MATH 229: Calculus III for Engineers

Takahiro Sakai

Martin Gong / 七海喬介

Jan 8 – ??, 2024

Contents

Chapter 1

Vector and the Geometry of Space

1.1 3-Dimensional Space

1.1.1 2D Coordinates

$$\mathbb{R}^2 = \left\{(x, y) \mid x, y \in \mathbb{R}\right\} \tag{1.1.1}$$

1.1.2 3D Coordinates

$$\mathbb{R}^3 = \left\{(x, y, z) \mid x, y, z \in \mathbb{R}\right\} \tag{1.1.2}$$

Lemma 1.1.1 (Distance Between 2 Points)

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{1.1.3}$$

Proof. Easily proven by using the Pythagorean Theorem twice. ∎

Lemma 1.1.2 (Spherical Surface)
Given point $C(a, b, c)$ and $P(x, y, z)$ where P is a point on the spherical surface and r is the radius of the sphere.

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \tag{1.1.4}$$

To define a solid spherical space

$$\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} \leq r \tag{1.1.5}$$

1.2 Vectors

Definition 1.2.1 (Vector)
Vector is a quantity that has a **magnitude** and a **direction**.

We say that two vectors $\vec{u}$ and $\vec{v}$ are equal if they have the same length and direction.

1.2.1 Vector Operation

Omitted

1.2.2 Components

In $\mathbb{R}^2$

$$\vec{a} \equiv \langle a_1, a_2 \rangle \tag{1.2.1}$$

In $\mathbb{R}^3$

$$\begin{cases} \vec{a} & \equiv \langle a_1, a_2, a_3 \rangle \\ \vec{0} & \equiv \langle 0, 0, 0, \rangle \end{cases} \tag{1.2.2}$$

Definition 1.2.2
Length of $\vec{a} \equiv \langle a_1, a_2, a_3 \rangle$ is

$$|\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2} \tag{1.2.3}$$

1.2.3 Standard Basis Vectors

$$\begin{cases} \hat{i} & = \langle 1, 0, 0 \rangle \\ \hat{j} & = \langle 0, 1, 0 \rangle \\ \hat{k} & = \langle 0, 0, 1 \rangle \end{cases} \tag{1.2.4}$$

1.3 The Dot Products

Definition 1.3.1

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle \tag{1.3.1}$$

Then, the dot product is

$$\vec{a} \cdot \vec{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.3.2}$$

Properties

1. $\vec{a} \cdot \vec{a} = {a_1}^2 + {a_2}^2 + {a_3}^2 = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

Theorem 1.3.1

$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\theta \tag{1.3.3}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\left|\vec{b}\right|}, 0 \le \theta \le \pi \tag{1.3.4}$$

Lemma 1.3.2
- If $\vec{a} \cdot \vec{b} > 0$ then $\cos\theta > 0 \implies \theta < \frac{\pi}{2}$
- If $\vec{a} \cdot \vec{b} < 0$ then $\cos\theta < 0 \implies \theta > \frac{\pi}{2}$
- If $\vec{a} \cdot \vec{b} = 0$, then $\theta = \frac{\pi}{2}, \vec{a} \perp \vec{b}$

1.3.1 Law of Cosine

$$\left|\vec{a} - \vec{b}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2|\vec{a}|\left|\vec{b}\right| \cos\theta \tag{1.3.5}$$

Proof.

$$\left|\vec{a} - \vec{b}\right|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \tag{1.3.6}$$

$$= |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + \left|\vec{b}\right|^2 \tag{1.3.7}$$

$$= |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2ab\cos(\theta) \tag{1.3.8}$$

