# PHYS 161 Lecture Notes

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# Contents

# Chapter 1

# Mathematical Interlude

**Definition 1.0.1**
Kinematics is the study of motion without regard to its cause.

## 1.1 Units & Dimensions

In *Classical Mechanics* all quantities are expressed in terms of three dimensions, and we use SI units to define them:

- length – meters, m
- time – seconds, s
- mass – kilograms, kg

How do we measure distance? Sometimes it is easier to use the **point particle** approximation where we think of an object just as a point object with all of its mass concentrated at that point.

## 1.2 Coordinate System

A **coordinate system** is a collection of coordinate axis & a point called the origin.

A coordinate system is often called a **frame of reference**.

Physics should apply in whatever coordinate system (**covariant**), so scalars, vectors, tensors, ...

### 1.2.1 Cartesian Coordinates

$$\{(x, y, z)|x, y, z \in \mathbb{R}\} \tag{1.2.1}$$

### 1.2.2 Spherical Coordinates

$$\{(r, \theta, \phi)|0 \leq r \leq \infty, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\} \tag{1.2.2}$$

where $\phi$ is the angle of the radius deviating from the $z$-axis and $\theta$ is the deviation from the $x$-axis.

The coordinate converstions are

$$\begin{cases} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan\left({}^{y}\!/_{x}\right) \\ \phi &= \arctan\left({}^{\sqrt{x^2+y^2}}\!/_{z}\right) \end{cases} \tag{1.2.3}$$

### 1.2.3 Cylindrical Coordinates

$$\{(s,\theta,z)|0 \le s \le \infty, 0 \le \theta \le 2\pi, -\infty \le z \le \infty\} \tag{1.2.4}$$

$$\begin{cases} s &= \sqrt{x^2+y^2} \\ \theta &= \arctan\left({}^{y}\!/_{x}\right) \\ z &= z \end{cases} \tag{1.2.5}$$

## 1.3 Position Vectors

The position of a particle can be specified by its *unique* coordinates or by a **position vector**, $\vec{r}$.

A vector is just an arrow, an arrow is a vector – a geometric quantity.

**Definition 1.3.1** (Vector)
A **vector** is a directed line segment, i.e. an arrow.

A vector has both **magnitude** and **direction**.

## 1.4 Vector Algebra

**Notation**

$\vec{A}$ the vector

$A = \left|\vec{A}\right|$ the magnitude

$\hat{A} = {}^{\vec{A}}\!/_{A}$ direction / unit vector

**Remark**
Technically, magnitude cannot be negative, but notation wise we do that anyways. $-\vec{A} = A(-\hat{A})$

### 1.4.1 Vector Addition

$$\vec{C} = \vec{A} + \vec{B} \tag{1.4.1}$$

Note that addition is commutative and associative.

### 1.4.2 Vector Subtraction

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \tag{1.4.2}$$

Final - Initial

### 1.4.3 Vector Multiplication

**Dot product**

$$\vec{A} \cdot \vec{B} = AB\cos(\theta) \tag{1.4.3}$$

Facts:

- if $\vec{A} \perp \vec{B} \iff \vec{A} \cdot \vec{B} = 0$
- if $\vec{A} \parallel \vec{B} \iff \vec{A} \cdot \vec{B} = AB$ is maximal
- $$\begin{cases} \vec{A} \cdot \vec{B} > 0 & \implies \text{point in similar directions} \\ \vec{A} \cdot \vec{B} < 0 & \implies \text{point in opposite directions} \end{cases}$$
- $\vec{A} \cdot \vec{A} = A^2$

Also defined component wise

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i \tag{1.4.4}$$

**Example 1.4.1**
Prove the law of cosines.
Consider the triangle, $ABC$ where $\theta$ is the angle between vectors $\vec{A}$ and $\vec{B}$.

$$c^2 = a^2 + b^2 - 2ab\cos(\theta) \tag{1.4.5}$$

*Proof.* Define $\vec{A}, \vec{B}, \vec{C}$ by $A = a, B = b, C = c; \vec{C} = \vec{A} - \vec{B}$

Then,

$$\begin{aligned} \vec{C} \cdot \vec{C} = C^2 &= (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) && (1.4.6) \\ &= A^2 - 2\vec{A} \cdot \vec{B} + B^2 && (1.4.7) \\ &= a^2 + b^2 - 2ab\cos(\theta) && (1.4.8) \end{aligned}$$

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**Cross Product**

$$\vec{A} \times \vec{B} \equiv AB \sin(\theta)\hat{n} \tag{1.4.9}$$

Facts:

- If $\vec{A} \parallel \vec{B}$ or antiparallel $\implies \vec{A} \times \vec{B} = 0$
- If $\vec{A} \perp \vec{B} \implies \vec{A} \times \vec{B}$ is maximal.
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- $\vec{A} \times \vec{A} = \vec{0}$

Also defined component wise as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.4.10}$$

## 1.5 Components of Vectors Basis Vectors

Say we have in Cartesian coordinates $(x, y)$

$$\vec{A} = \vec{A}_x + \vec{A}_y \tag{1.5.1}$$

Then,

$$\begin{cases} A_x & = A\cos\theta \\ A_y & = A\sin\theta \end{cases} \tag{1.5.2}$$

$$\begin{cases} \vec{A}_x & = A\cos\theta\vec{x} \\ \vec{A}_y & = A\sin\theta\vec{y} \end{cases} \tag{1.5.3}$$

$$\begin{cases} \hat{x} & = \langle 1, 0, 0\rangle \\ \hat{y} & = \langle 0, 1, 0\rangle \\ \hat{z} & = \langle 0, 0, 1\rangle \end{cases} \tag{1.5.4}$$

## 1.6 Vectors in Different Basis

### 1.6.1 Cartesian Coordinates

This is to say the same vectors but different components represented in different coordinates.

We can express them in the same way where $\theta$ is the original relative angle and $\theta'$ is the new relative angle:

$$\begin{cases} \vec{A} &= A\cos\theta\hat{x} + A\sin\theta\hat{y} \\ \vec{A}' &= A\cos\theta'\hat{x} + A\sin\theta'\hat{y} \end{cases} \tag{1.6.1}$$

Now, say we want to express our components in a different basis that rotates our standard basis by an angle of $\phi$ in the counterclockwise direction.

$$\begin{bmatrix} A'_x \\ B'_x \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} A_x \\ B_x \end{bmatrix} \tag{1.6.2}$$

### 1.6.2 Polar Coordinates

We have two basis vectors defined by the following

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} \tag{1.6.3}$$

$\hat{r}$ is in the direction

The conversion between the bases of Cartesian and Polar are the following:

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \tag{1.6.4}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} \tag{1.6.5}$$

**Remark**
It can be useful because coordinates will be much easier to express with

$$\vec{A} = A(\theta, r)\hat{r} \tag{1.6.6}$$

## 1.7 Calculus with Vectors

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \equiv \lim_{\Delta t\to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t} \tag{1.7.1}$$

$\vec{A}(t)$ generally changes in magnitude and direction and this does capture both.

There are two cases:

**Case 1: $\vec{A}(t)$ changes in magnitude only**

Then $\mathrm{d}\vec{A}$ is parallel to $\vec{A}(t)$ (or antiparallel).

Let $\frac{\mathrm{d}\vec{A}_{\parallel}}{\mathrm{d}t}$ the component of $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \parallel \vec{A}$.

then here

$$\left|\frac{\mathrm{d}\vec{A}_{\parallel}}{\mathrm{d}t}\right| = \frac{\mathrm{d}A}{\mathrm{d}t} \tag{1.7.2}$$

**Case 2: $\vec{A}(t)$ changes in direction only**

Then $\mathrm{d}\vec{A}$ is perpendicular to $\vec{A}(t)$ (Almost, if we see the angle as small enough, the $\mathrm{d}\vec{A}$ would be at a right angle).

Call $\frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t}$ the component of $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \perp \vec{A}(t)$.

then here

$$\left|\frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t}\right| = A\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{1.7.3}$$

**Generally**

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{A}_{\parallel}}{\mathrm{d}t} + \frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t} \tag{1.7.4}$$

But $\vec{A} = A\hat{A}$ is naively

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\hat{A} + A\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} \tag{1.7.5}$$

and

$$\frac{\mathrm{d}\vec{A}_{\parallel}}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\hat{A} \qquad \frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t} = A\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} \tag{1.7.6}$$

### 1.7.1 With Cartesian Components

**Derivative**

$$\vec{A}(t) = A_x(t)\hat{x} + A_y(t)\hat{y} \rightarrow \frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}A_x}{\mathrm{d}t}\hat{x} + \frac{\mathrm{d}A_y}{\mathrm{d}t}\hat{y} \tag{1.7.7}$$

*Notation*

$$\dot{f} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} \qquad f' = \frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{space derivative} \tag{1.7.8}$$

Hence

$$\dot{\vec{A}} = \dot{A}_x\hat{x} + \dot{A}_y\hat{y} \tag{1.7.9}$$

**Integral**

$$\int \vec{A}(t)\,\mathrm{d}t \equiv (\int A_x\,\mathrm{d}t)\hat{x} + (\int A_y\,\mathrm{d}t)\hat{y} \tag{1.7.10}$$

Note that the fundamental theorem of calculus still applies.

### 1.7.2 With Polar Components

$$\vec{A}(t) = A_r(t)\hat{r}(t) + A_\theta(t)\hat{\theta}(t) \tag{1.7.11}$$

Then

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}A_r}{\mathrm{d}t}\hat{r} + A_r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} + \frac{\mathrm{d}A_\theta}{\mathrm{d}t}\hat{\theta} + A_\theta\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} \tag{1.7.12}$$

If we derive Eq. (1.6.4), we obtain

$$\begin{cases} \dot{\hat{r}} & = (-\sin\theta)\dot{\theta}\hat{x} + (\cos\theta)\dot{\theta}\hat{y} = \dot{\theta}\hat{\theta} \\ \dot{\hat{\theta}} & = (-\cos\theta)\dot{\theta}\hat{x} + (-\sin\theta)\dot{\theta}\hat{y} = -\dot{\theta}\hat{r} \end{cases} \tag{1.7.13}$$

which means that

$$\dot{\hat{r}} = \dot{\theta}\hat{\theta} \qquad \dot{\hat{\theta}} = -\dot{\theta}\hat{r} \tag{1.7.14}$$

which makes sense if we think about it.

And if we put it together

$$\dot{\vec{A}} = \dot{A}_r\hat{r} + A_r\dot{\theta}\hat{\theta} + \dot{A}_\theta\hat{\theta} - A_\theta\dot{\theta}\hat{r} \tag{1.7.15}$$

$$\implies \dot{\vec{A}} = (\dot{A}_r - A_\theta\dot{\theta})\hat{r} + (A_r\dot{\theta} + \dot{A}_\theta)\hat{\theta} \tag{1.7.16}$$

# Chapter 2

# Kinematics

We have our position vector

$$\vec{r}(t) = (x(t), y(t)) \tag{2.0.1}$$

We use $\vec{r}$ because it seems natural, it is the direction we are pointing in.

**Remark**
Sometimes when reference to radial $\vec{r}$ is misleading, we use $\vec{x}(t)$.

The change of the vector in space across time sweeps over some **trajectory**.

## 2.1 Displacement

**Definition 2.1.1** (Displacement)
The *displacement vector* $\Delta\vec{r}$ is a measure of where the particle went (which depends on the origin!).

$$\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = \vec{r}(t_f) - \vec{r}(t_i) \tag{2.1.1}$$

1. $\|\Delta\vec{r}\| \neq$ distance travelled in general
   - distance traveled = arc length of trajectory
2. $\Delta\vec{r}$ is coordinate independent.

   Take two coordinate systems $S$ and $S'$. Let them be defined with the relation $\vec{r} = \vec{r}' + \vec{R}$ where $\vec{r}$ and $\vec{r}'$ are vectors in the respective coordinate systems.

$$\begin{cases} S: & \Delta\vec{r} = \vec{r}_f - \vec{r}_i \\ S': & \Delta\vec{r}' = \vec{r}_f{}' - \vec{r}_i{}' \end{cases} \tag{2.1.2}$$

   If we plug in the relation, we realize that they are the same, $\Delta\vec{r} = \Delta\vec{r}'$

## 2.2 Velocity

**Definition 2.2.1** (Average Velocity)

$$\vec{v}_{\text{avg}} \equiv \frac{\Delta \vec{r}}{\Delta t} \tag{2.2.1}$$

Let $\mathrm{d}\vec{r}$ be the infinitesimal displacement.

