The Effect of Image Interpolation on the Generalization of Generative and Discriminative Models

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Abstract

In statistical classification, there are two main approaches to learning: *generative* and *discriminative*. In computer vision, *image interpolation* is a method to rescale images analogous to dimensionality reduction. We explore the generalization and tractability of *generative-discriminative pairs* when interpolating images to smaller sizes.

1 Introduction

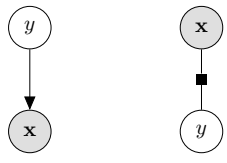
Probabilistic graphical models have many applications in computer science, ranging from image segmentation to named entity recognition. However, these models can become unwieldy when our feature space $\mathbf{x}$ grows. In the case of images, as we increase the number of pixels per row and column, the total number of features grows exponentially. For large images, if we treat the input space as a rasterized image, then the total number of features is $3 \times size^2$.

This leads to issues of tractability for both parameter estimation and inference. This paper concerns itself with using image interpolation as a dimensionality reduction technique to make learning tractable. We vary the dimensionality to see both the generalization ability of our models and the efficiency of parameter estimation. We describe our data in Section 4, our experimental procedure in Section 7, and image interpolation (Section 3).

We experiment with this concept through two approaches: *generative* and *discriminative* learning. Both approaches are outlined in Section 2. Our models are formally described in Section 5 and Section 6. Our results are presented for all models experiments (Section 8 and 9).

2 Statistical Classification

Classification is the task of assigning a label y to a set of observed features $\mathbf{x}$. Yet the way we approach modeling these relationships can be broken down into either the *generative* or the *discriminative* approach.



Generative Discriminative

The generative approach models the joint distribution $p(y, \mathbf{x})$, and can assign labels through Bayes rule [8].

$$\hat{y} = \underset{y}{\operatorname{argmax}}\ p(y, \mathbf{x}) = \underset{y}{\operatorname{argmax}}\ p(y) \cdot p(\mathbf{x}|y) \tag{1}$$

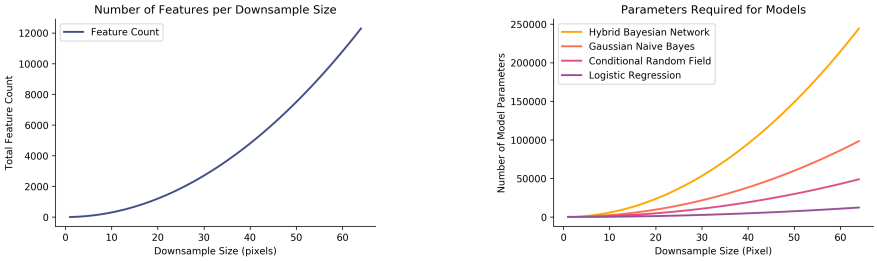
Generative models are attractive in that distributions are learned over the feature space for each individual class. However, an important limitation mentioned by McCallum et. al [7] is that modeling a distribution *per feature* quickly becomes intractable as the dimensionality of our observed variables $\mathbf{x}$ grows. While simple models can mitigate these issues by assuming independence among the features, allowing complex dependencies between inputs offers the ability to increase performance.

An alternative approach is to model the conditional probabilities directly, ignoring the feature distributions. This is the discriminative approach, and is sometimes referred to as a *distribution-free classifier*. By ignoring the feature distributions, parameters are learned only on the conditional likelihood $p(y|\mathbf{x})$, resulting in a compact model that can handle large feature spaces, as dependencies are ignored [7].

To compare generative and discriminative modeling, one can experiment with pairs of classifiers that can be considered analogous to each other. More formally, a *generative-discriminative pair* is a parametric family of probabilistic models that can either be fit to maximize the joint probability $p(y, \mathbf{x})$ or the conditional likelihood $p(y|\mathbf{x})$ [8]. The simplest pairing is the Naive Bayes classifier (generative) and Logistic Regression (discriminative). This paradigm can be extended to sequential and general models to form more pairs. We discuss our experimental pairs in Sections 5 and 6.

