N, t) = \sum_{i,j=1}^{N} x_i^T P_{ij}(x_1, \ldots, x_j, t)x_j \tag{2.20}$$

where for each $i = 1, \ldots, N$ there exists $\xi_i > 0$ such that

$$\xi_i \|x_i\|^2 \leqslant V(x_1, \ldots, x_N, t)$$

(ii) $$\dot{V}(x_1, \ldots, x_N, t) \leqslant -\sum_{i \in I_1}\left(\alpha_i - \sum_{j \in I_{2i}} \gamma_{ij}\|x_j(t)\|^{k_{ij}}\right)\|x_i(t)\|^2 \tag{2.21}$$

where $\alpha_i, \gamma_{ij}, k_{ij} > 0$, $I_{2i} \subset I_1 \subset \{1, \ldots, N\}$.

Let $V_0 \triangleq V(x_1(0), \ldots, x_N(0), 0)$. If $\forall i \in I_1$

$$\alpha_i > \sum_{j \in I_{2i}} \gamma_{ij}\left(\frac{V_0}{\xi_j}\right)^{k_{ij}/2} \tag{2.22}$$

then $\forall \lambda_i \in \left[0, \alpha_i - \sum_{j \in I_{2i}} \gamma_{ij}\left(\frac{V_0}{\xi_j}\right)^{k_{ij}/2}\right]$, the following inequality holds

$$\dot{V}(x_1, \ldots, x_N, t) \leqslant -\sum_{i \in I_1} \lambda_i \|x_i\|^2 \; \forall t \geqslant 0$$

Proof

From (2.22), $\exists \beta_j > 0, j \in \bigcup_{i \in I_1} I_{2i}$, such that

$$\left(\frac{V_0}{\xi_j}\right)^{1/2} < \beta_j \tag{2.23}$$

and

$$\gamma_{ij} \sum_{j \in I_{2_i}} \beta_j^{k_{ij}} < \alpha_i \tag{2.24}$$

By (2.21)

$$\|x_j(0)\| \leqslant \left(\frac{V_0}{\xi_j}\right)^{1/2} < \beta_j \tag{2.25}$$

Assume $x_j(t)$ eventually escapes the β_j-ball for some $j \in \bigcup_{i \in I_1} I_{2_i}$. The assumption on f_i implies that $x_i(t)$ is unique and continuous at least up to the escape time of $x_i\ \forall_i \in \{1, \ldots, N\}$. Without loss of generality, say $x_j(t)$ is the first one that escapes its β_j-ball, i.e. $\exists\, T \in (0, \infty)$ such that

$$\|x_l(t)\| < \beta_l\ \forall l \in \bigcup_{l \in I_1} I_{2_i}\ \forall t \in (0,\ T)$$

$$\|x_j(T)\| = \beta_j$$

Since $(V_0/\xi_j)^{1/2}$-ball is properly contained in β_j-ball by (2.23), $\exists\, T_1 \in (0, T)$ such that

$$\left(\frac{V_0}{\xi_j}\right)^{1/2} < \|x_j(t)\| < \beta_j\ \forall t \in (T_1,\ T)$$

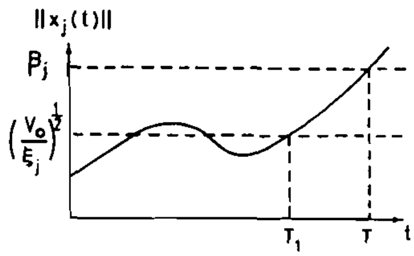

Figure 1.

For $t \in [0, T]$, (2.21) implies

$$\dot{V}(x_1, \ldots, x_N, t) \leqslant -\sum_{i \in I_1} \left(\alpha_i - \sum_{j \in I_{2_i}} \gamma_{ij} \beta_j^{k_{ij}}\right) \|x_i(t)\|^2 \leqslant 0 \tag{2.26}$$

The second inequality follows from (2.24). Define $V_t \triangleq V(x_1(t), \ldots, x_N(t), t)$. For $t \in (T_1, T)$

$$\left(\frac{V_0}{\xi_j}\right)^{1/2} < \|x_j(t)\| \leqslant \left(\frac{V_t}{\xi_j}\right)^{1/2} \leqslant \left(\frac{V_0}{\xi_j}\right)^{1/2} \tag{2.27}$$

which is a contradiction. Hence, all x_j values are within β_j-ball for $t \geqslant 0$. Since

$\gamma_{ij} \sum_{j \in I_{2_i}} \beta_j^{k_{ij}}$ can be made arbitrarily close to

$$\gamma_{ij} \sum_{j \in I_{2_i}} \left(\frac{V_0}{\xi_j}\right)^{k_{ij}/2}$$

the stated result follows directly. □

In the above lemma, we choose to bound over

$$\sum_{j \in I_{2_i}} \gamma_{ij} \|x_j(t)\|^{k_{ij}}$$

(Condition (2.22) reflects that choice.) This choice is arbitrary; in fact, we can extract any quadratic cross-term from

$$\sum_{j \in I_{2_i}} \gamma_{ij} \|x_j(t)\|^{k_{ij}} \|x_i(t)\|^2$$

and overbound the rest. After completing the square, a stability condition similar to (2.22) can be stated. We do not pursue this generalization here.

2.4. *Recent results*

Some of the recent results relating to Lyapunov analysis of robot systems are reviewed in this section. For the set point control problem, (Golla 1979, Takegaki and Arimoto 1981, Koditschek 1984, 1987, Miyazaki and Arimoto 1985, Paden 1986) have applied the result that linear negative feedback of generalized position and velocities globally and asymptotically stabilizes a natural system to robot manipulators. We restate this result, mention work for the tracking problem (Slotine and Li 1986, Koditschek 1987, Paden 1986) and state some open issues that will be addressed in the remainder of this paper and in Part 2 of this work (Bayard and Wen 1988).

Theorem 2.1

Consider (2.2) with the control law

$$u = -K_p \Delta q_1 - K_v q_2 + k(q_1), \quad K_p > 0, K_v > 0 \tag{2.28}$$

The null state of the $(\Delta q_1, q_2)$-system is a globally asymptotically stable equilibrium.

Proof

Since $M(q_1) > 0$ and $K_p > 0$, the following is a valid Lyapunov function candidate

$$V(\Delta q_1, q_2) = \tfrac{1}{2} q_2^{\mathrm{T}} M(q_1) q_2 + \tfrac{1}{2} \Delta q_1^{\mathrm{T}} K_p \Delta q_1 \tag{2.29}$$

It is understood that $M(q_1)$ means $M(\Delta q_1 + q_{1d})$. Take two derivatives along the solution

$$\dot V(\Delta q_1, q_2) = q_2^{\mathrm{T}}(M(q_1)\dot q_2 + \tfrac{1}{2}\dot M(q_1, q_2) q_2) + q_2^{\mathrm{T}} K_p \Delta q_1$$

By (2.2) and Identity 1

$$\dot V(\Delta q_1, q_2) = q_2^{\mathrm{T}}(-C(q_1, q_2) q_2 - k(q_1) + \tfrac{1}{2} M_D(q_1, q_2) q_2 + K_p \Delta q_1 + u)$$

Substituting in the control law (2.11) and applying Identity 2

$$\begin{aligned} \dot V(\Delta q_1, q_2) &= -q_2^{\mathrm{T}} K_v q_2 + \tfrac{1}{2} q_2^{\mathrm{T}} J(q_1, q_2) q_2 \\ &= -q_2^{\mathrm{T}} K_v q_2 \end{aligned}$$

since J is asymmetric. The invariance principle (LaSalle 1960) can now be applied since the $(\Delta q_1, q_2)$-system is time-invariant. Then $(\Delta q_1(t), q_2(t))$ approaches the largest invariant set contained in

$$E \triangleq \{(\Delta q_1, q_2) : q_2 = 0\} \subset R^{2n}$$

On E, $\dot{q}_2 = 0$. From (2.2), this implies $u = 0$, which means

$$K_p \Delta q_1 = 0, \quad \forall (q_1, q_2) \in E$$

Since $K_p > 0$, $\Delta q_1 = 0$ on E. In other words, (0, 0) is a globally asymptotically stable equilibrium point. □

