

Decision and Estimation in Data Processing

## Chapter III. Elements of Estimation Theory

### III.1 Introduction

## What means “Estimation”?

- ▶ A sender transmits a signal  $s_\Theta(t)$  which depends on an **unknown** parameter  $\Theta$
- ▶ The signal is affected by noise, we receive  $r(t) = s_\Theta(t) + \text{noise}$
- ▶ We want to **find out** the correct value of the parameter
  - ▶ based on samples from the received signal, or the full continuous signal
  - ▶ available data is noisy => we “estimate” the parameter
- ▶ The found value is  $\hat{\Theta}$ , **the estimate** of  $\Theta$  (“estimatul”, rom)
  - ▶ there will always be some estimation error  $\epsilon = \hat{\Theta} - \Theta$

## What means “Estimation”?

- ▶ Examples:
  - ▶ Unknown amplitude of constant signal:  $r(t) = A + \text{noise}$ , estimate  $A$
  - ▶ Unknown phase of sine signal:  $r(t) = \cos(2\pi ft + \phi)$ , estimate  $\phi$
  - ▶ Even complicated problems:
    - ▶ Record speech signal, estimate/decide what word is pronounced

- ▶ Consider the following estimation problem:

We receive a signal  $r(t) = A + \text{noise}$ , estimate  $A$

- ▶ For detection, we have to choose between **two known values** of  $A$ :
  - ▶ i.e.  $A$  can be 0 or 5 (hypotheses  $H_0$  and  $H_1$ )
- ▶ For estimation,  $A$  can be anything  $\Rightarrow$  we choose between **infinite number of options** for  $A$ :
  - ▶  $A$  might be any value in  $\mathbb{R}$ , in general

## Estimation vs Decision

- ▶ Detection = Estimation constrained to **only a few** discrete options
- ▶ Estimation = Detection with an **infinite number** of options available
- ▶ The statistical methods used are quite similar
  - ▶ In practice, distinction between Estimation and Detections is somewhat blurred
  - ▶ (e.g. when choosing between 1000 hypotheses, do we call it “Detection” or “Estimation”?)

## Available data

- ▶ The available data is the received signal  $r(t) = s_\Theta(t) + \text{noise}$ 
  - ▶ it is affected by noise
  - ▶ it depends on the unknown parameter  $\Theta$
- ▶ We consider **N samples** from  $r(t)$ , taken at some sample times  $t_i$

$$\mathbf{r} = [r_1, r_2, \dots, r_N]$$

- ▶ The samples depend on the value of  $\Theta$

## Available data

- ▶ Each sample  $r_i$  is a random variable that depends on  $\Theta$  (and the noise)

- ▶ Each sample has a distribution that depends on  $\Theta$

$$w_i(r_i; \Theta)$$

- ▶ The whole sample vector  $\mathbf{r}$  is a N-dimensional random variable that depends on  $\Theta$  (and the noise)

- ▶ It has a N-dimensional distribution that depends on  $\Theta$

$$w(\mathbf{r}; \Theta)$$

- ▶ Equal to the product of all  $w_i(r_i|\Theta)$

$$w(\mathbf{r}|\Theta) = w_1(r_1|\Theta) \cdot w_2(r_2|\Theta) \cdot \dots \cdot w_N(r_N|\Theta)$$

## Two types of estimation

- ▶ We consider two types of estimation:
  1. **Maximum Likelihood Estimation (MLE)**: Besides  $r$ , nothing else is known about the parameter  $\Theta$ , except maybe some allowed range (e.g.  $\Theta > 0$ )
  2. **Bayesian Estimation**: Besides  $r$ , we know a **prior** distribution  $p(\Theta)$  for  $\Theta$ , which tells us the values of  $\Theta$  that are more likely than others
    - ▶ this is more general than BE

## II.2 Maximum Likelihood estimation

- ▶ When no distribution is known except  $\mathbf{r}$ , we use a method known as **Maximum Likelihood estimation (MLE)**
- ▶ We define the **likelihood** of a parameter value  $\Theta$ , given the available observations  $\mathbf{r}$  as:

$$L(\Theta|\mathbf{r}) = w(\Theta|\mathbf{r})$$

- ▶  $L(\Theta|\mathbf{r})$  is the likelihood function
- ▶ Compare with formula in Chapter 2, slide 20
  - ▶ it is the same
  - ▶ here we try to “guess”  $\Theta$ , there we “guessed”  $H_i$

## Maximum Likelihood definition

Maximum Likelihood (ML) Estimation:

- ▶ The estimate  $\hat{\Theta}_{ML}$  is **the value that maximizes the likelihood, given the observed data**

- ▶ i.e. the value that maximizes  $L(\Theta|\mathbf{r})$ , i.e. maximize  $w(\mathbf{r}|\Theta)$

$$\hat{\Theta}_{ML} = \arg \max_{\Theta} L(\Theta|\mathbf{r}) = \arg \max_{\Theta} w(\mathbf{r}|\Theta)$$

- ▶ If  $\Theta$  is allowed to live only in a certain range, restrict the maximization only to that range.

