MATH 225: Linear Algebra and Differential Equations

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Induction

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Matrices

2.1 Definitions and Examples

Definition 2.1.1
A $m \times n$ matrix A is a rectangular array with m rows and n columns.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Remark
If $m = n$, matrix A is called a **square matrix**.

Example 2.1.1 (Three Matrices)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & 7 \end{bmatrix} \qquad 2 \times 3 \tag{2.1}$$

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 4 \\ 1 & 0 & 2 \end{bmatrix} \qquad 5 \times 3 \tag{2.2}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \text{Identity Matrix} \tag{2.3}$$

Definition 2.1.2 (Equality)
Two matrices A and B are equal if they have the same dimension and all entries are the same

$$\forall (i,j) \in [\![1,m]\!] \times [\![1,n]\!] \quad a_{ij} = b_{ij}$$

where

$$[\![x,y]\!] = \left\{x | n \in \mathbb{Z}, x \leq n \leq y\right\}$$

Definition 2.1.3 (Transpose)
The **transpose** of a matrix $A = \left[a_{ij}\right]_{1\leq i\leq m, 1\leq i\leq n}$ is the $n \times m$ matrix $A^T = \left[a_{ij}\right]_{1\leq i\leq n, 1\leq i\leq m}$ defined by

$$\forall (i,j) \in [\![1,n]\!] \times [\![1,m]\!], c_{ij} := a_{ji}$$

Definition 2.1.4 (Diagonal)
Let A be a n-dimensional square matrix.
The diagonal of A is the list

$$\text{diag}(A) = (a_{11}, a_{22}, \ldots, a_{nn})$$

Chapter 3

Gauss' Pivot Algorithm & Applications

Chapter 4

Determinant of a Square Matrix

Chapter 5

The Vector Space

5.1 Definition and Equations

Definition 5.1.1
$\mathbb{R}^n$ denotes the set of $n \times 1$ matrices with real coefficients.

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \right\} \tag{5.1}$$

Example 5.1.1
$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -5 \\ 3 \end{bmatrix}, w = \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix}$ are vectors in $\mathbb{R}^2$
We can express $\mathbb{R}^2$ on the plane and $\mathbb{R}^3$ in a 3D space.

Remark
An immediate, yet fundamental feature of the elements of $\mathbb{R}^n$ is that we can add them, multiply them by scalars, and still obtain a vector in $\mathbb{R}^n$.

In other words, $\mathbb{R}^n$ is closed **under linear combinations of vectors**

i.e.
if

$$\begin{cases} u_1, u_2, \ldots, u_p \in \mathbb{R}^n \\ \alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R} \end{cases} \tag{5.2}$$

then $\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_p u_p \in \mathbb{R}^n$
which can also be expressed also

$$\sum_{i=0}^{p} \alpha_i u_i \tag{5.3}$$

are linear combinations of u_i

Note that $\mathbb{R}^n$ contains a **zero** element (zero vector), namely $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Multiplying the zero vector by any vector should be the zero vector.

5.2 Vector Equations in $\mathbb{R}^n$

Definition 5.2.1
Let $a_1, \ldots, a_p, b \in \mathbb{R}^n$
A **vector equation** in $\mathbb{R}^n$ is an equation of the form:

$$x_1 a_1 + x_2 a_2 + \cdots + x_p a_p = b, x_i \in \mathbb{R} \tag{5.4}$$

The scalars x_i are the **unknowns**

Remark
Solving the vector equation Eq. (5.4) is checking whether the vector b can be obtained as a linear combination of the vectors a_i

Remark
Solving Eq. (5.4) $\iff$ Solving the linear system $Ax = b$

where

$$A = \begin{bmatrix} a_1 \mid a_2 \mid \cdots \mid a_p \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \in \mathbb{R}^n \tag{5.5}$$

Conversely, the linear system $Ax = b$ has solutions if b can be obtained as a linear combination of the columns.

Example 5.2.1
Let

$$a_i = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, a_2 = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ -8 \\ -9 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix} \tag{5.6}$$

Solve the vector equation

$$x_1 a_1 + x_2 a_2 + x_3 a_3 = b, x_i \in \mathbb{R} \tag{5.7}$$

Solution:

Eq. (5.7) is equivalent to the linear system $Ax = b$ with

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & -9 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$$

$\det(A) = -34$ which means that Eq. (5.7) has a **unique** solution.

Let us find x_1 using Cramer's Rule.

$$\det\big(A_1(b)\big) = \begin{vmatrix} 0 & -2 & 1 \\ 8 & 2 & -8 \\ 9 & 5 & -9 \end{vmatrix} = 22 \tag{5.8}$$

Thus,

$$x_1 = \frac{\det\big(A_1(b)\big)}{\det(A)} = -\frac{11}{17} \tag{5.9}$$

5.3 Span of a Family of Vectors

Definition 5.3.1
Let $v_1, \ldots, v_p$ be a family of vectors in $\mathbb{R}^n$.
The **span** of $v_1, \ldots, v_p$ is the set of all linear combinations of $v_1, \ldots, v_p$.

$$\operatorname{span}\{v_1, \ldots, v_p\} = \left\{ \begin{matrix} \alpha_1 v_1 + \ldots + \alpha_p v_p \\ \alpha_i \in \mathbb{R} \end{matrix} \right. = \left\{ \sum_{i=1}^{p} \alpha_i v_i \mid \alpha_i \in \mathbb{R} \right\} \tag{5.10}$$

Given a vector $v \in \mathbb{R}^n$ one may want to know, if v lies in $\operatorname{span}\{\mu_1, \ldots, \mu_p\}$. This means solving the vector equation

$$v = \alpha_1 \mu_1 + \ldots + \alpha_p \mu_p \tag{5.11}$$

In particular $v \in \text{span}\{\mu_1, \ldots, \mu_p\}$ if Eq. (5.11) has a solution.

Example 5.3.1

$\mu_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mu_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, does $v = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ lie in $\text{span}\{\mu_1, \mu_2\}$?

Chapter 6

Vector Spaces

6.1 Definitions and Examples

Definition 6.1.1
Let V be a nonempty set equipped with an addition operation $(+)$, and a scalar multiplication $(\cdot)$, scalars being in $\mathbb{R}$ (or in $\mathbb{C}$).
V is a $\mathbb{R}$-vector space (or a $\mathbb{C}$-vector space) if[a]

1. $\forall u, v \in V, u + v \in V$
2. $\forall \lambda \in \mathbb{R}, \forall u \in V, \lambda u \in V$[b]
3. $\forall u, v \in V, u + v = v + u$
4. $\forall u, v, w \in V, u + v + w = (u + v) + w = u + (v + w)$
5. $\exists 0_v \in V, \forall u \in V, u + 0_v = u$
6. $\forall u \in V, \exists - u \in V : u + (-u) = 0_v$[c]
7. $\forall u \in V, 1 \cdot u = u$
8. $\forall u \in V, \forall \lambda, \mu, \lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u$
9. $\forall \lambda \in \mathbb{R}, \forall u, v \in V, \lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$
10. $\forall \lambda, \mu \in \mathbb{R}, \forall u, (\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$

Note: Statements 3 - 6 defines a commutative group (Abelian Group)

[a] To check a vector space, all 10 of these conditions need to be checked. Good to be memorized.
[b] This is called being closed under addition and scalar multiplication.
[c] Try: Show that $-u = (-1) \cdot u$

Examples

Example 6.1.1
$\mathbb{R}^n$ is a $\mathbb{R}$-vector space

Example 6.1.2
$\mathbb{C}^n$ is a $\mathbb{C}$-vector space, it can also be seen as a $\mathbb{R}$-vector space.

Example 6.1.3
$M_{mn}(\mathbb{R})$[a] is a $\mathbb{R}$-vector space

[a]Set of all $m \times n$ matrices with real entries

Example 6.1.4
$C^k(I, \mathbb{R})$[a] is a $\mathbb{R}$-vector space.

[a]Set of all real-valued functions that are C_k on the interval $I \subset \mathbb{R}$

Example 6.1.5
$P_n(\mathbb{R}) = \left\{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1_0 \mid a_i \in \mathbb{R}\right\}$[a] is a $\mathbb{R}$-vector space.

[a]All polynomial functions of degree at most n with real coefficients.

Remark

$$f \qquad f(x)$$

f is technically the function, and $f(x)$ is not the function but the image of f at x.

Example 6.1.6
$V = \left\{A \in M_n(\mathbb{R}) \mid \operatorname{tr}(A) = 0\right\}$ is a $\mathbb{R}$-vector space.

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_i i$$

Example 6.1.7
$V = \{0_v\}$ is a vector space.

Example 6.1.8
$V = \left\{y \in C^\infty(\mathbb{R}, \mathbb{R}) \mid y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y^{(0)} = 0\right\}$[a] where a_i are given coefficients.
0 is the zero function, because it is the linear combination of a series of functions.

[a](linear) differential equation of order n

6.2 Subspaces

Definition 6.2.1
Let V be a vector space. A subset S of V is a subspace of the vector space V if S is itself a vector space.

Theorem 6.2.1
Let S be a subset of a vector space V. Then, S is a subspace of V

$$\iff (\forall u, v \in S, \forall \lambda \in \mathbb{R}, u + \lambda v \in S) \tag{6.1}$$

Example 6.2.1
$S = \left\{A \in M_n(\mathbb{R}) | \operatorname{tr}(A) = 0\right\}$ is a subspace of the vector space $V = M_n(\mathbb{R})$

Proof. Let $A, B \in S$, let $\lambda \in \mathbb{R}$

Is $A + \lambda B \in S$?

$$\begin{aligned}
\operatorname{tr}(A + \lambda B) &= \sum_{i=1}^{n} a_{ii} + \lambda b_{ii} && (6.2)\\
&= \sum_{i=1}^{n} (a_{ii}) + \lambda \sum_{i=1}^{n} (b_{ii}) && (6.3)\\
&= 0 + \lambda 0 && (6.4)\\
&= 0 \implies \boxed{A + \lambda B \in S} && (6.5)
\end{aligned}$$

by Theorem, S is a subspace of the vector space $V = M_n(\mathbb{R})$ ■

Example 6.2.2
$S = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R} \mid x_1 - 2x_2 + 3x_3 = 0 \right\}$ is a subapce of the vector space $V = \mathbb{R}^3$

Example 6.2.3
$S = \left\{ y \in C^\infty(\mathbb{R}, \mathbb{R}) \mid y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y^{(0)} = 0 \right\}$ is a subspace of the vector space $V = C^\infty(\mathbb{R}, \mathbb{R})$

Proof. Let $y_1, y_2 \in S$, let $\lambda \in \mathbb{R}$

Does $y_1 + \lambda y_2 = S$?

$$\begin{aligned}
&(y_1 + \lambda y_2)^{(n)} + \cdots + a_0 (y_1 + \lambda y_2)^{(0)} && (6.6)\\
&= (y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \cdots + a_0 y_1) + \lambda (y_2^{(n)} + a_{n-1} y_2^{(n-1)} + \cdots + a_0 y_2) && (6.7)\\
&\implies y_1 + \lambda y_2 \in S && (6.8)
\end{aligned}$$

$\implies$, by the theorem, S is a subspace of the vector space $v \in C^\infty(\mathbb{R}, \mathbb{R})$ ■

Example 6.2.4
$S = \left\{ A \in M_n(\mathbb{R}) \mid A^T = A \right\}$ is a subspace of the vector space $V = M_n(\mathbb{R})$

Example 6.2.5
$$S = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 - 2x_2 + 3x_3 = 1 \right\}$$

Solution:

This is NOT a subspace of $V = \mathbb{R}^3$ since $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S$

Example 6.2.6
$S = \{0_v\}$ is a subspace of the vector space V

Solution:

It is indeed the smallest subspace

Example 6.2.7 (Quiz 4)
Determine whether $S = \{A \in M_3(\mathbb{R}) \mid A(I_3 + A) = 2A\}$ is a subspace.

