Matrix and tensor notation in the theory of elasticity

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Abstract

This summary[1] focuses on the connection between the tensor and the matrix notation in the theory of elasticity. Many theories in this field are based on the manipulation of tensorial equations, e.g. the rotation of the stiffness or the compliance tensor, or calculating the inverse of fourth order symmetric tensors. The focus of the second part of this summary is the application of the matrix notation to the theories of porous materials.

Contents

1 The Basics

The theory of elastic constants is a theory of two second–order symmetric tensors (strain ε and stress σ) and four fourth–order symmetric tensors (the stiffness tensor **C**, the compliance tensor **D**, the Eshelby tensor **S** and its counterpart **T**.

Second–order symmetric tensors and fourth–order symmetric tensors can be constructed from second–order symmetric base tensors $\mathbf{e}^s_{ij}$

$$\mathbf{e}^s_{ij} = \frac{1}{2}(\mathbf{e}_{ij} + \mathbf{e}^T_{ij}) = \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i) \tag{1}$$

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[1]Corrections, Patches and Comments are welcome.

which are defined via the dyads $\mathbf{e}_i \otimes \mathbf{e}_j$ and are only orthogonal, but not orthonormal

$$\mathbf{e}^s_{ij} : \mathbf{e}^s_{kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2}$$

where : denotes the double contraction (or inner product/dot–product) of the two tensors. The double contraction is defined by

$$(\mathbf{e}_i \otimes \mathbf{e}_j) : (\mathbf{e}_k \otimes \mathbf{e}_l) \equiv (\mathbf{e}_i \cdot \mathbf{e}_k)(\mathbf{e}_j \cdot \mathbf{e}_l) = \delta_{ik}\delta_{jl} \tag{3}$$

or more general for an arbitrary number of dimensions

$$(\mathbf{a}_1 \otimes \ldots \mathbf{a}_k) : (\mathbf{b}_1 \otimes \ldots \mathbf{b}_l) = (a_{k-1} \ldots b_1)(a_k \cdot b_2)(\mathbf{a}_1 \otimes \ldots \mathbf{a}_{k-2}) : (\mathbf{b}_3 \otimes \ldots \mathbf{b}_l) \tag{4}$$

The result of equation (2) can thus be easily derived from these definitions:

$$\begin{aligned}
&\mathbf{e}^s_{ij} : \mathbf{e}^s_{kl} = \\
&\frac{1}{4}[(\mathbf{e}_i \otimes \mathbf{e}_j) : (\mathbf{e}_k \otimes \mathbf{e}_l) + (\mathbf{e}_i \otimes \mathbf{e}_j) : (\mathbf{e}_l \otimes \mathbf{e}_k) + (\mathbf{e}_j \otimes \mathbf{e}_i) : (\mathbf{e}_k \otimes \mathbf{e}_l) + (\mathbf{e}_j \otimes \mathbf{e}_i) : (\mathbf{e}_l \otimes \mathbf{e}_k] = \\
&\frac{1}{4}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}] = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
\end{aligned} \tag{5}$$

The set of all second–order and fourth–order symmetric tensors can then be defined [2] as

$$T^{(2s)} \equiv \{\mathbf{S}|\mathbf{S} = S_{ij}\mathbf{e}^s_{ij}, S_{ij} = S_{ji}\} \tag{6}$$

$$T^{(4s)} \equiv \{\mathbf{T}|\mathbf{T} = T_{ijkl}\mathbf{e}^s_{ij} \otimes \mathbf{e}^s_{kl}, T_{ijkl} = T_{jikl} = T_{ijlk}\} \tag{7}$$