## Notations

- ▶ General mathematical notations:
  - ▶  $\arg \max_x f(x)$  = “the value  $x$  which maximizes the function  $f(x)$ ”
  - ▶  $\max_x f(x)$  = “the maximum value of the function  $f(x)$ ”

## Maximum Likelihood estimation vs decision

- ▶ Very similar with decision problem!
- ▶ ML decision criterion:
  - ▶ “pick the hypothesis with a higher likelihood”:

$$\frac{L(H_1|r)}{L(H_0|r)} = \frac{w(r|H_1)}{w(r|H_0)} \stackrel{H_1}{\gtrless} 1$$

- ▶ ML estimation
  - ▶ “pick the value which maximizes the likelihood”

$$\hat{\Theta}_{ML} = \arg \max_{\Theta} L(\Theta|r) = \arg \max_{\Theta} w(r|\Theta)$$

## How to solve

- ▶ How to solve the maximization problem?
  - ▶ i.e. how to find the estimate  $\hat{\Theta}_{ML}$  which maximizes  $L(\Theta|\mathbf{r})$
- ▶ Find maximum by setting derivative to 0

$$\frac{dL(\Theta|\mathbf{r})}{d\Theta} = 0$$

- ▶ We can also maximize the **natural logarithm** of the likelihood function (“log-likelihood function”)

$$\frac{d \ln(L(\Theta))}{d\Theta} = 0$$

## Solving procedure

Solving procedure:

1. Find the function

$$L(\Theta|\mathbf{r}) = w(\mathbf{r}|\Theta)$$

2. Set the condition that derivative of  $L(\Theta|\mathbf{r})$  or  $\ln(L(\Theta))$  is 0

$$\frac{dL(\Theta|\mathbf{r})}{d\Theta} = 0, \text{ or } \frac{d \ln(L(\Theta))}{d\Theta} = 0$$

3. Solve and find the value  $\hat{\Theta}_{ML}$

4. Check that second derivative at point  $\hat{\Theta}_{ML}$  is negative, to check that point is a maximum

- ▶ because derivative = 0 for both maximum and minimum points
- ▶ we'll sometimes skip this, for brevity

## Examples:

- ▶ Estimating a constant signal in gaussian noise:

Find the ML estimate of a constant value  $s_\Theta(t) = A$  from 5 noisy measurements  $r_i = A + \text{noise}$  with values [5, 7, 8, 6.1, 5.3]. The noise is AWGN  $\mathcal{N}(\mu = 0, \sigma^2)$ .

- ▶ Solution: at whiteboard.
- ▶ The estimate  $\hat{A}_{ML}$  is the average value of the samples
  - ▶ not surprisingly, what other value would have been more likely?
  - ▶ that's literally what "expected value" means

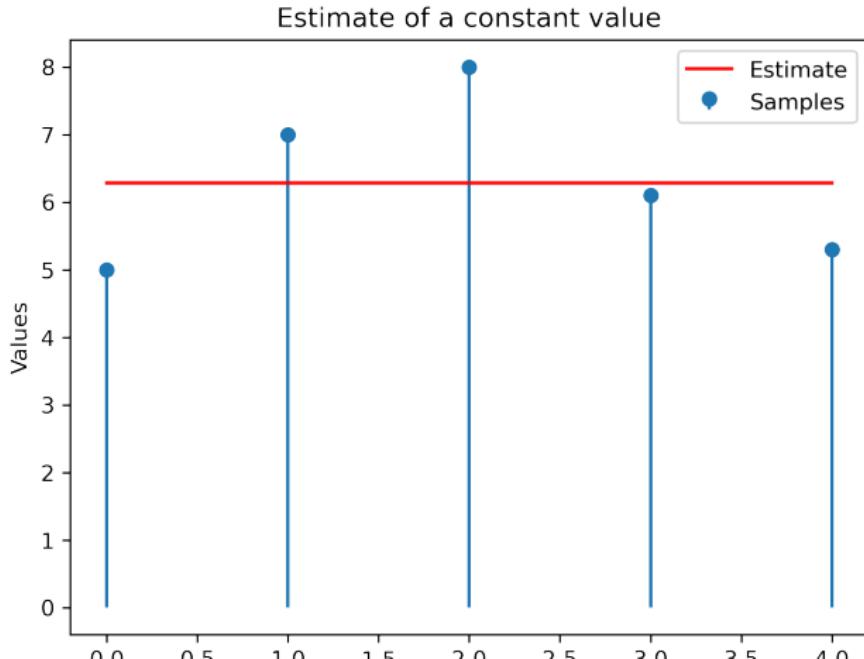
## Numerical simulation

Traceback (most recent call last):

```
  File "source.py", line 1, in <module>
```

```
    import matplotlib.pyplot as plt, numpy as np, math;
```

```
ModuleNotFoundError: No module named 'matplotlib'
```



- ▶ **Estimation = curve fitting**
  - ▶ we're finding the best fitting of  $s_\Theta(t)$  through the data  $\mathbf{r}$
- ▶ From the previous graphical example:
  - ▶ we have some data  $\mathbf{r} = \text{some points}$
  - ▶ we know the shape of the signal = a line (constant A)
  - ▶ we're fitting the best line through the data

## General signal in AWGN

- ▶ Consider that the true underlying signal is  $s_\Theta(t)$
- ▶ Consider **AWGN noise**  $\mathcal{N}(\mu = 0, \sigma^2)$ .
- ▶ The samples  $r_i$  are taken at sample moments  $t_i$
- ▶ The samples  $r_i$  have normal distribution with average value  $\mu = s_\Theta(t_i)$  and variance  $\sigma^2$
- ▶ Overall likelihood function = product of likelihoods for each sample  $r_i$

$$\begin{aligned}L(\Theta | \mathbf{r}) = w(\mathbf{r} | \Theta) &= \prod_{i=1}^N \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(r_i - s_\Theta(t_i))^2}{2\sigma^2}} \\&= \left( \frac{1}{\sigma \sqrt{2\pi}} \right)^N e^{-\frac{\sum (r_i - s_\Theta(t_i))^2}{2\sigma^2}}\end{aligned}$$