3 Image Interpolation

The most common problem encountered in machine learning is the *curse of dimensionality*. Essentially, as our feature space $\mathbf{x}$ grows in size, we require a larger number of parameters to estimate. This can quickly become intractable in complex models, and reasonably difficult in simple ones. We can see the growth in features as we increase the image size in Figure 2. One method to reduce the number of features for a data set is Principle Component Analysis, which projects data to a smaller dimension while maximizing the variance. However, interpretability is lost, and the structure of the painting is unintelligible.



(a) Growth in the Number of Features

(b) Growth in the Number of Model Parameters

Figure 2: As the image size increases, the problem quickly becomes intractable

Image interpolation offers a convenient way to downsample images according to their local neighborhood, aggregating otherwise noisy estimates into a single value. This may be beneficial for classification tasks, as we combine a method analogous to dimensionality reduction while preserving the fundamental structure of the original content. We use OpenCV's implementation of interpolation [1].

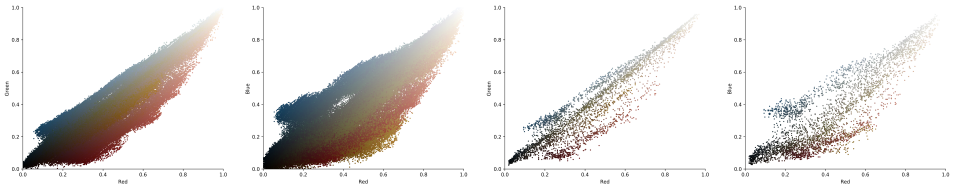
Formally, image interpolation is the task of rescaling an image of one size to another. This can be used to both decimate and expand the image. Several methods exists for this task, including linear, bicubic, nearest neighbor, and *area relation*.

The preferred method for image decimation, or downsampling, is to resampling using pixel area relation as it gives moire'-free results. We can see in Figure 3 that the general space of the pixels in relation to their values is replicated, albeit with information loss.

Figure 3: Results of Interpolating a 249,000 pixel image to a 4,096 pixel image.

(a) Original Image (498×500)

(b) Decimated Image (64×64)



(c) Pixel Plot for the Original Image

(d) Pixel Plot for the Decimated Image

4 Data

Our data is drawn from a subset of the Rijkmuseum Dataset [4]. In order to have enough samples for a challenging classification task, we elected to perform type categorization. We used the top 4 art types: paintings, photos, drawings, and prints. Each category is limited to the size of the smallest set, which is $2,269$ photos. Hence we have a dataset of $9,076$ images split evenly between the 4 classes. For training purposes, we use 10% of the data as our test set ($8,168$ training, 908 test).

Figure 4: Class Examples

(a) Sample Painting

(b) Sample Photo

(c) Sample Drawing

(d) Sample Print

From Figure 4, we can see that paintings are vastly different from the other classes, while photos and prints may be difficult to distinguish between. Photos and prints share a similar hue and tone, and drawings do not always have the vibrant colors found in paintings. Hence, our 4-way classification task should be difficult to learn, allow ample room to experiment with complex models.

5 Baseline Models

5.1 Gaussian Naive Bayes

Naive Bayes is a simple generative classifier based on applying Bayes' theorem to extract the conditional likelihood $p(y \mid \mathbf{x})$. Its' fundamental assumption for the features is "naive" in that it assumes that every pair of features is conditionally independent given then class label y for K features.

We can formulate our model from the joint distribution:

$$p(y, \mathbf{x}) \;=\; p(y) \cdot \prod_{k=1}^{K} p(x_k \mid y) \tag{2}$$

Our task is to find the class label y that maximizes the posterior conditional likelihood:

$$\begin{aligned}\hat{y} &= \underset{y}{\operatorname{argmax}}\ p(y|\mathbf{x}) \\ &= \underset{y}{\operatorname{argmax}}\ \frac{p(y)\cdot\prod_{k=1}^{K} p(x_k \mid y)}{p(\mathbf{x})} \\ &= \underset{y}{\operatorname{argmax}}\ p(y)\cdot\prod_{k=1}^{K} p(x_k \mid y)\end{aligned} \tag{3}$$

We make use of the conditional independence assumption that all features are solely dependent on y alone, and that $p(\mathbf{x})$ is constant.