1.1 The matrix representation

Tensorial operations between $T^{(2s)}$ and $T^{(4s)}$ can be related to six–dimensional matrix operations, since the symmetric order basis tensors $\mathbf{e}^s_{ij}$ and their tensor product $\mathbf{e}^s_{ij} \otimes \mathbf{e}^s_{kl}$ span $T^{(2s)}$ and $T^{(4s)}$, respectively. The second–order symmetric basis tensors are written in this case as

$$\begin{array}{lll}
\mathbf{b}_1 = \mathbf{e}^s_{11} & \mathbf{b}_2 = \mathbf{e}^s_{22} & \mathbf{b}_3 = \mathbf{e}^s_{33} \\
\mathbf{b}_4 = \mathbf{e}^s_{23} & \mathbf{b}_5 = \mathbf{e}^s_{13} & \mathbf{b}_6 = \mathbf{e}^s_{12}
\end{array} \tag{8}$$

$\{\mathbf{b}_a\}$ is, as $\{\mathbf{e}^s_{ij}\}$, an orthogonal set, and the dot–product, which is equivalent to the double contraction of elements of $\{\mathbf{e}^s_{ij}\}$, can be written as

$$\mathbf{b}_a \cdot \mathbf{b}_b = \frac{1}{W_{ab}}\delta_{ab} = \left[W_{ab}\right]^{-1}, \; a \text{ and } b \text{ are not summed} \tag{9}$$

with W being the matrix

$$\left[W_{ab}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{10}$$

[2]Please note, that the definition of a second–order symmetric tensor demands the symmetry $S_{ij} = S_{ji}$ and the definition of a fourth–order symmetric tensors demands $T_{ijkl} = T_{jikl} = T_{ijlk}$. The symmetry $T_{ijkl} = T_{klij}$ is not a requisite for a fourth–order symmetric tensor, although both the stiffness tensor C and the compliance tensor D feature this symmetry in addition to the aforementioned symmetries.

The tensor $\mathbf{S}$ from $T^{(2s)}$ can, for example, now be written as

$$\mathbf{S} = S_{11}\mathbf{b}_1 + S_{22}\mathbf{b}_2 + S_{33}\mathbf{b}_3 + 2S_{23}\mathbf{b}_4 + 2S_{13}\mathbf{b}_5 + 2S_{12}\mathbf{b}_6 \tag{11}$$

where the factors 2 originate in the symmetry of the tensor. $\mathbf{S}$ can thus be written as

$$\mathbf{S} = \mathbf{b}_a[W_{ab}][S_b]\, a \text{ not summed} \tag{12}$$

with a column vector $[S_b]$ defined as $[S_a] = [S_{11}, S_{22}, S_{33}, S_{23}, S_{13}, S_{12}]$.

1.2 Contractions and the matrix "multiplication"

The product

$$\mathbf{S} : \mathbf{S}' = (\mathbf{b}_a[W_{ab}][S_b])^T : (\mathbf{b}_c[W_{cd}][S'_d]) \tag{13}$$

can then be written as

$$\begin{aligned}\mathbf{S} : \mathbf{S}' &= (\left[S_b\right]\left[W_{b_a}\right]\mathbf{b}_a) \cdot (\mathbf{b}_c\left[W_{cd}\right]\left[S'_d\right])) = \\ &= \left[S_b\right]\left[W_{ba}\right]\left[W_{ac}\right]^{-1}\left[W_{cd}\right]\left[S'_d\right] = \left[S_b\right]\left[W_{ba}\right]\left[S'_a\right]\end{aligned} \tag{14}$$

The second order contraction of two $T^{(2s)}$ tensors can thus, with the aid of the matrix $[W]$, be expressed as two matrix multiplications. The same is true for fourth–order symmetric tensors: $\mathbf{T} : \mathbf{T}' \to [T_{ab}][W_{bc}][T'_{cd}]$, where

$$[T_{ab}] = \begin{bmatrix} T_{1111} & T_{1122} & T_{1133} & T_{1123} & T_{1113} & T_{1112} \\ T_{2211} & T_{2222} & T_{2233} & T_{2223} & T_{2213} & T_{2212} \\ T_{3311} & T_{3322} & T_{3333} & T_{3323} & T_{3313} & T_{3312} \\ T_{2311} & T_{2322} & T_{2333} & T_{2323} & T_{2313} & T_{2312} \\ T_{1311} & T_{1322} & T_{1333} & T_{1323} & T_{1313} & T_{1312} \\ T_{1211} & T_{1222} & T_{1233} & T_{1223} & T_{1213} & T_{1212} \end{bmatrix} \tag{15}$$

