Decision and Estimation in Data Processing

Chapter III. Elements of Estimation Theory

III.1 Introduction

What means “Estimation”?

- A sender transmits a signal $s_{\Theta}(t)$ which depends on an **unknown** parameter Θ
- The signal is affected by noise, we receive $r(t) = s_{\Theta}(t) + noise$
- We want to **find out** the correct value of the parameter
  - based on samples from the received signal, or the full continuous signal
  - available data is noisy $=>$ we “estimate” the parameter
- The found value is $\hat{\Theta}$, **the estimate** of Θ (“estimatul”, rom)
  - there will always be some estimation error $\epsilon = \hat{\Theta} - \Theta$

What means “Estimation”?

- Examples:
  - Unknown amplitude of constant signal: $r(t) = A + \mathit{noise}$, estimate A
  - Unknown phase of sine signal: $r(t) = \cos(2\pi ft + \phi)$, estimate ϕ
  - Even complicated problems:
    - Record speech signal, estimate/decide what word is pronounced

Estimation vs Decision

- Consider the following estimation problem:

  We receive a signal $r(t) = A + \mathit{noise}$, estimate A

- For detection, we have to choose between **two known values** of A:
  - i.e. A can be 0 or 5 (hypotheses H_0 and H_1)
- For estimation, A can be anything $=>$ we choose between **infinite number of options** for A:
  - A might be any value in $\mathbb{R}$, in general

Estimation vs Decision

- Detection = Estimation constrained to **only a few** discrete options
- Estimation = Detection with an **infinite number** of options available
- The statistical methods used are quite similar
  - In practice, distinction between Estimation and Detections is somewhat blurred
  - (e.g. when choosing between 1000 hypotheses, do we call it “Detection” or “Estimation”?)

Available data

- The available data is the received signal $r(t) = s_\Theta(t) + noise$
  - it is affected by noise
  - it depends on the unknown parameter Θ
- We consider **N samples** from $r(t)$, taken at some sample times t_i

$$t = [t_1, t_2, \ldots t_N]$$

$$\mathbf{r} = [r_1, r_2, \ldots r_N] \Rightarrow \hat{\Theta}$$

- The samples depend on the value of Θ

Available data

- Each sample r_i is a random variable that depends on Θ (and the noise)
  - Each sample has a distribution that depends on Θ

$$w_i(r_i; \Theta)$$

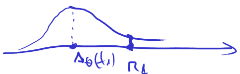
- The whole sample vector $\mathbf{r}$ is a N-dimensional random variable that depends on Θ (and the noise)
  - It has a N-dimensional distribution that depends on Θ

$$w(\mathbf{r}; \Theta)$$

  - Equal to the product of all $w_i(r_i|\Theta)$

$$\boxed{w(\mathbf{r}|\Theta) = w_1(r_1|\Theta) \cdot w_2(r_2|\Theta) \cdot \ldots \cdot w_N(r_N|\Theta)}$$

$t_1 \rightarrow s_\Theta(t_1)$



Two types of estimation

- We consider two types of estimation:
    1. **Maximum Likelihood Estimation (MLE)**: Besides $\mathbf{r}$, nothing else is known about the parameter Θ, except maybe some allowed range (e.g. $\Theta > 0$)
    2. **Bayesian Estimation**: Besides $\mathbf{r}$, we know a **prior** distribution $p(\Theta)$ for Θ, which tells us the values of Θ that are more likely than others

II.2 Maximum Likelihood estimation

Maximum Likelihood definition

- When no distribution is known except $\mathbf{r}$, we use a method known as **Maximum Likelihood estimation (MLE)**
- We define the **likelihood** of a parameter value Θ, given the available observations $\mathbf{r}$ as:

$$L(\Theta|\mathbf{r}) = w(\mathbf{r}|\Theta)$$

- $L(\Theta|\mathbf{r})$ is the likelihood function
- "The plausibility of a parameter value Θ given some measurements $\mathbf{r}$ = = the probability density of generating $\mathbf{r}$ if the true value would be Θ"
- Compare with formula in Chapter 2, slide 20
  - it is the same
  - here we try to "guess" Θ, there we "guessed" H_i

Maximum Likelihood definition

Maximum Likelihood (ML) Estimation:

- The estimate $\hat{\Theta}_{ML}$ is **the value that maximizes the likelihood, given the observed data r**
  - i.e. the value that maximizes $L(\Theta|\mathbf{r})$, i.e. maximize $w(\mathbf{r}|\Theta)$

$$\boxed{\hat{\Theta}_{ML} = \arg\max_{\Theta} L(\Theta|\mathbf{r}) = \arg\max_{\Theta} w(\mathbf{r}|\Theta)}$$

Find Θ such that this is max!

- If Θ is allowed to live only in a certain range, restrict the maximization only to that range.