When we consider a smaller interval:

$$\lim_{\Delta t \to 0} \implies \| \mathrm{d}\vec{r} \| = \mathrm{d}r \quad \text{(distance traveled)} \tag{2.2.2}$$

A small change to $t$ results in a small change in $\mathrm{d}S$ (the distance / speed), proportionally

$$\mathrm{d}S \propto \mathrm{d}t \tag{2.2.3}$$

$$\implies \mathrm{d}S = \left( \frac{\mathrm{d}S}{\mathrm{d}t} \right) \mathrm{d}t \tag{2.2.4}$$

**Definition 2.2.2** (Velocity)
AKA the *instantaneous velocity*

$$\vec{v}(t) \equiv \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \tag{2.2.5}$$

- $\|\vec{v}\|$ = speed
- $\hat{v}$ = direction of motion

**Remark**
A note on average velocity:

$$\vec{v}_{\text{avg}} = \frac{1}{\Delta t} \int_{t_i}^{t_f} \vec{v}(t)\,\mathrm{d}t = \frac{1}{\Delta t} \int_{t_i}^{t_f} \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t = \frac{\Delta \vec{r}}{\Delta t} \tag{2.2.6}$$

Note also if we find the magnitude, it would not be the same as the average speed since the norm would go over the integrals instead of what is being integrated.

- $\vec{v}$ a vector, so write $\vec{v}(t) = \dot{x}\hat{x} + \dot{y}\hat{y} = \dot{\vec{r}}$
- Compare to frames of reference, $S$ & $S'$

  Suppose $\dot{\vec{R}} \neq 0$.

  Then we have

$$\begin{cases} \vec{r} &= \vec{r}' + \vec{R} \\ \vec{v} &= \vec{v}' + \vec{V} \end{cases} \tag{2.2.7}$$

This is known as the Galilean transformations, which, at higher velocities, "translates" to the Lorentz transformations.

We can also obtain $\vec{r}(t)$ given $\vec{v}(t)$

$$\Delta\vec{r} = \int \mathrm{d}\vec{r} = \int_{t_i}^{t_f} \vec{v}\,\mathrm{d}t \tag{2.2.8}$$

and

$$\vec{r}(t) = \vec{r}_i + \vec{v}_i(t - t_i) \tag{2.2.9}$$

## 2.3 Acceleration

**Definition 2.3.1**

$$\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}^2 t} \tag{2.3.1}$$

Similar to what is mentioned in section 1.7, $\vec{a}_{\parallel}$ is change in speed, $\vec{a}_{\perp}$ is change in direction of motion.

**Remark**
We do have the *jerk*, but it just seems that it never really matters, and acceleration is fully sufficient.

### 2.3.1 Cartesian Coordinates

$$\begin{cases} \vec{r}(t) & = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \\ \vec{v}(t) & = \dot{x}(t)\hat{x} + \dot{y}(t)\hat{y} + \dot{z}(t)\hat{z} \\ \vec{a}(t) & = \ddot{x}(t)\hat{x} + \ddot{y}(t)\hat{y} + \ddot{z}(t)\hat{z} \end{cases} \tag{2.3.2}$$

**Example 2.3.1**
Suppose particle's position is $\vec{r}(t) = A(e^{\alpha t}\hat{x} + e^{-\alpha t}\hat{y})$ with $A$ and $\alpha$ constants. ($[A] = \mathrm{m}, [\alpha] = \mathrm{m}^{-1}$) Find $\vec{v}(t)$ and $\vec{a}(t)$ and sketch trajectory.

***Solution:***

**Velocity**

$$\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \tag{2.3.3}$$

$$= A(\alpha e^{\alpha t}\hat{x} - \alpha e^{-\alpha t}\hat{y}) \tag{2.3.4}$$

$$= \alpha A(e^{\alpha t}\hat{x} - e^{-\alpha t}\hat{y}) \tag{2.3.5}$$

**Acceleration**

$$\begin{aligned} \vec{a}(t) &= \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} && (2.3.6)\\ &= \alpha^2 A(e^{\alpha t}\hat{x} + e^{-\alpha t}\hat{y}) && (2.3.7)\\ &= \alpha^2 \vec{r}(t) && (2.3.8) \end{aligned}$$

**Speed**

$$\begin{aligned} |\vec{v}| &= \sqrt{\vec{v}\cdot\vec{v}} && (2.3.9)\\ &= \sqrt{(\alpha A)^2\left[e^{2\alpha t} + e^{-2\alpha t}\right]} && (2.3.10)\\ &= \alpha A\sqrt{2\cosh(2\alpha t)} && (2.3.11) \end{aligned}$$

Note that (by definition)

$$\begin{cases} x(t) &= Ae^{\alpha t} \\ y(t) &= Ae^{-\alpha t} \end{cases} \tag{2.3.12}$$

We can try to find $y(x)$ by eliminating $t$, which is the equation for the trajectory, we obtain:

$$y(x) = \frac{A^2}{x} \qquad y \propto \frac{1}{x} \tag{2.3.13}$$

So although the velocity and acceleration changes at a exponential rate, the trajectory that it produces exhibits the inverse curve.

**Example 2.3.2**
A particle moves in the plane with trajectory of a circle of radius $R$. The particle sweeps out the circle at a uniform and constant rate. That is, it undergoes uniform circular motion. Find $\vec{r}(t), \vec{v}(t)$, and $\vec{a}(t)$.

***Solution:***

We know that the magnitude of the position vector $|\vec{r}| = R$ and that

$$\vec{r} = R\cos\theta(t)\hat{x} + R\sin\theta(t)\hat{y} \tag{2.3.14}$$

**Remark**
$\vec{v}$ changes direction, but with uniform rate $|\vec{v}| = c$.

From our $\vec{r}(t)$ we have that

$$\vec{v}(t) = -R\sin\theta(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{x} + R\cos\theta(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{y} \tag{2.3.15}$$

$$= R\dot{\theta}\left[-\sin\theta\hat{x} + \cos\theta\hat{y}\right] \tag{2.3.16}$$

We know that $v$ is constant and that $v = R\dot{\theta}$, so $R\dot{\theta}$ must also be constant. Since $R$ is constant, $\dot{\theta}$ is constant.

$$\dot{\theta} \equiv \omega \implies \theta(t) = \omega t \tag{2.3.17}$$

This is assuming $\theta(0) = 0$.

As a result of our derivation, we find

$$\begin{cases} \vec{r}(t) &= R\cos(\omega t)\hat{x} + R\sin(\omega t)\hat{y} \\ \vec{v}(t) &= -\omega R\sin(\omega t)\hat{x} + \omega R\cos(\omega t)\hat{y} \end{cases} \tag{2.3.18}$$

Now, noting the magnitude:

$$\begin{cases} r &= R \\ v &= \omega R \\ a &= \omega^2 R = \frac{v^2}{R} \end{cases} \tag{2.3.19}$$

**Acceleration**

$$\vec{a}(t) = -\omega^2 R\cos(\omega t)\hat{x} - \omega^2 R\sin(\omega t)\hat{y} \tag{2.3.20}$$

$$= -\omega^2\vec{r}(t) \tag{2.3.21}$$

**Remark**
Because $\hat{a}$ points towards the origin $[\hat{a} = -\hat{r}]$, we call it "centripetal" (← central seeking).

## 2.4 Formal Solution of Kinematic Equations

We want to obtain $\vec{v}(t)$ and $\vec{r}(t)$ given $\vec{a}(t)$.

### 2.4.1 $\vec{v}$ from $\vec{a}$

$$\int_0^t \vec{a}(t)\,\mathrm{d}t' = \int_{\vec{v}_0}^{\vec{v}} \frac{\mathrm{d}\vec{v}}{\mathrm{d}t'}\,\mathrm{d}t' \tag{2.4.1}$$

$$= \vec{v}(t) - \vec{v}_0 \tag{2.4.2}$$

$$\vec{v}(t) = \boxed{\vec{v}_0 + \int_0^t \vec{a}(t')\,\mathrm{d}t'} \tag{2.4.3}$$

### 2.4.2 $\vec{r}$ from $\vec{v}$ (from $\vec{a}$)

$$\int_0^t \vec{v}(t')\,\mathrm{d}t' = \int_{\vec{r}_0}^{\vec{r}} \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t \tag{2.4.4}$$

$$= \vec{r}(t) - \vec{r}_0 \tag{2.4.5}$$

$$\vec{r}(t) = \boxed{\vec{r}_0 + \int_0^t \vec{v}(t')\,\mathrm{d}t'} \tag{2.4.6}$$

$$= \vec{r}_0 + \int_0^t \left[\vec{v}_0 + \int_0^{t'} \vec{a}(t'')\,\mathrm{d}t''\right]\mathrm{d}t' \tag{2.4.7}$$

$$\vec{r}(t) = \boxed{\vec{r}_0 + \vec{v}_0 t + \int_0^t \int_0^{t'} \vec{a}(t'')\,\mathrm{d}t''\,\mathrm{d}t'} \tag{2.4.8}$$

**Remark**
We need to know $\vec{r}_0$.
To find $\vec{r}(t)$ given $\vec{a}(t)$ we need also to know the initial conditions, $\vec{r}_0$ and $\vec{v}_0$.

## 2.5 Constant Acceleration Motion

**Theorem 2.5.1** (Kinematic Equations with Constant $\vec{a}$)
There are many cases of constant $\vec{a}$ motion. With our previous analysis, the cases of when $\vec{a} = \text{const}$ gives:

$$\begin{cases} \vec{r}(t) &= \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \\ \vec{v}(t) &= \vec{v}_0 + \int_0^t \vec{a}\,\mathrm{d}t' = \vec{v}_0 + \vec{a}t \\ v^2 &= {v_0}^2 + 2\vec{a}\cdot\Delta\vec{r} \end{cases} \tag{2.5.1}$$

**Remark**
if $t_0 \neq 0$, the $t \to \Delta t$ in formulas.

Let's eliminate $t$ from these equations:

From $\vec{v} = \vec{v}_0 + \vec{a}t$, compute $v^2 = \vec{v}\cdot\vec{v}$

$$v^2 = {v_0}^2 + 2\vec{v}_0\cdot\vec{a}t + a^2t^2 \tag{2.5.2}$$

$$\frac{1}{2}v^2 = \frac{1}{2}{v_0}^2 + \vec{v}_0\cdot\vec{a}t + \frac{1}{2}a^2t^2 \tag{2.5.3}$$

Now, from $\vec{r}$ compute

$$\vec{a}\cdot\vec{r} = \vec{a}\cdot\vec{r}_0 + \vec{a}\cdot\vec{v}_0 t + \frac{1}{2}a^2t^2 \tag{2.5.4}$$

Then, we take the difference, we have

$$\frac{1}{2}v^2 - \vec{a}\cdot\vec{r} = \frac{1}{2}{v_0}^2 - \vec{a}\cdot\vec{r}_0 \tag{2.5.5}$$

$$\frac{1}{2}v^2 = \frac{1}{2}{v_0}^2 + \vec{a}\cdot(\vec{r}-\vec{r}_0) \tag{2.5.6}$$

$$\boxed{v^2 = {v_0}^2 + 2\vec{a}\cdot\Delta\vec{r}} \tag{2.5.7}$$

### 2.5.1 Components of the Equations

**Remark**
These laws are also applicable in components.

## 2.6 Two-Dimensional Motion

### 2.6.1 Free Fall

All objects regardless of mass, shape, composition, etc., fall downward towards earth with same motion – *free fall*.

Free fall is vertical motion subject *only* to earth's gravity, which is constant acceleration motion.

The acceleration due to gravity, $g$, is

$$g = 9.8\,\mathrm{m/s^2} \qquad \vec{a} = -g\hat{z}^{1} \tag{2.6.1}$$

### 2.6.2 Projectile Motion

*Projectile motion* is motion subject only to gravity, that is, motion for which $\vec{a} = -g\hat{z}$.