Here, $p(y)$ is the frequency of class y is the training set. For our purposes we assume the features are normally distributed per class. This is beneficial for learning, as we learn class-wise means and variances, in contrast to the discrete case where each variable can take on 256 states. Hence, our conditional feature likelihood is:

$$p(x_i \mid y) = \frac{1}{\sqrt{2\pi\sigma_y^2}}\exp\left(-\frac{(x_i-\mu_y)^2}{2\sigma_y^2}\right) \tag{4}$$

To learn a Gaussian Naive Bayes model, we use Maximum Likelihood Estimates for the parameters μ_{kc} and σ_{kc} per feature k and class c, and class weights θ_c. Hence the MLE estimates for N samples are:

$$\begin{aligned}\hat{\theta}_c &= \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}\left\{y_i = y_c\right\} = \frac{N_c}{N} \\ \hat{\mu}_{kc} &= \frac{1}{N_c}\sum_{i=1}^{N}\mathbb{1}\left\{y_i = y_c\right\}\cdot x_{ik} \\ \hat{\sigma}_{kc} &= \frac{1}{N_c}\sum_{i=1}^{N}\mathbb{1}\left\{y_i = y_c\right\}\cdot\left(x_{ik}-\mu_{kc}\right)^2\end{aligned} \tag{5}$$

where $N_c = \sum_{i=1}^{N}\mathbb{1}\left\{y_i = y_c\right\}$, or the number of data points belonging to class c [5].

5.2 Multinomial Logistic Regression

Logistic Regression is the discriminative analog to the Naive Bayes classifier, sometimes referred to as the *maxmimum-entropy classifer* or a normalized *log-linear model*. As a discriminative model, we can optimize directly for the conditional likelihood $p(y \mid \mathbf{x})$, with $Z(\mathbf{x})$ as the normalizing partition function.

$$p(y \mid \mathbf{x}) = \frac{1}{Z(\mathbf{x})}\exp\left\{\sum_{j=1}^{K}\theta_k f_k\left(y,\mathbf{x}\right)\right\} \tag{6}$$

McCallum et. al formulates the Logistic Regression model, parameterized by weights $\theta \in \mathbb{R}^K$, in which a single set of weights is shared across all the classes [7]. This is achieved through a set of feature functions that are nonzero for a single class and is defined as follows:

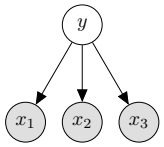
$$\begin{aligned}f_{y',j}\left(y,\mathbf{x}\right) &= \mathbb{1}\{y'=y\}\cdot x_j \\ f_{y'}\left(y,\mathbf{x}\right) &= \mathbb{1}\{y'=y\}\end{aligned} \tag{7}$$

This notation, while not standard, mirrors our later formulation of Conditional Random Fields in Section 6.2.

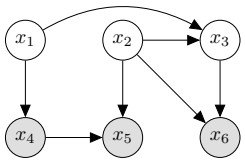
For parameter estimation, we opted to use scikit-learn's implementation of the L-BFGS algorithm, which approximates the Broyden-Fletcher-Goldfarb-Shanno algorithm [6]. This is a quasi-Newton optimization method that approximates the Hessian matrix to reduce memory usage. Scikit-learn optimizes the L2 penalized multinomial loss [6] [5].

$$\min_{\theta}\ \frac{1}{2}\cdot\theta^T\theta - \sum_{i=1}^{N}\left[\sum_{c=1}^{C}\left[\mathbb{1}\{y_i=y_c\}\cdot\sum_{k=1}^{K}\theta_k f_k\left(y_i,\mathbf{x}_i\right)\right] - \log Z(\mathbf{x}_i)\right] \tag{8}$$

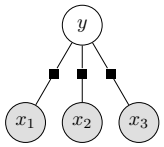
Generative: Naive Bayes



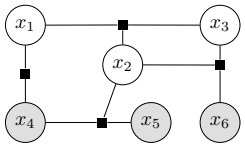
Generative: General Bayesian Networks



Discriminative: Logistic Regression



Discriminative: Conditional Random Fields



where y_i is the true label and y_c could be any label.