The main idea of this approach is to shape the potential field in such a way that it is globally convex and attains a global minimum at $\Delta q_1 = 0$. Complete damping (in the terminology of Meirovitch 1970) is added through the derivative feedback to drive the system to the minimum potential energy state which by design is the desired state. To be specific, suppose the desired potential field is $U^*(\Delta q_1)$. The total energy under this potential is

$$V = T + U^* \tag{2.30}$$

Rewrite V as

$$V = T + U^\circ + U^* - U^\circ = V^\circ + U^* - U^\circ$$

where U° is the original potential energy *without* external force fields, and V° is the corresponding total energy. Let $p = M(q_1)q_2$ be the generalized momentum. From Hamilton's equation

$$\begin{aligned} \dot{V} &= \left(\frac{\partial V^\circ}{\partial p}\right)^{\mathrm{T}} u + q_2^{\mathrm{T}}\left(\frac{\partial U^*}{\partial \Delta q_1} - \frac{\partial U^\circ}{\partial \Delta q_1}\right) \\ &= q_2^{\mathrm{T}}\left(u + \frac{\partial U^*}{\partial \Delta q_1} - \frac{\partial U^\circ}{\partial \Delta q_1}\right) \end{aligned} \tag{2.31}$$

Hence, to drive the desired total energy to its minimum state, we can select (Takegaki and Arimoto 1981)

$$u = -K_v q_2 - \frac{\partial U^*}{\partial \Delta q_1} + \frac{\partial U^\circ}{\partial \Delta q_1} \tag{2.32}$$

Then $\dot{V} = -q_2^{\mathrm{T}} K_v q_2$. From the fact that $-2\dot{q}_2^{\mathrm{T}} K_v q_2$ is uniformly bounded ($\dot{V} \leqslant 0$), $q_2(t) \to 0$ as $t \to \infty$ (Hale 1969).

$$\begin{aligned} \dot{p} &= -\frac{\partial V^\circ}{\partial q_1} + u = -\frac{\partial T}{\partial q_1} - \frac{\partial U^\circ}{\partial \Delta q_1} + u \\ &= -\frac{\partial T}{\partial q_1} - K_v q_2 - \frac{\partial U^*}{\partial \Delta q_1} \end{aligned}$$

Since $(\partial T/\partial q_1) \to 0$, $(\partial U^*/\partial \Delta q_1) \to 0$, also. Hence, if $U^*(\Delta q_1)$ is globally convex with a minimum at $\Delta q_1 = 0$, $\Delta q_1(t) \to 0$ as $t \to \infty$. If $U^*(\Delta q_1) = \frac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1$, Theorem 2.1 is immediately obtained.

Obviously, any other potential field convex in Δq_1 that has global minimum at $\Delta q_1 = 0$ (and no local minima) can be used. We will use this idea in the next section to address the joint stop issue.

This control law is very appealing in its simplicity and obvious robustness with respect to modelling error in mass matrix, and centrifugal and coriolis terms. Generalization to the tracking problem is partially addressed by Koditschek (1987) and Paden (1986). A control algorithm is given by Koditschek (1987) but it lacks stability analysis. Metrosov's theorem (Rouche *et al.* 1977) is used by Paden (1986). A question remains on the necessity and applicability of Metrosov's theorem to the tracking problem. One version of the tracking control law in §4 is the same as that of Paden (1986) but the stability issue is resolved more completely. The non-adaptive version of the tracking control laws of Paden (1986) and Slotine and Li (1986) do yield global asymptotic stability. However, the simple structure of (2.28) is lost; even for set point control, full model information is needed.

Based on the above very brief review of the currently available pertinent results, it is evident that the following issues remain open.

(*a*) Can we get away with no gravity information, thus achieving a 'universal' (arm independent) set point control law?

(*b*) Computed torque achieves exponential stability. Are schemes based on energy Lyapunov functions inherently inferior (e.g. only asymptotic stability is possible) or have we not been clever enough in choosing the Lyapunov functions?

(*c*) The tracking problem produces a time-varying system. Can the invariance principle still be applied?

(*d*) How far can we reduce the on-line computation requirement (thus allow increasing performance) for both set point and tracking problems? What is the price to be paid?

(*e*) How does one incorporate joint stop constraints?

(*f*) Would these schemes (set point and tracking controls) still work if unknown parameters are adapted?

The rest of this paper will be devoted to answering issues (*a*)–(*e*). The last item is addressed in Part 2 of this work (Bayard and Wen 1988).

2.5. *Computed torque from Lyapunov perspective*

In § 2.4, we introduced the total energy Lyapunov function (2.29) to derive a simple set point control law. The computed torque method can also be motivated in the same manner with a different Lyapunov function. For generality, we will consider the tracking case. Let

$$V(\Delta q_1, \Delta q_2) = \tfrac{1}{2}\|\Delta q_2\|^2 + \tfrac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1 \tag{2.33}$$

Calculate the derivative along solution

$$\dot{V} = \Delta q_2^{\mathrm{T}}(M^{-1}(q_1)(-C(q_1, q_2)q_2 - k(q_1) + u) - \dot{q}_{2d} + K_p \Delta q_1)$$

If the computed torque control is used

$$u = k(q_1) + C(q_1, q_2)q_2 + M(q_1)(\dot{q}_{2d} - K_p \Delta q_1 - K_v \Delta q_2) \tag{2.34}$$

then

$$\dot{V} \leqslant -\Delta q_2^{\mathrm{T}} K_v \Delta q_2$$

From the same line of reasoning as before, the closed-loop system is globally asymptotically stable. However, we know that the closed-loop system is linear, and therefore is exponentially stable. This means that we should look for a better Lyapunov function. An obvious choice is to add in a cross-term into (2.33): Then

$$V(\Delta q_1, \Delta q_2) = \tfrac{1}{2}\|\Delta q_2\|^2 + \tfrac{1}{2}\Delta q_1^{\mathrm{T}}(K_p + cK_v)\Delta q_1 + c\Delta q_1^{\mathrm{T}}\Delta q_2 \tag{2.35}$$

where c is a small constant so that V is positive-definite. Take the derivative and apply (2.34)

$$\begin{aligned}\dot{V} &= -\Delta q_2^{\mathrm{T}} K_v \Delta q_2 + c\Delta q_2^{\mathrm{T}} K_v \Delta q_1 + c\|\Delta q_2\|^2 - c\Delta q_1^{\mathrm{T}} K_p \Delta q_1 - c\Delta q_1^{\mathrm{T}} K_v \Delta q_2 \\ &\leqslant -(\sigma_{\min}(K_v) - c)\|\Delta q_2\|^2 - c\Delta q_1^{\mathrm{T}} K_p \Delta q_1\end{aligned}$$

which shows closed-loop exponential stability.

Note that in (2.34), in contrast to (2.28), even for the set point case, full model non-linearity cancellation is needed. The approach in this paper is to use the energy Lyapunov function instead of (2.33) to generate control laws. We will see in later sections that this affords a much larger class of controls which contains much simpler structure in certain cases (especially for set point control).

