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The abstract should be only one paragraph long. It must provide the required information so that it is not necessary to read the other parts or even the complete work. It must also include key words to outline the subject for indexing and for bibliographic reference.

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1 Introduction

Differential equations are a very powerful mathematical tool to describe the universe. From fluid dynamic and heat flow in mechanical engineering or electrodynamics and circuit-design in electrical engineering to the fluctuation of stock market prices and even the movement of astronomical objects - most dynamic processes can be formulated by them. But only a tiny fraction of them are actually analytically solvable. The rise of the computer paved the way for approximating the solution of a differential equation numerically. There are plenty of different solver methods ranging from simple single step solvers like the Forward-Euler (FE) method over more complex multistep solvers like the Adams–Bashforth (AB) method to whole FEM solver packages. Most of these solvers are specialised for one kind of differential equation, leveraging some special properties of the problem to be most efficient in time and error. But there are very few generalised methods that can solve many different types of equations. This lack of a universal solver is the main motivation of the present master thesis.

There is already a small, yet steadily growing research community interested in tying together the concepts of computational intelligence and numerical solver for differential equation. In chapter 2.2 a brief overview of related work done within the last 20 years is given. The main idea of all listed papers is to reformulate the original problem into an optimisation problem, which in turn is then solved using an evolutionary optimisation algorithm. The main focus of this thesis lies two-dimensional PDE. The results are then compared to the analytical solution as well as a state of the art FEM solver.

This master thesis tries to answer the following questions: Is it possible to improve the results obtained in other research papers by using a modern variant of the differential evolution (DE) optimisation algorithm? How well does this method stack up against classical FEM package for solving PDEs considering time and memory usage as well as numerical error? Is there a meaningful way to reduce the time consumption by making use of a parallel computation architecture?

2 State of the Art

This chapter provides an overview of the current state of the art in solving PDE. Included are the widely used FEM as well as heuristic optimisation methods. Further, an introduction to the DE framework is given, which provides the basis for the algorithms described in this thesis.

2.1 Finite Element Method

Currently the finite element method is the go-to approach to solve partial differential equations. The domain Ω on which the PDE is posed, is discretised into multiple smaller elements - as the name suggests. Thus, FEM counts to the category of meshed methods. The underlying solution function $u(\mathbf{x})$ to the PDE is then approximated by so called “basis-functions” $\Phi(\mathbf{x})$ limited to these finite elements. This thesis uses the open-source Netgen/NGSolve FEM package (Schöberl, Lackner, and Hochsteger 2020).

The general steps taken to solve a PDE with an FEM solver are:

- Setup

At first a Hilbertspace V is needed. The principal idea is to find a bilinear form $a : V \times V \rightarrow \mathbf{R}$ and a linear functional $F : V \rightarrow \mathbf{R}$ so that for

$$u \in V \quad a(u, v) = F(v) \quad \forall v \in V \quad (2.1)$$

This can be achieved by reformulating the strong form into the weak form as seen in equation 2.3. There, the left-hand side represents the $a(u, v)$ and the right hand side (RHS) is $F(v)$. In this case u is often called the trial-function and v is named the test-function.

- Strong Form

This is the standard formulation of the PDE with a Dirichlet boundary condition:

$$\begin{aligned} u, f, g &: \Omega \rightarrow \mathbf{R} \\ \mathbf{L}u(\mathbf{x}) &= f(\mathbf{x}) \text{ on } \Omega \\ u(\mathbf{x})|_{\partial\Omega} &= g(\mathbf{x}) \end{aligned} \quad (2.2)$$

- Weak Form

This is equivalent to the strong form but written in an integral formulation.

$$\begin{aligned} \int_{\Omega} -(\nabla^T A \nabla) u(\mathbf{x}) v(\mathbf{x}) dV - \int_{\Omega} b^T \nabla u(\mathbf{x}) v(\mathbf{x}) dV \\ + \int_{\Omega} c u(\mathbf{x}) v(\mathbf{x}) dV = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) dV \end{aligned} \quad (2.3)$$

In this equation, the A , b and c correspond to the constant factors of the derivatives in strong form. The derivatives are hidden in the differential operator \mathbf{L} .

- Discretisation of Ω

Create a mesh of finite elements that span the whole domain. Usually these are triangles. Thus, this step is sometimes called “triangulation”.

- Basis functions

Choose a basis function $\Phi(\mathbf{x})$ that can be used to approximate the solution $u(\mathbf{x}) \approx u_h(\mathbf{x}) = \sum_{i=1}^N u_i \Phi_i(\mathbf{x})$. A common choice are Lagrange or Chebyshev polynomials.

- Solution

Solve the resulting linear system of equations to determine the factors u_i . In the simple case where only second order derivatives are used in the strong form, the resulting linear system looks like this:

$$\begin{aligned} \sum_{j=1}^N v_j \sum_{i=1}^N u_i a(\Phi_i, \Phi_j) &= \sum_{j=1}^N v_j \mathbf{L}(\Phi_j) \\ \underbrace{\sum_{i=1}^N u_i a(\Phi_i, \Phi_j)}_{\mathbf{A}\mathbf{u}} &= \underbrace{\mathbf{L}(\Phi_j)}_{\mathbf{b}} \end{aligned} \quad (2.4)$$

Modern solvers include more complex and advanced techniques to further improve the solution error and the computation time. Some of the most important concepts that are also available in NGSolve are listed here.

- Static Condensation:

Depending on the number of discrete elements, the \mathbf{A} matrix can be very large. Inverting large matrices is very time consuming. Condensation reduces this dimensionality by exploiting the structure of \mathbf{A} .

- Adaptive Mesh Refinement:

The accuracy of a FEM-approximated solution mainly depends on the granularity of the mesh. Typically, finer meshes tend to produce more accurate solutions, but the computation time takes longer. This trade-off can be overcome by a self-adaptive mesh. NGSolve implements that in an adaptive loop that executes:

- Solve PDE (with coarse mesh)
- Estimate Error (for every element)
- Mark Elements (that have the greatest error)
- Refine Elements (that were previously marked)
- \circlearrowleft until degrees of freedom exceed a specified N
- Preconditioner:
Instead of solving the \mathbf{A}^{-1} exactly, this can also be approximated by a matrix that is similar to \mathbf{A}^{-1} . The actual inverse can be iteratively approximated. NGSolve implements multiple different preconditioners and it even allows to create your own method.

2.2 Computational Intelligence Methods

The research community interested in computational intelligence solvers for differential equations has been steadily growing over the past 20 years. This chapter summarises the most important works done in the general field of development and application of such statistical numerical solvers. The following table 2.1 gives a brief overview of these papers and sorts them by relevance.

In general, all of these papers from the table use the weighted residual method (WRM), or some variant of that concept, to transform their differential equation into an optimisation problem. This serves as the fitness function and is necessary to evaluate a possible candidate solution and perform the evolutionary selection. The WRM method is further described in chapter 3.1.

In their paper Chaquet and Carmona 2019 describe an algorithm that approximates a solution with a linear combination of gaussian radial basis function (RBF) as kernels:

$$u(\mathbf{x})_{apx} = \sum_{i=1}^N \omega_i e^{\gamma_i (\|\mathbf{x} - \mathbf{c}_i\|^2)} \quad (2.5)$$

The approximated function $u(\mathbf{x})_{approx}$ can be fully determined by a finite number of parameters: ω_i , γ_i , $c0_i$, and $c1_i$. These are stacked together into a vector \vec{u}_{apx} and called the decision variables which are optimised by the algorithm. The objective function can be seen in equation 2.6.

$$F(\vec{u}_{apx}) = \frac{\sum_{i=1}^{n_C} \xi(\mathbf{x}_i) \|\mathbf{L}u(\mathbf{x}_i) - f(\mathbf{x}_i)\|^2 + \phi \sum_{j=1}^{n_B} \|\mathbf{B}u(\mathbf{x}_j) - g(\mathbf{x}_j)\|^2}{m(n_C + n_B)} \quad (2.6)$$

The limit of the sum n_C denotes the number of inner collocation points \mathbf{x}_i within the boundary Ω , whereas n_B is the number of discrete points \mathbf{x}_j on the boundary

$\partial\Omega$. The first term represents the differential equation itself where \mathbf{L} is a differential operator and $\mathbf{f}(\mathbf{x})$ is the inhomogeneous part. Similar, the second term stands for the boundary condition, where \mathbf{B} is the differential operator and $\mathbf{g}(\mathbf{x})$ is the value on the boundary. The multipliers ξ and ϕ are weighting factors for either the inner or the boundary term. The whole term is normalised with the number of collocation points. The parameter of the kernels are determined via a Covariance Matrix Adaption Evolution Strategy (CMA-ES) (Hansen 2006). To further improve the solution, the evolutionary algorithm is coupled with a Downhill-Simplex (DS) method (Nelder and Mead 1965) to carry out the local search. The authors can show empirically that the local search significantly improves the performance by testing the algorithm on a set of 32 differential equations.

Babaei 2013 takes a different approach. They approximate a solution using a partial fourier series. The main advantage of this candidate representation is, that it is backed up with a solid foundation of convergence theory. The optimal parameters for the candidates are found using a particle swarm optimisation (PSO) algorithm (Kennedy and Eberhart 1995). The fitness function consists of two parts, one for the inner area (equation 2.7) and one for the boundary (equation 2.8). These are added together resulting in $F(u(\mathbf{x})_{apx}) = WRF + PFV$.

$$WRF = \int_{\Omega} |\mathbf{W}(\mathbf{x})| |\mathbf{R}(\mathbf{x})| dx \quad (2.7)$$

This is exactly the formulation of the WRM, where \mathbf{R} is the residual that originates directly from the differential equation $\mathbf{R} = \mathbf{L}u(\mathbf{x}_i) - f(\mathbf{x}_i)$ and \mathbf{W} is an arbitrary weighting function. Instead of using a sum of collocation points, the integral is evaluated using a numerical integration scheme.

$$PFV = WRF \cdot \sum_{m=1}^{n_B} K_m g_m \quad (2.8)$$

In the penalty function from equation 2.8 the g_m stands for the boundary condition and K_m are penalty multipliers. The concept of this penalty function originates from Rajeev S. and Krishnamoorthy C. S. 1992.

Sobester, Nair, and Keane 2008 tried a radical different approach to incorporate the boundary condition into the solution. They found, that using genetic programming (GP) (Koza 1992) for the inner domain is only effective if the algorithm does not have to consider the boundary. They split the solution $u(\mathbf{x})_{apx}$ into two parts where $u(\mathbf{x})_{GP}$ represents the solution for the inner domain and $u(\mathbf{x})_{RBF}$ ensures the boundary condition, as seen in equation 2.9.

$$u(\mathbf{x})_{apx} = u(\mathbf{x})_{GP} + u(\mathbf{x})_{RBF} \quad (2.9)$$

At first, the GP step produced a trial solution according to the objective function

2.10.

$$F(u(\mathbf{x})_{apx}) = \sum_{i=1}^{n_C} \|\mathbf{L}u(\mathbf{x}_i) - f(\mathbf{x}_i)\|^2 \quad (2.10)$$

After the GP procedure, a linear combination of radial basis functions $u(\mathbf{x})_{RBF} = \sum_{i=1}^{n_b} \alpha_i \Phi(||\mathbf{x} - \mathbf{x}_{i+n_d}||)$ is specifically tailored to $u(\mathbf{x})_{GP}$, that ensures the boundary condition at all \mathbf{x}_{i+n_d} points on $\partial\Omega$. Finding the parameters α_i can be formulated as a least squares problem.

Chaque and Carmona 2012 use a simple self-adaptive Evolution Strategy (ES) (as developed by Schwefel 1977 and Rechenberg 1978) to evolve the coefficients of a partial Fourier series. The fitness function from equation 2.6 is reused. To reduce the search dimension (represented by the number of harmonics), they developed a scheme that only optimises one harmonic at a time and freezes the other coefficients. This scheme is based on the often observed principle that lower frequencies are more important in reconstructing a signal than higher ones. Albeit this concept might not be valid for all possible functions, it worked on all differential equations of their testbed.

Sadollah et al. 2017 compares three different optimisation algorithms to approximate differential equations: PSO, Harmony Search (HS) (Geem, Kim, and Loganathan 2001) and Water Cycle Algorithm (WCA) (Eskandar et al. 2012). The objective function for all algorithms is the same. They use the formulation in equation 2.7, where the weighting function is the same as the residual $|\mathbf{W}(\mathbf{x})| = |\mathbf{R}(\mathbf{x})| \implies WRF = \int_{\Omega} |\mathbf{R}(\mathbf{x})|^2 dx$. The integral is again approximated using a numerical integration scheme. They find that the PSO is slightly better at producing low error solutions, however the WCA is better at satisfying the boundary condition.

Fateh et al. 2019 use a simple variant of DE (Storn and Price 1997) where the candidates are extended to matrices. They take discrete function value points within the domain to appximate a solution of elliptic partial differential equations. This is a radical brute force approach that results in a massive search space dimension. Yet the main advantage is, that the solution is not limited to a decomposition of kernel functions and thus even non-smooth functions can be approximated. Since this approach does not produce an analytical solution, the boundary condition can be easily incorporated into the fitness function by simply taking the sum of squared residuals at every grid point, as seen in equation 2.11.

$$F(u(\mathbf{x})_{apx}) = \sqrt{\sum_{i=0}^n R(\mathbf{x}_i)^2} \quad (2.11)$$

Howard, Brezulianu, and Kolibal 2011 use a GP scheme to find the solution to a specific set of simplified convection-diffusion equations. They also represent a candidate as function value points over the domain. The function between these points is interpolated. The fitness function is similar to equation 2.10 with the

exception that the n_C points are not predetermined. These points are sampled randomly in the domain, thus allowing the algorithm to approximate the solution aside from the base points.

Panagant and Bureerat 2014 use polynomials as a candidate representation. They did not specify the order or the type of the polynomial. Five different simple versions of DE were tested. Further, they introduce a so called DE-New that increases the population size after every generation. Their proposition is that greater population sizes are better at finding good solutions.