**Remark**
Projectile motion lies in the plane formed by $\vec{v}_0$ and $\vec{a}$. This implies 2D motion.

Now, the equations:

But a lot of times what we do is to consider the two components in Cartesian.

$$x \text{ component} \implies x(t) = x_0 + v_{0x}t \tag{2.6.2}$$

$$y \text{ component} \implies 0 \tag{2.6.3}$$

$$z \text{ component} \implies \begin{cases} z(t) & = z_0 + v_{0z}t - \frac{1}{2}gt^2 \\ v_z(t) & = v_{0z} - gt \\ {v_z}^2 & = {v_{0z}}^2 - 2g\Delta z \end{cases} \tag{2.6.4}$$

[1]True near earth's surface

**Example 2.6.1**
Consider a projectile launched with initial velocity $\vec{v}_0$ that makes angle $\theta$ with the horizontal. Choose coordinates s.t. $(x_0, y_0, z_0) = (0, 0, h)$ with the plane of motion the $xz$-plane.
Find:

a) the trajectory of the projectile, $z = z(x)$
b) the maximum height and horizontal distance (i.e. range) of the projectile
c) the velocity of the projectile when it hits the ground
d) the launch angle, $\phi$, that maximizes the range. Here, let $h = 0$.

***Solution:***

a) Equations for the motion are:

$$\begin{cases} z(t) &= h + v_0 \sin\theta t + \frac{1}{2} g t^2 \\ v_z(t) &= v_0 \sin\theta + gt \\ {v_z}^2 &= {v_0}^2 \sin^2\theta - 2g(z - h) \\ x(t) &= v_0 \cos\theta t \end{cases} \tag{2.6.5}$$

We simply have to find $z$ in terms of $x$, notice how $z(x) = z(t(x))$. We just need $t(x)$. We find that

$$x = v_0 \cos\theta t \tag{2.6.6}$$

$$t = \frac{x}{v_0 \cos\theta} \tag{2.6.7}$$

Now we substitute

$$z(t) = h + v_0 \sin\theta t + \frac{1}{2} g t^2 \tag{2.6.8}$$

$$z(t) = h + v_0 \sin\theta \left( \frac{x}{v_0 \cos\theta} \right) + \frac{1}{2} g \left( \frac{x}{v_0 \cos\theta} \right)^2 \tag{2.6.9}$$

$$= \boxed{h + x \tan\theta + \frac{g x^2}{2 {v_0}^2 \cos^2\theta}} \tag{2.6.10}$$

b) **Maximum Heigh** $z_{\text{max}}$

Obtained when $v_z = 0$

$$\implies 0 = v_0 \sin\theta - g t_{\text{max}} \tag{2.6.11}$$

$$t_{\text{max}} = \frac{v_0 \sin\theta}{g} \tag{2.6.12}$$

Sub into $z$-equation

$$z_{\max} = h + v_0 \sin\theta \left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{v_0 \sin\theta}{g}\right)^2 \tag{2.6.13}$$

$$= \boxed{h + \frac{{v_0}^2 \sin^2\theta}{2g}} \tag{2.6.14}$$

Alternatively,

$$0 = {v_0}^2 \sin^2\theta - 2g(z_{\max} - h) \tag{2.6.15}$$

$$z_{\max} = \boxed{h + \frac{{v_0}^2 \sin^2\theta}{2g}} \tag{2.6.16}$$

**Range** $x_{\max}$

Occurs when $z = 0$

$$0 = h + v_0 \sin\theta t_f - \frac{1}{2} g {t_f}^2 \tag{2.6.17}$$

$$t_f = \frac{-v_0 \sin\theta \pm \sqrt{{v_0}^2 \sin^2\theta + 2gh}}{-g} \tag{2.6.18}$$

$$= \frac{v_0 \sin\theta}{g} \mp \sqrt{\left(\frac{v_0 \sin\theta}{g}\right)^2 + \frac{2h}{g}} \tag{2.6.19}$$

**Remark**
We have to chose the positive of the $\mp$ because larger time.

We notice that if $h = 0$ (more generally, $\Delta z = z_f - z_0 = 0$)

$$t_f = \frac{2v_0 \sin\theta}{g} = 2t_{\max} \implies \text{symmetry of } z(t) \text{ parabola} \tag{2.6.20}$$

From $x$-equation:

$$x_{\max} = \frac{{v_0}^2 \sin\theta\cos\theta}{g} + v_0 \cos\theta \sqrt{\left(\frac{v_0 \sin\theta}{g}\right)^2 + \frac{2h}{g}} \tag{2.6.21}$$

Use the identity $2\sin\theta\cos\theta = \sin(2\theta)$

Which gives us

$$x_{\max} = \boxed{\frac{{v_0}^2\sin(2\theta)}{2g} + \sqrt{\left(\frac{{v_0}^2\sin(2\theta)}{2g}\right)^2 + \frac{2h{v_0}^2\cos^2\theta}{g}}} \tag{2.6.22}$$

$$= \frac{{v_0}^2\sin(2\theta)}{2g}\left[1 + \sqrt{1 + \frac{2gh}{{v_0}^2\sin^2\theta}}\right] \tag{2.6.23}$$

Now, if we solve for the case wehre $h = 0$, we get

$$x_{\max} = \frac{{v_0}^2\sin(2\theta)}{g} \tag{2.6.24}$$

c) We want $\vec{v}_f$, which is $\vec{v}_f = \vec{v}_0 - gt_f\hat{z}$

$$\vec{v}_f = v_0\cos\theta\hat{x} - \left(v_0\sin\theta\sqrt{1 + \frac{2gh}{{v_0}^2\sin^2\theta}}\right)\hat{z} \tag{2.6.25}$$

We can also write it in terms of magnitude and angle:

First to find the magnitude

$${v_f}^2 = {v_0}^2\cos^2\theta + {v_0}^2\sin^2\theta\left(1 + \frac{2gh}{{v_0}^2\sin^2\theta}\right) \tag{2.6.26}$$

$$= {v_0}^2 + 2gh \tag{2.6.27}$$

$$\implies v_f = \sqrt{{v_0}^2 + 2gh} \tag{2.6.28}$$

Now, for the angle of the projectile when it hits the ground

$$\tan\theta_f = \left|\frac{v_{fz}}{v_{fx}}\right| = \tan\theta\sqrt{1 + \frac{2gh}{{v_0}^2\sin^2\theta}} \tag{2.6.29}$$

$$\theta_f = \arctan\left[\tan\theta\sqrt{1 + \frac{2gh}{{v_0}^2\sin^2\theta}}\right] \tag{2.6.30}$$

**Remark**
Notice now when $h = 0$, $\theta_f = \theta$.

d) Since $h = 0$, the range is $x_{\max} = \frac{{v_0}^2 \sin(2\theta)}{g}$

We want to maximize, so we can think that $x_{\max} = x_{\max}(\theta)$ and find $\theta = \phi$ s.t. $\left.\frac{\mathrm{d}x_{\max}}{\mathrm{d}\theta}\right|_{\phi} = 0$

$$\left.\frac{2{v_0}^2 \cos(2\theta)}{g}\right|_{\phi} = \frac{2{v_0}^2}{g}\cos(2\phi) = 0 \tag{2.6.31}$$

$$\implies \cos(2\phi) = 0 \tag{2.6.32}$$

$$\phi = \frac{\pi}{4} = 45\,\mathrm{deg} \tag{2.6.33}$$

**Example 2.6.2**
A hunter is trying to hunt a bear on a tree with heigh $h$ distance $d$ away. The moment the hunter shoots, the bear is scared and drops from the tree. What angle relative to the bear should the hunter aim at to hit the bear?

***Solution:***

We can consider the vertical component, which must match for the hunter's arrow to hit

$$y + 0 + v_{0_y} t - \frac{1}{2} g t^2 \tag{2.6.34}$$

$$v_0 \sin\theta t - \frac{1}{2} g t^2 = h - \frac{1}{2} g t^2 \tag{2.6.35}$$

$$v_0 \sin\theta t = h \tag{2.6.36}$$

$$t = \frac{h}{v_0 \sin\theta} \tag{2.6.37}$$

then we plug the vertical to horizontal

$$\frac{h}{v_0 \sin\theta} \cos\theta = d \tag{2.6.38}$$

$$d = h \cot\theta \tag{2.6.39}$$

$$\theta = \boxed{\operatorname{arccot}\left(\frac{d}{h}\right)} \tag{2.6.40}$$

We notice that $\theta$ then is essentially directly at the bear.

**Remark**
Another way of thinking about it, is if we consider $g = 0$, then consider the problem, we would come to the conclusion that we should aim at the bear too. Adding g to both bodies shouldn't change that fact.

Since we also want the hunder to hit the bear before it hits the ground, we can find that

$$h = \frac{1}{2}gt^2 \tag{2.6.41}$$
$$t = \sqrt{\frac{2h}{g}} \tag{2.6.42}$$
$$t < \sqrt{\frac{2h}{g}} \tag{2.6.43}$$

Consequently

$$\sqrt{\frac{2h}{g}}v_0\cos\theta = \sqrt{\frac{2h}{g}}\frac{d}{d^2+h^2}v_0 \tag{2.6.44}$$
$$v_0 > \boxed{\sqrt{\frac{g(d^2+h^2)}{2d}}} \tag{2.6.45}$$

**Remark**
The **Frenet-Serret Formulas** gives a way of finding motion only based on the particle's current motion relative to itself.

## 2.7 Kinematics in Plane Polar Coordinates

**Definition 2.7.1**

$$\vec{r}(t) = r\hat{r} = r(t)\hat{r}(t) \tag{2.7.1}$$
$$\dot{\vec{r}}(t) = \dot{r}\hat{r} + r\dot{\hat{r}} \tag{2.7.2}$$
$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{2.7.3}$$
$$= \dot{r}\hat{r} + r\omega\hat{\theta} \tag{2.7.4}$$
$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} \tag{2.7.5}$$
$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \tag{2.7.6}$$

**Remark**
If we have the trajectory being a circle, we have velocity $\vec{v} = r\omega\hat{\theta}$.

$\ddot{r}$ is radial acceleration, and $-r\dot{\theta}^2 = -r\omega^2$ is the centripetal acceleration.
$r\ddot{\theta}$ is angular acceleration ($\alpha \equiv \ddot{\theta}, r\alpha$), and $2\dot{r}\dot{\theta}$ is the Coriolis Acceleration[a]

[a] This is related to the Coriolis Effect in non-inertial frames.

Once again, we can see for circular motion

$$\dot{r} = \ddot{r} = 0 \tag{2.7.7}$$

$$\implies \vec{a} = (-r\dot{\theta}^2)\hat{r} + (r\ddot{\theta})\hat{\theta} = -r\omega^2\hat{r} + r\alpha\hat{\theta} \tag{2.7.8}$$

Note the above is from constant speed only

**Remark**
Three examples from the notes are not included.

# Chapter 3

# Newton's Laws

## 3.1 Dynamics

Newton's Laws provide the framework for the dynamics of classical particle motion.

Question of Classical Mechanics:

*Given $\vec{r}_0$ and $\vec{v}_0$ of the particle, with mass $m$, determine its subsequent motion, $\vec{r}(t)$, for all time $t$.*

### 3.1.1 Within Context

Newton originally formulated the laws to solve the question of gravity – along the way, he formulated concepts like forces and momentum.

## 3.2 Newton's Laws

**Definition 3.2.1** (Newton's Laws)
The three laws of motion:

**Law of Invertia** A particle remains at rest or moving with constant velocity unless influenced by a force.

**$\mathbf{F} = \mathbf{ma}$** The change in a particle's motion (i.e. its acceleration) is proportional to the force impressed, as vectors.

**Action / Reaction** Forces come in pairs: to every action by one particle on another, there is an equal and opposite force in return.

### 3.2.1 First Law

There exist *intertial frames of reference*, that is, a frame in which a *free particle*[1] has constant velocity.

**Remark**
Essentially, a frame at rest and a frame with constant velocity are the same.

[1] particle subject to absolutely no influences

Mathematically, this is expressed as

$$\frac{d^2\vec{r}}{d^2t} = 0 \tag{3.2.1}$$

### 3.2.2 Second Law

Denote the force by $\vec{F}$

Two different particels subject to the same force (e.g. a spring). After the unfluence of the force (e.g. left spring), particle 1 has speed $v_1$ and particle 2 has speed $v_2$.