3. New results on PD set point control

3.1. *Simple PD controls*

In this section, we explore the use of different U^* in the controller design. The following has been suggested by Takegaki and Arimoto (1981)

$$U^*(\Delta q_1) = \tfrac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1 + g(\Delta q_1 + q_{1d}) - g(q_{1d}) - \Delta q_1^{\mathrm{T}} k(q_{1d}) \tag{3.1}$$

To verify the convexity of U^*, we check the higher order conditions

$$\frac{\partial U^*}{\partial \Delta q_1} = K_p \Delta q_1 + k(\Delta q_1 + q_{1d}) - k(q_{1d})$$

$$\frac{\partial^2 U^*}{\partial \Delta q_1^2} = K_p + \frac{\partial k(\Delta q_1 + q_{1d})}{\partial \Delta q_1}$$

Clearly, if K_p is sufficiently large, $\Delta q_1 = 0$ is the global minimum of $U(\Delta q_1)$. Hence, a simpler control law can be used

$$u = -K_p \Delta q_1 - K_v q_2 + k(q_{1d}) \tag{3.2}$$

Suppose each joint is constrained between joint stops

$$q_{1i}^{(l)} \leqslant q_{1i} \leqslant q_{1i}^{(h)} \tag{3.3}$$

and the set point is in the interior of the joint inputs

$$q_{1i}^{(l)} < \underline{q}_{1i} \leqslant q_{1id} \leqslant \bar{q}_{1i} < q_{1i}^{(h)} \tag{3.4}$$

Introduce the upper and lower barrier potential functions for joint i

$$H_i(q_{1i}) = \left.\begin{cases} \left(\dfrac{1}{q_{1i}^{(h)} - \bar{q}_{1i}} - \dfrac{1}{q_{1i}^{(h)} - q_{1i}}\right)^3, & \text{if } q_{1i} \in [\bar{q}_{1i}, q_{1i}^{(h)}) \\ 0 & \text{if } q_{1i} \in (q_{1i}^{(l)}, \bar{q}_{1i}) \end{cases}\right\} \tag{3.5}$$

$$L_i(q_{1i}) = \left.\begin{cases} \left(\dfrac{1}{\underline{q}_{1i} - q_{1i}^{(l)}} - \dfrac{1}{q_{1i} - q_{1i}^{(l)}}\right)^3, & \text{if } q_{1i} \in (q_{1i}^{(l)}, \underline{q}_{1i}] \\ 0 & \text{if } q_{1i} \in (\underline{q}_{1i}, q_{1i}^{(h)}) \end{cases}\right\} \tag{3.6}$$

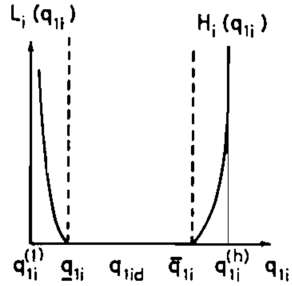

Figure 2.

Let the desired potential function be

$$U^*(\Delta q_1) = \frac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1 + \sum_{i=1}^{n} [H_i(\Delta q_{1i} + q_{1id}) + L_i(\Delta q_{1i} + q_{1id})] \tag{3.7}$$

Checking higher order conditions

$$\frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) = K_p \Delta q_1 + \tilde{H}(\Delta q_1) + \tilde{L}(\Delta q_1)$$

$$\tilde{H}(\Delta q_1) \triangleq \mathrm{Col}\left\{\frac{\partial H_1}{\partial \Delta q_{11}}(\Delta q_{11} + q_{11d}), \ldots, \frac{\partial H_n}{\partial \Delta q_{1n}}(\Delta q_{1n} + q_{1nd})\right\}$$

$$\tilde{L}(\Delta q_1) \triangleq \mathrm{Col}\left\{\frac{\partial L_1}{\partial \Delta q_{11}}(\Delta q_{11} + q_{11d}), \ldots, \frac{\partial L_n}{\partial \Delta q_{1n}}(\Delta q_{1n} + q_{1nd})\right\}$$

$$\frac{\partial H_i}{\partial \Delta q_{1i}}(\Delta q_{1i} + q_{1id}) = \begin{cases} -3\left(\dfrac{1}{q_{1i}^{(h)} - \bar{q}_{1i}} - \dfrac{1}{q_{1i}^{(h)} - \Delta q_{1i} - q_{1id}}\right)^2 \dfrac{1}{(q_{1i}^{(h)} - \Delta q_{1i} - q_{1id})^2}, & \text{if } q_{1i} \in [\bar{q}_{1i}, q_{1i}^{(h)}) \\ 0 & \text{if } q_{1i} \in (q_{1i}^{(l)}, \bar{q}_{1i}) \end{cases}$$

$$\frac{\partial L_i}{\partial \Delta q_{1i}}(\Delta q_{1i} + q_{1id}) = \begin{cases} 3\left(\dfrac{1}{\underline{q}_{1i} - q_{1i}^{(l)}} - \dfrac{1}{\Delta q_{1i} + q_{1id} - q_{1i}^{(l)}}\right)^2 \dfrac{1}{(\Delta q_{1i} + q_{1id} - q_{1i}^{(l)})^2}, & \text{if } q_{1i} \in (q_{1i}^{(l)}, \underline{q}_{1i}] \\ 0 & \text{if } \underline{q}_{1i} \in (q_{1i}, q_{1i}^{(h)}) \end{cases}$$

$$\frac{\partial^2 U^*}{\partial \Delta q_1^2}(\Delta q_1) = \tilde{K}_p + \frac{\partial \tilde{H}}{\partial \Delta q_1}(\Delta q_1) + \frac{\partial \tilde{L}}{\partial \Delta q_1}(\Delta q_1)$$

$\partial \tilde{H}/\partial \Delta q_1$, $\partial \tilde{L}/\partial \Delta q_1$ are diagonal matrices with elements

$$\frac{\partial^2 H_i}{\partial \Delta q_{1i}}(\Delta q_{1i} + q_{1id}) = \begin{cases} 6\left(\dfrac{1}{q_{1i}^{(h)} - \bar{q}_{1i}} - \dfrac{1}{q_{1i}^{(h)} - \Delta q_{1i} - q_{1id}}\right)\dfrac{1}{(q_{1i}^{(h)} - \Delta q_{1i} - q_{1id})^4} \\ -6\left(\dfrac{1}{q_{1i}^{(h)} - \bar{q}_{1i}} - \dfrac{1}{q_{1i}^{(h)} - \Delta q_{1i} - q_{1id}}\right)^2 \dfrac{1}{(q_{1i}^{(h)} - \Delta q_{1i} - q_{1id})^3}, & \text{if } q_{1i} \in [\bar{q}_{1i}, q_{1i}^{(h)}) \\ 0 & \text{if } q_{1i} \in (q_{1i}^{(l)}, \bar{q}_{1i}) \end{cases}$$

$$\frac{\partial^2 L_i}{\partial \Delta q_{1i}}(\Delta q_{1i} + q_{1id}) = \begin{cases} 6\left(\dfrac{1}{\underline{q}_{1i} - q_{1i}^{(l)}} - \dfrac{1}{\Delta q_{1i} + q_{1id} - q_{1i}^{(l)}}\right)\dfrac{1}{(\Delta q_{1i} + q_{id} + q_{1i}^{(l)})^4} \\ -6\left(\dfrac{1}{\underline{q}_{1i} - q_{1i}^{(l)}} - \dfrac{1}{\Delta q_{1i} + q_{1id} - q_{1i}^{(l)}}\right)^2 \dfrac{1}{(\Delta q_{1i} + q_{1d} - q_{1i}^{(l)})^3}, & \text{if } q_{1i} \in (q_{1i}^{(l)}, \underline{q}_{1i}] \\ 0 & \text{if } q_{1i} \in (q_{1i}, q_{1i}^{(h)}) \end{cases}$$

Note that

$$\frac{\partial U^*}{\partial \Delta q_1}(0) = 0 \quad \text{and} \quad \frac{\partial^2 U^*}{\partial \Delta q_1^2}(0) > 0$$

