# Decision and Estimation in Data Processing

# Chapter II. Elements of Signal Detection Theory

# II.1 Introduction

# Introduction

- Signal detection = the problem of deciding which signal is present from 2 or more possibilities
  - one possibility may be that there is no signal
- Based on **noisy** observations
  - signals are affected by noise
  - noise is additive (added to the original signal)

# The context for signal detection

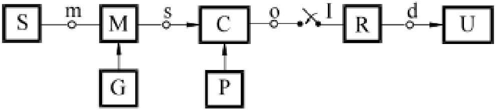



Figure 1: Block scheme of a communication system

- Block scheme of a communication system:
  - Information source: generates messages $a_n$ with probabilities $p(a_n)$
  - Generator: generates different signals $s_1(t), \dots s_n(t)$
  - Modulator: transmits a signal $s_n(t)$ for message $a_n$
  - Channel: adds random noise
  - Sampler: takes samples from the signal $s_n(t)$
  - Receiver: **decides** what message $a_n$ has been transmitted
  - User receives the recovered messages

# Problem formulation

- There are two messages $a_0$ and $a_1$ (e.g. logical 0 and 1)
- Messages are encoded as signals $s_0(t)$ and $s_1(t)$
  - for $a_0$: send $s(t) = s_0(t)$
  - for $a_1$: send $s(t) = s_1(t)$
- The signal is affected by additive white noise $n(t)$
- Receiver receives noisy signal $r(t) = s(t) + n(t)$
- **Decision problem**: based on $r(t)$, decide which signal was received, $s_0(t)$ or $s_1(t)$?

# Practical scenarios

- Data transmission with various binary modulations:
  - Constant voltage levels (e.g. $s_n(t) =$ constant $= 0$ or 5V)
  - PSK modulation (Phase Shift Keying): $s_n(t) =$ cosine with same frequency but various initial phases
  - FSK modulation (Frequency Shift Keying): $s_n(t) =$ cosines with different frequencies
  - OFDM modulation (Orthogonal Frequency Division Multiplexing): particular case of FSK
  - The receiver gets some noisy signal, has to decide when it is 0 and when it is 1

# Practical scenarios

- Radar detections:
  - a signal is emitted; if there is an obstacle, the signal gets reflected back
  - the receiver waits for possible reflections of the signal and must **decide**:
    - no reflection is present -> no object
    - reflected signal is present -> object detected

# Generalizations

- Decide between more than two signals
- Number of observations:
  - use only one sample
  - use multiple samples
  - observe the whole continuous signal for some time $T$

# II.2 Detection of signals based on 1 sample

# Problem formulation

- There are two messages $a_0$ and $a_1$ (e.g. logical 0 and 1)
- Messages are encoded as signals $s_0(t)$ and $s_1(t)$
  - for $a_0$: send $s(t) = s_0(t)$
  - for $a_1$: send $s(t) = s_1(t)$
- The signal is affected by additive white noise $n(t)$
- Receiver receives noisy signal $r(t) = s(t) + n(t)$
- **Decision problem**: based on $r(t)$, decide which signal was received, $s_0(t)$ or $s_1(t)$?
- Simplest case: receiver **takes just 1 sample** at time $t_0$, value is $r = r(t_0)$

# Hypotheses and decisions

- There are **two hypotheses**:
  - $H_0$: true signal is $s(t) = s_0(t)$ ($a_0$ has been transmitted)
  - $H_1$: true signal is $s(t) = s_1(t)$ ($a_1$ has been transmitted)
- Receiver can take **two decisions**:
  - $D_0$: receiver decides that signal was $s(t) = s_0(t)$
  - $D_1$: receiver decides that signal was $s(t) = s_1(t)$

# Possible outcomes

- There are 4 possible outcomes:
  1. **Correct rejection**: true hypothesis is $H_0$, decision is $D_0$
     - Probability is $P_r = P(D_0 \cap H_0)$
     - Also known as **True Negative**
  2. **False alarm**: true hypothesis is $H_0$, decision is $D_1$
     - Probability is $P_{fa} = P(D_1 \cap H_0)$
     - Also known as **False Positive**
  3. **Miss**: true hypothesis is $H_1$, decision is $D_0$
     - Probability is $P_m = P(D_0 \cap H_1)$
     - Also known as **False Negative**
  4. **Correct detection** ("*hit*"): true hypothesis is $H_1$, decision $D_1$
     - Probability is $P_d = P(D_1 \cap H_1)$
     - Also known as **True Positive**

# Origin of terms

- The terms originate from radar applications:
  - a signal is emitted from source
  - received signal = possible reflection from a target, with lots of noise
  - $H_0$ = no target is present, no reflected signal (only noise)
  - $H_1$ = target is present, there is a reflected signal
  - hence the names “miss”, “hit” etc.

# The noise

- In general we consider **additive**, **white**, **stationary** noise
  - additive = the noise is added to the signal
  - white = two samples from the noise are uncorrelated
  - stationary = has same statistical properties at all times
- The noise signal $n(t)$ is unknown
  - it's random
  - we just know it's distribution, but not the actual values

# The sample

- The receiver receives:

$$r(t) = s(t) + n(t)$$

  - $s(t)$ = original signal, either $s_0(t)$ or $s_1(t)$
  - $n(t)$ = unknown noise

- The value of the sample taken at $t_0$ is:

$$r(t_0) = s(t_0) + n(t_0)$$

  - $s(t_0)$ = the true signal = either $s_0(t_0)$ or $s_1(t_0)$
  - $n(t_0)$ = a sample from the noise

# The sample

- The sample $n(t_0)$ is a **random variable**
  - since it is a sample of noise (a sample from a random process)
  - assume is a continuous r.v., i.e. range of possible values is continuous
- $r(t_0) = s(t_0) + n(t_0) =$ a constant + a random variable
  - it is also a random variable
  - $s(t_0)$ is a constant, either $s_0(t_0)$ or $s_1(t_0)$
- What distribution does $r(t_0)$ have?
  - a constant + a r.v. = has same distribution as r.v., but shifted with the constant

# The conditional distributions

- Assume the noise has known distribution $w(x)$
- The distribution of $r = w(x)$ shifted by $s(t_0)$
- In hypothesis $H_0$, the distribution is $w(r|H_0) = w(x)$ shifted by $s_0(t_0)$
- In hypothesis $H_1$, the distribution is $w(r|H_1) = w(x)$ shifted by $s_1(t_0)$
- $w(r|H_0)$ and $w(r|H_1)$ are known as **conditional distributions** or **likelihood functions**
  - "|" means "conditioned by", "given that"
  - i.e. considering one hypothesis or the other one
  - $r$ is the unknown of the function

# The conditional distributions

Example:

- A constant signal $s(t)$ can have two values, 0 or 4. The signal is affected by noise $\mathcal{N}(\mu = 0, \sigma^2 = 2)$. What is the distribution of a sample $r$, in both hypotheses?

# Decision problem

The problem of decision:

- We have two possible distributions (one in each hypothesis)
- We have a sample $r = r(t_0)$, which could have come from either one
- Which hypothesis do we **decide** is the correct one?

# The likelihood of a parameter

- In general, the **likelihood** of a some parameter $P$ based on some **observation** $O$ = the probability density of $O$, if the parameter has value $P$:

$$L(P|O) = w(O|P)$$

- In our case:
  - the unknown parameter = which hypothesis $H$ is the true one
  - the observation = the sample $r$ that we got
- The **likelihood of a hypothesis H** based on the **observation** $r$ is:

$$L(H_0|r) = w(r|H_0)$$

$$L(H_1|r) = w(r|H_1)$$

# Maximum Likelihood decision criterion

- **Maximum Likelihood (ML) criterion**: choose the hypothesis that has the **highest likelihood** of having generated the observed sample value $r = r(t_0)$
  - "pick the most likely hypothesis"
  - "pick the hypothesis with a higher likelihood"

$$\frac{L(H_1|r)}{L(H_0|r)} = \frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

- We choose the higher value between $w(r(t_0)|H_0)$ and $w(r(t_0)|H_1)$
- This is known as a **likelihood ratio** test

# Example: gaussian noise

Example (follow-up):

- A constant signal $s(t)$ can have two values, 0 or 4. The signal is affected by noise $\mathcal{N}(\mu = 0, \sigma^2 = 2)$.
- What is the decision taken with the ML criterion, if $r = 1.6$?
- At blackboard:
    - plot the two conditional distributions for $w(r|H_0)$, $w(r|H_1)$
    - discuss the decision taken for different values of $r$
    - discuss the choice of the threshold value $T$ for taking decisions

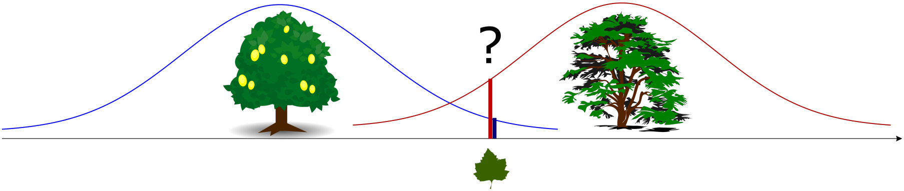
# Example: Trees

From what tree did the leaf fall?

?