Tsoulos and Lagaris 2006 use a grammatical evolution (GE) (Ryan, Collins, and Neill 1998) to find solutions to various differential equations. In contrary to GP, GE uses vectors instead of trees to represent the candidate string. The solution is evaluated as an analytical string, constructed of the functions *sin*, *cos*, *exp* and *log*, as well as all digits and all four basic arithmetic operations. The solution is measured by the same idea as displayed in equation 2.10. The algorithm was tested on multiple problems of ordinary differential equation (ODE), system of ODEs and PDE. Only the results for ODEs were promising.

Mastorakis 2006 couples a genetic algorithm (GA) (Holland 1962) with a DS method for the local solution refinement. The candidates are represented as polynomials of the order 5 where the coefficients are optimised. The boundary condition is directly incorporated into the candidate, thus simplifying the objective function to equation 2.10. The focus here lays on unstable ODEs, that can not be solved with finite difference methods.

Kirstukas, Bryden, and Ashlock 2005 proposes a three-step procedure. The first step is time consuming and employs GP techniques to find basis functions that span the solution space. The second step is faster and uses a Gram–Schmidt algorithm to compute the basis function multipliers to develop a complete solution for a given set of boundary conditions. Using linear solver methods, a set of coefficients is found that produces a single function that both satisfies the differential equation and the boundary or initial conditions at all collocation points.

Howard and Roberts 2001 is one of the first advances in this field. They approximate a subset of the convection-diffusion equations with GP. Their main idea is to use a polynomial of variable length as the candidate representation. They use the same fitness function as already described in equation 2.9.

2.3 Differential Evolution

The differential evolution optimisation framework was first introduced in Storn and Price 1997. Due to its simple and flexible structure, it quickly became one of the most successful evolutionary algorithm. Over the years, several adaptations to the original framework have been proposed and some of them currently count to the best

performing algorithms, as the 100-Digit Challenge at the GECCO 2019 (Ponnuthurai Nagaratnam Suganthan 2020) shows.

The main DE framework consists of three necessary steps, that continuously update a population of possible solutions. The population can be interpreted as a matrix, where each vector is a possible candidate for the optimum of the fitness function $f : \mathbf{R}^n \rightarrow \mathbf{R}$. These steps are performed in a loop until some termination condition is reached. Each of these steps are controlled by a user-defined parameter:

- mutation:

mutation strength parameter F ;
uses the information from within the population to create a trial vector v_i ;
this is done by scaling the difference between some vectors in the population - hence the name *differential* evolution; the */current-to-pbest/1* mutation operator can be seen in equation 2.12; x_i is the current individual, x_{best}^p is one random vector of the p% top vectors, x_{r1} is a random vector of the population while \tilde{x}_{r2} is randomly chosen from the population and the archive;

$$v_i = x_i + F_i(x_{best}^p - x_i) + F_i(x_{r1} - \tilde{x}_{r2}) \quad (2.12)$$

- crossover:

crossover probability parameter CR;
randomly mix the information between the trial vector v_i and a random candidate from the population x_i to a new trial vector u_i ; the binomial crossover from equation 2.13 randomly takes elements from both vectors, where K is a random index to ensure that at least one element from the trial vector v_i is taken;

$$u_{ij} = \begin{cases} v_{ij}, & \text{if } j = K \vee \text{rand}[0, 1] \leq CR \\ x_{ij}, & \text{otherwise} \end{cases} \quad (2.13)$$

- selection:

population size N ;
replace the old candidate x_i if the trial candidate u_i is better (according to the fitness function); this is performed for every individual in the population;

In modern DE variants, these parameters are self-adapted during the evolutionary process. This means that the algorithms can balance out between exploration of the search-space and exploitation of promising locations.

A prominent example of a modern DE with self-adaption is JADE, which was developed by Zhang and Sanderson 2009. The adaption is performed by taking successful F and CR parameter of the last generation into account. If a certain setting is successful in generating better candidates, newly selected F and CR tend towards that setting. The pseudocode is represented in the appendix A.1.

This idea was later refined by Tanabe and A. Fukunaga 2013. They propose a similar self-adaptive scheme but extend the “memory” for good F and CR parameters over multiple generations. This idea should improve the robustness as compared to JADE. The pseudocode in appendix A.2 shows the outline of this so called SHADE algorithm.

The latest iteration of SHADE is called L-SHADE (Tanabe and A. S. Fukunaga 2014), which further improves the performance by including a deterministic adaptive concept for the population size. At first, L-SHADE starts with a big population size, and reduces the number of individuals in a linear fashion by deleting bad candidates. This has the effect of reducing the number of unnecessary function evaluation. The code is displayed in the appendix A.3.

| Paper | Algorithm | Coding | Problems |
|--------------------------------------|-----------------------------------|--|--|
| Chaque and Carmona 2019 | CMA-ES (global); DS (local) | linear combination of Guassian kernals | testbench of ODEs system of ODEs and PDEs |
| Babaei 2013 | PSO | partial sum of Fourier series | integro-differential equation systme of linear ODEs Brachistochrone nonlinear Bernoulli |
| Sobester, Nair, and Keane 2008 | GP and RBF-NN | algebraic term for inner; RBF for boundary | Elliptic PDEs |
| Chaque and Carmona 2012 | self-adaptive ES | partial sum of Fourier series | testbench of ODEs system of ODEs and PDEs |
| Sadollah et al. 2017 | PSO HS WCA | partial sum of Fourier series | singular BVP |
| Fateh et al. 2019 | DE | function value gird | Elliptic PDEs |
| Howard, Brezulianu, and Kolibal 2011 | GP | function value grid | convection–diffusion equation at different Peclet numbers |
| Panagant and Bureerat 2014 | DE | polynomial of unspecified order | set of 6 different PDEs |
| Tsoulos and Lagaris 2006 | GE | algebraic term | set of ODEs system of ODEs and PDEs |
| Mastorakis 2006 | GA (global); DS (local) | 5th order polynomial | unstable ODEs |
| Kirstukas, Bryden, and Ashlock 2005 | GP | algebraic expression | heating of thin rod heating by current |
| Howard and Roberts 2001 | GP | polynomial of arbitrary lenght | one-dimensional steady-state model of convection-diffusion equation |

Table 2.1: Literature research on the general topic of stochastic slover and their application. The papers are sorted by relevance to the present work.

3 Problem Definition

This chapter describes the broader idea of the WRM, which is used to reformulate any differential equation into an optimisation problem, and its application in the field of spectral methods (Shen, Tang, and Wang 2011). The fitness function originates from these approaches. Further, the different approximation schemes and numerical solution representation are depicted. The following paragraphs also introduce the mathematical notation used for this work.

3.1 Theoretical Foundation

The most general description of a PDE is displayed in equation 3.1.

$$\begin{aligned} \mathbf{L}u(\mathbf{x}) &= f(\mathbf{x}) \\ \text{subjected to: } \mathbf{B}u(\mathbf{x}) &= g(\mathbf{x}) \end{aligned} \tag{3.1}$$

This is similar to the formulation of the strong form in equation 2.2, but not limited to a Dirichlet boundary problem. The matrix \mathbf{B} can include differentiation, effectively allowing i.e. Neumann boundary condition.

An approximate solution to this problem is expressed as a finite sum of basis functions $\phi_k(\mathbf{x})$, as denoted in equation 3.2. The coefficients a_k are to be determined. In classical approximation schemes, the basis functions are orthogonal, such as trigonometric functions or Lagrange basis polynomials.

$$u(\mathbf{x}) \approx u_{apx}(\mathbf{x}) = \sum_{k=0}^N a_k \phi_k(\mathbf{x}) \tag{3.2}$$

The residual of a PDE, as shown in equation 3.3, is formulated as the difference between the left and the right side of the equation. The residual of a solved PDE is zero. This is only possible for the $u_{ext}(x, y)$ analytical solution. For a numerical approximation $u_{apx}(x, y)$, the residual should be "small enough".

$$\mathbf{R}(\mathbf{x}) = \mathbf{L}u_{apx}(\mathbf{x}) - f(\mathbf{x}) \tag{3.3}$$

The WRM (Shen, Tang, and Wang 2011) tries to minimise this residual of a numerical candidate solution over the whole domain Ω . Therefore, R at every $\mathbf{x}_i \in \Omega$ is evaluated and added up, resulting in the following integral of equation 3.4. The residual at every \mathbf{x} can be scaled by a weighting function over the domain as denoted with $W(\mathbf{x})$. The choice of the test function $\psi(\mathbf{x})$ is distinct for different methods. The actual optimum does not change, since the zero-product property holds. The WRM builds the basis for many solving strategies, including FEM.

$$WRF = \int_{\Omega} \mathbf{R}(\mathbf{x})\psi(\mathbf{x})W(\mathbf{x})dx \quad (3.4)$$

This integral must be evaluated numerically. There are many different integration schemes available, but they are typically computational expensive. A less extensive approach is to evaluate the integral argument at different “collocation points” and add up the results. This can be seen in equation 3.5. Although it is not an integral per se, it does assign a numerical “score” to the residual.

$$\sum_{k=0}^N \mathbf{R}(\mathbf{x}_k)\psi(\mathbf{x}_k)W(\mathbf{x}_k) \quad (3.5)$$

To ensure, that negative and positive residuals do not get canceled out, a common choice in heuristic methods from table 5.1, is to choose $\psi(\mathbf{x}_k)$ as the residual itself, effectively squaring it. The resulting “sum of squared residuals” forms the basis for nearly all fitness function in the current literature, as represented in equation 3.6.

$$\sum_{k=0}^N \mathbf{R}(\mathbf{x}_k)^2 W(\mathbf{x}_k) \quad (3.6)$$

These methods are often called “spectral methods” (Shen, Tang, and Wang 2011). Their main advantage over finite difference methods is, that they take the whole domain into account. The derivative of the function at one point is influenced by the function at every other point in the domain. This global approach can lead to a better accuracy.

3.2 Fitness Function

As mentioned above, the basis of the fitness function used in Chaquet and Carmona 2019 is the squared weighted residual in equation 3.6. They split the summation over the collocation points into two parts: the points on the boundary (nb) and the points on the inner domain (nc). Further, they devide the fitness value by the number of points used. This fitness function, as seen in equation 3.7, is adopted in this work.

$$F(u_{apx}) = \frac{\sum_{i=1}^{n_C} \xi(\mathbf{x}_i) \|\mathbf{L}u(\mathbf{x}_i) - f(\mathbf{x}_i)\|^2 + \phi \sum_{j=1}^{n_B} \|\mathbf{B}u(\mathbf{x}_j) - g(\mathbf{x}_j)\|^2}{(n_C + n_B)} \quad (3.7)$$

The fitness function has two important properties:

- The fitness value of the exact solution must be 0:
 $F(u_{ext}(\mathbf{x})) \equiv 0$
- The fitness value of an approximate solution is greater or equal to 0:
 $F(u_{apx}(\mathbf{x})) \geq 0$

The weighting function is also split into two parts: ξ for the inner collocation points and ϕ as penalty factor for the boundary points. The weighting function ξ can be adapted by a parameter κ . For a $\kappa > 1$, ξ assigns larger values for collocation points closer to the boundary, effectively shifting the relative importance towards the boundary. These weights can be calculated apriori, so no computational effort is added to the fitness function. Again, the weights are directly extracted from Chaquet and Carmona 2019.

$$\xi(\mathbf{x}_i) = \frac{1 + \kappa \left(1 - \frac{\min_{\forall \mathbf{x}_j \in n_B} \|\mathbf{x}_i - \mathbf{x}_j\|}{\max_{\forall \mathbf{x}_k \in n_C} (\min_{\forall \mathbf{x}_j \in n_B} \|\mathbf{x}_k - \mathbf{x}_j\|)} \right)}{1 + \kappa} \quad (3.8)$$

3.3 Candidate Representation

As Chaquet and Carmona 2019 suggest, an approximate solution should be represented as a summation of N scaled and shifted radial basis functions ϕ , as denoted in equation 3.2 RBF are functions that only are point symmetric to a centre c and their function value only depend on the radius r .

cite approximation theorem

$$\mathbf{r} = \|\mathbf{x} - \mathbf{c}\| \quad (3.9)$$

This thesis mainly works with two different types of RBF. The first one is the classical Gauss RBF (*gak*). The second one, further called GSin RBF (*gsk*), is more complex. A candidate solution is represended as the parameters that shift and deform these RBF.

