

# Review of Classical Electromagnetism

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## 1 Maxwell's Equations and Lorentz Force Law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \quad (\text{Gauss's Law for electricity}) \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Gauss's Law for magnetism}) \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{Faraday's Law}) \quad (3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (\text{Ampere's Law with Maxwell's Correction}) \quad (4)$$

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \quad (\text{Lorentz Force Law}) \quad (5)$$

These 4+1 equations are the foundations of *everything* in classical electromagnetism.

## 2 Light is Electromagnetic Wave

Taking **curl** of Eq. (3) and Eq. (4).

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t} \quad (6)$$

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 (\nabla \times \mathbf{J}) + \mu_0 \epsilon_0 \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \quad (7)$$

By the identity:  $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$ , we simplify Eq. (6) and (7) as follow.

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t} \quad (8)$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 (\nabla \times \mathbf{J}) + \mu_0 \epsilon_0 \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \quad (9)$$

For Eq. (8), we simplify it using Maxwell's equation Eq. (1) and Eq. (4). For Eq. (9), we simplify it using Maxwell's equation Eq. (2) and Eq. (3). **Thus, we obtain the wave equations, showing that electric and magnetic field can propagate as a wave.**

$$\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = -\frac{1}{\epsilon_0} \nabla \rho - \mu_0 \frac{\partial \mathbf{J}}{\partial t} \quad (10)$$

$$\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \mu_0 \nabla \times \mathbf{J} \quad (11)$$

Recall that wave equation (e.g. a vibrating string with displacement  $s$  and wave speed  $v$ ) without source term is in the form of

$$\nabla^2 s = \frac{1}{v^2} \frac{\partial^2 s}{\partial t^2} \quad (12)$$

Therefore, we conclude that the speed of electromagnetic wave in vacuum is

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3 \times 10^8 \text{ ms}^{-1} \quad (\text{the speed of light}) \quad (13)$$

Optical phenomena, such as reflection, refraction, can be derived by solving the wave equations Eq. (10) and (11) with boundary conditions determined by Maxwell's equations.

### 3 Charge Conservation

Consider the Ampere's Law with Maxwell's Correction, Eq. (4).

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (14)$$

Taking divergence of both sides and note that  $\nabla \cdot \nabla \times (\dots) = 0$ , we have

$$0 = \nabla \cdot \mathbf{J} + \epsilon_0 \frac{\partial(\nabla \cdot \mathbf{E})}{\partial t} \quad (15)$$

Together with Gauss's Law of electricity, Eq. (1), we obtain the conservation of charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (16)$$

$$\Rightarrow \frac{dQ}{dt} + \oint_S \mathbf{J} \cdot d\mathbf{a} = 0 \quad (\text{divergence theorem}) \quad (17)$$

### 4 Energy Conservation

Consider the Lorentz force density acting on a charge distribution

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \quad (18)$$

The rate of work done (i.e. power) by the force is

$$\mathbf{f} \cdot \mathbf{v} = \rho \mathbf{E} \cdot \mathbf{v} + (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{v} \quad (19)$$

$$= \rho \mathbf{v} \cdot \mathbf{E} \quad (\because \mathbf{J} = \rho \mathbf{v}) \quad (20)$$

$$= \mathbf{J} \cdot \mathbf{E} \quad (21)$$

Now, we want to establish the relation to  $E$ -field and  $B$ -field. By Eq. (4), we have

$$\mathbf{J} \cdot \mathbf{E} = \frac{1}{\mu_0} \left[ \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right] \cdot \mathbf{E} \quad (22)$$

Using the vector calculus identity:  $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ , we have

$$\mathbf{J} \cdot \mathbf{E} = -\epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \left[ -\nabla \cdot (\mathbf{E} \times \mathbf{B}) + \mathbf{B} \cdot (\nabla \times \mathbf{E}) \right] \quad (23)$$

$$= -\epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \left[ -\nabla \cdot (\mathbf{E} \times \mathbf{B}) - \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \right] \quad \text{By Eq. (3)} \quad (24)$$

$$= -\frac{\partial}{\partial t} \underbrace{\left( \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right)}_u - \nabla \cdot \underbrace{\left( \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right)}_{\mathbf{S}} \quad (25)$$

$u$  is the electromagnetic field energy density.  $\mathbf{S}$  is the Poynting vector, which is the energy flux transferred in the direction of  $\mathbf{E} \times \mathbf{B}$ .