# Example: Trees

Pick the tree with the **highest likelihood**:



# Gaussian noise (AWGN)

- Particular case: the noise has normal distribution $\mathcal{N}(0, \sigma^2)$
  - i.e. it is AWGN
- Likelihood ratio is $\frac{w(r|H_1)}{w(r|H_0)} = \frac{e^{-\frac{(r-s_1(t_0))^2}{2\sigma^2}}}{e^{-\frac{(r-s_0(r_0))^2}{2\sigma^2}}} \underset{H_0}{\overset{H_1}{\gtrless}} 1$
- For normal distribution, it is easier to apply **natural logarithm** to the terms
  - logarithm is a monotonic increasing function, so it won't change the comparison
  - if $A < B$, then $\log(A) < \log(B)$

# Gaussian noise (AWGN)

- Applying natural logarithm to both sides leads to:

$$-(r - s_1(t_0))^2 + (r - s_0(t_0))^2 \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

- Which means

$$|r - s_0(t_0)| \underset{H_0}{\overset{H_1}{\gtrless}} |r - s_1(t_0)|$$

- Note that $|r - A|$ = **distance** from $r$ to $A$
  - $|r|$ = distance from $r$ to 0
- So we choose the **smallest distance** between $r(t_0)$ and $s_1(t_0)$ vs $s_0(t_0)$

# Maximum Likelihood for gaussian noise

- ML criterion **for gaussian noise**: choose the hypothesis based on whichever of $s_0(t_0)$ or $s_1(t_0)$ is **nearest** to our observed sample $r = r(t_0)$
  - also known as **nearest neighbor** principle / decision
  - very general principle, encountered in many other scenarios
  - because of this, a receiver using ML is also known as **minimum distance receiver**

# Steps for ML decision

1. Sketch the two conditional distributions $w(r|H_0)$ and $w(r|H_1)$
2. Find out which function is higher at the observed value $r = r(t_0)$ given.

# Steps for ML decision in case of gaussian noise

- Only if the noise is Gaussian, identical for all hypotheses:
    1. Find $s_0(t_0) =$ the value of the original signal, in absence of noise, in case of hypothesis $H_0$
    2. Find $s_1(t_0) =$ the value of the original signal, in absence of noise, in case of hypothesis $H_1$
    3. Compare with observed sample $r(t_0)$ and choose **the nearest**

# Thresholding based decision

- Choosing the nearest value = same thing as **comparing** $r$ **with a threshold** $T = \frac{s_0(t_0)+s_1(t_0)}{2}$
  - i.e. if the two values are 0 and 5, decide by comparing with 2.5 (like in laboratory)
- For the **ML criterion**, the threshold = the **cross-over point** between the conditioned distributions

# Exercise

- A signal can have two possible values, 0 or 5. The signal is affected by white gaussian noise $\mathcal{N}\left(\mu = 0, \sigma^2 = 2\right)$. The receiver takes one sample with value $r = 2.25$.

  a. Write the expressions of the conditional probabilities and sketch them
  b. What is the decision based on the Maximum Likelihood criterion?
  c. What if the signal 0 is affected by gaussian noise $\mathcal{N}(0, 0.5)$, while the signal 5 is affected by uniform noise $\mathcal{U}[-4, 4]$?
  d. Repeat b. and c. assuming the value 0 is replaced by $-1$

# Decision regions

- The **decision regions** = the range of values of $r$ for which a certain decision is taken
- Decision regions $R_0$ = all the values of $r$ which lead to decision $D_0$
- Decision regions $R_1$ = all the values of $r$ which lead to decision $D_1$
- The decision regions cover the whole $\mathbb{R}$ axis
- Example: indicate the decision regions for the previous exercise:
  - $R_0 = [-\infty, 2.5]$
  - $R_1 = [2.5, \infty]$

# The likelihood function

- The subtle distinction in terms: “probability” vs “likelihood”
- Consider the conditional distribution $w(r|H_i)$ in the previous example:

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r-s_i(t_0))^2}{2\sigma^2}}$$

- Which is the unknown in this function?
  - in general, the unknown is $r$
  - but for our decision problem it is $i$, and $r$ is known

# Terminology: probability vs likelihood

- In the same mathematical expression of a distribution function:
  - if we know the parameters (e.g. $\mu$, $\sigma$, $H_i$), and the unknown is the value (e.g. $r$, $x$) we call it **probability density function** (distribution)
  - if we know value (e.g. $r$, $x$), and the unknown is some statistical parameter (e.g. $\mu$, $\sigma$, $i$), we call it a **likelihood function**

# Generalizations

- What if the noise has another distribution?
  - Sketch the conditional distributions
  - Locate the given $r = r(t_0)$
  - ML criterion = choose the highest function $w(r|H_i)$ in that point
- The decision regions are defined by the **cross-over points**
  - There can be more cross-overs, so multiple thresholds

# Generalizations

- What if the noise has a different distribution in hypothesis $H_0$ than in hypothesis $H_1$?
- Same thing:
  - Sketch the conditional distributions
  - Locate the given $r = r(t_0)$
  - ML decision = choose **the highest function** $w(r|H_i)$ in that point

# Generalizations

- What if the two signals $s_0(t)$ and $s_1(t)$ are constant / not constant?
- We **don't care about the shape** of the signals
- All we care about are **the two values at the sample time** $t_0$:
  - $s_0(t_0)$
  - $s_1(t_0)$

# Generalizations

- What if we have more than two hypotheses?
- Extend to $n$ hypotheses
  - We have $n$ possible signals $s_0(t)$, ... $s_{n-1}(t)$
  - We have $n$ different values $s_0(t_0)$, ... $s_{n-1}(t_0)$
  - We have $n$ conditional distributions $w(r|H_i)$
  - We **choose the highest function** $w(r|H_i)$ in the point $r = r(t_0)$

# Generalizations

- What if we take more than 1 sample?
- Patience, we'll treat this later as a separate sub-chapter

# Multiple separate detection

- In a binary communications setup, each detection/decision reads 1 bit
- We have a different detection for the next bit, and so on

# Exercise

- A signal can have four possible values: -6, -2, 2, 6. Each value lasts for 1 second. The signal is affected by white noise with normal distribution. The receiver takes 1 sample per second. Using ML criterion, decide what signal has been transmitted, if the received samples are:

$$4,\ 6.6,\ -5.2,\ 1.1,\ 0.3,\ -1.5,\ 7,\ -7,\ 4.4$$

# Conditional probabilities

- We compute the **conditional probabilities** of the 4 possible outcomes
- Consider the decision regions:
  - $R_0$: when $r \in R_0$, decision is $D_0$
  - $R_1$: when $r \in R_1$, decision is $D_1$
- Conditional probability of correct rejection
  - $=$ probability to take decision $D_0$ in the case that hypothesis is $H_0$
  - $=$ probability that $r$ is in $R_0$ computed from the distribution $w(r|H_0)$

$$P(D_0|H_0) = \int_{R_0} w(r|H_0)dx$$

- Conditional probability of false alarm
  - $=$ probability to take decision $D_1$ in the case that hypothesis is $H_0$
  - $=$ probability that $r$ is in $R_1$ computed from the distribution $w(r|H_0)$

$$P(D_1|H_0) = \int_{R_1} w(r|H_0)dx$$

# Conditional probabilities

- Conditional probability of miss
  - $=$ probability to take decision $D_0$ in the case that hypothesis is $H_1$
  - $=$ probability that $r$ is in $R_0$ computed from the distribution $w(r|H_1)$

$$P(D_0|H_1) = \int_{R_0} w(r|H_1)dx$$

- Conditional probability of correct rejection
  - $=$ probability to take decision $D_1$ in the case that hypothesis is $H_1$
  - $=$ probability that $r$ is in $R_1$ computed from the distribution $w(r|H_1)$

$$P(D_1|H_1) = \int_{R_1} w(r|H_1)dx$$

# Conditional probabilities

- Relation between them:
  - $P(D_0|H_0) + P(D_1|H_0) = 1$ (correct rejection + false alarm)
  - $P(D_0|H_1) + P(D_1|H_1) = 1$ (miss + correct detection)
  - Why? Prove this.

# Computing conditional probabilities

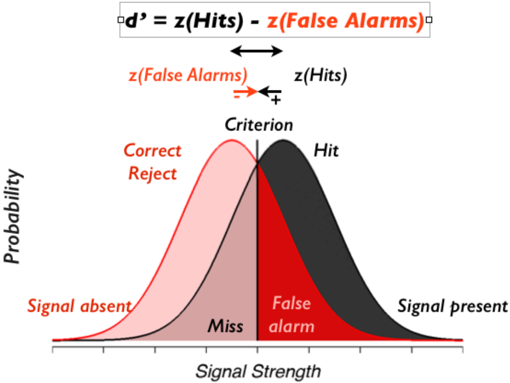



Figure 2: Conditional probabilities

- Ignore the text, just look at the nice colors
- [image from hhttp://gru.stanford.edu/doku.php/tutorials/sdt]*

# ML criterion optimality

**Theorem:**

The ML criterion **minimizes the total conditioned probability of error** $P(D_1|H_0) + P(D_0|H_1)$

**Proof:**

Informal: on the previous pivture, if $T$ is moved either to the right or to the left, the sum of the two areas of false alarm + misses increases.

