Decision and Estimation in Data Processing

Chapter III. Elements of Estimation Theory

III.1 Introduction

What means “Estimation”?

- A sender transmits a signal $s_{\Theta}(t)$ which depends on an **unknown** parameter Θ
- The signal is affected by noise, we receive $r(t) = s_{\Theta}(t) + noise$
- We want to **find out** the correct value of the parameter
  - based on samples from the received signal, or the full continuous signal
  - available data is noisy => we “estimate” the parameter
- The found value is $\hat{\Theta}$, **the estimate** of Θ (“estimatul”, rom)
  - there will always be some estimation error $\epsilon = \hat{\Theta} - \Theta$

What means “Estimation”?

- Examples:
  - Unknown amplitude of constant signal: $r(t) = A + \mathit{noise}$, estimate A
  - Unknown phase of sine signal: $r(t) = \cos(2\pi ft + \phi)$, estimate ϕ
  - Even complicated problems:
    - Record speech signal, estimate/decide what word is pronounced

Estimation vs Decision

- Consider the following estimation problem:

  We receive a signal $r(t) = A + \textit{noise}$, estimate A

- For detection, we have to choose between **two known values** of A:
  - i.e. A can be 0 or 5 (hypotheses H_0 and H_1)
- For estimation, A can be anything $=>$ we choose between **infinite number of options** for A:
  - A might be any value in $\mathbb{R}$, in general

Estimation vs Decision

- Detection = Estimation constrained to **only a few** discrete options
- Estimation = Detection with an **infinite number** of options available
- The statistical methods used are quite similar
  - In practice, distinction between Estimation and Detections is somewhat blurred
  - (e.g. when choosing between 1000 hypotheses, do we call it "Detection" or "Estimation"?)

Available data

- The available data is the received signal $r(t) = s_{\Theta}(t) + \mathit{noise}$
  - it is affected by noise
  - it depends on the unknown parameter Θ
- We consider **N samples** from $r(t)$, taken at some sample times t_i

$$\mathbf{r} = [r_1, r_2, ... r_N]$$

- The samples depend on the value of Θ

Available data

- Each sample r_i is a random variable that depends on Θ (and the noise)
  - Each sample has a distribution that depends on Θ

$$w_i(r_i; \Theta)$$

- The whole sample vector $\mathbf{r}$ is a N-dimensional random variable that depends on Θ (and the noise)
  - It has a N-dimensional distribution that depends on Θ

$$w(\mathbf{r}; \Theta)$$

  - Equal to the product of all $w_i(r_i|\Theta)$

$$w(\mathbf{r}|\Theta) = w_1(r_1|\Theta) \cdot w_2(r_2|\Theta) \cdot ... \cdot w_N(r_N|\Theta)$$

Two types of estimation

- We consider two types of estimation:
    1. **Maximum Likelihood Estimation (MLE)**: Besides $\mathbf{r}$, nothing else is known about the parameter Θ, except maybe some allowed range (e.g. $\Theta > 0$)
    2. **Bayesian Estimation**: Besides $\mathbf{r}$, we know a **prior** distribution $p(\Theta)$ for Θ, which tells us the values of Θ that are more likely than others

II.2 Maximum Likelihood estimation

Maximum Likelihood definition

- When no distribution is known except $\mathbf{r}$, we use a method known as **Maximum Likelihood estimation (MLE)**
- We define the **likelihood** of a parameter value Θ, given the available observations $\mathbf{r}$ as:

$$L(\Theta|\mathbf{r}) = w(\mathbf{r}|\Theta)$$

- $L(\Theta|\mathbf{r})$ is the likelihood function
- "The plausibility of a parameter value Θ given some measurements $\mathbf{r}$ = = the probability density of generating $\mathbf{r}$ if the true value would be Θ"
- Compare with formula in Chapter 2, slide 20
  - it is the same
  - here we try to "guess" Θ, there we "guessed" H_i

Maximum Likelihood definition

Maximum Likelihood (ML) Estimation:

- The estimate $\hat{\Theta}_{ML}$ is **the value that maximizes the likelihood, given the observed data r**
  - i.e. the value that maximizes $L(\Theta|\mathbf{r})$, i.e. maximize $w(\mathbf{r}|\Theta)$

$$\hat{\Theta}_{ML} = \arg\max_{\Theta} L(\Theta|\mathbf{r}) = \arg\max_{\Theta} w(\mathbf{r}|\Theta)$$

- If Θ is allowed to live only in a certain range, restrict the maximization only to that range.