1.3 The "inverse" of a tensor

The fourth–order symmetric identity tensor $\mathbf{1}^{(4a)}$ is defined such that it maps any second–order tensor in $T^{(2s)}$ to itself.

$$\mathbf{1}^{(4a)} \equiv \mathbf{e}^s_{ij} \otimes \mathbf{e}^s_{kl} = \frac{1}{2}(\mathbf{e}_{ij} \otimes \mathbf{e}_{ij} + \mathbf{e}_{ij} \otimes \mathbf{e}^T_{ij}) \tag{16}$$

with the components

$$1^{(4s)}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{17}$$

which can be written as

$$\mathbf{1}^{(4s)} = \mathbf{b}_1 \otimes \mathbf{b}_1 + \mathbf{b}_2 \otimes \mathbf{b}_2 + \mathbf{b}_3 \otimes \mathbf{b}_3 + 2\mathbf{b}_4 \otimes \mathbf{b}_4 + 2\mathbf{b}_5 \otimes \mathbf{b}_5 + 2\mathbf{b}_6 \otimes \mathbf{b}_6 \tag{18}$$

The corresponding matrix $[1^{(4s)}_{ab}]$ is equal to $[W_{ab}]^{-1}$, which is the inverse of the the matrix $[W_{ab}]$. A tensor $\mathbf{T}$ in $T^{(4s)}$ is, due to its symmetries, always singular. It may nevertheless exist another element in $T^{(4s)}$, that is the inverse tensor in the sense, that the second order contraction of the two tensors yields the fourth–order symmetric identity tensor $\mathbf{1}^{(4s)}$.

The matrix notation of this inverse tensor[3] must therefore satisfy

$$\left[T_{ab}^{-1}\right]\left[W_{bc}\right]\left[T_{cd}\right] = \left[T_{ab}\right]\left[W_{bc}\right]\left[T_{cd}^{-1}\right] = \left[1_{ab}^{(4s)}\right] \tag{19}$$

Multiplying with $[W_{ab}]$ from the left respectively the right gives

$$\left[W_{bc}\right]\left[T_{ab}^{-1}\right]\left[W_{bc}\right]\left[T_{cd}\right] = \left[T_{ab}\right]\left[W_{bc}\right]\left[T_{cd}^{-1}\right]\left[W_{bc}\right] = \left[1_{ab}^{(4s)}\right]\left[W_{bc}\right] = \left[1_{ab}\right] \tag{20}$$

where $[1_{ab}]$ is the six dimensional identity matrix. This means, that $[W_{bc}][T_{ab}^{-1}][W_{bc}]$ is equal to $[T_{ab}]^{-1}$, which is the matrix inverse to the matrix notation of the tensor $\mathbf{T}$, $[T_{ab}]$. The inverse tensor is then $[T_{ab}^{-1}] = [W_{bc}]^{-1}[T_{ab}]^{-1}[W_{bc}]^{-1}$. The existence of such an inverse tensors is necessary for the following calculations and it is being assumed, that the corresponding inverse tensors exist.