## General signal in AWGN

- The log-likelihood is

$$\ln(L(\Theta|\mathbf{r})) = \underbrace{\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)}_{constant} - \frac{\sum(r_i - s_\Theta(t_i))^2}{2\sigma^2}$$

## General signal in AWGN

- ▶ The maximum of the function = the minimum of the exponent

$$\hat{\Theta}_{ML} = \arg \max_{\Theta} L(\Theta | \mathbf{r}) = \arg \min \sum (r_i - s_{\Theta}(t_i))^2$$

- ▶ The term  $\sum (r_i - s_{\Theta}(t_i))^2$  is the **squared distance**  $d(\mathbf{r}, s_{\Theta})$

$$d(\mathbf{r}, s_{\Theta}) = \sqrt{\sum (r_i - s_{\Theta}(t_i))^2}$$

$$(d(\mathbf{r}, s_{\Theta}))^2 = \sum (r_i - s_{\Theta}(t_i))^2$$

## General signal in AWGN

- ▶ ML estimation can be rewritten as:

$$\hat{\Theta}_{ML} = \arg \max_{\Theta} L(\Theta | \mathbf{r}) = \arg \min_{\Theta} d(\mathbf{r}, \mathbf{s}_{\Theta})^2$$

- ▶ ML estimate  $\hat{\Theta}_{ML}$  = the value that makes  $s_{\Theta}(t_i)$  **closest to the received values r**
  - ▶ closer = better fit = more likely
  - ▶ closest = best fit = most likely = maximum likelihood

## General signal in AWGN

- ▶ ML estimation in AWGN noise = **minimization of distance**
- ▶ Hey, we had the same interpretation with ML decision!
  - ▶ but for decision, we choose the minimum out of 2 options
  - ▶ here, we choose the minimum out of all possible options
- ▶ Same interpretation applies for all kinds of vector spaces
  - ▶ vectors with N elements, continuous signals, etc
  - ▶ just change the definition of the distance function

## General signal in AWGN

Procedure for ML estimation in AWGN noise:

1. Write the expression for the (squared) distance:

$$D = (d(\mathbf{r}, s_\Theta))^2 = \sum (r_i - s_\Theta(t_i))^2$$

2. We want it minimal, so set derivative to 0:

$$\frac{dD}{d\Theta} = \sum 2(r_i - s_\Theta(t_i))(-\frac{ds_\Theta(t_i)}{d\Theta}) = 0$$

3. Solve and find the value  $\hat{\Theta}_{ML}$

4. Check that second derivative at point  $\hat{\Theta}_{ML}$  is positive, to check that point is a minimum

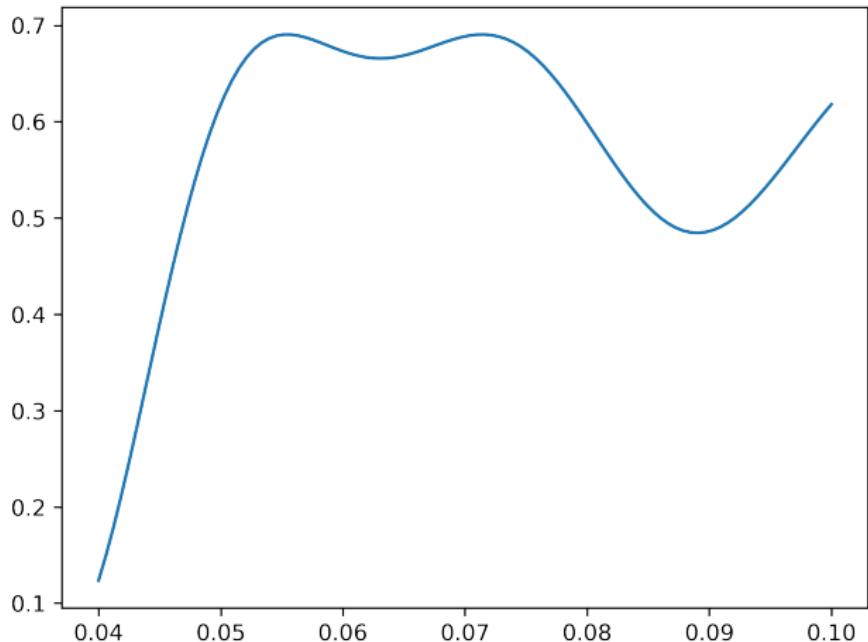
- ▶ we'll sometimes skip this, for brevity

