Distribution function of the total stopping time of the Collatz (3x+1) problem

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Let $a := \ln \frac{3}{2}$ and $b := -\ln 2$. We approximate the total stopping time distribution of the Collatz problem by the Brownian motion with drift and a variance parameter σ. We define scaled parameters by

$$v := \frac{a+b}{2\sigma}, \quad s := \frac{a-b}{2\sigma},$$

and let

$$\alpha := \frac{9}{2}s^2 - (3s-v)\sigma, \quad \beta := 2s^2.$$

The total stopping time distribution of the Brownian motion model for numbers $n \in [n_s, n_e]$ is given by

$$C(T; n_s, n_e) = \phi(T; n_e) - \phi(T; n_s), \tag{1}$$

$$\phi(T; n) := \sqrt{2} s\sigma \exp(g(x,T)) \Big[\Big(-\frac{s}{\sqrt{\beta}} - \frac{3s-v}{2\sqrt{\alpha}}\Big)\mathrm{erfcx}(\mathrm{X}) + \Big(\frac{s}{\sqrt{\beta}} - \frac{3s-v}{2\sqrt{\alpha}}\Big)\mathrm{erfcx}(\mathrm{Y})\Big], \tag{2}$$

where

$$\begin{aligned}
x &:= \frac{1}{\sigma}\ln n, \\
t &:= \frac{x+2sT}{3s-v}, \\
X &:= T\sqrt{\frac{\beta}{t}} + \sqrt{\alpha t}, \quad Y := T\sqrt{\frac{\beta}{t}} - \sqrt{\alpha t}, \\
g(x,T) &:= (6s^2 - 2s\sigma)T - \alpha t - \frac{\beta}{t}T^2,
\end{aligned}$$

and $\mathrm{erfcx}(x)$ is the scaled complementary error function defined as $\mathrm{erfcx}(x) := \exp(x^2)\mathrm{erfc}(x)$. We can estimate the longest total stopping time $T_{\max}$ for numbers $n \in [n_s, n_e]$ by solving the following equation.

$$C(T_{\max}; n_s, n_e) \sim \phi(T_{\max}; n_e) = 1. \tag{3}$$

In our implementation, we put $\sigma = \frac{a-b}{2}$ and we use $\log\phi$ instead of ϕ to avoid overflow.