Finally, we obtain the conservation of energy.

$$\frac{\partial u}{\partial t} = -\mathbf{J} \cdot \mathbf{E} - \nabla \cdot \mathbf{S} \quad (26)$$

$$\Rightarrow \frac{\partial U}{\partial t} = -\int_V \mathbf{J} \cdot \mathbf{E} d\tau - \oint_S \mathbf{S} \cdot d\mathbf{a} \quad (\text{divergence theorem}) \quad (27)$$

## 5 Momentum Conservation

Consider the Lorentz Force Law

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \quad (28)$$

Making use of Maxwell's Equations Eq. (1) and (4), we replace  $\rho$  and  $\mathbf{J}$  obtain

$$\mathbf{f} = \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \left( \frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \times \mathbf{B} \quad (29)$$

To introduce Poynting vector  $\mathbf{S} = (\mathbf{E} \times \mathbf{B})/\mu_0$  in the last term, consider

$$\frac{\partial \mathbf{S}}{\partial t} = \frac{1}{\mu_0} \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0} \left( \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} - \mathbf{E} \times (\nabla \times \mathbf{E}) \right) \quad \text{By. Eq. (3)} \quad (30)$$

Hence, Eq. (29) becomes

$$\mathbf{f} = \epsilon_0 \left[ (\nabla \cdot \mathbf{E}) \mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E}) \right] - \frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}) - \mu_0 \epsilon_0 \frac{\partial \mathbf{S}}{\partial t} \quad (31)$$

Making use of the identity,  $\nabla(\mathbf{v} \cdot \mathbf{v}) = 2[\mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v}]$ , we simplify Eq. (31) as

$$\mathbf{f} = \underbrace{-\nabla \left( \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) + \epsilon_0 [(\nabla \cdot \mathbf{E}) \mathbf{E} + (\mathbf{E} \cdot \nabla) \mathbf{E}] + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{S}}{\partial t}}_{\nabla \cdot \mathbb{T}} \quad (32)$$

where  $\mathbb{T}$  is the **Maxwell stress tensor** given by

$$\mathbb{T} = \left( \epsilon_0 \mathbf{E} \otimes \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \otimes \mathbf{B} \right) - \left( \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) \mathbb{I} \quad (33)$$

Finally, we obtain the conservation of momentum.

$$\mathbf{f} = \nabla \cdot \mathbb{T} - \frac{1}{c^2} \frac{\partial \mathbf{S}}{\partial t} \quad (34)$$

$$\Rightarrow \mathbf{F} = \oint_S \mathbb{T} \cdot d\mathbf{a} - \frac{d}{dt} \underbrace{\frac{1}{c^2} \int_V \mathbf{S} d\tau}_{\mathbf{p}_{\text{em}}} \quad (\text{divergence theorem}) \quad (35)$$

$$\Rightarrow \frac{d}{dt} (\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{em}}) = \oint_S \mathbb{T} \cdot d\mathbf{a} \quad (36)$$

## 6 Potential

Since  $\nabla \cdot \mathbf{B} = 0$  from Eq. (2), and mathematically  $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ , we define a **vector potential**  $\mathbf{A}$  such that

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (37)$$

Now, we make use of Eq. (3) and Eq. (37)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( -\frac{\partial \mathbf{A}}{\partial t} \right) \implies \nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = \mathbf{0} \quad (38)$$

Mathematically,  $\nabla \times (\nabla v) = 0$ . Therefore, we can define a **(scalar) potential**  $\phi$  such that

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \quad (39)$$

Note that the potentials  $\mathbf{A}$  and  $\phi$  are not unique and they admit the **gauge transformation**:

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla f \quad \text{and} \quad \phi \rightarrow \phi - \frac{\partial f}{\partial t} \quad (40)$$

such that  $\mathbf{E}$  and  $\mathbf{B}$  remain unchanged.

Using Eq. (1), Gauss's Law for electricity, and Eq. (39) we have

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \quad (41)$$

Using Eq. (4), Ampere's Law with Maxwell's equation, and Eq. (37) and Eq. (39), we have

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = -\mu_0 \mathbf{J} \quad (42)$$

If we choose **Coulomb's Gauge**,

$$\nabla \cdot \mathbf{A} = 0 \quad (43)$$

Eq. (41) reduces to Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \implies \phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \quad (44)$$

However,  $\phi$  alone cannot determine  $\mathbf{E}$ , we also need to find  $\mathbf{A}$  by solving Eq. (42).

If we choose **Lorenz's Gauge**,

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t} \quad (45)$$

then  $\phi$  and  $\mathbf{A}$  are decoupled.

$$-\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = \frac{\rho}{\epsilon_0} \implies \phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \quad (46)$$

$$-\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \implies \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \quad (47)$$

where  $t_r = t - |\mathbf{r} - \mathbf{r}'|/c$  is the retarded time.