2 Theory of elasticity: definitions

It should be noted, that certain definitions exist in connection with the stiffness, compliance, strain and stress tensors: The aim of these definitions is to be able to write the anisotropic version of Hooke's law

$$\sigma = \mathbf{C} : \varepsilon \text{ or } \varepsilon = \mathbf{D} : \sigma \tag{21}$$

in a similar way as matrix equations. The matrix notation of equation (21) is

$$\left[\sigma_a\right] = \left[C_{ab}\right]\left[W_{bc}\right]\left[\varepsilon_c\right] \tag{22}$$

respectively

$$\left[\varepsilon_a\right] = \left[D_{ab}\right]\left[W_{bc}\right]\left[\sigma_c\right] \tag{23}$$

If equation (22) or (23) are multiplied from the left with $[W_{ab}]$, then the matrix notation of Hooke's law is

$$\left[\tau_a\right] = \left[\mathsf{C}_{ab}\right]\left[\gamma_c\right] \tag{24}$$

respectively

$$\left[\gamma_a\right] = \left[\mathsf{D}_{ab}\right]\left[\tau_c\right] \tag{25}$$

with the following definitions: $[\mathsf{C}_{ab}] \equiv [C_{ab}]$, $[\mathsf{D}_{ab}] \equiv [W_{bc}][D_{ab}][W_{bc}] = [C_{ab}]^{-1}$, $[\tau_a] \equiv [\sigma_a]$ and $[\gamma_a] \equiv [W_{ab}][\varepsilon_b]$.
Other definitions, that move the factor 2 to $[\mathsf{C}_{ab}]$ and $[\tau_a]$ instead, or allocate the factor $\sqrt{2}$ symmetric to all four matrices $[\mathsf{C}_{ab}]$, $[\mathsf{D}_{ab}]$, $[\tau_a]$, and $[\gamma_a]$ are in sporadic use.

3 Porous materials

Several of the theories employed for porous marterials are derived as tensorial equations. The general appearance of such a theory is the following one

$$\overline{\mathbf{C}} = \mathbf{C} + \sum_{\alpha=1}^{n} f_\alpha(\mathbf{C}^\alpha - \mathbf{C})\mathbf{E}^\alpha \tag{26}$$

[3]The $^{-1}$ inside the square brackets denotes that the matrix notation of the inverse tensor and not the inverse of the matrix notation of the respective tensor, which is denoted by a $^{-1}$ outside the square brackets, is meant. $[T_{ab}^{-1}]$ is the matrix notation of the inverse tensor and $[T_{ab}]^{-1}$ the inverse matrix of the matrix notation of the tensor T.

where $\mathbf{C}$ is the stiffness tensor of the matrix material, $\mathbf{C}^\alpha$ is the stiffness tensor of the inclusion α, $\overline{\mathbf{C}}$ is the stiffness tensor of the composite material, f_α is the volume fraction of the inclusion α, and $\mathbf{E}^\alpha$ is the so called concentration tensor of the inclusion α and its exact definition depends on the theory used.

The theory of the compliance tensor is very similar to the theory for the stiffness tensor, but involves different boundary conditions.

$$\overline{\mathbf{D}} = \mathbf{D} + \sum_{\alpha=1}^{n} f_\alpha (\mathbf{D}^\alpha - \mathbf{D})\mathbf{F}^\alpha \tag{27}$$

The derivation of these equations and of the various concentration tensors is not subject of this appendix and can be found in the book by Nemat–Nasser[1].

4 The concentration tensors

The tensor A of the homogeneous linearly elastic inclusion α is defined ([1], equation (7.3.17a,b)) as

$$\mathbf{A}^\alpha \equiv (\mathbf{C} - \mathbf{C}^\alpha)^{-1} : \mathbf{C} \tag{28}$$

With [1], equation (15.5.23a), it is then possible to write equation (28) as

$$\left[A^\alpha_{ab}\right] \equiv \left[(C_{ap} - C^\alpha_{ap})^{-1}\right]\left[W_{pq}\right]\left[C_{qb}\right] \tag{29}$$

where the square brackets denote the matrix expression for the respective fourth–order tensors. Equation (29) can be rewritten using

$$\left[T^{-1}_{ab}\right] = \left[W_{ap}\right]^{-1}\left[T_{pq}\right]^{-1}\left[W_{qb}\right]^{-1} \tag{30}$$

as

$$\begin{aligned}\left[A^\alpha_{ab}\right] &\equiv \left[W_{ap}\right]^{-1}\left[(C_{pq} - C^\alpha_{pq})\right]^{-1}\left[W_{qr}\right]^{-1}\left[W_{rs}\right]\left[C_{sb}\right] = \\ &= \left[W_{ap}\right]^{-1}\left[(C_{pq} - C^\alpha_{pq})\right]^{-1}\left[C_{qb}\right]\end{aligned} \tag{31}$$