6 General Models

6.1 Hybrid Bayesian Networks

General Bayesian Networks (GBNs) are directed graphical models that are comprised of a set of nodes and edges. The nodes are our random variables, and they interrelate through directed edges. This allows us to encode complex relations between our features $\mathbf{x}$ that are not captured by the Naive Bayes classifier. We can exploit these relationships to decompose the joint distribution into factors conditioned on the node's parents [5].

$$p(\mathbf{x}) \;=\; \prod_{k=1}^{K} p(x_k \mid pa(x_k)) \tag{9}$$

Since our labels are discrete and the data is continuous, we use Hybrid Bayesian Networks. As defined in Koller & Friedman [2], these networks are known as *Conditional Linear Gaussian (CLG)* models if every discrete variable has only discrete parents and every continuous variable is represented as a CLG Conditional Probability Distribution (CPD).

Continuous nodes make use of the multivariate gaussian distribution, parameterized by a mean vector μ and a symmetric square covariance matrix $\mathbf{\Sigma}$.

$$p(\mathbf{x}; \mu, \mathbf{\Sigma}) \;=\; \frac{1}{(2\pi)^{n/2} \mid \Sigma \mid^{1/2}} \exp\Big\{ -\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu) \Big\} \tag{10}$$

The conditional linear Gaussian CPD for a continuous variable x_j, discrete parents $\mathbf{U} = \{U_1, \ldots, U_M\}$, continuous parents $\mathbf{C} = \{C_1, \ldots, C_K\}$ is, for every value $\mathbf{u} \in Val(\mathbf{U})$ with $k+1$ coefficients $a_{\mathbf{u},0}, \ldots, a_{\mathbf{u},k}$ and variance $\sigma^2_{\mathbf{u}}$:

$$p(x_j \mid \mathbf{u}, \mathbf{c}) \;=\; \mathcal{N}\left(a_{\mathbf{u},0} + \sum_{k=1}^{K} a_{\mathbf{u},k} \cdot c_k \;;\; \sigma^2_{\mathbf{u}} \right) \tag{11}$$

The log likelihood function ℓ for a continuous node X_i is:

$$\ell \;=\; \log \mathcal{L} \;=\; \sum_{i=1}^{N} \left\{ -\frac{1}{2}\log(2\pi\sigma^2_{\mathbf{u}}) - \frac{1}{2} \cdot \frac{1}{\sigma^2_{\mathbf{u}}} \left[a_{\mathbf{u},0} + \sum_{k=1}^{K} a_{\mathbf{u},k} \cdot c^i_k - x^i_j \right]^2 \right\} \tag{12}$$

The MLE parameters are obtained by ordinary linear regression of each node x_j on its continuous parents $c \in \mathbf{C}$, for each class value $\mathbf{u} \in \{0, 1, 2, 3\}$ [2].

In contrast, our discrete node is the class label y, which is just the frequency. The maximum likelihood estimate is:

$$\hat{\theta}_{y_c} = \frac{\sum_{i=1}^{N} \mathbb{1}\{y^i = y_c\}}{\sum_{i=1}^{N} \sum_{y_k} \mathbb{1}\{y^i = y_k\}} = \frac{\sum_{i=1}^{N} \mathbb{1}\{y^i = y_c\}}{N} \tag{13}$$

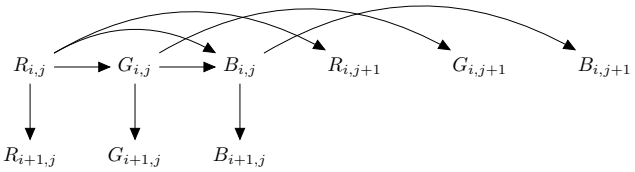


Figure 6: Hybrid Bayesian Network Model Architecture, for Pixel ij with color channels RGB

Here, y_c is the class being estimated, y^i is the label for the i-th sample, and y_k are the other classes. N is the number of samples.

We elected to use a hybrid model as the discrete case led to *data fragmentation*, and not every state configuration can be seen in the training set.