Notations

- General mathematical notations:
  - $\arg\max_x f(x) =$ "the value x which maximizes the function f(x)"
  - $\max_x f(x) =$ "the maximum value of the function f(x)"

Maximum Likelihood estimation vs decision

- Very similar with decision problem!
- ML decision criterion:
  - "pick the hypothesis with a higher likelihood":

$$\frac{L(H_1|r)}{L(H_0|r)} = \frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

- ML estimation
  - "pick the value which maximizes the likelihood"

$$\hat{\Theta}_{ML} = \arg\max_{\Theta} L(\Theta|\mathbf{r}) = \arg\max_{\Theta} w(\mathbf{r}|\Theta)$$

How to solve

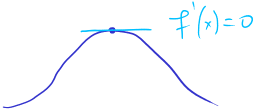
- How to solve the maximization problem?
  - i.e. how to find the estimate $\hat{\Theta}_{ML}$ which maximizes $L(\Theta|\mathbf{r})$
- Find maximum by setting derivative to 0

$$\frac{dL(\Theta|\mathbf{r})}{d\Theta} = 0$$

- We can also maximize the **natural logarithm** of the likelihood function ("log-likelihood function")

$$\frac{d\ln\left(L(\Theta)\right)}{d\Theta} = 0$$

Solving procedure



Solving procedure:

1. Find the function

$$L(\Theta|\mathbf{r}) = \boxed{w(\mathbf{r}|\Theta)}$$

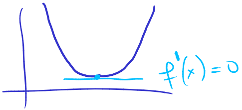
2. Set the condition that derivative of $L(\Theta|\mathbf{r})$ or $\ln((L(\Theta|\mathbf{r}))$ is 0

$$\frac{dL(\Theta|\mathbf{r})}{d\Theta} = 0, \text{ or } \frac{d\ln\left(L(\Theta)\right)}{d\Theta} = 0$$

3. Solve and find the value $\hat{\Theta}_{ML}$

4. Check that second derivative at point $\hat{\Theta}_{ML}$ is negative, to check that point is a maximum
   - because derivative $= 0$ for both maximum and minimum points
   - we'll sometimes skip this, for brevity

Examples:



- Estimating a constant signal in gaussian noise:

  Find the ML estimate of a constant value $s_\Theta(t) = A$ from 5 noisy measurements $r_i = A + noise$ with values $[5, 7, 8, 6.1, 5.3]$. The noise is AWGN $\mathcal{N}(\mu = 0, \sigma^2)$.

  $r_1 \quad r_2 \quad r_3 \quad r_4 \quad r_5$

- Solution: at whiteboard.

  $\hat{A}_{ML} = \frac{\sum r_i}{N}$

- The estimate $\hat{A}_{ML}$ is the average value of the samples
  - not surprisingly, what other value would have been more likely?
  - that's literally what "expected value" means

$$\boxed{\hat{A}_{ML} = \arg\min_A d(r, s_\Theta)^2}$$

$r = [5 \quad 7 \quad 8 \quad 6.1 \quad 5.3]$

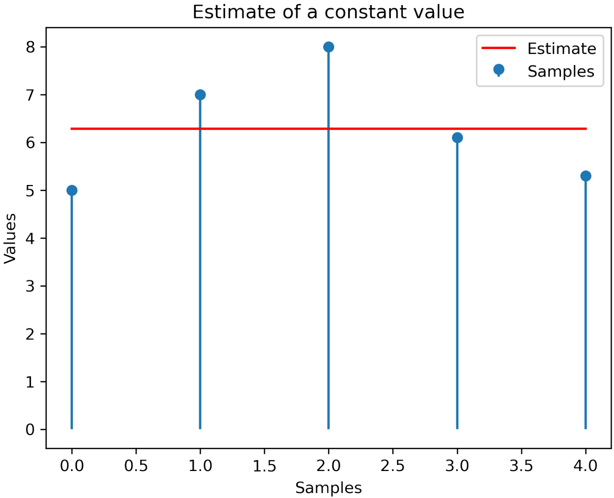
$s_\Theta = [A \quad A \quad A \quad A \quad A]$

$d(r, s_\Theta)^2 = (A-5)^2 + (A-7)^2 + \dots + (A-5.3)^2$

$\frac{d}{dA} d(r, s_\Theta)^2 = 0 \Leftrightarrow 2(A-5) + 2(A-7) + \dots + 2(A-5.3) = 0$

$\Rightarrow \hat{A}_{ML} = \frac{5+7+8+6.1+5.3}{5}$

Numerical simulation

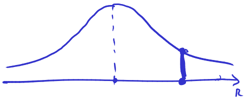


Curve fitting

- **Estimation = curve fitting**
  - we're finding the best fitting of $s_\Theta(t)$ through the data $\mathbf{r}$
- From the previous graphical example:
  - we have some data $\mathbf{r}$ = some points
  - we know the shape of the signal = a line (constant A)
  - we're fitting the best line through the data

General signal in AWGN

R_i



- Consider that the true underlying signal is $s_\Theta(t)$
- Consider **AWGN noise** $\mathcal{N}(\mu = 0, \sigma^2)$.
- The samples r_i are taken at sample moments t_i

$$w(r_i|\theta) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(r_i - s_\theta(t_i))^2}{2\sigma^2}}$$

$s_\theta(t_i)$ r_i

- The samples r_i have normal distribution with average value $\mu = s_\Theta(t_i)$ and variance σ^2
- Overall likelihood function = product of likelihoods for each sample r_i

a vector

$$L(\Theta|\mathbf{r}) = w(\mathbf{r}|\Theta) = \prod_{i=1}^{N} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r_i - s_\Theta(t_i))^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \cdot e^{-\frac{\sum_{i=1}^{N}(r_i - s_\Theta(t_i))^2}{2\sigma^2}}$$

Want max!