The concentration tensor E, which is defined as

$$\mathbf{E}^\alpha = \mathbf{A} : (\mathbf{A} - \mathbf{S})^{-1} \tag{32}$$

can then be written in the same notation as above as

$$\left[E^\alpha_{ab}\right] = \left[A_{ap}\right]\left[(A_{pq} - S^\alpha_{pq})\right]^{-1}\left[W_{qb}\right]^{-1} \tag{33}$$

Equation (26) can then be formulated in the matrix notation as

$$\left[\,\overline{C_{ab}}\,\right] = \left[C_{ab}\right] + \sum_{\alpha=1}^{n} f_\alpha \left[(C^\alpha_{ap} - C_{ap})\right]\left[W_{pq}\right]\left[E^\alpha_{qb}\right] \tag{34}$$

the same holds true for the compliance tensor, with

$$\left[B^\alpha_{ab}\right] \equiv \left[(D_{ap} - D^\alpha_{ap})^{-1}\right]\left[W_{pq}\right]\left[D_{qb}\right] \tag{35}$$

$$\left[T^\alpha_{ab}\right] = \left[W_{ap}\right]^{-1} - \left[W_{ap}\right]^{-1}\left[D_{pq}\right]^{-1}\left[S^\alpha_{qr}\right]\left[W_{ra}\right]\left[D_{sb}\right] \tag{36}$$

$$\left[F^\alpha_{ab}\right] = \left[B_{ap}\right]\left[(B_{pq} - T^\alpha_{pq})\right]^{-1}\left[W_{qb}\right]^{-1} \tag{37}$$

and

$$\left[\,\overline{D_{ab}}\,\right] = \left[D_{ab}\right] + \sum_{\alpha=1}^{n} f_\alpha \left[(D^\alpha_{ap} - D_{ap})\right]\left[W_{pq}\right]\left[F^\alpha_{qb}\right] \tag{38}$$

5 Porous materials

The equations (33) and (37) can be simplified for porous material, as $\mathbf{C}^{\alpha} = 0$ and thus

$$\left[A^{\alpha}_{ab}\right] \equiv \left[W_{ap}\right]^{-1}\left[(C_{pq})\right]^{-1}\left[C_{qb}\right] = \left[W_{ab}\right]^{-1} = \left[1^{(4s)}_{ab}\right] \tag{39}$$

and

$$\left[E^{\alpha}_{ab}\right] = \left[W_{ap}\right]^{-1}\left[(W^{-1}_{pq} - S^{\alpha}_{pq})\right]^{-1}\left[W_{qb}\right]^{-1} \tag{40}$$

Porous materials cannot be represented in the compliance notation, as $\mathbf{D}^{\alpha}$ would contain elements with the value ∞.

6 Rotations of the coordinate system

Sometimes it is necessary to calculate the elastic constants or strain and shear in a rotated coordinate system. The transformation

$$S'_{kl} = S_{ij} L_{ik} L_{jl} \tag{41}$$

converts the symmetric second–order tensor S_{lm} into its representation in the new coordinate system S'_{ik}. The L_{ij} are orthogonal symmetric second–order tensors, whose components are the directions cosines between the *i-th* axis of the new and the *j-th* axis of the old coordinate axes. The direction cosines are equal to the scalar product of a unit vector in the direction of the *i-th* axis of the old system with a unit vector in the direction of the *j-th* axis of the new system. If $\vec{x}_i$ are the unit vectors of the old system and $\vec{y}_j$ are the unit vectors of the new system, then $L_{ij} = \vec{x}_i \cdot \vec{y}_j$.

