Review of Classical Electromagnetism

1 Maxwell's Equations and Lorentz Force Law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \qquad \text{(Gauss's Law for electricity)} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss's Law for magnetism)} \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law)} \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampere's Law with Maxwell's Correction)} \tag{4}$$

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \qquad \text{(Lorentz Force Law)} \tag{5}$$

These 4+1 equations are the foundations of *everything* in classical electromagnetism.

2 Light is Electromagnetic Wave

Taking **curl** of Eq. (3) and Eq. (4).

$$\nabla \times \Big(\nabla \times \mathbf{E} \Big) = -\frac{\partial \left(\nabla \times \mathbf{B} \right)}{\partial t} \tag{6}$$

$$\nabla \times \Big(\nabla \times \mathbf{B} \Big) = \mu_0 \Big(\nabla \times \mathbf{J} \Big) + \mu_0 \epsilon_0 \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \tag{7}$$

By the identity: $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$, we simplify Eq. (6) and (7) as follow.

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial \left(\nabla \times \mathbf{B} \right)}{\partial t} \tag{8}$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \Big(\nabla \times \mathbf{J} \Big) + \mu_0 \epsilon_0 \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \tag{9}$$

For Eq. (8), we simplify it using Maxwell's equation Eq. (1) and Eq. (4). For Eq. (9), we simplify it using Maxwell's equation Eq. (2) and Eq. (3). **Thus, we obtain the wave equations, showing that electric and magnetic field can propagate as a wave.**

$$\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = -\frac{1}{\epsilon_0} \nabla \rho - \mu_0 \frac{\partial \mathbf{J}}{\partial t} \tag{10}$$

$$\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \mu_0 \, \nabla \times \mathbf{J} \tag{11}$$

Recall that wave equation (e.g. a vibrating string with displacement s and wave speed v) without source term is in the form of

$$\nabla^2 s = \frac{1}{v^2} \frac{\partial^2 s}{\partial t^2} \tag{12}$$

Therefore, we conclude that the speed of electromagnetic wave in vacuum is

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3 \times 10^8 \, \mathrm{ms}^{-1} \quad (\textbf{the speed of light}) \tag{13}$$

Optical phenomena, such as reflection, refraction, can be derived by solving the wave equations Eq. (10) and (11) with boundary conditions determined by Maxwell's equations.

3 Charge Conservation

Consider the Ampere's Law with Maxwell's Correction, Eq. (4).

$$\nabla \times \mathbf{B} \;=\; \mu_0 \mathbf{J} \;+\; \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{14}$$

Taking divergence of both sides and note that $\nabla \cdot \nabla \times (\cdots) = 0$, we have

$$0 \;=\; \nabla \cdot \mathbf{J} \;+\; \epsilon_0 \frac{\partial (\nabla \cdot \mathbf{E})}{\partial t} \tag{15}$$

Together with Gauss's Law of electricity, Eq. (1), we obtain the conservation of charge

$$\frac{\partial \rho}{\partial t} \;+\; \nabla \cdot \mathbf{J} \;=\; 0 \tag{16}$$

$$\Longrightarrow \frac{dQ}{dt} \;+\; \oint_{\mathrm{S}} \mathbf{J} \cdot d\mathbf{a} \;=\; 0 \qquad \text{(divergence theorem)} \tag{17}$$

4 Energy Conservation

Consider the Lorentz force density acting on a charge distribution

$$\mathbf{f} \;=\; \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{18}$$

The rate of work done (i.e. power) by the force is

$$\begin{aligned} \mathbf{f} \cdot \mathbf{v} \;&=\; \rho \mathbf{E} \cdot \mathbf{v} \;+\; (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{v} && (19) \\ &=\; \rho \mathbf{v} \cdot \mathbf{E} \qquad (\because \mathbf{J} = \rho \mathbf{v}) && (20) \\ &=\; \mathbf{J} \cdot \mathbf{E} && (21) \end{aligned}$$

Now, we want to establish the relation to E-field and B-field. By Eq. (4), we have

$$\mathbf{J} \cdot \mathbf{E} \;=\; \frac{1}{\mu_0} \Big[\nabla \times \mathbf{B} \;-\; \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \Big] \cdot \mathbf{E} \tag{22}$$