Solution:

It is not a subspace. It is easier to show counterexamples since intuitively it is not a subspace.

Example 6.2.8
Let V be a vecto space. Let $\{v_1, v_2, \cdots, v_n\} \subset V$
Then: $\text{span}\{v_i\}_{i=1,\cdots,p}$ is a subspace of the vector space V

Example 6.2.9
Proposition: If S_1, S_2 be subspaces of a vector spac V
Then: $S_1 \cap S_2$ is a subspace of V

Proof. Let $u, v \in S_1 \cap S_2$, let $\lambda \in \mathbb{R}$

$$u, v \in S_1 \cap S_2 \implies \begin{cases} u, v \in S_1 & \implies u + \lambda v \in S_1 \text{ since } S_1 \text{ is a subspace} \\ u, v \in S_2 & \implies u + \lambda v \in S_2 \text{ since } S_2 \text{ is a subspace} \end{cases} \tag{6.9}$$

Which implies $u + \lambda v \in S_1 \cap S_2 \implies S_1 \cap S_2$ is a subspace ■

Remark
S_1, S_2 subspaces of a $v = V \not\Longrightarrow S_1 \cup S_2$ is a subspace of V

Example 6.2.10

$V = \mathbb{R}^2$
$S_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y = x \right\}$
$S_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y = -x \right\}$
$S_1 \cup S_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y^2 = x^2 \right\}$
Given two vectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ there will be a disagreement

6.3 Spanning Set, Linear Independency, Basis

6.3.1 Spanning Set

Definition 6.3.1
Let V be a vector space. A family $\{v_i\}_{i=1,2,\ldots,p} \subset V$ is a spanning set of V if $V = \text{span}\{v_1, v_2, \ldots, v_p\}$

Example 6.3.1

$$\mathbb{R}^n = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \tag{6.10}$$

Solution:

Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, x_i \in \mathbb{R}$

Then, $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

$\Longrightarrow x \subseteq \text{span}\{e_1, \ldots, e_n\}$

$\mathbb{R}^n \subseteq \text{span}\{e_i\}_i$

Note that $\{e_i\}_i \subset \mathbb{R}^n \implies \text{span}\{e_i\}_i \subset \mathbb{R}^n$

$\mathbb{R}^n$ is a vector space hence closed under combination.

Thus, $\text{span}\{e_i\}_i = \mathbb{R}^n$

Example 6.3.2

$$P_n(\mathbb{R}) = \left\{p \in C^\infty(\mathbb{R},\mathbb{R}) \mid \exists a_i \in \mathbb{R}, \forall x \in \mathbb{R}, p(x) = a_n x^n + \cdots + a_1 x + a_0\right\} \tag{6.11}$$

is spanned by the family

$$1, x, x^2, \ldots, x^n$$

Example 6.3.3

The plane $P : x_1 + 3x_2 - 2x_3 = 0$ is a subspace of $\mathbb{R}^3$ spanned by $\left\{ v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$

P is a subspace of $\mathbb{R}^3$ (Homework)

Let $x \in P : x_1 = -3x_2 + 2x_3 \implies x = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, x_2, x_3 \in \mathbb{R}$

i.e. $x \in \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$, $P \subset \text{span}\{w_1, w_2\}$

Note that also $w_1, w_2 \in P$

$\text{span}\{w_1, w_2\} \subset P$

$\therefore \text{span}\{w_1, w_2\} = P$

Now show that $\text{span}\{w_1, w_2\} = \text{span}\{v_1, v_2\}$

i) $v_1 = w_2 \implies v_1 \in \text{span}\{w_i\}_i$

$v_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = 2w_1 + 3w_2 \implies v_2 \in \text{span}\{w_i\}_i$

$\text{span}\{w_i\}_i \subset \therefore \text{span}\{v_i\}_i$

ii) $w_2 = v_1 \implies w_2 \in \text{span}\{v_i\}_i$

$w_1 = -\frac{3}{2}v_1 + \frac{1}{2}v_2 \implies w_1 \in \text{span}\{v_i\}_i$

$\therefore \mathit{span}\{v_i\}_i \subset \text{span}\{w_i\}_i$

$\therefore P = \text{span}\{v_1, v_2\}$

Example 6.3.4

The space of solutions of the homogenous LDE

$$y'' - 3y' + 2y = 0 \tag{6.12}$$

is a vector space spanned by

$$\left\{y_1(t) = e^t, y_2(t) = e^{2t}\right\} \tag{6.13}$$

Definition 6.3.2
Let $\{v_i\}_{i=1,2,\ldots,p}$
The family $\{v_1, \ldots, v_p\}$ is linearly independent if the vector equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p = 0 \tag{6.14}$$

only has the zero solution, namely $a_1 = a_2 = \ldots = a_3 = 0$

Example 6.3.5
[a]

$$\{e_i\}_{i=1,\ldots,n} \tag{6.15}$$

in $\mathbb{R}^n$ is linearly independent
Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = 0 \implies \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

Therefore $\{e_i\}_i$ is linearly independent in $\mathbb{R}^n$

[a] A reference example for proof of linear independence

Example 6.3.6

$$y_1(t) = e^t, y_2(t) = e^{2t}$$

is linearly independent in $C^\infty(\mathbb{R}, \mathbb{R})$
Let $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1 y_1 + \alpha_2 y_2 = 0 \tag{6.16}$$

$\iff \forall t \in \mathbb{R}, \alpha_1 y_1(t) + \alpha_2 y_2(t) = 0$
$t = 0$ **in Eq. (6.16)**[a]: $\alpha_1 + \alpha_2 = 0$
$t = \ln(2)$ **in Eq. (6.16)**: $2\alpha_1 + 4\alpha_2 = 0$
Solve to obtain that $\alpha_1 = \alpha_2 = 0$
$\therefore \{y_1, y_2\}$ is linearly independent in $C^\infty(\mathbb{R}, \mathbb{R})$

[a] Sometimes it can be that we chose two t and we get the exact same equation. Get a feel for whether they are linearly independent first

Example 6.3.7

$$\left\{\cos(2x), \cos^2(x), \sin^2(x)\right\}$$

is linearly independent in $C^\infty(\mathbb{R}, \mathbb{R})$?
No, because $\cos(2x) = \cos^2(x) - \sin^2(x)$.

$$(1)\cos(2x) + (-1)\cos^2(x) + (1)\sin^2(x) = 0$$

6.3.2 Basis

Definition 6.3.3
Let V be a vector space.
A family of vectors $\{v_1, \dots, v_n\}$ is a finite basis of V if

1. $\{v_i\}_i$ is a spanning set of V
2. $\{v_i\}_i$ is linearly independent in V

A vector space V that has a finite basis is said ot be finite dimensional.

Example 6.3.8
$\{e_i\}_{i=1,\dots,n}$[a] is a basis of $\mathbb{R}^n$ as it is both a spanning set of $\mathbb{R}^n$ and is linearly independent in $\mathbb{R}^n$.

[a] This is know as the standard basis of $\mathbb{R}^n$

Example 6.3.9
[a] $P_n(\mathbb{R}^n) = \left\{a_n x^n + a_{n-1}x^{x-1} + \cdots + a_1 x + a_0 | a_i \in \mathbb{R}\right\}$
The family $\left\{1, x, x^2, \dots, x^n\right\}$ is a basis of $P_n(\mathbb{R})$ since it is a spanning set of $P_n(\mathbb{R})$ *by the definition of polynomial.* And, is linearly independent in $P_n(\mathbb{R})$

[a] **EXERCISE**

Theorem 6.3.1
Let V be a vector space that has a finite basis.
Then any basis of V contains the same *number* of elements.
The *number* of vectors in any basis of V is called the *dimension of* V, denoted by

$$\dim(V)$$

Proof. **LATER**, he said. ∎

Theorem 6.3.2
Let V be a finite dimensional vector space, $\dim(V) = n$
Let $\{v_1, v_2, \dots, v_n\}$ be a family of n vectors in V.
The following statements are all equivalent.[a]

1. $\{v_1, v_2, \ldots, v_n\}$ is a basis of V
2. $\{v_1, v_2, \ldots, v_n\}$ is a spanning set of V
3. $\{v_1, v_2, \ldots, v_n\}$ is linearly independent in V

[a] Proving that a set of vectors are linearly independent is enough to prove that it is a basis.

Example 6.3.10
Given three vectors $\{v_1, v_2, v_3\} = \left\{x^2 - 2x - 3, x^2 + 1, x - 1\right\}$ is a basis of $P_2(\mathbb{R})$.
We want to prove that the three are linearly independent.
Consider $\alpha_1, \alpha_2, \alpha_3$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \tag{6.17}$$

$x = 0$ in Eq. (6.17): $-3\alpha_1 + \alpha_2 - \alpha_2 = 0$
$x = 1$ in Eq. (6.17): $-4\alpha_1 + 2\alpha_2 = 0$
$x = -1$ in Eq. (6.17): $2\alpha_2 - 2\alpha_3 = 0$
And the above three implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, which implies that $\{v_1, v_2, v_3\}$ is linearly independent. Then, by the theorem, $\dim(P_2(\mathbb{R})) = 3$. $\{v_i\}_i$ is a basis of $P_2(\mathbb{R})$

Theorem 6.3.3
Let S be a subspace of a finite dimensional vector space V.
Then, S is finite dimensional as well and $\dim(S) \leq \dim(V)$.
Moreover, any basis $\{v_1, \ldots, v_p\}$ of S can be extended to a basis $\{v_1, \ldots, v_p, v_{p+1}, \ldots v_n\}$ of V.

Corollary 6.3.4 i) If S is a subspace of a finite dimensional vector space V, and $\dim(S) = \dim(V)$, then $S = V$

ii) If $\{v_1, v_2, \ldots, v_p\}$ be a family of non-zero vectors in V with $p > \dim(V)$, then the family $\{v_i\}_i$ is linearly dependent.

Example 6.3.11
Let $V = \text{span}\{\cos^2(x), cos(2x), sin^2(x)\}$
1. Find the $\dim(V)$
2. Show that $\text{span}\{\cos(2x), 1\} = V$

Solution:

i) Note that $\cos(2x) = (1)\cos^2(x) + (-1)\sin^2(x) \implies \cos(2x) \in \text{span}\{\cos^2(x), \sin^2(x)\} \implies V = \boxed{\text{span}\{\cos^2(x), \sin^2(x)\}}$

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\forall x, \alpha_1 \cos^2(x) + \alpha_2 \sin^2(x) = 0$

$$\begin{cases} x = 0: & \alpha_1 + 0 = 0 \implies \alpha_1 = 0 \\ x = \frac{\pi}{2}: & 0 + \alpha_2 = 0 \implies \alpha_2 = 0 \end{cases}$$

$\alpha_1 = \alpha_2 = 0$ implies that $\{\cos^2(x), \sin^2(x)\}$ is linearly independent. Thus $V = \text{span}\{\cos^2(x), \sin^2(x)\}$ is 2-dimensional.

ii) $\cos(2x) \in V$ since i)

$1 = \cos^2 + \sin^2 \implies \boxed{1 \in V}$

$$\text{span}\{\cos(2x), 1\} \subseteq V$$

Let α_1, α_2 such that $\forall x, \alpha_1 \cos(2x) + \alpha_2(1) = 0$

$$\begin{cases} x = \frac{\pi}{4}: & 0 + \alpha_2 = 0 \implies \alpha_2 = 0 \\ x = 0: & \alpha_1 + \alpha_2 = 0 \implies \alpha_1 = 0 \end{cases}$$

$\{\cos(2x), 1\}$ is linearly independent, thus $H = \text{span}\{\cos(2x), 1\}$ is 2 dimensional.