Estimating the frequency  $f$  of a cosine signal

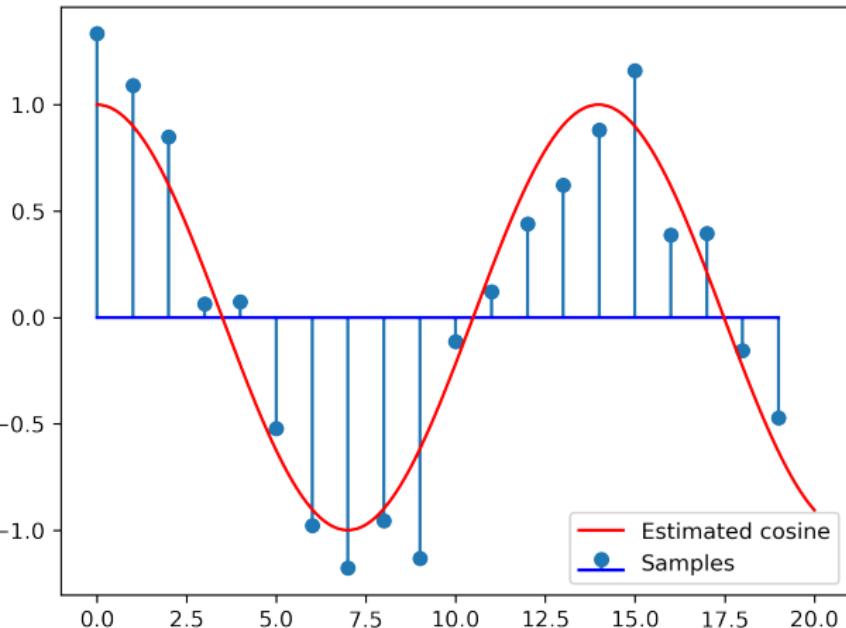
- ▶ Find the Maximum Likelihood estimate of the frequency  $f$  of a cosine signal  $s_\Theta(t) = \cos(2\pi ft_i)$ , from 10 noisy measurements  
 $r_i = \cos(2\pi ft_i) + \text{noise}$  with values [...]. The noise is AWGN  $\mathcal{N}(\mu = 0, \sigma^2)$ . The sample times  $t_i = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]$
- ▶ Solution: at whiteboard.

## Numerical simulation

The likelihood function is:



# Numerical simulation



## Multiple parameters

- ▶ What if we have more than one parameter?
  - ▶ e.g. unknown parameters are the amplitude, frequency and the initial phase of a cosine:
$$s(t) = A \cos(2\pi f t + \phi)$$
- ▶ We can consider the parameter  $\Theta$  to be a vector:

$$\Theta = [\Theta_1, \Theta_2, \dots, \Theta_M]$$

- ▶ e.g.  $\Theta = [\Theta_1, \Theta_2, \Theta_3] = [A, f, \phi]$

## Multiple parameters

- ▶ We solve with the same procedure, but instead of one derivative, we have  $M$  derivatives
- ▶ We solve the system:

$$\begin{cases} \frac{\partial L}{\partial \Theta_1} = 0 \\ \frac{\partial L}{\partial \Theta_2} = 0 \\ \dots \\ \frac{\partial L}{\partial \Theta_M} = 0 \end{cases}$$

- ▶ sometimes difficult to solve

# Gradient Descent

- ▶ How to estimate the parameters  $\Theta$  in complicated cases?
  - ▶ e.g. in real life applications
  - ▶ usually there are many parameters ( $\Theta$  is a vector)
- ▶ Typically it is impossible to get the optimal values directly by solving the system
- ▶ Improve them iteratively with **Gradient Descent** algorithm or its variations

## Gradient Descent procedure

1. Start with some random parameter values  $\Theta^{(0)}$
2. Repeat for each iteration  $k$ :
  - 2.1 Compute function  $L(\Theta^{(k)} | \mathbf{r})$
  - 2.2 Compute derivatives  $\frac{\partial L}{\partial \Theta_i^{(k)}}$  for each  $\Theta_i$  ("gradient")
  - 2.3 Update all values  $\Theta_i$  by subtracting the derivative ("descent")
3. Until termination criterion (e.g. parameters don't change much)

$$\Theta_i^{(k+1)} = \Theta_i^{(k)} - \mu \frac{\partial L}{\partial \Theta_i^{(k)}}$$

► or, in vector form:

$$\Theta^{(k+1)} = \Theta^k - \mu \frac{\partial L}{\partial \Theta^{(k)}}$$

## Gradient Descent explained

- ▶ Explanations at blackboard
- ▶ Simple example: logistic regression on 2D-data
  - ▶ maybe do example at blackboard

- ▶ The most prominent example is **Artificial Neural Networks** (a.k.a. Neural Networks, Deep Learning, etc.)
  - ▶ Can be regarded as ML estimation
  - ▶ Use Gradient Descent to update parameters
  - ▶ State-of-the-art applications: image classification/recognition, automated driving etc.
- ▶ More info on neural networks / machine learning:
  - ▶ look up online courses, books
  - ▶ join the IASI AI Meetup

- ▶ How good is an estimator?
- ▶ An estimator  $\hat{\Theta}$  is a **random variable**
  - ▶ can have different values, because it is computed based on the received samples, which depend on noise
  - ▶ example: in lab, try on multiple computers => slightly different results
- ▶ As a random variable, it has:
  - ▶ an average value (expected value):  $E\left\{\hat{\Theta}\right\}$
  - ▶ a variance:  $E\left\{(\hat{\Theta} - \Theta)^2\right\}$