Hence, $\Delta q_1 = 0$ is a global minimum of $U^*(\Delta q_1)$. From (2.32)

$$u = -K_v q_2 - K_p \Delta q_1 - \tilde{H}(\Delta q_1) - \tilde{L}(\Delta q_1) + k(q_1) \tag{3.8}$$

Similarly, if K_p is sufficiently large $(K_p + (\partial k/\partial \Delta q_1)(\Delta q_1 + q_{1d}) > 0)$, the following control law also achieves global asymptotic stability

$$u = -K_v q_2 - K_p \Delta q_1 - \tilde{H}(\Delta q_1) - \tilde{L}(\Delta q_1) + k(q_{1d}) \tag{3.9}$$

Control laws (3.2), (3.8) and (3.9) still require information on the gravity load. It is interesting to ask if this last piece of model information can be removed. This case corresponds to the desired potential energy

$$U^*(\Delta q_1) = \tfrac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1 + g(\Delta q_1 + q_{1d}) \tag{3.10}$$

The corresponding control law is

$$u = -K_p \Delta q_1 - K_v \Delta q_2 \tag{3.11}$$

From § 2.4

$$\frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) = K_p \Delta q_1 + k(\Delta q_1 + q_{1d}) \to 0$$

This implies

$$\limsup_{t\to\infty} \|\Delta q_1(t)\| \leqslant \sigma_{\min}(K_p) \sup_{q_1 \in R^n} \|k(q_1)\|$$

If $K_p + (\partial k(q_1)/\partial q_1) > 0 \ \forall q_1 \in R^n$, $-K_p^{-1} k(\Delta q_1 + q_{1d})$ is a contraction map in Δq_1. Then $\exists! q_1^*$ such that

$$\left.\begin{aligned} K_p(q_1^* - q_{1d}) + k(q_1^*) = 0 \\ \lim_{t\to\infty} q_1(t) = q_1^* \end{aligned}\right\} \tag{3.12}$$

This result suggests a very simple, robust and practical control scheme. The feedback gain K_p can be chosen large enough to justify the use of PID control (Hale 1969 Chap. 3, Arimoto and Miyazaki 1983) which is locally stable. Typically, $k(q_1)$ and $\partial k(q_1)/\partial q_1$ are composed of trigonometric functions, therefore, they are uniformly bounded.

3.2. *PD control with exponential convergence rate*

The use of the invariance principle in § 2.4 shows only asymptotic stability. Some guaranteed rate of convergence is highly desirable not just for performance reasons but also for robustness analysis and adaptive control. In § 2.5, a Lyapunov function with cross-term has been used to show exponential stability. This suggests a similar modification here. The result is summarized below.

Theorem 3.1

Given the control law (2.32)

$$u = -K_v q_2 - \frac{\partial U^*}{\partial q_1} + \frac{\partial U^\circ}{\partial q_1} \tag{2.32}$$

Suppose $\exists\, v > 0$ such that

$$\Delta q_1^{\mathrm{T}} \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) > v \|\Delta q_1\|^2 \tag{3.13}$$

and $U^*(\Delta q_1)$ has a global minimum at $\Delta q_1 = 0$, then the closed-loop $(\Delta q_1, q_2)$ system is exponentially stable.

Proof

Modify the total energy Lyapunov function (2.30) to

$$V = T + U^* + c\Delta q_1^{\mathrm{T}} p + \tfrac{1}{2} c \Delta q_1^{\mathrm{T}} K_v \Delta q_1$$

where c is a small constant so that V is positive-definite in p and q_1. Without loss of generality, U^* can be considered positive-definite in q_1 (by adding an appropriate constant). Then from (2.31)

$$\begin{aligned}\dot V &= q_2^{\mathrm{T}}\left(u + \frac{\partial U^*}{\partial q_1} - \frac{\partial U^\circ}{\partial q_1}\right) + c q_2^{\mathrm{T}} p + c\Delta q_1^{\mathrm{T}}\left(-\frac{\partial T}{\partial q_1} - \frac{\partial U^\circ}{\partial q_1} + u\right) + c q_2^{\mathrm{T}} K_v \Delta q_1 \\ &= -q_2^{\mathrm{T}} K_v q_2 + c q_2^{\mathrm{T}} M(q_1) q_2 - c\Delta q_1^{\mathrm{T}} \frac{\partial T}{\partial q_1} - c \Delta q_1^{\mathrm{T}} \frac{\partial U^*}{\partial q_1}\end{aligned}$$

Let

$$\mu \triangleq \sup_{q_1 \in R^n} \|M(q_1)\| \tag{3.14}$$

Note

$$\frac{\partial T}{\partial q_1} = \frac{1}{2}\sum_{i=1}^{n} e_i q_2^{\mathrm{T}} M_i(q_1) q_2 = \frac{1}{2} M_D^{\mathrm{T}}(q_1, q_2) q_2 \tag{3.15}$$

Then

$$\dot V \leqslant -(\sigma_{\min}(K_v) - c\mu)\|q_2\|^2 - cv\|\Delta q_1\|^2 - \frac{c}{2}\Delta q_1^{\mathrm{T}} M_D^{\mathrm{T}}(q_1, q_2) q_2$$

Define

$$\eta_1 = \sup_{q_1} \sum_{i=1}^{n} \|M_i(q_i)\| \tag{3.16}$$

Then

$$\dot V \leqslant -\alpha_1 \|\Delta q_1\|^2 - \alpha_2 \|\Delta q_2\|^2 + \gamma_{21} \|\Delta q_1\| \, \|\Delta q_2\|^2$$

where

$$\alpha_1 = cv, \quad \alpha_2 = \sigma_{\min}(K_v) - c\mu \quad \text{and} \quad \gamma_{21} = \tfrac{1}{2} c \eta_1$$

Choose

$$c < \sigma_{\min}(K_v)\left[\mu + \frac{1}{2}\eta_1\left(\frac{V_0}{\xi_1}\right)^{1/2}\right]^{-1} \tag{3.17}$$

where $V_0 = V\big|_{t=0}$ and

$$\xi_1 = \inf_{\|\Delta q_1\| = 1}\left[U^*(\Delta q_1) + \frac{1}{2}c\Delta q_1^{\mathrm{T}} K_v \Delta q_1 - \frac{1}{2}c\mu l^2\right] > 0 \quad \text{for some constant } l \tag{3.18}$$

From Lemma 2.1, $\forall \lambda_2 \in (0, \alpha_2 - \gamma_{21}(V_0/\xi_1)^{1/2})$

$$\dot{V} \leqslant -\alpha_1\|\Delta q_1\|^2 - \lambda_2\|\Delta q_2\|^2 \leqslant -\lambda V$$

for some $\lambda > 0$. Hence, the closed-loop system is exponentially stable. □

Given any U^* according to (3.13), $K_v > 0$ and initial condition, there always exists c that satisfies (3.17). Even though c is not needed in the implementation, its maximum allowable size affects the convergence rate. The artificial potentials $U^* = 1/2\Delta q_1^{\mathrm{T}} K_p \Delta q_1$, (3.1) and (3.7) all satisfy the assumptions of Theorem 3.1. Therefore, the corresponding closed-loop systems are exponentially stable. For the potential given by (3.10) and K_p large enough ($K_p + (\partial k/\partial q_1) > 0$), we can add a constant to U^* so that U^* is positive-definite in $q_1 - q_1^*$ and (3.13) is satisfied for $\Delta q_1 = q_1 - q_1^*$, where q_1 solves (3.12). Then Theorem 3.1 implies exponential convergence of q_1 to q_1^* which is within $\sigma_{\min}(K_p) \sup_{q_1} \|k(q_1)\|$ from the true q_{1d}.