Consider the ratio

$$\frac{v_1}{v_2} \equiv \frac{m_2}{m_1} \tag{3.2.2}$$

where $m_i$ is an intrinsic property of the $i$-th particle we call its mass [unit: kg].

*Assumption:* $m$ is independent of $\vec{F}$ and $\vec{v}$.

So we can write a relation:

$$m_1 v_1 = m_2 v_2 \tag{3.2.3}$$

Assume we start from rest and apply some force for some duration, then we have

$$m_1 \Delta v_1 = m_2 \Delta v_2 = F \Delta t \tag{3.2.4}$$

And thus we have

$$F\Delta t = m\Delta v \implies F = \frac{m\Delta v}{\Delta t} = \frac{\Delta(mv)}{\Delta t} \tag{3.2.5}$$

**Definition 3.2.2**
Define the (physical) **momentum** of a particle to be

$$\vec{p} = m\vec{v} \tag{3.2.6}$$

As so we have with Eq. (3.2.5) the following

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} \tag{3.2.7}$$

1. As $\Delta t \to 0$, we have that

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{3.2.8}$$

2. Forces (empirical) obey the *principle of superposition.*

$$\vec{F}_{\text{net}} = \sum_i \vec{F}_i \tag{3.2.9}$$

Altogether we have that

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \tag{3.2.10}$$

If $m$ is constant, then we have

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = m\vec{a} \implies \boxed{\vec{F}_{\text{net}} = m\vec{a}} \tag{3.2.11}$$

mass is a measure of an object's inertia – *tendency to persist in its state of motion.*

### 3.2.3 Third Law

**Definition 3.2.3**
A force is a directed influence between pairs of particles.

If force of 1 on 2 is $\vec{F}_{12}$,

then force of 2 on 1 is $\vec{F}_{21} = -\vec{F}_{12}$.

**IMPORTANT:** Forces always come in pairs! (e.g. When we are sitting on our seats, its us pushing on the seat, and the seat pushing on us. The force of us pushing on the seat comes from gravity.)

**Example 3.2.1**
Given $\vec{r}_0$, $\vec{v}_0$, and $m$, fint $\vec{r}(t)$
Newton's laws:

1. go to an intertial frame: $\vec{r}(t)$
2. Identify forces acting on particle: $\vec{F}$
3. Then we just solve the differential equation.

$$\vec{F}_{\text{net}} = m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} \tag{3.2.12}$$

Our initial conditions are the two givens.

## 3.3 Forces

There are two types of forces:

1. Contact forces
2. Long range forces

### 3.3.1 Contact Forces

arises due to contact between bodies.

Deconstruct the force into components parallel and perpendicular to the surface of contact.

- The component $\perp$ is called the **normal force**, $\vec{F}_N$.
- The component $\parallel$ is called the **frictional force**, $\vec{F}_f$.

**Remark**
Normal forces are constraint[a] forces.

[a] They generally constraint the motion rather than "generating" the motion.

### 3.3.2 Tension Forces

arise due to internal elastic forces of a one-dimensional string (rope / chain / etc.)

An ideal massless string has a uniform tension force throughout. (Otherwise parts of the string can have infinite acceleration.)

**Remark**
If any body is considered to be massless, we assume automatically $F_{\text{NET}} = 0$ for that body.

The direction of the force is:

- Directed away from the string for the string
- Directed away from the body if the string is attached to some

### 3.3.3 Long-Range Forces

Forces exerted over a distance between bodies not in contact.

e.g. gravity, electromagnetic, (strong nuclear, weak nuclear)

- **Weight** $\vec{F}_g = -mg$ downward – the downward force exerted by a body near earth's surface.

  **Remark**
  Why does the specific force $F_g$ involved $m$, when $m$ is part of the 2nd law and independent of forces?
  Perhaps, $F_g = m_g g$, then free fall means $m_g g = m_I a = m_I g$ which means

  $$m_g = m_I \tag{3.3.1}$$

  The above is called the **Principle of Equivalence**

### 3.3.4 Friction

Component of contact force parallel to surface of contact.

Based on experiment, friction has two behaviors:

- **Static** – when *no* relative motion between objects in contact. Acts to balance forces to ensure constant relative velocity. Has max value.
- **Kinetic** – *is* relative motion between objects in contact. Acts in opposition to relative motion (i.e. to decelerate the object). Constant in magnitude (independent of relative speed & surface area).

The phenomenological models are:

**Static Friction**

$$F_{f_s} \leq \mu_s F_N \tag{3.3.2}$$

- $\mu_s$ is coefficient of static friction.
- $F_N$ is between the objects in contact.

$$F_{f_k} = \mu_k F_N \tag{3.3.3}$$

- $\mu_k$ is coefficient of kinetic friction
- $F_N$ is between the objects in contact.

**Remark**
Generally, $\mu_s \geq \mu_k$.

### 3.3.5 Comments

These forces are *phenomenological* in character.

That is, models based on empirical observation disregarding their fundamental origin.

However, so far as we know, there are only 4 fundamental forces in nature:

- Gravity
- Electromagnetic
- Strong Nuclear
- Weak Nuclear

All these 4 forces are long range and position dependent.

In general, if we characterize the behavior of forces, there are four categories

1. Constant force
2. Variable force with time (e.g. This could be anything, say an electric field formed by an EMF that decreases in voltage over time moving a charge.)
3. Variable force with position (e.g. Gravity)
4. Variable force with velocity (e.g. Drag force)

## 3.4 Algorithm for Solving Constant-Force Newton's Laws Problems

1) Isolate relevant bodies for analysis
2) For each body in 1), draw a free body diagram (FBD) which includes
   (a) *all* forces acting on body (may on occassion ignore some forces)
   (b) an inertial coordinate system for analysis
3) Write down the equations of motion for each body in 1) using the FBD in 2); i.e. write Newton's 2nd Law in component form.
4) Impose any kinematic constraints on the bodies in your equations from 3), along with Newton's 3rd law relation.
5) Solve for desired unknowns. Treat the equations from 4) as a system of algebraic equations, regardless of the origin.

## 3.5 Pulleys

Mechanical device used to redirect tension forces. An ideal pulley is massless and frictionless. If pulley doesn't rotate, then $F_{T_1} = F_{T_2}$. Redirects $F_T$. If they are not equal, the pulley rotates.

## 3.6 Newton's Laws in Polar Coordinates

When we reconsider our forces in polar coordinates, they just manifest in (we have already discussed this in polar kinematics)

$$\begin{cases} \vec{r} &= r\hat{r} \\ \vec{v} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \end{cases} \tag{3.6.1}$$

**Example 3.6.1**
Consider a case where $\dot{r} = \ddot{r} = \ddot{\theta} = 0$ and $\dot{\theta} = \omega$

***Solution:***

We apply Newton's Law in this situation, we obtain

$$\sum \vec{F} = m\vec{a} \implies \begin{cases} \sum F_r &= m(\ddot{r} - r\dot{\theta}^2) \\ \sum F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \end{cases} \tag{3.6.2}$$

we apply our conditions to obtain

$$F_R = -m(r\omega^2) \tag{3.6.3}$$
$$F_\theta = 0 \tag{3.6.4}$$

If we apply the condition that $v = \omega r$, we obtain

$$F_R = --mr\left(\frac{v}{r}\right)^2 = -\frac{mv^2}{r} \tag{3.6.5}$$

## 3.7 Simple Harmonic Motion

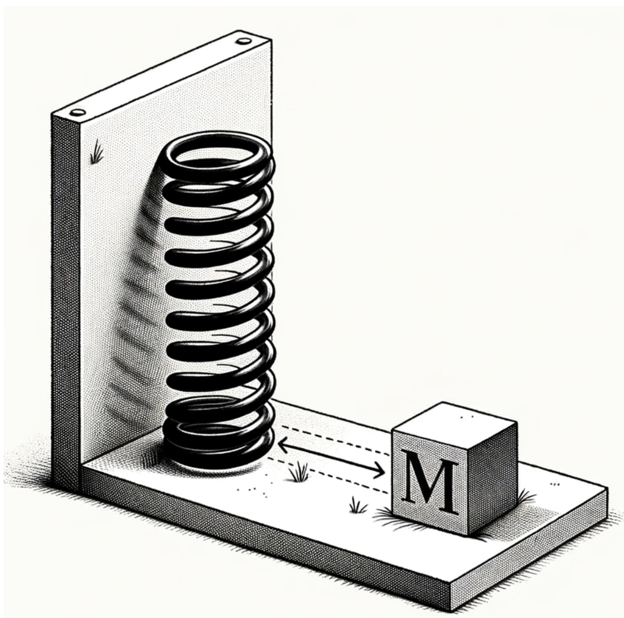



Figure 3.1: DALLE is Smart

A spring (massless) exerts a linear restoring force, $\vec{F}_s$, given by (Hooke's Law):

$$\vec{F}_S = -k\Delta\vec{r} \tag{3.7.1}$$

where $k$ is the spring constant (N m) and $\Delta\vec{r}$ is displacement of spring from equilibrium length.

> **Example 3.7.1**
> We have a block of mass $M$ connected to a wall by a spring of constant $k$.

If we setup Newton's second law:

$$\sum F = ma \implies -kx = m\ddot{x} \tag{3.7.2}$$

Let $\omega \equiv \sqrt{\frac{k}{m}}$, the this equation is

$$\boxed{\ddot{x} = -\omega^2 x} \tag{3.7.3}$$

Our guess ansatz for the solution:

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \tag{3.7.4}$$

Aside: Euler identity $e^{i\theta} = \cos\theta + i \sin\theta$ allows us to construct complex solutions

$$\tilde{x} = \tilde{c}_1 e^{i\alpha t} + \tilde{c}_2 e^{i\beta t} \tag{3.7.5}$$

If we have initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$, then we have:

$$x(t) = x_0 \cos(\omega t) \tag{3.7.6}$$

We have $\cos(\theta + 2\pi) = \cos(\theta)$,

here, $\cos(\omega t + 2\pi) = \cos(\omega t)$

let $t = t_0 - T$

$$\cos(\omega t_0 - \omega T + 2\pi) = \cos(\omega t_0) \tag{3.7.7}$$
$$\omega T - 2\pi = 0 \tag{3.7.8}$$
$$T = \frac{2\pi}{\omega} \tag{3.7.9}$$

$T$ is the **period**: time to complete one oscillation.

Then, $T = \frac{2\pi}{\omega}$

$f$ is the **frequency**, $f \equiv \frac{1}{T}$: number of oscillations per unit time ($[f] = s^{-1} \equiv \text{Hz}$)

$$\omega = 2\pi f \tag{3.7.10}$$

One is angular, one is frequency.

**Example 3.7.2**
We have a mass $m$ hung from the ceiling by a spring with spring constant $k$.

***Solution:***

This situation is actually really easy to consider. Consider when the mass is at the equilibrium position (currently at rest), this position, is displacement $y_0$ from the equilibrium position of the massless spring. Let's take downwards as the positive direction, then the forces at play here are:

$$\sum F_y = 0 \tag{3.7.11}$$
$$mg + (-ky_0) = 0 \tag{3.7.12}$$
$$y_0 = \frac{mg}{k} \tag{3.7.13}$$

which is what we expect.

Then, define our new coordinate system where $y_n = y - y_0$, and let find the solution for this situation:

We would realize that any displacement from the new coordinate $y_n$ causes a net force of

$$\sum F_y = -ky_n \tag{3.7.14}$$

on the hanging block, which is the same for our normal harmonic oscillator. If we apply Newton's 2nd law and rearrange, we obtain

$$m\ddot{y}_n = -ky_n \tag{3.7.15}$$
$$\ddot{y}_n = -\left(\sqrt{\frac{k}{m}}\right)^2 y_n \tag{3.7.16}$$

**Example 3.7.3**
The simple pendulum.

***Solution:***

We find our force in terms of polar coordinates:

$$F_r = mg\cos\theta - F_T = m(\ddot{r} - r\dot{\theta}^2) = 0 \tag{3.7.17}$$
$$F_\theta = -mg\sin\theta = mr\ddot{\theta} \tag{3.7.18}$$

If $\theta \ll 1$, $\sin\theta \approx \theta$, so we have

$$-mg\theta = ml\ddot{\theta} \tag{3.7.19}$$
$$\ddot{\theta} = -\frac{g}{l}\theta \qquad \omega = \sqrt{\frac{g}{l}} \tag{3.7.20}$$
$$\ddot{\theta} = -\omega^2\theta \tag{3.7.21}$$

so we have motion of a simple harmonic oscillator.

