Parameter estimation for gravitational-wave astronomy

Viola De Renzis
University of Milano Bicocca

Gravitational-wave signals

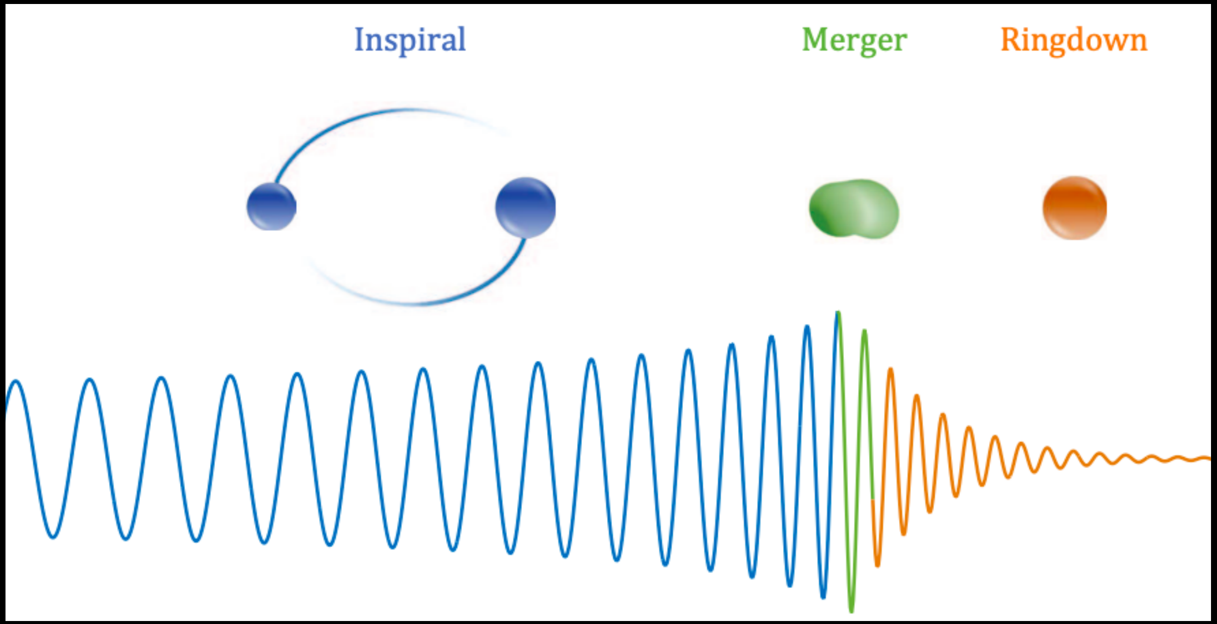
$$\boxed{d = h(\theta) + n}$$

data

waveform

(astrophysical contribution)

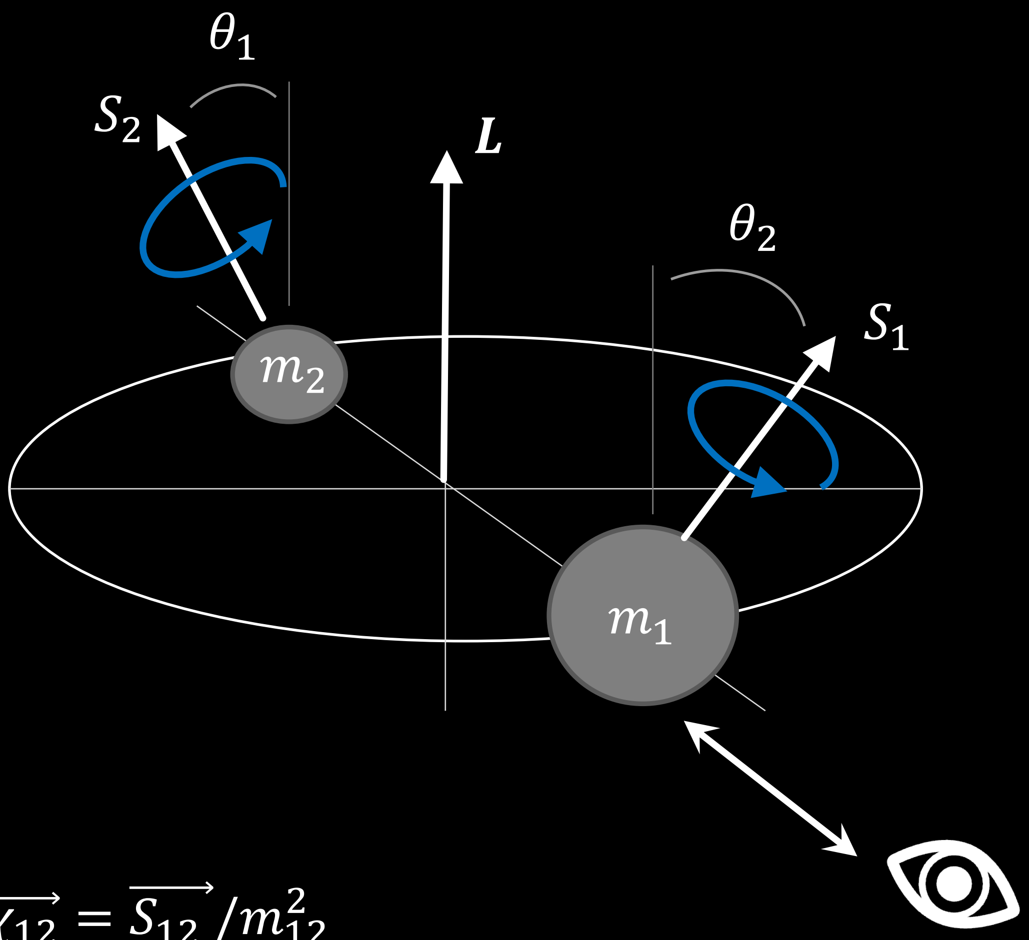
detector noise



- Gaussian and stationary noise and can be estimated through the power spectral density (PSD)

- The astrophysical contribution depends on the parameters θ of the binary system

Astrophysical contribution $h(\vec{\theta})$



$\overrightarrow{\chi_{12}} = \overrightarrow{S_{12}}/m_{12}^2$

- The GW signals carries the information about the extrinsic and intrinsic properties of the source

- 15 parameters $\vec{\theta}$ describing a BBH merger:
  $\vec{\theta} = \{m_1, m_2, \chi_1, \chi_2, \theta_1, \theta_2, \phi_{12}, \phi_{JL}, d_L, ra, dec, \psi, \phi, \theta_{JN}, t_C\}$

Intrinsic	Extrinsic
• Masses • Spins • Tilt angles	• Inclination • Distance • Sky localization • Coalescence time

From data to astrophysics

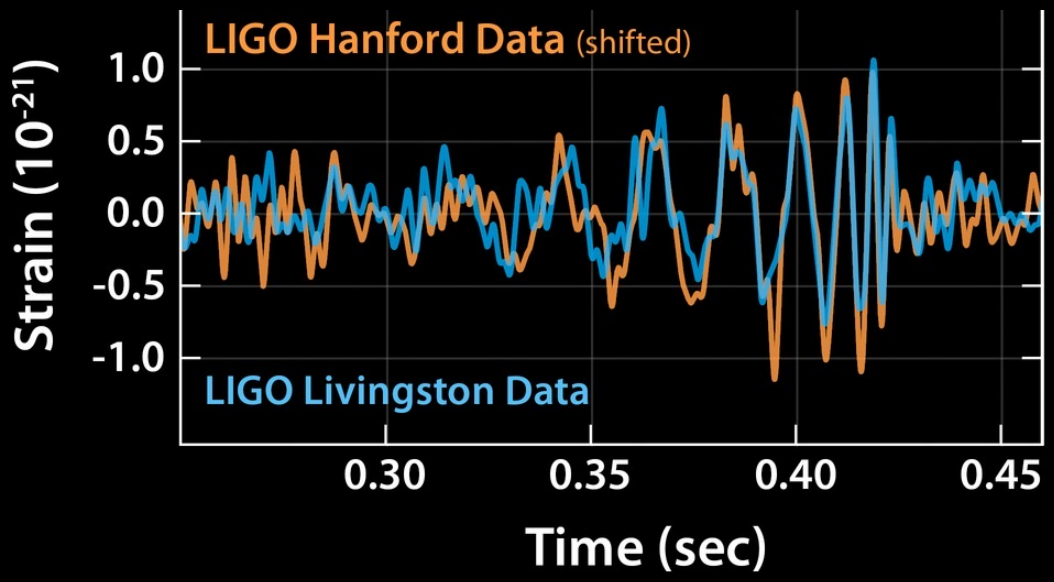
$$\boxed{d = h(\theta) + n}$$

data

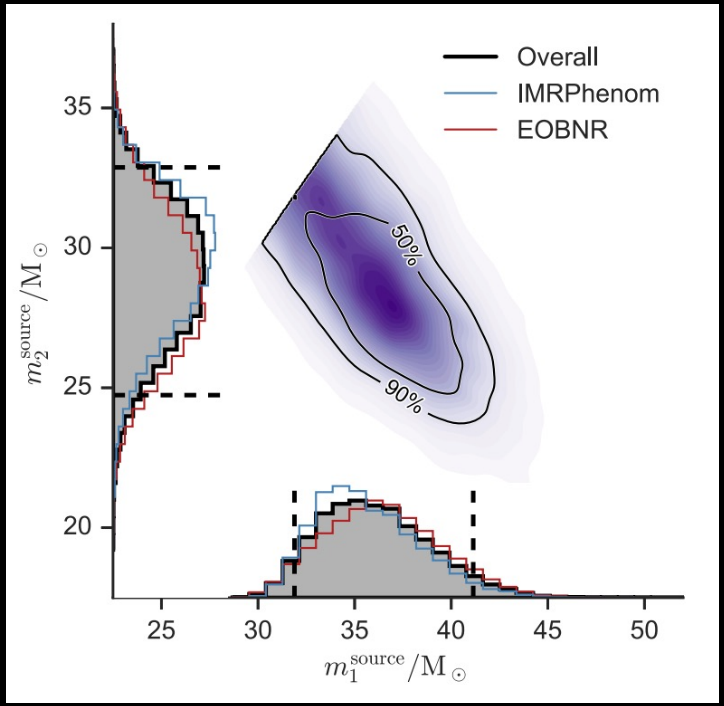
waveform

(astrophysical contribution)

detector noise



?



(LVC, 2016)

Parameter estimation for GW astronomy

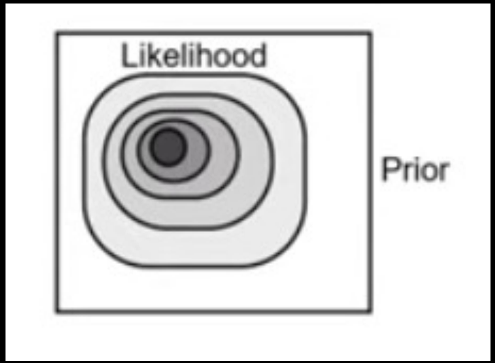
Bayesian Inference Library

- Parameter estimation tools (like **BILBY**) exploit Bayesian inference to extract source properties from GW signals

Bayes' Theorem

Posterior / Likelihood / Prior / Evidence

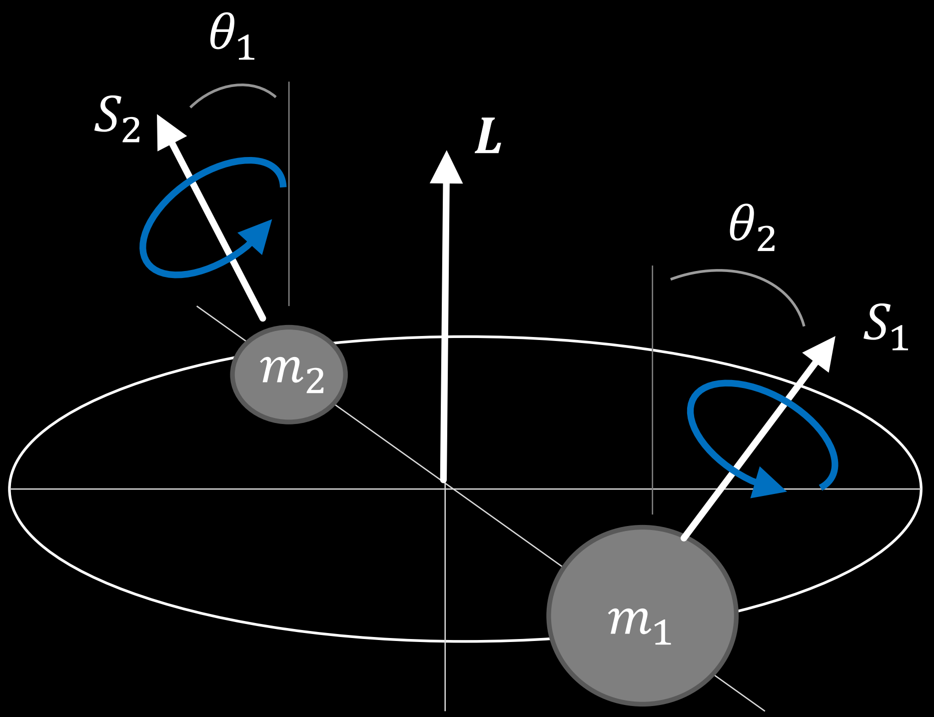
$$P(\vec{\theta} \mid d, M) = \frac{\mathcal{L}(d|\vec{\theta}, M)\, \pi(\vec{\theta}, M)}{Z(d, M)}$$



Recover the posterior distribution of the parameters **θ** through **stochastic sampling algorithms** (MCMC or nested sampling)

Influence of masses and spins

No hair theorem:
Kerr BHs are uniquely described by their **mass M** and their **spin S**



- We can measure with great accuracy the **chirp mass**:

$$\mathcal{M} = \frac{(m_1 m_2)^{3\backslash 5}}{(m_1 + m_2)^{1\backslash 5}}$$

The amplitude $\mathcal{A}_{GW}$ of the waveform is proportional to the chirp mass

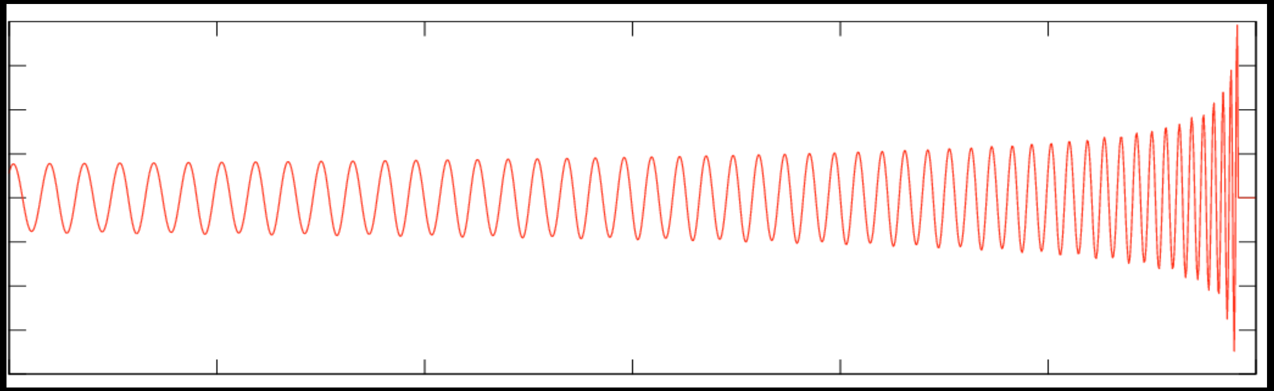
$$\mathcal{A}_{GW} \propto \frac{\mathcal{M}^{5/6} f^{-7/6}}{D_L}$$

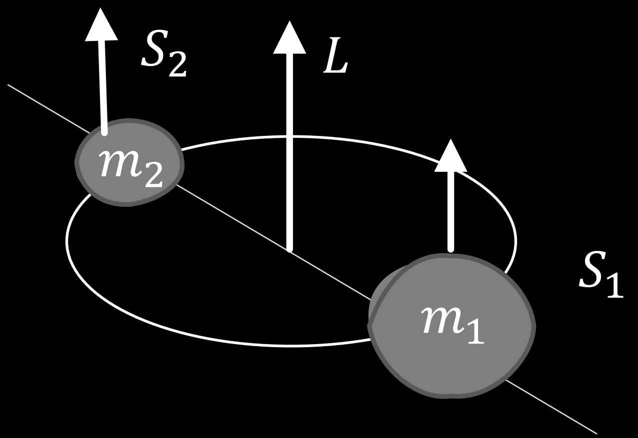
- **Spins** provide an highly subdominant contribution to the emitted radiation