Notations

- General mathematical notations:
  - $\arg\max_x f(x) =$ "the value x which maximizes the function f(x)"
  - $\max_x f(x) =$ "the maximum value of the function f(x)"

Maximum Likelihood estimation vs decision

- Very similar with decision problem!
- ML decision criterion:
  - "pick the hypothesis with a higher likelihood":

$$\frac{L(H_1|r)}{L(H_0|r)} = \frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

- ML estimation
  - "pick the value which maximizes the likelihood"

$$\hat{\Theta}_{ML} = \arg\max_{\Theta} L(\Theta|\mathbf{r}) = \arg\max_{\Theta} w(\mathbf{r}|\Theta)$$

How to solve

- How to solve the maximization problem?
  - i.e. how to find the estimate $\hat{\Theta}_{ML}$ which maximizes $L(\Theta|\mathbf{r})$
- Find maximum by setting derivative to 0

$$\frac{dL(\Theta|\mathbf{r})}{d\Theta} = 0$$

- We can also maximize the **natural logarithm** of the likelihood function ("log-likelihood function")

$$\frac{d\ln\left(L(\Theta)\right)}{d\Theta} = 0$$

Solving procedure

Solving procedure:

1. Find the function
$$L(\Theta|\mathbf{r}) = w(\mathbf{r}|\Theta)$$

2. Set the condition that derivative of $L(\Theta|\mathbf{r})$ or $\ln((L(\Theta|\mathbf{r}))$ is 0
$$\frac{dL(\Theta|\mathbf{r})}{d\Theta} = 0, \text{ or } \frac{d\ln\left(L(\Theta)\right)}{d\Theta} = 0$$

3. Solve and find the value $\hat{\Theta}_{ML}$

4. Check that second derivative at point $\hat{\Theta}_{ML}$ is negative, to check that point is a maximum
   - because derivative $= 0$ for both maximum and minimum points
   - we'll sometimes skip this, for brevity

Examples:

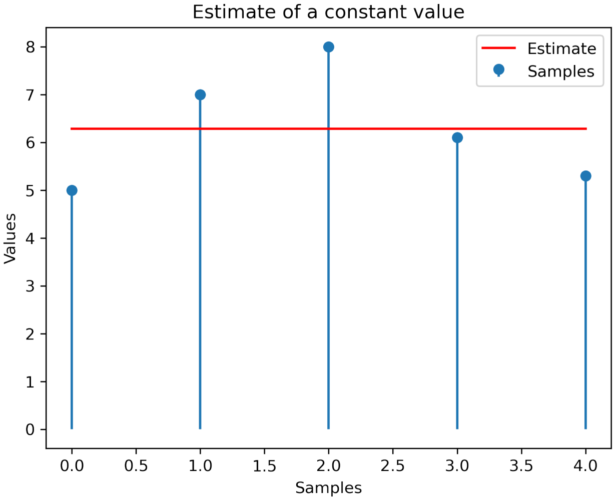
- Estimating a constant signal in gaussian noise:

  Find the ML estimate of a constant value $s_\Theta(t) = A$ from 5 noisy measurements $r_i = A + \textit{noise}$ with values $[5, 7, 8, 6.1, 5.3]$. The noise is AWGN $\mathcal{N}(\mu = 0, \sigma^2)$.

- Solution: at whiteboard.

- The estimate $\hat{A}_{ML}$ is the average value of the samples
  - not surprisingly, what other value would have been more likely?
  - that's literally what "expected value" means

Numerical simulation



Curve fitting

- **Estimation = curve fitting**
  - we're finding the best fitting of $s_\Theta(t)$ through the data $\mathbf{r}$
- From the previous graphical example:
  - we have some data $\mathbf{r}$ = some points
  - we know the shape of the signal = a line (constant A)
  - we're fitting the best line through the data

General signal in AWGN

- Consider that the true underlying signal is $s_\Theta(t)$
- Consider **AWGN noise** $\mathcal{N}(\mu = 0, \sigma^2)$.
- The samples r_i are taken at sample moments t_i
- The samples r_i have normal distribution with average value $\mu = s_\Theta(t_i)$ and variance σ^2
- Overall likelihood function = product of likelihoods for each sample r_i

$$
\begin{aligned}
L(\Theta|\mathbf{r}) = w(\mathbf{r}|\Theta) &= \prod_{i=1}^{N} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r_i - s_\Theta(t_i))^2}{2\sigma^2}} \\
&= \left(\frac{1}{\sigma\sqrt{2\pi}} \right)^N e^{-\frac{\sum (r_i - s_\Theta(t_i))^2}{2\sigma^2}}
\end{aligned}
$$

General signal in AWGN

- The log-likelihood is

$$\ln\left(L(\Theta|\mathbf{r})\right) = \underbrace{\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)}_{constant} - \frac{\sum(r_i - s_\Theta(t_i))^2}{2\sigma^2}$$

General signal in AWGN

- The maximum of the function = the minimum of the exponent

$$\hat{\Theta}_{ML} = \arg\max_{\Theta} L(\Theta|\mathbf{r}) = \arg\min \sum (r_i - s_\Theta(t_i))^2$$

- The term $\sum (r_i - s_\Theta(t_i))^2$ is the **squared distance** $d(\mathbf{r}, s_\Theta)$

$$d(\mathbf{r}, s_\Theta) = \sqrt{\sum (r_i - s_\Theta(t_i))^2}$$

$$(d(\mathbf{r}, s_\Theta))^2 = \sum (r_i - s_\Theta(t_i))^2$$

General signal in AWGN

- ML estimation can be rewritten as:

$$\hat{\Theta}_{ML} = \arg\max_{\Theta} L(\Theta|\mathbf{r}) = \arg\min_{\Theta} d(\mathbf{r}, \mathbf{s}_{\Theta})^2$$