# Chapter 4

# Energy

## 4.1 Derivation from Newton's Laws

Work and energy can actually be thought as a consequence of Newton's laws and is another way of looking at the motion of objects. For more on the derivation, check provided lecture notes.

**Theorem 4.1.1** (Work-Energy Theorem)
In simple form:

$$W = \Delta K \tag{4.1.1}$$

In general form:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{l} = \frac{1}{2}mv^2 - \frac{1}{2}m{v_0}^2 \tag{4.1.2}$$

## 4.2 Work & Energy

We know $\vec{F} \propto \vec{a} \implies \vec{F} \parallel \mathrm{d}\vec{r}$ changes speed; $\vec{F} \perp \mathrm{d}\vec{r}$ changes direction.

In our process of deriving $W = \Delta K$, we turned a vector equation into a scalar equation. Hence, here we are concerned only with $\vec{F}_{\parallel}$ (moreover, if I draw trajectory I know what $\vec{F}_{\perp}$ does but not $\vec{F}_{\parallel}$)

Evidently, $\vec{F}_{\parallel} \implies \vec{F} \cdot \mathrm{d}\vec{r} \propto$ change in speed.

$$\mathrm{d}\vec{v} \cdot \vec{v} \sim \frac{1}{2}\,\mathrm{d}\left(v^2\right) \tag{4.2.1}$$

More precisely, the 2nd law of motion:

$$\vec{F} \cdot \mathrm{d}\vec{r} = \frac{1}{2}m\,\mathrm{d}\left(v^2\right) \tag{4.2.2}$$

Infinitesimally, this would be:

$$\vec{F}(\vec{r}_j) \cdot \mathrm{d}\vec{r}_j = \frac{1}{2} m \left[ {v_{j+1}}^2 - {v_j}^2 \right] \tag{4.2.3}$$

We can informally sum all these parts as

$$\sum_{\vec{r}=\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}_j) \cdot \mathrm{d}\vec{r}_j = \frac{1}{2} m \left( v^2 - v_0^2 \right) = \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r} \tag{4.2.4}$$

**Conclusion**

$W$ is adding up all the infinitesimal contributions of the force tangent to the trajectory (i.e. ones that change the speed) along the trajectory from start to finish.

$K$ measures the change in speed due to $\vec{F}$ as required by Newton's 2nd law.

$W$ is a sum of infinitesimal scalar quantities along a curve which is a line integral.

In cartesian:

$$\int \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r} = \int \vec{F}(x, y, z) \cdot [\, \mathrm{d}x\hat{x} + \mathrm{d}y\hat{y} + \mathrm{d}z\hat{z}] \tag{4.2.5}$$

### 4.2.1 Kinetic Energy

$$K \equiv \frac{1}{2} m v^2 \tag{4.2.6}$$

- Kinetic energy is, well, energy associated with motion.
- $K$ is frame-dependent.
- And the units are Joules.

### 4.2.2 Work

$$W = \int_C \vec{F} \cdot \mathrm{d}\vec{l} \tag{4.2.7}$$

Since $W = \Delta K$ is the change in energy of our particle/system.

- If $W > 0 \implies K > K_0 \implies$ gained energy/speed
- If $W < 0 \implies K < K_0 \implies$ loses energy/speed
- If $W = 0 \implies K = K_0 \implies$ no change in energy/speed

Work is the energy transferred into/out-of a system by mechanical means (i.e. application of a fore over a displacement)

We say "Work is done by force $\vec{F}$ along the path" when writing Eq. (4.2.7).

It is useful to measure rate at which work is done – called the **power** [units: Watts 1 W = 1 J/s]

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(W) \tag{4.2.8}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}') \cdot \mathrm{d}\vec{r}'\right] \tag{4.2.9}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_0}^{t} \vec{F}(\vec{v}) \cdot \vec{v}(t')\,\mathrm{d}t'\right] \tag{4.2.10}$$

And so we have:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \vec{F} \cdot \vec{V} \tag{4.2.11}$$

in this case, $\mathrm{d}W$ refers to an infinitesimal amount of work instead of it's change. It is technically $\text{đ}W$ − inexact differential.

**Example 4.2.1**
Consider constant force $\vec{F} = F_0\hat{n}$, $F_0 = \text{const}$, $\hat{n}$ constant unit vector.
Compute work done over displacement $\Delta\vec{r} = \vec{f} - \vec{f_0}$

***Solution:***

$$W = \vec{F} \cdot \Delta\vec{r} \tag{4.2.12}$$

**Example 4.2.2**
Consider a central force, $\vec{F}(\vec{r}) = f(r)\hat{r}$ work in 2-dim
(a) Show W is independent of path
(b) Let $f(r) = -A/r^2$ for $A > 0$ a constant. Find $v(r)$ if $v(r = r_0) = 0$.

***Solution:***

(a) Work in polar coords

$\mathrm{d}\vec{l} = \mathrm{d}r\hat{r} + r\,\mathrm{d}\theta\hat{\theta}$

Thus,

$$W = \int_C \vec{F} cdot\,\mathrm{d}\vec{l} \tag{4.2.13}$$

$$= \int_C f(r)\hat{r} \cdot (\,\mathrm{d}r\hat{r} + r\,\mathrm{d}\theta\hat{\theta}) \tag{4.2.14}$$

$$= \int_{r_0}^{r} f(r)\,\mathrm{d}r \tag{4.2.15}$$

Since it only requires the endpoints, it does not require the path, it is independent.

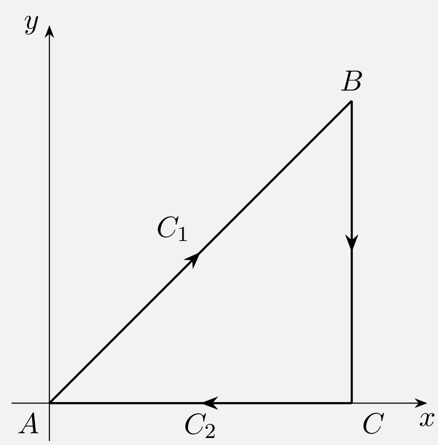
(b) $f(r) = -\frac{A}{r^2}$

$$W = \int_{r_0}^{r} -\frac{A}{r^2}\,\mathrm{d}r = A(\frac{1}{r} - \frac{1}{r_0}) \tag{4.2.16}$$

We can apply $W = \Delta K$

$$A\left(\frac{1}{r} - \frac{1}{r_0}\right) = \frac{1}{2}mv^2(r) \tag{4.2.17}$$

$$\boxed{v(r) = \pm\sqrt{\frac{2A}{m}\left(\frac{1}{r} - \frac{1}{r_0}\right)}} \tag{4.2.18}$$




**Example 4.2.3**
A particle, mass $m$, is pulled across a horizontal force with coefficient of kinetic friction $\mu_k$. First it is posted along $C_1$, then pushed along $C_2$, bringing it back to where it started. Compute $W$ along total path by friction.
where $B = (4d, 4d)$.

***Solution:***

Find $W_{C_1}$:

We know that

$$\vec{F}_{fk} = \mu_k mg\left[-\alpha(\hat{x} + \hat{y})\right] \tag{4.2.19}$$

then

$$\vec{F}_{fk} \cdot \mathrm{d}\vec{l} = \left[-\frac{\mu_k mg}{\sqrt{2}}(\hat{x} + \hat{y})\right] \cdot \left[\frac{1}{\sqrt{2}}\,\mathrm{d}s\hat{x} + \frac{1}{\sqrt{2}}\,\mathrm{d}s\hat{y}\right] = -\mu_k mg\,\mathrm{d}s \tag{4.2.20}$$

Then we calculate

$$W_{C_1} = \int_{C_1} \vec{F}_{fk} \cdot \mathrm{d}\vec{l} \tag{4.2.21}$$
$$= -4\sqrt{2}\mu_k mgd \tag{4.2.22}$$

$W_{C_2}$ is trivial.

$$W_{C_2} = -8\mu_k mgd \tag{4.2.23}$$

If we sum it up, we realize that it is not equal to 0.

**Conclusion:**

Friction is not a nonconservative force.

## 4.3 Conservative Force Fields

**Theorem 4.3.1**
The following statements are equivalent:

1. The work done by $\vec{F}$ is path-independent
2. The work done by $\vec{F}$ along any closed path is 0.

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{l} = 0 \tag{4.3.1}$$

3. $\nabla \times \vec{F} = 0$
4. There exists a scalar function $u(\vec{r})$ s.t. $\vec{F} = -\nabla u$

$$u(\vec{r}_a) - u(\vec{r}_b) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot \mathrm{d}\vec{l} \tag{4.3.2}$$

We call $u(\vec{r})$ the **potential energy** associated with force $\vec{F}$. Moreover:

$$W = -\Delta u \tag{4.3.3}$$

If $u$ exist for $\vec{F}$ we say $\vec{F}$ is a **conservative force**.

Recall $\forall \vec{F}, W = \Delta K$

If $\vec{F}$ is conservative, then $W = -\Delta u = \Delta K$

Thus,

$$\Delta K + \Delta u = 0 \tag{4.3.4}$$

Three examples of conservative forces are:

1. Constant force
2. Spring force
3. Central force

## 4.4 Different Potential Energies

### 4.4.1 Gravitational Potential Energy

**Definition 4.4.1** (Gravitational Potential Energy)
The gravitational potential energy is defined as

$$U_g(y) = mgy \tag{4.4.1}$$

obtained via

$$\begin{aligned} U_g(y) - U_g(y_0) &= -\int_{y_0}^{y} \vec{F}_g \cdot (\,\mathrm{d}y\hat{y}) && (4.4.2) \\ &= mg(y - y_0) && (4.4.3) \\ &= mgy - mgy_0 && (4.4.4) \end{aligned}$$

Note that

- It is typical to take $U_g = 0$ at $y = 0$; i.e. reference point $y_0$ is $y_0 = 0$
- Physically, only the difference in $u$ matters, so shifting $u$ by a consstant leaves physics unaltered.

### 4.4.2 Spring/Elastic Potential Energy

**Definition 4.4.2** (Spring/Elastic Potential Energy)
The spring potential energy is defined as

$$U_s(x) = \frac{1}{2}k\Delta x^2 \tag{4.4.5}$$

but it is usually referred to as

$$U_s(x) = \frac{1}{2}kx^2 \tag{4.4.6}$$

where it is assumed $x_0 = l_0$ where $x = 0$ is rest length.

We know that $\vec{F}_s = -k\Delta\vec{r}$, so we choose coordinates so that $\Delta\vec{r} = \Delta x\hat{x} = (x - l_0)\hat{x}$.

$$U_s(x) - U_s(x_0) = -\int_{x_0}^{x} \vec{F}_s \cdot (\,\mathrm{d}x\hat{x}) \tag{4.4.7}$$

$$= +k\int_{x_0}^{x} (x - l_0)\,\mathrm{d}x \tag{4.4.8}$$

$$= \frac{1}{2}k(x-l_0)^2 - \frac{1}{2}k(x_0 - l_0)^2 \tag{4.4.9}$$

### 4.4.3 Central Force

**Definition 4.4.3** (Central Force Potential Energy)
The general form for potential energy related to central forces is

$$U_c(r) - U_c(r_0) = -\frac{A}{r} + \frac{A}{r_0} \tag{4.4.10}$$

which means

$$U_c(r) = -\frac{A}{r} \tag{4.4.11}$$

We have $\vec{F} = f(r)\hat{r} = -{}^{A}\!/_{r^2}\hat{r}$

$$U_c(r) - U_c(r_0) = -\int_{r_0}^{r} \vec{F} \cdot (\,\mathrm{d}r\hat{r}) \tag{4.4.12}$$

$$= -\int_{r_0}^{r} f(r)\,\mathrm{d}r \tag{4.4.13}$$

$$= A\int_{r_0}^{r} \frac{1}{r^2}\,\mathrm{d}r \tag{4.4.14}$$

Typically speaking, we take $r_0 = \infty$ s.t. $U_c(r_0 = \infty) = 0$.