By the corollary, since $\dim(H) = \dim(V)$, $H = V$.

Example 6.3.12

Find a basis of the set of solutions of $Ax = 0$ where $A = \begin{bmatrix} 4 & 2 & 2 & -2 \\ 3 & -1 & 0 & 2 \\ 7 & 1 & 2 & 0 \\ -2 & 4 & 2 & -6 \end{bmatrix}$

Solution:

$$\text{rref}\left(\begin{bmatrix} 4 & 2 & 2 & -2 \\ 3 & -1 & 0 & 2 \\ 7 & 1 & 2 & 0 \\ -2 & 4 & 2 & -6 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1/5 & 1/5 \\ 0 & 1 & 3/5 & -7/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

After listing the solution set, and thus the two vectors that form the basis. Check the definition of the basis: 1) Whether they span the set 2) Whether they are linearly independent.

Chapter 7

System of Coordinates

7.1 Coordinate of a Vector Relative to a Basis

Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of a vector space V.

Then $\forall v \in V, v = x_1 v_1 + x_2 v_2 + \cdots x_n v_n, x_i \in \mathbb{R}$

The set of scalars $x_1, x_2, \ldots, x_n$ are unique.

Definition 7.1.1

$[v]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the coordinate vector of v relative to the basis B.

Example 7.1.1

$V = \mathbb{R}^2, B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

Let $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

One verifies that

$$v = \left(-\frac{1}{4}\right) v_1 + \left(\frac{5}{4}\right) v_2 + \left(\frac{7}{4}\right) v_3$$

$$[v]_{B'} = \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{4} \\ \frac{7}{4} \end{bmatrix}$$

Example 7.1.2
$V = P_2(\mathbb{R}), B' = \left\{1, x+1, (x+1)^2\right\}$
Let $p(x) = x^2 + 2x - 1$
Then, $[p(x)]_{B'} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Example 7.1.3
$V = P_n(\mathbb{R}), B = \left\{1, x, x^2, \dots, x^n\right\}$
Let $p(x) = a_n x^n + a_{-1} x^{n-1} + \cdots + a_1 x + a_0$ be a vector in $P_n(\mathbb{R})$
Then, $[p(x)]_{B'} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$

Example 7.1.4
$V = \mathbb{R}^n, B = \{e_1, e_2, \dots, e_n\}$ canonical basis of $\mathbb{R}^n$
Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$
Then $[x]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

7.2 Change of Basis

Let $B = \{v_1, \dots, v_n\}$ and $\mathcal{C} = \{w_1, \dots, w_n\}$ be two bases of a vector space V.

Then, $\forall v \in V$, $[v]_B$: coordinate vector of V relative to B, $[v]_{\mathcal{C}}$: $\mathcal{C}$

We wish to relate the quantities $[v]_B$ and $[v]_{\mathcal{C}}$

Definition 7.2.1
Let $P_{\mathcal{C}\leftarrow B}$ be the $n \times n$ matrix defined by

$$P_{\mathcal{C}\leftarrow B} = \left[[v_1]_{\mathcal{C}} \mid [v_2]_{\mathcal{C}} \mid \cdots \mid [v_n]_{\mathcal{C}} \right] \tag{7.1}$$

$P_{\mathcal{C}\leftarrow B}$ is the *change of basis matrix* from B to $\mathcal{C}$; change of coordinate.

Theorem 7.2.1

$$\forall v \in V, [v]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow B} \cdot [v]_B \tag{7.2}$$

Example 7.2.1
$V = \mathbb{R}^n, B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

i) Find $P_{B'\leftarrow B}$

ii) If $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $[x]_{B'}$

Solution:

i) $P_{B'\leftarrow B} = \begin{bmatrix} 3/4, 1/4, -1/2 \\ 1/4, -1/4, 1/2 \\ -1/4, 1/4, 1/2 \end{bmatrix}$

ii) $[x]_{B'\leftarrow B}[x]_B = \begin{bmatrix} -1/4 \\ 5/4 \\ 7/4 \end{bmatrix}$

Theorem 7.2.2

$$P_{B\leftarrow \mathcal{C}} = (P_{B\leftarrow \mathcal{C}})^{-1} \tag{7.3}$$

Example 7.2.2
$V = P_2(\mathbb{R}), B = \{1, x, x^2\}, \mathcal{C} = \{1, x+1, (x+1)^2\}$
Let $p(x) = ax^2 + bx + c$. Find $[p(x)]_{\mathcal{C}}$ Then write down $p(x)$ as a linear combination of the elements in $\mathcal{C}$

Solution:

$[p(x)]_B = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$

$P_{B\leftarrow \mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$P_{B\leftarrow \mathcal{C}} = (P_{\mathcal{C}\leftarrow B})^{-1}$

$p(x) = (c - b + a)(1) + (b - 2a)(x+1) + a(x+1)^2$

Theorem 7.2.3
$P_{\mathcal{D} \leftarrow B} = P_{\mathcal{D} \leftarrow \mathcal{C}} \cdot P_{\mathcal{C} \leftarrow B}$

Chapter 8

Rank-Nullity Theorem

8.1 The Rowspace and the Colspace of a $m \times n$ matrix A

Definition 8.1.1
$\text{Rowspace}(A) = \text{span}\{\text{rows of } A\} \subseteq \mathbb{R}^n$

Definition 8.1.2
$\text{Colspace}(A) = \text{span}\{\text{columns of } A\} \subseteq \mathbb{R}^n$
$\text{Col}(A) = \text{Row}(A^T)$

Theorem 8.1.1 (Rowspace)
1. If A and B are matrices that are row-eqivalent:
   $\text{Row}(A) = \text{Row}(B)$
2. In particular, $\text{Row}(A) = \text{Row}(\text{ref}(A)) = \text{Row}(\text{rref}(A)$[1]$)$
3. The rows containing the leading "1" form a basis of $\text{Row}(A)$
   As a resulve, $\dim(\text{Row}(A)) = \text{rank}(A)$

[1] yields an optimally simplified basis of $\text{Row}(A)$

Theorem 8.1.2 (Colspace)
1. $\dim(\text{Col}(A)) = \dim(\text{Row}(A)) = \text{rank}(A)$
2. A basis of $\text{Col}(A)$ is given by the columns of A containing the leading "1".

Remark
$\text{Col}(A) \neq \text{Col}(\text{ref}(A)) \neq \text{Col}(\text{rref}(A))$

Example 8.1.1
Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix}$
Find
i) A basis of $\text{Row}(A)$

ii) An optimally simplified basis of $\text{Row}(A)$
iii) A basis of $\text{Col}(A)$
iv) An optimally simplified basis of $\text{Col}(A)$

Solution:

$$\text{Row}(A) = \text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}9\\10\\11\end{bmatrix}\right\}$$

i) See below, since they are the same

ii) $\text{ref}(A) = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$ $\text{rref}(A) = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$

$$\text{Row}(A) = \text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}5\\6\\7\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\} = \text{spam}\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$$

$\dim(\text{Row}(A)) = 2$

iii) $\text{ref}(A) = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$

$$\text{Row}(A^T) = \text{Col}(A) = \text{span}\left\{\begin{bmatrix}1\\5\\9\end{bmatrix}, \begin{bmatrix}2\\6\\10\end{bmatrix}\right\}$$

$\dim(\text{Col}(A)) = 2$

Remark i) Check that the redundant vector is actually a linear combination of the other vectors.

ii) $\text{Col}(A) = \text{Row}(A^T) = \text{Row}(\text{rref}(A^T))$[2]

8.1.1 Applications

1. Given a family $\{v_i\}_{i=1,\ldots,p}$ of vectors in $\mathbb{R}^n$, one can consider the $n \times p$ matrix $A = \begin{bmatrix} v_1 \mid v_2 \mid \cdots \mid v_p \end{bmatrix}$, then apply either the colspace theorem (for A) or the rowspace theorem (for A^T) to extract a subfamily $\{v_{i_1}, v_{i_2}, \ldots, v_{i_q}\}(q \leq p)$ of linear independent set of maximal order, that is to say a basis of $\text{span}\{v_1, \ldots, v_p\}$. $(q = \text{rank}(A))$

2. If $\{v_1, \ldots, v_p\}$ in a abstract vector space V, ten choose a basis B of V and apply colspace/rowspace Theorem to the family $\left\{[v_i]_B\right\}_{i=1,\ldots,p} \subseteq \mathbb{R}^n$

[2] This yields a optimally simplified basis of $\text{Col}(A)$

3. # of linearly independent columns = # of linearly independent rows
4. If A is a square matrix, all columns are linearly independent $\iff$ all rows are linearly independent.

8.2 Rank-Nullity Theorem

Definition 8.2.1
Given a $m \times n$ matrix A

$$\text{Nullspace}(A) = \left\{x \in \mathbb{R}^n \mid Ax = 0\right\} : \text{Nullspace of } \mathbb{R}^n \tag{8.1}$$

We already know that: $\dim(\text{Null}(A)) = n - \text{rank}(A)$

Theorem 8.2.1
Let A be a $m \times n$ matrix

$$\text{rank}(A) + \dim(\text{Null}(A)) = n \tag{8.2}$$

Chapter 9

Linear Transformations

9.1 Definitions and Examples

Definition 9.1.1
Let V and W be vector spaces. A **linear transformation** is a map $T : V \to W$ that satisfies the following properties:
1) $\forall u, v \in V, T(u+v) = T(u) + T(v)$
2) $\forall u \in V, \forall \lambda \in \mathbb{R}, T(\lambda u) = \lambda T(u)$

Remark 1) The two conditions $\iff \forall u, v \in V, \forall \lambda \in \mathbb{R}, T(u + \lambda v) = T(u) + \lambda T(v)$

2) More generally: $\forall v_1, v_2, \ldots, v_p \in V, \forall \alpha_1, \ldots, \alpha_p \in \mathbb{R}$

$$T\left(\sum_{i=1}^{p} \alpha_i v_i\right) = \sum_{i=1}^{p} \alpha_i T(v_i) \tag{9.1}$$

3) $T(0_v) = 0_w$

$T(0_v) = T(0_v + 0_v) = T(0_v) + T(0_v) \implies T(0_v) = 0_w$

Example 9.1.1
Let A be a $m \times n$ matrix
$T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear map: it is the linear map associated with the $m \times n$ matrix A.
Let $x, y \in \mathbb{R}^n, \lambda \in \mathbb{R}$
Then,

$$\begin{aligned} T_A(x + \lambda y) &= A(x + \lambda y) && (9.2) \\ &= Ax + \lambda Ay && (9.3) \\ &= T_A(x) + \lambda T_A(y) && (9.4) \end{aligned}$$

T_A is linear by theorem.