General signal in AWGN

- The log-likelihood is

$$\ln\left(L(\Theta|\mathbf{r})\right) = \underbrace{\underbrace{\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{N}}_{constant} - \frac{\sum (r_i - s_\Theta(t_i))^2}{2\sigma^2}}_{\text{Want max!}}$$

Want minimum!

General signal in AWGN

- The maximum of the function = the minimum of the exponent

$$\hat{\Theta}_{ML} = \arg\max_{\Theta} L(\Theta|\mathbf{r}) = \arg\min_{\Theta} \sum_{i=1}^{N} (r_i - s_\Theta(t_i))^2$$

- The term $\sum (r_i - s_\Theta(t_i))^2$ is the **squared distance** $d(\mathbf{r}, s_\Theta)$

$$d(\mathbf{r}, s_\Theta) = \sqrt{\sum (r_i - s_\Theta(t_i))^2}$$

$$(d(\mathbf{r}, s_\Theta))^2 = \sum (r_i - s_\Theta(t_i))^2$$

$r = \{r_1, r_2, \dots r_N]$

$s_\Theta = [s_\Theta(t_1), s_\Theta(t_2), \dots s_\Theta(t_N)]$

$d(r, s_\Theta)^2$

General signal in AWGN

When noise is Gaussian:

- ML estimation can be rewritten as:

$$\hat{\Theta}_{ML} = \arg\max_{\Theta} L(\Theta|\mathbf{r}) = \boxed{\arg\min_{\Theta} d(\mathbf{r}, \mathbf{s}_{\Theta})^2}$$

- ML estimate $\hat{\Theta}_{ML}$ = the value that makes $s_{\Theta}(t_i)$ **closest to the received values r**
  - closer = better fit = more likely
  - closest = best fit = most likely = maximum likelihood

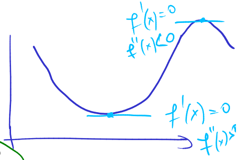
General signal in AWGN

ML decision : $d(r, s_0)^2 \gtrless d(r, s_1)^2$

- ML estimation in AWGN noise = **minimization of distance**
- Hey, we had the same interpretation with ML decision!
  - but for decision, we choose the minimum out of 2 options
  - here, we choose the minimum out of all possible options
- Same interpretation applies for all kinds of vector spaces
  - vectors with N elements, continous signals, etc
  - just change the definition of the distance function

General signal in AWGN

Procedure for ML estimation in AWGN noise:



1. Write the expression for the (squared) distance:

vectors

$$D = (d(\mathbf{r}, s_\Theta))^2 = \sum_{i=1}^{N} (r_i - s_\Theta(t_i))^2$$

2. We want it minimal, so set derivative to 0:

$$\frac{dD}{d\Theta} = \sum 2(r_i - s_\Theta(t_i))(-\frac{ds_\Theta(t_i)}{d\Theta}) = 0$$

3. Solve and find the value $\hat{\Theta}_{ML}$

4. Check that second derivative at point $\hat{\Theta}_{ML}$ is positive, to check that point is a minimum
   - we'll sometimes skip this, for brevity

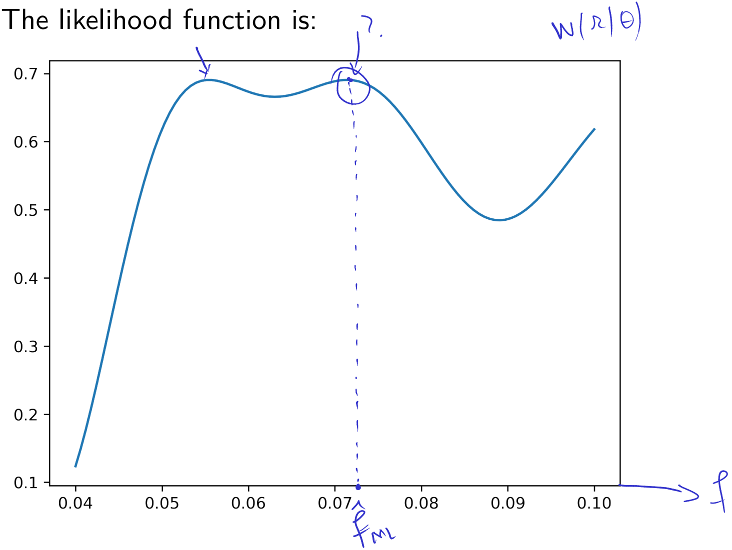
Numerical simulation

Estimating the frequency f of a cosine signal

- Find the Maximum Likelihood estimate of the frequency f of a cosine signal $s_{\Theta}(t) = cos(2\pi f t_i)$, from 10 noisy measurements $r_i = cos(2\pi f t_i) + noise$ with values [...]. The noise is AWGN $\mathcal{N}(\mu = 0, \sigma^2)$. The sample times $t_i = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]$
- Solution: at whiteboard.