$\overrightarrow{\chi_{12}} = \overrightarrow{S_{12}} / m_{12}^2$

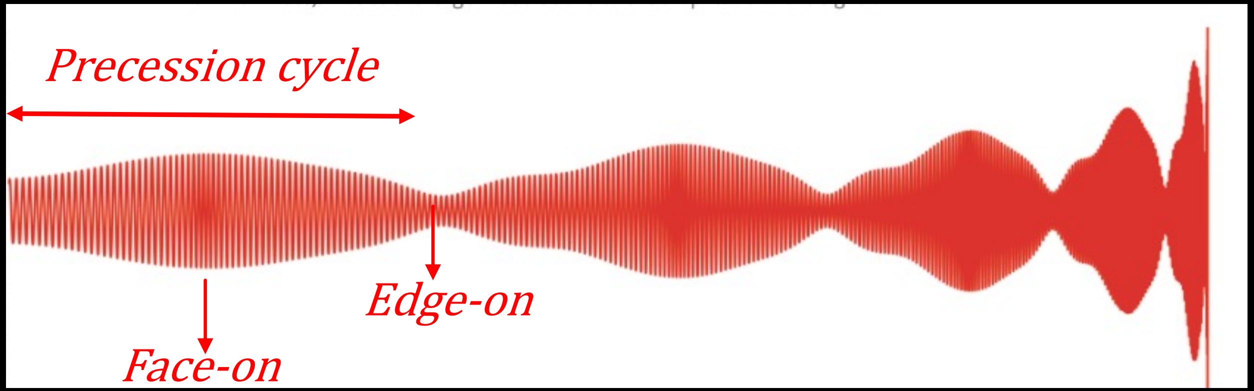
Effects of spins on the waveform

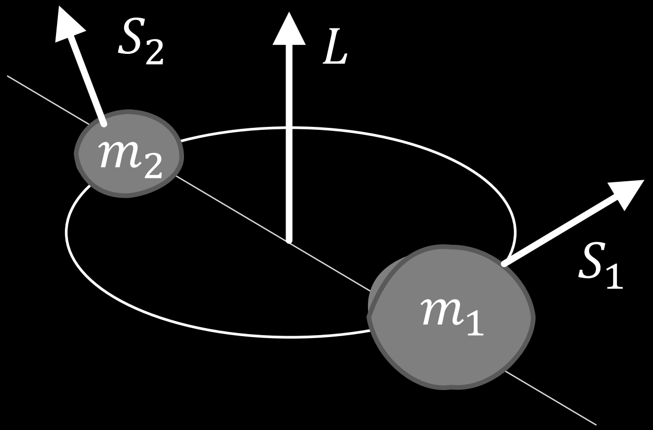
Aligned spins→ non precessing



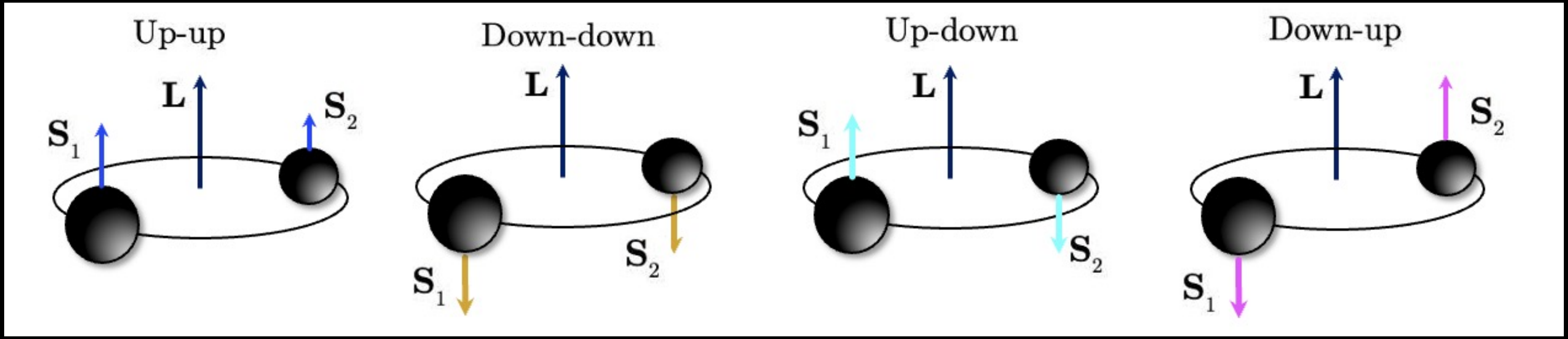


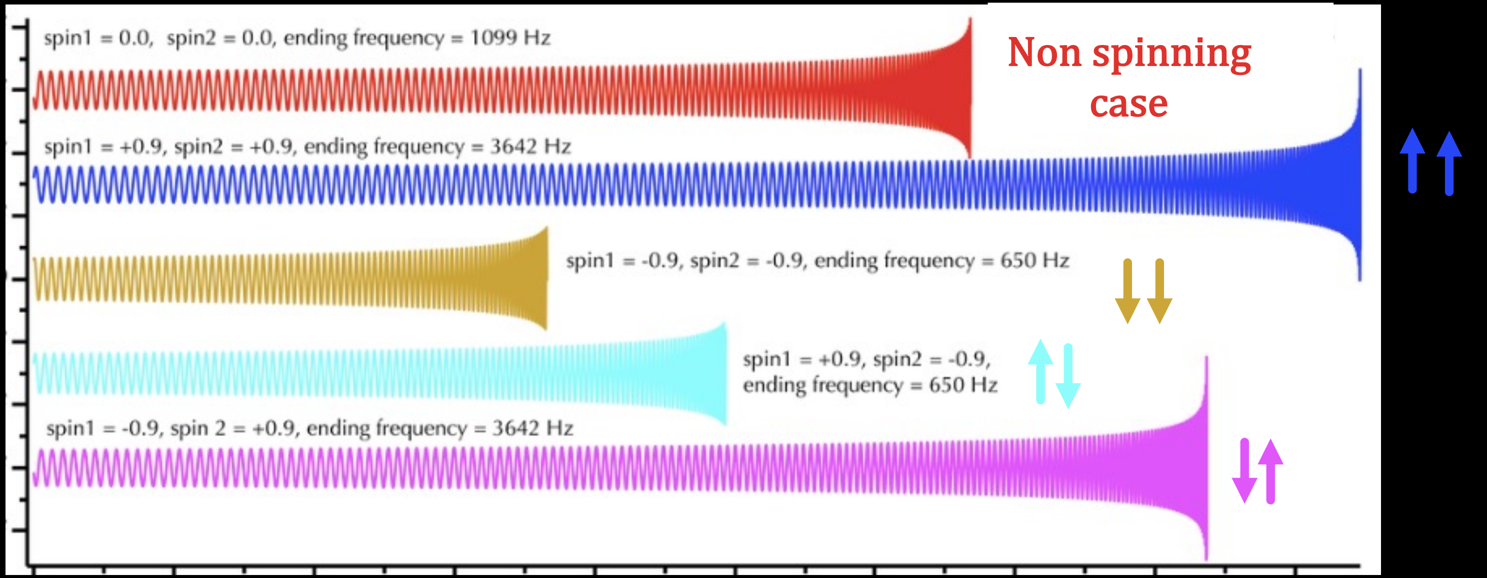
Misaligned spins→ spin precession





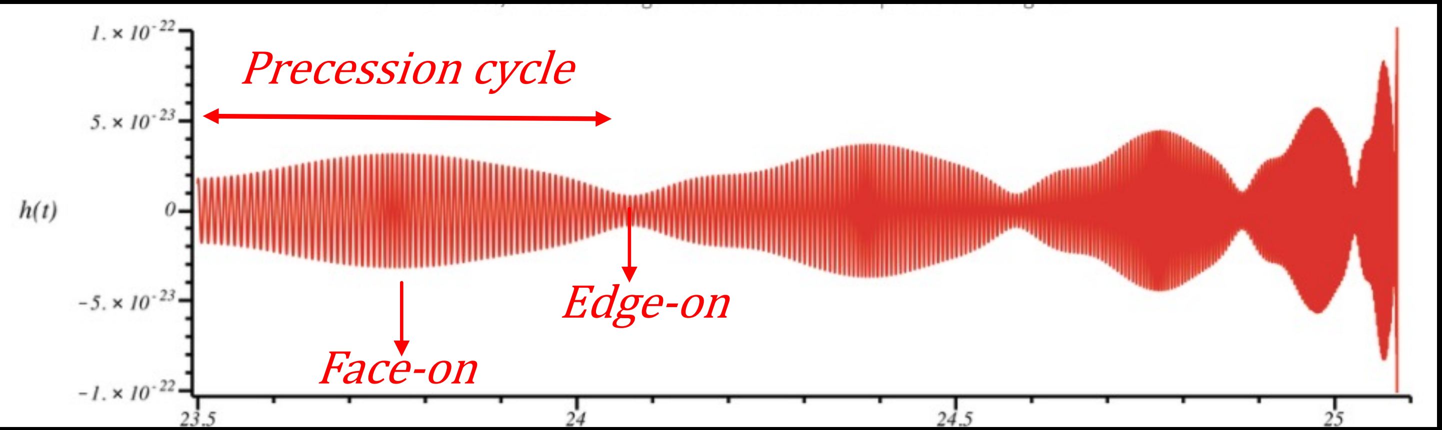
Aligned spins → no precession



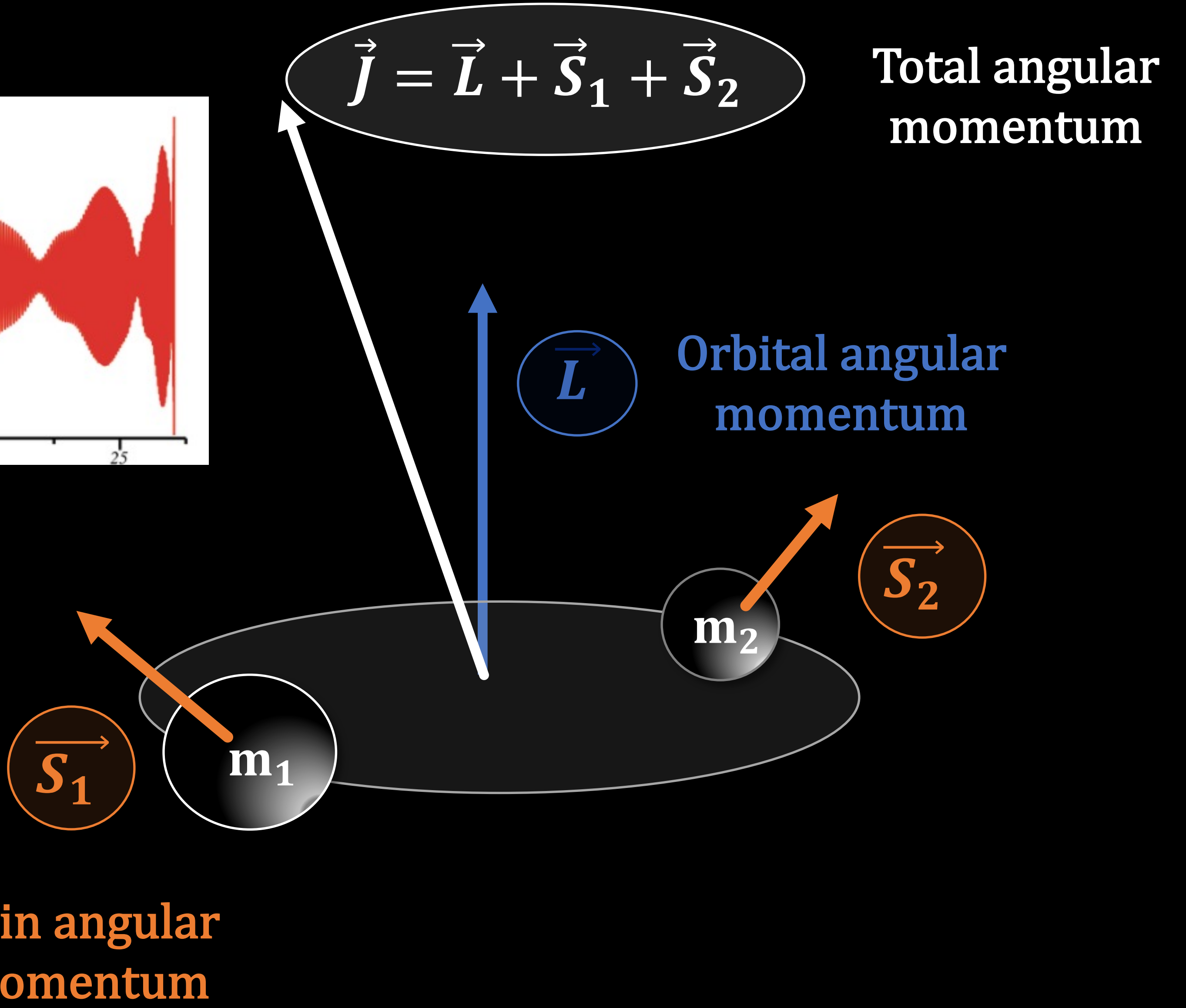


- Similar morphologies
- Different merger frequencies and signal durations

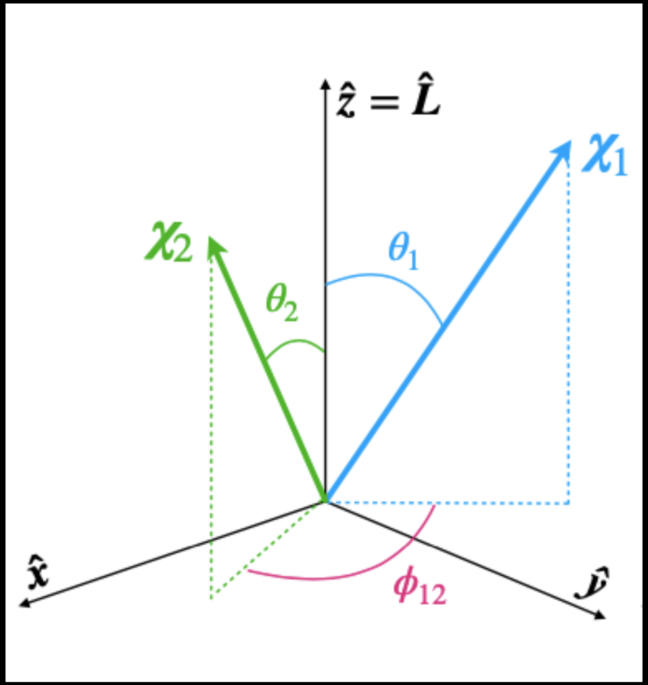
Misaligned spins→ spin precession



- Modulation in amplitude and phase
- Change in direction of the GW emission



Looking for precession



I. Measurements of component **spin magnitudes** and **tilt angles** (low accuracy!)

or

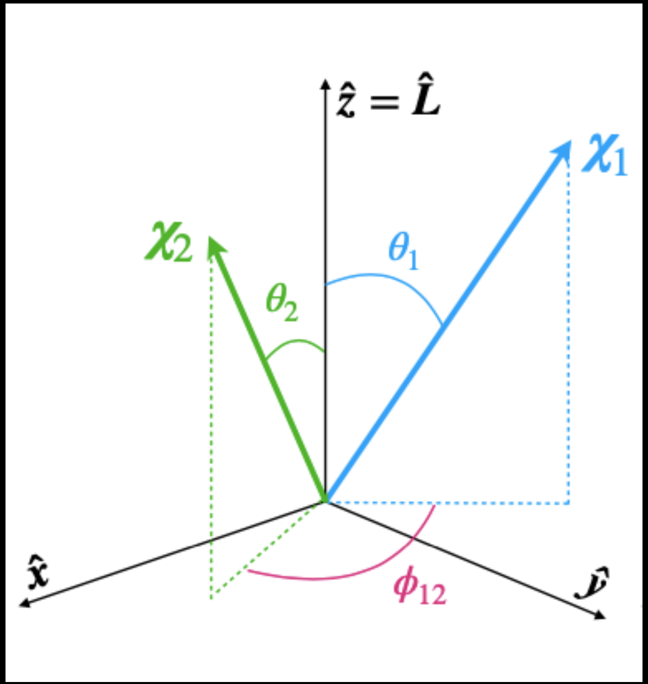
II. Measurement of the **precessing spin parameter χ_p**

$$\chi_p = \left[(\chi_1 \sin\theta_1)^2 + \left(q\frac{4q+3}{(4+3q)}\chi_2\sin\theta_2\right)^2 + 2q\frac{4q+3}{(4+3q)}\chi_1\chi_2\sin\theta_1\sin\theta_2\cos\phi_{12}\right]^{1/2}$$

(Gerosa+ 2020)

$$\overrightarrow{\chi_{12}} = \overrightarrow{S_{12}}/m_{12}^2$$
$$q = m_2/m_1$$

Looking for precession



$$\overrightarrow{\chi_{12}} = \overrightarrow{S_{12}} / m_{12}^2$$
$$q = m_2/m_1$$

I. Measurements of component **spin magnitudes** and **tilt angles** (low accuracy!)

or

II. Measurement of the **precessing spin parameter χ_p**

$$\chi_p = \left[(\chi_1 \sin\theta_1)^2 + \left(q \frac{4q+3}{(4+3q)} \chi_2 \sin\theta_2 \right)^2 + 2q \frac{4q+3}{(4+3q)} \chi_1 \chi_2 \sin\theta_1 \sin\theta_2 \cos\phi_{12} \right]^{1/2}$$

(Gerosa+ 2020)

- Better measured than the individual spin components
- Domain: $0 < \chi_p < 2$
- $1 < \chi_p < 2$ → **Two precessing spins**

Characterization of merging black holes with two precessing spins

Viola De Renzis[1,2,*] Davide Gerosa[1,2,3] Geraint Pratten[3] Patricia Schmidt[3] and Matthew Mould[3]

[1] *Dipartimento di Fisica "G. Occhialini", Universitá degli Studi di Milano-Bicocca, Piazza della Scienza 3, 20126 Milano, Italy*

[2] *INFN, Sezione di Milano-Bicocca, Piazza della Scienza 3, 20126 Milano, Italy*

[3] *School of Physics and Astronomy & Institute for Gravitational Wave Astronomy, University of Birmingham, Birmingham, B15 2TT, United Kingdom*

(Dated: October 24, 2022)

If an incoming LIGO/Virgo source is composed of merging BHs with two precessing spins, will we able to tell?

Characterization of merging black holes with two precessing spins

Viola De Renzis,[1,2,*] Davide Gerosa,[1,2,3] Geraint Pratten,[3] Patricia Schmidt,[3] and Matthew Mould[3]

[1] *Dipartimento di Fisica "G. Occhialini", Universitá degli Studi di Milano-Bicocca, Piazza della Scienza 3, 20126 Milano, Italy*
[2] *INFN, Sezione di Milano-Bicocca, Piazza della Scienza 3, 20126 Milano, Italy*
[3] *School of Physics and Astronomy & Institute for Gravitational Wave Astronomy, University of Birmingham, Birmingham, B15 2TT, United Kingdom*
(Dated: October 24, 2022)

If an incoming LIGO/Virgo source is composed of merging BHs with two precessing spins, will we able to tell?

1) Generate synthetic GW signals with BILBY
2) Perform a parameter estimation analysis
3) Recover χ_p in post processing

1) Generation of the simulated signals

- **15 injected parameters:**

$M_{1,detector} = 27.6$,
$M_{1,detector} = 26.5$
$\boldsymbol{\theta_1 = \pi/2}$
$\boldsymbol{\theta_2 = \pi/2}$
$\phi_{12} = 0.1$
$\chi_1 = 0.56$
$\chi_1 = 0.7$
$\phi_{JL} = 1.0$

$d_L = 200$ Mpc
$\theta_{JN} = 1.0$
$dec = 0.75$
$ra = 0.5$
$\psi = 1.0$
$t_{geocent} = 0.0$
$\phi = \pi/4$

- **Choose a waveform model for the injection:** IMRPhenomXPHM

- **Initialize the interferometers:** LIGO H, LIGO L, Virgo (O4)

- **Detector arguments:** $f_{ref} = 20.0$ Hz
$f_{min} = 20.0$ Hz
$f_{sampling} = 2048.0$ Hz
$duration = 4$s

2) Parameter estimation analysis

- **Standard BBH priors**

- **Gaussian noise likelihood**

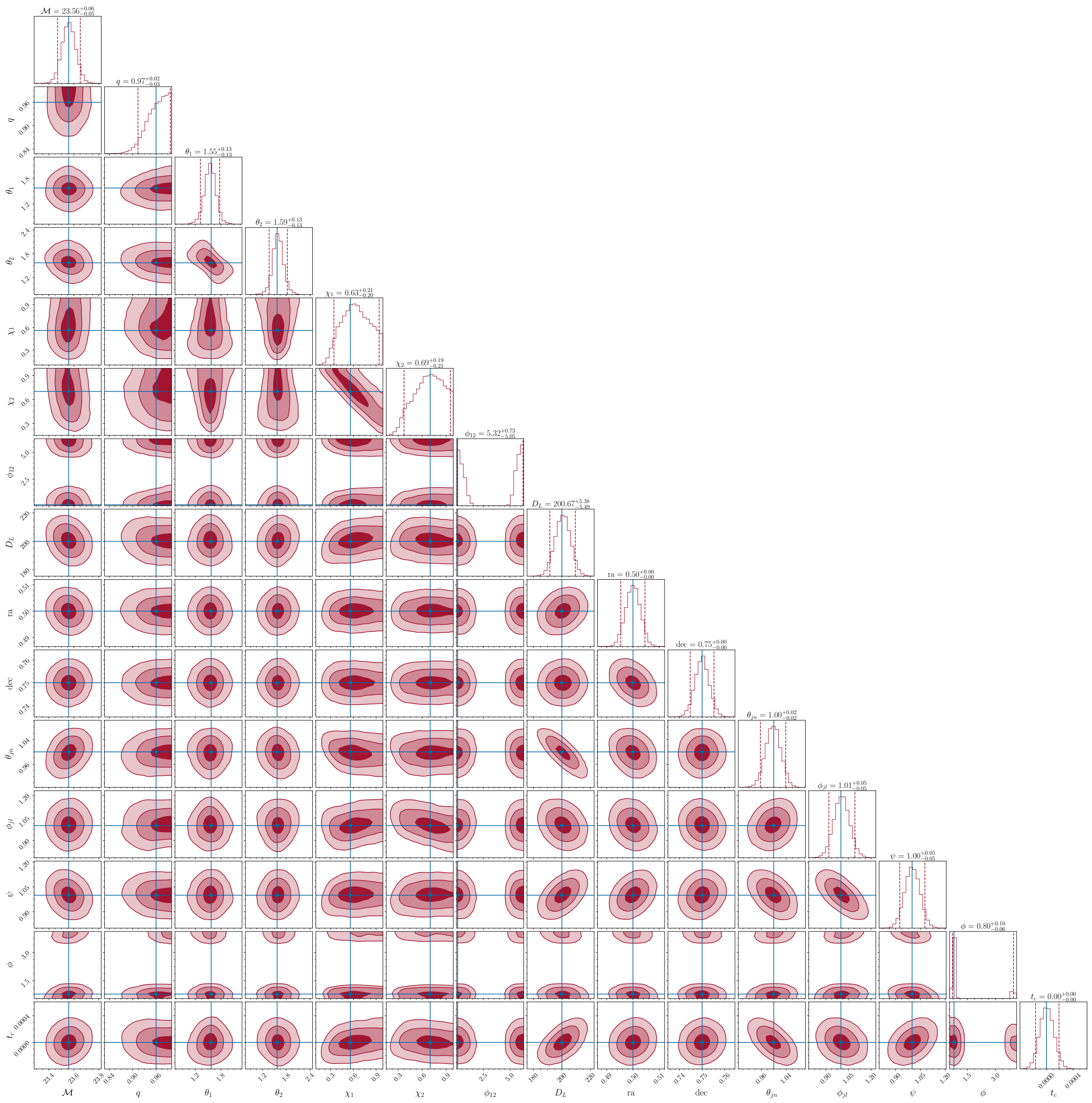
$$\ln\mathcal{L}(d|\vec{\theta},M) \propto -\sum_i \frac{\left|d_i - h_i(\vec{\theta})\right|^2}{TS_i}$$

Duration of the analysis segment

PSD

Recovery waveform: IMRPhenomXPHM

- **Run the sampler** (DYNESTY)



