

MATH 229: Calculus III for Engineers
Takahiro Sakai

Martin Gong / 七海喬介

Jan 8 – ??, 2024

Contents

1	Vector and the Geometry of Space	1
1.1	3-Dimensional Space	1
1.1.1	2D Coordinates	1
1.1.2	3D Coordinates	1
1.2	Vectors	1
1.2.1	Vector Operation	2
1.2.2	Components	2
1.2.3	Standard Basis Vectors	2
1.3	The Dot Products	2
1.3.1	Law of Cosine	3
1.3.2	Projection	3
1.3.3	Work	4
1.4	The Cross Product	4
1.5	Lines and Planes	6
1.6	Cylinders and Quadric Surfaces	8
1.6.1	Cylinders	8
1.6.2	Quadric Surface	9
1.7	Vector Functions	10

Chapter 1

Vector and the Geometry of Space

1.1 3-Dimensional Space

1.1.1 2D Coordinates

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\} \quad (1.1)$$

1.1.2 3D Coordinates

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} \quad (1.2)$$

Lemma 1.1.1 (Distance Between 2 Points)

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \quad (1.3)$$

Proof. Easily proven by using the Pythagorean Theorem twice. ■

Lemma 1.1.2 (Spherical Surface)

Given point $C(a, b, c)$ and $P(x, y, z)$ where P is a point on the spherical surface and r is the radius of the sphere.

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \quad (1.4)$$

To define a solid spherical space

$$\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} \leq r \quad (1.5)$$

1.2 Vectors

Definition 1.2.1 (Vector)

Vector is a quantity that has a **magnitude** and a **direction**.

We say that two vectors \vec{u} and \vec{v} are equal if they have the same length and direction.

1.2.1 Vector Operation

Omitted

1.2.2 Components

In \mathbb{R}^2

$$\vec{a} \equiv \langle a_1, a_2 \rangle \quad (1.6)$$

In \mathbb{R}^3

$$\begin{cases} \vec{a} & \equiv \langle a_1, a_2, a_3 \rangle \\ \vec{0} & \equiv \langle 0, 0, 0 \rangle \end{cases} \quad (1.7)$$

Definition 1.2.2

Length of $\vec{a} \equiv \langle a_1, a_2, a_3 \rangle$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad (1.8)$$

1.2.3 Standard Basis Vectors

$$\begin{cases} \hat{i} & = \langle 1, 0, 0 \rangle \\ \hat{j} & = \langle 0, 1, 0 \rangle \\ \hat{k} & = \langle 0, 0, 1 \rangle \end{cases} \quad (1.9)$$

1.3 The Dot Products

Definition 1.3.1

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \quad \vec{b} = \langle b_1, b_2, b_3 \rangle \quad (1.10)$$

Then, the dot product is

$$\vec{a} \cdot \vec{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (1.11)$$

Properties

1. $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

Theorem 1.3.1

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \quad (1.12)$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}, 0 \leq \theta \leq \pi \quad (1.13)$$

Lemma 1.3.2 • If $\vec{a} \cdot \vec{b} > 0$ then $\cos \theta > 0 \implies \theta < \frac{\pi}{2}$

- If $\vec{a} \cdot \vec{b} < 0$ then $\cos \theta < 0 \implies \theta > \frac{\pi}{2}$
- If $\vec{a} \cdot \vec{b} = 0$, then $\theta = \frac{\pi}{2}$, $\vec{a} \perp \vec{b}$

1.3.1 Law of Cosine

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}| |\vec{b}| \cos \theta \quad (1.14)$$

Proof.

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \quad (1.15)$$

$$= |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \quad (1.16)$$

$$= |\vec{a}|^2 + |\vec{b}|^2 - 2ab \cos(\theta) \quad (1.17)$$

■

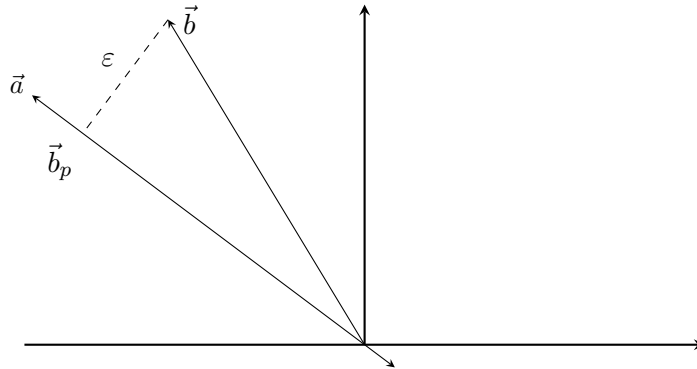
1.3.2 Projection

Figure 1.1: Projection

Add to this.

$$|\vec{b}| \quad (1.18)$$

Example 1.3.1

$$\vec{u} = \langle 1, 1, 2 \rangle \quad \vec{v} = \langle -2, 3, 1 \rangle \quad (1.19)$$