Using the vector calculus identity: $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$, we have

$$\begin{aligned} \mathbf{J} \cdot \mathbf{E} \;&=\; -\epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \;+\; \frac{1}{\mu_0} \Big[-\nabla \cdot (\mathbf{E} \times \mathbf{B}) \;+\; \mathbf{B} \cdot (\nabla \times \mathbf{E}) \Big] && (23) \\ &=\; -\epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \;+\; \frac{1}{\mu_0} \Big[-\nabla \cdot (\mathbf{E} \times \mathbf{B}) \;-\; \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \Big] \qquad \text{By Eq. (3)} && (24) \\ &=\; -\frac{\partial}{\partial t} \underbrace{\left(\frac{1}{2} \epsilon_0 E^2 \;+\; \frac{1}{2\mu_0} B^2 \right)}_{u} - \nabla \cdot \underbrace{\left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right)}_{\mathbf{S}} && (25) \end{aligned}$$

u is the electromagnetic field energy density. $\mathbf{S}$ is the Poynting vector, which is the enery flux transferred in the direction of $\mathbf{E} \times \mathbf{B}$.

Finally, we obtain the conservation of energy.

$$\frac{\partial u}{\partial t} \;=\; -\mathbf{J} \cdot \mathbf{E} - \nabla \cdot \mathbf{S} \tag{26}$$

$$\Longrightarrow \frac{\partial U}{\partial t} \;=\; -\int_{\mathcal{V}} \mathbf{J} \cdot \mathbf{E} \, d\tau \;-\; \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a} \qquad \text{(divergence theorem)} \tag{27}$$

5 Momentum Conservation

Consider the Lorentz Force Law

$$\mathbf{f} \;=\; \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} \tag{28}$$

Making use of Maxwell's Equations Eq. (1) and (4), we replace ρ and $\mathbf{J}$ obtain

$$\mathbf{f} \;=\; \epsilon_0\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} \;+\; \left(\frac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B} \tag{29}$$

To introduce Poynting vector $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu_0$ in the last term, consider

$$\frac{\partial\mathbf{S}}{\partial t} \;=\; \frac{1}{\mu_0}\frac{\partial}{\partial t}\Big(\mathbf{E}\times\mathbf{B}\Big) \;=\; \frac{1}{\mu_0}\left(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} \;-\; \mathbf{E}\times\big(\nabla\times\mathbf{E}\big)\right) \qquad \text{By. Eq. (3)} \tag{30}$$

Hence, Eq. (29) becomes

$$\mathbf{f} \;=\; \epsilon_0\Big[(\nabla\cdot\mathbf{E})\mathbf{E} \;-\; \mathbf{E}\times\big(\nabla\times\mathbf{E}\big)\Big] \;-\; \frac{1}{\mu_0}\mathbf{B}\times\big(\nabla\times\mathbf{B}\big) \;-\; \mu_0\epsilon_0\frac{\partial\mathbf{S}}{\partial t} \tag{31}$$

Making use of the identity, $\nabla(\mathbf{v}\cdot\mathbf{v}) \;=\; 2\big[\mathbf{v}\times(\nabla\times\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{v}\big]$, we simplify Eq. (31) as

$$\mathbf{f} \;=\; \underbrace{-\nabla\left(\frac{1}{2}\epsilon_0 E^2 \;+\; \frac{1}{2\mu_0}B^2\right) \;+\; \epsilon_0\Big[(\nabla\cdot\mathbf{E})\mathbf{E} \;+\; (\mathbf{E}\cdot\nabla)\mathbf{E}\Big] \;+\; \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}}_{\nabla\cdot\mathbb{T}} \;-\; \frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t} \tag{32}$$

where $\mathbb{T}$ is the **Maxwell stress tensor** given by

$$\mathbb{T} \;=\; \left(\epsilon_0\mathbf{E}\otimes\mathbf{E} \;+\; \frac{1}{\mu_0}\mathbf{B}\otimes\mathbf{B}\right) - \left(\frac{1}{2}\epsilon_0 E^2 \;+\; \frac{1}{2\mu_0}B^2\right)\mathbb{I} \tag{33}$$