# Estimator bias and variance

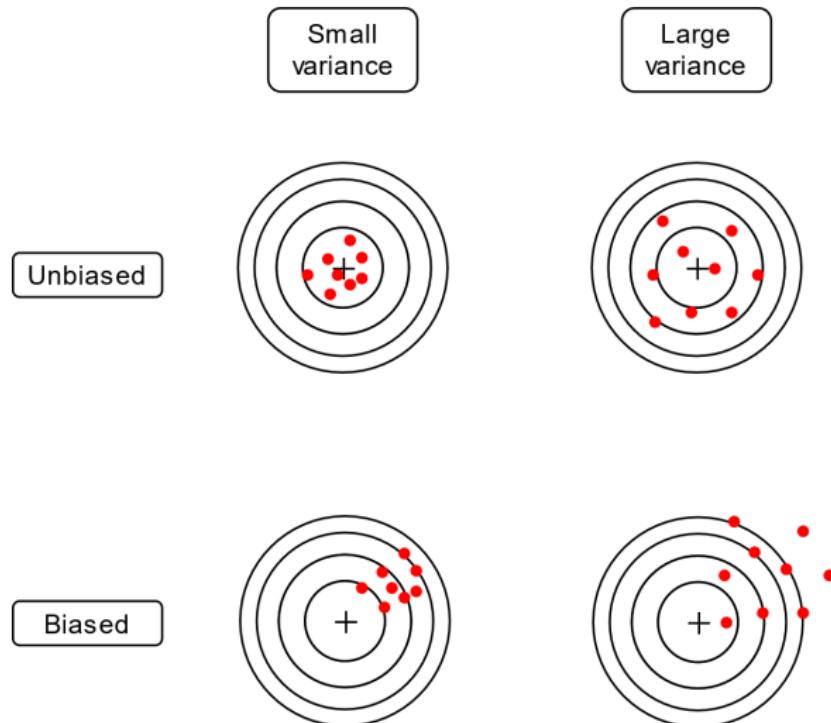


Figure 1: Estimator bias and variance

## Estimator bias

- ▶ The **bias** of an estimator  $\hat{\Theta}$  = difference between the estimator's average value and the true value

$$Bias = E\{\hat{\Theta}\} - \Theta$$

- ▶ Estimator is **unbiased** = the average value of the estimator is the true value of  $\Theta$

$$E\{\hat{\Theta}\} = \Theta$$

- ▶ Estimator is **biased** = the average value of the estimator is different from the true value  $\Theta$

- ▶ the difference  $E\{\hat{\Theta}\} - \Theta$  is **the bias** of the estimator

## Estimator bias

- ▶ Example: for constant signal  $A$  with AWGN noise (zero-mean), ML estimator is  $\hat{A}_{ML} = \frac{1}{N} \sum_i r_i$
- ▶ Then:

$$\begin{aligned} E\left\{\hat{A}_{ML}\right\} &= \frac{1}{N} E\left\{\sum_i r_i\right\} \\ &= \frac{1}{N} \sum_{i=1}^N E\{r_i\} \\ &= \frac{1}{N} \sum_{i=1}^N E\{A + \text{noise}\} \\ &= \frac{1}{N} \sum_{i=1}^N A \\ &= A \end{aligned}$$

- ▶ This estimator is unbiased

- ▶ The **variance** of an estimator measures the “spread” of the estimator around its average
  - ▶ that’s the definition of variance
- ▶ Unbiased estimators are good, but if the **variance** of the estimator is large, then estimated values can be far from the true value
- ▶ We prefer estimators with **small variance**, even if maybe slightly biased

## II.3 Bayesian estimation

- ▶ **Bayesian estimation** considers extra factors alongside  $w(\mathbf{r}|\Theta)$  in the estimation:
  - ▶ a prior distribution  $w(\Theta)$
  - ▶ possibly some cost function
- ▶ This makes it the estimation version of the MPE and MR decision criteria

- ▶ We define the **posterior** probability density of  $\Theta$ , given the known observations  $\mathbf{r}$ , using the **Bayes rule**:

$$w(\Theta|\mathbf{r}) = \frac{w(\mathbf{r}|\Theta) \cdot w(\Theta)}{w(\mathbf{r})}$$

- ▶ Explanation of the terms:
  - ▶  $\Theta$  is the unknown parameter
  - ▶  $\mathbf{r}$  are the observations that we have
  - ▶  $w(\Theta|\mathbf{r})$  is the probability of a certain value  $\Theta$  to be the correct one, given our current observations  $\mathbf{r}$ ;
  - ▶  $w(\mathbf{r}|\Theta)$  is the likelihood function
  - ▶  $w(\Theta)$  is the **prior distribution** of  $\Theta$ , i.e. what we know about  $\Theta$  even in the absence of evidence
  - ▶  $w(\mathbf{r})$  is the prior distribution of  $\mathbf{r}$ ; it is assumed constant

- ▶ The Bayes rule shows that the estimate of  $\Theta$  depends on two things:
  1. The observations that we have, via the term  $w(\mathbf{r}|\Theta)$
  2. The prior knowledge (or prior belief) about  $\Theta$ , via the term  $w(\Theta)$
  - ▶ (the third term  $w(\mathbf{r})$  is considered a constant, and plays no role)
- ▶ Known as “Bayesian estimation”
  - ▶ Thomas Bayes = discovered the Bayes rule
  - ▶ Stuff related to Bayes rule are often named “Bayesian”

## The prior distribution

- ▶ Suppose we know beforehand a distribution of  $\Theta$ ,  $w(\Theta)$ 
  - ▶ we know beforehand how likely it is to have a certain value
  - ▶ known as *a priori* distribution or *prior* distribution
- ▶ The estimation must take it into account
  - ▶ the estimate will be slightly “moved” towards more likely values

## The MAP estimator

- ▶ Suppose we know  $w(\Theta|\mathbf{r})$ . What is our estimate?
- ▶ Let's pick the value with the highest probability
- ▶ The **Maximum A Posteriori (MAP)** estimator:

$$\hat{\Theta}_{MAP} = \arg \max_{\Theta} w(\Theta|\mathbf{r}) = \arg \max_{\Theta} \Theta w(\mathbf{r}|\Theta) \cdot w(\Theta)$$