Find projection of \vec{u} onto \vec{v}

Solution:

$$\text{comp}_{\vec{v}} \vec{u} = \vec{u} \cdot \frac{\vec{v}}{|\vec{v}|} \quad (1.20)$$

$$= \frac{-2 + 3 + 2}{\sqrt{14}} = \frac{3}{\sqrt{14}} \quad (1.21)$$

$$\text{proj}_{\vec{v}} \vec{u} = (\text{comp}_{\vec{v}} \vec{u}) \frac{\vec{v}}{|\vec{v}|} = \frac{3}{\sqrt{14}} \cdot \frac{\vec{v}}{\sqrt{14}} = \frac{3}{14} \vec{v} \quad (1.22)$$

1.3.3 Work

Move an object from P to Q with a force \vec{F} forming an angle θ with the displacement vector \vec{D} .

$$\text{Work} \equiv \text{Force} \times \text{Dist} \quad (1.23)$$

$$W = \left| \vec{F} \right| \cos \theta \left| \vec{D} \right| \quad (1.24)$$

$$= \left| \vec{F} \right| \left| \vec{D} \right| \cos \theta \quad (1.25)$$

$$= \vec{F} \cdot \vec{D} \quad (1.26)$$

$$\implies W = \vec{F} \cdot \vec{D} \quad (1.27)$$

Example 1.3.2

Move a particle from $P(2, 1, 0)[m]$ to $Q(4, 6, 2)$ with a force $\vec{F} = \langle 3, 4, 5 \rangle [N]$.

What is the work done by \vec{F} ?

Solution:

$$W = \vec{F} \cdot \vec{PQ} \quad (1.28)$$

$$= \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \quad (1.29)$$

$$= 36 \text{ N m} \quad (1.30)$$

1.4 The Cross Product

Definition 1.4.1

Given the vectors

$$\vec{a} = \langle a_1, a_2, a_3 \rangle, \vec{b} = \langle b_1, b_2, b_3 \rangle \quad (1.31)$$

The cross product is defined as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \quad (1.32)$$

Properties of the Dot Product

1. $(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp \vec{b}$ and the direction follows the right-hand rule.
2. $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta, 0 \leq \theta \leq \pi$
3. $|\vec{a} \times \vec{b}|$ = the area of the parallelogram formed by the two vectors.
4. If $\vec{a} \parallel \vec{b}$, then $\vec{a} \times \vec{b} = \vec{0}$
5. Cross product of basis vectors

$$\begin{cases} \hat{i} \times \hat{j} &= \hat{k} \\ \hat{j} \times \hat{k} &= \hat{i} \\ \hat{k} \times \hat{i} &= \hat{j} \end{cases} \quad (1.33)$$

6. The cross product is not commutative
7. The cross product is not associative

Example 1.4.1

$$\begin{cases} \hat{i} \times (\hat{i} \times \hat{j}) &= \hat{i} \times \hat{k} = -\hat{j} \\ (\hat{i} \times \hat{i}) \times \hat{j} &= \vec{0} \times \hat{j} = \vec{0} \end{cases} \quad (1.34)$$

8. You can find the normal vector to a plane by applying the cross product to two non-parallel vectors on that plane.

Example 1.4.2

Given points

$$P(1, 4, 6), Q(-2, 5, 1), R(1, -1, 1)$$

that lie on a plane

- a) Find the vector normal to the plane
- b) Find the area of $\triangle PQR$

Solution:

TBA

Definition 1.4.2 (Triple Products)

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) \quad (1.35)$$

Eq. (1.35) shows the scalar triple product. This is also the volume of the parallelepiped.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \quad (1.36)$$

Eq. (1.36) shows the vector triple product.

Lemma 1.4.1

If \vec{a} , \vec{b} , and \vec{c} are on the same plane (*coplanar*), then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

1.5 Lines and Planes

Definition 1.5.1 (Line)

We define a line with a direction vector $\vec{v} = \langle a, b, c \rangle$

$$\vec{r} = \vec{v}_0 + t\vec{v} \quad (1.37)$$

Parametric Form

$$\begin{cases} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{cases} \quad (1.38)$$

Symmetric Form

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad (1.39)$$

Notice how the symmetric form does not require parameters, it tells the relationship between the coordinates.

Example 1.5.1

Intersection problem

Definition 1.5.2 (Plane)

Given a point $P_0 \equiv \vec{r}_0$ and another point $P \equiv \vec{r}$ on the plane, along with the normal vector $\hat{n} = \langle a, b, c \rangle$.

Now, we see that $\vec{r} - \vec{r}_0$ is always on the plane, so that it follows that

$$\hat{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad (1.40)$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (1.41)$$

$$ax + by + cz = d \quad (1.42)$$

where $d = ax_0 + by_0 + cz_0$

Example 1.5.2

Given $A(2, 0, 3)$, $B(0, -4, 6)$, $C(-3, 6, 0)$, on a plane, find the equation of the plane.