The matrix representation of the second–order tensors L_{ij} cannot be easily used to transform the matrix notation of S_{lm}. What is needed is a symmetric fourth–dimensional tensor, whose double contraction is equivalent to the two single contraction with the tensors L_{ik} and L_{jl}. We therefore define the symmetric fourth–order tensor A_{ijkl}, which is in $T^{(4s)}$, as

$$A_{ijkl} \equiv \frac{1}{2}\left(L_{ik}L_{jl} + L_{il}L_{jk}\right) \tag{42}$$

and write equation (41) as

$$S'_{kl} = S_{ij} A_{ijkl} \tag{43}$$

This is equivalent to equation (41), because

$$S'_{kl} = S_{ij} A_{ijkl} = \frac{1}{2}\left(S_{ij}L_{ik}L_{jl} + S_{ij}L_{il}L_{jk}\right) \tag{44}$$

which is, because S_{lm} is in $T^{(2s)}$, equal to

$$\frac{1}{2}\left(S_{ji}L_{ik}L_{jl} + S_{ij}L_{il}L_{jk}\right) = \frac{1}{2}\left(S_{ij}L_{jk}L_{il} + S_{ij}L_{il}L_{jk}\right) = S_{ij}L_{il}L_{jk} \tag{45}$$

A_{ijkl} is in $T^{(4s)}$ and can thus be represented as a matrix

$$[A_{ab}] = \begin{bmatrix} L_{11}^2 & L_{12}^2 & L_{13}^2 & L_{12}L_{13} & L_{11}L_{13} & L_{11}L_{12} \\ L_{21}^2 & L_{22}^2 & L_{23}^2 & L_{22}L_{23} & L_{21}L_{23} & L_{21}L_{22} \\ L_{31}^2 & L_{32}^2 & L_{33}^2 & L_{32}L_{33} & L_{31}L_{33} & L_{31}L_{32} \\ L_{21}L_{31} & L_{22}L_{32} & L_{23}L_{33} & \begin{matrix}\frac{1}{2}(L_{23}L_{32} \\ +L_{22}L_{33})\end{matrix} & \begin{matrix}\frac{1}{2}(L_{23}L_{31} \\ +L_{21}L_{33})\end{matrix} & \begin{matrix}\frac{1}{2}(L_{22}L_{31} \\ +L_{21}L_{32})\end{matrix} \\ L_{11}L_{31} & L_{12}L_{32} & L_{13}L_{33} & \begin{matrix}\frac{1}{2}(L_{13}L_{32} \\ +L_{12}L_{33})\end{matrix} & \begin{matrix}\frac{1}{2}(L_{13}L_{31} \\ +L_{11}L_{33})\end{matrix} & \begin{matrix}\frac{1}{2}(L_{12}L_{31} \\ +L_{11}L_{32})\end{matrix} \\ L_{11}L_{21} & L_{12}L_{22} & L_{13}L_{23} & \begin{matrix}\frac{1}{2}(L_{13}L_{22} \\ +L_{12}L_{23})\end{matrix} & \begin{matrix}\frac{1}{2}(L_{13}L_{21} \\ +L_{11}L_{23})\end{matrix} & \begin{matrix}\frac{1}{2}(L_{12}L_{21} \\ +L_{11}L_{22})\end{matrix} \end{bmatrix} \tag{46}$$

The rotation of the symmetric second–order tensor S_{lm} in equation (41) can thus be reduced to

$$[S'_a] = [A_{ab}]\,[W_{bc}]\,[S_c] \tag{47}$$

The matrix notation $[T_{ab}]$ of a symmetric fourth–order tensor T_{ijlk} can be rotated in the same manner

$$[T'_{af}] = [A_{ab}]\,[W_{bc}]\,[T_{cd}]\,[W_{de}]\,[A_{ef}] \tag{48}$$

References

[1] S. Nemat-Nasser and M. Hori, *Micromechanics: Overall properties of heterogenous materials*, second ed., Elsevier, 1999.