■

1.3.2 Projection

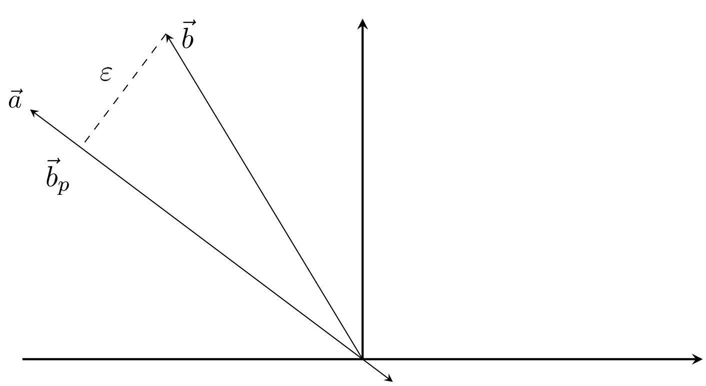

Figure 1.1: Projection

Add to this.

$$\left|\vec{b}\right| \tag{1.3.9}$$

Example 1.3.1

$$\vec{u} = \langle 1, 1, 2 \rangle \qquad \vec{v} = \langle -2, 3, 1 \rangle \tag{1.3.10}$$

Find projection of $\vec{u}$ onto $\vec{v}$

Solution:

$$\text{comp}_{\vec{c}}\vec{u} = \vec{u} \cdot \frac{\vec{v}}{|\vec{v}|} \tag{1.3.11}$$

$$= \frac{-2+3+2}{\sqrt{14}} = \frac{3}{\sqrt{14}} \tag{1.3.12}$$

$$\text{proj}_{\vec{v}}\vec{u} = (\text{comp}_{\vec{v}}\vec{u})\frac{\vec{v}}{|\vec{v}|} = \frac{3}{\sqrt{14}} \cdot \frac{\vec{v}}{\sqrt{v}} = \frac{3}{14}\vec{v} \tag{1.3.13}$$

1.3.3 Work

Move an an object from P to Q with a force $\vec{F}$ forming an angle θ with the displacement vector $\vec{D}$.

$$\text{Work} \equiv \text{Force} \times \text{Dist} \tag{1.3.14}$$

$$W = (\left|\vec{F}\right| \cos\theta)\left|\vec{D}\right| \tag{1.3.15}$$

$$= \left|\vec{F}\right|\left|\vec{D}\right| \cos\theta \tag{1.3.16}$$

$$= \vec{F} \cdot \vec{D} \tag{1.3.17}$$

$$\implies W = \vec{F} \cdot \vec{D} \tag{1.3.18}$$

Example 1.3.2
Move a particle from $P(2,1,0)[m]$ to $Q(4,6,2)$ with a force $\vec{F} = \langle 3, 4, 5\langle [N]$.
What is the work done by $\vec{F}$?

Solution:

$$W = \vec{F} \cdot \vec{PQ} \tag{1.3.19}$$

$$= \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \tag{1.3.20}$$

$$= 36\,\mathrm{N\,m} \tag{1.3.21}$$

1.4 The Cross Product

Definition 1.4.1
Given the vectors

$$\vec{a} = \langle a_1, a_2, a_3 \rangle, \vec{b} = \langle b_1, b_2, b_3 \rangle \tag{1.4.1}$$

The cross product is defined as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \tag{1.4.2}$$

Properties of the Dot Product

1. $(\vec{a} \times \vec{b}) \perp \vec{a} \& \vec{b}$ and the direction follows the right-hand rule.
2. $\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \left|\vec{b}\right| \sin\theta, 0 \leq \theta \leq \pi$
3. $\left|\vec{a} \times \vec{b}\right|$ = the area of the parallelogram formed by the two vectors.
4. If $\vec{a} \parallel \vec{b}$, then $\vec{a} \times \vec{b} = \vec{0}$
5. Cross product of basis vectors

$$\begin{cases} \hat{i} \times \hat{j} & = \hat{k} \\ \hat{j} \times \hat{k} & = \hat{i} \\ \hat{k} \times \hat{i} & = \hat{j} \end{cases} \tag{1.4.3}$$

6. The cross product is not commutative
7. The cross product is not associative

Example 1.4.1

$$\begin{cases} \hat{i} \times (\hat{i} \times \hat{j}) & = \hat{i} \times \hat{k} = -\hat{j} \\ (\hat{i} \times \hat{i}) \times \hat{j} & = \vec{0} \times \hat{j} = \vec{0} \end{cases} \tag{1.4.4}$$