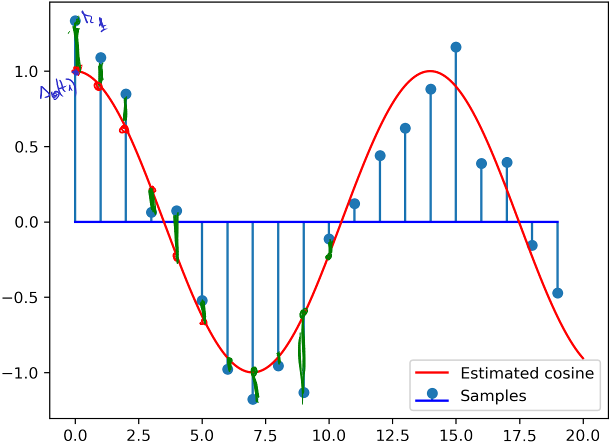
Numerical simulation

The likelihood function is:



Numerical simulation

Minimize $\sum_{i=1}^{N}$ (green segments)^2



Multiple parameters

- What if we have more than one parameter?
  - e.g. unknown parameters are the amplitude, frequency and the initial phase of a cosine:

$$\underset{s_\theta}{\cancel{s}}(t) = \overset{?}{A}\cos(2\pi \overset{?}{f}t + \overset{?}{\phi})$$

- We can consider the parameter Θ to be a vector:

$$\overset{\text{vector}}{\boldsymbol{\Theta}} = [\Theta_1, \Theta_2, ...\Theta_M]$$

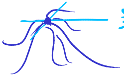
  - e.g. $\boldsymbol{\Theta} = [\Theta_1, \Theta_2, \Theta_3] = [A, f, \phi]$

Multiple parameters

$$\hat{A}_{ML}, \hat{f}_{ML}, \hat{\varphi}_{ML} = \arg\max_{A,f,\varphi} L(\theta|r) = \arg\max_{A,f,\varphi} \boxed{w(r|\theta)}$$

Gaussian: $\arg\min_{A,f,\varphi} d(r, s_\theta)^2$

- We solve with the same procedure, but instead of one derivative, we have M derivatives



$\frac{\partial f}{\partial \theta_i} = 0$

- We solve the system:

$$\begin{cases} \frac{\partial L}{\partial \Theta_1} = 0 \\ \frac{\partial L}{\partial \Theta_2} = 0 \\ \dots \\ \frac{\partial L}{\partial \Theta_M} = 0 \end{cases}$$

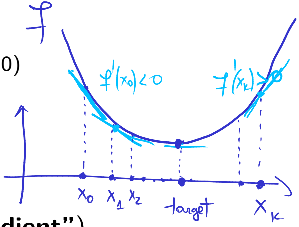
System of M equations, M unknowns => solve

  - sometimes difficult to solve

Gradient Descent

- How to estimate the parameters $\mathbf{\Theta}$ in complicated cases?
  - e.g. in real life applications
  - usually there are many parameters ($\mathbf{\Theta}$ is a vector)
- Typically it is impossible to get the optimal values directly by solving the system
- Improve them iteratively with **Gradient Descent** algorithm or its variations

Gradient Descent procedure



1. Start with some random parameter values $\boldsymbol{\Theta}^{(0)}$
2. Repeat for each iteration k:
   - 2.1 Compute function $L(\boldsymbol{\Theta}^{(k)}|\mathbf{r})$ $\quad d(r, s_\theta)^2$
   - 2.2 Compute derivatives $\frac{\partial L}{\partial \Theta_i^{(k)}}$ for each Θ_i (**"gradient"**)
   - 2.3 Update all values Θ_i by subtracting the derivative ("**descent**")

$$\Theta_i^{(k+1)} = \Theta_i^{(k)} - \mu \frac{\partial L}{\partial \Theta_i^{(k)}} \qquad X_{k+1} = X_k - \mu \cdot f'(x_k)$$

- or, in vector form:

$$\boldsymbol{\Theta}^{(k+1)} = \boldsymbol{\Theta}^{k} - \mu \frac{\partial L}{\partial \boldsymbol{\Theta}^{(k)}}$$

3. Until termination criterion (e.g. parameters don't change much)

Gradient Descent explained

- Explanations at blackboard
- Simple example: logistic regression on 2D-data
  - maybe do example at blackboard

Neural Networks

- The most prominent example is **Artificial Neural Networks** (a.k.a. Neural Networks, Deep Learning, etc.)
  - Can be regarded as ML estimation
  - Use Gradient Descent to update parameters
  - State-of-the-art applications: image classification/recognition, automated driving etc.
- More info on neural networks / machine learning:
  - look up online courses, books
  - join the IASI AI Meetup

Estimator bias and variance

- How good is an estimator?
- An estimator $\hat{\Theta}$ is a **random variable**
  - can have different values, because it is computed based on the received samples, which depend on noise
  - example: in lab, try on multiple computers => slightly different results
- As a random variable, it has:
  - an average value (expected value): $E\left\{\hat{\Theta}\right\}$
  - a variance: $\cancel{E\left\{(\hat{\Theta}-\Theta)^2\right\}}$

$$= E\left\{\left(\hat{\Theta} - E\{\hat{\Theta}\}\right)^2\right\}$$

individual $\hat{\Theta}$'s

average $\bar{\hat{\Theta}}$

Θ

Estimator bias and variance

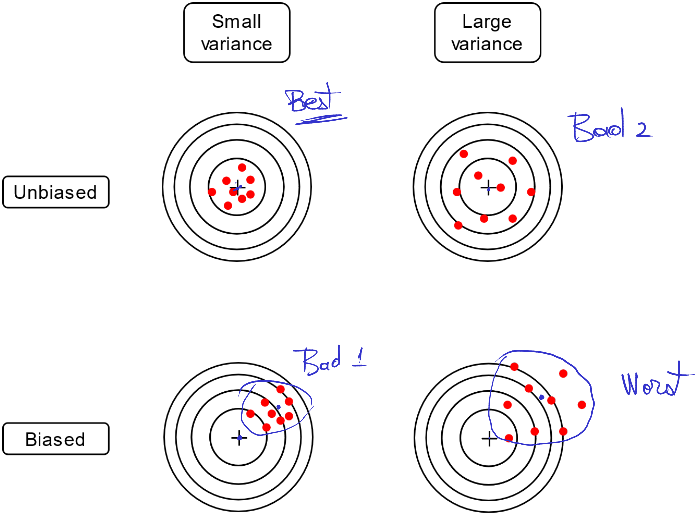


Figure 1: Estimator bias and variance

Estimator bias

- The **bias** of an estimator $\hat{\Theta}$ = difference between the estimator's average value and the true value