3.3.1 Gauss Kernel

The general Gauss kernel is the classical choice to approximate functions. It was first introduced in Broomhead and Lowe 1988. These kernels have been used in many different works, including Chaquet and Carmona 2019. Equation 3.10 shows the formulation of such a kernel. One kernel has 4 parameters, that change its shape and location. ω is a scaling factor for each and γ describes how sharp the e-function is. Depending on the space the solution is defined in, the number of \mathbf{c} could change - one for each dimension. However, in this work only \mathbf{R}^2 is of interest. In figure 3.1 a single standard Gauss kernel is plotted.

cite approximation proof

$$gak(\mathbf{x}) = \omega e^{-\gamma \mathbf{r}^2} \quad (3.10)$$

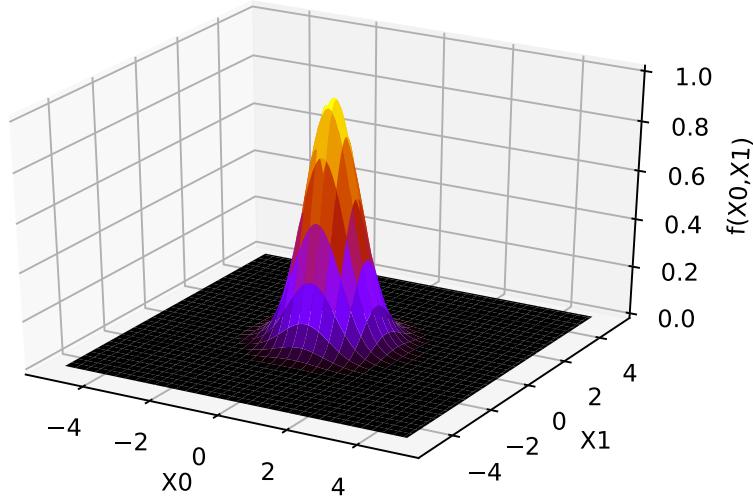


Figure 3.1: 3D plot of a single Gauss kernel with the parameters $\omega = 1$, $\gamma = 1$, $c_0 = 0$ and $c_1 = 0$

An approximate solution to the PDE in question is described as the superposition of N kernels as shown in equation 3.11.

$$u_{apx}(\mathbf{x}) = \sum_{i=0}^N \omega_i e^{-\gamma_i \mathbf{r}_i^2} \quad (3.11)$$

A solution is encoded as a vector of these $4 \cdot N$ parameters stacked together. This is shown in equation 3.12.

$$\vec{u}_{apx} = \left[\underbrace{\omega_0, \gamma_0, c_{00}, c_{01}}_{\text{kernel 0}}, \dots \underbrace{\omega_i, \gamma_i, c_{i0}, c_{i1}}_{\text{kernel i}}, \dots \underbrace{\omega_N, \gamma_N, c_{N0}, c_{N1}}_{\text{kernel N}} \right]^T \quad (3.12)$$

To evaluate the residual of a PDE, the derivatives must be known. To that extend, the derivative of u_{apx} from equation 3.2 is calculated. The first order derivative with respect to x_0 is seen in equation 3.13. To solve the proposed testbed, also the second order derivatives are needed, as described in equation 3.14. Although the mixed term derivative is not used in this work, for the sake of completeness it is displayed in equation 3.15.

$$\frac{\partial u_{apx}(\mathbf{x})}{\partial x_0} = -2 \sum_{i=0}^N \omega_i \gamma_i (x_0 - c_{i0}) e^{-\gamma_i \mathbf{r}_i^2} \quad (3.13)$$

$$\frac{\partial^2 u_{apx}(\mathbf{x})}{\partial x_0^2} = \sum_{i=0}^N \omega_i \gamma_i [4\gamma_i(x_0 - c_{i0})^2 - 2] e^{-\gamma_i \mathbf{r}_i^2} \quad (3.14)$$

$$\frac{\partial^2 u_{apx}(\mathbf{x})}{\partial x_0 \partial x_1} = 4 \sum_{i=0}^N \omega_i \gamma_i^2 (x_0 - c_{i0})(x_1 - c_{i1}) e^{-\gamma_i \mathbf{r}_i^2} \quad (3.15)$$

3.3.2 GSin Kernel

Additional to the Gauss kernel, a “GSin” kernel is proposed. In essence, this is a multiplication of a Gauss kernel with a sine function, as seen in equation 3.16. It can be thought of a sine function, where its influence declines with the radius r . Both “sub-functions” are centered at the same \mathbf{c} . The plot in figure 3.2 shows how the first half period is much larger than the second half. With the sine, two new parameters are introduced: f for the frequency of the sine wave and φ represents the phase shift.

approximation proof possible?

$$gsk(\mathbf{x}) = \omega e^{-\gamma \mathbf{r}^2} \sin(f \mathbf{r}^2 - \varphi) \quad (3.16)$$

It is important to notice that, with the correct parameter choice, both traits - the sine and the e-function - can be retrieved. When $\gamma = 0$ the e-function evaluates to 1 and leaves the sine on its own. By setting $f = 0$ and $\varphi = \frac{\pi}{2}$, the sine vanishes and the e-function part is obtained. Thus, with this kernel, it should be possible to approximate the PDEs 0A and 0B from the testbed arbitrarily close.

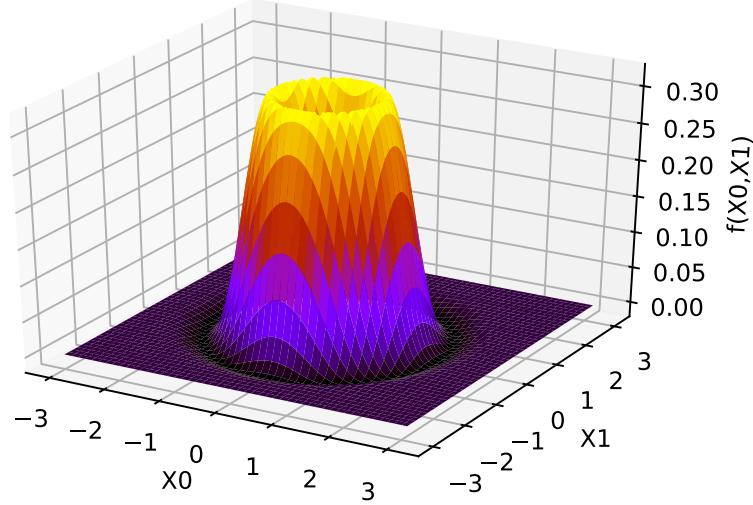


Figure 3.2: 3D plot of a single GSin kernel with the parameters $\omega = 1$, $\gamma = 1$, $c_0 = 0$, $c_1 = 0$, $f = 1$, $\varphi = 0$

Similar to the Gauss kernel, the approximate solution is described as the superposition of N GSin kernels, as seen in equation 3.17.

$$u_{apx}(\mathbf{x}) = \sum_{i=0}^N \omega_i e^{\gamma_i \mathbf{r}_i^2} \sin(f_i \mathbf{r}_i^2 - \varphi_i) \quad (3.17)$$

Equation 3.18 shows the solution representation in stacked vector notation, including the two new parameters f and φ .

$$\vec{u}_{apx} = \left[\underbrace{\omega_0, \gamma_0, c_{00}, c_{01}, f_0, \varphi_0}_{\text{kernel 0}}, \cdots \underbrace{\omega_i, \gamma_i, c_{i0}, c_{i1}, f_i, \varphi_i}_{\text{kernel i}}, \cdots \underbrace{\omega_N, \gamma_N, c_{N0}, c_{N1}, f_N, \varphi_N}_{\text{kernel N}} \right]^T \quad (3.18)$$

Again, the derivatives of the solution are needed, which are shown in the equations 3.19 through 3.21.

$$\frac{\partial u_{apx}(\mathbf{x})}{\partial x_0} = \sum_{i=0}^N 2\omega_i(x_0 - c_{i0})e^{-\gamma_i \mathbf{r}_i^2} (\gamma_i \sin(\varphi_i - f_i \mathbf{r}_i^2) + f_i \cos(\varphi_i - f_i \mathbf{r}_i^2)) \quad (3.19)$$

$$\begin{aligned} \frac{\partial^2 u_{apx}(\mathbf{x})}{\partial x_0^2} &= \sum_{i=0}^N 2\omega_i e^{-\gamma_i \mathbf{r}_i^2} \\ &[(2c_{i0}^2(f_i^2 - \gamma_i^2) + 4c_{i0}x(\gamma_i^2 - f_i^2) + 2f_i^2x^2 - 2\gamma_i^2x^2 + \gamma_i) \sin(\varphi_i - f_i \mathbf{r}_i^2) + \\ &f_i(-4c_{i0}^2\gamma_i + 8c_{i0}\gamma_i x - 4\gamma_i x^2 + 1) \cos(\varphi_i - f_i \mathbf{r}_i^2)] \end{aligned} \quad (3.20)$$

$$\begin{aligned} \frac{\partial^2 u_{apx}(\mathbf{x})}{\partial x_0 x_1} &= \sum_{i=0}^N 4\omega_i(c_{i0} - x)(c_{i1} - y)e^{-\gamma_i \mathbf{r}_i^2} \\ &[(f_i^2 - \gamma_i^2) \sin(\varphi_i - f_i \mathbf{r}_i^2) - 2f_i\gamma_i \cos(\varphi_i - f_i \mathbf{r}_i^2)] \end{aligned} \quad (3.21)$$

4 Experimental Design

This chapter serves as an introduction to the experiments and gives an overview on how these are conducted. An integral part of solver comparison is to define a testbed that holds example problems with a known analytical solution. Further, quality measurements must be defined that compares the numerical to the analytical solution. The baseline for all experiments is the FEM solver NGSolve (Mitchell 2018).

4.1 Testbed

The testbed is a collection of multiple different 2-dimensional scalar PDEs that are analytically solved such that $u(\mathbf{x}) : \mathbf{R}^2 \rightarrow \mathbf{R}$ is a solution to the underlaying PDE. These can be used to demonstrate the correct implementation of a solver. The testbed can also be used to compare the performance of the classical FEM solver (NGSolve) with the Computational Intelligence (CI) solver. The actual equations are displayed in the appendix B. The equations used here are a mixture of multiple different testbeds. Specifically, the equations 2 and 3 are picked from the testbed in Chaquet and Carmona 2019. These problems were also used by Tsoulos and Lagaris 2006 and Panagant and Bureerat 2014. Preliminary tests have shown that the equations used in these papers are rather simple to approximate. Thus, more complicate equations are added to test the solvability over a wider variety of functions. The more complex equations are taken from the National Institute of Standards and Technology (NIST) website (Mitchell 2018) that provides benchmarking problems for FEM slovers with adaptive mesh refinement methods. The other equations 0A, 0B and 8 are specially created to show different properties of the solver.

PDE 0A: Gauss Kernel (equation B.1) The analytic solution of this problem can be approximated arbitrarily close, since the solution is a sum of 5 different Gauss Kernel (GaK) as represented in equation 3.10. The purpose of this PDE is to show that the algorithm converges towards the analytical solution.

PDE 0B: GSin Kernel (equation B.3) Similar to the PDE 0A, this problem is a sum of Gauss Sine Kernel (GSK), as seen in equation 3.16. To keep the search dimensionality roughly the same as in PDE 0A, the problem can be solved with 3 kernels. Again, the purpose of this equation is to determine the convergence with other kernel types.

PDE 1: Polynomial 2D (equation B.5) The solution of this equation is a polynomial of order 20. The function is 0 on the boundary of the unit square. (Mitchell 2018)

PDE 2: Chaquet PDE 1 (equation B.7) This is the problem of the Chaquet testbed, that is also used by several other authors. Its main purpose is to build the bridge to those papers so that the results can be compared. (Chaquet and Carmona 2019, Chaquet and Carmona 2012, Tsoulos and Lagaris 2006, Sobester, Nair, and Keane 2008, Panagant and Bureerat 2014)

PDE 3: Chaquet PDE 3 (equation B.9) This equation is solved by a polynomial of order 2. Again, the purpose is to compare the results to other papers. (Chaquet and Carmona 2019, Chaquet and Carmona 2012, Tsoulos and Lagaris 2006, Panagant and Bureerat 2014)

PDE 4: Sine Bump 2D (equation B.11) The sine bump occurs in a similar fashion in both, the Chaquet testbed (PDE 8 in Chaquet and Carmona 2019) as well as the NIST (Mitchell 2018) testbed. This means that the underlying solution function is the same, but the PDE is posed differently. Preliminary results have shown that the formulation of the NIST testbed is harder to solve. Thus, the formulation of Chaquet is disregarded and the NIST problem is implemented.

PDE 5: Arctan Circular Wave Front (equation B.13) The main difficulty of this problem is the transition from the flat plateaus to the steep gradient of the circular wave front. Preliminary tests have shown that this equation is one of the most difficult problems in this testbed. (Mitchell 2018)

PDE 6: Peak 2D (equation B.15) The solution to this problem is described by a single Gaussian “peak” at $(0.5, 0.5)$ with a large exponent. This solution could be approximated by a single GaK. The difficulty here is the steep gradient and a small region of interest. (Mitchell 2018)

PDE 7: Boundary Line Singularity (equation B.17) This equation is only determined on $x \in \mathbf{R}^+$, which results in a singularity line at $x = 0$. Towards this line the gradient increases. (Mitchell 2018)

PDE 8: Interior Point Singularity (equation B.19) The idea of this PDE is similar to the problem 7, but the singularity is located on the inner domain. The solution to this problem is not defined at $(0.5, 0.5)$, resulting in a very difficult problem.

PDE 9: Arctan Wave Front Homogeneous Boundary Conditions 2D (equation B.21) Similar to PDE 5, the difficulty of this problem is the steep gradient. Additionally, the boundary condition is zero which results in sharp corners on the boundary, that are hard to approximate. (Mitchell 2018)

4.2 Software Architecture

To simplify the preparation, execution and evaluation of the experiments, a comprehensive software architecture is defined. The UML class diagram can be seen in the appendix C. The architecture is organised in 4 main segments.

optimisation algorithm in red

The `IOptAlgoBase` interface must be implemented by every `OptAlgo` class to ensure the compatibility with `CiPdeBase` class of the testbed. A nice side-effect is that it reduces the number of user-defined parameters. An optimisation algorithm of this class must only take an initial guess (e.g. the starting population) as well as two stopping criteria: the maximum number of function evaluation or a minimum error to reach. Also a fitness function (i.e. the function to be optimised) must be provided. Four lists of the same length are returned: the optimum-guess, the function value, the crossover probability and the scale factor per each generation. The actual implementation of the algorithm is not predefined.

kernels in blue

As described in chapter 3.3, a candidate solution is defined as a sum of RBF. In order to test different candidate representations, different classes must be implemented. Again, to ensure compatibility with the `CiPdeBase` class, all representations must implement the `IKernelBase` interface. This assures that all classes have a method that can calculate the solution as well as first and second order derivatives. Here, only two kernels are implemented. A typical Gauss kernel as described in chapter 3.3.1 and a so called GSin kernel shown in chapter 3.3.2.

testbed in orange

The testbed holds the 11 differential equations used in all experiments. The testbed is abstracted in such a way that an experiment is as simple as creating an PDE object and calling its solve method. All testbed classes must implement the `ITestbenchBase` interface. This ensures the minimal functionality of every subsequent class. Currently two classes implement this interface, the `FemPdeBase` and the `CiPdeBase`. These are the base classes that provide the specific attributes and methods needed for the FEM solver and the CI solver. The actual PDE problems are implemented in the classes `FemPde0` or `CiPde0` which inherit from the `FemPdeBase` and the `CiPdeBase`, respectively. The number in their name is representative for all different testbench problems and every PDE has its own class. Since all PDE classes have the same methods and attributes and only differ in their implementation and name, they do not have to be displayed separately. Therefore, they are symbolised together by a “stacked notation” used in the class diagram. Some methods in these classes must be overridden and adapted to the current PDE problem, which is indicated by the \wedge character.

post processing in purple

Although the post processing block is not actually a class, it is still represented in

this diagram. This module provides functions that take `FemPdeN` or `CiPdeN` objects and performs actions with them.