Solution:

We find that

$$\overrightarrow{AB} \times \overrightarrow{AC} = -8\langle 2, 3, 4 \rangle \quad (1.43)$$

We take any point and compute d

$$d = 2 \cdot 2 + 0 \cdot 3 + 3 \cdot 4 = 16 \quad (1.44)$$

so the equation is

$$2x + 3y + 4z = 16 \quad (1.45)$$

What if we want to sketch the plane?

We simply find the x, y, z -intersection of the plane, label them on a skeleton, then connect they for a triangle.

Example 1.5.3

Given two planes

$$\begin{cases} x + y + z &= 1 \\ x - 2y + 3z &= 1 \end{cases} \quad (1.46)$$

- a) Find the angle between the two planes
- b) Find the equation of the intersecting line

Solution:

- a) We have the normal vectors

$$\begin{cases} \hat{n}_1 &= \langle 1, 1, 1 \rangle \\ \hat{n}_2 &= \langle 1, -2, 3 \rangle \end{cases} \quad (1.47)$$

We simply find the angle between them using the dot product.

$$\arccos\left(\frac{\vec{a} \cdot \vec{b}}{ab}\right) = \arccos\left(\frac{2}{\sqrt{42}}\right) \quad (1.48)$$

b) We need the direction vector and a point on the line.

We can find a point on the line by defining either x, y , or z for the two equations and solve for the other variables. (e.g. A point here on the line is $P(1, 0, 0)$)

For the direction vector, we can cross the normal vectors $\vec{n}_1 \times \vec{n}_2$ to find the vector.

Definition 1.5.3 (Distance Between a Point and a Plane)

Given some point P and a random point A on the plane, we can have some vector \overrightarrow{AP} , which, if we project onto the normal vector \hat{n} of the plane, will give us the component of the vector \overrightarrow{AP} parallel to the normal vector.

$$d = |\overrightarrow{AP}| |\cos \theta| = |\overrightarrow{AP} \cdot \hat{n}| \quad (1.49)$$

where θ is the angle between the \overrightarrow{AP} and \hat{n}

Example 1.5.4

We want to find the distance between two parallel planes.

Solution:

Simply find a vector that “connects” the two planes, let that vector be \vec{v} . Then, calculate $|\vec{v} \cdot \hat{n}|$ where \hat{n} is the normal vector of the plane.

1.6 Cylinders and Quadric Surfaces

1.6.1 Cylinders

The perimeter of different cross sections of a cylinder are called **traces**, and the lines parallel to the cylindrical axis are called ???.

Example 1.6.1

$$x = z^2 \quad (1.50)$$

This is a parabolic cylinder; we see a parabolic curve on the xz -plane.

Example 1.6.2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1.51)$$

This is a cylindrical surface with intersect a and b on x and y respectively.

1.6.2 Quadric Surface

Remark

Spherical surfaces are a type of quadric surface.

Example 1.6.3 (Ellipsoid)

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 \quad (1.52)$$

Solution:

Let $z = k$ some constant.

Then,

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \quad (1.53)$$

We observe that only $-2 \leq k \leq 2$ do we see a surface. So the z intersects are -2 and 2 . Now we can solve for $x|_{y=0}$ and $y|_{x=0}$ and we find $\pm 1, \pm 3$

We can keep finding the range by turning the equation into the form $c_1x_1^2 + c_2x_2^2 = c_3$.

When $x = k$ we get

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \quad (1.54)$$

When $y = k$ we get

$$x^2 + \frac{z^2}{4} = 1 - \frac{y^2}{9} \quad (1.55)$$

This is called an ellipsoid.

Example 1.6.4 (Elliptic Parabola)

$$z = 4x^2 + y^2 \quad (1.56)$$

Solution:

We notice that this surface only exists when $z \geq 0$.

Then, we can just sketch traces and connect.

Let $z = k$

$$4x^2 + y^2 = k \quad (1.57)$$

We see an ellipse cross section that grows as k increases.

Let $x = 0$ or $y = 0$

We see a parabolic cross section.

This is an **elliptic parabola**.

Example 1.6.5 (Hyperbolic Parabola)

$$z = y^2 - x^2 \quad (1.58)$$

Example 1.6.6 (Elliptic Cone)

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad (1.59)$$

1.7 Vector Functions

Definition 1.7.1 (Vector Function)

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \quad (1.60)$$

Example 1.7.1

The vector function of a line.

$$\vec{r}_0 + \vec{v}t = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle \quad (1.61)$$

Definition 1.7.2 (Derivative of Vector Functions)

$$\frac{d\vec{r}}{dt} \equiv \vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle \quad (1.62)$$

Definition 1.7.3 (Rules of Differentiation)

$$\frac{d}{dt} [a\vec{u}(t) + b\vec{v}(t)] = a\vec{u}'(t) + b\vec{v}'(t) \quad (1.63)$$

$$\frac{d}{dt} [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \quad (1.64)$$

$$\frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t) \quad (1.65)$$

$$\frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t) \quad (1.66)$$