$$Bias = E\left\{\hat{\Theta}\right\} - \Theta$$

- Estimator is **unbiased** = the average value of the estimator is the true value of Θ

$$E\left\{\hat{\Theta}\right\} = \Theta$$

θ

- Estimator is **biased** = the average value of the estimator is different from the true value Θ
  - the difference $E\left\{\hat{\Theta}\right\} - \Theta$ is **the bias** of the estimator

θ

Estimator bias

- Example: for constant signal A with AWGN noise (zero-mean), ML estimator is $\boxed{\hat{A}_{ML} = \frac{1}{N}\sum_i r_i}$

$\mu = 0$

$$\boxed{\hat{A}_{ML} = \frac{1}{N} \cdot \sum r_i} \Rightarrow$$

$$E\{\hat{A}_{ML}\} = E\{\frac{1}{N} \cdot \sum r_i\}$$

$$\boxed{\begin{gathered} E\{x+y\} = E\{x\} + E\{y\} \\ E\{c \cdot x\} = c \cdot E\{x\} \end{gathered}}$$

- Then:

$$\begin{aligned} E\left\{\hat{A}_{ML}\right\} &= \frac{1}{N} E\left\{\sum_i r_i\right\} \\ &= \frac{1}{N}\sum_{i=1}^{N} E\left\{r_i\right\} \\ &= \frac{1}{N}\sum_{i=1}^{N} \underbrace{E\left\{A + noise\right\}}_{E\{A\} + E\{noise\}} \\ &= \frac{1}{N}\sum_{i=1}^{N} A \\ &= A \end{aligned}$$

A

$\mu = 0$

$$E\{\hat{A}_{ML}\} = A$$

Unbiased!

- This estimator in unbiased

Estimator variance

- The **variance** of an estimator measures the "spread" of the estimator around its average
  - that's the definition of variance
- Unbiased estimators are good, but if the **variance** of the estimator is large, then estimated values can be far from the true value
- We prefer estimators with **small variance**, even if maybe slightly biased

II.3 Bayesian estimation

Bayesian estimation

- **Bayesian estimation** considers extra factors alongside $w(\mathbf{r}|\Theta)$ in the estimation:
  - a prior distribution $w(\Theta)$
  - possibly some cost function
- This makes it the estimation version of the MPE and MR decision criteria

Bayesian estimation

- Conceptually, Bayesian estimation consists of two major steps:
  1. Finding the **posterior distribution** $w(\Theta|\mathbf{r})$
  2. Estimating a value from the distribution, based on a **cost function**

Bayesian estimation

- We define the **posterior** probability density of Θ, given the known observations $\mathbf{r}$, using the **Bayes rule**:

posterior distribution

$$\boxed{w(\Theta|\mathbf{r}) = \frac{w(\mathbf{r}|\Theta) \cdot w(\Theta)}{w(\mathbf{r})}}$$

constant

- Explanation of the terms:
  - Θ is the unknown parameter
  - $\mathbf{r}$ are the observations that we have
  - $w(\Theta|\mathbf{r})$ is the probability of a certain value Θ to be the correct one, given our current observations $\mathbf{r}$;
  - $w(\mathbf{r}|\Theta)$ is the likelihood function
  - $w(\Theta)$ is the **prior distribution** of Θ, i.e. what we know about Θ even in the absence of evidence
  - $w(\mathbf{r})$ is a scaling constant, which makes the integral of the resulting function be 1 (like for any distribution)

Bayesian estimation

have

- With MLE estimation, we only have the term $w(\mathbf{r}|\Theta)$. When viewed as a function of Θ, this is not a distribution of Θ. It's just something we want to maximize.

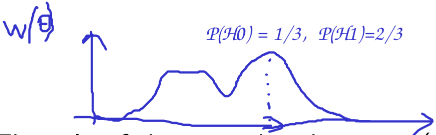
want

- Bayesian estimation, however, uses $w(\Theta|\mathbf{r})$, which **is** the actual probability distribution of the possible values of Θ

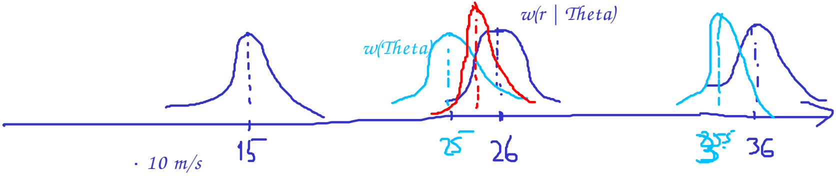
Bayes rule

- The Bayes rule shows that the probability of a value Θ depends on two things:
  1. The observations that we have, via the term $w(\mathbf{r}|\Theta)$
  2. The prior knowledge (or prior belief) about Θ, via the term $w(\Theta)$
  - (the third term $w(\mathbf{r})$ is considered a constant, and plays no role)
- Known as "Bayesian estimation"
  - Thomas Bayes = discovered the Bayes rule
  - Stuff related to Bayes rule are often named "Bayesian"