Theorem 9.1.1
Let

$$T : \mathbb{R}^n \to \mathbb{R}^m \tag{9.5}$$

be a linear map. There exists a (unique) $m \times n$ matrix A such that $T = T_A$

Proof. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map

Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$$T(n) = T\left(\sum_{i=1}^{n} x_i e_i\right) \tag{9.6}$$

$$= \sum_{i=1}^{n} x_i T(e_i) \tag{9.7}$$

$$= \Big[T(e_0) \mid \cdots \mid T(e_n)\Big] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{9.8}$$

■

Example 9.1.2
Given

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x + y + 2z \end{bmatrix}$$

is a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}, A\begin{bmatrix} x \\ y \\ z \end{bmatrix} \implies T = T_A$$

9.2 Kernel and Range of a Linear Map

Let $T : V \to W$ be a linear map

Definition 9.2.1
The **Kernel** of T is $\ker(T) = \big\{v \in V \mid T(v) = 0\big\}$

Definition 9.2.2
The **Range** of T is $\text{rng}(T) = \{w \in W \mid \exists v \in V, w = T(v)\}$

Remark 1. $\ker(T)$ is a subspsace of V

Proof. Let $u, v \in \ker(T)$, let $\lambda \in \mathbb{R}$

$$\begin{aligned} T(u + \lambda v) &= T(u) + \lambda T(v) \\ &= 0 \end{aligned}$$

We know that T is linear,

V being a vector space, $\ker(T)$ is a subspace by the theorem. ∎

2. $\text{rng}(T)$ is a subspace of W

Proof. Let $w_1, w_2 \in \text{rng}(T)$, let $\lambda \in \mathbb{R}$

$$\begin{aligned} w_1 + \lambda w_2 &= T(v_1) + \lambda T(v_2) \\ &= T(v_1 + \lambda v_2) \end{aligned}$$

Since given our assumption $v_1, v_2 \in V$ and V is a vector space, by properties of a vector space, $v_1 + \lambda v_2 \in V$

This means that $w_1 + \lambda w_2 \in \text{rng}(T)$

By theorem, W being a vector space, $\text{rng}(T)$ is a subspace of W ∎

3. If $B = \{v_i\}_{i \in [1,n]}$ is a basis of V, then $\text{rng}(T) = \text{span}\{T(v_i)\}_{i \in [1,n]}$

Proof. Let $\{v_i\}_i$ be a basis of V

Let $w \in \text{rng}(T) \iff \exists v \in V$ such that $w\ T(v)$

Let $\{\alpha_i\}_i \subseteq \mathbb{R}$ such that $v \in \sum_i \alpha_i v_i$ since $\{v_i\}_i$ basis of V.

We have: $w = T(v) = T(\sum_i \alpha_i v_i) = \sum_i \alpha_i T(v_i) \in \text{span}\{T(v_i)\}_i$

Therefore, $\text{rng}(T) \subseteq \text{span}\{T(v_i)\}_i$

We also know that $\text{rng}(T) \supseteq \text{span}\{T(v_i)\}_i$ because we can go backwards in the equality above, and see that any vector of the span must be a $w \in T(v)$ ∎

Theorem 9.2.1 (Rank-Nullity)
Let $T : V \to W$ a linear map, where V, W are finite dimensional, $\dim(V) < +\infty$

$$\dim(V) = \dim(\ker(T)) + \dim(\text{rng}(T))$$

Example 9.2.1
Determine a basis for $\ker(T)$ and $\operatorname{rng}(T)$ where $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 2x - z \\ x + y + 2z \end{bmatrix}$$

Solution:

Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \ker(T)$ then $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Row reduce and find that $\ker(T) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}\right\}$

then find the range

$\dim(\ker(T)) = 1$

$3 = 1 + \dim(\operatorname{rng}(T)) \implies \dim(\operatorname{rng}(T)) = 2$

Example 9.2.2
Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$
$p(x) \mapsto p(x) - (x+1)p'(x)$
1. Show that T is linear
2. Find a basis of $\ker(T)$
3. Find a basis of $\operatorname{rng}(T)$

Solution:

1. Let $p, q \in P_3(\mathbb{R})$, let $\lambda \in \mathbb{R}$

$$\begin{aligned} T(p + \lambda q) &= (p + \lambda q)(x) - (x+1)(p + \lambda q)'(x) \\ &= \big[p(x) - (x+1)p'(x)\big] + \lambda\big[q(x) - (x+1)q'(x)\big] \end{aligned}$$

   Thus, T is linear.

2. Let $p \in \ker(T)$, $p(x) = ax^3 + bx^2 + cx + d$

   $T(p) = 0 \iff \forall x, p(x) = (x+1)p'(x) = 0$

   We can expand it to the form

$$\begin{aligned}p(x) - (x+1)p'(x) &= 0\\ ax^3 + bx^2 + cx + d + (x+1)(3ax^2 + 2bx + c) &= 0\\ (-2a)x^3 + (-b-3a)x^2 + (-2b)x + (d-c) &= 0\end{aligned}$$

which means

$$\begin{cases}-2a = 0 & \implies a = 0\\ -b - 3a = 0 & \\ -2b = 0 & \implies b = 0\\ d - c = 0 & \implies c = d\end{cases}$$

> We know that if two polynomials are equivalent, their coefficients are the same because $\{1, x, x^2, x^3\}$ is linearly independent.

$p(x) = dx + d = d(x+1), d \in \mathbb{R}$

$\therefore \ker(T) = \text{span}\{(x+1)\}$

3. $\dim(\text{rng}(T)) = \dim(P_3(\mathbb{R})) - \dim(\text{rng}(T)) = 4 - 1 = 3$

   We have

$$\begin{aligned}\text{rng} &= \text{span}\left\{T(1), T(x), T(x^2), T(x^3)\right\}\\ &= \text{span}\left\{1, -1, -x^2 - 2x, -2x^3 - 3x^2\right\}\\ &= \text{span}\left\{1, -x^2 - 2x, -2x^3 - 3x^2\right\}\end{aligned}$$

   It is a spanning set of $\text{rng}(T)$ and $\dim(\text{rng}(T)) = 3$

   Thus, it is a basis.

9.3 Properties of Linear Maps

Definition 9.3.1
Let $T : V \to W$ be a linear map

1. T is **one-to-one / injective** if it satisfies the following condition:

$$\forall v_1, v_2, \text{if } T(v_1) = T(v_2), \text{then } v_1 = v_2$$

2. T is **onto / surjective** if $\text{rng}(T) = W$

$$\forall w \in W, \exists v \in V \text{ such that } T(v) = w$$

3. T is bijective if it is both injective and surjective.
4. $\forall w \in W, \exists! v \in V$ such that $T(v) = w$

Definition 9.3.2
Let $T : V \to W$ be a linear map that is bijective. (**isomorphism**) The inverse map $T^{-1} : W \to V$ is defined by:

$$\forall (v, w) \in V \times W, T^{-1}(W) = V \text{ if } T(v) = w$$

Properties

1. $\forall v \in V, T^{-1}\big(T(v)\big) = v$
2. $\forall w \in W, T\big(T_{-1}(w)\big) = w$

Theorem 9.3.1
If $T : V \to W$ is an isomorphism, then so is $T^{-1} : W \to V$

Proof. Let $w_1, w_2 \in W, \lambda \in \mathbb{R}$

$$\begin{aligned} T^{-1}(w_1 + \lambda w_2) &= T^{-1}(T(v_1) + \lambda T(v_2))^{1} \\ &= T^{-1}(T(v_1 + \lambda v_2)) \\ &= v_1 + \lambda v_2 \\ &= T^{-1}(w_1) + \lambda T^{-1}(w_2) \\ &\therefore T \text{ is linear} \end{aligned}$$

■

Theorem 9.3.2
Let $T : V \to W$ be a linear map.

1. T is injective iff $\ker(T) = \{0\}$
2. If $\dim(W) < +\infty$, then T is surjective iff $\dim(\text{rng}(T)) = \dim(W)$

Proof. 1. Assume T is injective

Let $v \in \ker(T) \implies T(v) = 0 = T(0)$. Then, by the definition of injective maps, $v = 0$, thus $\ker(T) = \{0\}$

Assum $\ker(T) = \{0\}$

Let $v_1, v_2 \in V$ such that

$$T(v_1) = T(v_2) \implies T(v_1) - T(v_2) = T(v_1 - v_2) = 0 \implies v_1 - v_2 \in \ker(T) = \{0\}$$

Thus $v_1 = v_2$

■

[1]Where uniquely, $T(v_1) = w_1, T(v_2) = w_2$

Theorem 9.3.3
Let $T : V \to W$ be a linear map
Let $B = \{v_1, \ldots, v_n\}$ be a basis of the domain V.
1. T is injective *iff* $\{T(v_1), \ldots, T(v_n)\}$ is a family of linearly independent vectors in the codomain W
2. T is surjective *iff* $\{T(v_1), \ldots, T(v_n)\}$ is a spanning set of the codomain V.
3. T is an isomorphism *iff* $\{T(v_1), \ldots, T(v_n)\}$ is a basis of the codomain W

Corollary 9.3.4
Let $T : V \to W$ be a linear map, V, W finite dimensional.

1. If T is injective, then $\dim(V) \leq \dim(W)$
2. If T is surjective, then $\dim(V) \geq \dim(W)$
3. If T is bijective/isomorphism, then $\dim(V) = \dim(W)$

Example 9.3.1
Let $T : \mathbb{R}^3 \to \mathbb{R}^4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - x_2 \\ x_2 + 2x_1 \\ x_2 + x_1 - 2x_3 \\ x_3 - x_1 \end{bmatrix}$$

Show is one-to-one

Solution:

To show that the map is injective, show that the kernel space has a dimension of 0.

Example 9.3.2
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 2x_1 - x_3 \\ x_1 + x_2 + 2x_3 \end{bmatrix}$$

Show that is onto.

Solution:

To show that the it is surjective, find the dimension of the range space, which is defined by

$$\dim(\operatorname{rng}(T)) = \dim(\mathbb{R}^3) - \dim(\ker(T))$$

We find that the kernel spaces has dimension of 1, thus dimension of the range space is 2.

$$\dim(\text{rng}(T)) = 2 = \dim(\mathbb{R}^2)$$

Thus we've shown it is onto.