Finally, we obtain the conservation of momentum.

$$\mathbf{f} \;=\; \nabla\cdot\mathbb{T} \;-\; \frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t} \tag{34}$$

$$\Longrightarrow \mathbf{F} \;=\; \oint_{\mathcal{S}}\mathbb{T}\cdot d\mathbf{a} \;-\; \frac{d}{dt}\underbrace{\frac{1}{c^2}\int_{\mathcal{V}}\mathbf{S}\,d\tau}_{\mathbf{p}_{\text{em}}} \qquad \text{(divergence theorem)} \tag{35}$$

$$\Longrightarrow \frac{d}{dt}\Big(\mathbf{p}_{\text{mech}} \;+\; \mathbf{p}_{\text{em}}\Big) \;=\; \oint_{\mathcal{S}}\mathbb{T}\cdot d\mathbf{a} \tag{36}$$

6 Potential

Since $\nabla \cdot \mathbf{B} = 0$ from Eq. (2), and mathematically $\nabla \cdot (\nabla \times \mathbf{v}) = 0$, we define a **vector potential A** such that

$$\mathbf{B} \;=\; \nabla \times \mathbf{A} \tag{37}$$

Now, we make use of Eq. (3) and Eq. (37)

$$\nabla \times \mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t} \;=\; \nabla \times \Big(-\frac{\partial \mathbf{A}}{\partial t} \Big) \quad \Longrightarrow \quad \nabla \times \Big(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \Big) \;=\; \mathbf{0} \tag{38}$$

Mathematically, $\nabla \times (\nabla v) = 0$. Therefore, we can define a **(scalar) potential** ϕ such that

$$\mathbf{E} \;=\; -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{39}$$

Note that the potentials $\mathbf{A}$ and ϕ are not unique and they admit the **gauge transformation**:

$$\mathbf{A} \;\to\; \mathbf{A} + \nabla f \qquad \text{and} \qquad \phi \;\to\; \phi - \frac{\partial f}{\partial t} \tag{40}$$

such that **E and B remain unchanged.**

Using Eq. (1), Gauss's Law for electricity, and Eq. (39) we have

$$\nabla^2 \phi \;+\; \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) \;=\; -\frac{\rho}{\epsilon_0} \tag{41}$$

Using Eq. (4), Ampere's Law with Maxwell's equation, and Eq. (37) and Eq. (39), we have

$$\nabla^2 \mathbf{A} \;-\; \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \;-\; \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \right) \;=\; -\mu_0 \mathbf{J} \tag{42}$$

If we choose **Coulomb's Gauge**,

$$\nabla \cdot \mathbf{A} \;=\; 0 \tag{43}$$

Eq. (41) reduces to Poisson's equation

$$\nabla^2 \phi \;=\; -\frac{\rho}{\epsilon_0} \quad \Longrightarrow \quad \phi(\mathbf{r}, t) \;=\; \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}\, d\mathbf{r}' \tag{44}$$

However, ϕ alone cannot determine $\mathbf{E}$, we also need to find $\mathbf{A}$ by solving Eq. (42).
If we choose **Lorenz's Gauge**,

$$\nabla \cdot \mathbf{A} \;=\; -\frac{1}{c^2}\frac{\partial \phi}{\partial t} \tag{45}$$

then ϕ and $\mathbf{A}$ are decoupled.

$$-\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} \;+\; \nabla^2 \phi \;=\; \frac{\rho}{\epsilon_0} \quad \Longrightarrow \quad \phi(\mathbf{r}, t) \;=\; \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|}\, d\tau' \tag{46}$$

$$-\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \;+\; \nabla^2 \mathbf{A} \;=\; -\mu_0 \mathbf{J} \quad \Longrightarrow \quad \mathbf{A}(\mathbf{r}, t) \;=\; \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|}\, d\tau' \tag{47}$$

where $t_r = t - |\mathbf{r} - \mathbf{r}'|/c$ is the retarded time.