8. You can find the normal vector to a plane by applying the cross product to two non-parallel vectors on that plane.

Example 1.4.2
Given points

$$P(1, 4, 6), Q(-2, 5, 1), R(1, -1, 1)$$

that lie on a plane

a) Find the vector normal to the plane
b) Find the area of $\triangle PQR$

Solution:

TBA

Definition 1.4.2 (Triple Products)

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) \tag{1.4.5}$$

Eq. (1.4.5) shows the scalar triple product. This is also the volume of the parallelepiped.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{1.4.6}$$

Eq. (1.4.6) shows the vector triple product.

Lemma 1.4.1
If $\vec{a}, \vec{b}$, and $\vec{c}$ are on the same plane (*coplanar*), then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

1.5 Lines and Planes

Definition 1.5.1 (Line)
We define a line with a direction vector $\vec{v} = \langle a, b, c \rangle$

$$\vec{r} = \vec{v}_0 + t\vec{v} \tag{1.5.1}$$

Parametric Form

$$\begin{cases} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{cases} \tag{1.5.2}$$

Symmetric Form

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{1.5.3}$$

Notice how the symmetric form does not require parameters, it tells the relationship between the coordinates.

Example 1.5.1
Intersection problem

Definition 1.5.2 (Plane)
Given a point $P_0 \equiv \vec{r}_0$ and another point $P \equiv \vec{r}$ on the plane, along with the normal vector $\hat{n} = \langle a, b, c \rangle$.
Now, we see that $\vec{r} - \vec{r}_0$ is always on the plane, so that it follows that

$$\hat{n} \cdot (\vec{r} - \vec{r}_0) = 0 \tag{1.5.4}$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{1.5.5}$$
$$ax + by + cx = d \tag{1.5.6}$$

where $d = ax_0 + by_0 + cz_0$

Example 1.5.2
Given $A(2, 0, 3), B(0, -4, 6), C(-3, 6, 0)$, on a plane, find the equation of the plane.

Solution:

We find that

$$\overrightarrow{AB} \times \overrightarrow{AC} = -8\langle 2, 3, 4\rangle \tag{1.5.7}$$

We take any point and compute d

$$d = 2 \cdot 2 + 0 \cdot 3 + 3 \cdot 4 = 16 \tag{1.5.8}$$

so the equation is

$$2x + 3y + 4z = 16 \tag{1.5.9}$$

What if we want to sketch the plane?

We simply find the x, y, z-intersection of the plane, label them on a skeleton, then connect they for a triangle.

Example 1.5.3
Given two planes

$$\begin{cases} x + y + z & = 1 \\ x - 2y + 3z & = 1 \end{cases} \tag{1.5.10}$$

a) Find the angle between the two planes
b) Find the equation of the intersecting line

Solution:

a) We have the normal vectors

$$\begin{cases} \hat{n}_1 & = \langle 1, 1, 1\rangle \\ \hat{n}_2 & = \langle 1, -2, 3\rangle \end{cases} \tag{1.5.11}$$

We simply find the angle between them using the dot product.

$$\arccos\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right) = \arccos\left(\frac{2}{\sqrt{42}}\right) \tag{1.5.12}$$

b) We need the direction vector and a point on the line.

We can find a point on the line by defining either x, y, or z for the two equations and solve for the other variables. (e.g. A point here on the line is $P(1, 0, 0)$)

For the direction vector, we can cross the normal vectors $\vec{n_1} \times \vec{n_2}$ to find the vector.

Definition 1.5.3 (Distance Between a Point and a Plane)
Given some point P and a random point A on the plane, we can have some vector $\overrightarrow{AP}$, which, if we project onto the normal vector $\hat{n}$ of the plane, will give us the component of the vector $\overrightarrow{AP}$ parallel to the normal vector.

$$d = \left|\overrightarrow{AP}\right||\cos\theta| = \left|\overrightarrow{AP}\cdot\hat{n}\right| \tag{1.5.13}$$

where θ is the angle between the $\overrightarrow{AP}$ and $\hat{n}$

Example 1.5.4
We want to find the distance between two paralle planes.