TODO: rigorous proof

# Probabilities of the 4 outcomes

- Conditional probabilities are computed **given that** one or the other hypothesis is true
- They do not account for the probabilities **of the hypotheses themselves**
  - i.e. $P(H_0) =$ how many times does $H_0$ happen?
  - $P(H_1) =$ how many times does $H_1$ happen?
- To account for these, multiply with $P(H_0)$ or $P(H_1)$
  - $P(H_0)$ and $P(H_1)$ are known as the **prior** (or **a priori**) probabilities of the hypotheses

# Reminder: the Bayes rule

- Reminder: **the Bayes rule**

$$P(A \cap B) = P(B|A) \cdot P(A)$$

- Interpretation:
  - The probability $P(A)$ is taken out from $P(B|A)$
  - $P(B|A)$ gives no information on $P(A)$, the chances of $A$ actually happening
  - Example: $\mathrm{P}(\text{score} \mid \text{shoot}) = \frac{1}{2}$. How many goals are scored?
- In our case:

$$P(D_i \cap H_j) = P(D_i|H_j) \cdot P(H_j)$$

  - for all $i$ and $j$, i.e. all 4 cases

# Exercise

- A constant signal can have two possible values, $-2$ or 5. The signal is affected by gaussian noise $\mathcal{N}(\mu = 0, \sigma^2 = 2)$. The receiver performs ML decision based on a single sample.

  a. Compute the conditional probability of a false alarm
  b. Compute the conditional probability of a miss
  c. If $P(H_0) = \frac{1}{3}$ and $P(H_1) = \frac{2}{3}$, compute the actual probabilities of correct rejection and correct detection (not conditional)

# Pitfalls of ML decision criterion

- The ML criterion is based on comparing **conditional** distributions
  - conditioned by $H_0$ or by $H_1$
- Conditioning by $H_0$ and $H_1$ **ignores the prior probabilities of $H_0$ or $H_1$**
  - Our decision doesn't change if we know that $P(H_0) = 99.99\%$ and $P(H_1) = 0.01\%$, or vice-versa
- But if $P(H_0) > P(H_1)$, we may want to move the threshold towards $H_1$, and vice-versa
  - because it is more likely that the true signal is $s_0(t)$
  - and thus we want to "encourage" decision $D_0$
- Looks like we want a more general criterion ...

# Example: Football fields

TODO

# Minimum error probability criterion

- Takes into account the probabilities $P(H_0)$ and $P(H_1)$
- **The minimum probability of error** criterion (MPE):

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$$

$$\frac{P(H_1) \cdot w(r|H_1)}{P(H_0) \cdot w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

# The minimum error probability criterion

**Theorem:**

The MPE decision criterion **minimizes the total probability of errors**:

$$P_e = P_{fa} + P_m = P(D_1 \cap H_0) + P(D_0 \cap H_1)$$

- errors $=$ false alarms and misses

# Minimum error probability criterion

**Proof:**

- The probability of false alarm is:

$$\begin{aligned} P(D_1 \cap H_0) =& P(D_1|H_0) \cdot P(H_0) \\ =& \int_{R_1} w(r|H_0)dx \cdot P(H_0) \\ =& (1 - \int_{R_0} w(r|H_0 dx) \cdot P(H_0) \end{aligned}$$

- The probability of miss is:

$$\begin{aligned} P(D_0 \cap H_1) =& P(D_0|H_1) \cdot P(H_1) \\ =& \int_{R_0} w(r|H_1)dx \cdot P(H_1) \end{aligned}$$

- The total error probability (their sum) is:

$$P_e = P(H_0) + \int_{R_0} [w(r|H_1) \cdot P(H_1) - w(r|H_0) \cdot P(H_0)]dx$$

# Minimum probability of error

- An integral is always minimal when you integrate the function on all the domain where it is negative
- The term $w(r|H_1) \cdot P(H_1) - w(r|H_0) \cdot P(H_0)$ is negative whenever

$$w(r|H_1) \cdot P(H_1) < w(r|H_0) \cdot P(H_0)$$

  i.e. when we take decision $D_0$, i.e. on region $R_0$
- Therefore the integral on $R_0$ is minimal,
- $P(H_0)$ is a constant $=>$ $P_e$ is minimal

# Interpretation

- MPE criterion is more general than ML, depends on probabilities of the two hypotheses
  - Also expressed as a likelihood ratio test
- When one hypothesis has higher probability than the other, the threshold is **pushed in its favor**, towards the other one
- The ML criterion is a particular case of the MPE criterion, for $P(H_0) = P(H_1) = \frac{1}{2}$

# Minimum probability of error - Gaussian noise

- Assuming the noise has normal distribution $\mathcal{N}(0, \sigma^2)$

$$w(r|H_1) = e^{-\frac{(r-s_1(t_0))^2}{2\sigma^2}}$$

$$w(r|H_0) = e^{-\frac{(r-s_0(t_0))^2}{2\sigma^2}}$$

- Apply natural logarithm

$$-\frac{(r-s_1(t_0))^2}{2\sigma^2} + \frac{(r-s_0(t_0))^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

- Equivalently

$$(r-s_0(t_0))^2 \underset{H_0}{\overset{H_1}{\gtrless}} (r-s_1(t_0))^2 + 2\sigma^2 \cdot \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

  - or, after further processing:

$$r \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s_0(t_0)+s_1(t_0)}{2} + \frac{\sigma^2}{s_1(t_0)-s_0(t_0)} \cdot \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

# Interpretation 1: Comparing distance

- For ML criterion, we compare the (squared) distances:

$$\left|r - s_0(t_0)\right| \underset{H_0}{\overset{H_1}{\gtrless}} \left|r - s_1(t_0)\right|$$

$$(r - s_0(t_0))^2 \underset{H_0}{\overset{H_1}{\gtrless}} (r - s_1(t_0))^2$$

- For MPE criterion, we compare the squared distances, but a supplementary term appears in favour of the most probable hypothesis:

$$(r - s_0(t_0))^2 \underset{H_0}{\overset{H_1}{\gtrless}} (r - s_1(t_0))^2 + 2\sigma^2 \cdot \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

  - term depends on the ratio $\frac{P(H_0)}{P(H_1)}$

# Interpretation 2: The threshold value

- For ML criterion, we compare $r$ with a threshold $T$

$$r \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s_0(t_0) + s_1(t_0)}{2}$$

- For MPE criterion, the threshold is moved towards the less probable hypothesis:

$$r \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s_0(t_0) + s_1(t_0)}{2} + \frac{\sigma^2}{s_1(t_0) - s_0(t_0)} \cdot \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

  - depending on the ratio $\frac{P(H_0)}{P(H_1)}$

# Exercises

- Consider the decision between two constant signals: $s_0(t) = -5$ and $s_1(t) = 5$. The signals are affected by gaussian noise $\mathcal{N}(0, \sigma^2 = 3)$ The receiver takes one sample $r$.
  a. Find the decision regions $R_0$ and $R_1$ according to the MPE criterion
  b. What are the probabilities of false alarm and of miss?
  c. Repeat a) and b) considering that $s_1(t)$ is affected by uniform noise $\mathcal{U}[-4, 4]$

# Minimum risk criterion

- What if we care more about one type of errors (e.g. false alarms) than other kind (e.g. miss)?
  - MPE criterion treats all errors the same
  - Need a more general criterion
- Idea: assign a **cost** to each scenario, minimize average cost
- $C_{ij}$ = cost of decision $D_i$ when true hypothesis was $H_j$
  - $C_{00}$ = cost for good detection $D_0$ in case of $H_0$
  - $C_{10}$ = cost for false alarm (detection $D_1$ in case of $H_0$)
  - $C_{01}$ = cost for miss (detection $D_0$ in case of $H_1$)
  - $C_{11}$ = cost for good detection $D_1$ in case of $H_1$
- The idea of assigning "costs" and minimizing average cost is very general
  - e.g. IT: Shannon coding: "cost" of each message is the length of its codeword, we want to minimize average cost, i.e. minimize average length

# Minimum risk criterion

- Define the **risk = the average cost** value

$$R = C_{00}P(D_0 \cap H_0) + C_{10}P(D_1 \cap H_0) + C_{01}P(D_0 \cap H_1) + C_{11}P(D_1 \cap H_1)$$

- Minimum risk criterion: **minimize the risk R**
  - i.e. minimize the average cost
  - also known as "minimum cost criterion"

# Computations

- Proof on blackboard: (sorry, no time to put in on slides)
  - Use Bayes rule
  - Notations: $w(r|H_j)$ (*likelihood*)
  - Probabilities: $\int_{R_i} w(r|H_j)dV$
- Conclusion, **decision rule is**

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)}$$

# Minimum risk criterion

**Minimum risk criterion** (MR):

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)}$$

# Interpretation

- MR is a generalization of MPE criterion (which was itself a generalization of ML)
  - also expressed as a likelihood ratio test
- Both **probabilities** and the assigned **costs** can influence the decision towards one hypothesis or the other
- If $C_{10} - C_{00} = C_{01} - C_{11}$, MR reduces to MPE:
  - e.g. if $C_{00} = C_{11} = 0$, and $C_{10} = C_{01}$

# Minimum Risk - gaussian noise

- If the noise is gaussian (normal), do like for the other criteria, apply logarithm
- Obtain:

$$(r - s_0(t_0))^2 \underset{H_0}{\overset{H_1}{\gtrless}} (r - s_1(t_0))^2 + 2\sigma^2 \cdot \ln\left(\frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)}\right)$$

  - or

$$r \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s_0(t_0) + s_1(t_0)}{2} + \frac{\sigma^2}{s_1(t_0) - s_0(t_0)} \cdot \ln\left(\frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)}\right)$$

# Interpretation 1: Comparing distance

- For MPE criterion, we compare the squared distances, but a supplementary term appears in favour of the most probable hypothesis:

$$(r - s_0(t_0))^2 \underset{H_0}{\overset{H_1}{\gtrless}} (r - s_1(t_0))^2 + 2\sigma^2 \cdot \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

  - term depends on the ratio $\frac{P(H_0)}{P(H_1)}$
- For MR criterion, besides the probabilities we also are influenced by the costs

$$(r - s_0(t_0))^2 \underset{H_0}{\overset{H_1}{\gtrless}} (r - s_1(t_0))^2 + 2\sigma^2 \cdot \ln\left(\frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)}\right)$$

# Interpretation 2: The threshold value

- For MPE criterion, the threshold is moved towards the less probable hypothesis:

$$r \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s_0(t_0) + s_1(t_0)}{2} + \frac{\sigma^2}{s_1(t_0) - s_0(t_0)} \cdot \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

  - depending on the ratio $\frac{P(H_0)}{P(H_1)}$
- For MR criterion, besides the probabilities we also are influenced by the costs

$$r \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s_0(t_0) + s_1(t_0)}{2} + \frac{\sigma^2}{s_1(t_0) - s_0(t_0)} \cdot \ln\left(\frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)}\right)$$

# Influence of costs

- The MR criterion pushes the decision towards **minimizing the high-cost scenarios**
- Example: from the equations:
  - what happens if cost $C_{01}$ increases, while the others are unchanged?
  - what happens if cost $C_{10}$ increases, while the others are unchanged?
  - what happens if both costs $C_{01}$ and $C_{10}$ increase, while the others are unchanged?