- `bool saveExpObj(obj, filename)`

Save an `CiPdeN` object as a JavaScript Object Notation (JSON) file. The filename parameter can include a path, but it must end with `.json`. The results: execution time, memory usage, solution quality and all intergenerational data of the optimisation algorithm are stored in the file.

update implemented function description and class diagram in appendix

- `dict loadExpObject(filename)`

Loads the JSON file located at the specified filename, which again can include a path. A dictionary with the saved `CiPdeN` parameters is returned.

- `dict loadExpObjectFast(filename)`

When solving a PDE with more function evaluation, the result file can become very large ($>400\text{Mb} @ 10^6 \#FE$). For evaluating such large files, this function can be used. It does not load the generation data (meaning population, function value, F and CR). Since the standard json interpreter in python loads files to be serialized, a new interpreter is needed. To that extent, *bigjson* from Heino 2020 is used. This package accesses only those parts of a json file, that are actually needed.

- `bool drawGaussKernel(parameter, ggb)`

Draws a solution approximated by Gauss kernels and with the specified parameters to a GeoGebra file. If the filename provided in the `ggb` argument does not exist, the function searches for a template and prints to a copy of that file.

- `bool drawGSinKernel(parameter, ggb)`

This is similar to the method `drawGaussKernel` method - but it takes parameters for a GSin kernel.

- `float calcRSME(solve_dict)`

To compare the obtained results with previous work, the Root-Mean-Square Error (RMSE) quality metric (as described in the chapter 4.3.3) must be computed. This is done from a single dictionary, as obtained by the functions `loadExpObject` or `loadExpObjectFast`.

- `None plotApprox3D(kernel, parameter, lD, uD)`

The approximate solution of a PDE can be plotted over the with this function. Only square sized domains can be plotte, as specified by the lower and the upper domain parameters `lD` and `uD`.

- `string statsWilcoxon(a, b, alpha=0.05)`

This function is a wrapper for the `scipy.stats.wilcoxon` (SciPy 2020). A string is returned, that describes if the mean/median of `a` is significantly smaller than the mean/median of `b`. The default significance level is set to 0.05.

4.3 Metric

In order to scientifically compare the results produced by the different solvers, some metrics are necessary. Three important solver-properties are measured: the execution time, the memory usage and the quality of the numerical solution. The following chapters describe the measurement process in greater detail.

4.3.1 Solving Time

The solving time is measured within the `solve()` method of either class. The time module of the Python Standard Library 2020 is used to interact with the system clock. The resolution, that the time module can access, depends on the system it is running on. Specifically, on the machine used in all further experiments, `time.time()` returns a 24 byte float that represents the time passed since 1st of January 1970. Usually, consecutive calls of this function return increasing values - changing the system time could interfere with the correctness of this value.

As the execution time of a program depends on many other factors, such as the current system load, the CPU temperature and the process scheduler, it is necessary to view it as a random variant. Thus, multiple replications have to be done before trying to interpret the results. These replications are not done within the `solve()` function and must be applied during the experiment. To reduce the random effects and prevent possible outlier, the Python garbage collector is switched off during the time measurement. For a step-by-step description, the pseudocode is displayed in the appendix D.

4.3.2 Memory Usage

Similar to the solving time measurement, the memory usage is determined within the `solve()` method. The *psutil* module (Rodola 2020) provides the functionality to read the amount of memory attached to a process at a given time. The function call `process.memory_info()` returns an object with multiple attributes about the current state of the process. Of special interest is the Virtual Memory Size (VMS) field. This includes the Resident Set Size (RSS), the memory that is currently held within the main memory, and the memory that is currently swapped out to the harddrive.

Without assuming anything about the inner workings of the process, the memory usage is also a random variant. Thus, similar to the time measurement, replications have to be performed. The same pseudocode as for the time measurement (appendix D) also applies here. To remove outliers, it is helpful to create and solve one testbed object before recording the experiment. This sets up the necessary references and

which are then not mingled into the actual solving process. This holds true for both, the FEM and the CI solver.

4.3.3 Quality Measurement

Although the fitness function is the criterion that is optimised, it is not applicable as an objective quality measure. As Chaquet and Carmona 2019 describe, it depends on multiple factors:

- user-defined parameters ξ and ϕ
- the formulation of the PDE
- number of collocation points used
- number of kernels used

Thus, Chaquet and Carmona 2019 define a new quality measurement based on the RMSE over the collocation points as seen in equation 4.1.

$$RMSE^2 = \frac{\sum_{i=1, \mathbf{x}_i \in C}^{n_C} \|\mathbf{u}(\mathbf{x}_i) - \mathbf{u}_{\text{ext}}(\mathbf{x}_i)\|^2 + \sum_{j=1, \mathbf{x}_j \in B}^{n_B} \|\mathbf{u}(\mathbf{x}_j) - \mathbf{u}_{\text{ext}}(\mathbf{x}_j)\|^2}{m(n_C + n_B)} \quad (4.1)$$

This quality criterion has three inherent issues. At first, it firmly depends on the number of collocation points used. In an algorithm that uses self-adaptive collocation points, this measurement would be rendered useless. Further, the quality is only measured on the collocation points and not in between. However, a good solution fits not only these discrete points, but the whole domain. This is called an aliasing error. An approximation can fit the points, without correctly representing the space between them. This is shown in the following figure 4.1.

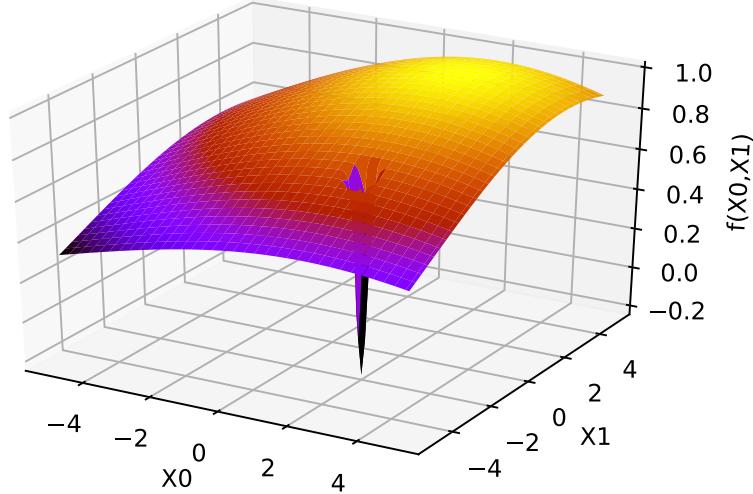


Figure 4.1: aliasing error: sharp inaccuracies in between collocation points

Finally, the FEM method doesn't use collocation points, so this quality measurement could not be calculated. A good quality measurement tool only uses the numerical and analytical solution, independently of the solving method.

This leads to the quality measurement formulation used in this thesis: the L2 norm defined for functions as denoted in equation 4.2. This actually measures the distance between the analytical solution and the numerical approximation.

$$\|u_{ext} - u_{apx}\| = \sqrt{\int_{\Omega} (u_{ext}(\mathbf{x}) - u_{apx}(\mathbf{x}))^2 d\mathbf{x}} \quad (4.2)$$

Although this integral is numerically evaluated, the discretisation is much finer than the resolution of the collocation points used in the fitness function - thus also regarding the areas between these points.

4.4 Baseline: NGSolve

As mentioned above, the NGSolve framework (Schöberl, Lackner, and Hochsteger 2020) is used as the baseline for all experiments. NGSolve is a state of the art FEM solver, that is in part developed and maintained by numerous well-known institutes such as Vienna University of Technology, University of Göttingen and Portland State

University. This chapter describes the results obtained by running NGSolve on the testbed. The metrics from chapter 4.3 are applied.

4.4.1 Setup

At first the PDEs must be transformed into their corresponding weak form. As all testbed problems are Poisson equations and only differ in their algebraic sign and inhomogeneous part, the weak form is similar for all problems. Referring to equation 2.3, the terms $\vec{b} = 0$ and $c = 0$, thus these parts vanish. The matrix A is either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or with a negative sign $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ as only the non-mixed second order derivatives occur in the equations. This results in weak form as presented in equation 4.3, where the f is the RHS of the problem.

$$\int_{\Omega} \nabla u^T \nabla v dV = \int_{\Omega} f v dV \quad (4.3)$$

To enhance the performance of NGSolve, static condensation is turned on. Further, a multigrid preconditioner is used. All problems are approximated by second order polynomials. The automatic mesh refinement is performed until a maximum of $5 \cdot 10^4$ Degree of Freedom (DOF) is reached. To properly interpret the results, 20 replications for each problem are performed. This is only needed for regarding time and memory, the solution itself is not a random variant. To further reduce the memory consumption, the GUI of NGSolve is switched off.

4.4.2 Result

With the parameters described above, the following results are produced. The solving time as well as the memory usage are displayed in the boxplots 4.2 and 4.3, respectively. In general, the solving time ranges from 2.5 to 5.0 seconds at about 50 to 80 MByte of memory. Only problem 3 stands out as a notable exception. To keep the diagram visually appealing, this PDE is omitted and plotted in a separate figure.

The table 4.1 presents the achieved distances between the exact and the approximated solutions. On PDE 3 the best numerical quality is achieved.

The problem PDE 7 only needs around 2.5 seconds to be solved. A reason could be that the solution only depends on one variable and the derivative with respect to y $\frac{\partial u}{\partial y} = 0$ vanishes. This does not effect the memory usage, since all $5 \cdot 10^4$ DOF must be created to terminate.

| Problem PDE | Distance |
|-------------|------------------------------------|
| 0A | $2.9670770746774782 \cdot 10^{-5}$ |
| 0B | $1.070854603999225 \cdot 10^{-5}$ |
| 1 | $8.004152462854497 \cdot 10^{-7}$ |
| 2 | $3.5013418621193666 \cdot 10^{-8}$ |
| 3 | $1.6795224037775289 \cdot 10^{-9}$ |
| 4 | $4.765830679060112 \cdot 10^{-7}$ |
| 5 | $6.056858428283682 \cdot 10^{-6}$ |
| 6 | $1.9078788449490833 \cdot 10^{-7}$ |
| 7 | $5.202739901395381 \cdot 10^{-5}$ |
| 8 | $3.237437258132996 \cdot 10^{-7}$ |
| 9 | $2.3655968008139198 \cdot 10^{-7}$ |

Table 4.1: These are the results obtained by the FEM solver in terms of distance to the analytical solution. The solver achieves the smallest deviation in PDE 3.

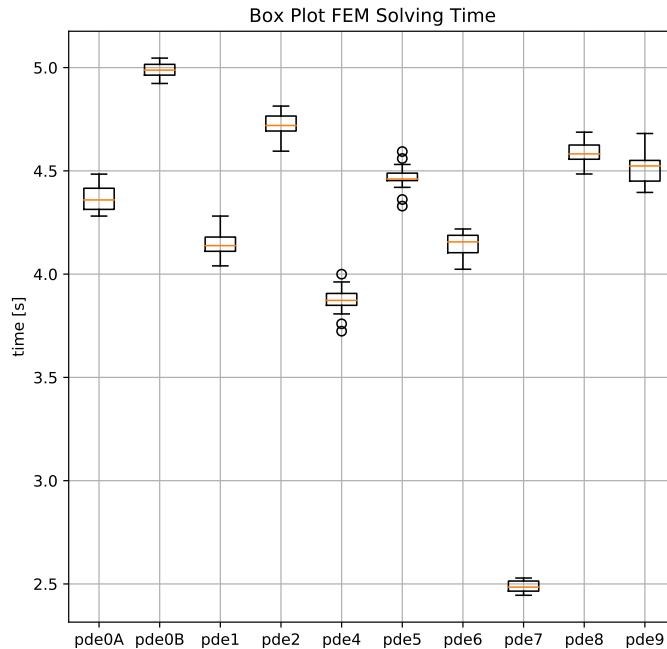


Figure 4.2: boxplot: time (in seconds) needed to solve the testbed PDE (without PDE3)

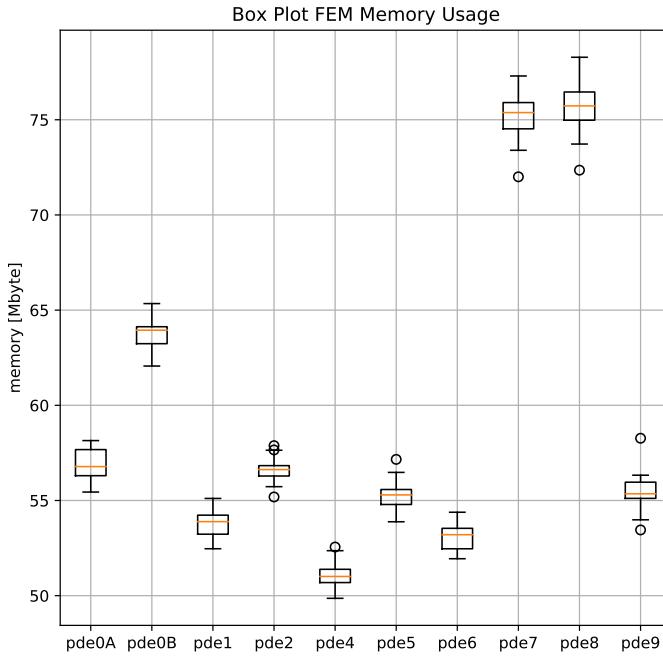


Figure 4.3: boxplot: memory (in Mbyte) needed to solve the testbed PDE (without PDE3)

The following images 4.4 and 4.5 show the boxplots of the time and memory consumption for the testbed PDE 3. Compared to the other equations, this problem takes longer to solve while also needing more memory: somewhere around 65.2 seconds at about 130 Mbyte. A possible explanation for that is the underlying structure of the PDE. As described in equation B.10, the exact solution to this problem is a polynomial of second order. This can be approximated perfectly by the FEM solver, since it also uses second order polynomials as basis functions. The mesh-refinement step takes more iteration to produce the $5 \cdot 10^4$ DOF as compared to the other testbed problems.

