Understanding Dirac Notation in QM

P. H. Tam

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1 Introduction

When I was an undergrad year 3 (final year) student, I start learning QM with Dirac notation (Braket notation). But I have a hard time in understanding $\psi(x)$ versus $|\psi\rangle$. I was confused. Is it $\psi(x) = |\psi\rangle$? or there is something I am missing? I self studied this topic with reference *Modern Quantum Mechanics, J. J. Sakurai* and finally get the picture right. In this note, I am going to demystify the Dirac notation, not involving rigorous math as I am not a math guy personally.

Prerequisite: **(1)** Familiar with Schrödinger equation in real-space; **(2)** Know momentum-space wavefunction; **(3)** Good understanding in eigenvalues and eigenvectors and idea of basis vectors; **(4)** Know basic Dirac notation but find it confusing!

2 Real-space vs Momentum-space

It all begins with which prespective we want to describe the quantum system.

	Real-space	Momentum-space
Wavefunction	$\psi(x)$	$\phi(k)$
Domain	position	momentum

Both $\psi(x)$ and $\phi(k)$ have a complete information in describing the system. It is a matter of taste or convenience. Let's think of a classical wave signal, it is natural to represent the signal in time domain, but usually engineers prefer working in frequency domain. Both representations contain equivalent information. This kind of switching in domain also applies to quantum system.

As $\psi(x)$ and $\phi(k)$ are actually representing the same quantum system, then why don't we invent a single mathematical object / symbol that is **independent of the choice of domain or basis**.

Now we introduce the Dirac notation for wavefunction: $|\psi\rangle$, which does not have argument of x or k. Unfortunately, we use the same symbol for the ket vector and the real-space wavefunction, but just bare with it and remember that they are saying a slightly different thing.

Then, the second question is how does $|\psi\rangle$ relate to $\psi(x)$ and $\phi(k)$. The way that I think of it is as follow (very personally, not sure if the math is meant to be like that, but very useful).

We project $|\psi\rangle$ to the real space (or momentum space) with the basis $|x\rangle$ (or $|k\rangle$)

$$\psi(x) = \langle x|\psi\rangle \tag{1}$$
$$\phi(k) = \langle k|\psi\rangle \tag{2}$$

Hopefully, you now have a better idea on what Dirac notation is.

3 Momentum Operator

Another confusing issue that I would like to talk is operator, especially momentum operator. When you first learn momentum operator $\hat{p}$ in 1D (in real space), you are told with this equality

$$\hat{p} \;=\; i\hbar\frac{\partial}{\partial x} \tag{3}$$

and I don't like it very much (so does J. J. Sakurai) (in his book, he prefer the notation := instead of =). Why is that the case? Just like the wavefunction $|\psi\rangle$ we discussed in the last section, operator can be represented in different domain (or basis). Therefore, using the equality sign is misleading.

Here, I am going to re-derive momentum operator in real-space representation, using some "formal" approach. We shall stick with 1D only as 2D or 3D generalisation are similar.

3.1 Position and Momentum Basis

We define $|x'\rangle$ to be the wavefunction with certainty of finding the particle at position x'. Also, we define $|p'\rangle$ to be the wavefunction with certainty of finding the particle at momentum p'.

Now that gives us a way to define position operator $\hat{x}$ and momentum operator $\hat{p}$ through the eigenvalue equations

$$\hat{x}\,|x'\rangle \;=\; x'\,|x'\rangle \tag{4}$$

$$\hat{p}\,|p'\rangle \;=\; p'\,|p'\rangle \tag{5}$$

We say that $|x'\rangle$ is the eigenstate of position operator $\hat{x}$ (or simply position eigenstate). Also, we say that $|p'\rangle$ is the eigenstate of momentum operator $\hat{p}$ (or simply momentum eigenstate). These eigenstate have the following orthogonal property

$$\langle x''|x'\rangle \;=\; \delta(x'-x'') \tag{6}$$

$$\langle p''|p'\rangle \;=\; \delta(p'-p'') \tag{7}$$