Inference is performed by averaging likelihood weighting simulations using $n = 500$ random samples. For discrete values, the prediction is the highest conditional probability. The predicted value of a continuous value is the expected value of the conditional distribution.

6.2 Conditional Random Fields (CRFs)

Conditional Random Fields (CRFs) are discriminative graphical models that can model output variables $\mathbf{y}$ conditioned on features $\mathbf{x}$ compactly, in contrast to generative models. CRFs are natural extensions to Logistic Regression for general graphs. We describe the model formulation, parameter estimation, and inference tasks as outlined by McCallum et. al [7], and introduced by Lafferty et. al [3].

CRFs model a conditional likelihood $p(\mathbf{y} \mid \mathbf{x})$ on a general graph $\mathcal{G}$, composed of a set of factors $F = \{\psi_a\}_{a=1}^{A}$. We reiterate McCallum's description of a CRF with parameter tying, where a set of factors can share a set of feature functions $\{f_{pk}\left(\mathbf{x}_c, \mathbf{y}_c\right)\}_{k=1}^{K(p)}$ and parameters $\theta_p \in \mathbb{R}^{K(p)}$ for a given *clique template* C_p. These clique templates come from partitioning the factors of graph $\mathcal{G}$ into $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$. Hence, our model is composed of the following:

$$
\begin{aligned}
p(\mathbf{y} \mid \mathbf{x}) &= \frac{1}{Z(\mathbf{x})} \prod_{C_p \in \mathcal{C}} \prod_{\psi_c \in C_p} \psi_c(\mathbf{x}_c, \mathbf{y}_c; \theta_p) \\
\psi_c(\mathbf{x}_c, \mathbf{y}_c; \theta_p) &= \exp\left\{ \sum_{k=1}^{K(p)} \theta_{pk} f_{pk}(\mathbf{x}_c, \mathbf{y}_c) \right\} \\
Z(\mathbf{x}) &= \sum_{\mathbf{y}} \prod_{C_p \in \mathcal{C}} \prod_{\psi_c \in C_p} \psi_c(\mathbf{x}_c, \mathbf{y}_c; \theta_p)
\end{aligned}
\tag{14}
$$

Where p is a factor index, f_{pk} is a feature function, and θ_{pk} is the weight associated with it. K is the number of features and $K(p)$ are the features found in factor p. Because our labels are discrete, we can write our feature functions f_{pk} in terms of observation functions q_{pk} to make *label-observation features*.

$$
f_{pk}(\mathbf{y}_c, \mathbf{x}_c) = \mathbb{1}\{\mathbf{y}_c = \widetilde{\mathbf{y}}_c\} \cdot q_{pk}(\mathbf{x}_c) \tag{15}
$$

Parameter estimation for CRFs can be done through maximum likelihood estimation, where we can minimize the conditional negative log likelihood:

$$
\ell(\theta) = -\sum_{C_p \in \mathcal{C}} \sum_{\psi_c \in C_p} \sum_{k=1}^{K(p)} \theta_{pk} f_{pk}(\mathbf{x}_c, \mathbf{y}_c) + \log Z(\mathbf{x}) \tag{16}
$$

Hence, the partial derivative with respect to our paramter θ_{pk} associated with the clique template C_p is:

$$
\frac{\partial \ell}{\partial \theta_{pk}} = -\sum_{\psi_c \in C_p} f_{pk}(\mathbf{x}_c, \mathbf{y}_c) + \sum_{\psi_c \in C_p} \sum_{\mathbf{y'}_c} f_{pk}(\mathbf{x}_c, \mathbf{y'}_c) \cdot p(\mathbf{y'}_c \mid \mathbf{x}) \tag{17}
$$

We can then minimize our cost function $\ell(\theta)$ using L-BFGS optimization algorithm as described in Section 5.2.