- ML estimate $\hat{\Theta}_{ML}$ = the value that makes $s_{\Theta}(t_i)$ **closest to the received values r**
  - closer = beter fit = more likely
  - closest = best fit = most likely = maximum likelihood

General signal in AWGN

- ML estimation in AWGN noise = **minimization of distance**
- Hey, we had the same interpretation with ML decision!
  - but for decision, we choose the minimum out of 2 options
  - here, we choose the minimum out of all possible options
- Same interpretation applies for all kinds of vector spaces
  - vectors with N elements, continous signals, etc
  - just change the definition of the distance function

General signal in AWGN

Procedure for ML estimation in AWGN noise:

1. Write the expression for the (squared) distance:

$$D = (d(\mathbf{r}, s_\Theta))^2 = \sum (r_i - s_\Theta(t_i))^2$$

2. We want it minimal, so set derivative to 0:

$$\frac{dD}{d\Theta} = \sum 2(r_i - s_\Theta(t_i))(-\frac{ds_\Theta(t_i)}{d\Theta}) = 0$$

3. Solve and find the value $\hat{\Theta}_{ML}$
4. Check that second derivative at point $\hat{\Theta}_{ML}$ is positive, to check that point is a minimum
    - we'll sometimes skip this, for brevity

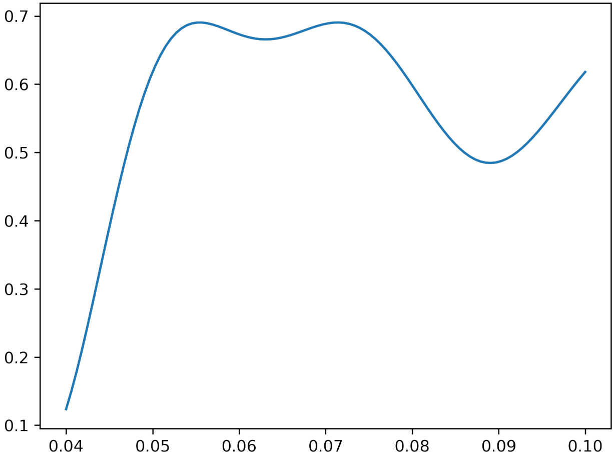
Numerical simulation

Estimating the frequency f of a cosine signal

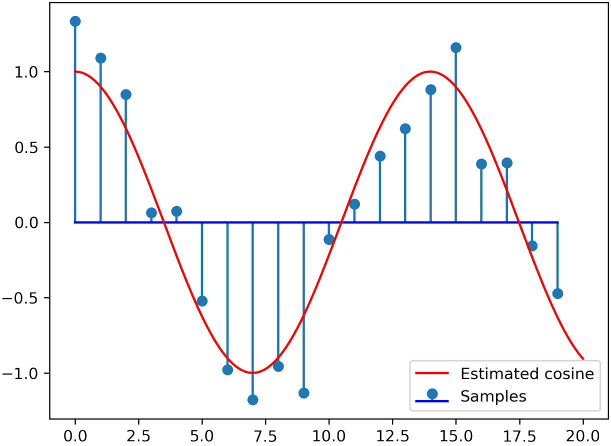
- Find the Maximum Likelihood estimate of the frequency f of a cosine signal $s_\Theta(t) = cos(2\pi f t_i)$, from 10 noisy measurements $r_i = cos(2\pi f t_i) + noise$ with values [...]. The noise is AWGN $\mathcal{N}(\mu = 0, \sigma^2)$. The sample times $t_i = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]$
- Solution: at whiteboard.

Numerical simulation

The likelihood function is:



Numerical simulation



Estimating parameters for certain distributions

- ML estimation can be used to estimate parameters of distributions
- Suppose we have a set of values r_i, which we model as samples coming from a distribution. How do we find the parameters of that distribution?
- For now, we consider only one unknown parameter

Estimating the parameters of the normal distribution

- Assume r_i are samples coming from a normal distribution $\mathcal{N}(\mu, \sigma^2)$
- The distribution has two parameters: mean μ and standard deviation σ (or variance σ^2)
- Estimating μ:

  Just like estimating a constant signal in AWGN noise of mean 0:

$$\hat{\mu}_{ML} = \frac{1}{N}\sum_{i=1}^{N} r_i$$

- Estimating σ^2:

  Can't be formulated as estimating a signal added with AWGN noise, but can still use ML:

$$\hat{\sigma}_{ML} = \arg\max_{\sigma} w(\mathbf{r}|\sigma)$$

Estimating the parameters of the normal distribution

$$
\begin{aligned}
\hat{\sigma}_{ML} &= \arg\max_{\sigma} w(\mathbf{r}|\sigma) \\
&= \arg\max_{\sigma} \left(\frac{1}{\sigma\sqrt{2\pi}} \right)^N e^{-\frac{\sum (r_i-\mu)^2}{2\sigma^2}} \quad \text{(apply ln())} \\
&= \arg\max_{\sigma} \left(-N \ln(\sigma\sqrt{2\pi}) - \frac{\sum (r_i-\mu)^2}{2\sigma^2} \right)
\end{aligned}
$$