**Remark**
Both Newton's law of universal gravitation and Coulomb's law for electrostatics are of the form $\vec{F} \propto \frac{1}{r^2}\hat{r} \implies u \propto -\frac{1}{r}$.

## 4.5 Definition of Energy

**Definition 4.5.1** (Mechanical Energy)
We define

$$E = K + U \qquad \Delta E = 0 \tag{4.5.1}$$

Then, $\Delta E = 0$ a dynamical quantity that does not change in time is called "conserved".
If $\vec{F}$ is conservative, then, energy is conserved.

Now, consider a system of particles interacting only via conservative forces.

A **system** is an arbitrary division of a collection of particles declared to be either in the system or not and hence part of the environment.

Supposed there are no external forces on the system, then:

$$W_{\text{total}} = \Delta K_{\text{total}} = -\Delta U_{\text{total}} \tag{4.5.2}$$

where *total* refers to sum over all particles & interactions.

This means, then,

$$\Delta K_{\text{total}} + \Delta U_{\text{total}} = 0 \tag{4.5.3}$$

Potential energy is energy *stored in a system* due to *conservative interactions* that is reversibly transmutable to other forms (i.e. kinetic energy).

In other terms, potential energy exist with regards to fields (e.g. EM fields, gravitational fields).

**Definition 4.5.2** (Law of Conservation of Mechanical Energy)
In a closed and isolated system, all of whose internal interactions are conservative, the total mechanical energy is constant in time, or conserved, along the motion.

*Proof.* Consider a closed and isolated system with only a conservative force. Then, $E = K + U = \frac{1}{2}mv^2 + U(\vec{r})$.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = m\vec{v} \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \nabla u \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \tag{4.5.4}$$

$$= \vec{v} \cdot \left[ m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \nabla u \right] \tag{4.5.5}$$

$$= \vec{v} \cdot \left[ \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} - \vec{F} \right] \tag{4.5.6}$$

but Newton's 2nd law says $\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F}$ i.e. $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$.

Note if $U = U(\vec{r}, t)$, then $\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial U}{\partial t}$ and so not conserved.

Think of my dynamical law as

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = -\nabla U \tag{4.5.7}$$

then if $U$ does not depend explicitly on time, $E$ is conserved. We say that law that time-translation symmetry. ∎

**Definition 4.5.3** (A Definition of Energy)
Energy is the quantity that is constant in time because the laws of physics has time-translation symmetry.
An extention from **Noether's Theorem**.

A slight extension:

- Symmetry in space/location – conservation of momentum
- Symmetry in angles/rotation – conservation of angular momentum
- Gauge symmetry – electric charge

## 4.6 Examples

**Example 4.6.1**
Segway to the average of a periodic function over time (i.e. $\sin(t), \cos(t)$) for integral number of periods.

***Solution:***

$$\overline{K} = \frac{1}{nT}\int_0^{nT} K \,\mathrm{d}t \tag{4.6.1}$$

$$= \frac{1}{nT}\cdot\frac{1}{2}m\int_0^{nT} \dot{x}^2 \,\mathrm{d}t \tag{4.6.2}$$

$$= \frac{m\omega^2}{2nT}A^2\int_0^{nT} \sin^2(\omega t+\phi)\,\mathrm{d}t \tag{4.6.3}$$

$$= \frac{m\omega^2}{2nT}A^2\int_0^{nT} \frac{1}{2}\Big(1-\cos\big[2(\omega t+\phi)\big]\Big)\,\mathrm{d}t \tag{4.6.4}$$

$$= \frac{m\omega^2}{2nT}A^2\left[\frac{1}{2}nT + \frac{\sin\big[2(\omega t+\phi)\big]}{2\omega}\Bigg|_0^{nT}\right] \tag{4.6.5}$$

$$= \left(\frac{1}{2}m\omega^2A^2\right)\cdot\frac{1}{2} + \frac{m\omega^2A^2}{4nT\omega}\big[\sin(2n\omega T+2\phi)-\sin(2\phi)\big]^{1} \tag{4.6.6}$$

$$= \frac{1}{2}\left(\frac{1}{2}kA^2\right) = \frac{1}{2}E \tag{4.6.7}$$

Similarly, $\overline{U_s} = \frac{1}{2}E = \overline{K}$.

**Conclusion:**

$K$ and $U$ are $\frac{\pi}{2}$ out of phase, and follows the above relation.

[1] Note that this, if we expand with sum of angles, evaluates to 0 (the expression in the square brackets)

**Example 4.6.2**
Example of central force

***Solution:***

omitted

**Example 4.6.3**
Use energy methods to show the motion of a simple pendulum (mass $m$, length $l$) is simple harmonic for small angles. What is the first correction to period if $\theta$ is not small?

***Solution:***

a) From the figure,[2]

$$U_g = mg(l - l\cos\theta) = mgl(1 - \cos\theta) \tag{4.6.8}$$

Let the initial angle be $\theta_0$, then $\theta = \theta_0 \implies K = 0$ and $U_g = mgl(1 - \cos\theta)$.

TBF

b) Given the original equation:

$$\dot{\theta}^2 = -2\left(\frac{g}{l}\right)[\cos\theta_0 - \cos\theta] \tag{4.6.9}$$

which becomes

$$\int_{\theta_0}^{\theta} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{2}\sqrt{\frac{g}{l}}t \tag{4.6.10}$$

For a period, $t = T$ change variables in integral: $\sin u \equiv \sin\bigl(\theta/2\bigr)/\sin\bigl(\theta_0/2\bigr)$ and $K \equiv \sin\bigl(\theta_0/2\bigr)$. Then show

$$\sqrt{2}\int_0^{2\pi} \frac{\mathrm{d}u}{\sqrt{1 - k^2\sin^2 u}} = \sqrt{2}\sqrt{\frac{g}{l}}T = \sqrt{2}\left(\frac{2\pi}{T_0}\right)T \tag{4.6.11}$$

[2] TBA

# Chapter 5

# Momentum

## 5.1 Introduction

Recall that we mentioned that what Newton actually defined for the 2nd law is:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \qquad \boxed{\vec{p} = m\vec{v}} \tag{5.1.1}$$

Consider a system of $N$ particles, unconstrained in their motion (i.e. do not necessarily have kinematic constraints)

For the $j$-th particle, the 2nd law says:

$$\vec{F}_j = \frac{\mathrm{d}\vec{v}_j}{\mathrm{d}t} \left( = m_j \frac{\mathrm{d}\vec{v}_j}{\mathrm{d}t} \right) \tag{5.1.2}$$

By superposition,

$$\vec{F}_j = \sum_{i \neq j}^{N} \vec{F}_{ij} + \vec{F}_j^{\text{ext}} \tag{5.1.3}$$

where $\vec{F}_{ij}$ is the force of the $i$-th particle (in system) on $j$-th particle, and $\vec{F}_j^{\text{ext}}$ is the (net) external (i.e. not in system) of $j$-th particle.

Consider the total force of the system $\sum_{j=1}^{N} \vec{F}_j$, then

$$\sum_j \vec{F}_j = \sum_j \sum_{i \neq j} \vec{F}_{ij} + \sum_j \vec{F}_j^{\text{ext}} \equiv \sum_j \sum_{i \neq j} \vec{F}_{ij} + \vec{F}^{\text{ext}} \tag{5.1.4}$$

where $\vec{F}^{\text{ext}}$ is the total external force on system.

If we look at first terms

$$\sum_j \sum_{i \neq j} \vec{F}_{ij} = \vec{F}_{12} + \vec{F}_{13} + \cdots + \vec{F}_{21} + \vec{F}_{23} + \cdots \tag{5.1.5}$$

Note that Newton's 3rd Law $\vec{F}_{ij} = -\vec{F}_{ji}$, so all the terms in the sum cancels out.

**Remark**
Since $\vec{F}_{ij}$ is a antisymmetric quantity, the sum over it is zero.

Hence,

$$\vec{F}_{\text{tot}} = \sum_j \vec{F}_j = \vec{F}^{\text{ext}} \tag{5.1.6}$$

We apply Newton's 2nd law:

$$\vec{F}^{\text{ext}} = \sum_j \frac{\mathrm{d}\vec{p}_j}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \vec{p}_j\right) \tag{5.1.7}$$

So we define the total momentum $\vec{p}_{\text{tot}}$ to be

$$\vec{p}_{\text{tot}} \equiv \sum_{j=1}^{N} \vec{p}_j \tag{5.1.8}$$

which means that

$$\vec{F}^{\text{ext}} = \frac{\mathrm{d}\vec{p}_{\text{tot}}}{\mathrm{d}t} \tag{5.1.9}$$

## 5.2 Center of Mass (COM)

### 5.2.1 Discrete Masses

The total external force on the system changes the total momentum of the system as if it were a point particle, let's try to write that in the form $\vec{F}^{\text{ext}} = M\vec{A}$.

$$\vec{F}^{\text{ext}} = \frac{\mathrm{d}\vec{p}_{\text{tot}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j m_j \vec{v}_j\right) = \sum_j m_j \frac{\mathrm{d}\vec{v}_j}{\mathrm{d}t} \tag{5.2.1}$$

$$= \sum_j m_j \frac{\mathrm{d}^2\vec{r}_j}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_j m_j \vec{r}_j \tag{5.2.2}$$

$$= \left(\frac{\sum m_j}{\sum m_j}\right) \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_j m_j \vec{r}_j \tag{5.2.3}$$

So we define the *total mass* $M$ by

$$M \equiv \sum_j m_j \tag{5.2.4}$$

and define the (position of) *center of mass*, $\vec{R}$ by

$$\vec{R} \equiv \frac{1}{M} \sum_j m_j \vec{r}_j \tag{5.2.5}$$

THen,

$$\vec{F} = M \frac{\mathrm{d}^2 \vec{R}}{\mathrm{d}t^2} \tag{5.2.6}$$

or if $\vec{V} = \frac{\mathrm{d}\vec{R}}{\mathrm{d}t}$ and $\vec{A} = \frac{\mathrm{d}\vec{V}}{\mathrm{d}t}$, that means $\vec{F}^{\text{ext}} = M\vec{A}$, and so there we have it.

**Example 5.2.1**
Say a combined particle of A ($m_A = \frac{1}{4}M$) and B ($m_B = \frac{3}{4}M$) is launched at some angle $\theta$ with initial velocity $v_0$ under the influence of gravity. At the apex of the trajectory, an internal explosion occurs, repelling the two particles in opposite direction horizontally. Relative to the launch position, if $x_B = d$, find $x_A$.