- ▶ The MAP estimator chooses  $\Theta$  as the value where the posterior distribution  $w(\Theta|\mathbf{r})$  is maximum
- ▶ The MAP estimator maximizes the likelihood of the observed data **but multiplied with the prior distribution  $w(\Theta)$**

## The MAP estimator

Image example here

## Relation with Maximum Likelihood Estimator

- ▶ The ML estimator:

$$\arg \max w(\mathbf{r}|\Theta)$$

- ▶ The MAP estimator:

$$\arg \max w(\mathbf{r}|\Theta) \cdot w(\Theta)$$

- ▶ The ML estimator is a particular case of MAP when  $w(\Theta)$  is a constant

- ▶  $w(\Theta) = \text{constant}$  means all values  $\Theta$  are equally likely
  - ▶ i.e. we don't have a clue where the real  $\Theta$  might be

- ▶ The MPE criterion  $\frac{w(r|H_1)}{w(r|H_0)} \stackrel{H_1}{\gtrless} \frac{P(H_0)}{P(H_1)}$
- ▶ It can be rewritten as  $w(r|H_1) \cdot P(H_1) \stackrel{H_1}{\gtrless} w(r|H_0)P(H_0)$ 
  - ▶ i.e. choose the hypothesis where  $w(r|H_i) \cdot P(H_i)$  is maximum
- ▶ **MPE decision criterion:** pick hypothesis which maximizes  $w(r|H_i) \cdot P(H_i)$ 
  - ▶ out of the two possible hypotheses
- ▶ **The MAP estimator:** pick value which maximizes  $w(\mathbf{r}|\Theta) \cdot w(\Theta)$ 
  - ▶ out of all possible values of  $\Theta$
- ▶ Same principle!

## Cost function

- ▶ Let's find an equivalent for the Minimum Risk criterion
- ▶ We need an equivalent for the costs  $C_{ij}$
- ▶ The **estimation error** = the difference between the estimate  $\hat{\Theta}$  and the true value  $\Theta$

$$\epsilon = \hat{\Theta} - \Theta$$

- ▶ The **cost function**  $C(\epsilon)$  = assigns a cost to each possible estimation error
  - ▶ when  $\epsilon = 0$ , the cost  $C(0) = 0$
  - ▶ small errors  $\epsilon$  have small costs
  - ▶ large errors  $\epsilon$  have large costs

## Cost function

- ▶ Usual types of cost functions:

- ▶ Quadratic:

$$C(\epsilon) = \epsilon^2 = (\hat{\Theta} - \Theta)^2$$

- ▶ Uniform (“hit or miss”):

$$C(\epsilon) = \begin{cases} 0, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| \leq E \\ 1, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| > E \end{cases}$$

- ▶ Linear:

$$C(\epsilon) = |\epsilon| = |\hat{\Theta} - \Theta|$$

- ▶ Draw them at whiteboard

## Cost function

- ▶ The cost function  $C(\epsilon)$  is the equivalent of the costs  $C_{ij}$  at detection
  - ▶ for detection we only had 4 costs:  $C_{00}$ ,  $C_{01}$ ,  $C_{10}$ ,  $C_{11}$
  - ▶ now we have a cost for all possible estimation errors  $\epsilon$
- ▶ The cost function guides which value to choose from  $w(\Theta|\mathbf{r})$

## The importance of the cost function

- ▶ Consider the following posterior distribution

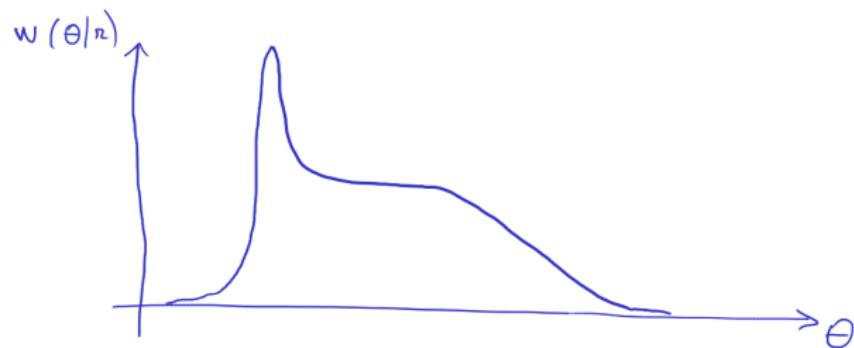


Figure 2: Unbalanced posterior distribution

- ▶ Which is the MAP estimate?
- ▶ Supposing we have the following cost function:
  - ▶ if your estimate  $\hat{\Theta}$  is  $<$  then the real  $\Theta$ , you pay 1000 \$
  - ▶ if your estimate  $\hat{\Theta}$  is  $>$  then the real  $\Theta$ , you pay 1 \$
  - ▶ does your estimate change ? :)

## The Bayesian risk

- ▶ The posterior distribution  $w(\Theta|\mathbf{r})$  tells us the probability of a certain value  $\hat{\Theta}$  to be the correct one of  $\Theta$
- ▶ Picking a certain estimate value  $\hat{\Theta}$  implies a certain error  $\epsilon$
- ▶ The error implies a certain cost  $C(\epsilon)$
- ▶ The **risk** = the average cost =  $C(\epsilon) \times$  the probability:

$$R = \int_{-\infty}^{\infty} C(\epsilon)w(\Theta|\mathbf{r})d\Theta$$

## The Bayes estimator

- We need to pick the value  $\hat{\Theta}$  which **minimizes the expected cost**  $R$

$$\hat{\Theta} = \arg \min_{\Theta} \int_{-\infty}^{\infty} C(\epsilon) w(\Theta | \mathbf{r}) d\Theta$$

- To find it, replace  $C(\epsilon)$  with its definition and derivate over  $\hat{\Theta}$ 
  - Attention: derivate with respect to  $\hat{\Theta}$ , not  $\Theta$ !