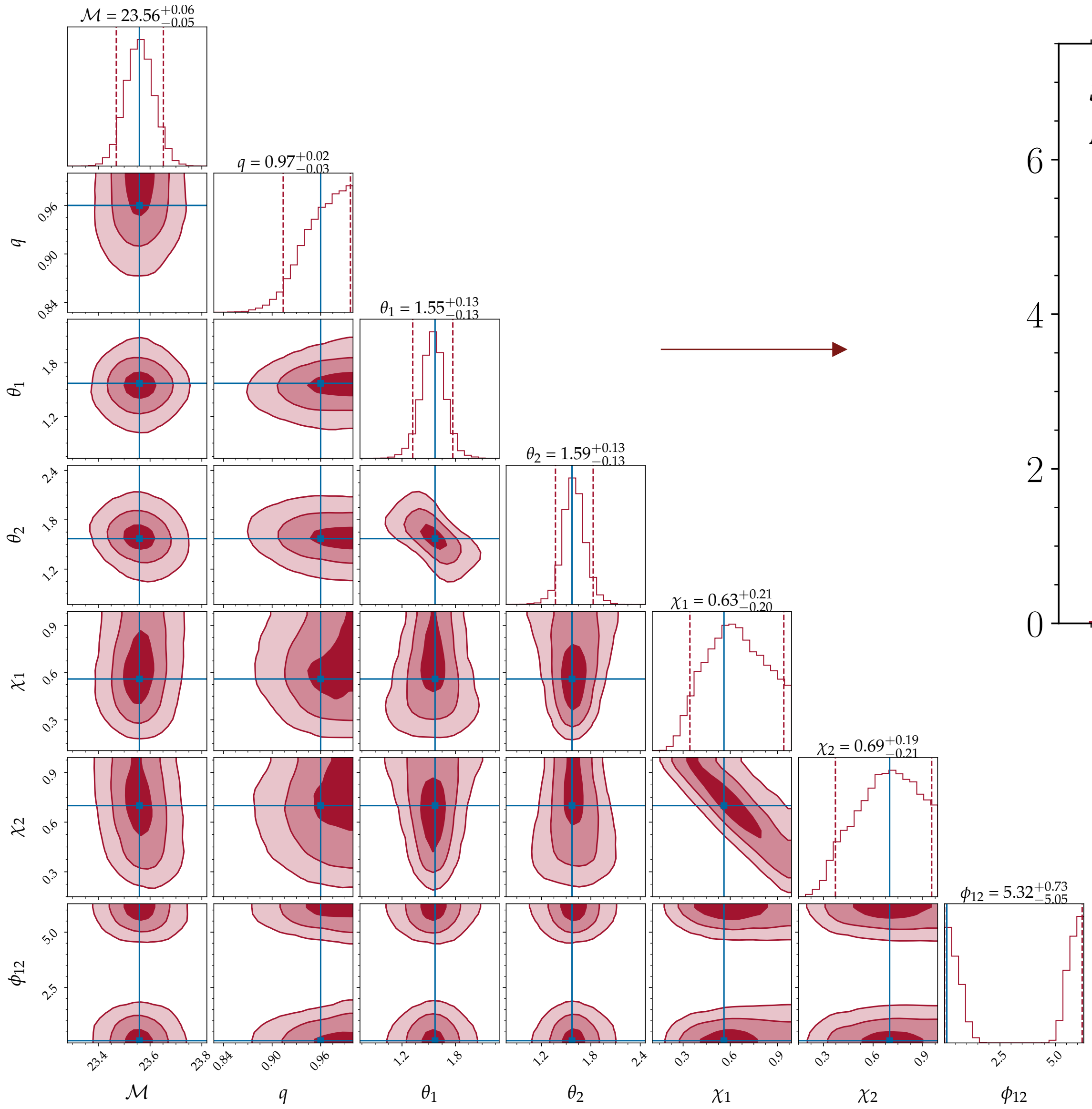
3) Recovery of χ_p

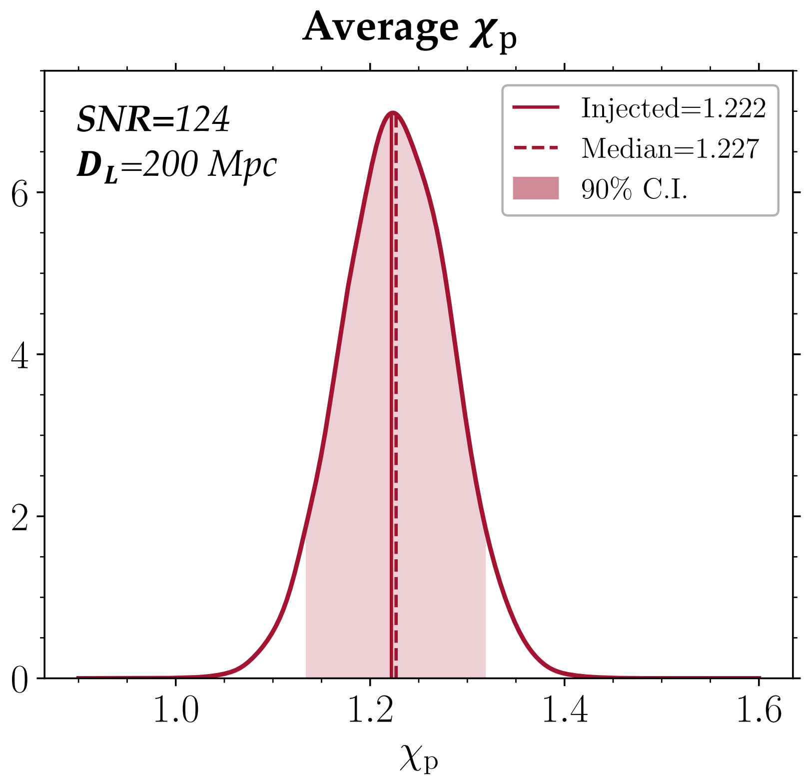
15 injected parameters

$M_{1,detector} = 27.6$
$M_{1,detector} = 26.5$
$\boldsymbol{\theta_1 = \pi/2}$
$\boldsymbol{\theta_2 = \pi/2}$
$\phi_{12} = 0.1$
$\chi_1 = 0.56$
$\chi_1 = 0.7$
$d_L = 200$ Mpc
$\theta_{JN} = 1.0$
$\phi_{JL} = 1.0$
$dec = 0.75$
$ra = 0.5$
$\psi = 1.0$
$t_{geocent} = 0.0$
$\phi = \pi/4$

q=0.96

IMRPhenomXPHM
Standard BBH priors





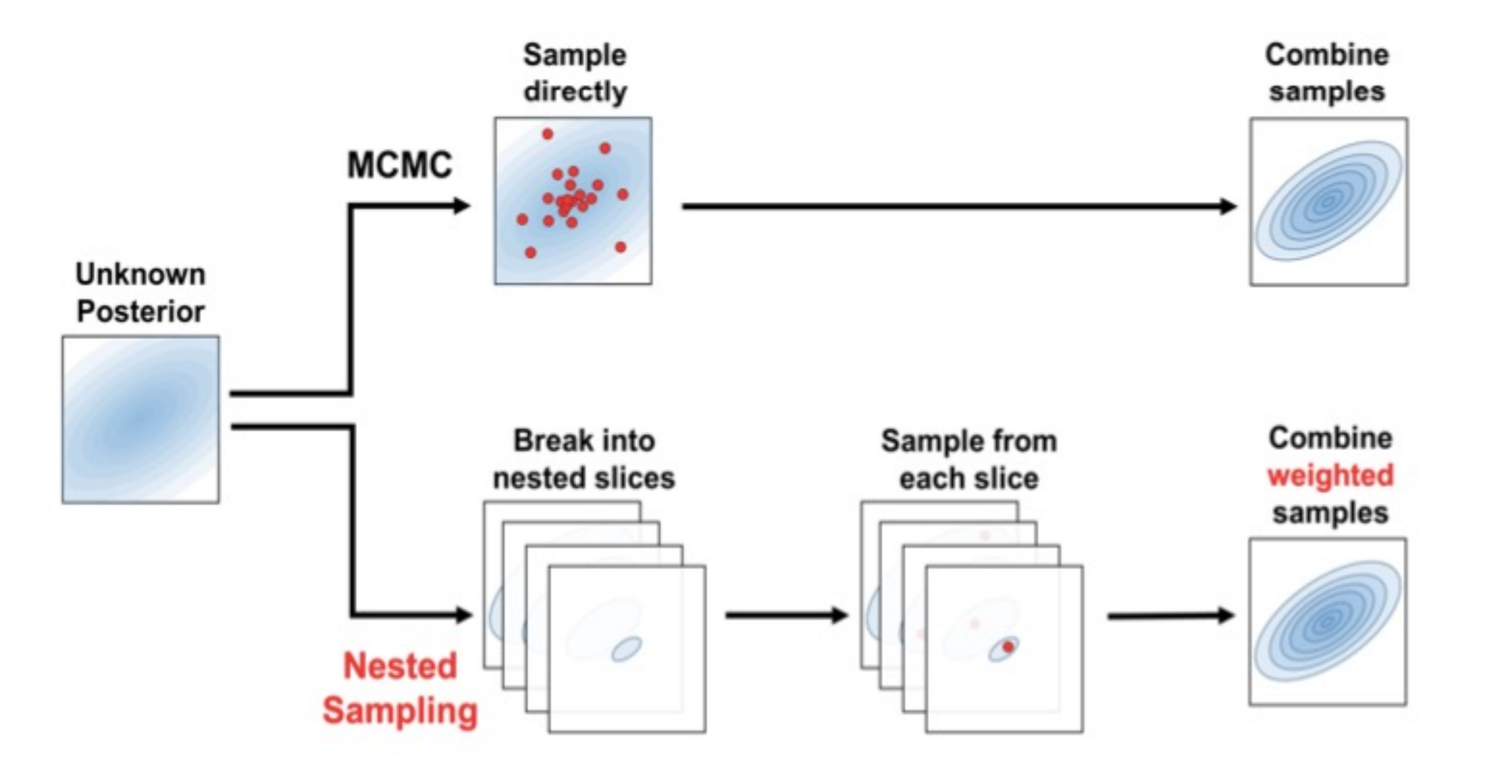
Two-spin effects!

Thanks for the attention!

BACK UP SLIDES

Sampling the posterior: MCMC vs nested sampling

We have likelihood and prior.
How do we calculate the posterior distribution in 15 dimensions?



(Speagle 2019)

MCMC

- Solving an hard problem **once**
- Does not calculate the evidence
- Can get stuck in multimodal distributions

NESTED SAMPLING

- Solving an hard problem **many times**
- Calculate directly the evidence

Parameter estimation for GW astronomy

Likelihood

$$\ln\mathcal{L}(d|\vec{\theta}, M) \propto -\sum_i \frac{\left|d_i - h_i(\vec{\theta})\right|^2}{TS_i}$$

Duration of the analysis segment

PSD

Understanding contributions to the GW signal

Prior

$$\pi(\vec{\theta}, M)$$

Initial beliefs about the distributions of the GW parameters

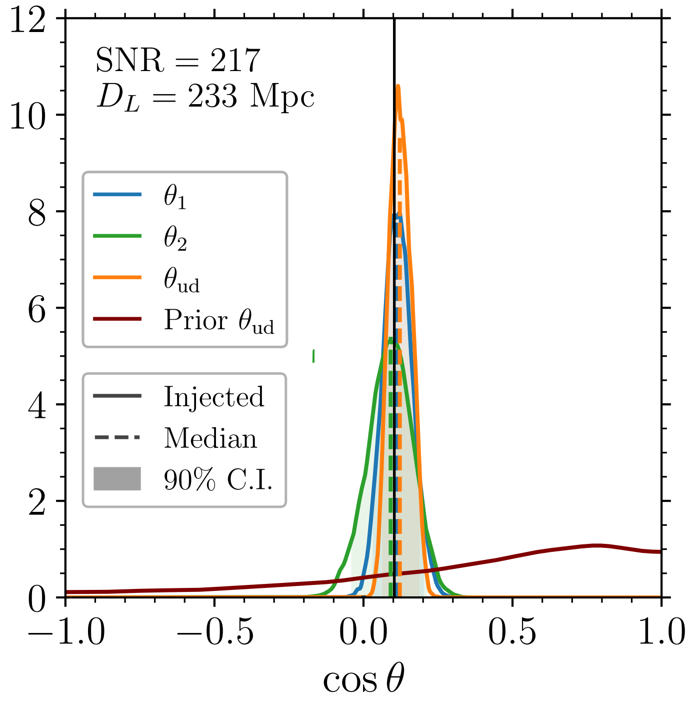
Evidence

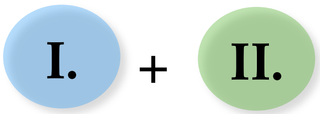
$$Z(d, M) = \int \mathrm{d}\theta P(d|\vec{\theta}, M_i)\pi(\vec{\theta}, M_i)$$

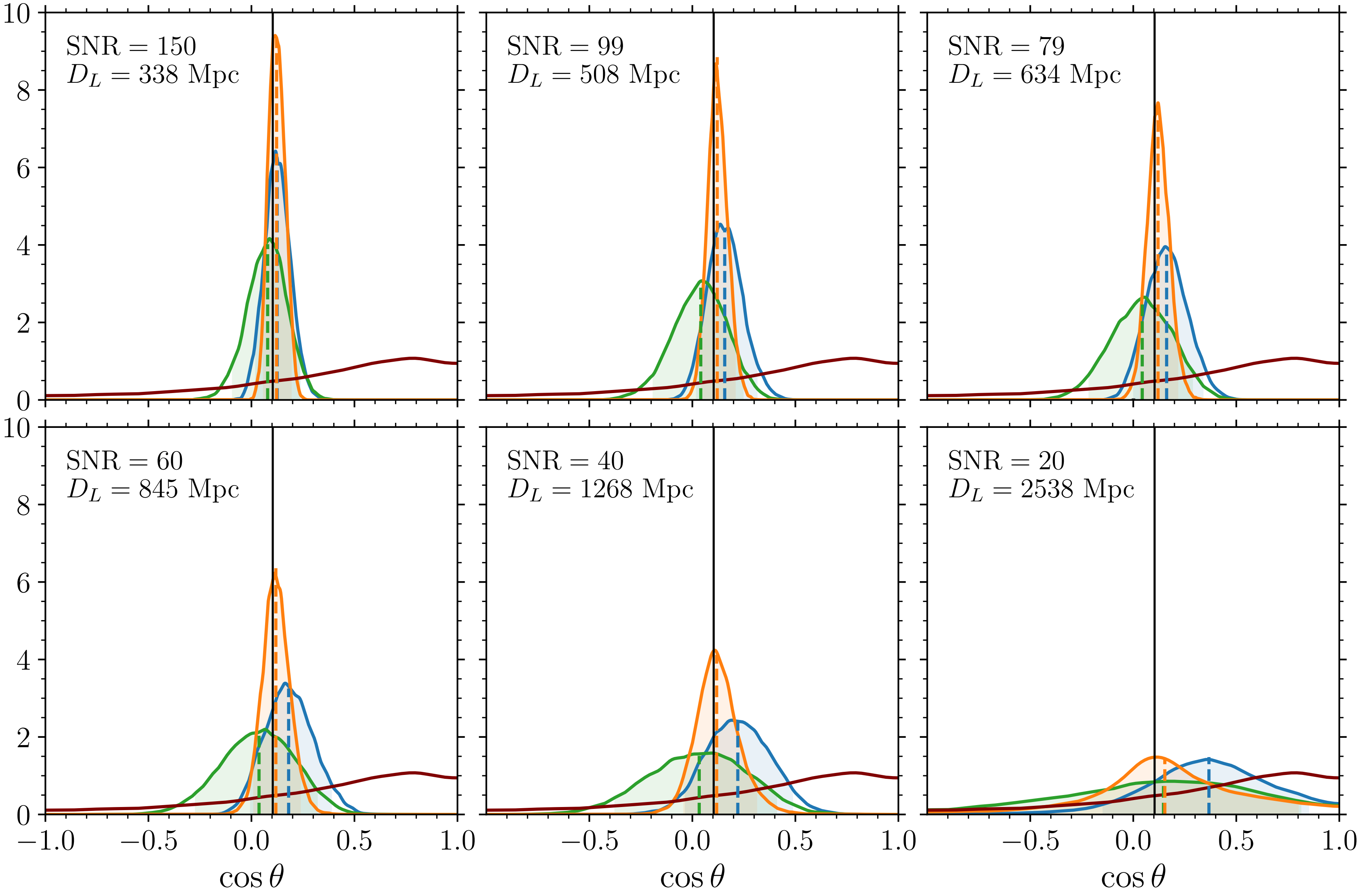
Model selection and calculation of the Bayes' factor