Example 9.3.3
Isomorphism problem, check image `img0.HEIC`

9.4 Matrix of a Linear Map

Fill in later

Example 9.4.1
Let $T : P_2(\mathbb{R}) to \mathbb{R}^3$
$p(x) \mapsto \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix}$

Show that T is an isomorphism. Then find the equivalent expression of $T^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$

Solution
Let $\begin{cases} \mathcal{B} & = \{1, x, x^2\} \text{ basis of } P_2(\mathbb{R}) \\ \mathcal{C} & = \{e_1, e_2, e_3\} \text{ standard basis of } \mathbb{R}^3 \end{cases}$

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \Big[[T(1)]_{\mathcal{C}}, [T(x)]_{\mathcal{C}}, [T(x^2)]_{\mathcal{C}}\Big] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = A$$

By the theorem we just have to prove that the transformation matrix is invertible, then we have shown that the transformation T is an isomorphism.

$$\det(A) = 2 \neq 0 : A \text{ is invertible} \iff \text{is isomorphic}$$

Then, we find

$$[T]_{\mathcal{C}}^{\mathcal{B}} = ([T]_{\mathcal{C}}^{\mathcal{B}})^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$T^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = aT^{-1}(e_1) + bT^{-1}(e_2) + cT^{-1}(e_3)$$

We should at least state once that

$$[T^{-1}(e_3)]_{\mathcal{B}} = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}$$

The answer is

$$T^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \left(1/2\right) x^2 + (b-c)x + (a-b+c/2)$$

Example 9.4.2

$$\begin{cases} T_1: & P_1(\mathbb{R}) \to M_2(\mathbb{R}) \\ & ax+b \mapsto \begin{bmatrix} a-b & 0 \\ -2b & 3b-a \end{bmatrix} \\ T_2: & M_2(\mathbb{R}) \to \mathbb{R} \\ & A \mapsto \operatorname{tr}(A) \end{cases} \qquad \begin{cases} \mathcal{B} & = \{1, x^2\} \\ \mathcal{C} & = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \right\} \\ \mathcal{D} & = \{\frac{1}{5}\} \end{cases}$$

1. Find $[T_2T_1]_{\mathcal{B}}^{\mathcal{D}}$
2. Verify that $[T_2T_1]_{\mathcal{B}}^{\mathcal{R}} = [T_2]_{\mathcal{C}}^{\mathcal{D}}[T_1]_{\mathcal{B}}^{\mathcal{C}}$

Solution:

1. $[T_2T_1]_{\mathcal{B}}^{\mathcal{D}} = \begin{bmatrix} 10 & 0 \end{bmatrix}$
2. $$[T_1]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} [T_1(1)]_{\mathcal{C}} \mid [T_1(x)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} \left[\begin{bmatrix} -1 & 0 \\ -2 & 3 \end{bmatrix}\right]_{\mathcal{C}} \mid [T_1(x)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ -2 & 0 \\ 3 & -1 \end{bmatrix} = B$$

   Then, using the same technique, we find

   $$[T_2]_{\mathcal{C}}^{\mathcal{D}} = \begin{bmatrix} 5 & 0 & 0 & 5 \end{bmatrix} = C$$

   Then we find that

   $$CB = \begin{bmatrix} 10 & 0 \end{bmatrix}$$

Chapter 10

Spectral Theory

10.1 Eigenvectors and Eigenvalues of a Matrix / Linear Map

Definition 10.1.1
Let A be a $n \times n$ matrix.
A non-zero vector $x \in \mathbb{R}^n$ is a (real) **eigenvector** of A if

$$Ax = \lambda_x x \tag{10.1}$$

for some scalar $\lambda_x \in \mathbb{R}$
Such a scalar $\lambda_x \in \mathbb{R}$ is called a (real) **eigenvalue** of A, associated with the eigenvector x.

Definition 10.1.2
Let $T : V \to V$ be an endomorphism
A non-zero vector $v \in V$ is an *eigenvector* of T if

$$T(v) = \lambda_x v \tag{10.2}$$

for some scalar $\lambda_x \in \mathbb{R}$
Such a scalar λ_x is called an *eigenvalue* of T, associated with the eigenvector v.

Remark
With definition 10.1.2, with $T = T_A : \mathbb{R}^n \to \mathbb{R}^n \implies$ definition 10.1.1

Example 10.1.1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Observe that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of A with associated eigenvalue $\lambda = 2$

Also observe that

$$A \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

i.e. $x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with associated eigenvalue $\lambda = 2$

Also observe that

$$A \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

i.e. $x = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$ is an eigenvector of A with associated eigenvalue $\lambda = -2$

Example 10.1.2

$$\begin{cases} T: & P_2(\mathbb{R}) \to P_2(\mathbb{R}) \\ & p(x) \mapsto p(x) - (x+1)p'(x) \end{cases}$$

Observe that

$T(x+1) = (x+1) - (x+1)(x+1)' = 0 = 0(x+1)$

Thus, the vector $x+1$ is an eigenvector with associated eigenvalue $\lambda = 0$

$T(1) = 1 = (1)1$

Thus, the vector 1 is an eigenvector with associated eigenvalue $\lambda = 1$

10.2 Tracking Eigenvalues and Eigenvectors of a Matrix A/Linear Map T

10.2.1 Finding the Eigenvalues

Definition 10.2.1
Let A be a $n \times n$ matrix.
The **characteristic polynomial** of A, denoted by $P_A(\lambda)$, is defined as

$$P_A(\lambda) = \det(A - \lambda I_n) \tag{10.3}$$

Theorem 10.2.1
The roots of $P_A(\lambda)$ are the eigenvalues of A.

Proof. Later... ∎

Example 10.2.1
Fking missing? WTF??? Did OneDrive just overwrite my local copy?
Missing:

1. Example of finding eigenvalues
2. Finding eigenvalues in different basis which doesn't matter. Even with a change of basis to another space, we see that we are left with the same characteristic polynomial

10.2.2 Finding the eigenvectors

Example 10.2.2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

-2 and 2 are the eigenvalues of A

Solution:

Basis of E_2

Solve the system

$$(A - 2I_3)x = 0$$

We observe that

$$\text{rref}(A - 2I_3) = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that the rank is 1, thus the dimension of the nullspace can be determined by the rank-nullity theorem.

$$\dim((A - 2I_3)) = \dim(\mathbb{R}^3) - \operatorname{rank}(A) = 3 - 1 = 2$$

We can then kind of guess

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in E_2$$

and that they are linearly independent, so given the dimension of E_2 we conclude

$$E_2 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Basis of E_{-2}

Solve $(A + 2I_3)x = 0$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, by theorem, we observe that

$$\dim(E_{-2}) = \dim(\mathbb{R}^3) - \operatorname{rank}(A + 2I_3) = 1$$

We can make out the vector $\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \in E_{-2}$ and given the dimension we calculated above, we can say

$$E_{-2} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

Example 10.2.3
Another example from discussion

10.3 Properties of Eigenvectors

Let A be a $n \times n$ matrix.

Let $P_A(\lambda)$ be its characteristic polynomial.

Let $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{C}$ be the distinct eigenvalues of A.

After full factorization:

$$P_A(\lambda) = (-1)^n(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p}$$

m_i is called the **multiplicity** of the eigenvalue λ_i

Remark
$n = \sum_{i=1}^{p} m_i$

Theorem 10.3.1

$$\dim(E_{\lambda_i}) \leq m_i$$

Definition 10.3.1
Let λ_i be an eigenvalue of a $n \times n$ matrix A. The eigenspace associated with λ_i is

i) $E_{\lambda_i} = \left\{x \in \mathbb{R}^n \mid Ax = \lambda_i x\right\} = (A - \lambda_i I_n)$ if $\lambda_i \in \mathbb{R}$
ii) $E_{\lambda_i} = \left\{x \in \mathbb{C}^n \mid Ax = \lambda_i x\right\}$ if $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$
iii) The theorem also applies

10.3.1 A Little Dispersion

Definition 10.3.2
Let $S_1, \ldots, S_m$ be the subspaces of a vector space V.

1. The sum of the S_i is the subspace of V defined as:

$$S_1 + S_2 + \cdots + S_m = \left\{v \in \mathbb{R}^n \mid v = v_1 + v_2 + \cdots + v_m, v_i \in S_i\right\}$$

2. The sum of S_i is said to be **direct** if $\forall i, j, i \neq j, S_i \cup S_j = \{0\}$
   When the sum of $S_1 + \cdots + S_n$ is direct, we use the notation

$$S_1 \oplus S_2 \oplus \cdots \oplus S_m$$

3. The vector space V is said to split as the direct sum of the S_i if

$$V = S_1 \oplus S_2 \oplus \cdots \oplus S_m$$

Example 10.3.1
Take $\mathbb{R}^2$ for example, say we have two vector spaces S_1, S_2 such that

$$\mathbb{R}^2 = S_1 \oplus S_2$$

Then any arbitrary vector $v \in \mathbb{R}^2$ can be reconstructed by a linear combination of the vectors $v_1 \in S_1, v_2 \in S_2$
This similarly applies to $\mathbb{R}^3$ and higher dimensions, with 3 subspaces.
We could also have a plane in $\mathbb{R}^3$ along with another 1 dimensional subspace in $\mathbb{R}^3$ and we still see how any arbitrary vector can be constructed with a linear combination of vectors in these spaces.

Theorem 10.3.2
Let V be a finite dimensional vector space.
Let $S_1, \ldots, S_m$ be subspaces of V. The following statements are equivalent:

1. $V = S_1 \oplus \cdots \oplus S_m$
2. $\begin{cases} V = S_1 + \cdots + S_m \\ \dim(V) = \sum_{i=1}^{m} \dim(S_i) \end{cases}$
3. The family of vectors $B_1 \cup B_2 \cup \cdots \cup B_m$, where B_i is a basis of S_i, form a basis of V.[a]

[a] The vectors that span V won't be a basis for V if any of them are linearly dependent.

Since projections, orthogonal or not, are linear, they can expressed in terms of a matrix.

Theorem 10.3.3
Let a be a $n \times n$ matrix ($T : V \to V$ be an end of a $\mathbb{R}$ vector space V)
Let $\lambda_1, \lambda_2, \ldots, \lambda_3 \in \mathbb{C}$ be the distinct eigenvalues of A (T, respectively)
Let $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_p}$ be the eigenspaces.
Let $S = E_{\lambda_1} + E_{\lambda_2} + \ldots + E_{\lambda_p}$: this is a subspace of $\mathbb{R}^n, \mathbb{C}^n, V, V^{\mathbb{C}}$.
Then,

$$S = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_p}$$

In particular:

- $\dim(S) = \sum_{i=1}^{p} \dim(E_{\lambda_i})$
- The family of vectors $B_1 \cup B_2 \cup \cdots \cup B_p$, where B_i is a basis of E_{λ_i} is linearly independent.

In other words, if $B_i = \{v_1^i, v_2^i, \ldots, v_{n_i}^i\}$, where $n_i = \dim(E_{\lambda_i})$ is a basis of E_{λ_i}, then $B_1 \cup B_2 \cup \cdots \cup B_p = \left\{v_1^1, v_2^1, \ldots, v_{n_i}^1, v_1^2, v_2^2, \ldots, v_{n_i}^2, \cdots, v_1^p, v_2^p, \ldots, v_{n_p}^p\right\}$

Definition 10.3.3
Let A be a $n \times n$ matrix ($T : V \to V$ be an endomorphism)
A (T, respectively) is said to be diagonalizable/nondefective if:

$$\mathbb{R}^n, \mathbb{R}^n, V, V^{\mathbb{C}} = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_p}$$

Theorem 10.3.4
Let A be a $n \times n$ matrix ($T : V \to V$ be an endomorphism, respectively)

A (T, respectively) is diagonalizable if any of the following condition are satisfied:

1. $\dim(\mathbb{R}^n, \mathbb{C}^n, V, V^{\mathbb{C}}) = \sum^p \dim(E_{\lambda_i})$
2. There exists a basis of $\mathbb{R}^n, \mathbb{C}^n, V, V^{\mathbb{C}}$ made of eigenvectors.[a]
3. $\forall i = 1, \ldots, p, \dim(E_{\lambda_i}) = m_i$ (m_i is the multiplicity in the characteristic polynomial $P_A(\lambda)$)

[a]This is probably the definition that we want to use most

Remark (Lecture Oct 25, 2023)
Given $A \in M_n(\mathbb{R})$, there are two cases for the eigenvectors/values of matrices:

- All eigenvalues are real: we stay in $\mathbb{R}^n$
- Otherwise, we need to switch to $\mathbb{C}^n$

Given $T : V \to V, V : \mathbb{R}$

$A = [T]_B \in M_n(\mathbb{R})$

- All eigenvalues λ_i of A are real: stick with the $\mathbb{R}$ vector space V.
- One of the eigenvalues is in $\mathbb{C} \setminus \mathbb{R}$: Need to switch to $\mathbb{C}$ vector space

$$V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$$

Theorem 10.3.5
Let A be in $M_n(\mathbb{C})$
A is diagonalizable *iff* $A = PDP^{-1}$, where P is invertible, and D is a **diagonal matrix**.
More precisely, let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the distinct eigenvalues of A.
Let C[1] $= \left\{ v_1^1, v_2^1, \ldots, v_{n_i}^1, v_1^2, v_2^2, \ldots, v_{n_i}^2, \cdots, v_1^p, v_2^p, \ldots, v_{n_p}^p \right\}$

$$P = P_{\mathcal{B}_{\text{canonical}}{}^{2} \leftarrow \mathcal{C}}$$

And, we know that the matrix D has eigenvalues along the diagonal, starting with a series of λ_1 on the top left, and ending with λ_p. The number of these eigenvalues on the diagonal is the multiplicity of the eigenvalue itself.