Derivate and set to 0 to obtain the minimum:

$$
\begin{aligned}
-N \frac{1}{\sigma\sqrt{2\pi}} \sqrt{2\pi} - \frac{\sum (r_i-\mu)^2}{2} (-2)\sigma^{-3} &= 0 \\
-\frac{N}{\sigma} + \frac{\sum (r_i-\mu)^2}{\sigma^3} &= 0 \\
\sigma^2 &= \frac{\sum (r_i-\mu)^2}{N}
\end{aligned}
$$

Estimating the parameters of the normal distribution

- Estimated parameters of the normal distribution are identical to the definitions of the mean and variance:

$$\hat{\mu}_{ML} = \frac{1}{N}\sum_{i=1}^{N} r_i$$

$$\hat{\sigma}_{ML} = \sqrt{\frac{\sum_{i=1}^{N}(r_i - \mu)^2}{N}}$$

- Note: estimating σ requires the value of μ
  - If μ is known, everything is fine
  - If μ is unknown, we can use $\hat{\mu}_{ML}$ beforehand, but then we're estimating based on another estimate, which is problematic (the estimator is biased, we'll see)

Estimating the parameters of the uniform distribution

- Assume r_i are samples from a uniform distribution $\mathcal{U}[a, b]$
- The distribution has two parameters: the limits a și b
- Estimating a and b:

$$\hat{a}_{ML} = \arg\max_a w(\mathbf{r}|a)$$
$$\hat{b}_{ML} = \arg\max_b w(\mathbf{r}|b)$$

  Which means:

$$\hat{a}_{ML} = \min(r_i)$$
$$\hat{b}_{ML} = \max(r_i)$$

- The interval must contain all the values r_i (otherwise, the likelihood would be 0) but should not extend more than necessary (otherwise, the probability would decrease)

Multiple parameters

- What if we have more than one parameter?
  - e.g. unknown parameters are the amplitude, frequency and the initial phase of a cosine:

$$s_{\hat{}}(t) = A\cos(2\pi f t + \phi)$$

- We can consider the parameter Θ to be a vector:

$$\mathbf{\Theta} = [\Theta_1, \Theta_2, ...\Theta_M]$$

  - e.g. $\mathbf{\Theta} = [\Theta_1, \Theta_2, \Theta_3] = [A, f, \phi]$

Multiple parameters

- We solve with the same procedure, but instead of one derivative, we have M derivatives
- We solve the system:

$$\begin{cases} \frac{\partial L}{\partial \Theta_1} = 0 \\ \frac{\partial L}{\partial \Theta_2} = 0 \\ \dots \\ \frac{\partial L}{\partial \Theta_M} = 0 \end{cases}$$

  - sometimes difficult to solve

Gradient Descent

- How to estimate the parameters $\mathbf{\Theta}$ in complicated cases?
  - e.g. in real life applications
  - usually there are many parameters ($\mathbf{\Theta}$ is a vector)
- Typically it is impossible to get the optimal values directly by solving the system
- Improve them iteratively with **Gradient Descent** algorithm or its variations
- Gradient Descent is a general method of finding the minimum or maximum of a function

Coborâre după gradient (Gradient Descent)

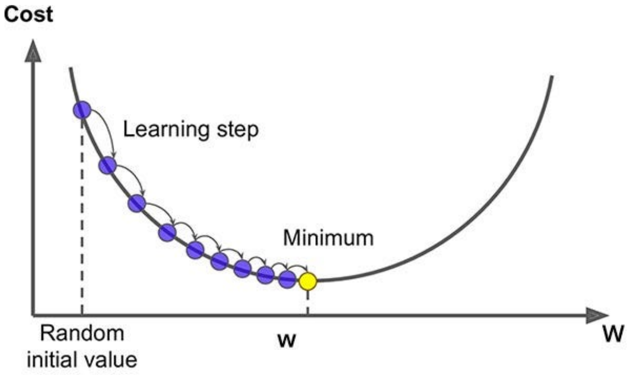


Figure 1: Coborâre după gradient[1]

[1]Imagine: Quick Guide to Gradient Descent and Its Variants, Sahdev Kansal, Towards Data Science, 2020

Gradient Descent procedure

1. Start with some random parameter values $\mathbf{\Theta}^{(0)}$
2. Repeat for each iteration k:
   2.1 Compute function $L(\mathbf{\Theta}^{(k)}|\mathbf{r})$
   2.2 Compute derivatives $\frac{\partial L}{\partial \Theta_i^{(k)}}$ for each Θ_i (**"gradient"**)
   2.3 Update all values Θ_i by subtracting the derivative ("**descent**")
   $$\Theta_i^{(k+1)} = \Theta_i^{(k)} - \mu \frac{\partial L}{\partial \Theta_i^{(k)}}$$
   - or, in vector form:
   $$\mathbf{\Theta}^{(k+1)} = \mathbf{\Theta}^{k} - \mu \frac{\partial L}{\partial \mathbf{\Theta}^{(k)}}$$
3. Until termination criterion (e.g. parameters don't change much)