$$B_1^0 = \frac{Z(d|M_0)}{Z(d|M_1)}$$

Recovery at different SNR

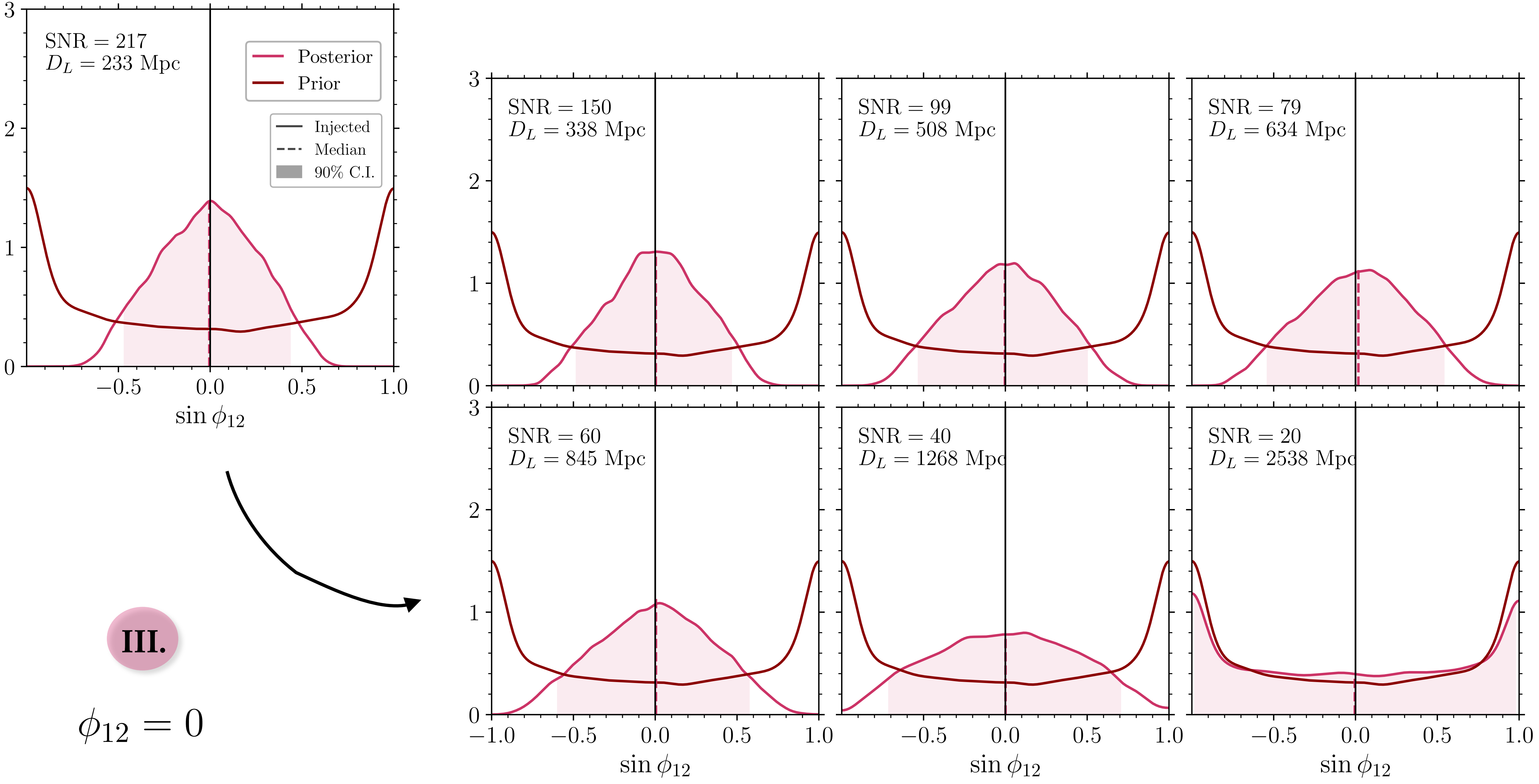






$$\cos\theta_1 = \cos\theta_2 = \frac{\chi_1 - q\chi_2}{\chi_1 + q\chi_2}$$

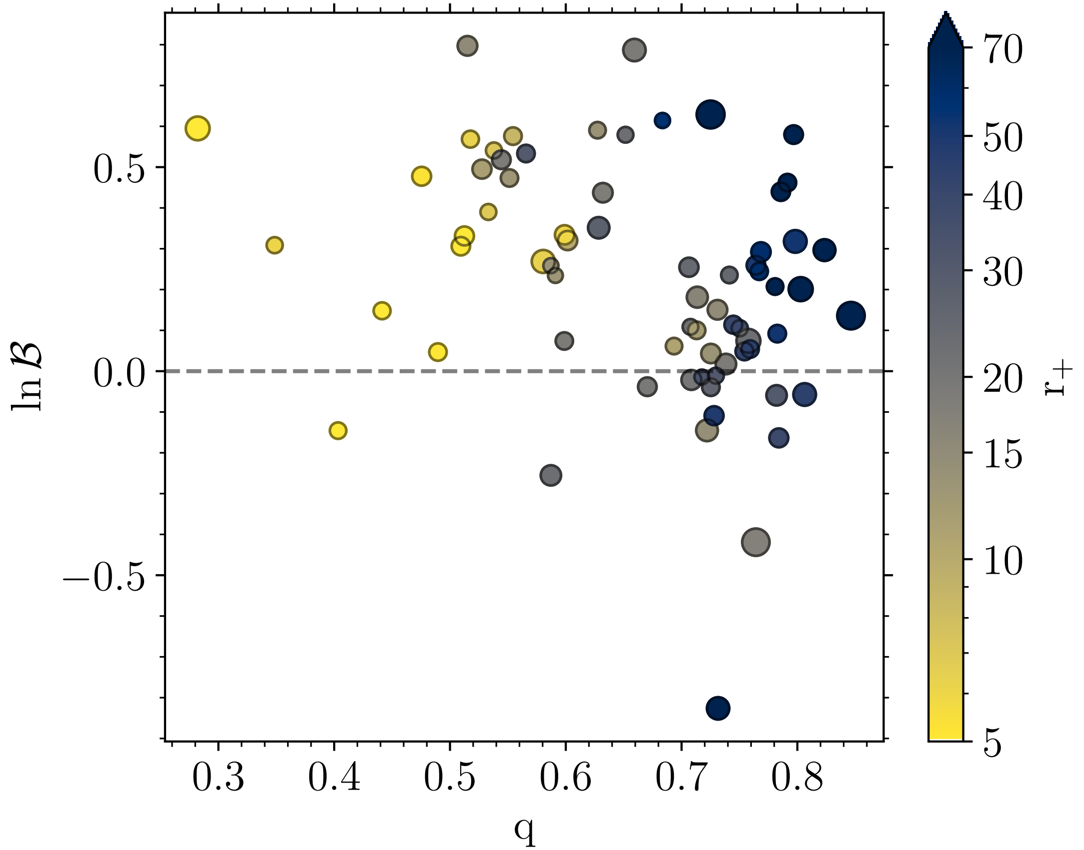
Recovery at different SNR

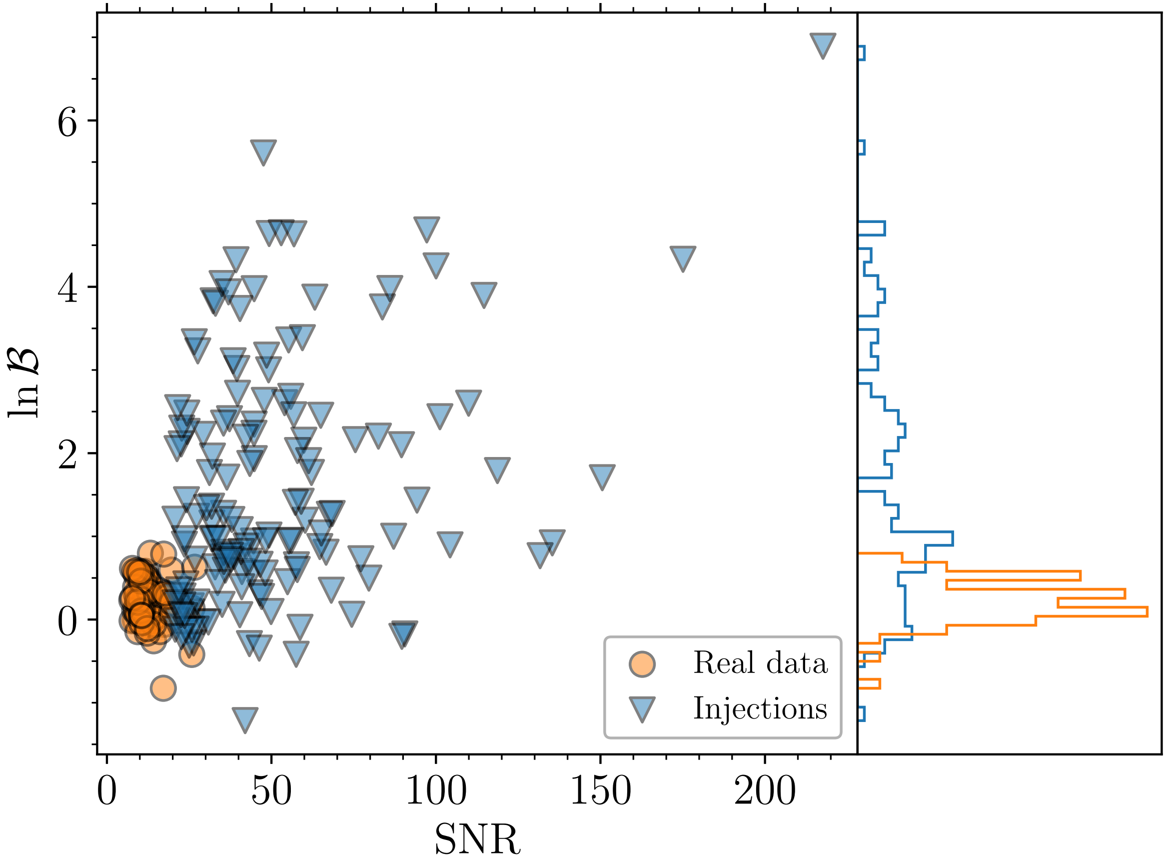


$\phi_{12} = 0$

Real data (GWTC-3)

- 69 events of BBH coalescences (BNS excluded)
- FAR $< 1yr^{-1}$ in at least one search





$|\ln\mathcal{B}| < 1 \rightarrow$ Inconclusive evidence

Multi-timescale analysis

- Orbital timescale: $t_{orb} \sim r^{3/2}$
- Precession timescale: $t_{pre} \sim r^{5/2}$
- Radiation-reaction timescale: $t_{RR} \sim r^4$

In the post-Newtonian regime (large separations):

$$r \gg M \longrightarrow t_{orb} \ll t_{pre} \ll t_{RR}$$

Each part of the binary dynamics can be addressed independently

Multi-timescale analysis

$$\boxed{t_{orb} \ll t_{pre}} \longrightarrow$$

Study precession in BBHs averaging the motion over the orbital period

- 2PN orbit-averaged spin precession equations

$$\frac{d\mathbf{S_i}}{dt} = \mathbf{\Omega_i} \times \mathbf{S_i}$$

$$\frac{d\mathbf{L}}{dt} = (\mathbf{\Omega_L} \times \hat{L})L + \frac{dL}{dt}\hat{\mathbf{L}}$$

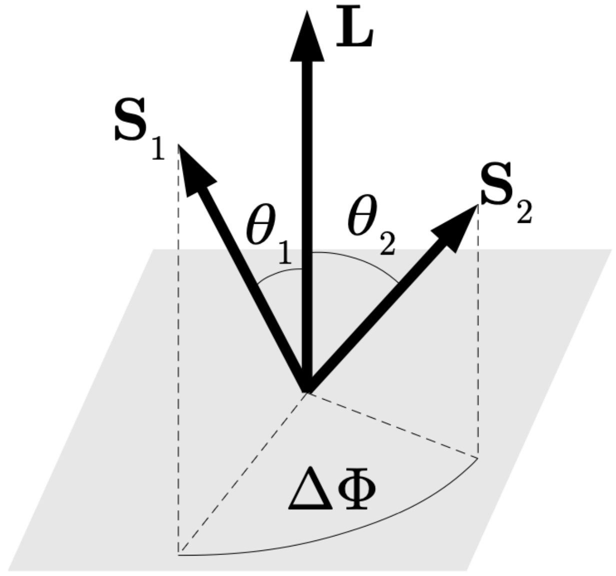
$$\left\{\begin{aligned} \mathbf{\Omega}_L &= \Omega_1\chi_1\hat{\mathbf{S}}_1 + \Omega_2\chi_2\hat{\mathbf{S}}_2 \\ \Omega_1 &= \frac{M^2}{2r^3(1+q)^2}\left[4+3q-\frac{3q\chi_{\text{eff}}}{(1+q)}\frac{M^2}{L}\right] \\ \Omega_2 &= \frac{qM^2}{2r^3(1+q)^2}\left[4q+3-\frac{3q\chi_{\text{eff}}}{(1+q)}\frac{M^2}{L}\right] \end{aligned}\right.$$

[Damour, 2008]

- **9D problem:** 2 spin vectors and 1 orbital vector

Dimensionality reduction

- **9D problem:** 2 spin vectors and 1 orbital vector
- **7D:** 2 BH spin magnitudes are conserved
- **4D**: choose a reference frame
- **3D:** χ_{eff} is a conserved quantity a 2PN



$$\boxed{t_{pre} \ll t_{RR}}$$

- **1D problem:** two additional conserved quantities on the short precessional timescale

$$\frac{dL}{dt} = 0 \rightarrow L = |\boldsymbol{L}|$$
$$J = |\boldsymbol{L} + \boldsymbol{S_1} + \boldsymbol{S_2}|$$

Motion on the precession timescale

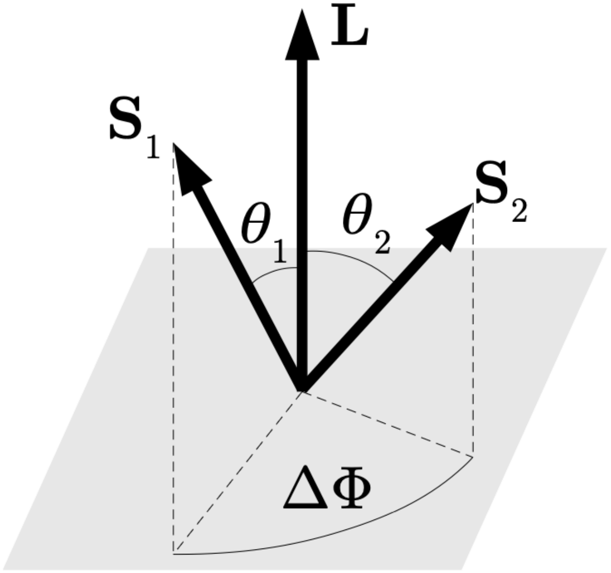
$$\boxed{t_{orb} \ll t_{pre} \ll t_{RR}}$$

$$\downarrow$$

The entire precessional dynamics can be parametrized with a single variable, the total spin magnitude

$$S = |\boldsymbol{S_1} + \boldsymbol{S_2}|$$

[Kesden, 2015]



Perturbation of the aligned configurations

- Small perturbations to aligned-spin configurations evolve as an **harmonic oscillator**

$$\boxed{\frac{d^2}{dt^2}(S^2 - S_*^2) + \omega^2 (S^2 - S_*^2) \simeq 0}$$

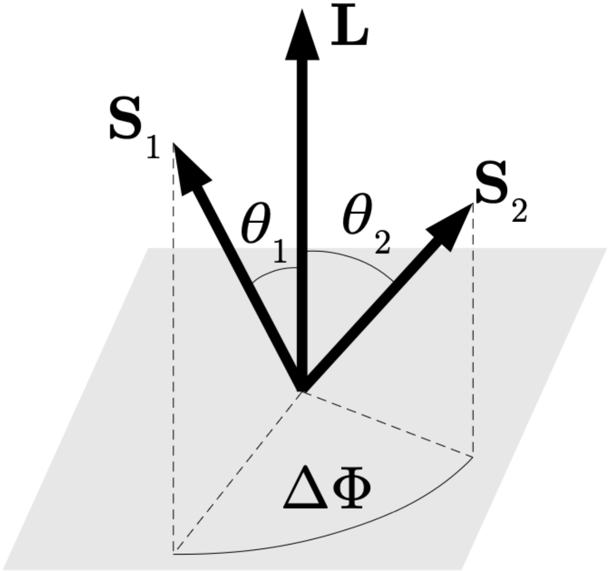
STABILITY?

- ➢ Real frequency $\omega^2 > 0$ → small amplitude oscillations (stable configuration)
- ➢ If $\omega^2 > 0 \rightarrow S^2 = \mathrm{const}$ → onset of the instability
- ➢ Imaginary frequency $\omega^2 < 0$ → **dynamical instability**

Spin-orbit resonances

- Spin-orbit resonances: the three angular momenta remain coplanar

$$\Delta\Phi = 0, \pi$$



Uninformative BBH priors

Intrinsic parameters

```
prior = {
'chirp_mass' : bilby.gw.prior.UniformInComponentsChirpMass(name='chirp_mass', minimum=10, maximum=60),
'mass_ratio' : bilby.gw.prior.UniformInComponentsMassRatio(name='mass_ratio', minimum=0.125, maximum=1),
'mass_1'     : bilby.gw.prior.Constraint(name='mass_1', minimum=5, maximum=100),
'mass_2'     : bilby.gw.prior.Constraint(name='mass_2', minimum=5, maximum=100),
'a_1'        : bilby.prior.analytical.Uniform(name='a_1', minimum=0, maximum=0.99),
'a_2'        : bilby.prior.analytical.Uniform(name='a_2', minimum=0, maximum=0.99),
'tilt_1'     : bilby.prior.analytical.Sine(name='tilt_1'),
'tilt_2'     : bilby.prior.analytical.Sine(name='tilt_2'),
'phi_12'     : bilby.prior.analytical.Uniform(name='phi_12', minimum=0, maximum=2 * np.pi, boundary='periodic'),
'phi_jl'     : bilby.prior.analytical.Uniform(name='phi_jl', minimum=0, maximum=2 * np.pi, boundary='periodic'),
```

Uniform mass prior $m_{1,2} \in [5,100] M_\odot$

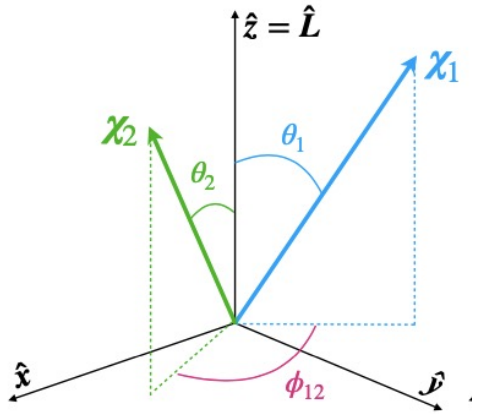
Uninformative BBH priors

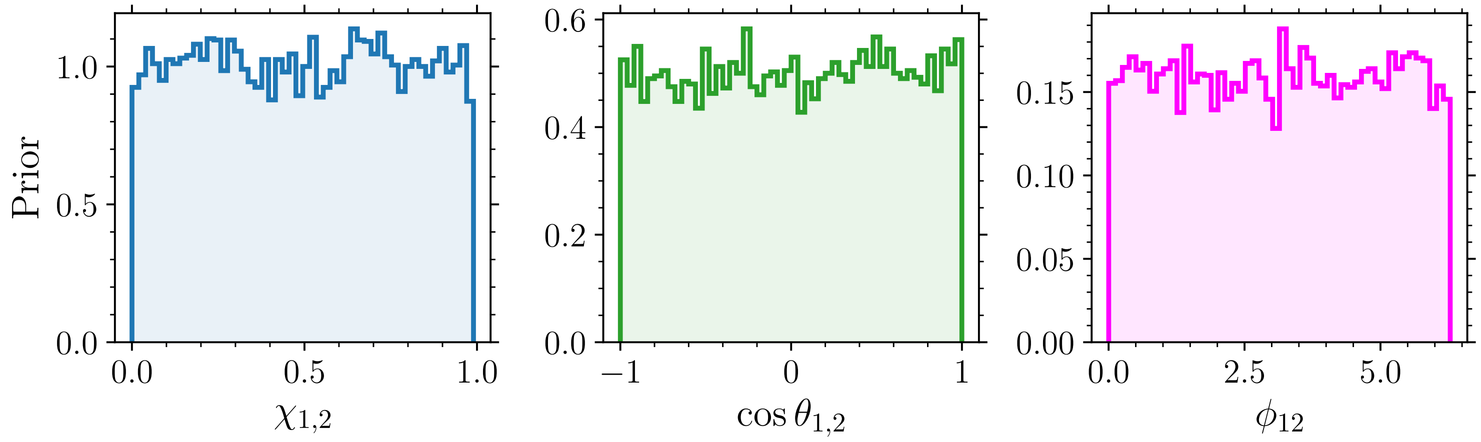
Intrinsic parameters

```
prior = {
'chirp_mass'  : bilby.gw.prior.UniformInComponentsChirpMass(name='chirp_mass', minimum=10, maximum=60),
'mass_ratio'  : bilby.gw.prior.UniformInComponentsMassRatio(name='mass_ratio', minimum=0.125, maximum=1),
'mass_1'      : bilby.gw.prior.Constraint(name='mass_1', minimum=5, maximum=100),
'mass_2'      : bilby.gw.prior.Constraint(name='mass_2', minimum=5, maximum=100),
'a_1'         : bilby.prior.analytical.Uniform(name='a_1', minimum=0, maximum=0.99),
'a_2'         : bilby.prior.analytical.Uniform(name='a_2', minimum=0, maximum=0.99),
'tilt_1'      : bilby.prior.analytical.Sine(name='tilt_1'),
'tilt_2'      : bilby.prior.analytical.Sine(name='tilt_2'),
'phi_12'      : bilby.prior.analytical.Uniform(name='phi_12', minimum=0, maximum=2 * np.pi, boundary='periodic'),
'phi_jl'      : bilby.prior.analytical.Uniform(name='phi_jl', minimum=0, maximum=2 * np.pi, boundary='periodic'),
```

$\boxed{p(\chi)\mathrm{d}\chi \propto \mathrm{d}\chi}$

Spins uniform in magnitudes and isotropic in directions





Uniform in $\chi_{1,2} \in [0, 0.99]$

Uniform in $\cos\theta_{1,2} \in [-1, 1]$

Uniform in $\phi_{12} \in [0, 2\pi]$

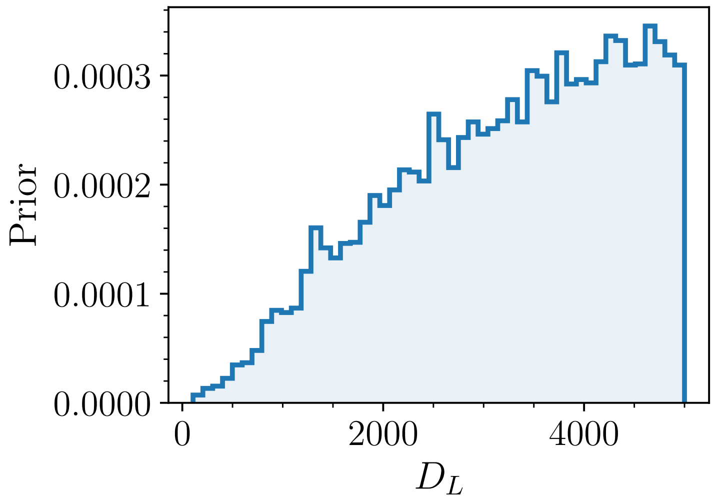
Uninformative BBH priors

Extrinsic parameters

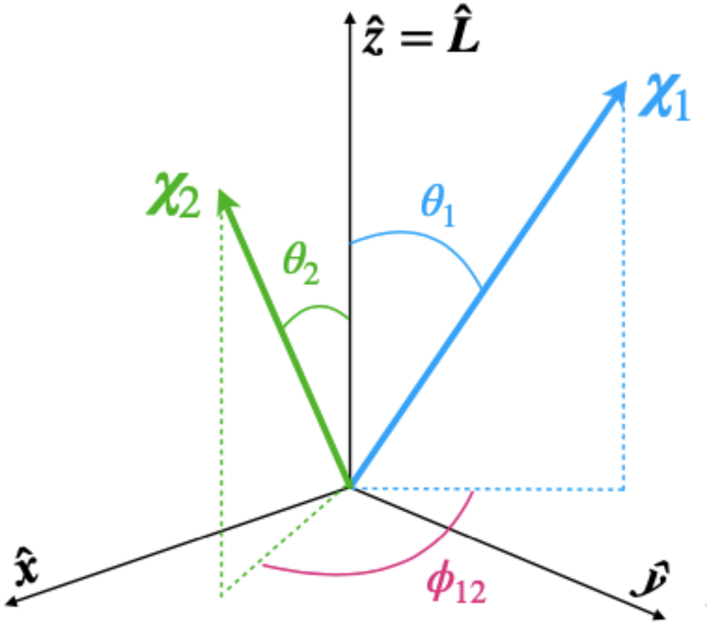
```
'luminosity_distance' : bilby.gw.prior.UniformSourceFrame(name='luminosity_distance', minimum=1e2, maximum=5e3,
unit='Mpc'),
'dec'       : bilby.prior.analytical.Cosine(name='dec'),
'ra'        : bilby.prior.analytical.Uniform(name='ra', minimum=0, maximum=2 * np.pi, boundary='periodic'),
'theta_jn'  : bilby.prior.analytical.Sine(name='theta_jn'),
'psi'       : bilby.prior.analytical.Uniform(name='psi', minimum=0, maximum=np.pi, boundary='periodic'),
'phase'     : bilby.prior.analytical.Uniform(name='phase', minimum=0, maximum=2 * np.pi, boundary='periodic')
}
```