# Pascal's wager

Reasoning of the French philosopher and mathematician Blaise Pascal (1623–1662):

> *God is, or God is not. Reason cannot decide between the two alternatives*
>
> *You must wager (it is not optional)*
>
> *If you gain, you gain all; if you lose, you lose nothing*
>
> *Wager, then, without hesitation that He is. There is here an infinity of an infinitely happy life to gain, against a finite number of chances of loss.*[1]

- A philosophical example of using the Minimum Risk criterion

[1] text source: Wikipedia

# General form of ML, MPE and MR criteria

- ML, MPE and MR criteria all have the following form

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

  - for ML: $K = 1$
  - for MPE: $K = \frac{P(H_0)}{P(H_1)}$
  - for MR: $K = \frac{(C_{10}-C_{00})p(H_0)}{(C_{01}-C_{11})p(H_1)}$

# General form of ML, MPE and MR criteria

In gaussian noise, all criteria reduce to:

- Comparing squared distances:

$$(r - s_0(t_0))^2 \underset{H_0}{\overset{H_1}{\gtrless}} (r - s_1(t_0))^2 + 2\sigma^2 \cdot \ln(K)$$

- Comparing the sample $r$ with a threshold $T$:

$$r \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\frac{s_0(t_0) + s_1(t_0)}{2} + \frac{\sigma^2}{s_1(t_0) - s_0(t_0)} \cdot \ln(K)}_{T}$$

# Exercise

- A vehicle airbag system detects a crash by evaluating a sensor which provides two values: $s_0(t) = 0$ (no crash) or $s_1(t) = 5$ (crashing)
- The signal is affected by gaussian noise $\mathcal{N}(\mu = 0, \sigma^2 = 1)$.
- The costs of the scenarios are: $C_{00} = 0$, $C_{01} = 100$, $C_{10} = 10$, $C_{11} = -100$
  a. Find the decision regions $R_0$ and $R_1$.

# Neyman-Pearson criterion

- An even more general criteria than all the others until now
- **Neyman-Pearson criterion**: maximize probability of correct detection ($P(D_1 \cap H_1)$) while keeping probability of false alarms smaller then a limit ($P(D_1 \cap H_0) \leq \lambda$)
  - Deduce the threshold $T$ from the limit condition $P(D_1 \cap H_0) = \lambda$
- ML, MPE and MR criteria are particular cases of Neyman-Pearson, for particular values of $\lambda$

# Exercise

- An information source provides two messages with probabilities $p(a_0) = \frac{2}{3}$ and $p(a_1) = \frac{1}{3}$.
- The messages are encoded as constant signals with values $-5$ ($a_0$) and 5 ($a_1$).
- The signals are affected by noise with uniform distribution $U[-5, 5]$.
- The receiver takes one sample $r$.
    a. Find the decision regions according to the Neymar-Pearson criterion, considering $P_{fa} \leq 10^{-2}$
    b. What is the probability of correct detection, in this case?

# Summary of criteria

- We have seen decision based on 1 sample $r$, between 2 signals (mostly)
- All decisions are based on a likelihood-ratio test

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- Different criteria differ in the chosen value of $K$ (likelihood threshold)
- Depending on the noise distributions, the real axis is partitioned into regions
  - region $R_0$: if $r$ is in here, decide $D_0$
  - region $R_1$: if $r$ is in here, decide $D_1$
- For gaussian noise, the boundary of the regions (threshold) is

$$T = \frac{s_0(t_0) + s_1(t_0)}{2} + \frac{\sigma^2}{s_1(t_0) - s_0(t_0)} \cdot \ln(K)$$

# Comparing two decision problems

- Suppose we have a decision problem with $s_0(t) = 0$, $s_1(t) = 10$, and noise $\mathcal{N}(\mu = 0, \sigma^2 = 4)$
- Suppose we have another totally different decision problem, with $s_0(t) = 10$, $s_1(t) = 16$, and noise $\mathcal{U}[-8, 8]$
- Which one is easier? How can we compare them?
- How to evaluate the overall performance in a decision problem?
  - We need to compare the "good" probabilities ($P_{cd}$, $P_{cr}$) and the "bad" probabilities ($P_{fa}$, $P_m$)

# Receiver Operating Characteristic

- A decision performance is usually represented with **"Receiver Operating Characteristic" (ROC)** graph
- It is a graph of $P_d = P(D_1|H_1)$ as a function of $P_{fa} = P(D_1|H_0)$,
  - obtained for different values of the threshold value $T$
  - i.e. for every $T$ you get a certain value of $P_{fa}$ and a certain value of $P_d$

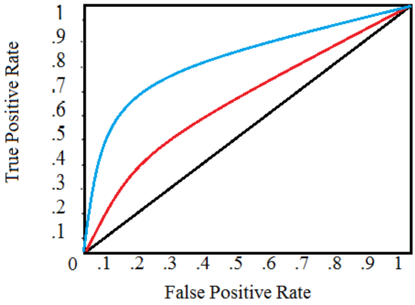



Figure 3: Sample ROC curves

# Receiver Operating Characteristic

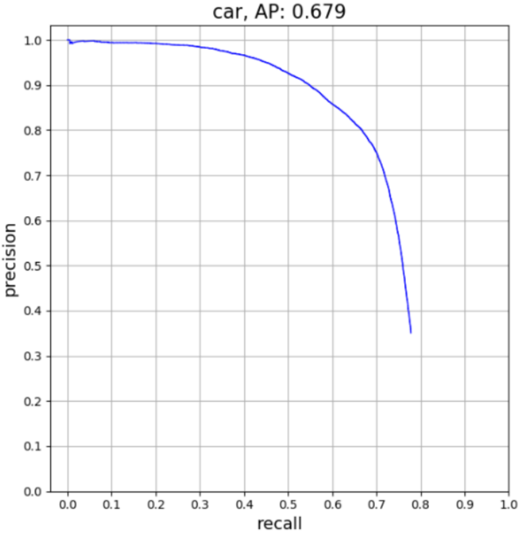
- ROC graph shows there is always a **tradeoff** between good $P_d$ and bad $P_{fa}$
  - to increase $P_d$ one must also increase $P_{fa}$
  - if we want to make sure we don't miss any real detections (increase P_d), we pay by increasing the chances of false alarms
- Different criteria = different likelihood thresholds $K$ = different points on the graph = different tradeoffs
- An overall performance measure is the total **Area Under the Curve** (AUC)
  - this doesn't depend on the choice of a particular threshold value
- We can compare two different decision situations (e.g. different signals, or different algorithms etc) by plotting their ROC and comparing their AUC

# The Precision-Recall curve

- A similar curve is the **Precision vs. Recall** curve
- **Precision** $= \frac{P(D_1 \cap H_1)}{P(D_1 \cap H_1) + P(D_1 \cap H_0)}$
  - $=$ True Positives / (True Positives $+$ False Positives)
- **Recall** $= \frac{P(D_1 \cap H_1)}{P(D_1 \cap H_1) + P(D_0 \cap H_1)} = P(D_1 | H_1)$
  - $=$ True Positives / (True Positives $+$ False Negatives)

# Precision-Recall curve

Example of a Precision vs Recall Curve



# Precision-Recall curve

Real-life app from which the preceding curve was taken:

# Signal-to-Noise Ratio

- How to improve the detection performance?
  - i.e. increase $P_D$ while keeping $P_{fa}$ the same
  - irrespective of what threshold is chosen
- Two solutions:
  - Increase the seperation between $s_0(t)$ and $s_1(t)$ (increase **signal power**)
  - Reduce the noise (decrease **noise power**)
  - i.e. increase **Signal-to-Noise ratio**

# 2020-2021 Exam

- 2020-2021 Exam: Skip next 3 slides (until Signal-to-noise ratio)

# Performance of likelihood-ratio decoding in AWGN

- WGN = "White Gaussian Noise"
- Assume equal probabilities $P(H_0) = P(H_1) = \frac{1}{2}$
  - Equivalently, consider only the conditional probabilities
- All decisions are based on a likelihood-ratio test

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- Conditional probability of correct detection is:

$$\begin{aligned}
P_d =& P(D_1|H_1) \\
=& \int_T^{\infty} w(r|H_1) \\
=& (F(\infty) - F(T)) \\
=& \frac{1}{2}\left(1 - erf\left(\frac{T - s_1(t_0)}{\sqrt{2}\sigma}\right)\right) \\
=& Q\left(\frac{T - s_1(t_0)}{\ldots}\right)
\end{aligned}$$