Gradient Descent explained

- The derivative always tells in which direction to advance
- To find the minimum of a function, subtract the derivative (gradient descent)

$$\Theta^{(k+1)} = \Theta^{(k)} - \mu \frac{\partial L}{\partial \Theta^{(k)}}$$

- TO find the maximum of a function, add the derivative (gradient ascent)

$$\Theta^{(k+1)} = \Theta^{(k)} + \mu \frac{\partial L}{\partial \Theta^{(k)}}$$

- The parameter μ is the **learning rate** and is empirically chosen, has a small value
- GD is sensitive to initial starting point, and can get trapped in local minima
- Other explanations: at whiteboard
- Exemplu practic: regresia logistică cu valori 2D

Neural Networks

- The most prominent example is **Artificial Neural Networks** (a.k.a. Neural Networks, Deep Learning, etc.)
  - Can be regarded as ML estimation
  - Use Gradient Descent to update parameters
  - State-of-the-art applications: image classification/recognition, automated driving etc.
- More info on neural networks / machine learning:
  - look up online courses, books
  - join the IASI AI Meetup

Estimator bias and variance

- How good is an estimator?
- An estimator $\hat{\Theta}$ is a **random variable**
  - can have different values, because it is computed based on the received samples, which depend on noise
  - example: in lab, try on multiple computers $=>$ slightly different results
- As a random variable, it has:
  - an average value (expected value): $E\left\{\hat{\Theta}\right\}$
  - a variance: $E\left\{(\hat{\Theta}-\Theta)^2\right\}$

Estimator bias and variance

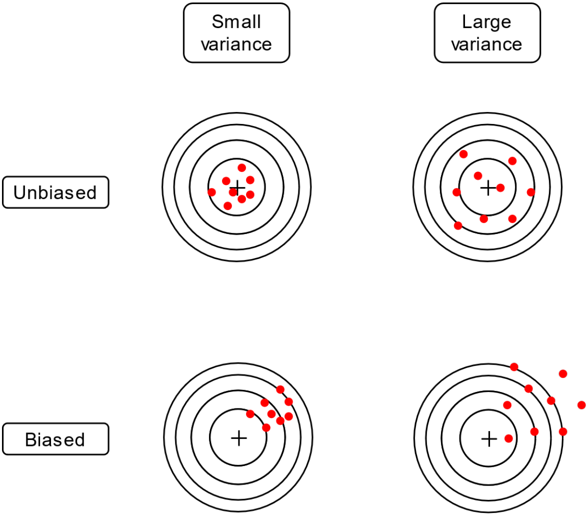


Figure 2: Estimator bias and variance

Estimator bias

- The **bias** of an estimator $\hat{\Theta}$ = difference between the estimator's average value and the true value

$$Bias = E\left\{\hat{\Theta}\right\} - \Theta$$

- Estimator is **unbiased** = the average value of the estimator is the true value of Θ

$$E\left\{\hat{\Theta}\right\} = \Theta$$

- Estimator is **biased** = the average value of the estimator is different from the true value Θ
  - the difference $E\left\{\hat{\Theta}\right\} - \Theta$ is **the bias** of the estimator

Estimator bias

- Example: for constant signal A with AWGN noise (zero-mean), ML estimator is $\hat{A}_{ML} = \frac{1}{N}\sum_i r_i$
- Then:

$$
\begin{aligned}
E\left\{\hat{A}_{ML}\right\} &= \frac{1}{N} E\left\{\sum_i r_i\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} E\left\{r_i\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} E\left\{A + noise\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} A \\
&= A
\end{aligned}
$$

- This estimator in unbiased

Estimator bias

- Example: the estimator of the variance of a normal distribution, using the estimated mean $\hat{\mu}_{ML}$:

$$\hat{\sigma}^2_{ML} = \frac{\sum_{i=1}^{N}(r_i - \hat{\mu}_{ML})^2}{N}$$

- This estimator is **biased**:

$$\begin{aligned} E\left\{\hat{\sigma}^2_{ML}\right\} &= E\left\{\frac{\sum_{i=1}^{N}(r_i - \hat{\mu}_{ML})^2}{N}\right\} \\ &= \dots \\ &= \frac{N-1}{N}\sigma^2 \end{aligned}$$

  where σ^2 is the real variance of the distribution

- Demonstrație: *Wikipedia* or "*Maximum Likelihood Estimator for Variance is Biased: Proof*", Dawen Liang, Carnegie Mellon University

The unbiased variance estimator

- The ML estimator of variance is biased, as it **underestimates** the real variance of the distribution by a factor (N-1)/N
- To obtain an unbiased estimator of the variance, we use:

$$\hat{\sigma}^2_{ML} = \frac{1}{N-1}\sum_{i=1}^{N}(r_i - \hat{\mu}_{ML})^2$$

- The difference: divide to $N-1$ instead of N
- Intuition: at whiteboard, with 2 point only; medie este la mijloc; the average is in middle, variance is minimized, so it underestimates the real one