Luminosity distance uniform in comoving volume

$D_L \in [100{,}5000]$ Mpc

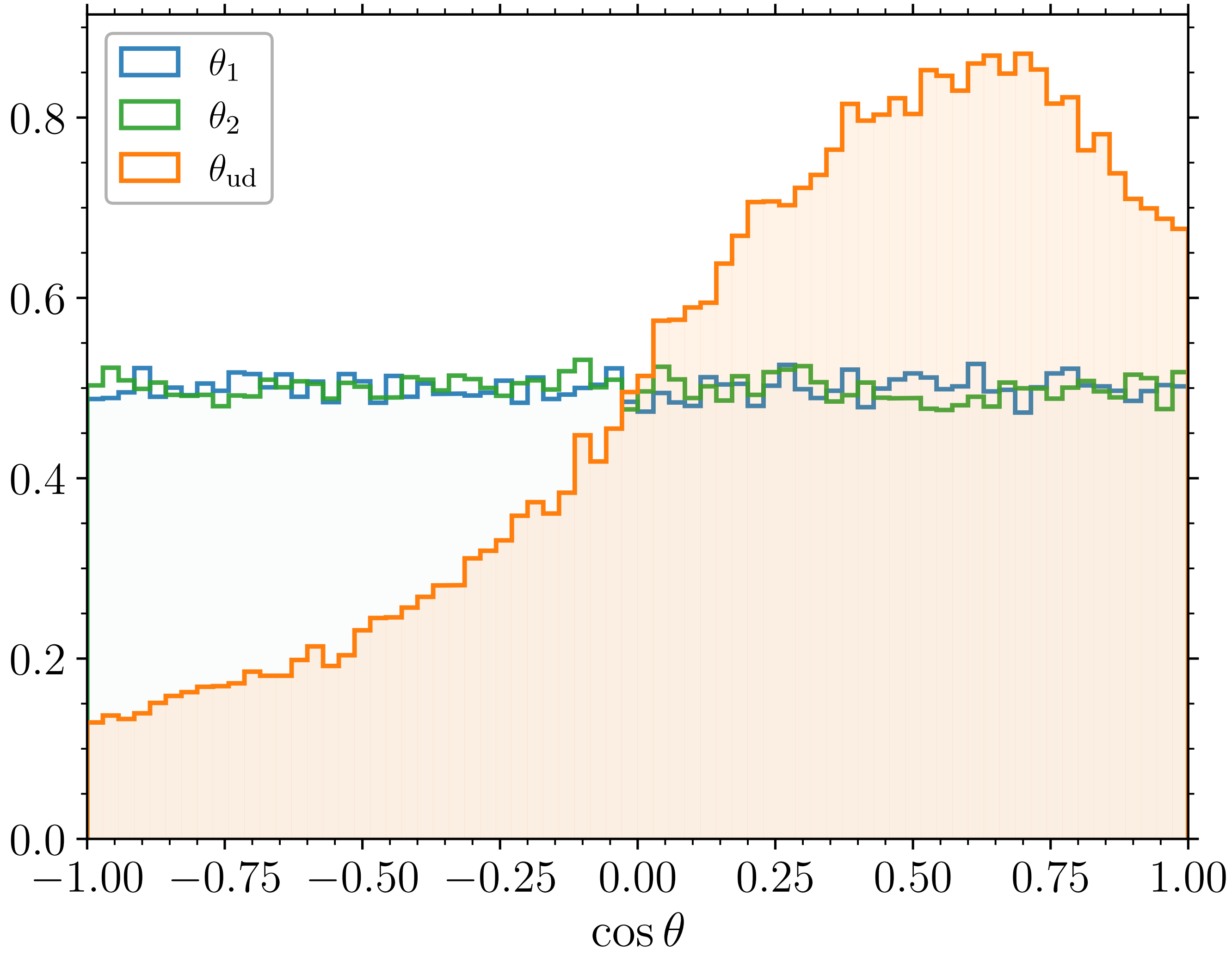


Priors



$$cos\theta_{ud} = \frac{\chi_1 - q\chi_2}{\chi_1 + q\chi_2}$$

$cos\theta_1$
$cos\theta_2$
→ Standard uniform priors

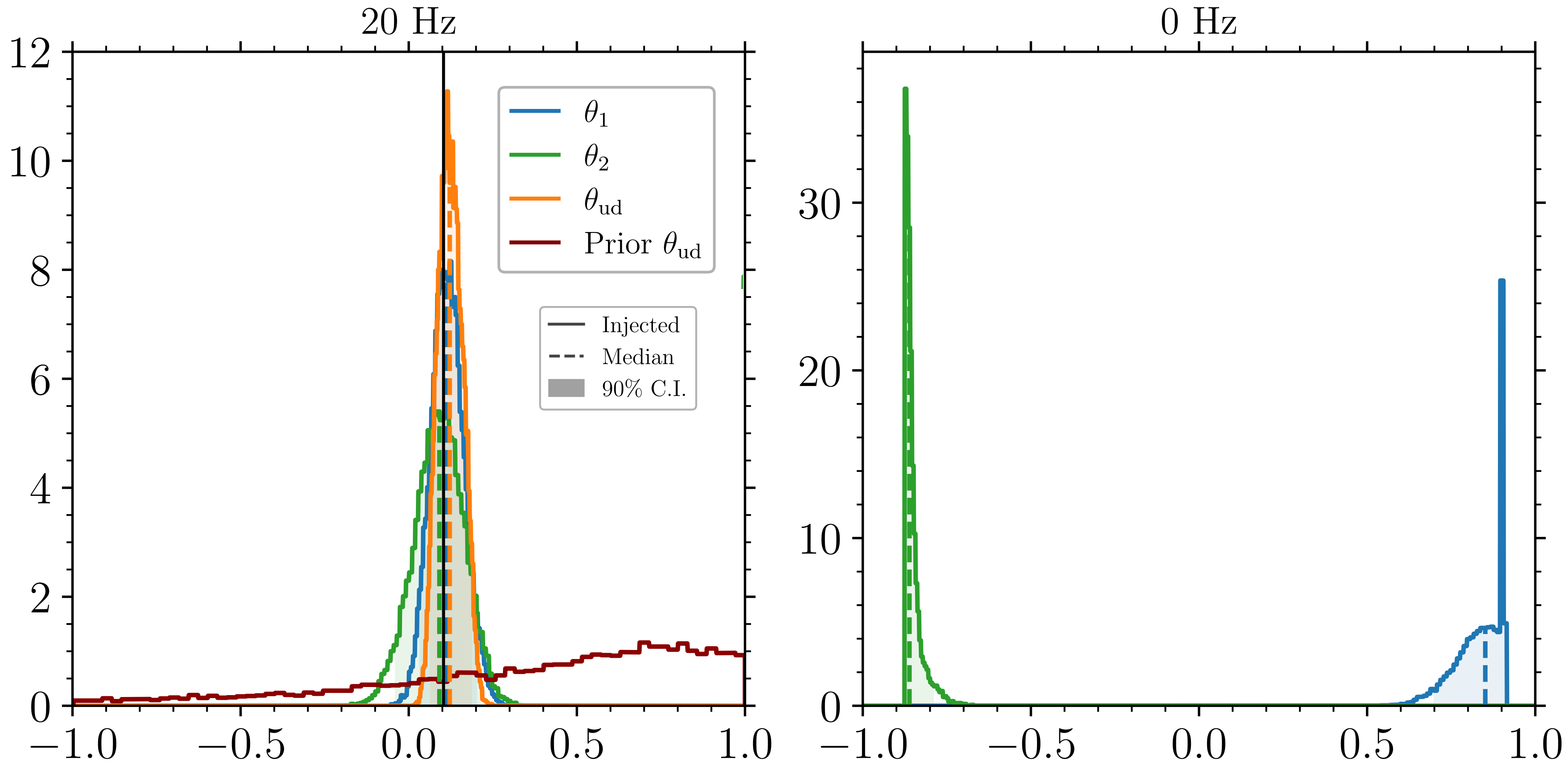


Backpropagation to 0 Hz

INJECTED VALUES

$M_{1,detector} = 49.5$
$M_{1,detector} = 39.4$
$\boldsymbol{\theta_1 = 1.47}$
$\boldsymbol{\theta_2 = 1.47}$
$\boldsymbol{\phi_{12} = 0.0}$
$\chi_1 = 0.92$
$\chi_1 = 0.94$
$d_L = 233.4$ Mpc
$\theta_{JN} = 0.37$
$\phi_{JL} = 5.71$
$dec = 0.24$
$ra = 6.11$
$\psi = 2.28$
$t_{geocent} = -0.069$
$\phi = 5.12$

q= 0.79
SNR = 277
$\ln\mathcal{B} = 6.9$
$r_+ = 266\ M \gg r = 10\ M$



Savage-Dickey Ratio for nested models

$$M_B \in M_A$$

Model M_A: 15 parameters

$$\overrightarrow{\theta_A} = \{\underbrace{m_1, m_2, \chi_1, \chi_2, \phi_{JL}, d_L, ra, dec, \psi, \phi, \theta_{JN}, t_C}_{\phi\ =\ \text{common parameters with } M_B}, \underbrace{\theta_1, \theta_2, \phi_{12}}_{\delta\ =\ \text{extra parameters of } M_A \text{ that are fixed to certain } \delta_0 \text{ in } M_B}\}$$

Model M_B: 12 parameters $\rightarrow \overrightarrow{\theta_B} = \phi$ and $\delta = \{\theta_1, \theta_2, \phi_{12}\} = \delta_0$

$$Z(d \mid M_B) = \int \mathrm{d}\phi\, P(d \mid \phi, M_B)\, \pi(\phi \mid M_B) =$$

$$= \int \mathrm{d}\phi\, P(d \mid \phi, \delta = \delta_0, M_A)\, \pi(\phi \mid \delta = \delta_0, M_A) = P(d \mid \delta = \delta_0, M_A)$$

Savage-Dickey Ratio for nested models

$$\boldsymbol{M_B \in M_A}$$

$$Z(d \mid M_B) = \int \mathrm{d}\phi \, P(d| \phi, M_B) \, \pi(\phi \mid M_B) =$$

$$= \int \mathrm{d}\phi \, P(d| \phi, \delta = \delta_0, M_A) \, \pi(\phi \mid \delta = \delta_0, M_A) = P(d|\delta = \delta_0, M_A)$$

By Bayes' theorem we can rewrite this last line (which has the shape of a likelihood) as

$$P(d|\delta = \delta_0, M_A) = \frac{P(\delta = \delta_0|d, M_A) Z(d|M_A)}{\pi(\delta = \delta_0|M_A)}$$

$$\longrightarrow \boxed{Z(d \mid M_B) = \frac{P(\delta = \delta_0|d, M_A) Z(d|M_A)}{\pi(\delta = \delta_0|M_A)}}$$

Savage-Dickey Ratio for nested models

$$M_B \in M_A$$

$$Z(d \mid M_B) = \frac{P(\delta = \delta_0|d, M_A)P(d|M_A)}{P(\delta = \delta_0|M_A)}$$

Bayes factor: $\mathrm{BF} = \frac{Z(\mathrm{d} \mid \mathrm{M_B})}{Z(\mathrm{d} \mid \mathrm{M_A})} =$

$$= Z(\mathrm{d} \mid \mathrm{M_B}) \cdot \frac{1}{Z(\mathrm{d} \mid \mathrm{M_A})} =$$

$$= Z(\mathrm{d} \mid \mathrm{M_B}) \cdot \frac{1}{Z(\mathrm{d} \mid \mathrm{M_A})} =$$

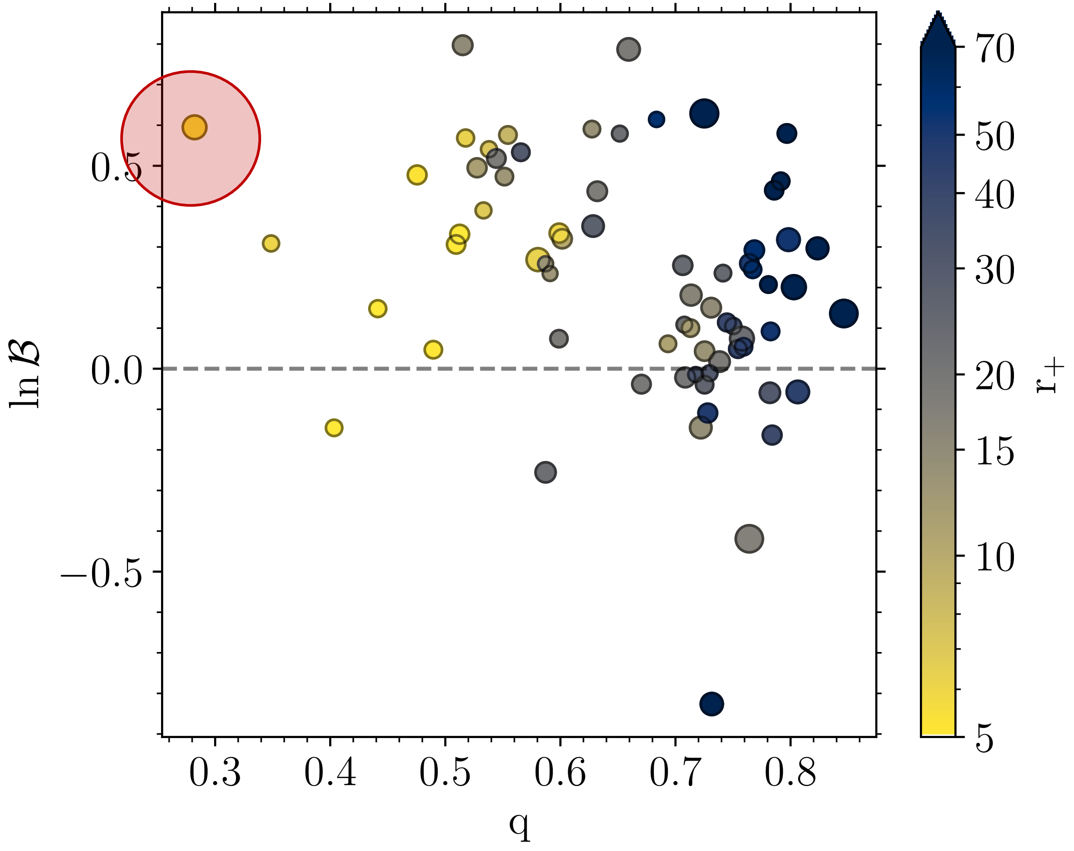
$$= \frac{P(\delta = \delta_0|d, M_A)\cancel{Z(d|M_A)}}{P(\delta = \delta_0|M_A)} \cdot \frac{1}{\cancel{Z(\mathrm{d} \mid \mathrm{M_A})}} =$$

$$= \frac{P(\delta = \delta_0|d, M_A)}{P(\delta = \delta_0|M_A)}$$

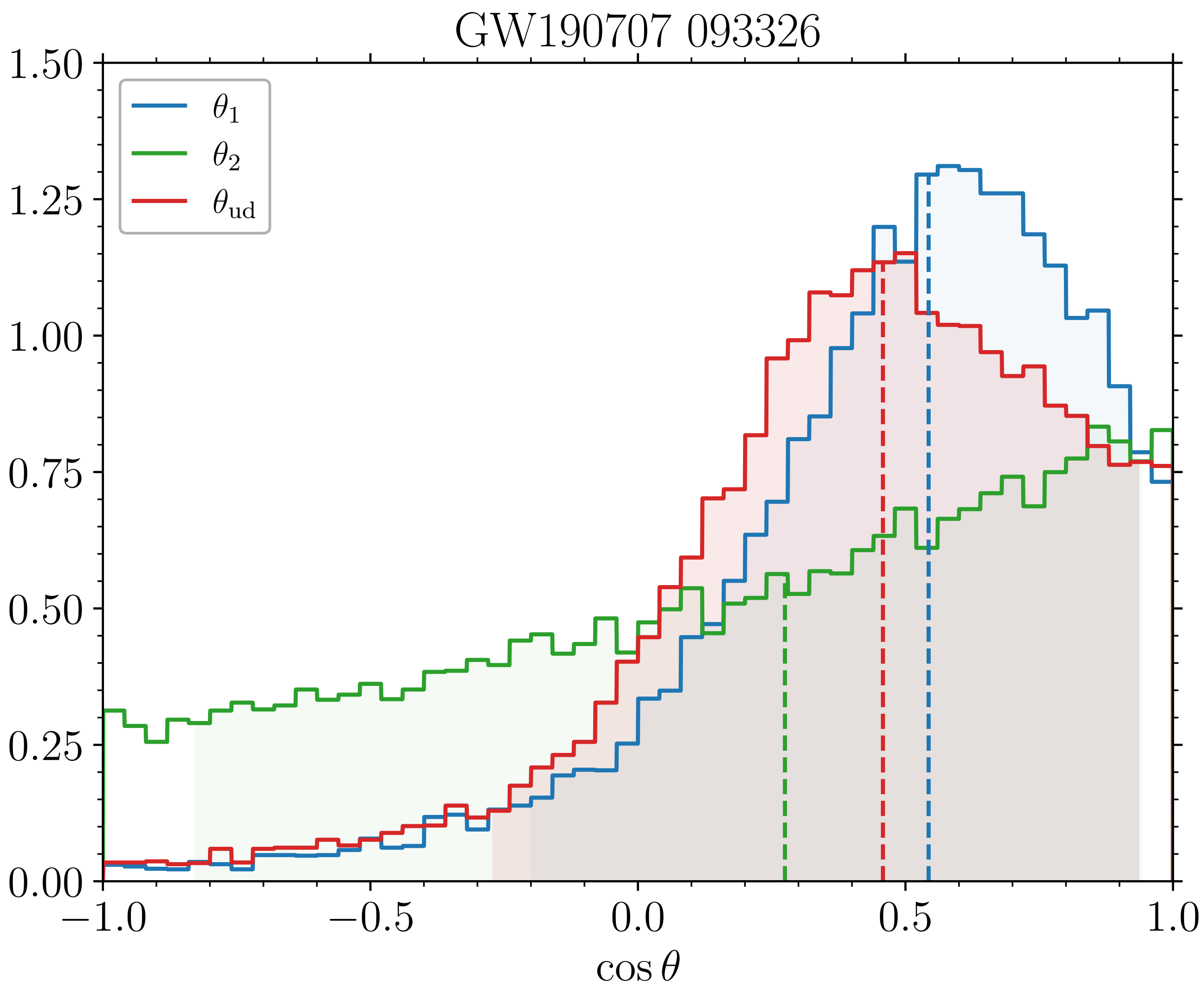
Real data

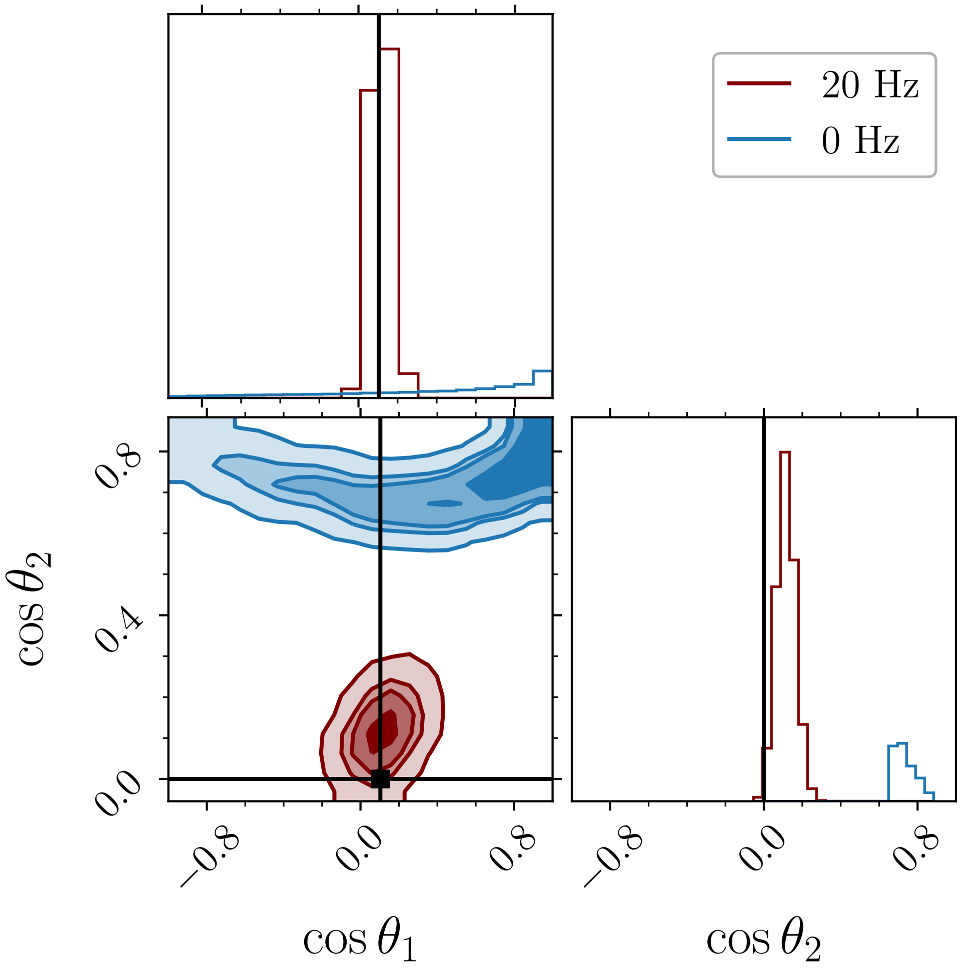
- 69 events of BBH coalescences (BNS excluded)
  - FAR $< 1yr^{-1}$ in at least one search