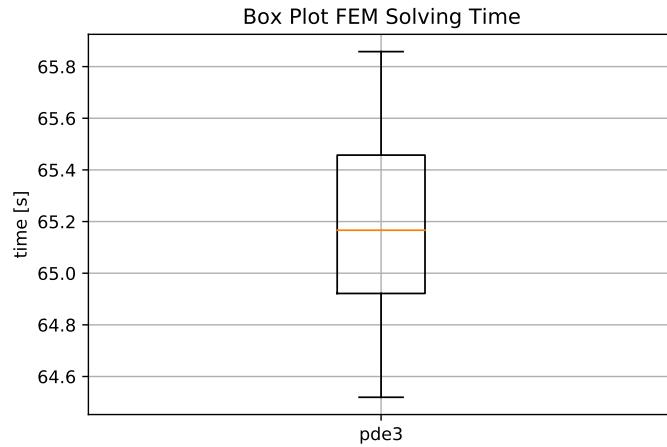


Figure 4.4: boxplot: time to solve testbed PDE3

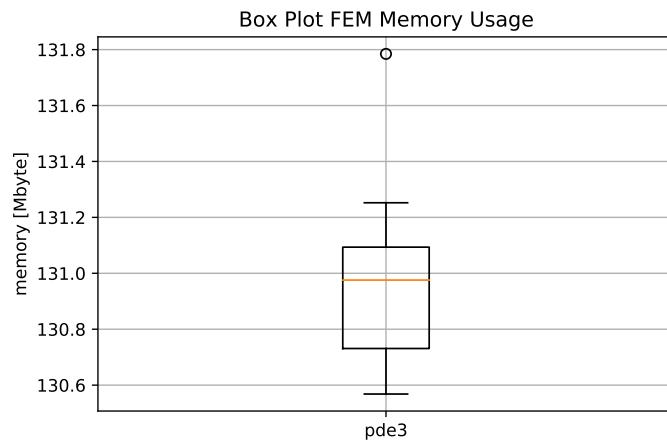


Figure 4.5: boxplot: memory to solve testbed PDE3

4.5 Default CI Parameter

Typically, heuristic optimisation algorithm use some parameters to tune the performance on specific testbed problems - for all further experiments this is the JADE algorithm form pseudocode A.1. Similary, the reported CI solver has many parameters that could be adapted. However, adjusting every parameter in order to find the best combination is not an option, since that would take an extensive amount of computation time. Some parameters, that might not have a great effect on the performance, can be predefined. These values can be determined by preliminary tests or intuition, without any rigerous scientific justification. If not stated otherwise, the following parameters from table 4.2 are used in the subsequent experiments.

| Parameter | Value | Chaquet |
|-----------------|---------------------------------------|--|
| φ | 100 | 300 |
| κ | 1 | 3 |
| population size | $2 \cdot \text{dim}$ | $\frac{3}{2}(4 + \lfloor 3 \cdot \ln(\text{dim}) \rfloor)$ |
| min error | 0 | - |
| p | 0.3 | - |
| c | 0.5 | - |
| replication | 20 | 50 |
| nb | 40 | 100 equally spaced points over the domain |
| nc | 81 | |
| | $121 = 11 \times 11$ | |
| initialisation | $\vec{u}_{apx} \in \mathcal{N}(0, 1)$ | $\omega_i \in \mathcal{U}[-0.01, 0.01]$ $\gamma_i \in \mathcal{U}(0, 1]$ $c_{ik} \in \mathcal{U}[2\Omega]$ |

Table 4.2: These predefined parameters are used for the following numerical experiments.

The parameters φ and κ are used in the fitness function (equation 3.7) for changing the relative importance of the boundary and the interior. Chaquet and Carmona 2019 take similar values. However, preliminary tests have shown that the current values perform slightly better on the present testbed. Figure 4.7 shows the values of ξ and φ . It further describes how the weighting factor emphasises the areas closer to the boundary.

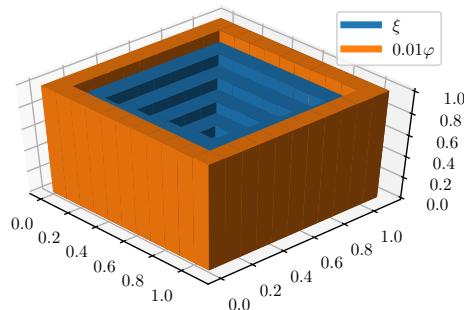


Figure 4.6: weighting factor φ and ξ on every collocation point

The population size used by Chaquet and Carmona 2019 is based on the original formulation of the CMA-ES. However, they observed a better convergence when scaling the recommended population size by 3. Typically, DE uses larger population

sizes. Mallipeddi and P. N. Suganthan 2008 describe an empirical study on choosing this parameter. They discuss the tradeoff between premature convergence and computational effort. The results suggest that population sizes of $2 \cdot \dim$ do get stuck in local optimas, but also converge faster. Since time is a critical resource, this population size is chosen.

The termination condition for DE is either a maximum number of function evaluation, or a minimal function value to reach. Since the fitness function (equation 3.7) has its optimum at 0, this value is used as the termination condition. A helpful side-effect is that it ensures the same amount of function evaluation in every run, without early termination. This prevents outliers in the time and memory measurement.

The parameters p and c are specific to JADE (algorithm A.1). The mutation operator *mutationCurrentToPBest1* uses p to select the best individuals and c weights the parameter adaption mechanism. For all experiments these values are set at $p = 0.3$ and $c = 0.5$.

To account for the statistical influence, all experiments are restarted 20 times with independant initial guesses.

Chaque and Carmona 2019 use 100 equally spaced collocation points over the domain to solve the PDE. Here, 121 points are created, as seen in figure 4.7.

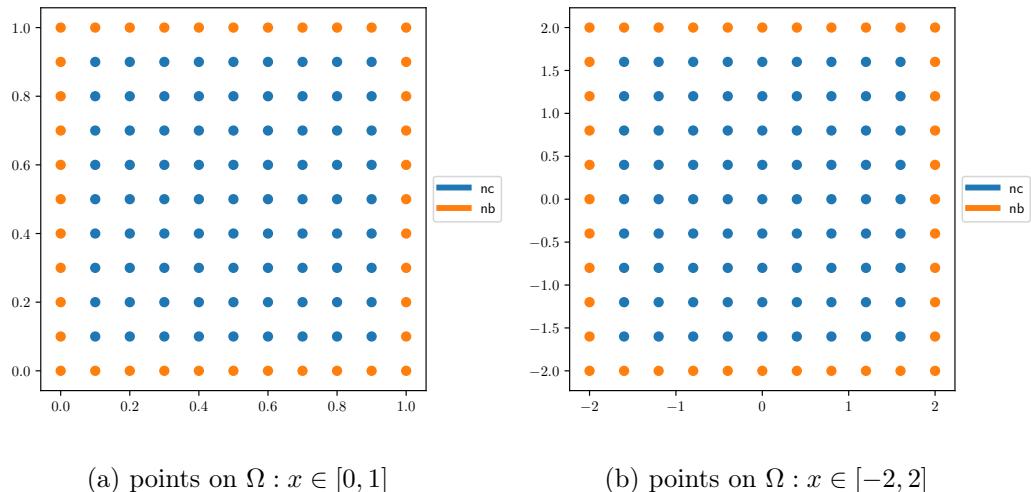


Figure 4.7: collocation points used on the two domains of the testbed

The initialisation is done by a standard normal distribution. This means, that every value in the \vec{u}_{apx} vector is drawn from a normal distribution. On the contrary, Chaquet and Carmona 2019 initialise the values specifically tailored to each parameter of the kernel. This process assumes a priori information about the solution and thus should be dropped.

5 Experiment 0: Serial Memetic JADE

This chapter describes the results obtained by the most basic adaption of JADE for solving PDEs. The algorithm used here, builds on the concepts described in Chaquet and Carmona 2019. Aside from substituting some parameters, the only main difference is the usage of JADE instead of a CMA-ES.

with 10^6 function evaluation:

hypothosis test in the difference of L2 norm reached statistically relevant vs. practically relevant -> show 3D plot of what the difference in L2 norm looks like (= effect size)
this servers as an experimental prove, that the objective function is multimodal and DS runs into premature convergence

show convergence on PDE0A

Compare to Chaquet Results PDE2 and PDE3

with 10^4 function evaluation:

box plot of time and memory usage -> compare to FEM

L2 norm table reached

how does time/memory scale with number of FE? -> $O(n)$

memory not needed: only for evaluation -> could be smaller

what can be done about time? -> transition to next chapter: parallel what are simpler and more complex testbed pdes?

5.1 Hypotheses

Algorithm 5.1: memeticJADE Pseudocode

```
1 Function memeticJADE(X, funct, minErr, maxFE):  
2     dim, popsize  $\leftarrow$  size(X)  
3     p  $\leftarrow$  0.3  
4     c  $\leftarrow$  0.5  
5     pop, FE, F, CR  $\leftarrow$  JADE(X, p, c, funct, minErr, maxFE - 2dim )  
6     bestIndex = argmin(FE)  
7     bestSol = pop[bestIndex]  
8     pop, FE = downhill simplex(funct, bestSol, minErr, 2dim) return  
      pop, FE, F, CR
```

5.2 Experiment Setup

5.3 Result

Table 5.1 shows a compariosn on the common testbed functions PDE 2 and PDE 3 with numerical results obtained in other research papers. It is important to notice, that the parameters, used to obtain the results, might not coincide. Tsoulos and Lagaris 2006 also use these two PDE, but their paper did not provide any usable error metric.

5.4 Interpretation

| Paper | parameter | RMSE PDE 2 | RMSE PDE 3 |
|--------------------------------|---|--------------------------------|--------------------------------|
| Chaque and Carmona 2019 | 4 kernel max #FE=10 ⁶ 50 replications | (1.75 ± 1.14)10 ⁻⁴ | (1.09 ± 0.846)10 ⁻⁵ |
| Chaque and Carmona 2012 | 10 harmonics max #FE = $G \cdot \lambda = 1.2 \cdot 10^6$ 10 replications | (6.37 ± 0.733)10 ⁻³ | (5.90 ± 0.799)10 ⁻³ |
| Sobester, Nair, and Keane 2008 | 50 max tree lenght 12 generations 20 replications | (6.9 ± 8.3)10 ⁻⁴ | - |
| Panagant and Bureerat 2014 | unknowns: N/A #FE=5 · 10 ⁵ replications: N/A | 7.25610 ⁻⁴ | 9.48910 ⁻⁶ |
| serial memetic JADE | 5 kernel max #FE = 10 ⁶ 20 replications | | |

Table 5.1: This table compares the results obtained with the serial memetic JADE to the numerical results obtained by similar work in literature. The same metric must be used, thus the RMSE as defined in equation 4.1 is calculated.

| #FE | 10 ⁴ | | 10 ⁶ | | Wilcoxon Test |
|--------|-----------------|--------|-----------------|--------|---------------|
| | stat | mean | median | mean | median |
| PDE 0A | 1.9415 ± 0.3321 | 1.8844 | 0.6596 ± 0.5510 | 0.9285 | sig. better |
| PDE 0B | 0.7137 ± 0.1979 | 0.6354 | 0.2027 ± 0.1302 | 0.1516 | sig. better |
| PDE 1 | 0.1874 ± 0.0408 | 0.1938 | 0.0149 ± 0.0049 | 0.0151 | sig. better |
| PDE 2 | 0.0890 ± 0.0334 | 0.0760 | 0.0257 ± 0.0140 | 0.0224 | sig. better |
| PDE 3 | 0.2409 ± 0.1051 | 0.2309 | 0.0328 ± 0.0169 | 0.0285 | sig. better |
| PDE 4 | 0.1102 ± 0.0367 | 0.0985 | 0.0378 ± 0.0083 | 0.0352 | sig. better |
| PDE 5 | 0.6645 ± 0.1930 | 0.6263 | 1.1968 ± 0.0286 | 1.2056 | sig. worse |
| PDE 6 | 1.9660 ± 1.3845 | 1.6540 | 0.4135 ± 1.2133 | 0.0018 | sig. better |
| PDE 7 | 0.0457 ± 0.0137 | 0.0452 | 0.0221 ± 0.0019 | 0.0223 | sig. better |
| PDE 8 | 0.2186 ± 0.0045 | 0.2191 | 0.2170 ± 0.0019 | 0.2175 | unsig. better |
| PDE 9 | 0.0525 ± 0.0147 | 0.0516 | 0.0451 ± 0.0119 | 0.0459 | unsig. better |

Table 5.2:

6 Experiment 1: Parallel Population JADE

with 10^6 FE perform hypothesis test to show that both reach the same resulting L2 norm

algorithm only a little bit different: information from population not present until the next generation

plot of significant relative speedup

able to add more workers - depends on the machine

greater speedup when a single FE takes longer -> on which testbed pdes? compare time to FEM

6.1 Hypotheses

6.2 Experiment Setup

6.3 Result

6.4 Interpretation

7 Experiment 2: Adaptive Number of Kernels

7.1 Hypotheses

Two ideas to perform adaption:

- half adaptive: increase dimension when better, decrease only once when worse
- fully adaptive: increase and decrease whenever a better or worse

How often does a reduction and restart step find a better solution?

sensitivity with the parameter “minerror”, increase minerror and observe L2 norm reached only on pde 2 and 3

7.2 Experiment Setup

7.3 Result

7.4 Interpretation

8 Experiment 3: GSin Kernel

8.1 Hypotheses

8.2 Experiment Setup

8.3 Result

8.4 Interpretation

9 Limitations

9.1 Testbed

Only PDEs with Laplace operator

9.2 Sine Bump 2D on Unit Square

Sensitive towards the boundary condition:

Chaquet PDE8 as described in dissertation vs. same PDE but with 0 on boundary

10 Theoretical Observations

10.1 Universial Approximation Theorem

10.2 Multimodality and Symmetry

11 Conclusion

12 Further Work

technology:

implemented in python -> precompile fitness function

theoretical:

aliasing error, tchebyshev collocation points

algorithmic:

13 Summary

Advantage: Can be applied to nearly any PDE without restrictions to a specific problem set
Problematic: Might be applied to a problem although, there is an algorithm that is faster and has better convergence

Acronyms

| | |
|---------------|---|
| AB | Adams–Bashforth. 1 |
| CI | Computational Intelligence. 18, 20, 23, 28 |
| CMA-ES | Covariance Matrix Adaption Evolution Strategy. 5, 10, 29 |
| DE | differential evolution. 1, 2, 6–8, 10, 29, 30 |
| DOF | Degree of Freedom. 25, 27 |
| DS | Downhill-Simplex. 5, 7, 10 |
| ES | Evolution Strategy. 6, 10 |
| FE | Forward-Euler. 1 |
| FEM | finite element method. VIII, 1, 2, 12, 18, 20, 23, 24, 26, 27 |
| GA | genetic algorithm. 7, 10 |
| GaK | Gauss Kernel. 18, 19 |
| GE | grammatical evolution. 7, 10 |
| GP | genetic programming. 5–7, 10 |
| GSK | Gauss Sine Kernel. 18 |
| HS | Harmony Search. 6, 10 |
| JSON | JavaScript Object Notation. 21 |
| ODE | ordinary differential equation. 7 |
| PDE | partial differential equation. VII, 1, 2, 7, 11, 14, 15, 18–21, 23, 25–28, 30, 32, 33 |
| PSO | particle swarm optimisation. 5, 6, 10 |

Acronyms

| | |
|-------------|--|
| RBF | radial basis function. 4, 13, 20 |
| RHS | right hand side. 2, 25 |
| RMSE | Root-Mean-Square Error. 21, 23 |
| RSS | Resident Set Size. 22 |
| UML | Unified Modeling Language. VII, 20, 67 |
| VMS | Virtual Memory Size. 22 |
| WCA | Water Cycle Algorithm. 6, 10 |
| WRM | weighted residual method. 4, 5, 11, 12 |

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Appendices

A Differential Evolution Pseudocodes

A.1 JADE Pseudocode

Algorithm A.1: JADE Pseudocode

```

1 Function JADE( $\mathbf{X}_{g=0}$ ,  $p$ ,  $c$ , function, minError, maxFE):
2    $fValue_{g=0} \leftarrow function(\mathbf{x}_{g=0})$ 
3    $\mu_{CR} \leftarrow 0.5$ 
4    $\mu_F \leftarrow 0.5$ 
5    $A \leftarrow \emptyset$ 
6   while  $fe \leq maxFE$  do
7      $g \leftarrow g + 1$ 
8      $S_F \leftarrow \emptyset$ 
9      $S_{CR} \leftarrow \emptyset$ 
10    for  $i = 1$  to  $NP$  do
11       $F_i \leftarrow randc_i(\mu_F, 0.1)$ 
12       $v_i \leftarrow mutationCurrentToPBest1(\mathbf{x}_{i,g}, A, fValue_g, F_i, p)$ 
13       $CR_i \leftarrow randn_i(\mu_{CR}, 0.1)$ 
14       $u_i \leftarrow crossoverBIN(\mathbf{x}_{i,g}, v_i, CR_i)$ 
15      if function( $\mathbf{x}_{i,g}$ )  $\geq function(\mathbf{u}_{i,g})$  then
16         $\mathbf{x}_{i,g+1} \leftarrow \mathbf{x}_{i,g}$ 
17      end
18      else
19         $\mathbf{x}_{i,g+1} \leftarrow \mathbf{u}_{i,g}$ 
20         $fValue_{i,g+1} \leftarrow function(\mathbf{u}_{i,g})$ 
21         $\mathbf{x}_{i,g} \rightarrow \mathbf{A}$ 
22         $CR_i \rightarrow S_{CR}$ 
23         $F_i \rightarrow S_F$ 
24      end
25    end
26    // resize  $A$  to size of  $\mathbf{x}_g$ 
27    if  $|A| > NP$  then
28       $A \leftarrow A \setminus A_{rand_i}$ 
29    end
30     $fe \leftarrow fe + size(\mathbf{X})$ 
31     $\mu_{CR} \leftarrow (1 - c) \cdot \mu_{CR} + c \cdot arithmeticMean(S_{CR})$ 
32     $\mu_F \leftarrow (1 - c) \cdot \mu_F + c \cdot lehmerMean(S_F)$ 
33  end

```

A.2 SHADE Pseudocode

Algorithm A.2: SHADE Pseudocode

```

1 Function SHADE( $\mathbf{x}_{G=0}$ ,  $p$ ,  $H$ , function, minError, maxFE):
2    $M_{CR} \leftarrow 0.5$ ;  $M_F \leftarrow 0.5$ ;  $A \leftarrow \emptyset$ ;  $G \leftarrow 0$ ;  $k \leftarrow 1$ 
3    $fValue_{G=0} \leftarrow function(\mathbf{x}_{G=0})$ 
4   while termination condition not met do
5      $S_{CR} \leftarrow \emptyset$ ;  $S_F \leftarrow \emptyset$ 
6     for  $i = 1$  to  $N$  do
7        $r_i \leftarrow rand_{int}(1, H)$ 
8        $CR_{i,G} \leftarrow randn_i(M_{CR,r_i}, 0.1)$ ;  $F_{i,G} \leftarrow randc_i(M_{F,r_i}, 0.1)$ 
9        $v_i \leftarrow mutationCurrentToPBest1(\mathbf{x}_{i,G}, A, fValue_G, F_i, p)$ 
10       $u_i \leftarrow crossoverBIN(pop, v_i, CR)$ 
11    end
12    for  $i = 1$  to  $N$  do
13      if function( $u_{i,G}$ )  $\leq$  function( $x_{i,G}$ ) then
14         $x_{i,G+1} \leftarrow u_{i,G}$ ;  $fValue_{i,G+1} \leftarrow function(\mathbf{u}_{i,G})$ 
15      end
16      else
17         $x_{i,G+1} \leftarrow x_{i,G}$ 
18      end
19      if function( $u_{i,G}$ )  $<$  function( $x_{i,G}$ ) then
20         $x_{i,G} \rightarrow A$ ;  $CR_{i,G} \rightarrow S_{CR}$ ;  $F_{i,G} \rightarrow S_F$ 
21      end
22    end
23    if  $|A| > N$  then
24       $A \leftarrow A \setminus A_{rand_i}$ 
25    end
26    if  $S_{CR} \neq \emptyset \wedge S_F \neq \emptyset$  then
27      
$$M_{CR,k,G+1} = \begin{cases} arithmeticMean(S_{CR}) & \text{if } S_{CR} \neq \emptyset \\ M_{CR,k,G} & \text{otherwise} \end{cases}$$

28      
$$M_{F,k,G+1} = \begin{cases} lehmerMean(S_F) & \text{if } S_F \neq \emptyset \\ M_{F,k,G} & \text{otherwise} \end{cases}$$

29       $k \leftarrow k + 1$ 
30      if  $k > H$  then
31         $k \leftarrow 1$ 
32      end
33    end
34     $G \leftarrow G + 1$ 
35  end

```

A.3 L-SHADE Pseudocode

Algorithm A.3: L-SHADE Pseudocode

```

1 Function LSHADE( $\mathbf{x}_{G=0}$ ,  $p$ ,  $H$ , function, minError, maxFE):
2    $M_{CR} \leftarrow 0.5$ ;  $M_F \leftarrow 0.5$ ;  $A \leftarrow \emptyset$ ;  $G \leftarrow 0$ ;  $k \leftarrow 1$ 
3    $fValue_{G=0} \leftarrow function(\mathbf{x}_{G=0})$ ;  $NG_{init} \leftarrow size(\mathbf{x}_{G=0})$ ;  $NG_{min} = \lceil 1/p \rceil$ 
4   while termination condition not met do
5      $S_{CR} \leftarrow \emptyset$ ;  $S_F \leftarrow \emptyset$ 
6     for  $i = 1$  to  $N$  do
7        $r_i \leftarrow rand_{int}(1, H)$ 
8        $CR_{i,G} \leftarrow randn_i(M_{CR,r_i}, 0.1)$ ;  $F_{i,G} \leftarrow randc_i(M_{F,r_i}, 0.1)$ 
9        $v_i \leftarrow mutationCurrentToPBest1(\mathbf{x}_{i,G}, A, fValue_G, F_i, p)$ 
10       $u_i \leftarrow crossoverBIN(pop, v_i, CR)$ 
11    end
12    for  $i = 1$  to  $N$  do
13      if function( $u_{i,G}$ )  $\leq$  function( $x_{i,G}$ ) then
14         $x_{i,G+1} \leftarrow u_{i,G}$ ;  $fValue_{i,G+1} \leftarrow function(\mathbf{u}_{i,G})$ 
15      end
16      else
17         $x_{i,G+1} \leftarrow x_{i,G}$ 
18      end
19      if function( $u_{i,G}$ )  $<$  function( $x_{i,G}$ ) then
20         $x_{i,G} \rightarrow A$ ;  $CR_{i,G} \rightarrow S_{CR}$ ;  $F_{i,G} \rightarrow S_F$ 
21      end
22    end
23    if  $|A| > N$  then
24       $A \leftarrow A \setminus A_{rand_i}$ 
25    end
26    if  $S_{CR} \neq \emptyset \wedge S_F \neq \emptyset$  then
27      
$$M_{CR,k,G+1} = \begin{cases} arithmeticMean(S_{CR}) & \text{if } S_{CR} \neq \emptyset \\ M_{CR,k,G} & \text{otherwise} \end{cases}$$

28      
$$M_{F,k,G+1} = \begin{cases} lehmerMean(S_F) & \text{if } S_F \neq \emptyset \\ M_{F,k,G} & \text{otherwise} \end{cases}$$

29       $k \leftarrow k + 1$ 
30      if  $k > H$  then
31         $k \leftarrow 1$ 
32      end
33    end
34     $\mathbf{x}_{G+1} \leftarrow popSizeRed(\mathbf{x}_{G+1}, fValue, G, maxGen, NG_{init}, NG_{min})$ 
35     $G \leftarrow G + 1$ 
36  end

```

B Testbed

The following pages describe the testbed that is used for all experiments. The problems are structured in these major points:

- differential equation
 - differential equation
 - domain Ω
 - Dirichlet boundary condition obtained by evaluating the solution on the boundary
- solution
- plot of the solution over the domain

PDE 0A: Gauss Kernel

Problem PDE:

$$\begin{aligned}
 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= (18x^2 - 6)e^{-1.5(x^2+y^2)} + (18y^2 - 6)e^{-1.5(x^2+y^2)} \\
 &+ 6(6x^2 + 12x + 5)e^{-3((x+1)^2+(y+1)^2)} + 6(6y^2 + 12y + 5)e^{-3((x+1)^2+(y+1)^2)} \\
 &+ 6(6x^2 - 12x + 5)e^{-3((x-1)^2+(y+1)^2)} + 6(6y^2 + 12y + 5)e^{-3((x-1)^2+(y+1)^2)} \\
 &+ 6(6x^2 + 12x + 5)e^{-3((x+1)^2+(y-1)^2)} + 6(6y^2 - 12y + 5)e^{-3((x+1)^2+(y-1)^2)} \\
 &+ 6(6x^2 - 12x + 5)e^{-3((x-1)^2+(y-1)^2)} + 6(6y^2 - 12y + 5)e^{-3((x-1)^2+(y-1)^2)}
 \end{aligned}$$

on the domain $\Omega : x, y \in [-2, 2]$
subjected to:

$$\begin{aligned}
 u(x, 2) &= 2e^{-1.5(x^2+4)} + e^{-3((x+1)^2+9)} + e^{-3((x+1)^2+1)} + e^{-3((x-1)^2+9)} + e^{-3((x-1)^2+1)} \\
 u(x, -2) &= 2e^{-1.5(x^2+4)} + e^{-3((x+1)^2+1)} + e^{-3((x+1)^2+9)} + e^{-3((x-1)^2+1)} + e^{-3((x-1)^2+9)} \\
 u(2, y) &= 2e^{-1.5(4+y^2)} + e^{-3(9+(y+1)^2)} + e^{-3(9+(y-1)^2)} + e^{-3(1+(y+1)^2)} + e^{-3(1+(y-1)^2)} \\
 u(-2, y) &= 2e^{-1.5(4+y^2)} + e^{-3(1+(y+1)^2)} + e^{-3(1+(y-1)^2)} + e^{-3(9+(y+1)^2)} + e^{-3(9+(y-1)^2)}
 \end{aligned}
 \tag{B.1}$$

Solution:

$$\begin{aligned}
 u_{ext}(x, y) &= 2e^{-1.5(x^2+y^2)} + e^{-3((x+1)^2+(y+1)^2)} + e^{-3((x+1)^2+(y-1)^2)} \\
 &+ e^{-3((x-1)^2+(y+1)^2)} + e^{-3((x-1)^2+(y-1)^2)}
 \end{aligned}
 \tag{B.2}$$