Estimator variance

- The **variance** of an estimator measures the "spread" of the estimator around its average
  - that's the definition of variance
- Unbiased estimators are good, but if the **variance** of the estimator is large, then estimated values can be far from the true value
- We prefer estimators with **small variance**, even if maybe slightly biased

II.3 Bayesian estimation

Bayesian estimation

- **Bayesian estimation** considers extra factors alongside $w(\mathbf{r}|\Theta)$ in the estimation:
  - a prior distribution $w(\Theta)$
  - possibly some cost function
- This makes it the estimation version of the MPE and MR decision criteria

Bayesian estimation

- Conceptually, Bayesian estimation consists of two major steps:
  1. Finding the **posterior distribution** $w(\Theta|\mathbf{r})$
  2. Estimating a value from the distribution, based on a **cost function**

Bayesian estimation

- We define the **posterior** probability density of Θ, given the known observations $\mathbf{r}$, using the **Bayes rule**:

$$w(\Theta|\mathbf{r}) = \frac{w(\mathbf{r}|\Theta) \cdot w(\Theta)}{w(\mathbf{r})}$$

- Explanation of the terms:
  - Θ is the unknown parameter
  - $\mathbf{r}$ are the observations that we have
  - $w(\Theta|\mathbf{r})$ is the probability of a certain value Θ to be the correct one, given our current observations $\mathbf{r}$;
  - $w(\mathbf{r}|\Theta)$ is the likelihood function
  - $w(\Theta)$ is the **prior distribution** of Θ, i.e. what we know about Θ even in the absence of evidence
  - $w(\mathbf{r})$ is a scaling constant, which makes the integral of the resulting function be 1 (like for any distribution)

Bayesian estimation

- With MLE estimation, we only have the term $w(\mathbf{r}|\Theta)$. When viewed as a function of Θ, this is not a distribution of Θ. It's just something we want to maximize.
- Bayesian estimation, however, uses $w(\Theta|\mathbf{r})$, which **is** the actual probability distribution of the possible values of Θ

Bayes rule

- The Bayes rule shows that the probability of a value Θ depends on two things:
  1. The observations that we have, via the term $w(\mathbf{r}|\Theta)$
  2. The prior knowledge (or prior belief) about Θ, via the term $w(\Theta)$

  - (the third term $w(\mathbf{r})$ is considered a constant, and plays no role)
- Known as "Bayesian estimation"
  - Thomas Bayes $=$ discovered the Bayes rule
  - Stuff related to Bayes rule are often named "Bayesian"

The prior distribution

- The role of the prior distribution $w(\Theta)$ is to express what we know beforehand about Θ
  - we know beforehand how likely it is to have a certain value
  - known as *a priori* distribution or *prior* distribution
- Bayesian estimation takes the prior information into account, alongside the measurements
  - the estimate will be slightly “moved” towards more likely values

The MAP estimator

- Suppose we know $w(\Theta|\mathbf{r})$. What is our estimate?
- Let's pick the value with the highest probability
- The **Maximum A Posteriori (MAP)** estimator:

$$\hat{\Theta}_{MAP} = \arg\max_{\Theta} w(\Theta|\mathbf{r}) = \arg\max_{\Theta} \{w(\mathbf{r}|\Theta) \cdot w(\Theta)\}$$

- The MAP estimator chooses Θ as the value where the posterior distribution $w(\Theta|\mathbf{r})$ is maximum
- The MAP estimator maximizes the likelihood of the observed data **but multiplied with the prior distribution** $w(\Theta)$

The MAP estimator

Image example here

Relation with Maximum Likelihood Estimator

- The ML estimator:

$$\arg\max w(\mathbf{r}|\Theta)$$

- The MAP estimator:

$$\arg\max\{w(\mathbf{r}|\Theta) \cdot w(\Theta)\}$$

- The ML estimator is a particular case of MAP when $w(\Theta)$ is a constant
  - $w(\Theta) = \text{constant}$ means all values Θ are equally likely
  - i.e. we don't have a clue where the real Θ might be

Relation with Detection

- The MPE criterion $\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$
- It can be rewritten as $w(r|H_1) \cdot P(H_1) \underset{H_0}{\overset{H_1}{\gtrless}} w(r|H_0)P(H_0)$
  - i.e. choose the hypothesis where $w(r|H_i) \cdot P(H_i)$ is maximum
- **MPE decision criterion**: pick hypothesis which maximizes $w(r|H_i) \cdot P(H_i)$
  - out of the two possible hypotheses
- **The MAP estimator**: pick value which maximizes $w(\mathbf{r}|\Theta) \cdot w(\Theta)$
  - out of all possible values of Θ
- Same principle!