This is the basis of eigenvectors, which is the basis of the entire vector space (since A is diagonalizable.)
This is the canonical basis of $\mathbb{R}^n$ or $\mathbb{C}^n$

Theorem 10.3.6
Let $T : V \to V$ be an endomorphism.
T is diagonalizable *iff* There exists a basis $\mathcal{C}$ such that $[T]_{\mathcal{C}}$ is a diagonal matrix.
iff There exists a basis of $\mathcal{C}$ made of eigenvectors.
(basis of $\begin{cases} V \\ V^{\mathbb{C}} \end{cases}$ depending on the ??)

Example 10.3.2

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

1. Find a basis of each eigenspace
2. Calculate $A^n, \forall n \geq 0$

Solution:

1. $E_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$; $E_{-2} = \text{span}\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$

2. Note that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$ made of eigenvectors $\implies A$ is diagonalizable.

   $P = P_{\mathcal{B}_{\text{can}} \leftarrow \mathcal{C}}$ and D is a diagonal matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ 1 & 0 & 1 \end{bmatrix}; D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

   We hshould note that the first column of P is $\begin{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$

   Now, when we consider

$$\begin{aligned} A^n &= (PDP^{-1})^n \\ &= (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) \\ &= PD^nP^{-1} \end{aligned}$$

   Now, for the actual calculation:

$$A^n = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & (-2)^n \end{bmatrix} \begin{bmatrix} 0 & 1/4 & 3/4 \\ 1 & 0 & 0 \\ 0 & -1/4 & 1/4 \end{bmatrix}$$

   To calculate we should simplify:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ 1 & 0 & 1 \end{bmatrix} (2^n) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^n \end{bmatrix} \frac{1}{4} \begin{bmatrix} 0 & 1 & 3 \\ 4 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

10.4 Exponential of a Matrix / Endomorphism

10.4.1 Definitions and Theorems

Definition 10.4.1
Let A be a $n \times n$ matrix. ($T : V \to V$, respectively)
We define the exponential of a matrix A as:

$$e^A = \sum_{k=0}^{+\infty} \frac{1}{k!} A^k$$

The expression $\frac{1}{k!}A^k$ is $n \times n$ since A is $n \times n$. Thus, e^A is also a $n \times n$ matrix.

Theorem 10.4.1
$\forall A \in M_n(\mathbb{C})$, the above series is convergent $\implies$ e^A is a well defined "value"

Theorem 10.4.2
If $A, B \in M_n(\mathbb{C})$ such that $AB = BA$
then $e^{A+B} = e^A \cdot e^B = e^B \cdot e^A$

Proof. If A, B commute, then

$$(A+B)^N = \sum_{k=0}^{N} \binom{N}{k} A^k B^{N-k}$$

where $\binom{N}{k} = \frac{N!}{k!(N-k)!}$

(aka the Binomial Formula)

$$\begin{aligned}
e^{A+B} &= \sum_{k=0}^{+\infty} \frac{1}{k!}(A+B)^k && (10.4)\\
&= \sum_{k=0}^{+\infty} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} A^l B^{k-l} && (10.5)\\
&= \sum_{k=0}^{+\infty} \sum_{l=0}^{k} \frac{1}{k!} \binom{k}{l} A^l B^{k-l} && (10.6)\\
&= \sum_{k=0}^{+\infty} \sum_{l=0}^{k} \left(\frac{1}{l!}A^l\right) \left(\frac{1}{(k-1)!}B^{k-l}\right)^{3} && (10.7)\\
&= \left(\sum_{k=0}^{+\infty} \frac{1}{l!}A^l\right) \left(\sum_{k=0}^{+\infty} \frac{1}{k!}B^k\right) && (10.8)\\
&= e^A \cdot e^B && (10.9)
\end{aligned}$$

All this is under the assumption that AB commute.

Since $e^{A+B} = e^{B+A}$, we have also proven the other case. (if $AB = BA$) ■

> **Theorem 10.4.3**
> $e^0 = I_n$ (0 is the zero-matrix)

Proof.

$$\begin{aligned}
e^A &= \sum_{k=0}^{+\infty} \frac{1}{k!}A^k && (10.10)\\
&= \frac{1}{0!}(0)^0 + \sum_{k=1}^{+\infty} \frac{1}{k!}A^k && (10.11)\\
&= I_n + 0 && (10.12)\\
&= I_n && (10.13)
\end{aligned}$$

■

> **Theorem 10.4.4**
> $\forall A \in M_n(\mathbb{C}), e^A$ is invertible, and $\left(e^A\right)^{-1} = e^{-A}$

Proof.

$$e^0 = e^{A-A} = e^A \cdot e^{-A}$$

since A and $-A$ commute.

Therefore, e^A is invertible, and $\left(e^A\right)^{-1} = e^{-A}$ by property of the inverse. ■

[3] The second summation is the general term of the product of 2 power series

10.4.2 Calculating e^A where A is Diagonalizable

If A is diagonalizable, i.e. $A = PDP^{-1}$ where D is a diagonal matrix.

Proof.

$$e^A = \sum_{k=0}^{+\infty} \frac{1}{k!} PD^k P^{-1} \tag{10.14}$$

$$= P \left(\sum_{k=0}^{+\infty} \frac{1}{k!} D^k \right) P^{-1} \tag{10.15}$$

$$= P \left(\sum_{k=0}^{+\infty} \frac{1}{k!} \begin{bmatrix} {\lambda_1}^k & & \\ & \ddots & \\ & & {\lambda_n}^k \end{bmatrix} \right) P^{-1} \tag{10.16}$$

$$= P \begin{bmatrix} \sum_{k=0}^{+\infty} \frac{1}{k!} {\lambda_1}^k & & \\ & \ddots & \\ & & \sum_{k=0}^{+\infty} \frac{1}{k!} {\lambda_n}^k \end{bmatrix} P^{-1} \tag{10.17}$$

$$= Pe^D P^{-1} \tag{10.18}$$

■

where $e^D = \text{diag}(e^\lambda_1, e^\lambda_2, \dots, e^\lambda_n)$

Example 10.4.1
Taking the last example, (can ref), where

$$D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & -2 \end{bmatrix}, e^D = \begin{bmatrix} e^2 & & \\ & e^2 & \\ & & e^{-2} \end{bmatrix}$$

Now what if the matrix is not diagonalizable? We will consider that case now.

10.5 Jordan Matrix

Take this 13×13 matrix for example

$$J = \left[\begin{array}{ccc|cc|cc|c|c|ccc|c}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7
\end{array}\right]$$

The different gridded sections separated along the diagonal are *Jordan Blocks*.

Theorem 10.5.1

$$\forall A \in M_n(\mathbb{C}), A = PJP^{-1}$$

where J is the *Jordan Matrix*.

Remark (Important Facts:)
If $A = PJP^{-1}$, where J is as above, we have:

i) $P_A(\lambda) = P_J(\lambda) = (-1)^{13}(\lambda+1)^5(\lambda-2)^4(\lambda-3)^3(\lambda-7)$

ii) Dimensions of the eigenspaces

- $\dim(E_{-1}) = 2 =$ number of Jordan Blocks associated with $\lambda = -1$
- $\dim(E_2) = 3$
- $\dim(E_3) = 1$
- $\dim(E_7) = 1$

iii) Dimension of the generalized eigenspace G_{λ_i}

$$E_{\lambda_i} \subset G_{\lambda_i}$$

$$\dim(G_{\lambda_i}) = m_i \implies \begin{cases} \dim(G_{-1}) & = 5 \\ \dim(G_2) & = 4 \\ \dim(G_3) & = 3 \\ \dim(G_7) & = 1 \end{cases}$$

Example 10.5.1

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Find the Jordan decomposition $A = PJP^{-1}$

Solution:

$$P_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -1 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 1 & -2-\lambda \end{vmatrix} = -(\lambda+2)^3$$

Thus, $\lambda = -2$ is the only eigenvalue.

Basis of E_{-2}

$$(A+2I)x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x = 0 \implies E_{-2} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis for G_{-2}

We know that $\dim(G_{-2}) = 3$ given its multiplicity, and that $E_{-2} \subset G_{-2}$.[4]

Now we try to find the third vector w that satisfies $(A+2I)w = v$ where $v \in E_{-2}$

Usually, we can try setting v to each of the bases of the eigenspace. In this case, however

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} w = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where w is a generalized eigenvector.

More formally:

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta \end{array}\right]$$

[4] Note that the generalized eigenspace is going to contain the eigenvectors. Thus, in this case, there are two eigenvectors for the basis of the generalized eigenspace.

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, we have:

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Respectively, we have v_1, v_3, w where v_3 is chosen because it is a pair[5] with w.

Now we create the PJP^{-1} decomposition.

$$P = P_{\mathcal{B}_{\text{canonical}} \leftarrow \mathcal{C}} = \Big[[v_1]_{\mathcal{B}} | [v_3]_{\mathcal{B}} | [w]_{\mathcal{B}} \Big] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$J = \left[\begin{array}{c|cc} -2 & 0 & 0 \\ \hline 0 & -2 & 1 \\ 0 & 0 & -2 \end{array} \right] = [T_A]_{\mathcal{C}}$$

Example 10.5.2

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & & \\ & -2 & 1 \\ & & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We notice that

$$J = D + N = \begin{bmatrix} -2 & & \\ & -2 & \\ & & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

where N is a nilpotent matrix. A special property we have is $DN = ND$ – they are commutative.

10.5.1 Applications

Power of a Matrix

$A^n = PJ^nP^{-1}$

What we are concerned with here is J^n

[5] need clarification

$$\begin{aligned}
J^n &= (D+N)^n \text{[6]} \\
&= \sum_{k=0}^{n} \binom{n}{k} D^{n-k} N^k \\
&= \binom{n}{0} D^n N^0 + \binom{n}{1} D^{n-1} N + \binom{n}{2} D^{n-2} N^2 + \cdots \\
&= (1) D^n I + n D^{n-1} N + 0 + \cdots \\
&= D^n + n D^{n-1} N
\end{aligned}$$

Because of the nilpotent matrix, in this case, all powers above 1 are 0, thus we get the result.