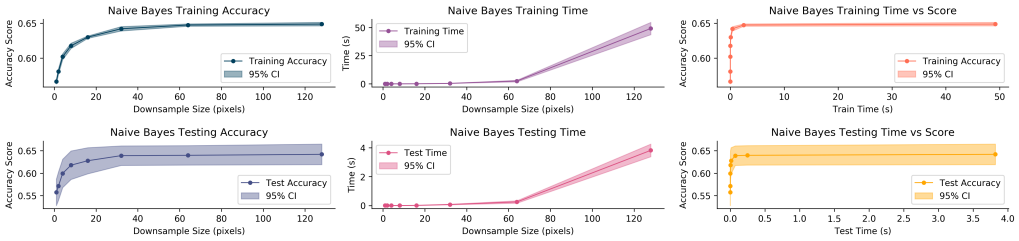


Figure 7: Generalization and Tractability Results for Gaussian Naive Bayes

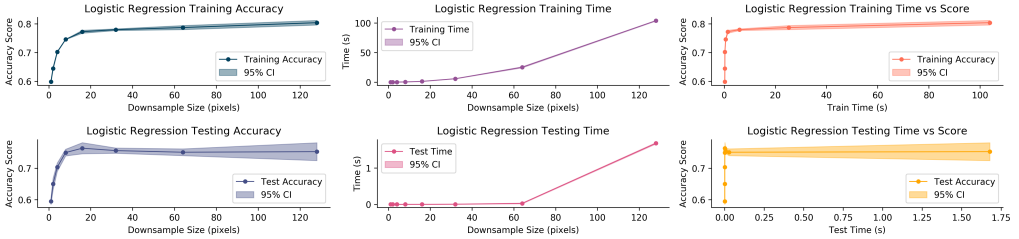


Figure 8: Generalization and Tractability Results for Multinomial Logistic Regression

7 Experimental Task

For a single downsample size, we performed a 3-fold cross validation on the full $9,076$ image set, using 908 images randomly as test. This gives us three points of accuracy, in which we can average and extract confidence intervals around. Variables measured for the cross-validation include: training accuracy, testing accuracy, fit time, and score time. These measures provide two fronts of analysis: generalization and tractability.

To effectively study these objective measures, we conduct this experiment per model per downsample size. Our choices of sizes included: 1, 2, 4, 8, 16, 32, 64, and 128. We did not test higher pixel sizes as 256×256 pixel images quickly became impossible due to memory constraints. For the general models, any test on 32×32 or larger images was intractable due to prediction costs.

All images were rescaled such that its pixel color channels are ranged from $[0, 1]$, and we assume our data is continuous instead of discrete to avoid data fragmentation.

8 Results: Baseline Models

The baseline models, while naive in its assumptions, did reasonably well in terms of generalization ability and tractability.

For the Gaussian Naive Bayes model (Figure 7), we see two distinct trends. First, the model performs equivalently in the training and testing phase, with similar accuracy scores. However, compared to the Logistic Regression (Figure 8), the Naive Bayes model takes longer to converge to a stationary accuracy, even when the number of features is growing exponentially.

The time to fit the Naive Bayes was also significantly smaller than the multinomial Logistic Regression. This effect stems from the lack of a quasi second-order optimization approach that Logistic Regression uses through L-BFGS. However, Logistic Regression was faster in inference, since Naive Bayes computes the gaussian likelihood for every feature x_{ijk} before multiplying all the conditional probabilities to make a prediction.

From Figures 7 and 8, we see that no noticeable gains are made for either model for downsampled images larger than 16×16, and in the case of Logistic Regression, can even overfit to noise. This stems from the fact that image interpolation creates robust, strong estimators of individual features, reducing noise and helping generalization. However, it is important to note the significant loss in accuracy when the image is decimated to smaller sizes than 16×16, as information is lost. While