GW190707_093326



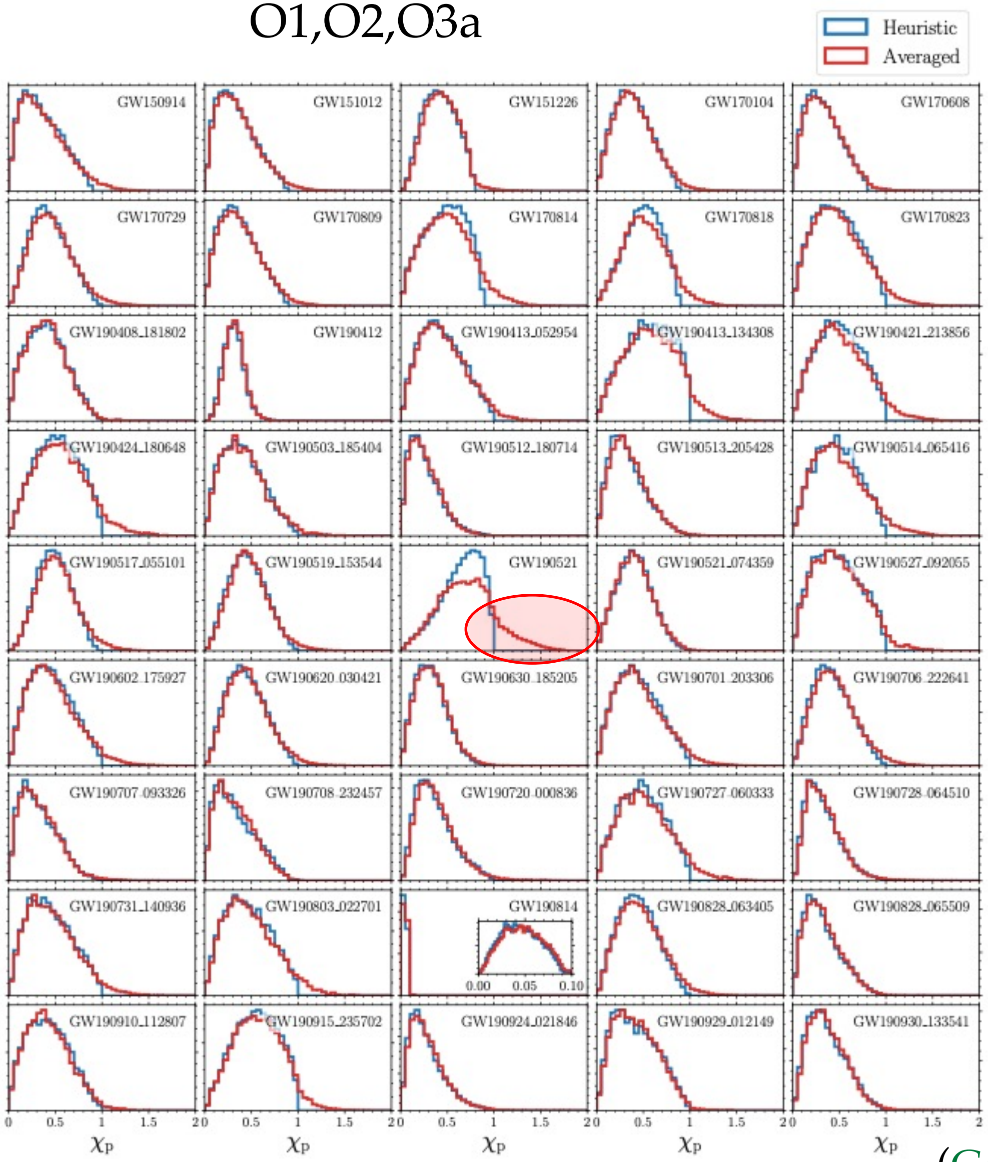
Real data


GW190707 093326
θ_1
θ_2
θ_{ud}
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−1.0
−0.5
0.0
0.5
1.0
$\cos\theta$


20 Hz
0 Hz
$\cos\theta_2$
0.8
0.4
0.0
−0.8
0.0
0.8
$\cos\theta_1$
−0.8
0.0
0.8
$\cos\theta_2$

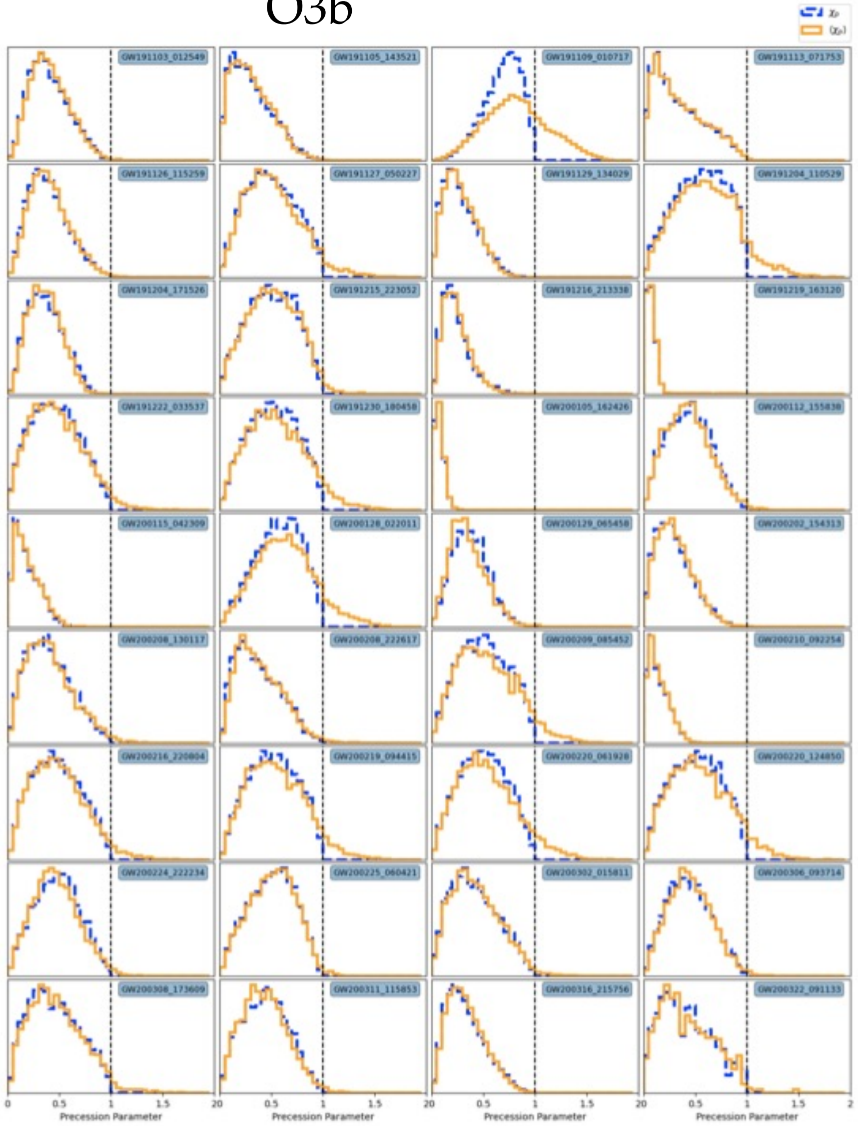
Two spin-effects in real data

O1,O2,O3a



(Gerosa+ 2020)

O3b



(Henshaw+ 2022)

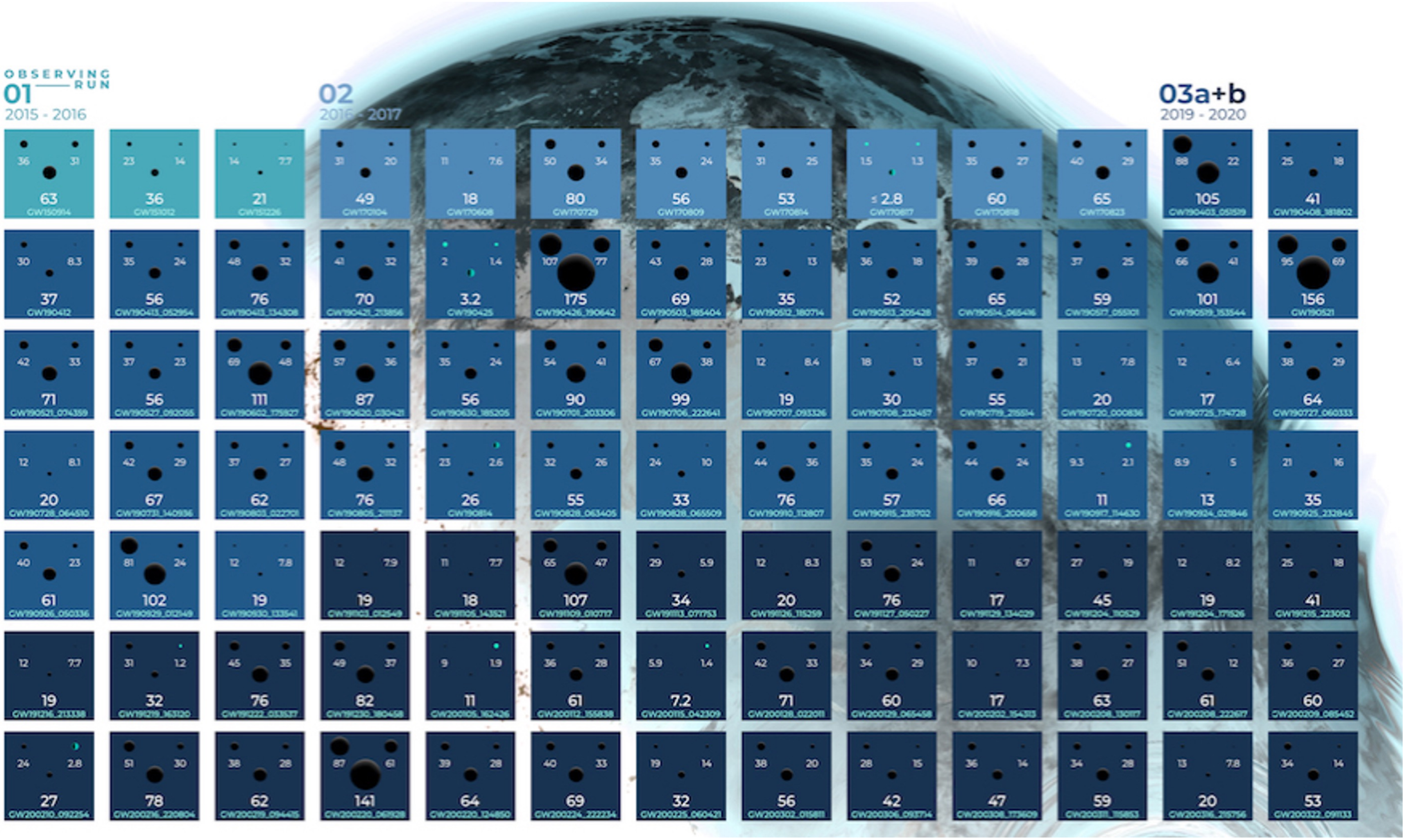
GW detections (GWTC-3)

2015: first GW detection

Observing runs
O1: 2015-2016
O2: 2016-2017
O3: 2019-2020
O4: March 2023

Total number of events is now **90**

- **Binary black holes** (BBH)
  Binary neutron stars (BNS)
- **Neutron star–black hole binaries** (NSBH)



(LVK Collaboration)

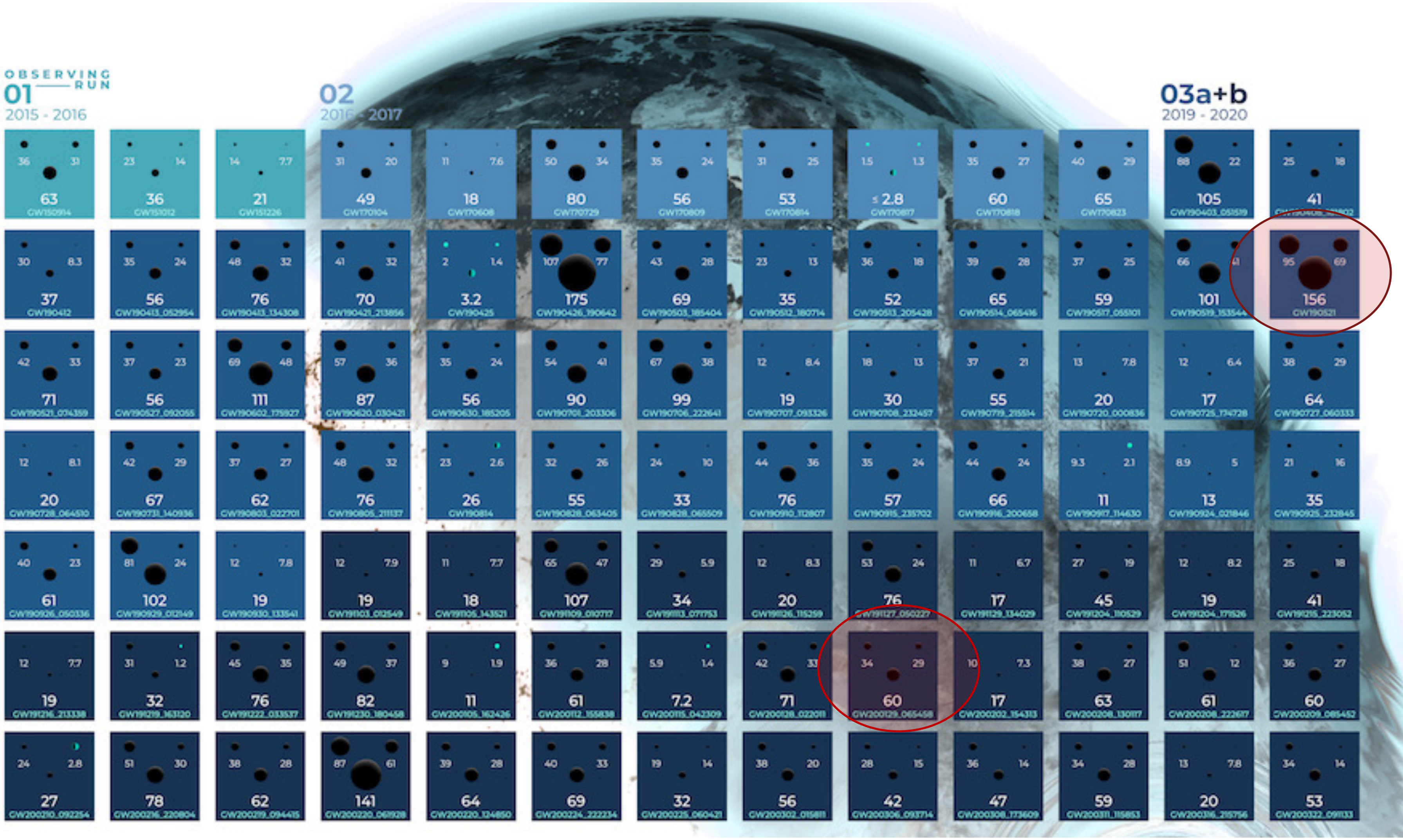
Spin precession in individual events

Moderate evidence for highly precessing spins in GWTC-3.

Promising candidates for spin precession:

- **GW190521**
  (potential degeneracies with the eccentricity [Romero-Shaw et al. 2020])

- **GW200129**
  [Hannam et al. 2021]
  (possible issues in the glitch mitigation analysis [Payne et al. 2021])

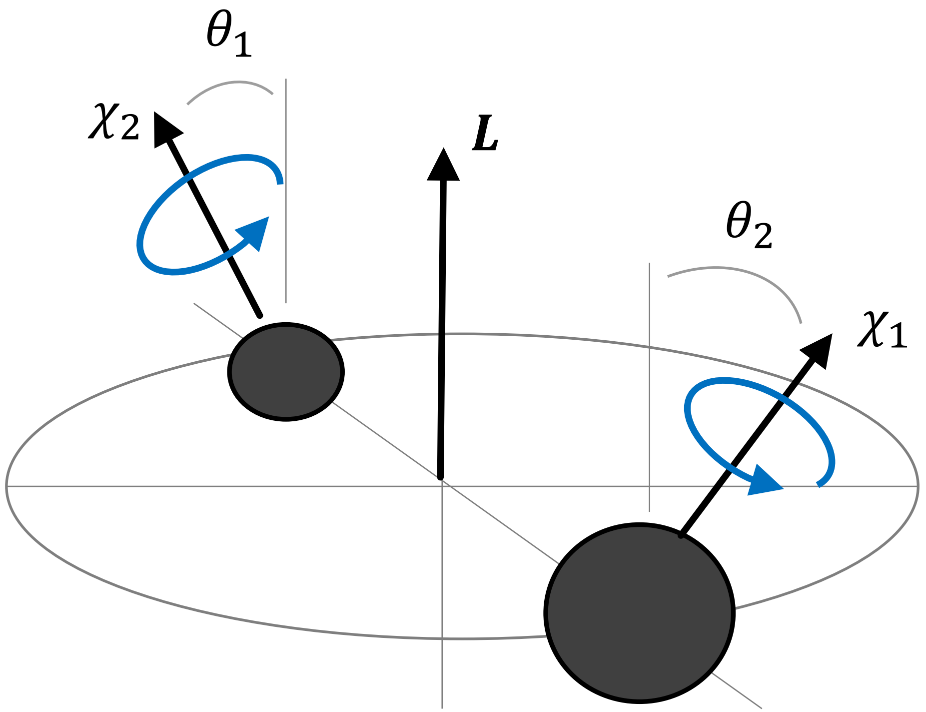


(LVK Collaboration)

Mass and spin measurements

No hair theorem:

Kerr BHs are uniquely described by their **mass M** and their **spin S**



- We can measure with great accuracy the **chirp mass**:

$$M_c = \frac{(m_1 m_2)^{3\backslash 5}}{(m_1 + m_2)^{1\backslash 5}}$$

- **Spins** provide an highly subdominant contribution to the emitted radiation

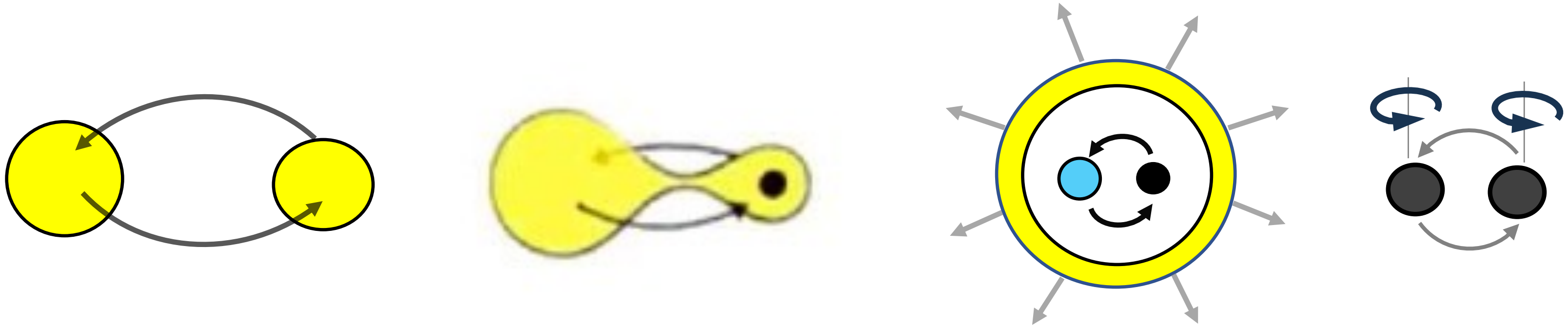
Measurements of component **spin magnitudes** and **tilt angles**