***Solution:***

Let's use $\vec{F}^{\text{ext}} = M \frac{\mathrm{d}^2 \vec{R}}{\mathrm{d}t^2}$

we have center of mass

$$\begin{cases} R_x &= \frac{1}{M}(m_A x_A + m_B x_B) \\ R_y &= \frac{1}{M}(m_A y_A + m_B y_B) \end{cases} \tag{5.2.7}$$

And $\vec{F}^{\text{ext}} = -Mg\hat{y}$

Looking at the 2nd law of the COM:

$$\ddot{R}_x = 0 \qquad \ddot{R}_y = -g \tag{5.2.8}$$

$$\begin{cases} 0 &= \frac{1}{M}(m_A \ddot{x}_A + m_B \ddot{x}_B) \\ -g &= \frac{1}{M}(m_A \ddot{y}_A + m_B \ddot{y}_B) \end{cases} \tag{5.2.9}$$

Our solution for COM is:

$$\vec{R}(t) = \vec{R}_0 + \vec{V}_0 t - \frac{1}{2}\vec{g}t^2 \tag{5.2.10}$$

$$\begin{cases} R_x & = v_0 \cos\theta t \\ R_Y & = v_0 \sin\theta t - \frac{1}{2} g t^2 \end{cases} \tag{5.2.11}$$

Particles hit the ground at

$$t_f = \frac{2 v_0 \sin\theta}{g} \tag{5.2.12}$$

Hence

$$R_{xf} = v_0 \cos\theta \left( \frac{2 v_0 \sin\theta}{g} \right) = \frac{v_0^2 \sin(2\theta)}{g} \tag{5.2.13}$$

Consequently:

$$R_{xf} = \frac{1}{M}(m_A x_{Af} + m_B x_{Bf}) \tag{5.2.14}$$

$$\frac{v_0^2 \sin(2\theta)}{g} = \frac{1}{M}(\frac{1}{4} M x_{Af} + \frac{3}{4} M d) = \frac{1}{4} x_{Af} + \frac{3}{4} d \tag{5.2.15}$$

$$x_{Af} = \boxed{\frac{4 v_0^2 \sin(2\theta)}{g} - 3d} \tag{5.2.16}$$

### 5.2.2 Continuous Distribution

For a body that is (to good approximation) continuous, i.e. its mass is continuously distributed throughout its volume $V$, its center of mass becomes

$$\vec{R} = \frac{1}{M} \int_V \vec{r}\, \mathrm{d}m \tag{5.2.17}$$

where $M = \int \mathrm{d}m$

Now we want to practically evaluate Eq. (5.2.17), we can replace $\mathrm{d}m$ by:

$$\mathrm{d}m = \begin{cases} \lambda(\vec{r})\, \mathrm{d}l & \Longrightarrow \text{1-dim distribution [mass/length]} \\ \sigma(\vec{r})\, \mathrm{d}\partial & \Longrightarrow \text{2-dim distribution [mass/area]} \\ \rho(\vec{r})\, \mathrm{d}V & \Longrightarrow \text{3-dim distribution [mass/vol]} \end{cases} \tag{5.2.18}$$

**Remark**
What if we have a point mass in 1-dim?

$$\int_V \mathrm{d}m = M \tag{5.2.19}$$

There's the **Dirac Delta** $\delta(\vec{r})$ s.t.

$$\int_{\mathbb{R}^3} \delta^3(\vec{r})\,\mathrm{d}^3\vec{r} = 1 \tag{5.2.20}$$

So, for a point pass

$$m = m\int_{-\infty}^{\infty} \delta(x-a)\,\mathrm{d}x \tag{5.2.21}$$

**Example 5.2.2**
Take some example 2D object with density $\sigma$ that is a triangle of height $h$ and length $b$ with the right angle side away from the origin.

***Solution:***

$$\vec{R} = \frac{1}{M}\int \vec{r}\,\mathrm{d}m \tag{5.2.22}$$
$$= \frac{1}{M}\vec{r}\sigma\,\mathrm{d}A \tag{5.2.23}$$
$$= \frac{\sigma}{M}\int \vec{r}\,\mathrm{d}A \tag{5.2.24}$$

Area is $\frac{1}{2}bh$ so $\sigma = \frac{2M}{bh}$

$$\vec{R} = \frac{2}{bh}\int \vec{r}\,\mathrm{d}A \tag{5.2.25}$$
$$= \frac{2}{bh}\left[\hat{x}\iint x\,\mathrm{d}x\,\mathrm{d}y + \hat{y}\iint y\,\mathrm{d}x\,\mathrm{d}y\right] \tag{5.2.26}$$
$$= \frac{2}{bh}\left[\hat{x}\int_0^b\int_0^{hx/b} x\,\mathrm{d}x\,\mathrm{d}y + \hat{y}\int_0^b\int_0^{hx/b} y\,\mathrm{d}x\,\mathrm{d}y\right] \tag{5.2.27}$$
$$= \frac{2}{3}b\hat{x} + \frac{1}{3}h\hat{y} \tag{5.2.28}$$

### 5.2.3 Center of Mass Frame

Given two particles $m_1$ and $m_2$, the center of mass is given by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{5.2.29}$$

The center of mass frame is the frame whose origin is $\vec{R}$.

$$\begin{cases} \vec{r}_{1_{\text{COM}}} &= \vec{r}_1 - \vec{R} \\ \vec{r}_{2_{\text{COM}}} &= \vec{r}_2 - \vec{R} \end{cases} \tag{5.2.30}$$

If $\ddot{\vec{R}} = 0$ that means that the frame is inertial (which is to also say that $\vec{F}^{\text{ext}} = 0$)

Note also that the **momentum in the COM frame is always 0!** It ends up much easier to analyze collisions in the COM frame.

## 5.3 Variable Mass Situations

**Example 5.3.1**
Let the speed oft he exhaust be $u$ relative to the rocket – this is an inertial frame. The rocket has mass $M_R$. The fuel is ejected at a rate $\mathrm{d}M_F$ for small interval $\mathrm{d}t$.

***Solution:***

Note that $\mathrm{d}M_F = -\mathrm{d}M_R$ where $M_R$ is the mass of the rocket.

This results in increase in rocket velocity: $v \to v + \mathrm{d}v$

Assume that $u$ is constant.

The initial momentum is then $P_0 = M_R v$.

Final momentum, which is

$$P_f = (M_R - \mathrm{d}M_F)(v + \mathrm{d}v) + \mathrm{d}M_F(v - u) \tag{5.3.1}$$

$$\begin{aligned}
\Delta P &= P_f - P_0 && (5.3.2)\\
&= M_R v + M_R\,\mathrm{d}v - V\,\mathrm{d}M_F - \mathrm{d}M_F\,\mathrm{d}v^{1} + V\,\mathrm{d}M_F - u\,\mathrm{d}M_F - M_R v && (5.3.3)\\
&= M_R\,\mathrm{d}v - u\,\mathrm{d}M_F && (5.3.4)\\
F &= M_R\frac{\mathrm{d}v}{\mathrm{d}t} - u\frac{\mathrm{d}M_F}{\mathrm{d}t} && (5.3.5)\\
& && (5.3.6)
\end{aligned}$$

which, given out initial inversion statement

$$F = M_R\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}M_R}{\mathrm{d}t} \tag{5.3.7}$$

**Example 5.3.2**
A rocket of mass $m_0$ at $t = 0$, exhausts mass backward, accelerating the rocket forward in free empty space. If exhaust velocity $u$ is constant, determine the rocket's velocity as function of it's mass.

***Solution:***

[1] Note that $\mathrm{d}M_F\,\mathrm{d}v$ we are considering as negligible since the two differentials are both rather small

We have that $F = 0$, and therefore

$$M_R \frac{\mathrm{d}v}{\mathrm{d}t} - u\frac{\mathrm{d}M_F}{\mathrm{d}t} = 0 \longrightarrow m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = 0 \tag{5.3.8}$$

Then, we can solve

$$m\,\mathrm{d}v = -u\,\mathrm{d}m \tag{5.3.9}$$
$$\frac{\mathrm{d}v}{u} = -\frac{\mathrm{d}m}{m} \tag{5.3.10}$$
$$\int_0^v \frac{\mathrm{d}v}{u} = -\int_{m_0}^{m} \frac{\mathrm{d}m}{m} \tag{5.3.11}$$
$$\frac{v}{u} = -\log\left({}^{m}/_{m_0}\right) \tag{5.3.12}$$
$$v(m) = \boxed{-u\log\left({}^{m}/_{m_0}\right)} \tag{5.3.13}$$

**Example 5.3.3**
A rocket of total mass $m_0$ in empty space has speed $v_0$ when it must slow down to speed ${}^{v_0}/_2$ to intercept an asteroid. how much fuel must be burned?

***Solution:***

In this case, the fuel must be ejected forward, so relative velocity of the fuel is $v + u$. We get an alter version

$$M_R \frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}M_F}{\mathrm{d}t} = 0 \longrightarrow m\frac{\mathrm{d}v}{\mathrm{d}t} = u\frac{\mathrm{d}m}{\mathrm{d}t} \tag{5.3.14}$$

So we have

$$m\,\mathrm{d}v = u\,\mathrm{d}m \tag{5.3.15}$$
$$\int_{m_0}^{m_f} \frac{\mathrm{d}m}{m} = \int_{v_0}^{v_0/2} \frac{\mathrm{d}v}{u} \tag{5.3.16}$$
$$\log\left({}^{m_f}/_{m_0}\right) = -\frac{v_0}{2u} \tag{5.3.17}$$
$$m_f = m_0 e^{-v_0/2u} \tag{5.3.18}$$

Amount of fuel is $m_0 - m_f$

$$\Delta m = m_0\left(1 - e^{-v_0/2u}\right) \tag{5.3.19}$$

This is also known as the Tsiolkovsky rocket equation

$$m_0 = m_f e^{\Delta v/u} \tag{5.3.20}$$

## 5.4 Impulse

We know that $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$, then

$$\Delta\vec{p} = \int_{t_0}^{t_f} \vec{F}\,\mathrm{d}t \tag{5.4.1}$$

**Definition 5.4.1**
We defined impulse to be

$$\vec{J} = \int_{t_0}^{t_f} \vec{F}\,\mathrm{d}t = \Delta\vec{p} \tag{5.4.2}$$

**Remark**
Consider $\vec{F} = \text{const}$, then $\vec{J} = \vec{F}\Delta t$

- So large $\vec{F}$ over short time means a large $\vec{J}$
- A large $\Delta t$ and small $\vec{F}$ also means a large $\vec{J}$

## 5.5 Conservation

For a system of particles

$$\vec{F}_{\text{ext}} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \tag{5.5.1}$$

If $\vec{F}_{\text{ext}} = 0$, then ${\mathrm{d}\vec{p}}/{\mathrm{d}t} = 0$, which means $\vec{p}$ is constant in time.

**Definition 5.5.1** (The Law of Conservation of Momentum)
For a system of particles such that the total external force on the system is zero, then the total linear momentum $\vec{P} = \sum_i \vec{p}_i$ is conserved – $\Delta\vec{p} = 0$.

Recalled that energy $E$ is conserved b/c $u$ is time-independent. Supposed $u = u(y, z)$ that is, $u$ is independent of $x$.

Then,

$$\vec{F} = -\nabla u = -\left[\frac{\partial u}{\partial x}\hat{x} + \frac{\partial u}{\partial y}\hat{y} + \frac{\partial u}{\partial z}\hat{z}\right] = \frac{\partial u}{\partial y}\hat{y} + \frac{\partial u}{\partial z}\hat{z} \tag{5.5.2}$$

Hence,

$$-\nabla u = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \implies \begin{cases} 0 & = \frac{\mathrm{d}\vec{p}_x}{\mathrm{d}t} \\ -\frac{\partial u}{\partial y} & = \frac{\mathrm{d}p_y}{\mathrm{d}t} \\ -\frac{\partial u}{\partial z} & = \frac{\mathrm{d}p_z}{\mathrm{d}t} \end{cases} \tag{5.5.3}$$

- If $u$ is space-translation invariant $\implies$ momentum is conserved.

- Because $\vec{p}$ is a vector, it may be that only some components are conserved.

## 5.6 Collisions

A collision is a short duration interaction between two objects, such that external forces are negligible in comparison to internal forces of interaction over that duration and hence momentum is conserved for this interaction.

We will classify collisions into three categories, based on the kinetic energy difference. Let $Q \equiv K_0 - K_f$ (total K.E.)

**$\mathbf{Q > 0}$** Inelastic Collision

Kinetic energy is "lost" to other forms.

*If particles stick together after collision, we have* ***perfectly inelastic collision***

**$\mathbf{Q = 0}$** Elastic Collision

Kinetic energy is conserved.

*note, typically the internal interaction is conservative, so $KE_0 \rightarrow PE \rightarrow KE_f$, meaning energy conserved.

**$\mathbf{Q < 0}$** Superelastic Collision

Kinetic energy is gained (i.e. an explosion)

> **Example 5.6.1**
> Two particles masses $m_1$ and $m_2$ with initial velocities $\vec{v}_1$ and $\vec{v}_2$ collide in a perfectly inelastic collision. Fidn the velocity $\vec{v}'$ of the combined mass after the collision and the kinetic energy lost.