# Performance of likelihood-ratio decoding in AWGN

- Conditional probability of false alarm is:

$$
\begin{aligned}
P_{fa} =& P(D_1|H_0) \\
=& \int_T^\infty w(r|H_0) \\
=& (F(\infty) - F(T)) \\
=& \frac{1}{2}\left(1 - erf\left(\frac{T - s_0(t_0)}{\sqrt{2}\sigma}\right)\right) \\
=& Q\left(\frac{T - s_0(t_0)}{\sqrt{2}\sigma}\right)
\end{aligned}
$$

- Therefore $\frac{T - s_0(t_0)}{\sqrt{2}\sigma} = Q^{-1}\left(P_{fa}\right)$,
- And: $\frac{T - s_1(t_0)}{\sqrt{2}\sigma} = Q^{-1}\left(P_{fa}\right) + \frac{s_0(t_0) - s_1(t_0)}{\sqrt{2}\sigma}$

# Performance of likelihood-ratio decoding in AWGN

- Replacing in $P_d$ yields:

$$P_d = Q\left(\underbrace{Q^{-1}\left(P_{fa}\right)}_{constant} + \frac{s_0(t_0) - s_1(t_0)}{\sqrt{2}\sigma}\right)$$

- Consider a simple case:
  - $s_0(t_0) = 0$
  - $s_1(t_0) = A = constant$
- We get:

$$P_d = Q\left(\underbrace{Q^{-1}\left(P_{fa}\right)}_{constant} - \frac{A}{\sqrt{2}\sigma}\right)$$

# Signal-to-noise ratio

- **Signal-to-noise ratio (SNR)** $= \frac{\text{power of original signal}}{\text{power of noise}}$
- Average power of a signal = average squared value = $\overline{X^2}$
  - Original signal power of $s(t)$ is $\frac{A^2}{2}$
  - Noise power is $\overline{X^2} = \sigma^2$ (when noise mean value $\mu = 0$)
- In our case, SNR $= \frac{A^2}{2\sigma^2}$

$$P_d = Q\left(\underbrace{Q^{-1}\left(P_{fa}\right)}_{constant} - \sqrt{SNR}\right)$$

- For a fixed $P_{fa}$, $P_d$ **increases with SNR**
  - Q is a monotonic decreasing function

# Performance depends on SNR

- Receiver performance increases with SNR increase
  - high SNR: good performance
  - poor SNR: bad perfomance

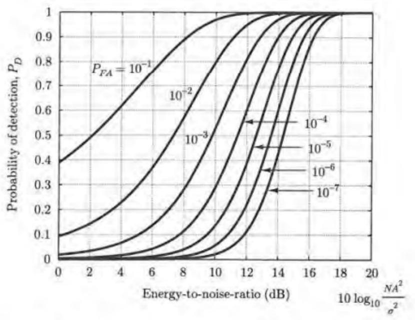



Figure 6: Detection performance depends on SNR

*[source: Fundamentals of Statistical Signal Processing, Steven Kay]*

# Applications of decision theory

- Can we apply these decision criteria in other engineering problems?
  - e.g. not for deciding between two signals, but for something else
- The core mathematical problem we solve is:
  - we have 2 (or more) possible distributions
  - we observe 1 value
  - we determine the most likely distribution, according to the value
- In our particular problem, we decide between two signals
- But this can be applied to many other statistical problems:
  - medicine: does this ECG signal look healthy or not?
  - business: will this client buy something or not?
  - Typically we use more than 1 value for these, but the mathematical principle is the same

# Applications of decision theory

Example (purely imaginary):

- A healthy person of weight = X kg has the concentration of thrombocytes per ml of blood distributed approximately as $\mathcal{N}\left(\mu = 10 \cdot X, \sigma^2 = 20\right)$.
- A person suffering from disease D has a much lower value of thrombocytes, distributed approximately as $\mathcal{N}\left(100, \sigma^2 = 10\right)$.
- The lab measures your blood and finds your value equal to $r = 255$. Your weight is 70 kg.
- Decide: are you most likely healthy, or ill?

# II.3 Signal detection with multiple samples

# Multiple samples from a signal

- The overall context stays the same:
  - A signal $s(t)$ is transmitted
  - There are **two hypotheses**:
    - $H_0$: true signal is $s(t) = s_0(t)$
    - $H_1$: true signal is $s(t) = s_1(t)$
  - Receiver can take **two decisions**:
    - $D_0$: receiver decides that signal was $s(t) = s_0(t)$
    - $D_1$: receiver decides that signal was $s(t) = s_1(t)$
  - There 4 possible outcomes

# Multiple samples from a signal

- The overall context stays the same:
  - There is noise on the channel (unknown)
  - The receiver receives $r(t) = s(t) + n(t)$
- Suppose we take N samples from $r(t)$, not just 1
  - Each sample is $r_i = r(t_i)$, taken at moment $t_i$
- The samples are arranged in a **sample vector**

$$\mathbf{r} = [r_1, r_2, ...r_N]$$

# Multiple samples from a signal

- Each sample $r_i$ is a **random variable**
  - since $r(t_i) = s(t_i) + n(t_i) =$ a constant $+$ a random variable
- The sample vector $\mathbf{r}$ is a set of $N$ random variables from a random process
- Considering the whole sample vector $\mathbf{r}$ as a whole, the values of $\mathbf{r}$ are described by the **distributions of order** $N$
- In hypothesis $H_0$:

$$w_N(\mathbf{r}|H_0) = w_N(r_1, r_2, ...r_N|H_0)$$

- In hypothesis $H_1$:

$$w_N(\mathbf{r}|H_1) = w_N(r_1, r_2, ...r_N|H_1)$$

# Likelihood of vector samples

- We can apply **the same criteria** based on likelihood ratio as for 1 sample

$$\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- Notes:
  - $\mathbf{r}$ is a vector; we consider the likelihood of all the sample vector as a whole
  - $w_N(\mathbf{r}|H_0)$ = likelihood of the whole vector $\mathbf{r}$ being obtained in hypothesis $H_0$
  - $w_N(\mathbf{r}|H_1)$ = likelihood of the whole vector $\mathbf{r}$ being obtained in hypothesis $H_1$
  - the value of $K$ is given by the actual decision criterion used
- Interpretation: we choose the hypothesis that is most likely to have produced the observed data
  - now the data = a set of samples, not just 1

# Separation

- Assuming the noise is white noise, the noise samples are independent, and therefore the samples $r_i$ are independent
- In that case the joint distribution $w_N(\mathbf{r}|H_i)$ can be decomposed as a **product of individual distributions**:

$$w_N(\mathbf{r}|H_i) = w(r_1|H_i) \cdot w(r_2|H_i) \cdot ... \cdot w(r_N|H_i)$$

  - e.g. the likelihood of obtaining $[5.1, 4.7, 4.9] =$ likelihood of obtaining $5.1 \times$ likelihood of getting $4.7 \times$ likelihood of getting $4.9$
- The $w(r_i|H_i)$ are just conditional distributions for each sample
  - we've seen them already

# Separation

- Then all likelihood ratio criteria can be written as:

$$\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} = \frac{w(r_1|H_1)}{w(r_1|H_0)} \cdot \frac{w(r_2|H_1)}{w(r_2|H_0)} \dots \frac{w(r_N|H_1)}{w(r_N|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- The likelihood ratio of a vector of samples = product of likelihood ratio for each sample
- We **multiply** the likelihood ratio **of each sample**, and then use the same criteria for the end result

# Criteria for decisions

- All likelihood ratio criteria can be written as:

$$\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} = \frac{w(r_1|H_1)}{w(r_1|H_0)} \cdot \frac{w(r_2|H_1)}{w(r_2|H_0)} \dots \frac{w(r_N|H_1)}{w(r_N|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- The value of $K$ is the same as for 1 sample:
  - for ML: $K = 1$
  - for MPE: $K = \frac{P(H_0)}{P(H_1)}$
  - for MR: $K = \frac{(C_{10} - C_{00}) p(H_0)}{(C_{01} - C_{11}) p(H_1)}$

# Particular case: AWGN

- AWGN = "Additive White Gaussian Noise"
- In hypothesis $H_1$: $w(r_i|H_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r_i - s_1(t_i))^2}{2\sigma^2}}$
- In hypothesis $H_0$: $w(r_i|H_0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r_i - s_1(t_i))^2}{2\sigma^2}}$
- Likelihood ratio for vector **r**

$$\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} = \frac{e^{-\frac{\sum (r_i - s_1(t_i))^2}{2\sigma^2}}}{e^{-\frac{\sum (r_i - s_0(t_i))^2}{2\sigma^2}}} = e^{\frac{\sum (r_i - s_0(t_i))^2 - \sum (r_i - s_1(t_i))^2}{2\sigma^2}}$$

# Decision criteria for AWGN

- The global likelihood ratio is compared with $K$:

$$\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} = e^{\frac{\sum(r_i - s_0(t_i))^2 - \sum(r_i - s_1(t_i))^2}{2\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- Applying the natural logarithm, this becomes:

$$\sum(r_i - s_0(t_i))^2 \underset{H_0}{\overset{H_1}{\gtrless}} \sum(r_i - s_1(t_i))^2 + 2\sigma^2 \ln(K)$$