Solution of PDE 0A

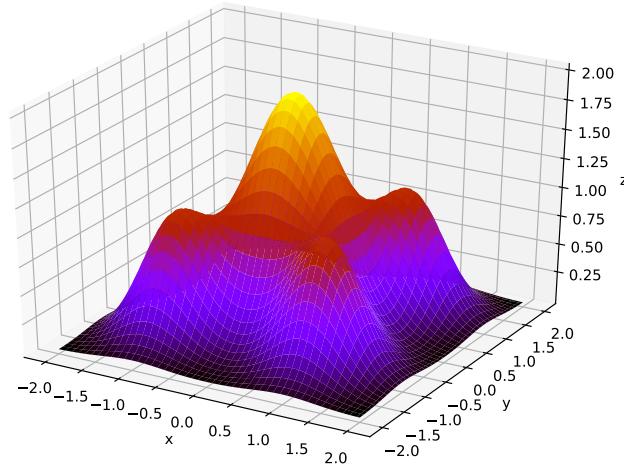


Figure B.1: PDE 0A Gauss Kernel solution plot

PDE 0B: GSin Kernel

Problem PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =$$

$$2e^{-2(x^2+y^2)}(2\sin(-2(x^2+y^2)) + 2(1-8x^2)\cos(-2(x^2+y^2))) +$$

$$2e^{-2(x^2+y^2)}(2\sin(-2(x^2+y^2)) + 2(1-8y^2)\cos(-2(x^2+y^2))) +$$

$$2e^{-1(x^2+y^2)}(1\sin(-1(x^2+y^2)) + 1(1-4x^2)\cos(-1(x^2+y^2))) +$$

$$2e^{-1(x^2+y^2)}(1\sin(-1(x^2+y^2)) + 1(1-4y^2)\cos(-1(x^2+y^2))) +$$

$$2e^{-0.1(x^2+y^2)}(0.1\sin(-0.1(x^2+y^2)) + 0.1(1-0.4x^2)\cos(-0.1(x^2+y^2))) +$$

$$2e^{-0.1(x^2+y^2)}(0.1\sin(-0.1(x^2+y^2)) + 0.1(1-0.4y^2)\cos(-0.1(x^2+y^2)))$$

on the domain $\Omega : x, y \in [-2, 2]$
subjected to:

$$u(x, 2) = e^{-2(x^2+4)}\sin(2((x^2+4))) + e^{-1(x^2+4)}\sin(1((x^2+4))) + e^{-0.1(x^2+4)}\sin(0.1((x^2+4)))$$

$$u(x, -2) = e^{-2(x^2+4)}\sin(2((x^2+4))) + e^{-1(x^2+4)}\sin(1((x^2+4))) + e^{-0.1(x^2+4)}\sin(0.1((x^2+4)))$$

$$u(2, y) = e^{-2(4+y^2)}\sin(2((4+y^2))) + e^{-1(4+y^2)}\sin(1((4+y^2))) + e^{-0.1(4+y^2)}\sin(0.1((4+y^2)))$$

$$u(-2, y) = e^{-2(4+y^2)}\sin(2((4+y^2))) + e^{-1(4+y^2)}\sin(1((4+y^2))) + e^{-0.1(4+y^2)}\sin(0.1((4+y^2)))$$

(B.3)

Solution:

$$\begin{aligned} u_{ext}(x, y) = & e^{-2(x^2+y^2)} \sin(2((x^2 + y^2))) \\ & e^{-1(x^2+y^2)} \sin(1((x^2 + y^2))) \\ & e^{-0.1(x^2+y^2)} \sin(0.1((x^2 + y^2))) \end{aligned} \quad (\text{B.4})$$

Solution of PDE 0B

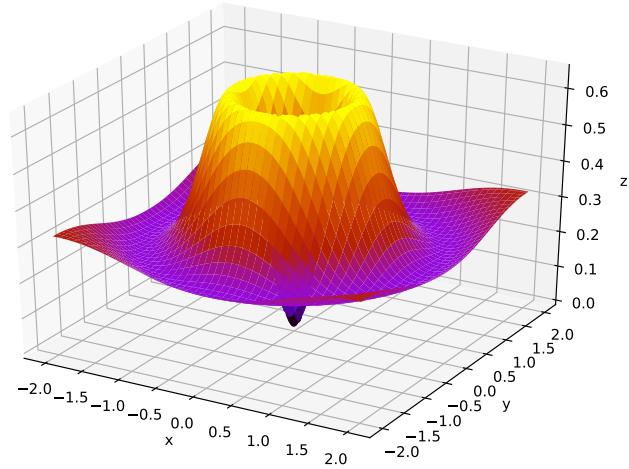


Figure B.2: PDE 0B Gsin Kernel solution plot

PDE 1: Polynomial 2D

Problem PDE:

$$\begin{aligned}
 & -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \\
 & -2^{40} y^{10} (1-y)^{10} [90x^8(1-x)^{10} - 200x^9(1-x)^9 + 90x^{10}(1-x)^8] \\
 & -2^{40} x^{10} (1-x)^{10} [90y^8(1-y)^{10} - 200y^9(1-y)^9 + 90y^{10}(1-y)^8] \\
 & \quad \text{on the domain } \Omega : x, y \in [0, 1] \\
 & \quad \text{subjected to:} \\
 & \quad u(x, 1) = 0 \\
 & \quad u(x, 0) = 0 \\
 & \quad u(1, y) = 0 \\
 & \quad u(0, y) = 0
 \end{aligned} \tag{B.5}$$

Solution:

$$u_{ext}(x, y) = 2^{40} x^{10} (1-x)^{10} y^{10} (1-y)^{10} \tag{B.6}$$

Solution of PDE 1

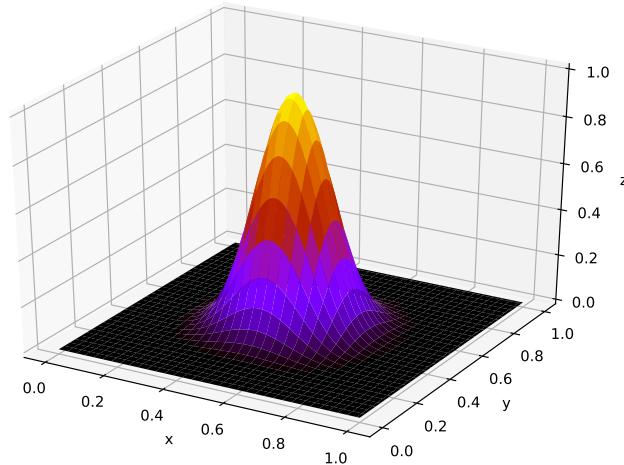


Figure B.3: PDE 1 Polynomial 2D solution plot

PDE 2: Chaquet PDE 1

Problem PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x}(x - 2 + y^3 + 6y)$$

on the domain $\Omega : \mathbf{x} \in [0, 1]$
subjected to:

$$u(x, 0) = xe^{-x}u(x, 1) = (x + 1)e^{-x}u(0, y) = y^3$$

$$u(1, y) = (1 + y^3)e^{-1}$$
(B.7)

Solution:

$$u_{ext}(x, y) = (x + y^3)e^{-x} \quad (B.8)$$

Solution of PDE 2

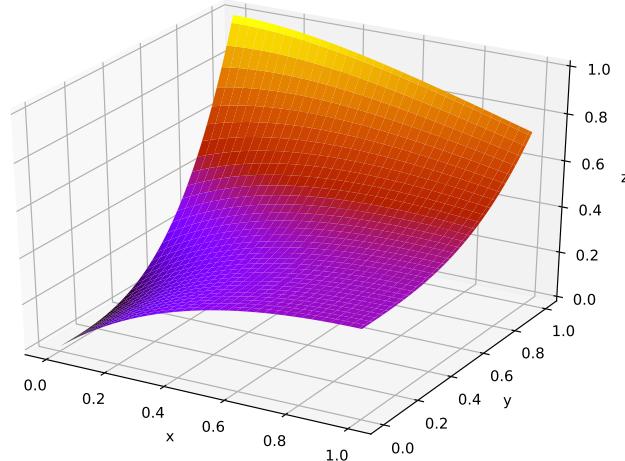


Figure B.4: PDE 2 Chaquet PDE 1 solution plot

PDE 3: Chaquet PDE 3

Problem PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4$$

on the domain $\Omega : \mathbf{x} \in [0, 1]$

subjected to:

$$u(x, 0) = x^2 + x + 1 \quad (\text{B.9})$$

$$u(x, 1) = x^2 + x + 3$$

$$u(1, y) = y^2 + y + 3$$

$$u(0, y) = y^2 + y + 1$$

Solution

$$u_{ext}(x, y) = x^2 + y^2 + x + y + 1 \quad (\text{B.10})$$

Solution of PDE 3

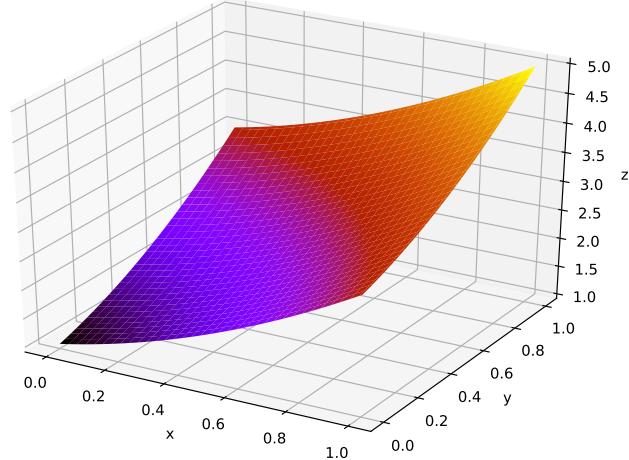


Figure B.5: PDE 3 Chaquet PDE 3 solution plot

PDE 4: Sine Bump 2D

Problem PDE:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 2\pi^2 \sin(\pi x) \sin(\pi y)$$

on the domain $\Omega : \mathbf{x} \in [0, 1]$

subjected to:

$$\begin{aligned} u(x, 0) &= 0 \\ u(x, 1) &= 0 \\ u(0, y) &= 0 \\ u(1, y) &= 0 \end{aligned} \tag{B.11}$$

Solution:

$$u_{ext}(x, y) = \sin(\pi x) \sin(\pi y) \tag{B.12}$$

Solution of PDE 4

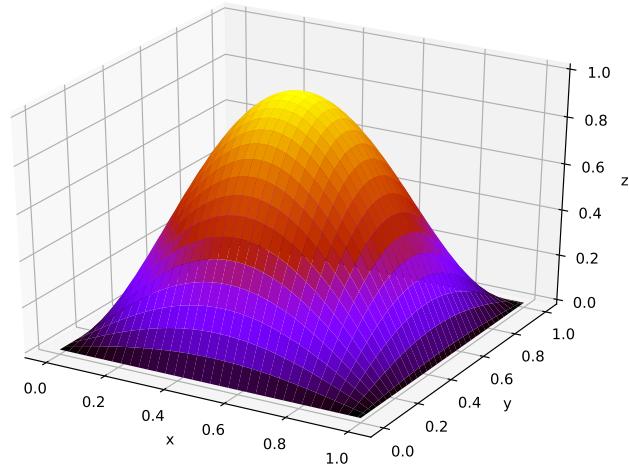


Figure B.6: PDE 4 Sine Bump 2D solution plot

PDE 5: Arctan Circular Wave Front

Problem PDE:

$$\begin{aligned}
 -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} &= \frac{16000(\sqrt{(x-0.05)^2 + (y-0.05)^2} - 0.7)}{(1+400(-0.7 + \sqrt{(x-0.05)^2 + (y-0.05)^2})^2)^2} \\
 + \frac{20(x-0.05)^2 + 20(y-0.05)^2}{(1+400(\sqrt{(x-0.05)^2 + (y-0.05)^2} - 0.7)^2)((x-0.05)^2 + (y-0.05)^2)^{3/2}} \\
 - \frac{40}{(1+400(\sqrt{(y-0.05)^2 + (x-0.05)^2} - 0.7)^2)\sqrt{(y-0.05)^2 + (x-0.05)^2}}
 \end{aligned}$$

on the domain $\Omega : \mathbf{x} \in [0, 1]$
subjected to:

$$\begin{aligned}
 u(x, 0) &= \tan^{-1} \left(20 \left(\sqrt{(x-0.05)^2 + 0.0025} - 0.7 \right) \right) \\
 u(x, 1) &= \tan^{-1} \left(20 \left(\sqrt{(x-0.05)^2 + 0.9025} - 0.7 \right) \right) \\
 u(0, y) &= \tan^{-1} \left(20 \left(\sqrt{0.0025 + (y-0.05)^2} - 0.7 \right) \right) \\
 u(1, y) &= \tan^{-1} \left(20 \left(\sqrt{0.9025 + (y-0.05)^2} - 0.7 \right) \right)
 \end{aligned} \tag{B.13}$$

Solution:

$$u_{ext}(x, y) = \tan^{-1} \left(20 \left(\sqrt{(x-0.05)^2 + (y-0.05)^2} - 0.7 \right) \right) \tag{B.14}$$

Solution of PDE 5

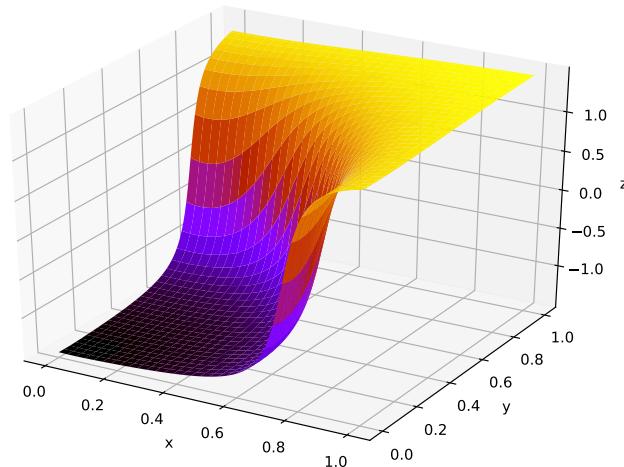


Figure B.7: PDE 5 Arctan Circular Wave Front solution plot

PDE 6: Peak 2D

Problem PDE:

$$\begin{aligned}
 & -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \\
 & -(4 \cdot 10^6 x^2 - 4 \cdot 10^6 x + 998 \cdot 10^3) e^{-1000((x-0.5)^2+(y-0.5)^2)} \\
 & -(4 \cdot 10^6 y^2 - 4 \cdot 10^6 y + 998 \cdot 10^3) e^{-1000((x-0.5)^2+(y-0.5)^2)} \\
 & \quad \text{on the domain } \Omega : \mathbf{x} \in [0, 1] \\
 & \quad \text{subjected to:} \tag{B.15} \\
 & u(x, 0) = e^{-1000((x-0.5)^2+0.25)} \\
 & u(x, 1) = e^{-1000((x-0.5)^2+0.25)} \\
 & u(0, y) = e^{-1000(0.25+(y-0.5)^2)} \\
 & u(1, y) = e^{-1000(0.25+(y-0.5)^2)}
 \end{aligned}$$