## MMSE estimator

- When the cost function is quadratic  $C(\epsilon) = \epsilon^2 = (\hat{\Theta} - \Theta)^2$

$$R = \int_{-\infty}^{\infty} (\hat{\Theta} - \Theta)^2 w(\Theta | \mathbf{r}) d\Theta$$

- We want the  $\hat{\Theta}$  that minimizes  $R$ , so we derivate

$$\frac{dR}{d\hat{\Theta}} = 2 \int_{-\infty}^{\infty} (\hat{\Theta} - \Theta) w(\Theta | \mathbf{r}) d\Theta = 0$$

- Equivalent to

$$\hat{\Theta} \underbrace{\int_{-\infty}^{\infty} w(\Theta | \mathbf{r}) d\Theta}_{1} = \int_{-\infty}^{\infty} \Theta w(\Theta | \mathbf{r}) d\Theta$$

- The **Minimum Mean Squared Error (MMSE)** estimator is

$$\hat{\Theta}_{MMSE} = \int_{-\infty}^{\infty} \Theta \cdot w(\Theta | \mathbf{r}) d\Theta$$

- ▶ **The MMSE estimator:** the estimator  $\hat{\Theta}$  is the **average value** of the posterior distribution  $w(\Theta|\mathbf{r})$

$$\hat{\Theta}_{MMSE} = \int_{-\infty}^{\infty} \Theta \cdot w(\Theta|\mathbf{r}) d\Theta$$

- ▶ MMSE = “Minimum Mean Squared Error”
- ▶ average value = sum (integral) of every  $\Theta$  times its probability  $w(\Theta|\mathbf{r})$
- ▶ The MMSE estimator is obtained from the posterior distribution  $w(\Theta|\mathbf{r})$  considering the quadratic cost function

# The MAP estimator

- When the cost function is uniform:

$$C(\epsilon) = \begin{cases} 0, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| \leq E \\ 1, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| > E \end{cases}$$

- Keep in mind that  $\Theta = \hat{\Theta} - \epsilon$
- We obtain

$$I = \int_{-\infty}^{\hat{\Theta}-E} w(\Theta|\mathbf{r})d\Theta + \int_{\hat{\Theta}+E}^{\infty} w(\Theta|\mathbf{r})d\Theta$$

$$I = 1 - \int_{\hat{\Theta}-E}^{\hat{\Theta}+E} w(\Theta|\mathbf{r})d\Theta$$

## The MAP estimator

- ▶ To minimize  $C$ , we must maximize  $\int_{\hat{\Theta}-E}^{\hat{\Theta}+E} w(\Theta|\mathbf{r})d\Theta$ , the integral around point  $\hat{\Theta}$
- ▶ For  $E$  a very small, the function  $w(\Theta|\mathbf{r})$  is approximately constant, so we pick the point where the function is maximum
- ▶ **The Maximum A Posteriori (MAP) estimator** = the value  $\hat{\Theta}$  which maximizes  $w(\Theta|\mathbf{r})$

$$\hat{\Theta}_{MAP} = \arg \max_{\Theta} w(\Theta|\mathbf{r}) = \arg \max_{\Theta} \Theta w(\mathbf{r}|\Theta) \cdot w(\Theta)$$

## Interpretation

- ▶ The MAP estimator chooses  $\Theta$  as the value where the posterior distribution is maximum
- ▶ The MMSE estimator chooses  $\Theta$  as average value of the posterior distribution

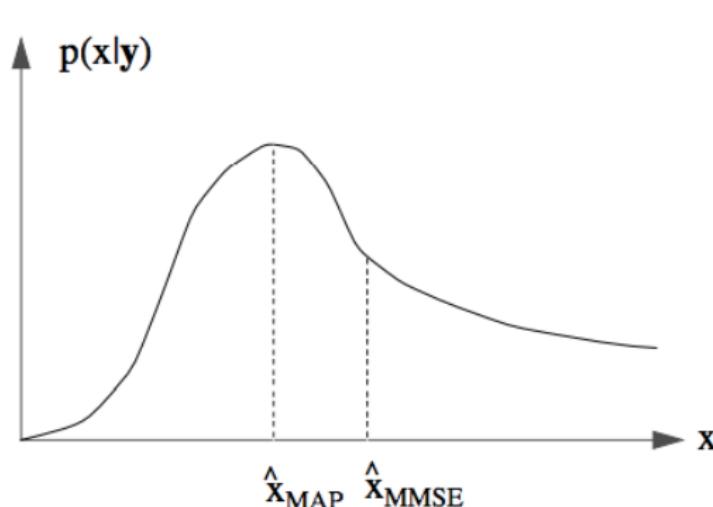


Figure 3: MAP vs MMSE estimators

## Relationship between MAP and MMSE

- ▶ The MAP estimator = minimizing the average cost, using the uniform cost function
  - ▶ similar with the MPE decision criteria = MR when all costs are same
- ▶ The MMSE estimator = minimizing the average cost, using the quadratic cost function
  - ▶ similar to MR decision criteria, but more general