Measurements of **effective spin parameters** χ_{eff} and χ_{p}

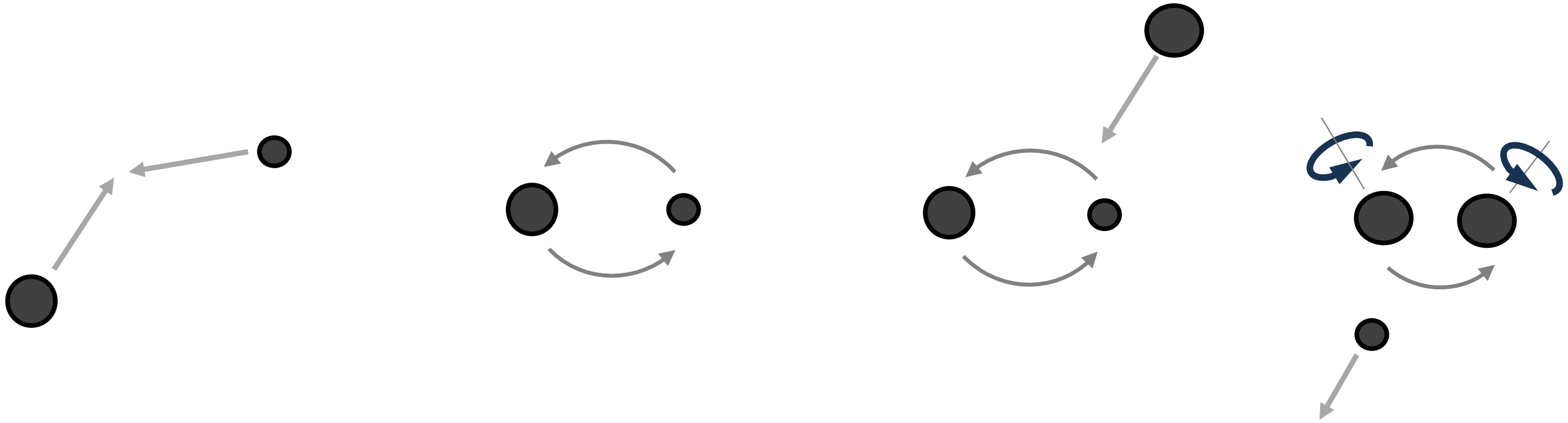
Spin distribution

The directions of the BH spins are believed to be clean tracers of the astrophysical formation pathway

Isolated formation channel:
preferentially aligned spins
(non precessing binaries)



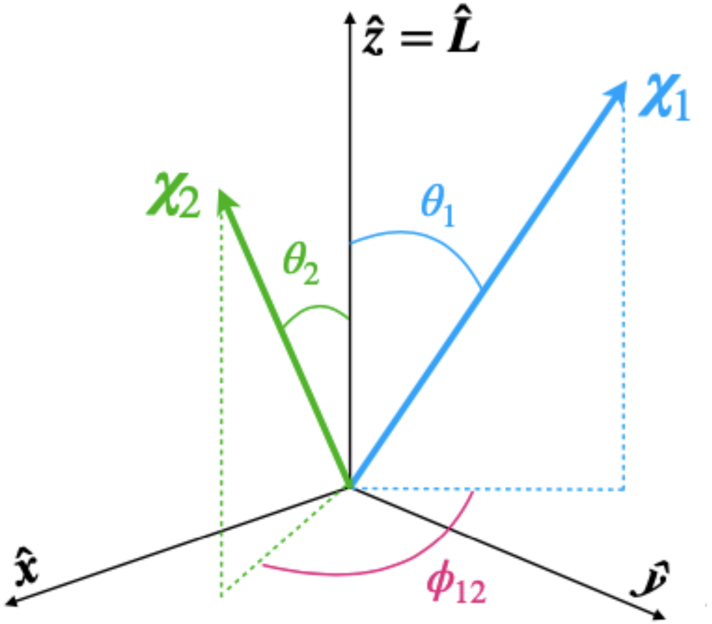
Dynamical formation channel:
isotropically oriented spins
(misalignment → spin
precession)



[Mandel and Farmer, 2022]

Effective spin parameters

(Biscoveanu+ 2021)



$$\overrightarrow{\chi_{12}} = \overrightarrow{S_{12}}/m_{12}^2$$
$$q = m_2/m_1$$

Alignment information:

$$\chi_{\text{eff}} = \frac{m_1\vec{\chi_1} + m_2\vec{\chi_2}}{m_1 + m_2} \cdot \hat{L}$$

- Only aligned spin components (Damour 2001)
- Constant of motion at 2PN (Racine 2006)

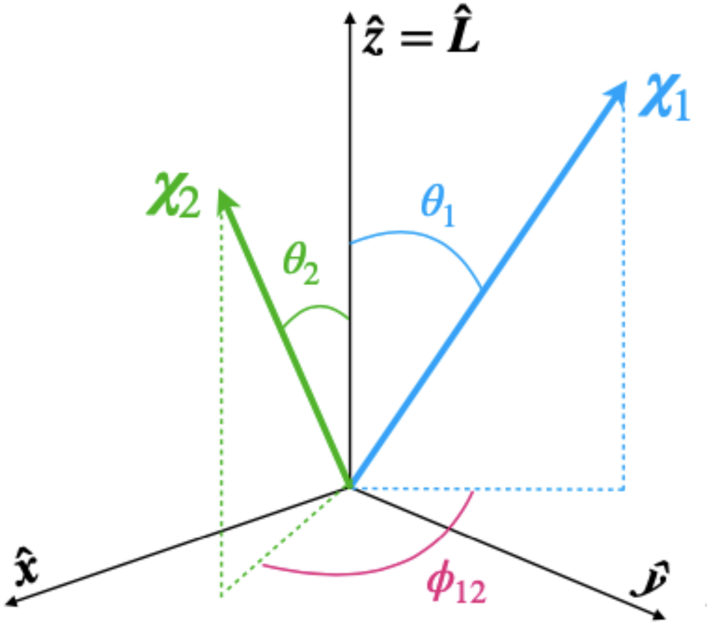
Precessing spin parameter

$$\chi_{\text{p}} = \max(\chi_1 \sin\theta_1, q\frac{4q+3}{4+3q}\chi_2 \sin\theta_2)$$

- In-plane spin components (Schmidt+ 2015)
- Used in current GW analysis

Effective spin parameters

(Biscoveanu+ 2021)



$$\overrightarrow{\chi_{12}} = \overrightarrow{S_{12}}/m_{12}^2$$
$$q = m_2/m_1$$

Alignment information:

$$\chi_{\text{eff}} = \frac{m_1\vec{\chi_1} + m_2\vec{\chi_2}}{m_1 + m_2} \cdot \hat{L}$$

- Only aligned spin components (Damour 2001)
- Constant of motion at 2PN (Racine 2006)

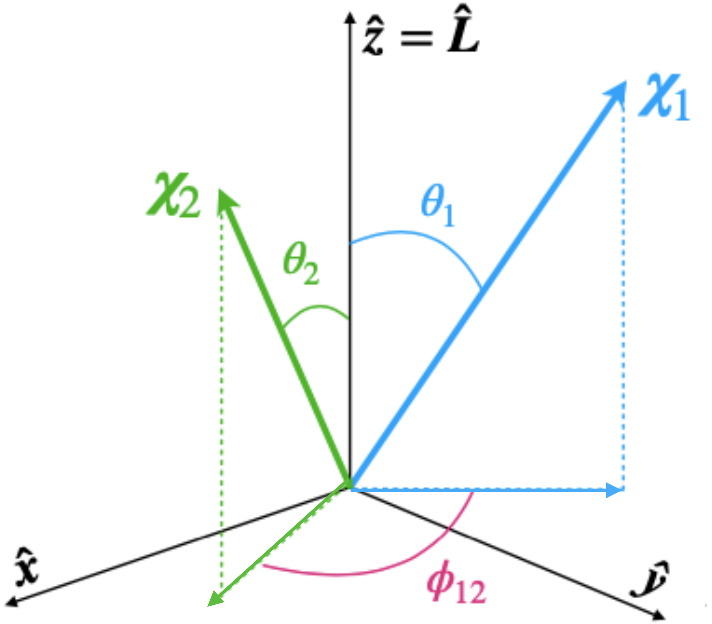
Precessing spin parameter

$$\chi_{\text{p}} = \max(\chi_1 \sin\theta_1, q\frac{4q+3}{4+3q}\chi_2 \sin\theta_2)$$

- In-plane spin components (Schmidt+ 2015)
- Used in current GW analysis

Effective spin parameters

(Biscoveanu+ 2021)



$$\overrightarrow{\chi_{12}} = \overrightarrow{S_{12}}/m_{12}^2$$
$$q = m_2/m_1$$

Alignment information:

$$\chi_{\mathrm{eff}} = \frac{m_1\vec{\chi_1} + m_2\vec{\chi_2}}{m_1 + m_2} \cdot \hat{L}$$

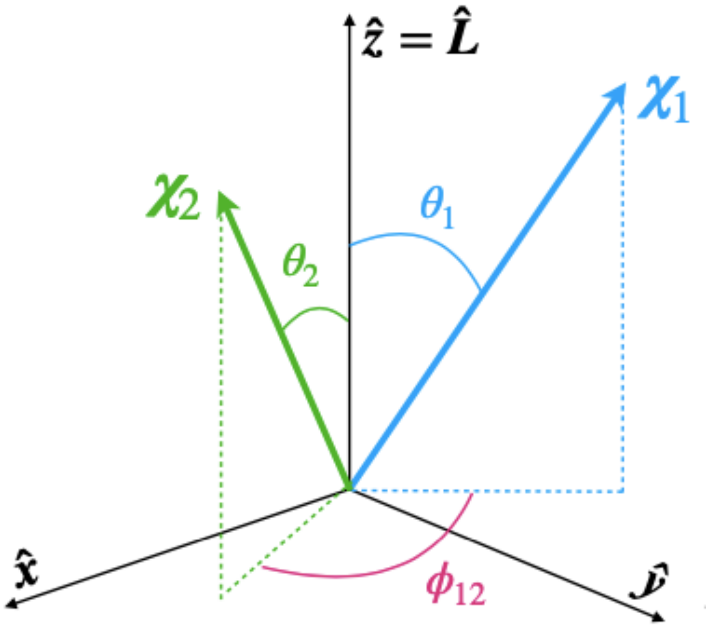
- Only aligned spin components (Damour 2001)
- Constant of motion at 2PN (Racine 2006)

Precessing spin parameter

$$\chi_{\mathrm{p}} = \max\left(\chi_1 \sin\theta_1, q\frac{4q+3}{4+3q}\chi_2 \sin\theta_2\right)$$

- In-plane spin components (Schmidt+ 2015)
- Used in current GW analysis

Averaged vs Heuristic χ_p



Heuristic

$$\chi_p^{\text{heu}} = \max\left(\chi_1 \sin\theta_1 \,, q\, \frac{4q+3}{(4+3q)} \chi_2 \sin\theta_2 \right)$$

- Takes information from only one of the two objects.
- Retains some, but not all the variations taking place on the precession timescale.
- Domain: $0 < \chi_p^{\text{heu}} < 1$

Averaged

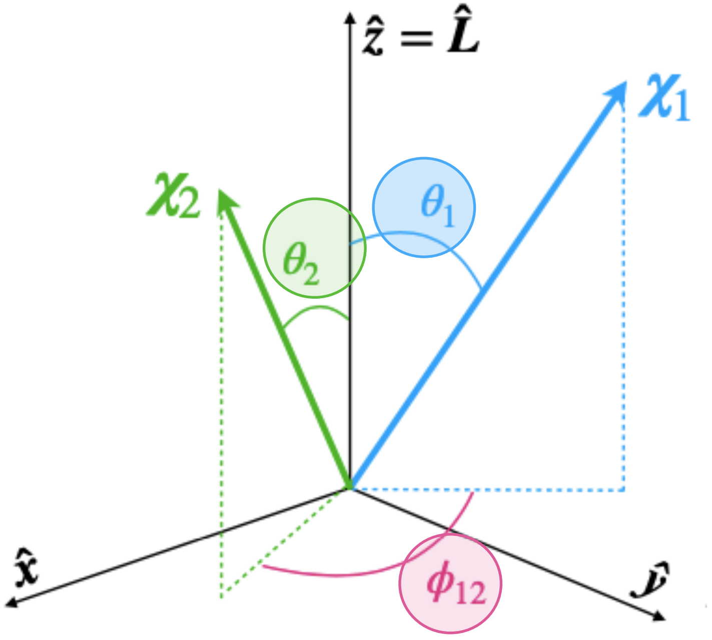
$$\langle \chi_p^{\text{av}} \rangle = \frac{\int \chi_p(\psi) \left(\frac{d\psi}{dt}\right)^{-1} d\psi}{\int \left(\frac{d\psi}{dt}\right)^{-1} d\psi}$$

(Gerosa+ 2020)

$$\left[(\chi_1 \sin\theta_1)^2 + \left(q\, \frac{4q+3}{(4+3q)} \chi_2 \sin\theta_2 \right)^2 + 2q\, \frac{4q+3}{(4+3q)} \chi_1 \chi_2 \sin\theta_1 \sin\theta_2 \cos\phi_{12} \right]^{1/2}$$

- Averages over all the variations occurring on the precession timescale.

Averaged vs Heuristic χ_p



$\theta_1, \theta_2, \phi_{12}$ all vary on the precession timescale!
(Kesden+ 2015, Gerosa+ 2015)

Heuristic

$$\chi_p^{\text{heu}} = \max\left(\chi_1 \sin\theta_1 \,,\, q\frac{4q+3}{(4+3q)}\chi_2 \sin\theta_2\right)$$

- Takes information from only one of the two objects.
- Retains some, but not all the variations taking place on the precession timescale.
- Domain: $0 < \chi_p^{\text{heu}} < 1$

Averaged

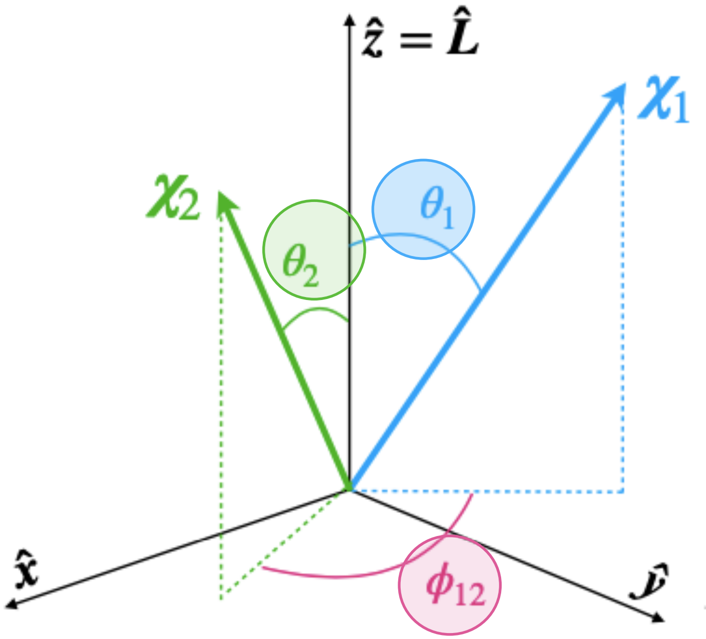
$$\langle \chi_p^{\text{av}} \rangle = \frac{\int \chi_p(\psi) \left(\frac{d\psi}{dt}\right)^{-1} d\psi}{\int \left(\frac{d\psi}{dt}\right)^{-1} d\psi}$$

(Gerosa+ 2020)

$$\left[(\chi_1 \sin\theta_1)^2 + \left(q\frac{4q+3}{(4+3q)}\chi_2 \sin\theta_2\right)^2 + 2q\frac{4q+3}{(4+3q)}\chi_1\chi_2 \sin\theta_1 \sin\theta_2 \cos\phi_{12}\right]^{1/2}$$

- Averages over all the variations occurring on the precession timescale.

Averaged vs Heuristic χ_p



$\theta_1, \theta_2, \phi_{12}$ all vary on the precession timescale!
(Kesden+ 2015,Gerosa+ 2015)

Heuristic	Averaged
$\chi_p^{\rm heu} = \max\left(\chi_1 \sin\theta_1, q\frac{4q+3}{(4+3q)}\chi_2 \sin\theta_2\right)$	$\langle\chi_p^{\rm av}\rangle = \frac{\int \chi_p(\psi)\left(\frac{d\psi}{dt}\right)^{-1} d\psi}{\int \left(\frac{d\psi}{dt}\right)^{-1} d\psi}$ (Gerosa+ 2020)
➢ Takes information from only one of the two objects.	➢ Take information from both objects.
➢ Retains some, but not all the variations taking place on the precession timescale.	➢ Averages over all the variations occurring on the precession timescale.
➢ Domain: $0 < \chi_p^{\rm heu} < 1$	➢ Domain: $0 < \chi_p^{\rm av} < 2$

Two spin-effects: $1 < \chi_p^{\rm av} < 2$

χ_p calculation

$t_{orb} \ll t_{pre} \ll t_{RR}$

$$\boxed{\frac{d\mathbf{L}}{dt} = \underbrace{\frac{d\hat{\mathbf{L}}}{dt}}_{\text{Precession}} L + \underbrace{\frac{dL}{dt}}_{\text{Radiation reaction}}\hat{\mathbf{L}} = (\mathbf{\Omega}_L \times \hat{\mathbf{L}})L + \frac{dL}{dt}\hat{\mathbf{L}}}$$

$$\begin{cases} \mathbf{\Omega}_L = \Omega_1\chi_1\hat{\mathbf{S}}_1 + \Omega_2\chi_2\hat{\mathbf{S}}_2 \\ \Omega_1 = \dfrac{M^2}{2r^3(1+q)^2}\left[4 + 3q - \dfrac{3q\chi_{\text{eff}}}{(1+q)}\dfrac{M^2}{L}\right] \\ \Omega_2 = \dfrac{qM^2}{2r^3(1+q)^2}\left[4q + 3 - \dfrac{3q\chi_{\text{eff}}}{(1+q)}\dfrac{M^2}{L}\right] \end{cases}$$

Amount of relativistic precession:

$$\left|\frac{d\hat{\mathbf{L}}}{dt}\right|^2 = (\Omega_1\chi_1\sin\theta_1)^2 + (\Omega_2\chi_2\sin\theta_2)^2 + 2\Omega_1\Omega_2\chi_1\chi_2\sin\theta_1\sin\theta_2\cos\Delta\Phi$$

Heuristic χ_p:

$$\chi_p^{\text{heu}} = \frac{1}{2\Omega_1}\left(\underbrace{\left|\frac{d\hat{\mathbf{L}}}{dt}\right|_+ + \left|\frac{d\hat{\mathbf{L}}}{dt}\right|_-}\right) = \max\left(\chi_1\sin\theta_1, q\frac{4q+3}{(4+3q)}\chi_2\sin\theta_2\right)$$

Aritmetic heuristic mean between two configurations: $\boldsymbol{\cos\Delta\Phi = \pm 1}$
normalized with Ω_1

Two problems:

I. The configurations are not always geometrically possible
II. The angles $\theta_1, \theta_2, \Delta\Phi$ all vary on the precession timescale.

How to fix the problems with heuristic χ_p?