4. New results in tracking control

4.1. *Exponentially stable algorithms*

Frequently a robot is required to follow a prespecified path for continuous action at the end effector (e.g. arc welding), tracking of a moving target (e.g. pick and place operation from conveyer belt) or other high level objectives (e.g. time optimality, collision avoidance, arm singularity avoidance). This can be posed as the problem of tracking the desired positions and velocities (q_{1d}, q_{2d}) by (q_1, q_2). In this section, we extend the basic ideas put forth in § 3 to the tracking problem. The error equation is now in the form

$$\left.\begin{aligned} \Delta\dot{q}_1 &= \Delta q_2 \\ M(q_1)\Delta\dot{q}_2 &= -C(q_1, q_2)q_2 - k(q_1) + u - M(q_1)\dot{q}_{2d} \end{aligned}\right\} \tag{4.1}$$

We will first state several direct generalizations of Theorem 3.1. An energy type Lyapunov function similar to (2.25) used in § 2.5 to motivate computed torque is used here.

Theorem 4.1

Consider (4.1) with the control law

$$u = -K_v\Delta q_2 + k(q_1) - \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + M(q_1)\dot{q}_{2d} - D(q_1, q_2, q_{2d}) \tag{4.2 a}$$

where D is given by any one of the following expressions

$$D(q_1, q_2, q_{2d}) = \tfrac{1}{2}(J(q_1, q_2)q_{2d} - M_D(q_1, q_{2d})q_2) \quad (4.2\ b)$$

$$D(q_1, q_2, q_{2d}) = \tfrac{1}{2}(J(q_1, q_{2d})q_2 - M_D(q_1, q_2)q_{2d}) \quad (4.2\ c)$$

$$D(q_1, q_2, q_{2d}) = \tfrac{1}{2}(J(q_1, q_{2d})q_{2d} - M_D(q_1, q_2)q_{2d}) \quad (4.2\ d)$$

$$D(q_1, q_2, q_{2d}) = \tfrac{1}{2}(J(q_1, q_2)q_2 - M_D(q_1, q_{2d})q_2) \quad (4.2\ e)$$

Assume $\exists\, v > 0$ such that

$$\Delta q_1^{\mathrm{T}} \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) > v\|q_1\|^2 \quad (4.3)$$

and $U^*(\Delta q_1)$ is positive-definite in Δq_1. Then the null state of the $(\Delta q_1, \Delta q_2)$ system is a globally exponentially stable equilibrium.

Proof

Use the following Lyapunov function

$$V(\Delta q_1, q_2) = \frac{1}{2}\Delta q_2^{\mathrm{T}} M(q_1)\Delta q_2 + U^*(\Delta q_1) + c\Delta q_1^{\mathrm{T}} M(q_1)\Delta q_2 + \tfrac{1}{2}c\Delta q_1^{\mathrm{T}} K_v \Delta q_1 \quad (4.4)$$

where c is a small constant, such that V is positive-definite in Δq_1 and Δq_2. Take the derivative along the solution

$$\dot{V}(\Delta q_1, \Delta q_2) = \Delta q_2^{\mathrm{T}}(M(q_1)\Delta \dot{q}_2 + \tfrac{1}{2}M_D(q_1, \Delta q_2)q_2 + \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + cM(q_1)\Delta q_2$$
$$+ cK_v\Delta q_1) + c\Delta q_1^{\mathrm{T}}(M(q_1)\Delta \dot{q}_2 + M_D(q_1, \Delta q_2)q_2)$$

Substitute (4.1) and (4.2) and use (2.7)

$$\begin{aligned}\dot{V}(\Delta q_1, \Delta q_2) &= \Delta q_2^{\mathrm{T}}(-C(q_1, q_2)q_2 - k(q_1) + u - M(q_1)\dot{q}_{2d} + \tfrac{1}{2}M_D(q_1, \Delta q_2)q_2 \\ &\quad + \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + cM(q_1)\Delta q_2 + cK_v\Delta q_1) \\ &\quad + c\Delta q_1^{\mathrm{T}}(-C(q_1, q_2)q_2 - k(q_1) + u - M(q_1)\dot{q}_{2d} + M_D(q_1, \Delta q_2)q_2) \\ &= r(q_1, q_2, q_{2d}) - \Delta q_2^{\mathrm{T}}(K_v - cM(q_1))\Delta q_2 - \Delta q_2^{\mathrm{T}} D(q_1, q_2, q_{2d}) \\ &\quad + c\Delta q_1^{\mathrm{T}}(M_D(q_1, \Delta q_2)q_2 - C(q_1, q_2)q_2) \\ &\quad - c\Delta q_1^{\mathrm{T}}\frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) - c\Delta q_1^{\mathrm{T}} D(q_1, q_2, q_{2d})\end{aligned}$$

Applying Identity 5, $r - \Delta q_2^{\mathrm{T}} D = 0$. Define η_1, η_2 as follows

$$\eta_1 = \sup_{q_1 \in R^n} \sum_{i=1}^{n} \|M_i(q_i)\|$$

$$\eta_2 = \sup_t \|q_{2d}\|\eta_1$$

From Identity 6

$$|c\Delta q_1^{\mathrm{T}}(M_D(q_1, \Delta q_2)q_2 - C(q_1, q_2)q_2 - D(q_1, q_2, q_{2d})|$$
$$\leqslant c\|\Delta q_1\|\left(a\eta_2\|\Delta q_2\| + \frac{\eta_1}{2}\|\Delta q_2\|^2\right)$$

where

$$a = \begin{cases} \frac{3}{2} & \text{for } (4.2\ b) \\ \frac{3}{2} & \text{for } (4.2\ c) \\ \frac{5}{2} & \text{for } (4.2\ d) \\ \frac{1}{2} & \text{for } (4.2\ e) \end{cases}$$

Hence,

$$\dot{V}(\Delta q_1, \Delta q_2) \leqslant -(\sigma_{\min}(K_v) - c\mu)\|\Delta q_2\|^2 - c\nu\|\Delta q_1\|^2 + c\|\Delta q_1\|\left(a\eta_2\|\Delta q_2\| + \frac{\eta_1}{2}\|\Delta q_2\|^2\right)$$

Completing the square for the cross-term

$$\dot{V}(\Delta q_1, \Delta q_2) \leqslant -\alpha_1\|\Delta q_1\|^2 - \alpha_2\|\Delta q_2\|^2 + \gamma_{21}\|\Delta q_1\|\,\|\Delta q_2\|^2 \tag{4.5}$$

where

$$\alpha_1 = c(\nu - \tfrac{1}{2}a\eta_2\rho^2)$$

$$\alpha_2 = \sigma_{\min}(K_v) - c\left(\mu + \frac{1}{2}\frac{a\eta_2}{\rho^2}\right)$$

$$\gamma_{21} = \tfrac{1}{2}c\eta_1$$

Given ν, choose $\rho^2 < 2\nu/a\eta_2$. By Lemma 2.1, for

$$c < \sigma_{\min}(K_v)\left(\mu + \frac{1}{2}\frac{a\eta_2}{\rho^2} + \frac{1}{2}\eta_1\left(\frac{V_0}{\xi_1}\right)^{1/2}\right)^{-1}$$

(V_0, ξ_1 are as defined from the proof of Theorem 3.1) and $\forall\lambda_2 \in (0, \alpha_1 - \gamma_{21}(V_0/\xi_1)^{1/2})$

$$\dot{V} \leqslant -\alpha_1\|\Delta q_1\|^2 - \lambda_2\|\Delta q_2\|^2 \leqslant -\gamma V$$

for some $\lambda > 0$. Hence, the closed-loop system is exponentially stable. □

A common Lyapunov function used for the tracking problem has been presented by Koditschek (1987) and Paden (1986)

$$V(\Delta q_1, \Delta q_2) = \tfrac{1}{2}\Delta q_2^{\mathrm{T}} M(q_1)\Delta q_2 + U^*(\Delta q_1)$$

In this case, a generalization of the invariance principle to the time-varying case is required. There are two possibilities. The result of Arstein (1976 Theorem A.7.6) appears promising but we must verify that (4.1) is positive pre-compact (in the sense defined by Arstein 1976). A more direct route is to use Lemma 1 of Yuan and Wonham (1977) which states that if $\Delta\dot{q}_2$ and $\Delta\ddot{q}_2$ are both bounded uniformly in t (which follows from $\dot{V} \leqslant 0$), then $\Delta q_2(t) \to 0$ implies $\Delta\dot{q}_2(t) \to 0$.