Cost function

- Let's find an equivalent for the Minimum Risk criterion. We need an equivalent for the costs C_{ij}
- The **estimation error** = the difference between the estimate $\hat{\Theta}$ and the true value Θ
$$\epsilon = \hat{\Theta} - \Theta$$
- The **cost function** $C(\epsilon)$ = assigns a cost to each possible estimation error
  - when $\epsilon = 0$, the cost $C(0) = 0$
  - small errors ϵ have small costs
  - large errors ϵ have large costs

Cost function

- Usual types of cost functions:
  - Quadratic:
$$C(\epsilon) = \epsilon^2 = \left(\hat{\Theta} - \Theta\right)^2$$
  - Uniform ("hit or miss"):
$$C(\epsilon) = \begin{cases} 0, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| \leq E \\ 1, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| > E \end{cases}$$
  - Linear:
$$C(\epsilon) = |\epsilon| = |\hat{\Theta} - \Theta|$$
- Draw them at whiteboard

Cost function

- The cost function $C(\epsilon)$ is the equivalent of the costs C_{ij} at detection
  - for detection we only had 4 costs: C_{00}, C_{01}, C_{10}, C_{11}
  - now we have a cost for all possible estimation errors ϵ
- The cost function guides which value to choose from $w(\Theta|\mathbf{r})$

The importance of the cost function

- Consider the following posterior distribution

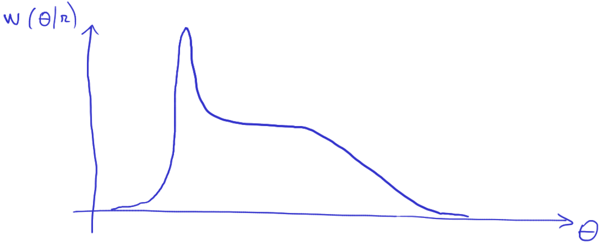


Figure 3: Unbalanced posterior distribution

- Which is the MAP estimate?
- Supposing we have the following cost function, does your estimate change ?:
  - if your estimate $\hat{\Theta}$ is $<$ then the real Θ, you pay 1000$
  - if your estimate $\hat{\Theta}$ is $>$ then the real Θ, you pay 1$

The importance of the cost function

- Choosing one particular value $\hat{\Theta}$ from the distribution of possible values is driven by the cost function
- The most probable value is not always the best
- The best value is the one which leads to the smallest average cost

The Bayesian risk

- The posterior distribution $w(\Theta|\mathbf{r})$ tells us the probability of a certain value $\hat{\Theta}$ to be the correct one of Θ
- Picking a certain estimate value $\hat{\Theta}$ implies a certain error ϵ
- The error implies a certain cost $C(\epsilon)$
- The **risk** $=$ the average cost $= C(\epsilon) \times$ the probability:

$$R = \int_{-\infty}^{\infty} C(\epsilon) w(\Theta|\mathbf{r}) d\Theta$$

The Bayes estimator

- We need to pick the value $\hat{\Theta}$ which **minimizes the expected cost** R

$$\hat{\Theta} = \arg\min_{\Theta} \int_{-\infty}^{\infty} C(\epsilon) w(\Theta|\mathbf{r}) d\Theta$$

- To find it, replace $C(\epsilon)$ with its definition and derivate over $\hat{\Theta}$
  - Attention: derivate with respect to $\hat{\Theta}$, not Θ!

MMSE estimator

- When the cost function is quadratic $C(\epsilon) = \epsilon^2 = \left(\hat{\Theta} - \Theta\right)^2$

$$R = \int_{-\infty}^{\infty} (\hat{\Theta} - \Theta)^2 w(\Theta|\mathbf{r}) d\Theta$$

- We want the $\hat{\Theta}$ that minimizes R, so we derivate

$$\frac{dR}{d\hat{\Theta}} = 2 \int_{-\infty}^{\infty} (\hat{\Theta} - \Theta) w(\Theta|\mathbf{r}) d\Theta = 0$$

- Equivalent to

$$\hat{\Theta} \underbrace{\int_{-\infty}^{\infty} w(\Theta|\mathbf{r})}_{1} d\Theta = \int_{-\infty}^{\infty} \Theta w(\Theta|\mathbf{r}) d\Theta$$

- The **Minimum Mean Squared Error (MMSE)** estimator is

$$\hat{\Theta}_{MMSE} = \int_{-\infty}^{\infty} \Theta \cdot w(\Theta|\mathbf{r}) d\Theta$$

Interpretation

- **The MMSE estimator**: the estimator $\hat{\Theta}$ is the **average value** of the posterior distribution $w(\Theta|\mathbf{r})$

$$\hat{\Theta}_{MMSE} = \int_{-\infty}^{\infty} \Theta \cdot w(\Theta|\mathbf{r}) d\Theta$$

  - MMSE = "Minimum Mean Squared Error"
  - average value = sum (integral) of every Θ times its probability $w(\Theta|\mathbf{r})$
- The MMSE estimator is obtained from the posterior distribution $w(\Theta|\mathbf{r})$ considering the quadratic cost function

The MAP estimator

- When the cost function is uniform:

$$C(\epsilon) = \begin{cases} 0, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| \leq E \\ 1, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| > E \end{cases}$$

- Keep in mind that $\Theta = \hat{\Theta} - \epsilon$
- We obtain

$$\begin{aligned} I &= \int_{-\infty}^{\hat{\Theta}-E} w(\Theta|\mathbf{r}) d\Theta + \int_{T\hat{h}eta+E}^{\infty} w(\Theta|\mathbf{r}) d\Theta \\ I &= 1 - \int_{\hat{\Theta}-E}^{\hat{\Theta}+E} w(\Theta|\mathbf{r}) d\Theta \end{aligned}$$

