DEDP Exam 2024-02-09 No.1

Exercises

1. (5p) Consider a continuous random variable X with uniform distribution $\mathcal{U}[-2, 2]$.

   a. (1p) Draw the distribution and compute its height;
   b. (1p) Compute the probability that X is $X > 0$;
   c. (1p) Compute the average squared value $\overline{X^2}$;
   d. (2p) Consider another random variable Y, also with uniform distribution, with the same average value as X, but with variance 2 times smaller than the variance of X. What is the distribution of Y?

1. (4p) Consider the decision problem for a constant signal $s(t)$ with two possible values, $s_0(t) = -2$ (hypothesis H_0) or $s_1(t) = 2$ (hypothesis H_1). The signal is perturbed by **uniform** noise with distribution $\mathcal{U}[-3, 3]$.

   The probabilities of the two hypotheses are $P(H_0) = 1/5$, $P(H_1) = 4/5$.

   The receiver takes a single sample r_0.

   a. (1p) Draw the two conditional distributions, $w(r|H_0)$ și $w(r|H_1)$, and write their mathematical expressions;
   b. (1p) Find the decision regions for the Minimum Probability of Error criterion;
   c. (2p) Compute the (unconditioned) probabilities of correct rejection and correct detection, for the same criterion.

2. (3p) Consider the detection of a signal that can be $s_0(t) = \cos(\pi t)$ (hypothesis H_0) or $s_1(t) = -\cos(\pi t)$ (hypothesis H_1).

   The signal is affected by gaussian noise $\mathcal{N}(\mu = 0, \sigma^2 = 5)$.

   The receiver takes 3 samples at time moments $t_1 = 0$, $t_2 = 1$ and $t_2 = 2$, with values $r_1 = -0.3$, $r_2 = 0.9$, and $r_3 = -0.7$.

   The probabilities of the two hypotheses are $P(H_0) = 1/4$, $P(H_1) = 3/4$.

   The costs of the four scenarios are $C_{00} = 0$, $C_{01} = 5$, $C_{10} = 25$, $C_{11} = 0$.

   a. (1p) What is the decision taken with the Minimum Risk criterion?
   b. (1p) Find the values $P(H_0), P(H_1)$ for which the Minimum Risk criterion is identical to the Maximum Likelihood criterion;

c. (1p) If a fourth sample would be taken for decision, which would be the worst sampling moment?

3. (3p) Consider a received signal $r(t) = \underbrace{2At+1}_{s(\Theta)} + noise$, where A is an unknown parameter. The signal is sampled at time moments $t_i = [0; 2; 4]$, and the values obtained are $r_i = [0.2; 9.2; 16.8]$. The noise has gaussian distribution $\mathcal{N}(\mu = 0, \sigma^2 = 4)$.

   Estimate the unknown parameter A using Maximum Likelihood estimation.

$$d(\mathbf{r}, \mathbf{s}_0)^2 \underset{H_0}{\overset{H_1}{\gtrless}} d(\mathbf{r}, \mathbf{s}_1)^2 + 2\sigma^2 \ln K \qquad K = \begin{cases} 1, & ML \\ \frac{P(H_0)}{P(H_1)}, & MPE \\ \frac{P(H_0)(C_{10}-C_{00})}{P(H_1)(C_{01}-C_{11})}, & MR \end{cases}$$

Theory

1. (2p) Let X be a random variable obtained by rolling a die with 10 faces (from 1 to 10). Draw the cumulative distribution function of X.
2. (2p) What does it mean for a random process to be ergodic?
3. (4p) Prove that the Minimum Probability of Error (MPE) criterion minimizes the total probability of error.
4. (2p) What is a matched filter? Explain how it is used, and for what purpose.
5. (2p) What is the relation between Maximum Likelihood estimation and Maximum A Posteriori estimation? Show that one of them is a particular case of the other.
6. (3p) Explain the role of the cost function in Bayesian estimation, and write the expressions of the cost functions used in Maximum A Posteriori (MAP) estimation and Minimum Mean Squared Error estimation.

Note: Obtain 30p for grade 10. 3p are granted by default. Time available: 2h