Note that $U^*(\Delta q_1)$ does not depend on time explicitly. This restriction eliminates some of the candidates used in the set point case. How to generalize to the case of $U^*(\Delta q_1, t)$ and $(\partial U^*/\partial t)(\Delta q_1, t)$ not necessarily negative-semidefinite is currently under investigation.

Control laws (4.2 *a–e*) all have the same stability property nominally. When q_2 is a very noisy measurement, as is typically the case, (4.2 *d*), which only uses q_2 once, may have better robustness.

Note that all the controllers have structures very similar to computer torque; in fact, if all occurrences of q_{2d} are replaced by q_2, then the non-linear compensation is exactly the same as the case of computer torque. However, in their present forms, (4.2 *a*)–(4.2 *e*) cannot take advantage of well-known recursive algorithms for inverse dynamics computation (Luh *et al.* 1981, Rodriguez 1987). Therefore, we next present slightly modified versions that can be implemented with these algorithms.

Corollary 4.1

Consider (4.1) with the control law

$$u = -K_v \Delta q_2 + k(q_1) - \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + M(q_1)\dot{q}_{2d} + C(q_1, q_{2d})q_{2d} \tag{4.6}$$

where $U^*(\Delta q_1)$ satisfies the same assumptions as in Theorem 4.1.

If

$$\sigma_{\min}(K_v) > \frac{\eta_2}{2}$$

then the null state of the $(\Delta q_1, \Delta q_2)$ system is a globally exponentially stable equilibrium.

Proof

Note that (4.6) is (4.2 *d*) substituted into (4.2 *a*) with the additional term $\frac{1}{2}M_D(q_1, \Delta q_2)q_{2d}$. Using the same Lyapunov function (4.4), the derivative along solution contains the extra terms

$$\tfrac{1}{2}\Delta q_2^{\mathrm{T}} M_D(q_1, \Delta q_2)q_{2d} + \frac{c}{2}\Delta q_1^{\mathrm{T}} M_D(q_1, \Delta q_2)q_{2d}$$

which can be overbounded by

$$\frac{\eta_2}{2}\|\Delta q_2\|^2 + \frac{c\eta_2}{2}\|\Delta q_1\|\,\|\Delta q_2\|$$

This is reduced to the case in Theorem 4.1 with $\sigma_{\min}(K_v)$ replaced by $\sigma_{\min}(K_v) - (\eta_2/2)$ (which is positive by assumption) and $a = 5/2$ by $a = 3$. The same conclusion then follows as before. □

Corollary 4.2

Consider (4.1) with the control law

$$u = -K_v \Delta q_2 + k(q_1) - \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + M(q_1)\dot{q}_{2d} + C(q_1, q_2)q_2 \tag{4.7}$$

where $U^*(\Delta q_1)$ satisfies the assumptions as in Theorem 4.1. Given a set of possible initial conditions, if K_v is sufficiently large, then the closed-loop system is exponentially stable with respect to that set.

Proof

Direct substitution into $\dot{V}$ yields

$$\dot{V}(\Delta q_1, \Delta q_2) \leqslant -\left(\sigma_{\min}(K_v) - \frac{\eta_2}{2} - c\mu\right)\|\Delta q_2\|^2 - cv\|\Delta q_1\|^2$$
$$+ c\eta_2 \|\Delta q_1\| \, \|\Delta q_2\| + c\eta_1 \|\Delta q_1\| \, \|\Delta q_2\|^2 + \frac{\eta_1}{2}\|\Delta q_2\|^3$$
$$\leqslant -\alpha_1 \|\Delta q_1\|^2 - \alpha_2 \|\Delta q_2\|^2 + \gamma_{21}\|\Delta q_1\| \, \|\Delta q_2\|^2 + \gamma_{22}\|\Delta q_2\|^3$$

where

$$\alpha_1 = c(v - \tfrac{1}{2}\eta_2\rho^2)$$
$$\alpha_2 = \delta_{\min}(K_v) - \frac{\eta_2}{2} - c\left(\mu + \frac{1}{2}\frac{\eta_2}{\rho^2}\right)$$
$$\gamma_{21} = c\eta_1$$
$$\gamma_{22} = \tfrac{1}{2}\eta_1$$

Select $\rho^2 < 2v/\eta_2$ so that $\alpha_1 > 0$. By Lemma 2.1, if $\sigma_{\min}(K_v)$ is chosen such that

$$\alpha_2 - \gamma_{21}\left(\frac{V_0}{\xi_1}\right)^{1/2} - \gamma_{22}\left(\frac{V_0}{\xi_2}\right)^{1/2} > 0$$

(This is possible since $(V_0/\xi_2)^{1/2} \sim \mathcal{O}(\|K_v\|^{1/2})$ and $(V_0/\xi_1)^{1/2}$ is bounded as $\|K_v\| \to \infty$.) then $\exists\, \lambda > 0$ such that

$$\dot{V} \leqslant -\lambda V \qquad \square$$

If $U^*(\Delta q_1) = \frac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1$, (4.7) is actually a modification of the computed torque method with K_p, K_v replacing $M(q_1)K_p$, $M(q_1)K_v$.

We have so far generated many control laws that are similar to computed torque. However, the ones just requiring $K_v > 0$, $v > 0$ are not easily implementable and the easily implementable ones need stronger conditions (K_v sufficiently large). The next control law that we shall present is of very appealing structure: the real time update computations are linear and the off-line computation can take advantage of efficient algorithms (e.g. Newton–Euler type). The trade-off is that K_v and v must both be large enough for a given set of initial conditions.

Theorem 4.2

Consider (4.1) with the control law

$$u = -K_v\Delta q_1 + k(q_{1d}) - \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + M(q_{1d})\dot{q}_{2d} + C(q_{1d}, q_{2d})q_{2d} \qquad (4.8)$$

where $U^*(\Delta q_1)$ satisfies the assumptions as in Theorem 4.1. Given a set of possible initial conditions, if K_v and v are sufficiently large, then the closed-loop system is exponentially stable with respect to that set.

Proof

Note that (4.8) is (4.6) with the additional terms

$$k(q_{1d}) - k(q_1) + (M(q_{1d}) - M(q_1))\dot{q}_{2d} + (C(q_{1d}, q_{2d}) - C(q_1, q_{2d}))q_{2d}$$

This produces the following additional terms in $\dot{V}$

$$(\Delta q_2 + c\Delta q_1)^{\mathrm{T}}(k(q_{1d}) - k(q_1) + (M(q_{1d}) - M(q_1))\dot{q}_{2d} + (C(q_{1d}, q_{2d}) - C(q_1, q_{2d}))q_{2d})$$

Note that k, M, C only contain trigonometric functions of q_1 hence the derivative of each element with respect to q_1 is bounded. Let

$$\eta_3 = \sup_{q_1 \in R^n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial \, \mathrm{Col}_j [M_i(q_1)]}{\partial q_1} \right\|$$

$$\eta_4 = \sup_{q_1 \in R^n} \sum_{j=1}^{n} \left\| \frac{\partial \, \mathrm{Col}_j [M(q_1)]}{\partial q_1} \right\|$$

$$\eta_5 = \sup_{q_1 \in R^n} \left\| \frac{\partial k(q_1)}{\partial q_1} \right\|$$

$$\eta_6 = \sup_{t \in R_+} \|q_{2d}(t)\|$$

$$\eta_7 = \sup_{t \in R_+} \|\dot{q}_{2d}(t)\|$$

The additional terms in $\dot{V}$ can be overbounded by

$$(\|\Delta q_2\| + c\|\Delta q_1\|)\|\Delta q_1\|(\eta_5 + \eta_4\eta_7 + \tfrac{3}{2}\eta_6^2\eta_3)$$