Solution:

Simply find a vector that "connects" the two planes, let that vector be $\vec{v}$. Then, calculate $|\vec{v}\cdot\hat{n}|$ where $\hat{n}$ is the normal vector of the plane.

1.6 Cylinders and Quadric Surfaces

1.6.1 Cylinders

The perimeter of different cross sections of a cylinder are called **traces**, and the lines parallel to the cylindrical axis are called **???**.

Example 1.6.1

$$x = z^2 \tag{1.6.1}$$

This is a parabolic cylinder; we see a parabolic curve on the xz-plane.

Example 1.6.2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.6.2}$$

This is a cylindrical surface with intersect a and b on x and y respectively.

1.6.2 Quadric Surface

Remark
Spherical surfaces are a type of quadric surface.

Example 1.6.3 (Ellipsoid)

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 \tag{1.6.3}$$

Solution:

Let $z = k$ some constant.

Then,

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \tag{1.6.4}$$

We observe that only $-2 \leq k \leq 2$ do we see a surface. So the z intersects are -2 and 2. Now we can solve for $x|_{y=0}$ and $y|_{x=0}$ and we find $\pm 1, \pm 3$

We can keep finding the range by turning the equation into the form $c_1 {x_1}^2 + c_2 {x_2}^2 = c_3$.

When $x = k$ we get

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \tag{1.6.5}$$

When $y = k$ we get

$$x^2 + \frac{z^2}{4} = 1 - \frac{y^2}{9} \tag{1.6.6}$$

This is called an ellipsoid.

Example 1.6.4 (Elliptic Parabola)

$$z = 4x^2 + y^2 \tag{1.6.7}$$

Solution:

We notice that this surface only exists when $z \geq 0$.

Then, we can just sketch traces and connect.

Let $z = k$

$$4x^2 + y^2 = k \tag{1.6.8}$$

We see an ellipse cross section that grows as k increases.

Let $x = 0$ or $y = 0$

We see a parabolic cross section.

This is an **elliptic parabola**.

Example 1.6.5 (Hyperbolic Parabola)

$$z = y^2 - x^2 \tag{1.6.9}$$

Example 1.6.6 (Elliptic Cone)

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{1.6.10}$$

1.7 Vector Functions

Definition 1.7.1 (Vector Function)

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \tag{1.7.1}$$

Example 1.7.1
The vector function of a line.

$$\vec{r}_0 + \vec{v}t = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle \tag{1.7.2}$$

Definition 1.7.2 (Derivative of Vector Functions)

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \equiv \vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h)\vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle \tag{1.7.3}$$

Definition 1.7.3 (Rules of Differentiation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[a\vec{u}(t) + b\vec{v}(t)\right] = a\vec{u}'(t) + b\vec{v}'(t) \tag{1.7.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \tag{1.7.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t) \tag{1.7.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t) \tag{1.7.7}$$

1.8 Arc Length and Curvature

Chapter 2

Partial Derivatives

2.1 Functions of Several Variables

Definition 2.1.1 (Function of 2 Variables)

$$z = f(x, y) \tag{2.1.1}$$

Where z is the dependent variable and x, y are independent variables.
Defines a point $(x, y, f(x, y))$ in $\mathbb{R}^3$
We have the domain of f in $\mathbb{R}^2$ and its range in $\mathbb{R}$

2.2 Continuity

Skipped

2.3 Partial Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x, y) \implies \frac{\partial f}{\partial x} \equiv f_x(x, y) \tag{2.3.1}$$

We simply hold all other variables as constant and derive with respect to x.

Definition 2.3.1 (Partial Derivative)

$$\frac{\partial f}{\partial x} \equiv f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \tag{2.3.2}$$