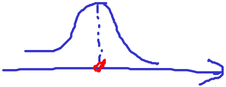
The prior distribution



- The role of the prior distribution $w(\Theta)$ is to express what we know beforehand about Θ
  - we know beforehand how likely it is to have a certain value
  - known as *a priori* distribution or *prior* distribution
- Bayesian estimation takes the prior information into account, alongside the measurements
  - the estimate will be slightly "moved" towards more likely values



The MAP estimator



- Suppose we know $w(\Theta|\mathbf{r})$. What is our estimate?
- Let's pick the value with the highest probability
- The **Maximum A Posteriori (MAP)** estimator:

$$\hat{\Theta}_{MAP} = \arg\max_{\Theta} w(\Theta|\mathbf{r}) = \arg\max_{\Theta} \left\{ w(\mathbf{r}|\Theta) \cdot w(\Theta) \right\}$$

- The MAP estimator chooses Θ as the value where the posterior distribution $w(\Theta|\mathbf{r})$ is maximum
- The MAP estimator maximizes the likelihood of the observed data **but multiplied with the prior distribution** $w(\Theta)$

The MAP estimator

Image example here

Relation with Maximum Likelihood Estimator

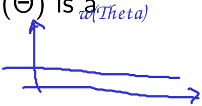
- The ML estimator:

$$\arg\max_{\Theta} \underline{w(\mathbf{r}|\Theta)}$$

- The MAP estimator:

$$\arg\max_{\Theta} \left\{ \underline{w(\mathbf{r}|\Theta)} \cdot \underline{w(\Theta)} \right\}$$

- The ML estimator is a particular case of MAP when $w(\Theta)$ is a constant
  - $w(\Theta) =$ constant means all values Θ are equally likely
  - i.e. we don't have a clue where the real Θ might be



Relation with Detection

k

- The MPE criterion $\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$
- It can be rewritten as $w(r|H_1) \cdot P(H_1) \underset{H_0}{\overset{H_1}{\gtrless}} w(r|H_0)P(H_0)$
  - i.e. choose the hypothesis where $w(r|H_i) \cdot P(H_i)$ is maximum
- **MPE decision criterion**: pick hypothesis which maximizes $w(r|H_i) \cdot P(H_i)$
  - out of the two possible hypotheses
- **The MAP estimator**: pick value which maximizes $w(\mathbf{r}|\Theta) \cdot w(\Theta)$
  - out of all possible values of Θ
- Same principle!

Cost function

C_{01}

- Let's find an equivalent for the Minimum Risk criterion. We need an equivalent for the costs C_{ij}
- The **estimation error** = the difference between the estimate $\hat{\Theta}$ and the true value Θ

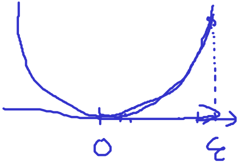
$$\epsilon = \hat{\Theta} - \Theta$$

- The **cost function** $C(\epsilon)$ = assigns a cost to each possible estimation error
  - when $\epsilon = 0$, the cost $C(0) = 0$
  - small errors ϵ have small costs
  - large errors ϵ have large costs

Cost function

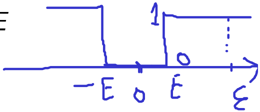
- Usual types of cost functions:
  - Quadratic:

$$C(\epsilon) \overset{A.}{=} \epsilon^2 = \left(\hat{\Theta} - \Theta\right)^2$$



  - Uniform ("hit or miss"):

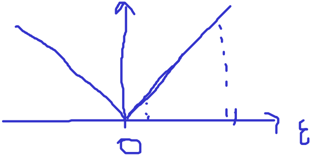
$$C(\epsilon) = \begin{cases} 0, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| \leq E \\ 1, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| > E \end{cases}$$



  - Linear:

$$C(\epsilon) = |\epsilon| = |\hat{\Theta} - \Theta|$$

- Draw them at whiteboard



Cost function

- The cost function $C(\epsilon)$ is the equivalent of the costs C_{ij} at detection
  - for detection we only had 4 costs: C_{00}, C_{01}, C_{10}, C_{11}
  - now we have a cost for all possible estimation errors ϵ
- The cost function guides which value to choose from $w(\Theta|\mathbf{r})$

The importance of the cost function

- Consider the following posterior distribution

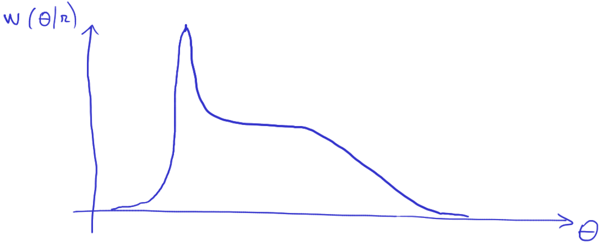


Figure 2: Unbalanced posterior distribution

- Which is the MAP estimate?
- Supposing we have the following cost function, does your estimate change ?:
  - if your estimate $\hat{\Theta}$ is $<$ then the real Θ, you pay 1000$
  - if your estimate $\hat{\Theta}$ is $>$ then the real Θ, you pay 1$