Solution:

$$u_{ext}(x, y) = e^{-1000((x-0.5)^2+(y-0.5)^2)} \tag{B.16}$$

Solution of PDE 6

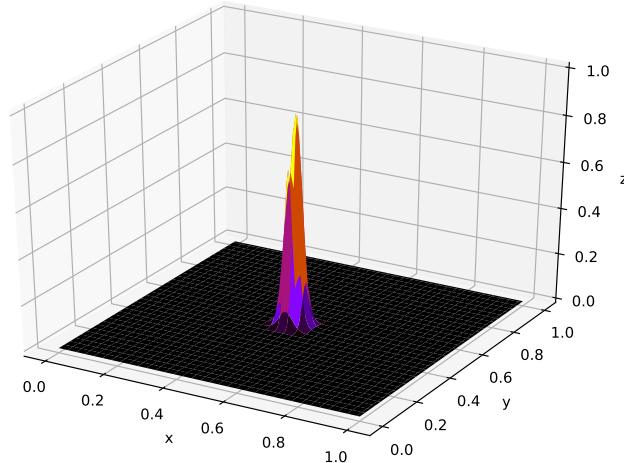


Figure B.8: PDE 6 Peak 2D solution plot

PDE 7: Boundary Line Singularity

Problem PDE:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.24x^{-1.4}$$

on the domain $\Omega : \mathbf{x} \in [0, 1]$

subjected to:

$$u(x, 0) = x^{0.6} \quad (\text{B.17})$$

$$u(x, 1) = x^{0.6}$$

$$u(0, y) = 0$$

$$u(1, y) = 1^{0.6}$$

Solution:

$$u_{ext}(x, y) = x^{0.6} \quad (\text{B.18})$$

Solution of PDE 7

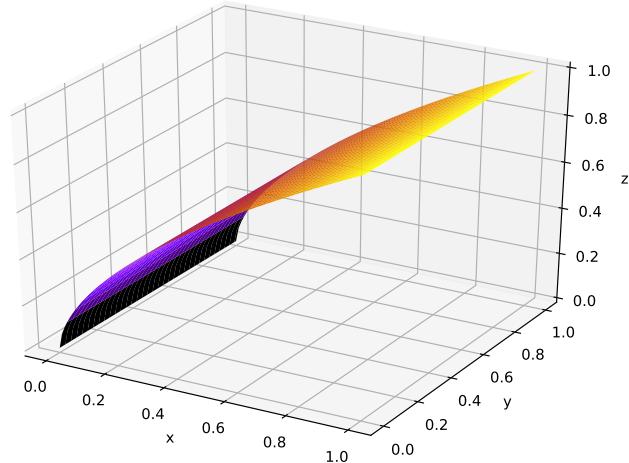


Figure B.9: PDE 7 Boundary Line Singularity solution plot

PDE 8: Interior Point Singularity

Problem PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\sqrt{x^2 - x + y^2 - y + 0.5}} \\ \Omega : \mathbf{x} \in [0, 1]$$

on the domain subjected to:

$$u(x, 0) = \sqrt{(x - 0.5)^2 + 0.25} \quad (\text{B.19}) \\ u(x, 1) = \sqrt{(x - 0.5)^2 + 0.25} \\ u(0, y) = \sqrt{0.25 + (y - 0.5)^2} \\ u(1, y) = \sqrt{0.25 + (y - 0.5)^2}$$

Solution:

$$u_{ext}(x, y) = \sqrt{(x - 0.5)^2 + (y - 0.5)^2} \quad (\text{B.20})$$

Solution of PDE 8

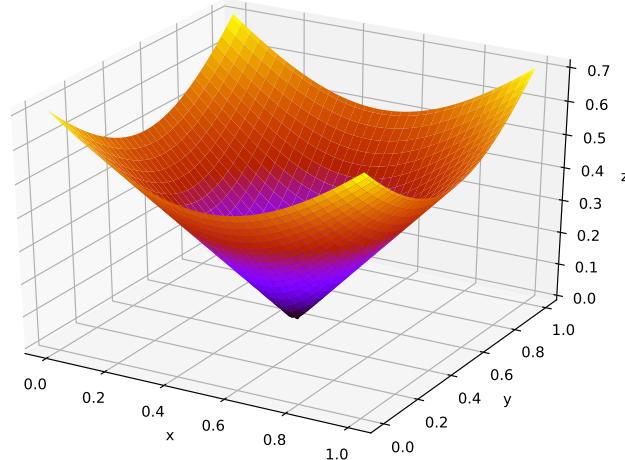


Figure B.10: PDE 8 Interior Point Singularity solution plot

PDE 9: Arctan Wave Front Homogeneous Boundary Conditions 2D

Problem PDE:

$$\begin{aligned}
 & -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \\
 & \frac{20\sqrt{2}(x^2 + y^2 - 2x^2y - 2xy^2 + 4xy - x - y)}{400(\frac{x+y}{\sqrt{2}} - 0.8)^2 + 1} \\
 & + \frac{16000(1-x)x(1-y)y(\frac{x+y}{\sqrt{2}} - 0.8)}{(400(\frac{x+y}{\sqrt{2}} - 0.8)^2 + 1)^2} \\
 & + \tan^{-1}\left(20\left(\frac{x+y}{\sqrt{2}} - 0.8\right)\right)(2(1-y)y + 2(1-x)x) \quad (\text{B.21})
 \end{aligned}$$

on the domain $\Omega : \mathbf{x} \in [0, 1]$
subjected to:

$$\begin{aligned}
 u(x, 0) &= 0 \\
 u(x, 1) &= 0 \\
 u(0, y) &= 0 \\
 u(1, y) &= 0
 \end{aligned}$$

Solution:

$$u_{ext}(x, y) = \tan^{-1}\left(20\left(\frac{(x+y)}{\sqrt{2}} - 0.8\right)\right)x(1-x)y(1-y) \quad (\text{B.22})$$

Solution of PDE 9

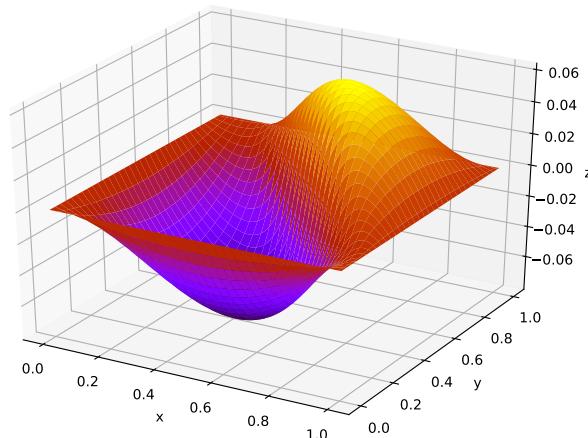


Figure B.11: PDE 9 Arctan Wave Front Homogeneous Boundary Conditions 2D solution plot

C Software Architecture

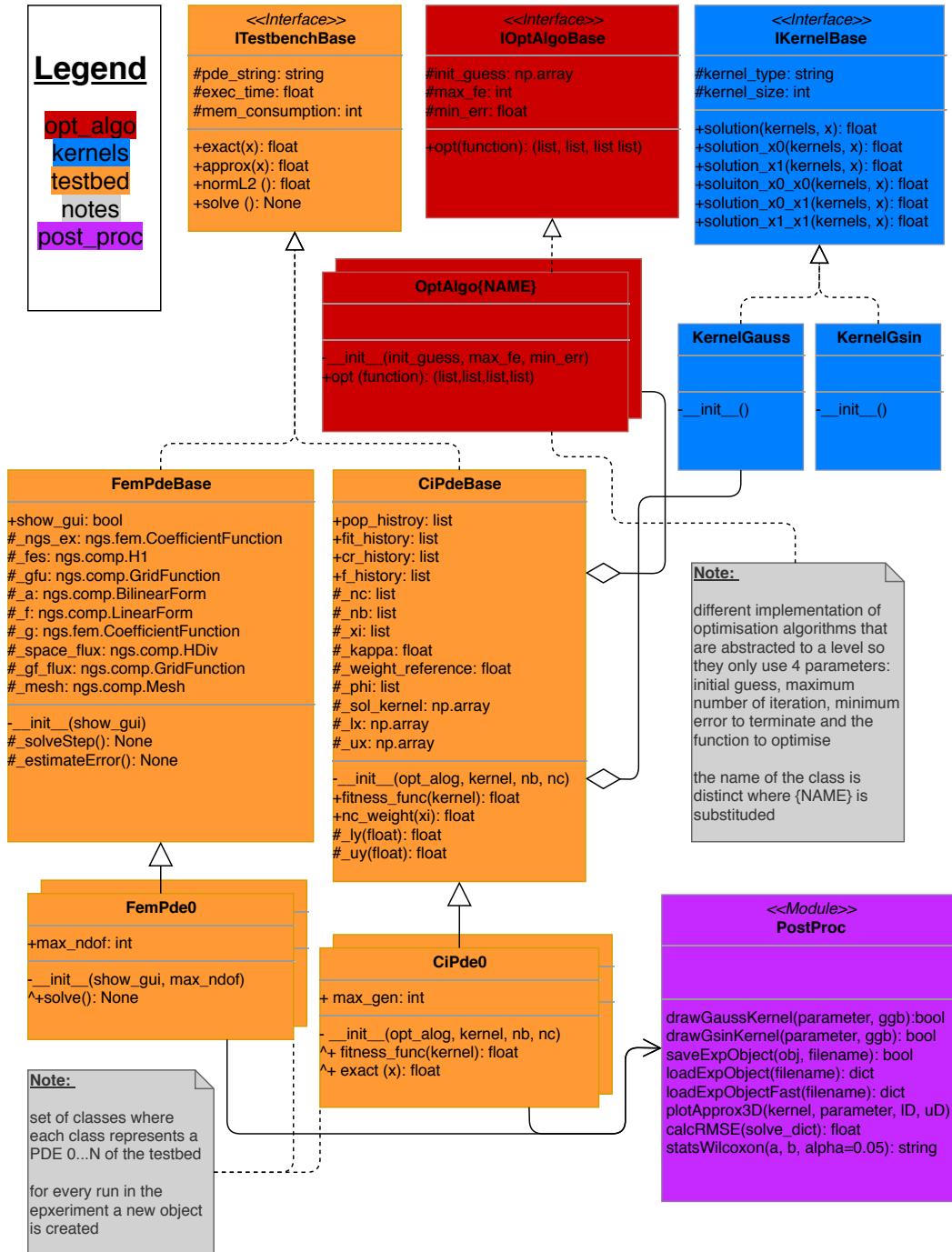


Figure C.1: This UML class diagram describes the software architecture defined to prepare, run and evaluate the experiments.

D Solve Method

Algorithm D.1: Solve Method Pseudocode

```
1 Function solve():
2     gc.disable()
3     while gc.isEnabled() do
4         time.sleep(0.1)
5     end
6     process = psutil.Process()
7     memstart = process.memory_info().vms
8     t_start = time.time()
9     // perform solver steps
10    // that are particular
11    // to FEM or CI solver
12    self._exec_time = time.time() - t_start
13    memstop = process.memory_info().vms - memstart
14    gc.enable()
15    gc.collect()
```

E pJADE

Algorithm E.1: pJADE Pseudocode

```

1 Function pJADE( $\mathbf{X}_{g=0}$ ,  $p$ ,  $c$ , function, minError, maxFE):
2    $fValue_{g=0} \leftarrow function(\mathbf{x}_{g=0})$ 
3    $\mu_{CR} \leftarrow 0.5$ 
4    $\mu_F \leftarrow 0.5$ 
5    $A \leftarrow \emptyset$ 
6   while  $fe \leq maxFE$  do
7      $g \leftarrow g + 1$ 
8      $S_F \leftarrow \emptyset$ 
9      $S_{CR} \leftarrow \emptyset$ 
10     $pResults \leftarrow \emptyset$ 
11    for  $i = 1$  to  $NP$  do parallel
12       $F_i \leftarrow randc_i(\mu_F, 0.1)$ 
13       $v_i \leftarrow mutationCurrentToPBest1(\mathbf{x}_{i,g}, A, fValue_g, F_i, p)$ 
14       $CR_i \leftarrow randn_i(\mu_{CR}, 0.1)$ 
15       $u_i \leftarrow crossoverBIN(\mathbf{x}_{i,g}, v_i, CR_i)$ 
16       $(u_i, function(u_i)) \rightarrow pResults$ 
17    end
18    for  $i = 1$  to  $NP$  do
19      if function( $\mathbf{x}_{i,g}$ )  $\geq pResults_{i,f,g}$  then
20         $\mathbf{x}_{i,g+1} \leftarrow \mathbf{x}_{i,g}$ 
21      end
22      else
23         $\mathbf{x}_{i,g+1} \leftarrow pResults_{i,u,g}$ 
24         $fValue_{i,g+1} \leftarrow pResults_{i,f,g}$ 
25         $\mathbf{x}_{i,g} \rightarrow \mathbf{A}$ 
26         $CR_i \rightarrow S_{CR}$ 
27         $F_i \rightarrow S_F$ 
28      end
29    end
30    // resize  $A$  to size of  $\mathbf{x}_g$ 
31    if  $|A| > NP$  then
32       $A \leftarrow A \setminus A_{rand_i}$ 
33    end
34     $fe \leftarrow fe + size(\mathbf{X})$ 
35     $\mu_{CR} \leftarrow (1 - c) \cdot \mu_{CR} + c \cdot arithmeticMean(S_{CR})$ 
36     $\mu_F \leftarrow (1 - c) \cdot \mu_F + c \cdot lehmerMean(S_F)$ 
37  end

```

Statement of Affirmation

I hereby declare that this thesis was in all parts exclusively prepared on my own, without using other resources than those stated. The thoughts taken directly or indirectly from external sources are properly marked as such. This thesis or parts of it were not previously submitted to another academic institution and have also not yet been published.

Dornbirn, 24.06.2020

Schwartzze Nicolai, BSc