Now we can write down $\dot{V}$ as

$$\begin{aligned}
\dot{V}(\Delta q_1, \Delta q_2) \leqslant & -\left(\sigma_{\min}(K_v) - \frac{\eta_2}{2} - c\mu\right)\|\Delta q_2\|^2 - cv\|\Delta q_1\|^2 \\
& + c\|\Delta q_1\|\left(3\eta_2\|\Delta q_2\| + \frac{\eta_1}{2}\|\Delta q_2\|^2\right) \\
& + \|\Delta q_1\|(\|\Delta q_2\| + c\|\Delta q_1\|)\left(\eta_5 + \eta_4\eta_7 + \frac{3}{2}\eta_6^2\eta_3\right) \\
\leqslant & -\left(\sigma_{\min}(K_v) - \frac{\eta_2}{2} - c\mu\right)\|\Delta q_2\|^2 - c\left(v - \left(\eta_5 + \eta_4\eta_7 + \frac{3}{2}\eta_6^2\eta_3\right)\right)\|\Delta q_1\|^2 \\
& + \left(3c\eta_2 + \eta_5 + \eta_4\eta_7 + \frac{3}{2}\eta_6^2\eta_3\right)\|\Delta q_1\|\,\|\Delta q_2\| + \frac{c\eta_1}{2}\|\Delta q_1\|\,\|\Delta q_2\|^2
\end{aligned}$$

Let

$$\eta_8 = \tfrac{1}{2}(3c\eta_2 + \eta_5 + \eta_4\eta_7 + \tfrac{3}{2}\eta_6^2\eta_3)$$

Then

$$\dot{V}(\Delta q_1, \Delta q_2) \leqslant -\alpha_1\|\Delta q_1\|^2 - \alpha_2\|\Delta q_2\|^2 + \gamma_{21}\|\Delta q_1\|\,\|\Delta q_2\|^2$$

where

$$\alpha_1 = c(v - (\eta_5 + \eta_4\eta_7 + \tfrac{3}{2}\eta_6^2\eta_3) - \rho^2\eta_8)$$

$$\alpha_2 = \sigma_{\min}(K_v) - \frac{1}{2}\eta_2 - \frac{\eta_8}{\rho^2} - c\mu$$

$$\gamma_{21} = \tfrac{1}{2}c\eta_1$$

Equation number	Control laws	Condition for exponential stability	Comments
(2.32)	$u = -K_v q_2 - \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + \frac{\partial U^\circ}{\partial \Delta q_1}(\Delta q_1)$	$U^*(\Delta q_1)$ has global minimum at $\Delta q_1 = 0$ $v > 0$ $\Delta q_1^{\mathrm{T}} \frac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) \geqslant v\|\Delta q_1\|^2, \quad K_v > 0$	General set point control
(2.28)	$u = -K_v q_2 - K_p \Delta q_1 + k(q_1)$	$K_p > 0, K_v > 0$	$U^*(\Delta q_1) = \frac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1$
(3.2)	$u = -K_v q_2 - K_p \Delta q_1 + k(q_{1d})$	$K_p > \sup_{q_1} \frac{\partial k(q_1)}{\partial q_1}, \quad K_v > 0$	$U^*(\Delta q_1) = \frac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1 + g(\Delta q_1 + q_{1d}) - g(q_{1d}) - \Delta q_1^{\mathrm{T}} k(q_{1d})$
(3.8)	$u = -K_v q_2 - K_p \Delta q_1 - \tilde{H}(\Delta q_1) - \tilde{L}(\Delta q_1) + k(q_1)$	$K_p > 0, \quad K_v > 0, \quad \underline{q}_{1i} \leqslant q_{1id} \leqslant \bar{q}_{1i}$	$\tilde{H}, \tilde{L}$ given in (3.5)–(3.6) $U^*(q_1) = \frac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1 + \sum_{i=1}^{n} [H_i(\Delta q_{1i} + q_{1id}) + L_i(\Delta q_{1i} + q_{1id})]$
(3.9)	$u = -K_v q_2 - K_p \Delta q_1 - \tilde{H}(\Delta q_1) - \tilde{L}(\Delta q_1) + k(q_{1d})$	$K_p > \sup_{q_1} \frac{\partial k(q_1)}{\partial \Delta q_1}, \quad K_v > 0, \underline{q}_{1i} \leqslant q_{1id} \leqslant \bar{q}_{1i}$ $\lim_{t\to\infty} \sup \|\Delta q_1(t)\| \leqslant \sigma_{\min}^{-1}(K_p) \sup_{q_1} \|k(q_1)\|$ $K_v > 0$	$U^*(\Delta q_1) = \frac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1 + g(\Delta q_1 + q_{1d}) - g(q_{1d}) - \Delta q_1^{\mathrm{T}} k(q_{1d}) + \sum_{i=1}^{n} [H_i(\Delta q_{1i} + q_{1id}) + L_i(\Delta q_{1i} + q_{1id})]$
(3.11)	$u = -K_v q_2 - K_p \Delta q_1$	If $K_p > \sup_{q_1} \frac{\partial k(q_1)}{\partial q_1}, \quad \lim_{t\to\infty} q_1(t) = q_1^*$ $K_p(q_1^* - q_{1d}) + k(q_1^*) = 0$	$U^*(\Delta q_1) = \frac{1}{2}\Delta q_1^{\mathrm{T}} K_p \Delta q_1 + g(\Delta q_1 + q_{1d})$ Global Lagrange stability
(2.34)	$u = M(q_1)(-K_p \Delta q_1 - K_v \Delta q_2 - \dot{q}_{2d}) + k(q_1) + C(q_1, q_2) q_2$	$K_p > 0, \quad K_v > 0$	Computed torque; Newton–Euler algorithm can be used to update control law

(4.2 *b*)	$u = -K_v \Delta q_2 - \dfrac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + M(q_1)\dot{q}_{2d} + k(q_1)$ $-\frac{1}{2}(J(q_1, q_2)q_{2d} - M_D(q_1, q_{2d})q_2)$	$K_v > 0$; U^* has global minimum at $\Delta q_1 = 0$; U^* time invariant; $\nu > 0 \quad \Delta q_1^T \dfrac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) \geqslant \nu \|\Delta q_1\|^2$	Convergence rate depends on initial condition, Newton–Euler algorithm not applicable
(4.2 *c*)	$u = -K_v \Delta q_2 - \dfrac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + M(q_1)\dot{q}_{2d} + k(q_1)$ $-\frac{1}{2}(J(q_1, q_{2d})q_2 - M_D(q_1, q_2)q_{2d})$	Same as (4.2 *a*)	Same as (4.2 *a*)
(4.2 *d*)	$u = -K_v \Delta q_2 - \dfrac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + M(q_1)\dot{q}_{2d} + k(q_1)$ $-\frac{1}{2}(J(q_1, q_{2d})q_{2d} - M_D(q_1, q_2)q_{2d})$	Same as (4.2 *a*)	Same as (4.2 *a*)
(4.2 *e*)	$u = -K_v q_2 - \dfrac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + M(q_1)\dot{q}_{2d} + k(q_1)$ $-\frac{1}{2}(J(q_1, q_2)q_2 - M_D(q_1, q_{2d})q_2)$	Same as (4.2 *a*)	Same as (4.2 *a*)
(4.6)	$u = -K_v \Delta q_2 - \dfrac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + k(q_1)$ $+ M(q_1)q_{2d} + C(q_1, q_{2d})q_{2d}$	Same condition on U^* as in (4.2 *a*); $K_v > \dfrac{\eta}{2}$	Newton–Euler algorithm can be applied to $(q_1, q_{2d}, \dot{q}_{2d})$; convergence rate depends on initial condition
(4.7)	$u = -K_v \Delta q_2 - \dfrac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + k(q_1)$ $+ M(q_1)q_{2d} + C(q_1, q_2)q_2$	$\nu > 0$, K_v sufficiently large with respect to a given set of initial conditions	Modified computed torque; Newton–Euler algorithm can be applied to $(q_1, q_2, \dot{q}_{2d})$; convergence rate depends on initial condition
(4.8)	$u = -K_v \Delta q_2 - \dfrac{\partial U^*}{\partial \Delta q_1}(\Delta q_1) + k(q_{1d})$ $+ M(q_{1d})q_{2d} + C(q_{1d}, q_{2d})q_{2d}$	ν, K_v sufficiently large with respect to a given set of initial conditions	Non-linear compensation can be computed off-line by applying Newton–Euler algorithm to $(q_{1d}, q_{2d}, \dot{q}_{2d})$; convergence rate depends on initial condition