The importance of the cost function

- Choosing one particular value $\hat{\Theta}$ from the distribution of possible values is driven by the cost function
- The most probable value is not always the best
- The best value is the one which leads to the smallest average cost

The Bayesian risk

- The posterior distribution $w(\Theta|\mathbf{r})$ tells us the probability of a certain value $\hat{\Theta}$ to be the correct one of Θ
- Picking a certain estimate value $\hat{\Theta}$ implies a certain error ϵ
- The error implies a certain cost $C(\epsilon)$
- The **risk** $=$ the average cost $= C(\epsilon) \times$ the probability:

$$R = \int_{-\infty}^{\infty} C(\epsilon) w(\Theta|\mathbf{r}) d\Theta$$

The Bayes estimator

- We need to pick the value $\hat{\Theta}$ which **minimizes the expected cost** R

$$\hat{\Theta} = \arg\min_{\Theta} \int_{-\infty}^{\infty} C(\epsilon) w(\Theta|\mathbf{r}) d\Theta$$

- To find it, replace $C(\epsilon)$ with its definition and derivate over $\hat{\Theta}$
  - Attention: derivate with respect to $\hat{\Theta}$, not Θ!

MMSE estimator

- When the cost function is quadratic $C(\epsilon) = \epsilon^2 = \left(\hat{\Theta} - \Theta\right)^2$

$$R = \int_{-\infty}^{\infty} (\hat{\Theta} - \Theta)^2 w(\Theta|\mathbf{r}) d\Theta$$

- We want the $\hat{\Theta}$ that minimizes R, so we derivate

$$\frac{dR}{d\hat{\Theta}} = 2\int_{-\infty}^{\infty} (\hat{\Theta} - \Theta) w(\Theta|\mathbf{r}) d\Theta = 0$$

- Equivalent to

$$\hat{\Theta} \underbrace{\int_{-\infty}^{\infty} w(\Theta|\mathbf{r})}_{1} d\Theta = \int_{-\infty}^{\infty} \Theta w(\Theta|\mathbf{r}) d\Theta$$

- The **Minimum Mean Squared Error (MMSE)** estimator is

$$\hat{\Theta}_{MMSE} = \int_{-\infty}^{\infty} \Theta \cdot w(\Theta|\mathbf{r}) d\Theta$$

Interpretation

- **The MMSE estimator**: the estimator $\hat{\Theta}$ is the **average value** of the posterior distribution $w(\Theta|\mathbf{r})$

$$\hat{\Theta}_{MMSE} = \int_{-\infty}^{\infty} \Theta \cdot w(\Theta|\mathbf{r}) d\Theta$$

  - MMSE = "Minimum Mean Squared Error"
  - average value = sum (integral) of every Θ times its probability $w(\Theta|\mathbf{r})$
- The MMSE estimator is obtained from the posterior distribution $w(\Theta|\mathbf{r})$ considering the quadratic cost function

The MAP estimator

- When the cost function is uniform:

$$C(\epsilon) = \begin{cases} 0, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| \leq E \\ 1, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| > E \end{cases}$$

- Keep in mind that $\Theta = \hat{\Theta} - \epsilon$
- We obtain

$$\begin{aligned} I &= \int_{-\infty}^{\hat{\Theta}-E} w(\Theta|\mathbf{r}) d\Theta + \int_{Th\hat{e}ta+E}^{\infty} w(\Theta|\mathbf{r}) d\Theta \\ I &= 1 - \int_{\hat{\Theta}-E}^{\hat{\Theta}+E} w(\Theta|\mathbf{r}) d\Theta \end{aligned}$$

The MAP estimator

- To minimize C, we must maximize $\int_{\hat{\Theta}-E}^{\hat{\Theta}+E} w(\Theta|\mathbf{r}) d\Theta$, the integral around point $\hat{\Theta}$
- For E a very small, the function $w(\Theta|\mathbf{r})$ is approximately constant, so we pick the point where the function is maximum
- **The Maximum A Posteriori (MAP) estimator** = the value $\hat{\Theta}$ which maximizes $w(\Theta|\mathbf{r})$

$$\hat{\Theta}_{MAP} = \arg\max_{\Theta} w(\Theta|\mathbf{r}) = \arg\max \Theta w(\mathbf{r}|\Theta) \cdot w(\Theta)$$

Interpretation

- The MAP estimator chooses Θ as the value where the posterior distribution is maximum
- The MMSE estimator chooses Θ as average value of the posterior distribution

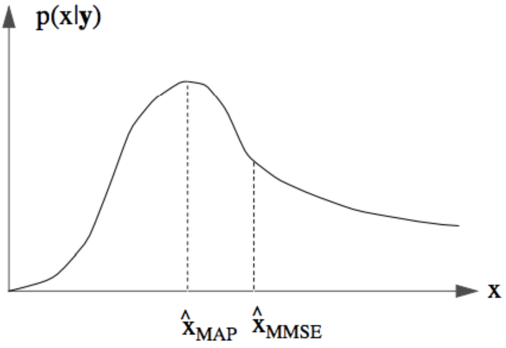


Figure 3: MAP vs MMSE estimators

Relationship between MAP and MMSE

- The MAP estimator = minimizing the average cost, using the uniform cost function
  - similar with the MPE decision criteria = MR when all costs are same
- The MMSE estimator = minimizing the average cost, using the quadratic cost function
  - similar to MR decision criteria, but more general