Exponential e^A of a Matrix A

We know that

$$e^A = P e^J P^{-1}$$

where

$$e^J = e^{D+N} = e^D \cdot e^N$$

we know that

$$e^D = \begin{bmatrix} e^{-2} & & \\ & e^{-2} & \\ & & e^{-2} \end{bmatrix}$$

so we are done with that.[7]

Now,

$$\begin{aligned}
e^N &= \sum_{k=0}^{+\infty} \frac{1}{k!} N^k \\
&= \frac{1}{0!} N^0 + N + \frac{1}{2!} N^2 + \cdots
\end{aligned}$$

i.e. $e^N = I + N$

[6] We can do this here because these two matrices are commutable

[7] Check previous section

A Few Questions

1. How come that, through the process that we went through of solving $(A - \lambda I)w = x$ where $x \in (A - \lambda I_n)$ (i.e. eigenvector), we are creating new linearly independent vectors that what we had before?
2. How exactly does the general eigenvector calculation on wikipedia relate to the way we are calculating it? (i.e. $(A - \lambda I_n)^k w_k = 0$ where w_k is called the "generalized eigenvector of rank k")
3. When I tried to continue the generalized eigenvector computation, I reached a degenerate matrix. Is this always the case? How does this work?
4. What exactly is a Jordan Chain? How do the 1s in the superdiagonal of a Jordan block act as a "chaining mechanism"?
5. Are we replacing the second eigenvector with the one that is linked because that is a part of the *Jordan Chain*?
6. If we see the S in SJS^{-1} as a change-of-basis matrix, how do we have multiple S that works with the same J? If we are changing from different bases, doesn't that mean that J is the same linear transformation in different bases that eventually end up being the same matrix that we are trying to decompose?
7. Given a 4×4 matrix of geometric multiplicity 2, how do we decided if we want a 1×1 and 3×3 Jordan block or two 2×2 Jordan blocks?

10.6 Generalized Eigenvectors

Definition 10.6.1

Let $A \in M_n(\mathbb{C})$; let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be its distinct eigenvalues.

1) The generalized eigenspace associated with λ_i is

$$G_{\lambda_i} = (A - \lambda_i I_n)^{m_i}$$

where m_i is the multiplicity of λ_i in $P_A(\lambda)$

2) A generalized eigenvector associated with λ_i is any non-zero vector $w \in G_{\lambda_i}$

Remark

Note that $E_{\lambda_i} \subseteq G_{\lambda_i}$

Theorem 10.6.1 (Cayley-Hamilton)

Let $A \in M_n(\mathbb{C})$, and let $P_A(\lambda)$ be its characteristic polynomial.

$$P_A(A) = 0$$

Theorem 10.6.2

$$\mathbb{R}^n, \mathbb{C}^n = G_{\lambda_1} \oplus \cdots \oplus G_{\lambda_p} \tag{10.19}$$

Theorem 10.6.3
$\forall i = 1, \ldots, p$, there exists a basis $\mathcal{C}_i$ of G_{λ_i} such that
next

$$J = [T_A]_{\mathcal{C}} = \left[\begin{array}{c|c|c|c} [T_A]_{C_1} & & & \\ \hline & [T_A]_{C_2} & & \\ \hline & & \ddots & \\ \hline & & & [T_A]_{C_p} \end{array}\right]$$

Chapter 11

Linear Differential Equations

11.1 Introduction

Example 11.1.1

$$y' = y$$

we know that the solution to this differential equation is

$$y(t) = c_0 e^t, t \in \mathbb{R}$$

we can rewrite this as $y' - y = 0$ – we observe here that we have a linear combination between y and y'.

Example 11.1.2

$$y' = ay$$

we know that the solution to this DE is

$$y(t) = c_0 e^{at}, c_0, a \in \mathbb{R}$$

Proof. Let $y(t)$ s.t.

$$
\begin{aligned}
y' &= ay \\
\int \frac{y'}{y}\,\mathrm{d}t &= \int a\,\mathrm{d}t \\
\ln\bigl|y(t)\bigr| &= at + c, c \in \mathbb{R} \\
\bigl|y(t)\bigr| &= e^c e^{at} \\
&= c_1 e^{at} \\
y(t) &= \pm c_1 e^{at}
\end{aligned}
$$

■

Example 11.1.3

$$
\begin{cases}
y_1' &= 4y_1 - 9y_2 \\
y_2' &= 4y_1 - 8y_2
\end{cases}
$$

Solution:

We can express this system as

$$
\begin{aligned}
\mathbf{y}' &= A\boldsymbol{y} \\
\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} &= \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\end{aligned}
$$

We can see the general solution as (this is in fact true for ALL LDE)

$$\mathbf{y} = e^{tA} \cdot C, C \in \mathbb{R}^2$$

Note that:

Let

$$e^{At} = \Bigl[\mathbf{y}_1(t)|\mathbf{y}_t(t)\Bigr]$$

then,

$$\Bigl[\mathbf{y}_1(t)|\mathbf{y}_t(t)\Bigr] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$$

Proof. By the theorem, we find

$$A = PJP^{-1}, A = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} J = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

$$e^{tA} = e^{tPJP^{-1}} = Pe^{tJ}P^{-1}$$

Now we apply this to the solution

$$\mathbf{y}(t) = e^{tA}c \tag{11.1}$$
$$= Pe^{tJ}P^{-1}c \tag{11.2}$$
$$= (Pe^{tJ})c \tag{11.3}$$
$$= c_1 y_1(t) + c_2 y_2(t) \tag{11.4}$$

■

Definition 11.1.1 (1st Order ODE)
A *1st order linear differential equation* is a differential equation of the form

$$\mathbf{y}' = A(t)\mathbf{y} + b(t), t \in I^{1} \tag{11.5}$$

where $A(t)$ is a function valued in $M_n(\mathbb{R})/M_n(\mathbb{C})$, and $b(t)$ is a function valued in $\mathbb{R}^n/\mathbb{C}^n$ Eq. (11.5) is said to be homogeneous if $b(t) \equiv 0$ on I. Nonhomogeneous otherwise.

This is the interval

Definition 11.1.2 (Initial Value Problem)
constraint of a differential equation and an initial condition.

$$\begin{cases} \mathbf{y}' & = A(t)\mathbf{y} + b(t), t \in I \\ \mathbf{y}(t_0) & = \mathbf{y}_0, \text{where } t_0 \in I, \mathbf{y}_0 \in \mathbb{R}^n/\mathbb{C}^n \end{cases} \tag{11.6}$$

Theorem 11.1.1 (Cuachy-Lipschitz)
There exists a unique solution $\mathbf{y}(t)$ defined on the interval I such that $\mathbf{y}(t_0) = \mathbf{y}_0$.

Corollary 11.1.2
The space of solutions $\mathcal{S}_H$, where H is Eq. (11.5), of the homogenous equation

$$\mathbf{y}' = A(t)\mathbf{y}$$

is a vector space of dimension n.

Proof. Note that $\mathcal{S}_H$ is a subset of the vector space $V = \{\text{all } \mathbb{R}^n/\mathbb{C}^n\text{– valued functions defined on } I\}$ Let y_1, y_2 be in $\mathcal{S}_H$; let $\lambda \in \mathbb{R}$. The following is true.

$$y_1 + \lambda y_2 \in \mathcal{S}_H$$

$$\begin{aligned}(\mathbf{y}_1 + \lambda\mathbf{y}_2)' &= \mathbf{y}_1' + \lambda\mathbf{y}_2' \\ &= A(t)\mathbf{y}_1 + \lambda A(t)\mathbf{y}_2 \\ &= A(t)(\mathbf{y}_1 + \lambda\mathbf{y}_2)\end{aligned}$$

Thus, $\mathbf{y}_1 + \lambda\mathbf{y}_2$ is in $\mathcal{S}_H$, and by the theorem, it is a vector space.

Next, by theorem 11.1.1, we have as many solutions in $\mathcal{S}_H$ as vector $\mathbf{y}_0$ in $\mathbb{R}^n/\mathbb{C}^n$. This implies $\dim(\mathcal{S}_H) = n$ ■

Theorem 11.1.3
If $A(t) = A$, i.e. $A(t)$ is constant, then any solution of the homogeneous linear differential system

$$\mathbf{y}' = A\mathbf{y}$$

is of the form

$$\mathbf{y}(t) = e^{tA} \cdot C, C \in \mathbb{R}^n/\mathbb{C}^n$$

Proof.

$$\{\mathbf{y}(t) = e^{tA} \cdot C, C \in \mathbb{R}^n/\mathbb{C}^n\} \subseteq \mathcal{S}_H$$

we see that both sets are n-dimensional. Thus, the two sets are equal. ■

Example 11.1.4
Find a fundamental set of solutions of

$$\mathbf{y}' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & -6 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{y}$$

Solution:

By theorem 11.1.3, since $A(t) = A$ is constant, any solution of the homogenous linear differential system is of the form

$$\mathbf{y}(t) = e^{tA} \cdot C, C \in \mathbb{R}^n/\mathbb{C}^n$$

We complete our jordan decomposition and obtain:

$$P = P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}, J = \left[\begin{array}{c|cc} 2 & & \\ \hline & -1 & 1 \\ & & -1 \end{array}\right] \tag{11.7}$$

Given the decomposition, the solution is

$$e^{tA} = \cdots = Pe^{tJ}P^{-1} \tag{11.8}$$

$$e^{tD} = \begin{bmatrix} e^{2t} & & \\ & e^{-t} & \\ & & e^{-t} \end{bmatrix}, e^{tN} = I + (tN) + \frac{1}{2!}(tN)^2 + \frac{1}{3!}(tN)^3 = \left[\begin{array}{c|cc} 1 & & \\ \hline & 1 & t \\ & & 1 \end{array}\right] \tag{11.9}$$

note that $N^k = 0, k \geq 2$

Thus, the solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{tA} \cdot C & (11.10) \\ &= Pe^{tD}e^{tN}P^{-1}C & (11.11) \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{-t} & \\ & & e^{-t} \end{bmatrix} \left[\begin{array}{c|cc} 1 & & \\ \hline & 1 & t \\ & & 1 \end{array}\right] C' & (11.12) \end{aligned}$$

11.2 Section 2?

11.3 Nonhomogeneous LDE/LDS

11.3.1 Undetermined Coefficients

Example 11.3.1

$$y'' + 4y' + 4y = t^2 - t + 3$$

Let's find a particular solution y_p
We can simply guess what the solution is – it has to be a polynomial.

Definition 11.3.1
Consider the following

$$y' = A(t)y + b(t) \tag{11.13}$$

A **particular/specific** solution $y_p(t)$ of Eq. (11.13) is a solution of Eq. (11.13).

Theorem 11.3.1
The general solution $y(t)$ is of the form:

$$\mathbf{y}(t) = \mathbf{y}_H(t) + \mathbf{y}_P(t) \tag{11.14}$$

where $\mathbf{y}_H(t)$ is the general solution of $\mathbf{y}' = A(t)\mathbf{y}$[a], and $\mathbf{y}_P(t)$ is a particular solution of the DE.