Table 1. Summary of control laws.

Choose $\nu > \eta_5 + \eta_4\eta_7 + \frac{3}{2}\eta_6^2\eta_3$. Then ρ^2 can be selected so that $\alpha_1 > 0$. Choose $\sigma_{\min}(K_v)$ large enough such that

$$\alpha_2 - \gamma_{21}\left(\frac{V_0}{\xi_1}\right)^{1/2} > 0$$

Then from Lemma 2.1, $\exists\,\lambda > 0$ such that

$$\dot{V} \leqslant -\lambda V \qquad \square$$

Typically, $q_2(0) - q_{2d}(0) = 0$ and $\Delta q_1(0)$ is always within 2π. Hence V_0 is bounded above and the result is essentially a global one. This scheme requires both ν and K_v large enough. This requirement is made easier by shifting the computational burden to off-line thus allowing very high sampling rates which in turn means high gains can be tolerated.

5. Summary

We have introduced a new class of exponentially stabilizing control laws for the joint level control of robot manipulators (summarized in Table 1). The stability result is achieved by making use of a particular class of energy-like Lyapunov functions (of the form (4.4)) in conjunction with a useful lemma (Lemma 2.1) for addressing higher order terms in Lyapunov function derivatives. This approach avoids the need for a generalization of the invariance principle to time-varying systems, which has been the major source of difficulty in the past (Koditschek 1987, Paden 1986).

In the set point control case, by incorporating artificial potential fields in the Lyapunov function we have derived a class of exponentially stabilizing, PD + potential shaping type control laws. Several useful potential fields have been examined resulting in simple structures: PD and PD + bias, and the ability to handle joint stop constraints with a PD + joint-stop-barrier controller.

In the tracking control case, the modified Lyapunov function leads to a new class of exponentially stable control laws. This class of control laws offers an alternative to the conventional computed torque method and provides trade-offs between on-line computation (which directly relates to performance through maximum sampling rate) and condition for stability. In one new design, (4.8), the non-linear structure is decoupled from the real-time measurements which completely removes the requirement for on-line non-linear computation. Figure 3 illustrates the trade-offs in the various tracking control laws.

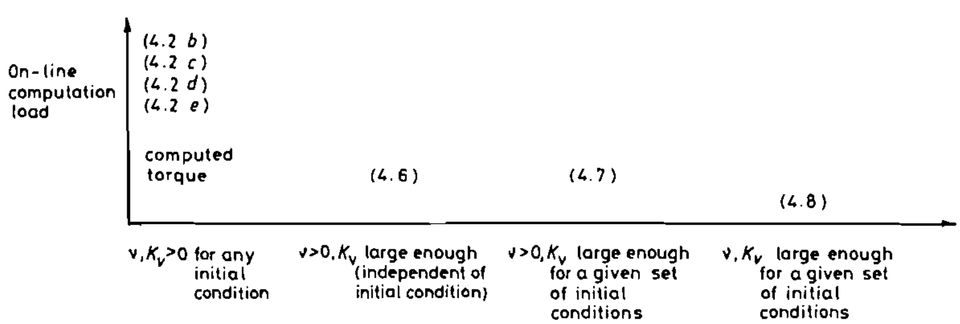

Figure 3. Tracking control law trade-offs.

The new stability analysis and controller design techniques presented in this paper open up many promising avenues for future research. In particular, our current research directions include ways to incorporate time-varying artificial potential fields in the tracking problem and the generalization of exponentially stabilizing joint-level control laws to the task space.

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References

Arimoto, S., and Miyazaki, F., 1983, Stability and robustness of PID feedback control for robot manipulators of sensory capability. *Proc. Int. Symp. on Robotics Research* (Cambridge, Mass: MIT Press).

Arstein, Z., 1976, Limiting equations and stability of nonautomous ordinary differential equations. *The Stability of Dynamical Systems*, edited by J. P. LaSalle, CBMS Regional Conference Series in Applied Math (Philadelphia, PA: SIAM Publishing).

Asada, H., and Slotine, J.-J. E., 1986, *Robot Analysis and Control* (New York: Wiley).

Bayard, D. S., and Wen, J. T., *Int. J. Control*, **47,** 1387.

Bejczy, A. K., 1974, Robot arm dynamics and control. Jet Propulsion Laboratory, California Institute of Technology, TM 33-669.

Golla, D. F., 1973, Linear state feedback control of rigid link manipulators. University of Toronto, Institute for Aerospace Studies.

Hale, J. K., 1969, *Ordinary Differential Equations* (New York: Wiley Interscience).

Koditschek, D. E., 1984, *Proc. I.E.E.E. Conf. on Decision and Control*, Las Vegas, NV, pp. 733–735; 1987, Natural control of robot arms. *I.E.E.E. J. Robotics Autom.*, (submitted for publication).

La Salle, J. P., 1960, *I.R.E. Trans. Circuit Theory*, **7,** 520.

Luh, J. Y. S., Walker, M. W., Paul, R. P. C., 1980, *A.S.M.E. Jl. dynam. Syst. Meas. Control*, **102,** 69.

Meirovitch, L., 1970, *Methods of Analytical Dynamics* (New York: McGraw-Hill).

Miyazaki, F., and Arimoto, S., 1985, *J. Robotic Syst.*, **2,** 53.

Paden, B., 1986, Private communication.

Pringle, Jr., R., 1966, *A.I.A.A. J.*, **4,** 1395.

Rodriguez, G., 1987, Kalman filtering, smoothing and recursive robot arm forward and inverse dynamics. Jet Propulsion Laboratory, California Institute of Technology, JPL Publication.

Rouche, N., Habets, P., and Laloy, M., 1977, *Stability Theory by Lyapunov's Direct Method* (New York: Springer-Verlag).

Siljak, D. D., 1978, *Large Dynamic Systems* (Amsterdam: North-Holland).

Slotine, J.-J. E., and Li, W., 1986, On the adaptive control of robot manipulators. A.S.M.E. Winter Meeting, Anaheim, CA.

Takegaki, M., and Arimoto, S., 1981, *A.S.M.E. Jl. dynam. Syst. Meas. Control*, **102,** 119.

Wen, J. T., 1986, On minimum time control of robot manipulators. *Recent Trends in Robotics, Modeling, Control and Education*, edited by M. Jamshidi, J. Y. S. Luh and M. Shahinpoor (Amsterdam: North-Holland).

Yuan, J. S. C., and Wonham, W. M., 1977, *I.E.E.E. Trans. autom. Control*, **4,** 530.