Exercise

Exercise: constant value, 3 measurement, Gaussian same σ

- We want to estimate today's temperature in Sahara
- Our thermometer reads 40 degrees, but the value was affected by Gaussian noise $\mathcal{N}(0, \sigma^2 = 2)$ (crappy thermometer)
- We know that this time of the year, the temperature is around 35 degrees, with a Gaussian distribution $\mathcal{N}(35, \sigma^2 = 2)$.
- Estimate the true temperature using ML, MAP and MMSE estimators

Exercise

Exercise: constant value, 3 measurements, Gaussian same σ

- What if he have three thermometers, showing 40, 38, 41 degrees

Exercise: constant value, 3 measurements, Gaussian different σ

- What if the temperature this time of the year has Gaussian distribution $\mathcal{N}(35, \sigma_2^2 = 3)$
  - different variance, $\sigma_2 \neq \sigma$

General signal in AWGN

- Consider that the true underlying signal is $s_\Theta(t)$
- Consider AWGN noise $\mathcal{N}(\mu = 0, \sigma^2)$.
- As in Maximum Likelihood function, overall likelihood function

$$w(\mathbf{r}|\Theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\sum (r_i - s_\Theta(t_i))^2}{2\sigma^2}}$$

- But now this function is also **multiplied with** $w(\Theta)$

$$w(\mathbf{r}|\Theta) \cdot w(\Theta)$$

General signal in AWGN

- MAP estimator is the argument that maximizes this product

$$\hat{\Theta}_{MAP} = \arg\max w(\mathbf{r}|\Theta)w(\Theta)$$

- Taking logarithm

$$\begin{aligned}\hat{\Theta}_{MAP} &= \arg\max \ln\left(w(\mathbf{r}|\Theta)\right) + \ln\left(w(\Theta)\right) \\ &= \arg\max -\frac{\sum(r_i - s_\Theta(t_i))^2}{2\sigma^2} + \ln\left(w(\Theta)\right)\end{aligned}$$

Gaussian prior

- If the prior distribution is also Gaussian $\mathcal{N}(\mu_\Theta, \sigma_\Theta^2)$

$$\ln\left(w(\Theta)\right) = -\frac{\sum(\Theta - \mu_\Theta)^2}{2\sigma_\Theta^2}$$

- MAP estimation becomes

$$\hat{\Theta}_{MAP} = \arg\min \frac{\sum(r_i - s_\Theta(t_i))^2}{2\sigma^2} + \frac{\sum(\Theta - \mu_\Theta)^2}{2\sigma_\Theta^2}$$

- Can be rewritten as

$$\hat{\Theta}_{MAP} = \arg\min d(\mathbf{r}, s_\Theta)^2 + \underbrace{\frac{\sigma^2}{\sigma_\Theta^2}}_{\lambda} \cdot d(\Theta, \mu_\Theta)^2$$

Interpretation

- MAP estimator with Gaussian noise and Gaussian prior

$$\hat{\Theta}_{MAP} = \arg\min d(\mathbf{r}, s_\Theta)^2 + \underbrace{\frac{\sigma^2}{\sigma_\Theta^2}}_{\lambda} \cdot d(\Theta, \mu_\Theta)^2$$

- $\hat{\Theta}_{MAP}$ is close to the expected value μ_Θ **and** it makes the true signal close to received data **r**
  - Example: "search for a house that is close to job and close to the Mall"
  - λ controls the relative importance of the two terms
- Particular cases
  - σ_Θ very small = the prior is very specific (narrow) = λ large = second term very important = $\hat{\Theta}_{MAP}$ close to μ_Θ
  - σ_Θ very large = the prior is very unspecific = λ small = first term very important = $\hat{\Theta}_{MAP}$ close to ML estimation

Applications

- In general, practical applications:
  - can use various prior distributions
  - estimate **multiple parameters** (a vector of parameters)
- Applications
  - denoising of signals
  - signal restoration
  - signal compression

Sample applications

1. Single object tracking with Kalman filtering

- estimating an object's position through successive noisy measurements (e.g. consecutive frames in a video)
- ata every new measurement, we have two distributions of the position:
  - one given by the measurement itself, $w(r|\Theta)$
  - one predicted based on position and speed from last moment
  - both are presumed Gaussian, described only through average value and variance
- the two are combined via the Bayes rule $=>$ a more precise distribution $w(\Theta|r)$, also Gaussian
- the exact position is estimated with MMSE (average value of $w(\Theta|r)$
- $w(\Theta|r) +$ speed is used to predict the position at the next time moment

Single object tracking

Single object tracking

Sample applications

2. Constrained Least Squares (CLS) image restoration

- We have an image I corrupted by noise (additive noise, missing pixels, blurring)

$$I_{noisy} = I_{true} + Z$$

- We can estimate the original image by solving:

$$\hat{I_{true}} = argmin_I \|I - I_{zg}\|_2 + \lambda \cdot \|HighPass\{I\}\|_2$$

- Examples:
  - https://www.mathworks.com/help/images/deblurring-images-using-a-regularized-filter.html
  - https://demonstrations.wolfram.com/ImageRestorationForDegradedImages
  - Google it

Constrained Least Squares (CLS) image restoration