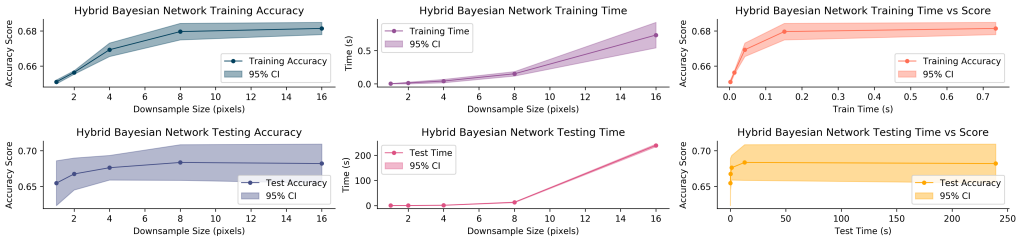


Figure 9: Generalization and Tractability Results for a Hybrid Bayesian Network

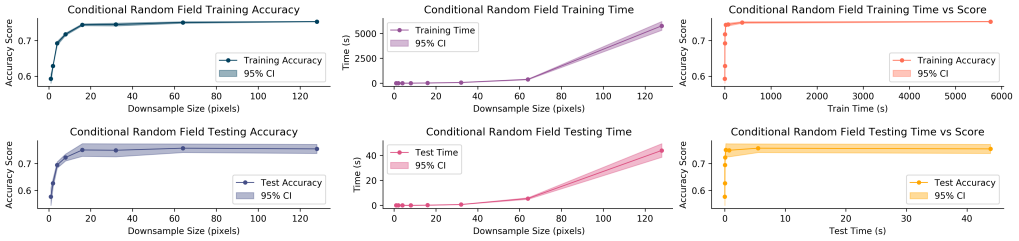


Figure 10: Generalization and Tractability Results for a Conditional Random Field

the models do surprising well with a single color, these results show that image interpolation can effectively improve generalization and tractability of raw feature training on images.

9 Results: General Models

The general models followed similar trends to our baselines.

For the Hybrid Bayesian Network (Figure 9), initial generalization ability was improved from inter-feature connections. However, this formulation proved to be intractable after 32×32. Hence, while we improve both train and test scores through complex interactions, detailed in Figure 6, this is offset by increased inference times. Parameter estimation proved to be effective, as each node learned a small subset of coefficients through linear regression on continuous parents and induced sparsity. However, this formulation hindered inference, and the time to predict is exponentially larger than either the Conditional Random Field or the baselines.

The Conditional Random Field (Figure 10) took longer to train than any other model, and yet had a comparable generalization curve to Logistic Regression, while outperforming the generative models. Logistic Regression performed better in train score, yet had similar test scores. The increased train and inference times result from a more complicated model and optimization problem. Since our CRF uses L-BFGS, the estimates to the hessian are more expensive, thus making parameter estimation difficult. Inference was faster than the Hybrid Bayesian Network.

10 Results: General Findings

Discriminative models learned better than their generative counterparts, as they model the conditional probability $p(y \mid \mathbf{x})$ directly. Hence, they are less concerned with modeling the features, and results in a lower parameter count.

Image interpolation can effectively reduce the size of the data and models by creating robust features invariant to noise. Caution must be exerted if too much information is loss, and our results may be different if the original images were of different sizes, i.e. 100×100 vs. 1000×1000.

Adding more complex interactions does not help our learning task, and in some cases hinders our performance, both in generalization and tractability. Perhaps image interpolation encodes all relevant interactions between pixels in the original image that we do not need to account for with general graphs over naive assumptions.

11 Conclusion

As we can see from our experiments, image interpolation can be a useful technique for dimensionality reduction when using graphical models for classification. As our original 500×500 images contained over $750,000$ features, a simple reduction use pixel sampling yields comparable results when reduced to 16×16 (768 features). However, image decimation can result in significant generalization errors from information loss. Using image interpolation as a dimensionality reduction technique thus preserves data intelligently and generalization ability while improving the time it takes to perform parameter estimation and inference in graphical models. Discriminative models made the most gains from this technique, in contrast to generative models.

All code for this project can be found here: https://github.com/nextBillyonair/PGMProject

12 Future Work

Over the course if this project, several ideas would have been interesting to explore, but where not pursued due to the scope of the research question. First, a natural area of exploration would be to rerun the experiments using color quantization, a technique used to reduce the number of colors in an image. This is reduces the total number of discrete states each color channel can take on.

A second proposal would be to experiment with structure learning algorithms to create networks that take the distribution of colors per class into account. Our model was created with the intention of injecting sparsity while still retaining relevant connections between pixels.

A third avenue would be to examine how interclass confusion in inference is influenced by increasing the number of features.

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