## Exercise

Exercise: constant value, 3 measurement, Gaussian same  $\sigma$

- ▶ We want to estimate today's temperature in Sahara
- ▶ Our thermometer reads 40 degrees, but the value was affected by Gaussian noise  $\mathcal{N}(0, \sigma^2 = 2)$  (crappy thermometer)
- ▶ We know that this time of the year, the temperature is around 35 degrees, with a Gaussian distribution  $\mathcal{N}(35, \sigma^2 = 2)$ .
- ▶ Estimate the true temperature using ML, MAP and MMSE estimators

## Exercise

Exercise: constant value, 3 measurements, Gaussian same  $\sigma$

- ▶ What if he have three thermometers, showing 40, 38, 41 degrees

Exercise: constant value, 3 measurements, Gaussian different  $\sigma$

- ▶ What if the temperature this time of the year has Gaussian distribution  $\mathcal{N}(35, \sigma_2^2 = 3)$ 
  - ▶ different variance,  $\sigma_2 \neq \sigma$

## General signal in AWGN

- ▶ Consider that the true underlying signal is  $s_\Theta(t)$
- ▶ Consider AWGN noise  $\mathcal{N}(\mu = 0, \sigma^2)$ .
- ▶ As in Maximum Likelihood function, overall likelihood function

$$w(\mathbf{r}|\Theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\sum(r_i - s_\Theta(t_i))^2}{2\sigma^2}}$$

- ▶ But now this function is also **multiplied with**  $w(\Theta)$

$$w(\mathbf{r}|\Theta) \cdot w(\Theta)$$

## General signal in AWGN

- ▶ MAP estimator is the argument that maximizes this product

$$\hat{\Theta}_{MAP} = \arg \max w(\mathbf{r}|\Theta)w(\Theta)$$

- ▶ Taking logarithm

$$\begin{aligned}\hat{\Theta}_{MAP} &= \arg \max \ln(w(\mathbf{r}|\Theta)) + \ln(w(\Theta)) \\ &= \arg \max -\frac{\sum(r_i - s_\Theta(t_i))^2}{2\sigma^2} + \ln(w(\Theta))\end{aligned}$$

## Gaussian prior

- If the prior distribution is also Gaussian  $\mathcal{N}(\mu_\Theta, \sigma_\Theta^2)$

$$\ln(w(\Theta)) = -\frac{\sum(\Theta - \mu_\Theta)^2}{2\sigma_\Theta^2}$$

- MAP estimation becomes

$$\hat{\Theta}_{MAP} = \arg \min \frac{\sum(r_i - s_\Theta(t_i))^2}{2\sigma^2} + \frac{\sum(\Theta - \mu_\Theta)^2}{2\sigma_\Theta^2}$$

- Can be rewritten as

$$\hat{\Theta}_{MAP} = \arg \min d(\mathbf{r}, s_\Theta)^2 + \underbrace{\frac{\sigma^2}{\sigma_\Theta^2}}_{\lambda} \cdot d(\Theta, \mu_\Theta)^2$$

- ▶ MAP estimator with Gaussian noise and Gaussian prior

$$\hat{\Theta}_{MAP} = \arg \min d(\mathbf{r}, s_\Theta)^2 + \underbrace{\frac{\sigma^2}{\sigma_\Theta^2} \cdot d(\Theta, \mu_\Theta)^2}_{\lambda}$$

- ▶  $\hat{\Theta}_{MAP}$  is close to the expected value  $\mu_\Theta$  **and** it makes the true signal close to received data  $\mathbf{r}$

- ▶ Example: “search for a house that is close to job and close to the Mall”
  - ▶  $\lambda$  controls the relative importance of the two terms

- ▶ Particular cases

- ▶  $\sigma_\Theta$  very small = the prior is very specific (narrow) =  $\lambda$  large = second term very important =  $\hat{\Theta}_{MAP}$  close to  $\mu_\Theta$
  - ▶  $\sigma_\Theta$  very large = the prior is very unspecific =  $\lambda$  small = first term very important =  $\hat{\Theta}_{MAP}$  close to ML estimation

# Applications

- ▶ In general, practical applications:
  - ▶ can use various prior distributions
  - ▶ estimate **multiple parameters** ( a vector of parameters)
- ▶ Applications
  - ▶ denoising of signals
  - ▶ signal restoration
  - ▶ signal compression

### 1. Single object tracking with Kalman filtering

- ▶ estimating an object's position through successive noisy measurements (e.g. consecutive frames in a video)
- ▶ at every new measurement, we have two distributions of the position:
  - ▶ one given by the measurement itself,  $w(r|\Theta)$
  - ▶ one predicted based on position and speed from last moment
  - ▶ both are presumed Gaussian, described only through average value and variance
- ▶ the two are combined via the Bayes rule => a more precise distribution  $w(\Theta|r)$ , also Gaussian
- ▶ the exact position is estimated with MMSE (average value of  $w(\Theta|r)$ )
- ▶  $w(\Theta|r) + \text{speed}$  is used to predict the position at the next time moment

## Single object tracking

## Single object tracking

### 2. Constrained Least Squares (CLS) image restoration

- ▶ We have an image  $I$  corrupted by noise (additive noise, missing pixels, blurring)

$$I_{noisy} = I_{true} + Z$$

- ▶ We can estimate the original image by solving:

$$\hat{I}_{true} = \operatorname{argmin}_I \|I - I_{zg}\|_2 + \lambda \cdot \|HighPass\{I\}\|_2$$

- ▶ Examples:

- ▶ <https://www.mathworks.com/help/images/deblurring-images-using-a-regularized-filter.html>
- ▶ <https://demonstrations.wolfram.com/ImageRestorationForDegradedImages>
- ▶ Google it

# Constrained Least Squares (CLS) image restoration