[a] This is the associated homogenous case

Example 11.3.2
Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$$

Solution:

First, we find the solution to the homogeneous case

$$\mathcal{S}_H = \text{span}\left\{ e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}, e^{-2t} \begin{bmatrix} 3t+2 \\ 2t+1 \end{bmatrix} \right\}$$

Then (non-homogeneous case), we can guess using the method called **Undetermined Coefficients**

$$\begin{bmatrix} -ae^{-t} \\ -be^{-t} \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} ae^{-t} \\ be^{-t} \end{bmatrix} + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} \tag{11.15}$$

$$\begin{cases} e^{-t}(5a - 9b) & = e^{-t} \\ e^{-t}(4a - 7b) & = 0 \end{cases} \longrightarrow \begin{cases} 5a - 9b & = -1 \\ 4a - 7b & = 0 \end{cases} \longrightarrow \begin{cases} a & = 7 \\ b & = 4 \end{cases} \tag{11.16}$$

$$\mathbf{y}(t) = \mathbf{y}_H(t) + \mathbf{y}_P(t) = \left(c_1 e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 3t+2 \\ 2t+1 \end{bmatrix} \right) + \left(\begin{bmatrix} 7 \\ 4 \end{bmatrix} e^{-t} \right) \tag{11.17}$$

Example 11.3.3
A couple of examples
Find the general solution of

$$y'' + 4y' + 4y = t^2 - t + 3 \tag{11.18}$$
$$y'' + 4y' + 4y = -52\sin(2t) \tag{11.19}$$
$$y'' + 4y' + 4y = e^{-3t} \tag{11.20}$$
$$y'' + 4y' + 4y = e^{-2t} \tag{11.21}$$
$$y'' + 4y' + 4y = -52\sin(2t) + e^{-2t} \tag{11.22}$$
$$y'' + 4y' + 4y = t^2 e^{-2t} \tag{11.23}$$

Solution:

First, find the solution to the homogeneous case

$$\mathcal{S}_H = \operatorname{span}\left\{e^{-2t}, te^{-2t}\right\} \tag{11.24}$$

Note here that we think of it as: the right hand side is a linear combination of the things on the left hand side. So, how do we structure it so that there is a specific solution?

$$\mathbf{y}_P(t) = at^2 + bt + c, a, b, c \in \mathbb{R} \tag{11.25}$$
$$\mathbf{y}_P(t) = a\cos(2t) + b\sin(2t), a, b \in \mathbb{R} \tag{11.26}$$
$$\mathbf{y}_P(t) = ae^{-3t}, a \in \mathbb{R} \tag{11.27}$$
$$\mathbf{y}_P(t) = ae^{-2t}, a \in \mathbb{R} \textbf{ This is not going to work...} \tag{11.28}$$

There is something tricky with the last case, since $e^{-2t} \in \mathcal{S}_H$.

We would then try something as $\mathbf{y}_P(t) = at^2e^{-2t}$ or $\mathbf{y}_P(t) = at^3e^{-2t}$

In this case, it should be $\mathbf{y}_P(t) = at^3e^{-2t}$.

Another tricky one is $b(t) = t^2e^{-2t}$.[2]

In this case we use $\mathbf{y}_P(t) = t^2(at^2 + bt + c)e^{-2t}$

11.3.2 Variation of Constant Method

Remark
MISSING NOV 27 LECTURE NOTES

MISSING NOV 28 DISCUSSION PROBLEMS

Example 11.3.4
Consider $y'' - 2y' + 5y = e^t\cos(2t)$.
Given that $\mathcal{S}_H = \operatorname{span}\left\{e^2\cos(2t), e^t\sin(2t)\right\}$, find a specific solution y_P of the DE.

[2] For the technique and what to choose, check page 524 ± 10 pages of the textbook.

Solution:

Set $\mathbf{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$

Because we need a first order differential equation, we need to first "reduce" it.

$$\mathbf{y}' = A\mathbf{y} + b(t) \tag{11.29}$$

$$\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \cos(2t) \end{bmatrix} \tag{11.30}$$

Note that $\left\{ \mathbf{y_1} = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}, \mathbf{y_2} = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} \right\}$ is a fundamental set of solution of $\mathcal{S}_H$

which means

$$X(t) = \begin{bmatrix} e^t \cos(2t) & e^t \sin(2t) \\ e^t \cos(2t) - 2e^t \sin(2t) & e^t \sin(2t) + 2e^t \cos(2t) \end{bmatrix} \tag{11.31}$$

$$= e^t \begin{bmatrix} \cos(2t) & \sin(2t) \\ \cos(2t) - 2\sin(2t) & \sin(2t) + 2\cos(2t) \end{bmatrix} \tag{11.32}$$

is a fundamental matrix for the DE.

Realize that

$$\det\left(\begin{bmatrix} \cos(2t) & \sin(2t) \\ \cos(2t) - 2\sin(2t) & \sin(2t) + 2\cos(2t) \end{bmatrix} \right) = 2\cos^2(2t) + 2\sin^2(2t) = 2$$

Thus,

$$X^{-1}(t) = e^{-t} \frac{1}{2} \begin{bmatrix} \sin(2t) + 2\cos(2t) & -\sin(2t) \\ 2\sin(2t) - \cos(2t) & \cos(2t) \end{bmatrix}$$

For what we have to integrate, we have

$$X^{-1}(t)b(t) = \begin{bmatrix} \sin(2t) + 2\cos(2t) & -\sin(2t) \\ 2\sin(2t) - \cos(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 0 \\ e^2 \cos(2t) \end{bmatrix} \tag{11.33}$$

$$= \frac{e^{-t}}{2} \begin{bmatrix} -e^t \cos(2t)\sin(2t) \\ e^t \cos^2(2t) \end{bmatrix} \tag{11.34}$$

$$= \frac{1}{2} \begin{bmatrix} -\cos(2t)\sin(2t) \\ \cos^2(2t) \end{bmatrix} \tag{11.35}$$

$$= \frac{1}{2} \begin{bmatrix} 1/2 \sin(4t) \\ 1/2 + 1/2 \cos(4t) \end{bmatrix} \tag{11.36}$$

And for our $C(t)$

$$C(t) = \int X^{-1}(t)b(t)\,\mathrm{d}t \tag{11.37}$$

$$= \frac{1}{4} \int \begin{bmatrix} -\sin(4t) \\ 1 + \cos(4t) \end{bmatrix} \mathrm{d}t \tag{11.38}$$

$$= \frac{1}{4} \begin{bmatrix} 1/4 \cos(4t) \\ t + 1/4 \sin(4t) \end{bmatrix} \tag{11.39}$$

$$= \frac{1}{16} \begin{bmatrix} \cos(4t) \\ 4t + \sin(4t) \end{bmatrix} \tag{11.40}$$

Now, by the variation of constant method:

$$\begin{bmatrix} y_P \\ y_P' \end{bmatrix} = \mathbf{y}_P = X(t)C(t) \tag{11.41}$$

$$= \left(e^t \begin{bmatrix} \cos(2t) & \sin(2t) \\ \cos(2t) - 2\sin(2t) & \sin(2t) + 2\cos(2t) \end{bmatrix} \right) \left(\frac{1}{16} \begin{bmatrix} \cos(4t) \\ 4t + \sin(4t) \end{bmatrix} \right) \tag{11.42}$$

$$= \frac{e^t}{16} \begin{bmatrix} \cos(2t)\cos(4t) + 4t\sin(2t) + \sin(2t)\sin(4t) \\ * \end{bmatrix} \tag{11.43}$$

The solution then is $y(t) = y_H(t) + y_P$.

Remark
We need to be able to integrate most functions. e.g. $\int e^t \cos(2t)\,\mathrm{d}t$

Regarding the final:
It will be like 30% differential equations

11.4 Higher Order LDE

Example 11.4.1
Find a fundamental set of solutions of the LDE of order 2

$$y'' + 4y' + 4y = 0$$

Solution:

Let y be a solution of this LDE.[3]

$$\mathbf{y} = \begin{bmatrix} y \\ y' \end{bmatrix} \implies \mathbf{y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix}$$

we notice the relation

$$\begin{bmatrix} y' \\ y'' \end{bmatrix} = A \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$$

which is

$$\mathbf{y}' = A\mathbf{y} \iff \begin{cases} y' & = y' \\ y'' & = -4y - 4y' \end{cases}$$

Now to solve the equation, we decompose the matrix A

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} P^{-1} \tag{11.44}$$

Since $A(t) = A$ *is constant*, any solution of the 1st order differential equation is of the form (according to theorem 11.1.3)

$$\mathbf{y}(t) = e^{tA}.C, C \in \mathbb{R}^2 \tag{11.45}$$

[3]This is important! That is the reason we can construct the relation between $\mathbf{y}'$ and $\mathbf{y}$.

$$\begin{aligned}
\mathbf{y}(t) &= e^{tA} \cdot C && (11.46)\\
&= Pe^{tJ}P^{-1} && \text{since } A = PJP^{-1} \quad (11.47)\\
&= Pe^{tD}e^{tN}P^{-1} && \text{since } J = D+N, DN = ND \quad (11.48)\\
&= \begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix} e^{-2t} I_2 \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} P^{-1}C && \text{since } N^2 = 0 \quad (11.49)\\
&= e^{-2t} \begin{bmatrix} 2 & 2t+1 \\ -4 & -4t \end{bmatrix} C' && \text{where } C' = \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} \quad (11.50)\\
&= c_1' e^{-2t} \begin{bmatrix} 2 \\ -4 \end{bmatrix} + c_2' e^{-2t} \begin{bmatrix} 2t+1 \\ -4t \end{bmatrix} && (11.51)
\end{aligned}$$

Thus, recall that we are trying to find y, not $\mathbf{y}$.

$$y \in \mathcal{S}_H \iff y(t) = 2c_1' e^{-2t} + c_2'(2t+1)e^{-2t} \tag{11.52}$$

Basis of $\mathcal{S}_H$ is fundamental set of solutions.

$$\begin{aligned}
\mathcal{S}_H &= \operatorname{span}\left\{2e^{-2t}, (2t+1)e^{-2t}\right\} && (11.53)\\
&= \operatorname{span}\left\{e^{-2t}, te^{-2t}\right\} && (11.54)
\end{aligned}$$

To prove that these two spans are indeed the same.

First we observer that the first span can be easily expressed as linear combination of elements in the second

$$\operatorname{span}\left\{2e^{-2t}, (2t+1)e^{-2t}\right\} \subseteq \operatorname{span}\left\{e^{-2t}, te^{-2t}\right\} \tag{11.55}$$

Since the dimension of the right-hand side span is at most 2, we just have to prove that the left-hand side span is 2 to show that they aveh the same dimensions and is thus the same.

11.5 Applications in Sequences

Example 11.5.1
Let $\{y_n\}_{n\geq 0}$ be the sequence defined by

$$\begin{cases} y_{n+2} &= 4u_{n+1} - 7u_n \\ u_0 &= 0 \\ u_1 &= 1 \end{cases}$$

1) Calculate $u_2, u_3, u_4, \ldots$

2) Find an equivalent expression for u_n.

Solution:

1) 4, 9, 8

2) Let us set $U_n = \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix}$, where u_n is the general term of the above sequence.

   Observe that

$$\begin{bmatrix} u_{n+2} \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix}$$

   Let $U_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   Let the matrix be A, then $U_n = A^n U_0$.[4]

   We find the decomposition of matrix $A = PDP^{-1}$

   where $P = \begin{bmatrix} 2 - i\sqrt{3} & 2 + i\sqrt{3} \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 - i\sqrt{3} & \\ & 2 + i\sqrt{3} \end{bmatrix}$

   After further simplification we find that

$$\forall n \geq 0, u_n = \frac{-1}{2\sqrt{3}} \left[(2 + i\sqrt{3})^n - (2 - i\sqrt{3})^n \right]$$

   It looks complex, but if we prove that $u_n = \bar{u_n}$, then we show that it is real. [5]

[4]This can be proven with induction.
[5]Worth noting is that the conjugate of the product is the product of the conjugate, etc. so it is very easy to do so.