I. Generalized χ_p → Retain all the variation occurring on the precession timescale

$$\chi_p^{\text{gen}} = \frac{1}{\Omega_1}\left(\left|\frac{\mathrm{d}\hat{\mathbf{L}}}{dt}\right|\right) = \Bigg[\underbrace{(\chi_1 \sin\theta_1)^2 + \left(q\,\frac{4q+3}{(4+3q)}\chi_2 \sin\theta_2\right)^2}_{\text{Heuristic } \chi_p} + 2q\,\frac{4q+3}{(4+3q)}\chi_1\chi_2 \sin\theta_1 \sin\theta_2 \underbrace{\cos\Delta\phi}_{\text{Retain the dependence on } \Delta\Phi}\Bigg]^{1/2}$$

II. Averaged χ_p → Average all the variation occurring on the precession timescale

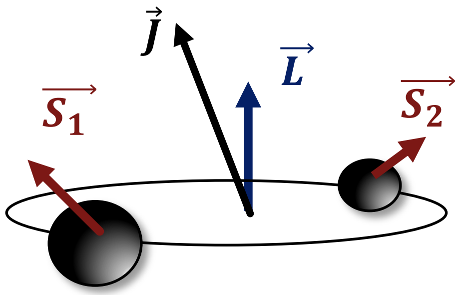
$$\langle \chi_p^{av} \rangle = \frac{\int \chi_p(\psi)\left(\frac{d\psi}{dt}\right)^{-1} d\psi}{\int \left(\frac{d\psi}{dt}\right)^{-1} d\psi}$$

ψ = quantity that parametrize the precession cycle

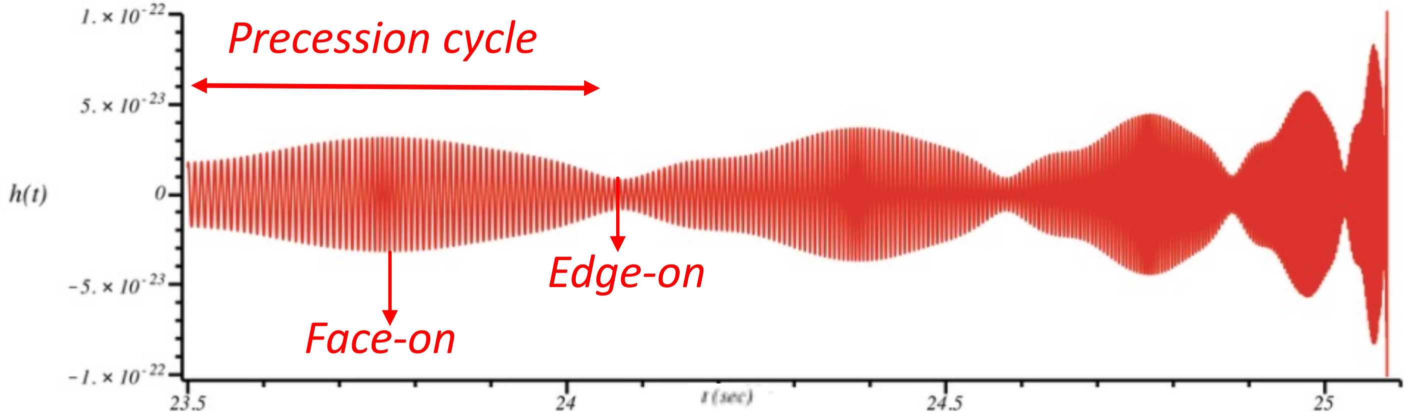
(Gerosa+ 2015)

Effects of spins in the waveform

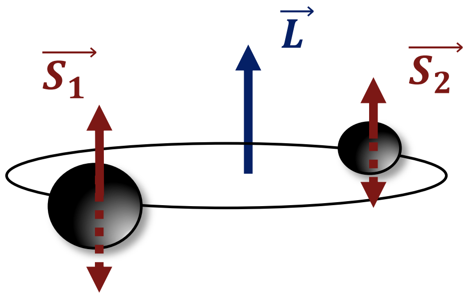
Misaligned spins→ precessing

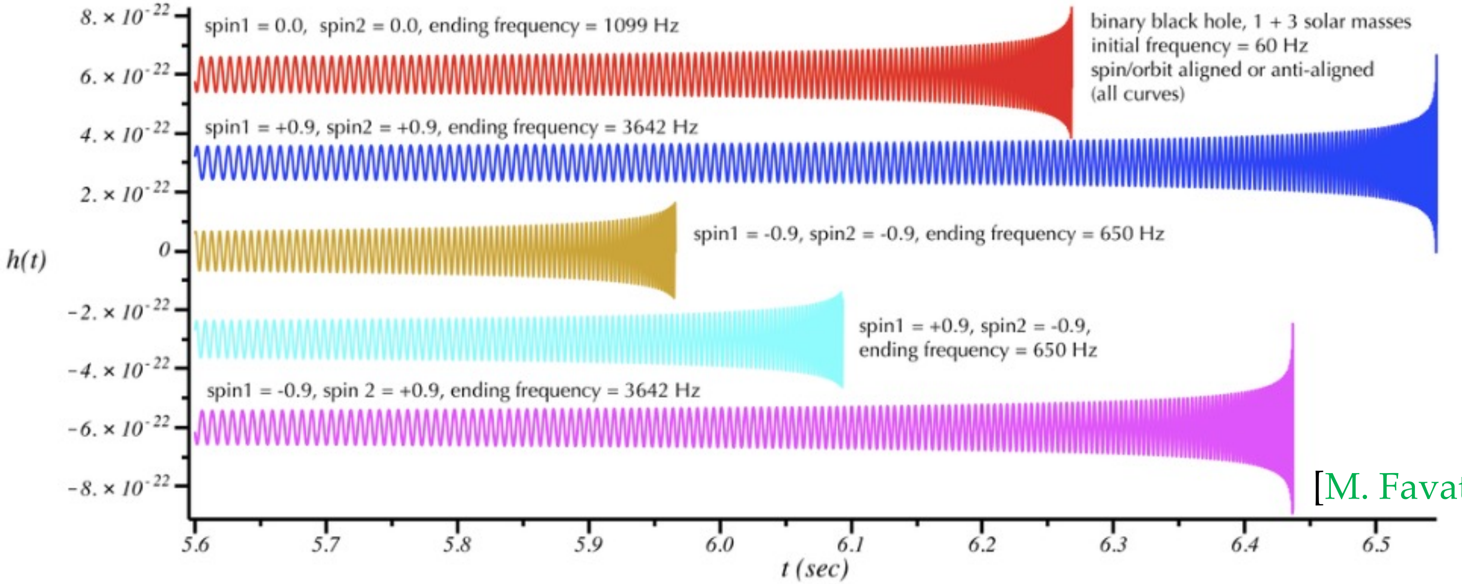


$t_{orb} \ll t_{pre} \ll t_{RR}$



Aligned spins → no precession



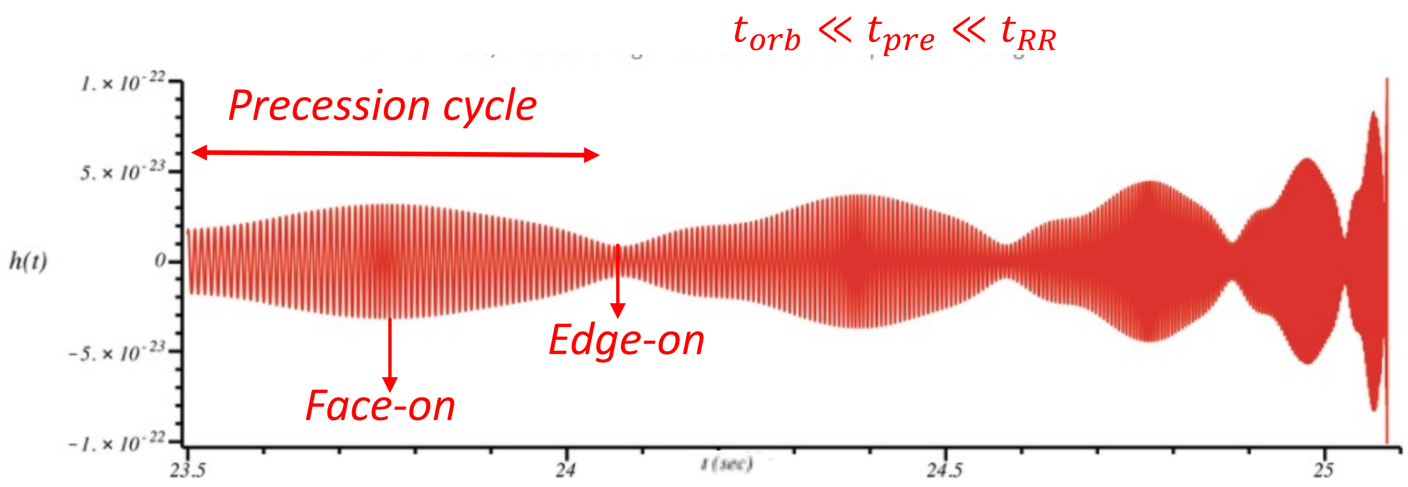


[M. Favata]

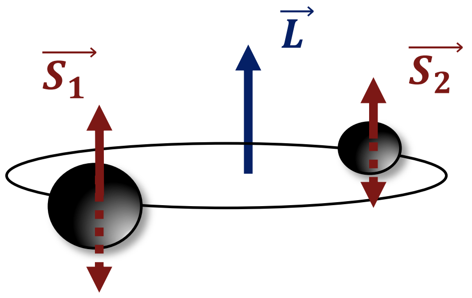
Effects of spins in the waveform

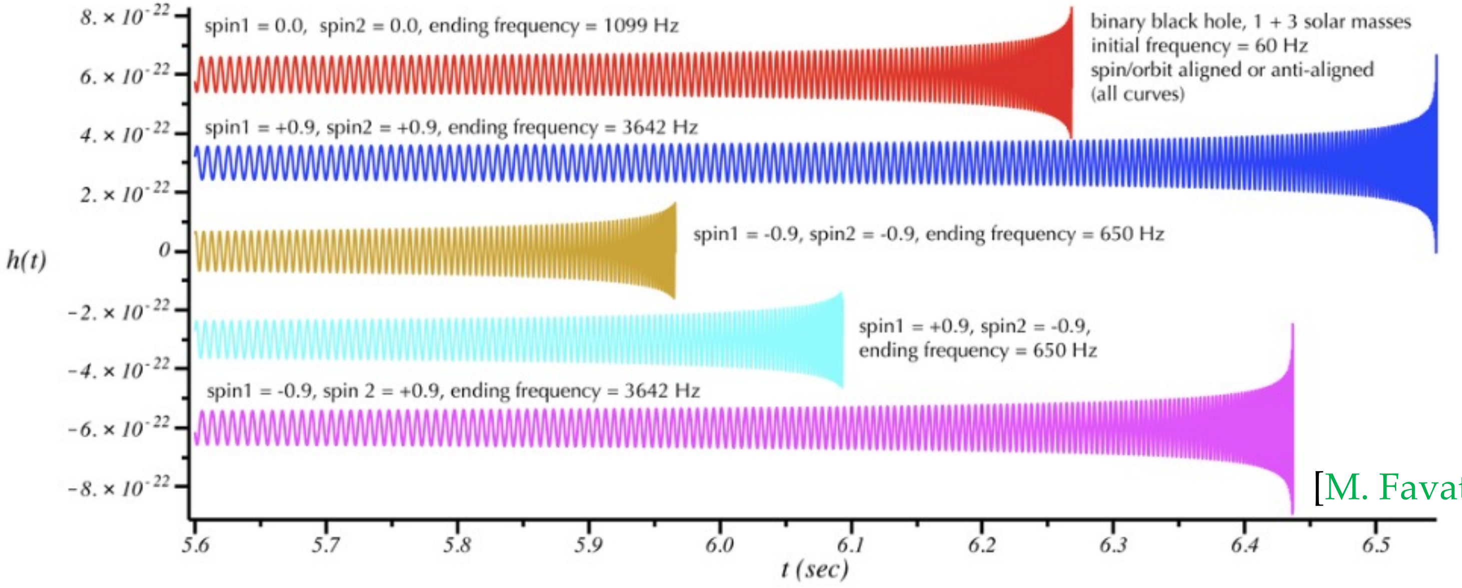
Misaligned spins→ precessing

- Modulation in amplitude and phase
- Change in direction of the GW emission



Aligned spins → no precession



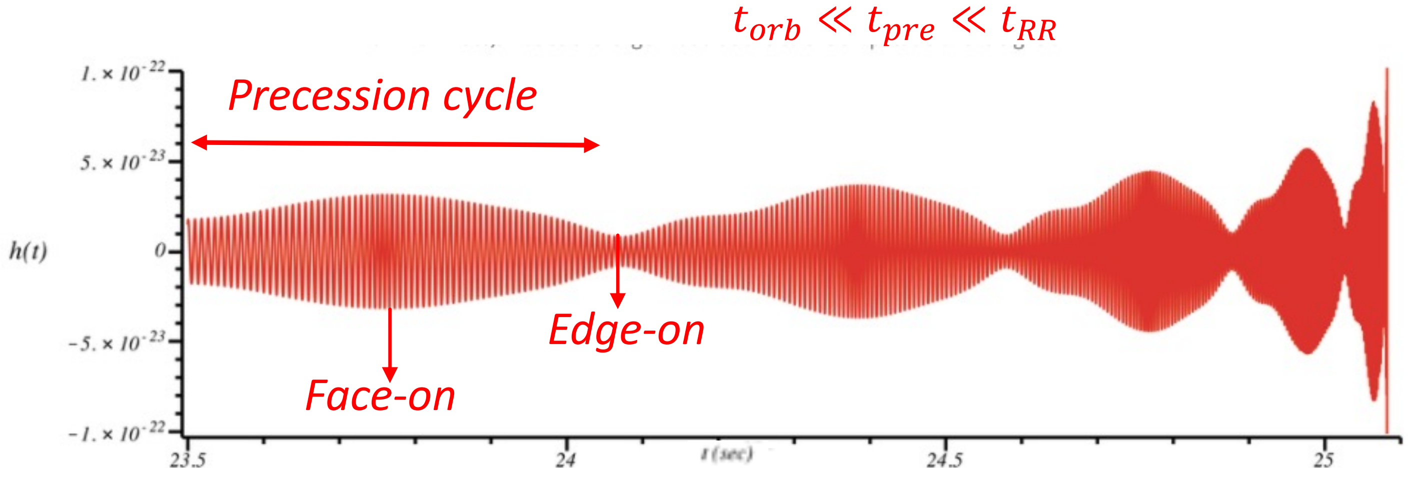


[M. Favata]

Effects of spins in the waveform

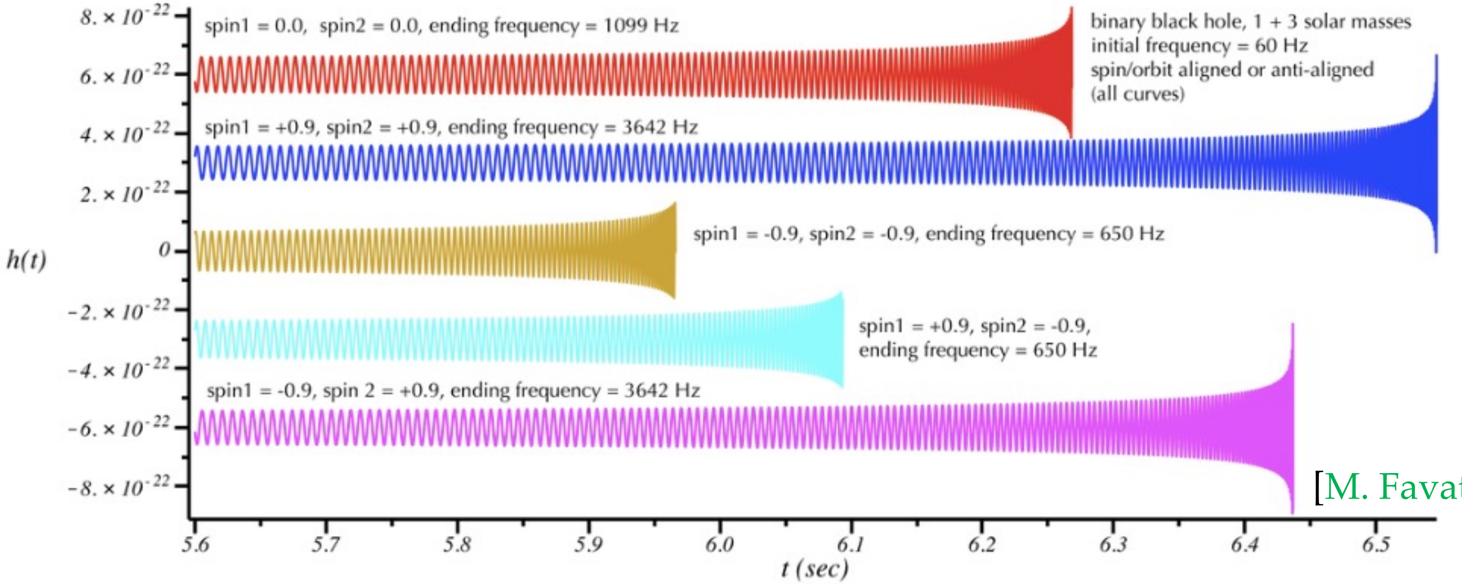
Misaligned spins→ precessing

- Modulation in amplitude and phase
- Change in direction of the GW emission



Aligned spins → no precession

- Similar morphologies
- Different merger frequencies and signal durations:
  - **Aligned spins** → orbital hung up
  - **Anti-aligned spins** → shorter signal



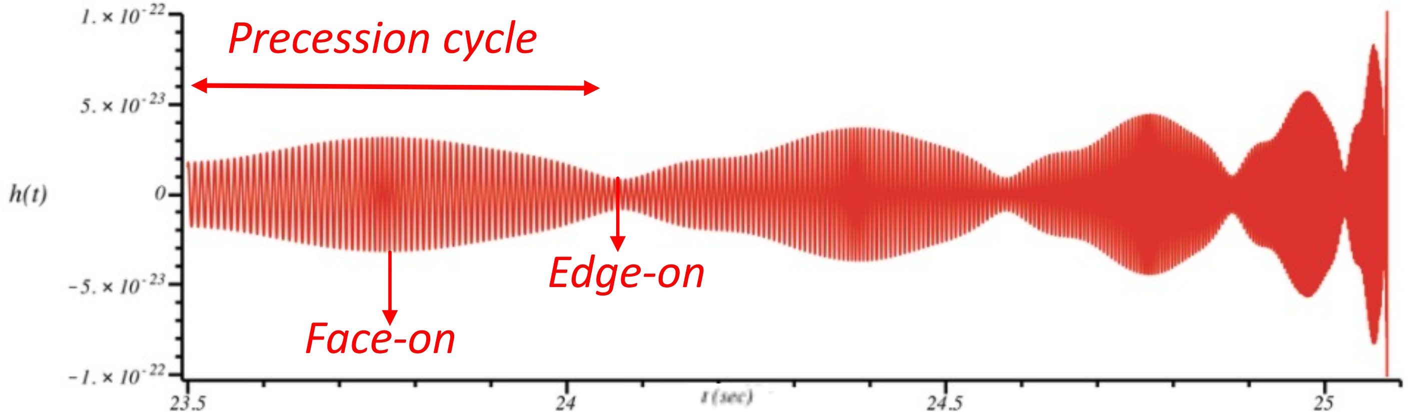
[M. Favata]

Effects of spins in the waveform

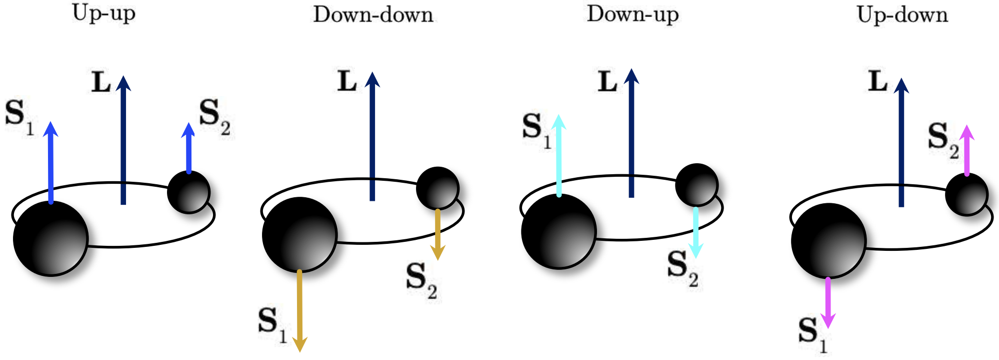
Misaligned spins→ precessing

$t_{orb} \ll t_{pre} \ll t_{RR}$

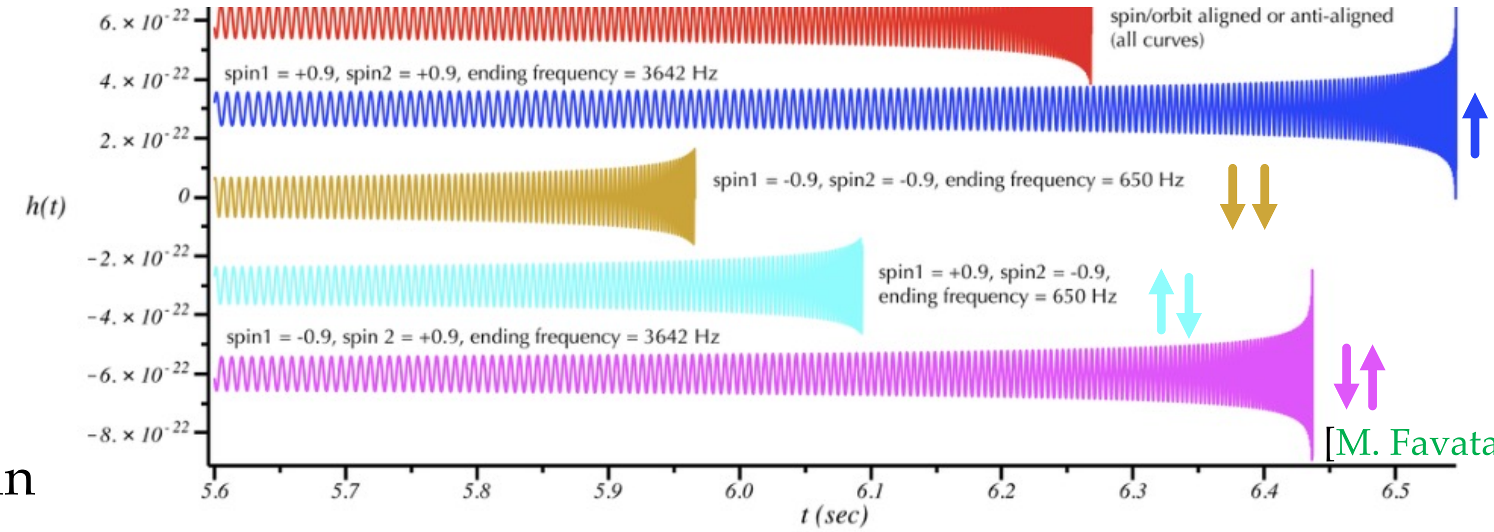
- Modulation in amplitude and phase
- Change in direction of the GW emission



Aligned spins → non precessing



4 equilibrium solution for the relativistic spin precession equation



[M. Favata