The MAP estimator

- To minimize C, we must maximize $\int_{\hat{\Theta}-E}^{\hat{\Theta}+E} w(\Theta|\mathbf{r})d\Theta$, the integral around point $\hat{\Theta}$
- For E a very small, the function $w(\Theta|\mathbf{r})$ is approximately constant, so we pick the point where the function is maximum
- **The Maximum A Posteriori (MAP) estimator** = the value $\hat{\Theta}$ which maximizes $w(\Theta|\mathbf{r})$

$$\hat{\Theta}_{MAP} = \arg\max_{\Theta} w(\Theta|\mathbf{r}) = \arg\max \Theta w(\mathbf{r}|\Theta) \cdot w(\Theta)$$

Interpretation

- The MAP estimator chooses Θ as the value where the posterior distribution is maximum
- The MMSE estimator chooses Θ as average value of the posterior distribution

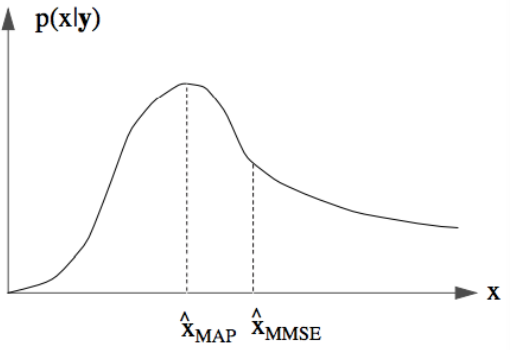


Figure 4: MAP vs MMSE estimators

Relationship between MAP and MMSE

- The MAP estimator = minimizing the average cost, using the uniform cost function
  - similar with the MPE decision criteria = MR when all costs are same
- The MMSE estimator = minimizing the average cost, using the quadratic cost function
  - similar to MR decision criteria, but more general

Sample applications

1. Kalman filter: estimate position of moving object

- Initial position and speed are known
- Take successive noisy measurements of the objects' position
- Estimate objects' position after every new measurement

For every new measurement, we have two distributions:

- from the measurement $w(r|\Theta)$ ("likelihood")
- the **prediction**, based on previous position and speed, $w(\Theta)$ ("prior")
- both assumed Gaussian, (characterized only by mean and variance)

Combine them with Bayes rule => a more precise distribution $w(\Theta|r)$, also Gaussian ("posterior")

- use MMSE to estimate the exact position (mean of $w(\Theta|r)$)
- $w(\Theta|r)$ is used to predict the next position

Kalman filter for estimating position of a moving object

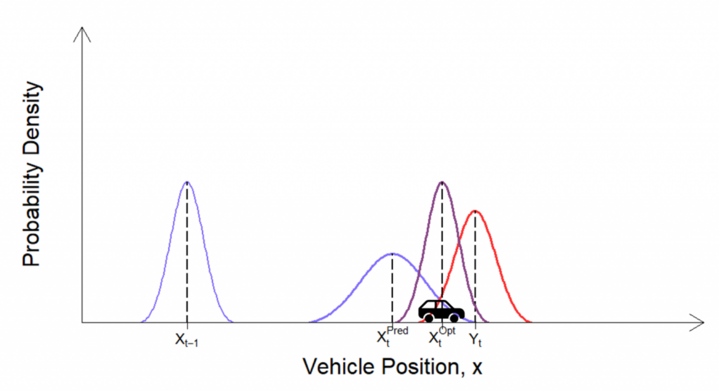


Figure 5: Position estimation with Kalman filter [2]

[2] Image source: https://www.lancaster.ac.uk/stor-i-student-sites/jack-trainer/how-nasa-used-the-kalman-filter-in-the-apollo-program/

Kalman filter for estimating position of a moving object

- Speed must be known, in order to predict next position of object
- In the previous example, speed is known (constant value, or known distribution)
- In general, Kalman filter also estimate **speed** of a moving object, based only on position measurements

Kalman filter for estimating position of a moving object

Kalman filter for joint position and speed estimation (moving along one axis only)

- Both the position and speed are estimated, i.e. the objects' **state**: $\mathbf{s} = \begin{bmatrix} x \\ v \end{bmatrix}$
- Work with 2D Gaussian distributions (position, speed) (movement along one axis only)

Steps:

1. At step k, we know the distribution of state s^k
2. Predict the distribution of the state at step $k+1$ ("prior distribution")
3. Take a measurement of position z^{k+1} ("likelihood")
4. Compute new distribution of the state ("posterior distibution") using the Bayes rule, by multiplication of the prior and likelihood
5. The mean of this distribution is the state estimate (position and speed) at step $k+1$ (MMSE estimation)

Kalman filter for estimating position of a moving object

Illustrations:

1. https://www.bzarg.com/p/how-a-kalman-filter-works-in-pictures/
2. https://towardsdatascience.com/what-i-was-missing-while-using-the-kalman-filter-for-object-tracking-8e4c29f6b795

Application:

- Trajectory smoothing: TrafAlert project