# Interpretation 1: geometrical distance

- The sums are squared **geometrical distances**:

$$\sum (r_i - s_1(t_i))^2 = \|\mathbf{r} - \mathbf{s_1}(\mathbf{t})\|^2 = d(\mathbf{r}, s_1(t))^2$$

$$\sum (r_i - s_0(t_i))^2 = \|\mathbf{r} - \mathbf{s_0}(\mathbf{t})\|^2 = d(\mathbf{r}, s_0(t))^2$$

  - the distance between the observed samples **r** and the true possible underlying signals $s_1(t)$ and $s_0(t)$
  - with N samples => distance between vectors of size $N$
- It comes down to a decision between distances

# Interpretation 1: geometrical distance

- Maximum Likelihood criterion:
  - $K = 1$, $\ln(K) = 0$
  - we choose the **minimum distance** between what is ($\mathbf{r}$) and what should have been in absence of noise ($s_1(t)$ and $s_0(t)$)
  - hence the name "minimum distance receiver"
- Minimum Probability of Error criterion:
  - $K = \frac{P(H_0)}{P(H_1)}$
  - An additional term appears in favor of the most probable hypothesis
- Minimum Risk criterion:
  - $K = \frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)}$
  - Additional term depends on both probabilities and costs

Exercise:

- A signal can have two values, 0 (hypothesis $H_0$) or 6 (hypothesis $H_1$). The signal is affected by AWGN $\mathcal{N}(0, \sigma^2 = 1)$. The receiver takes 5 samples with values $\{1.1, 4.4, 3.7, 4.1, 3.8\}$.
  a. What is decision according to Maximum Likelihood criterion?
  b. What is decision according to Minimum Probability of Error criterion, assuming $P(H_0) = 2/3$ and $P(H_1) = 1/3$?
  c. What is the decision according to Minimum Risk Criterion, assuming $P(H_0) = 2/3$ and $P(H_1) = 1/3$, and $C_{00} = 0$, $C_{10} = 10$, $C_{01} = 20$, $C_{11} = 5$?

# Another exercise

Another Exercise:

- Consider detecting a signal $s_1(t) = 3\sin(2\pi f_1 t)$ that can be present (hypothesis $H_1$) or not ($s_0(t) = 0$, hypothesis $H_0$). The signal is affected by AWGN $\mathcal{N}(0, \sigma^2 = 1)$. The receiver takes 2 samples.
  a. What are the best sample times $t_1$ and $t_2$ to maximize detection performance?
  b. The receiver takes 2 samples with values $\{1.1, 4.4\}$, at sample times $t_1 = \frac{0.125}{f_1}$ and $t_2 = \frac{0.625}{f_1}$. What is decision according to Maximum Likelihood criterion?
  c. What if we take the decision with Minimum Probability of Error criterion, assuming $P(H_0) = 2/3$ and $P(H_1) = 1/3$?
  d. What is the decision according to Minimum Risk Criterion, assuming $P(H_0) = 2/3$ and $P(H_1) = 1/3$, and $C_{00} = 0$, $C_{10} = 10$, $C_{01} = 20$, $C_{11} = 5$?
  e. What if the receiver takes an extra third sample at time $t_3 = \frac{0.5}{f_1}$. Will the detection be improved?

# Interpretation 2: inner-product

- Let's decompose the parentheses in the distances:

$$\sum (r_i - s_0(t_i))^2 \underset{H_0}{\overset{H_1}{\gtrless}} \sum (r_i - s_1(t_i))^2 + 2\sigma^2 \ln(K)$$

- Equivalent to:

$$\sum (r_i)^2 + \sum s_0(t_i)^2 - 2\sum r_i s_0(t_i) \underset{H_0}{\overset{H_1}{\gtrless}} \sum (r_i)^2 + \\ + \sum s_1(t_i)^2 - 2\sum r_i s_1(t_i) + 2\sigma^2 \ln(K)$$

- Equivalent to:

$$\sum r_i s_1(t_i) - \frac{\sum (s_1(t_i))^2}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \sum r_i s_0(t_i) - \frac{\sum (s_0(t_i))^2}{2} + \sigma^2 \ln(K)$$

# Interpretation 2: inner-product

- Linear algebra: **inner product** of vectors **a** and **b**:

$$\langle a, b \rangle = \sum_i a_i b_i$$

- $\sum r_i s_1(t_i) = \langle \mathbf{r}, \mathbf{s_1}(\mathbf{t}) \rangle$ is the inner product of vector $\mathbf{r} = [r_1, r_2, ...r_N]$ with $\mathbf{s_1}(\mathbf{t_i}) = [s_1(t_1), s_1(t_2), ...s_1(t_N)]$
- $\sum r_i s_0(t_i) = \langle \mathbf{r}, \mathbf{s_0}(\mathbf{t}) \rangle$ is the inner product of vector $\mathbf{r} = [r_1, r_2, ...r_N]$ with $\mathbf{s_0}(\mathbf{t_i}) = [s_0(t_1), s_0(t_2), ...s_0(t_N)]$
- $\sum (s_1(t_i))^2 = \sum s_1(t_i) \cdot s_1(t_i) = \langle \mathbf{s_1}(\mathbf{t}), \mathbf{s_1}(\mathbf{t}) \rangle = E_1$ is the **energy** of vector $s_1(t)$
- $\sum (s_0(t_i))^2 = \sum s_0(t_i) \cdot s_0(t_i) = \langle \mathbf{s_0}(\mathbf{t}), \mathbf{s_0}(\mathbf{t}) \rangle = E_0$ is the **energy** of vector $s_0(t)$

# Interpretation 2: inner-product

- The decision can be rewritten as:

$$\langle \mathbf{r}, \mathbf{s_1} \rangle - \frac{E_1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \langle \mathbf{r}, \mathbf{s_0} \rangle - \frac{E_0}{2} + \sigma^2 \ln(K)$$

- Interpretation: we **compare the inner-products**
  - also subtract the energies of the signals, for a fair comparison
  - also with a term depending on the criterion

# Interpretation 2: inner-product

- Particular case:
  - If the two signals have the same energy: $E_1 = \sum s_1(t_i)^2 = E_0 = \sum s_0(t_i)^2$
  - Examples:
    - BPSK modulation: $s_1 = A\cos(2\pi ft)$, $s_0 = -A\cos(2\pi ft)$
    - 4-PSK modulation: $s_{n=0,1,2,3} = A\cos(2\pi ft + n\frac{\pi}{4})$
  - Then it is simplified as:

$$\langle \mathbf{r}, \mathbf{s_1} \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \langle \mathbf{r}, \mathbf{s_0} \rangle + \sigma^2 \ln(K)$$

# Interpretation 2: inner-product

- Inner-product in signal processing measures **similarity** of two signals
- Interpretation: we check if the received samples $\mathbf{r}$ look **more similar to** $s_1(t)$ or to $s_0(t)$
  - Choose the one which shows more similarity to $\mathbf{r}$
  - There is also the subtraction of the energies, for a fair comparison (due to mathematical reasons)
- **Inner product** of vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\langle a, b \rangle = \sum_i a_i b_i$$

# Decision with correlator circuits

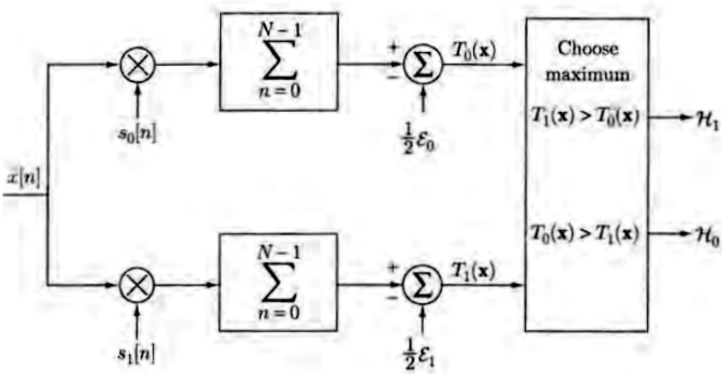



Figure 7: Decision between two signals

*[source: Fundamentals of Statistical Signal Processing, Steven Kay]*

# Example: BPSK

- BPSK demodulation:

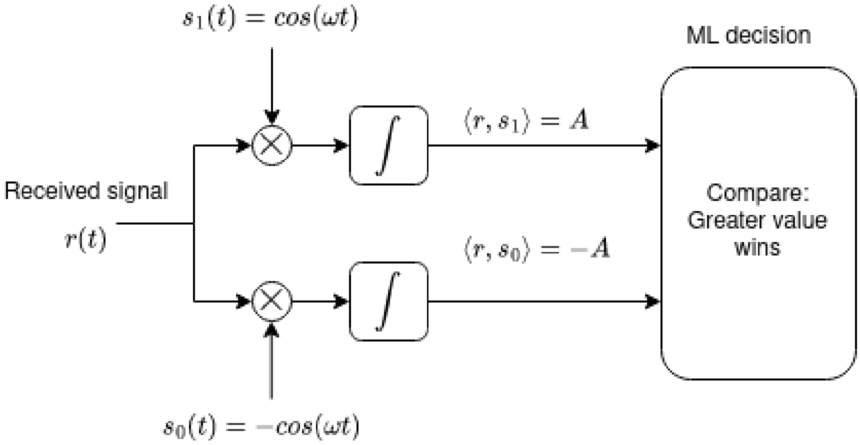



Figure 8: BPSK decision: naive implementation

# Example: BPSK

- BPSK demodulation:

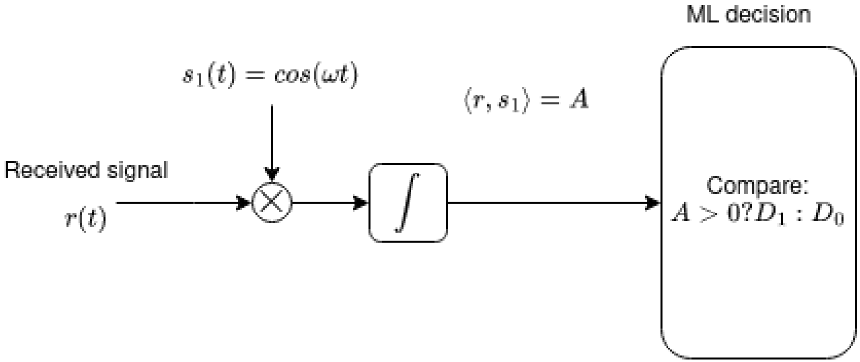



Figure 9: BKSP detection: usual implementation

# Example: QPSK

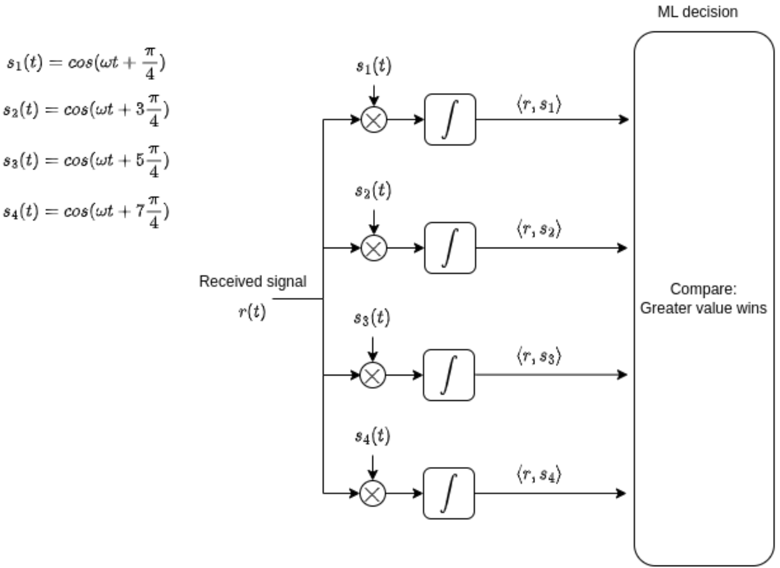



Figure 10: QPSK decision: naive implementation

# Example: QPSK

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$s_1(t) = \cos(\omega t + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}\cos(\omega t) + \frac{\sqrt{2}}{2}\sin(\omega t)$$

$$s_2(t) = \cos(\omega t + 3\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}\cos(\omega t) + \frac{\sqrt{2}}{2}\sin(\omega t)$$

$$s_3(t) = \cos(\omega t + 5\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}\cos(\omega t) - \frac{\sqrt{2}}{2}\sin(\omega t)$$

$$s_4(t) = \cos(\omega t + 7\frac{\pi}{4}) = \frac{\sqrt{2}}{2}\cos(\omega t) - \frac{\sqrt{2}}{2}\sin(\omega t)$$

$$s_n(t) = \cos(\omega t + n\frac{\pi}{4}) = A\cos(\omega t) + B\sin(\omega t)$$

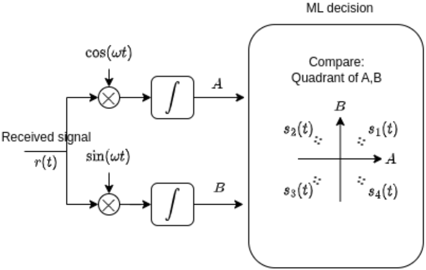



Figure 11: QKSP detection: usual implementation

# Matched filters

- How to compute the inner product of two signals $r[n]$ and $s[n]$ of length $N$?

$$\langle \mathbf{r}, \mathbf{s} \rangle = \sum r_i s(t_i)$$

- Let $h[n]$ be the signal $s[n]$ **flipped / mirrored** ("oglindit") and delayed with $N$
  - starts from time 0, goes up to time $N-1$, but backwards

$$h[n] = s[N-1-n]$$

- Example:
  - if $s[n] = [\underset{\uparrow}{1}, 2, 3, 4, 5, 6]$
  - then $h[n] = s[N-1-n] = [\underset{\uparrow}{6}, 5, 4, 3, 2, 1]$

# Matched filters

- The convolution of $r[n]$ with $h[n]$ is

$$y[n] = \sum_k r[k]h[n-k] = \sum_k r[k]s[N-1-n+k]$$

- The convolution sampled at the end of the signal, $y[N-1]$ (for $n = N-1$), is the inner product:

$$y[N-1] = \sum_k r[k]s[k]$$

# Matched filters

- To detect a signal $s[n]$ we can use a **filter with impulse response = mirrored version of** $s[n]$, and take the final sample of the output

$$h[n] = s[N - 1 - n]$$

  - it is identical to computing the inner product
- **Matched filter** = a filter designed to have the impulse response the flipped version of a signal we search for
  - the filter is *matched* to the signal we want to detect
  - rom. "filtru adaptat"

# Signal detection with matched filters

- Use one filter matched to signal $s_1(t_i)$
- Use another filter matched to signal $s_0(t_i)$
- Sample both filters at the end of the signal $n = N - 1$
  - obtain the values of the inner products
- Use the decision rule (with the inner products) to decide

# Signal detection with matched filters

- In case $s_0(t) = 0$, we need only one matched filter for $s_1(t)$, and compare the result to a threshold

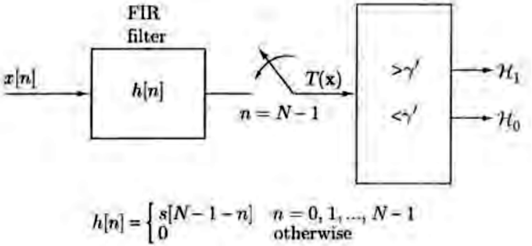



Figure 12: Signal detection with matched filter

*[source: Fundamentals of Statistical Signal Processing, Steven Kay]*

# II.4 Detection of general signals with continuous observations

# Continuous observation of a general signal

- Continuous observation = we don't take samples anymore, we use **all the continuous signal**
  - like taking $N$ samples but with $N \to \infty$
- Original signals are $s_0(t)$ and $s_1(t)$
- Signals are affected by noise
  - Assume **only Gaussian noise**, for simplicity
- Received signal is $r(t)$

# Euclidian space

- Extend from $N$ samples to the case a full continuous signal
- Each signal $r(t)$, $s_1(t)$ or $s_0(t)$ is a data point in an **infinite-dimensional Euclidean space**
- **Distance** between two signals is:

$$d(\mathbf{r}, \mathbf{s}) = \sqrt{\int \left(r(t) - s(t)\right)^2 dt}$$

- **Inner product** between two signals is:

$$\langle \mathbf{r}, \mathbf{s} \rangle = \int r(t)s(t)dt$$

- Similar with the N dimensional case, but with integral instead of sum

# Decision rule for AWGN: distances

- For AWGN, same decision rule as always:

$$d(\mathbf{r}, \mathbf{s_0})^2 \underset{H_0}{\overset{H_1}{\gtrless}} d(\mathbf{r}, \mathbf{s_1})^2 + 2\sigma^2 \ln(K)$$

- Distance = previous formula, with integral
- Same criteria:
  - Maximum Likelihood criterion: $K = 1$, $\ln(K) = 0$
    - we choose the **minimum distance**
  - Minimum Probability of Error criterion: $K = \frac{P(H_0)}{P(H_1)}$
  - Minimum Risk criterion: $K = \frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)}$

# Decision rule for AWGN: inner products

- For AWGN, same decision rule as always:

$$\langle \mathbf{r}, \mathbf{s_1} \rangle - \frac{E_1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \langle \mathbf{r}, \mathbf{s_0} \rangle - \frac{E_0}{2} + \sigma^2 \ln(K)$$

- Inner product = previous formula, with integral
- All interpretations remain the same
  - we only change the **type of signal** we work with

# Matched filters

- Inner product of signals can be computed with **matched filters**
- **Matched filter** = a filter designed to have the impulse response the flipped version of a signal we search for
  - if original signal $s(t)$ has length T
  - then $h(t) = s(T - t)$
  - filter is analogical, impulse response is continuous
- Output of a matched filter at time $t = T$ is equal to the inner product of the input $r(t)$ with $s(t)$

# Signal detection with matched filters

- Use one filter matched to signal $s_1(t)$
- Use another filter matched to signal $s_0(t)$
- Sample both filters at the end of the signal $t = T$
  - obtain the values of the inner products
- Use the decision rule (with the inner products) to decide

# Review of Euclidean vector spaces

- Review of Euclidean vector spaces
- Vector space
  - one thing $+$ another thing $=$ still in same space
  - constant $\times$ a vector $=$ still in same space
  - has basic arithmetic: sum, multiplication by a constant
  - Examples:
    - 1D $=$ a line
    - 2D $=$ a plane
    - 3D $=$ a 3-D space
    - N-D $= \ldots$
    - $\infty$-D $=$ ..

# Review of Euclidean vector spaces

- The fundamental function: **inner product**
  - for discrete signals
$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i$$
  - for continuous signals
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int x(t) y(t)$$
- Norm (length) of a vector = sqrt(inner product with itself)
$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$
- Distance between two vectors = norm of their difference
$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

# Review of Euclidean vector spaces

- Energy of a signal = squared norm

$$E_x = \|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$$

- Angle between two vectors

$$cos(\alpha) = \frac{\langle x, y \rangle}{||x|| \cdot ||y||}$$

  - value between -1 and 1
  - if $\langle x, y \rangle = 0$, the two vectors are **orthogonal** (perpendicular)

# Review of Euclidean vector spaces

- Bonus: the Fourier transform = inner product with $e^{j\omega t}$

$$\mathcal{F}\{x(t)\} = \langle x(t), e^{j\omega t} \rangle = \int x(t) e^{-j\omega t}$$

  - for complex signals, the second function is conjugated, hence $-j$ instead of $j$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i^*$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int x(t) y(t)^*$$

- Also same for discrete signals

# Review of Euclidean vector spaces

- Conclusion: expressing algorithms in a generic way, with inner products / distances / norms, is very powerful
  - they automatically apply to all vector spaces
  - work once, reuse in many places

# II.5 Decision with unknown distributions

# Knowing vs not knowing the distribution

- Until now, we always knew what samples we expect
  - We knew the signals:
    - $s_0(t) = ...$
    - $s_1(t) = ...$
  - We knew the noise type
    - gaussian, uniform, etc.
  - So we knew the sample distributions:
    - $w(r|H_0) = ...$
    - $w(r|H_1) = ...$
- In real life, things are more complicated

# Typical example

- What if the signals $s_0(t)$ and $s_1(t)$ do not exist / we do not know them?
- Example: face recognition
  - Task: identify person A vs B based on a face image
  - We have:
    - 100 images of person A, in various conditions
    - 100 images of person B, in various conditions

# Samples vs distributions

- Compare face recognition with our previous signal detection
- We still have:
  - two hypotheses $H_0$ (person A) and $H_1$ (person B)
  - a sample vector $\mathbf{r}$ = the test image we need to decide upon
  - we can take two decisions
  - 4 scenarios: correct rejection, false alarm, miss, correct detection
- What's different? We don't have formulas
  - there is no "true" data described by formulas $s_0(t) = ...$ and $s_1(t)$...
  - (faces of persons A and B are not signals)
  - instead, we have lots of examples of each distribution
    - 100 images of A = examples of $\mathbf{r}$ might look in hypotesis $H_0$
    - 100 images of B = examples of $\mathbf{r}$ might look in hypotesis $H_1$

# Machine learning terminology

- Terminology used in **machine learning**:
  - This kind of problem = signal **classification** problem
    - given one data vector, specify which class it belongs to
  - The **classes** = the two categories, hypotheses $H_i$, persons A/B etc
  - A **training set** = a set of known data
    - e.g. our 100 images of each person
    - it will be used in the decision process
  - Signal **label** = the class of a signal

# Samples vs distributions

- The training set gives us the same information as the conditional distributions $w(r|H_0)$ and $w(r|H_1)$
  - $w(r|H_0)$ tells us how $r$ looks like in hypothesis $H_0$
  - $w(r|H_1)$ tells us how $r$ looks like in hypothesis $H_1$
  - the training set shows the same thing, without formulas, but via many examples
- OK, so how to classify the data in these conditions?

# The k-NN algorithm

The k-Neareast Neighbours algorithm (k-NN)

- Input:
  - a labelled training set of vectors $\mathbf{x}_1 ... \mathbf{x}_N$, from $L$ possible classes $C_1 \dots C_L$
  - a test vector $\mathbf{r}$ we need to classify
  - a parameter $k$

1. Compute distance from $\mathbf{r}$ to each training vector $\mathbf{x}_i$
   - can use same Euclidean distance we used for signal detection with multiple samples
2. Choose the closest $k$ vectors to $\mathbf{r}$ (the $k$ *nearest neighbours*)
3. Determine class of $\mathbf{r}$ = the majority class among the $k$ nearest neighbours

- Output: the class of $\mathbf{r}$

# The k-NN algorithm

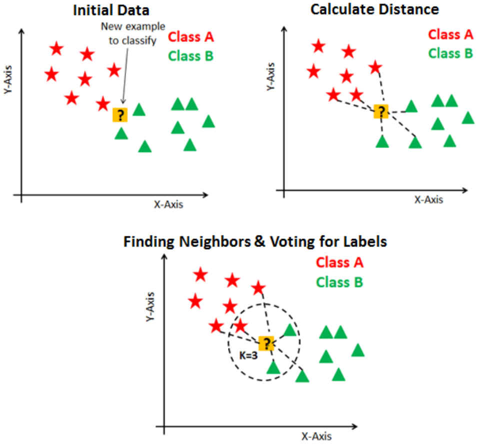



Figure 13: The k-NN algorithm illustrated [1]

[1] image from "KNN Classification using Scikit-learn", Avinash Navlani,

https://www.datacamp.com/community/tutorials/k-nearest-neighbor-classification-scikit-learn

# k-NN and ML decision

- If the training set is very large, the k-NN algorithm is a kind of ML decision
- The number of samples of a class in the vicinity of our point is proportional to $w(r|H_i)$
- More neighbors of class A than B $\Leftrightarrow w(r|H_A) > w(r|H_B)$

# k-NN and ML decision

- Example: leaves and trees

# Exercise

Exercise

1. Consider the k-NN algorithm with the following training set, composed of 5 vectors of class A and another 5 vectors from class B:
   - Class A:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \ \mathbf{v}_3 = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \ \mathbf{v}_4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \ \mathbf{v}_5 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

   - Class B:

$$\mathbf{v}_6 = \begin{bmatrix} 7 \\ 0 \end{bmatrix} \ \mathbf{v}_7 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \ \mathbf{v}_8 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \ \mathbf{v}_9 = \begin{bmatrix} -3 \\ 8 \end{bmatrix} \ \mathbf{v}_{10} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

   Compute the class of the vector $\mathbf{x} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$ using the k-NN algorithm, with $k = 1$, $k = 3$, $k = 5$, $k = 7$ and $k = 9$

# Discussion

- k-NN is a supervised learning algorithm
  - training data needs to be labelled
- Effect of $k$ is to smooth the decision boundary:
  - small $k$: lots of edges
  - large $k$: smooth boundary
- How to find $k$?

# Cross-validation

- How to find a good value for $k$?
  - by trial and error ("băbește")
- **Cross-validation** = use a small testing set for checking what parameter value is best
  - this data set is known as **cross-validation set**
  - use $k = 1$, test with cross-validation set and see how many vectors are classified correctly
  - repeat for $k = 2, 3, ...max$
  - choose value of $k$ with best results on the cross-validation set

# Evaluating algorithms

- How to evaluate the performance of k-NN?
  - Use a testing set to test the algorithm, check the percentage of correct classification
- Final testing set should be different from the cross-validation set
  - For final testing, use data that the algorithm has never seen, for fairness
- How to split the data into datasets?

# Datasets

- Suppose you have 200 face images, 100 images of person A and 100 of person B
- Split the data into:
  - Training set
    - data that shall be used by the algorithm
    - largest part (about 60% of the whole data)
    - i.e. 60 images of person A and 60 images of B
  - Cross-validation set
    - used to test the algorithm and choose best value of parameters ($k$)
    - smaller, about 20%, e.g. 20 images of A and 20 images of B
  - Testing set
    - used to evaluate the final algorithm, with all parameters set to a final value
    - smaller, about 20%, e.g. 20 images of A and 20 images of B

# The k-Means algorithm

- k-Means: an algorithm for data **clustering**
  - identifying groups of close vectors in data
- Is an example of unsupervised learning algorithm
  - "unsupervised learning" = we don't know the data classes of the signals beforehand

# The k-Means algorithm

The k-Means algorithm

- Input:
  - unlabelled training set of vectors $\mathbf{x}_1 ... \mathbf{x}_N$
  - number of classes C
- Initialization: randomly initialize the C centroids

$$\mathbf{c}_i \leftarrow \text{ random values}$$

- Repeat
  1. Classification: assign each data $\mathbf{x}$ to the nearest centroid $\mathbf{c}_i$:

  $$l_n = \arg\min_i d(\mathbf{x}, \mathbf{c}_i), \forall \mathbf{x}$$

  2. Update: update each centroids $\mathbf{c}_i =$ average of the $\mathbf{x}$ assigned to $\mathbf{c}_i$

  $$\mathbf{c}_i \leftarrow \text{ average of } \mathbf{x}, \forall \mathbf{x} \text{ in class } i$$

- Output: return the centroids $\mathbf{c}_i$, the labels $l_i$ of the input data $\mathbf{x}_i$

# The k-Means algorithm

Video explanations of the k-Means algorithm:

- Watch this, starting from time 6:28 to 7:08

  https: // www.youtube.com/watch?v=4b5d3muPQmA

- Watch this, starting from time 3:05 to end

  https: // www.youtube.com/watch?v=IuRb3y8qKX4

# The k-Means algorithm

- Not guaranteed that k-Means identifies good clusters
  - results depend on the random initialization of centroids
  - repeat many times, choose best result
  - smart initializations are possible (*k-Means++*)

# Exercise

Exercise

1. Consider the following data

$$\{\mathbf{v_n}\} = [1.3, -0.1, 0.5, 4.7, 5.1, 5.8, 0.4, 4.8, -0.7, 4.9]$$

   Use the k-Means algorithm to find the two centroids $\mathbf{c}_1$ and $\mathbf{c}_2$, starting from two random values $\mathbf{c}_1 = -0.5$ and $\mathbf{c}_2 = 0.9$. Perform 5 iterations of the algorithm.