***Solution:***

Conservation of momentum:

$$\vec{p} = \vec{p}' \tag{5.6.1}$$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}' \tag{5.6.2}$$

$$v' = \boxed{\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}} \tag{5.6.3}$$

Let's find $Q$:

$$
\begin{aligned}
Q &= K - K' && (5.6.4)\\
&= \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2 - \frac{1}{2}(m_1+m_2){v'}^2 && (5.6.5)\\
&= \frac{1}{2}\left[m_1 - \frac{{m_1}^2}{m_1+m_2}\right]{v_1}^2 + \frac{1}{2}\left[m_2 - \frac{{m_2}^2}{m_1+m_2}\right]{v_2}^2 - \left(\frac{m_1m_2}{m_1+m_2}\right)\vec{v}_1\cdot\vec{v}_2 && (5.6.6)\\
&= \boxed{\frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)(\vec{v}_1-\vec{v}_2)^2} && (5.6.7)
\end{aligned}
$$

There are several cases

- If $v_1 = v_2$, they never collide!
- If $v_1 = -v_2$, they collide with energy lost, which is the total energy.

# Chapter 6

# Rigid Body Motion

## 6.1 Introduction

When we talk about rigid body motion, at least at our current level, will be subject to several constraints:

- The axis of rotation will be fixed in direction, though it may translate.
- The body is rigid and thus do not deform.

**Theorem 6.1.1** (Chasles' Theorem)
Most general rigid body displacement can be produced by a translation along a line (called its screw axis or Mozzi axis) followed (or preceded) by a rotation about an axis parallel to that line.

## 6.2 Rotational Kinematics

We can categorize rotation into two categories:

**Orbiting** Rotation about an axis external to the body

**Spinning** Rotation about axis internal to the body

Consider coordinates s.t. the axis of rotation of rigid body is the $z$-axis. Then, the rotation view down the $z$-axis is either clockwise (CW) or counterclockwise (CCW).

Angular speed $\omega$ measures rate of rotation. To include direction of rotation, we promote $\omega$ to a vector: $\vec{\omega}$.

The direction of $\vec{\omega}$ is along axis of rotation (i.e. $\pm z$-axis) s.t. rotation follows the RHR.

We do a similar thing with $\alpha$ to $\vec{\alpha}$.

**Remark**
For fixed axis rotation i.e. 1-dim motion

- If $\alpha$ and $\omega$ have the same sign, the body is rotating faster

- If they have opposite signs, then the body is rotating slower.

Also, technically $\vec{\omega}$ and $\vec{\alpha}$ are "pseudovectors" and if we change around the axes, then there would be a sign change on the vector.

## 6.3 Rotational Dynamics

The rotational analogue of force is **torque**, $\vec{\tau}$, s.t.

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{6.3.1}$$

where $\vec{r}$ is the position vector

**orbiting** $\vec{r}$ is from origin to particle (which is the point of application of the force). If we change the origin, then we are changing the torque.

**spinning** $\vec{r}$ is from origin to point of application of the force.

and $\vec{F}$ is the force.

Thus, in conclusion, $\vec{t}au$ depends on the point of origin, i.e. the point about which it is computed (via $\vec{r}$).

Typically we choose the origin to lie on axis of rotation.

A body in **static equilibrium** requires that $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$.

**Example 6.3.1**
Show that the torque due to gravity about any point always acts at the center of mass: $\vec{\tau} = \vec{R} \times \vec{F}_g$, where $\vec{R}$ is the center of mass (measured from the point) and $\vec{F}_g$ is the weight.

***Solution:***

Torque due to gravity on $m_j$: $\vec{\tau}_j = \vec{r}_j \times \vec{F}_{gj} = \vec{r}_j \times m_j \vec{g}$.

Net torque on body is then

$$\vec{\tau}_g = \sum_j \vec{r}_j \times m_j \vec{g} \tag{6.3.2}$$

Given that

$$\vec{R} = \frac{1}{M} \sum m_j \cdot \vec{r}_j \tag{6.3.3}$$

we can rearrange

$$\vec{\tau}_g = \sum \vec{r}_j \times m_j \vec{g} \tag{6.3.4}$$
$$= \sum_j (m_j \vec{r}_j) \times \vec{g} \tag{6.3.5}$$
$$= M\vec{R} \times \vec{g} \tag{6.3.6}$$
$$= \vec{R} \times \vec{F}_g \tag{6.3.7}$$

Thus, in any coordinates, the torque due to gravity acts at the center of mass of the object, i.e. we can take the point of application of $\vec{F}_g$ to be at C.O.M. (sometimes center of mass is called center of gravity)

**Example 6.3.2**
A uniform of length $\frac{\pi R}{2}$ is bent into a quadrant of a circle of radius $R$.
It touches ground and wall – there is no friction with the wall but there is friction with the ground.
The rod is in static equilibrium. Find the force the wall exerts on the rod.

***Solution:***

Static equilibrium implies that $\sum \vec{\tau} = 0$ and $\sum \vec{F} = 0$. We choose to compute $\vec{\tau}$ about center point with the floor.

$$\vec{\tau} = \vec{r}_w \times \vec{F}_{NW} + \vec{R} \times \vec{F}_g \tag{6.3.8}$$

We can take care of direction of rotation automatically.

We weite $\left|\vec{r} \times \vec{F}\right| = r_\perp F = rF_\perp = rF \sin\phi$

Taking the perpendicular component of $r$ is easiest for this problem. Thus, we need the $y$-component of $\vec{r}_w$ and need the $x$-component of $\vec{R}$.

The $y$-component of $\vec{F}_W = R$

The $x$-component of $\vec{R} = R - R\cos\frac{\pi}{4}$

Taking everything together:

$$0 = -RF_{NW} + R(1 - \frac{\sqrt{2}}{2})Mg \tag{6.3.9}$$
$$F_{NW} = \boxed{(1 - \frac{\sqrt{2}}{2})Mg} \tag{6.3.10}$$

## 6.4 Angular Momentum & Rotational 2nd Law

We have established rotational analogue of force – torque.

We know the linear 2nd law and given $\vec{\tau} = \vec{r} \times \vec{F}$, we can try to $\vec{r} \times$ 2nd law.

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{6.4.1}$$
$$= \vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \tag{6.4.2}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times \vec{p}) - \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{p} \tag{6.4.3}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times \vec{p}) \tag{6.4.4}$$

Thus, we define the **angular momentum**

**Definition 6.4.1** (Angular Momentum & Rotational 2nd Law)
We defined angular momentum to be

$$\vec{L} = \vec{r} \times \vec{p} \tag{6.4.5}$$

and the 2nd law to be

$$\vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} \tag{6.4.6}$$

**Remark**
With regards to angular momentum
- Depends on the point about which it is computed/origin of coordinates
- A particle moving in a straight line may have angular momentum
- Direction of $\vec{L}$ is $\perp$ to $\vec{r}$ and $\vec{p}$, thus perpendicular plane of motion. The positive and negative sense of $\vec{L}$ is similar to what we mentioned with $\vec{\omega}$ and $\vec{\alpha}$.
- The magnitude is $L = rp\sin\phi = r_{\perp}p = rp_{\perp}$
- A userful expression to recognize:

$$\vec{L} = \vec{r} \times \vec{p} = \langle x, y \rangle \times \langle p_x, p_y \rangle \tag{6.4.7}$$
$$= (xp_y - yp_x)\hat{z} \tag{6.4.8}$$
$$= m(x\dot{y} - y\dot{x})\hat{z} \tag{6.4.9}$$

Thus,
*angular momentum gives a measure of the sense of rotation of the motion of a particle in given coordinates.*

**Example 6.4.1**
Consider a block of mass $m$ sliding along a straight line, say $+x$-axis, with velocity $\vec{v} = v\hat{x}$.

Suppose it is subject to a friction force $\vec{F}_f = -f\hat{x}$.
Find:

1. $\vec{L}_A$ and $\vec{\tau}_A$
2. $\vec{L}_B$ and $\vec{\tau}_B$
3. Show $\vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$ for both A and B.

***Solution:***

1. $\vec{L}_A = (x\hat{x}) \times (mv\hat{x}) = 0$

   and $\vec{\tau}_A = (x\hat{x}) \times (-f\hat{x}) = 0$

   Thus it is obvious that part 3 is true for this case.

2. In this case $\vec{r}_B = x\hat{x} - l\hat{y}$.

   Hence,

$$\vec{L}_B = \vec{r}_B \times \vec{p} = (x\hat{x} - l\hat{y}) \times (mv\hat{x}) \tag{6.4.10}$$

$$= mvl\hat{z} \tag{6.4.11}$$

$$\vec{\tau}_B = \vec{r}_B \times \vec{F}_f = -fl\hat{z} \tag{6.4.12}$$

3. We tested for part 1 (A), for B:

$$\frac{\mathrm{d}\vec{L}_B}{\mathrm{d}t} = ml\frac{\mathrm{d}v}{\mathrm{d}t}\hat{z} = l\frac{\mathrm{d}(mv)}{\mathrm{d}t}\hat{z} = l\frac{\mathrm{d}p}{\mathrm{d}t}\hat{z} \tag{6.4.13}$$

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \implies -f = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{6.4.14}$$

$$\frac{\mathrm{d}\vec{L}_B}{\mathrm{d}t} = l(-f)\hat{z} \tag{6.4.15}$$

## 6.5 Angular Momentum

Recall the rotational 2nd law, where

$$\sum \vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} \tag{6.5.1}$$

where

**Definition 6.5.1** (Angular Momentum)

$$\vec{L} = \vec{r} \times \vec{p} \tag{6.5.2}$$

is defined to be the angular momentum.

For a fixed axis rigid body rotation (i.e. spinning), $\vec{L}$ takes a simplified form. All points within a rigid body have some $\omega$ since it is a rigid body. The angular momentum of a particle $i$ on the rigid body is

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i \implies \left(\vec{L}_i\right)_z = (\vec{r}_i \times \vec{p}_i)_z \tag{6.5.3}$$

This means that (if we consider the linear momentum to tangential with radial $r_i$, i.e. $\vec{r}_i \perp \vec{p}_i$)

$$L_i = s_i m_i v_i = s_i m_i (\omega s_i) = m_i {s_i}^2 \omega \tag{6.5.4}$$

This is then defined to be the **moment of inertia** $I$ for a point particle.

**Definition 6.5.2** (Moment of Inertia)

$$I_{\text{body}} = \sum_i m_i {s_i}^2 \tag{6.5.5}$$

and to get a continuum description of body, we get

$$I = \int s^2 \, \mathrm{d}m \tag{6.5.6}$$

Now, coming back to angular momentum, we then have

**Definition 6.5.3** (Angular Momentum)

$$L = I\omega \tag{6.5.7}$$

1. Which is equivalent expression for $L$ for fixed axis rigid body rotation.
2. Analogous to $p = mv$ with $I \leftrightarrow m, \omega \leftrightarrow v$.

## 6.6 Moment of Inertia

1. $I$ is a measure of rotational inertia... rotational analogue of mass. Greater $I$ the greater "resistance" of body to changes in its rotational motion.

2. $I$ depends on axis of rotation (via $s^2$ factor), just like how $\tau$ and $L$ do as well. However, the relationships $L = I\omega$ and $\tau = \frac{\mathrm{d}l}{\mathrm{d}t}$ are **frame independent**.

3. $I$ depends not only on the mass, but how it is distributed within the body. If most of the mass is concentrated near axis of rotation, then $I$ is smaller compared to more mass concentrated further away.

4. Quantity of the form $\int s^n \, \mathrm{d}m$ is a "moment", so inertia is a "second moment" because $s^2$.

5. For non-fixed axis of rotation (rigid body), turns out that $v = \omega r$ no longer applies.

   Instead, consider $\vec{r}_i$ position of mass $m_i$ relative to center of mass of the body.

   Note that:

$$\dot{\vec{r}_i} = \vec{\omega} \times \vec{r}_i \implies \vec{v} = \vec{\omega} \times \vec{r} \tag{6.6.1}$$

To compute $\vec{L}$ with respect to C.O.M.

$$\vec{L}_{\text{CM}} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times \left(m_i \dot{\vec{r}_i}\right) = \sum_i \vec{r}_i \times m_i \left(\vec{\omega} \times \vec{r}_i\right) \tag{6.6.2}$$

$$\vec{L}_{\text{CM}} = \sum_i m_i \left(\vec{r}_i \times \vec{\omega} \times \vec{r}_i\right) = \sum_i m_i \left[{r_i}^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r}_i)\vec{r}_i\right] \tag{6.6.3}$$

$$= \sum_i \left[m_i {r_i}^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r}_i) m_i \vec{r}_i\right] \text{[1]} \tag{6.6.4}$$

$$= I\omega \tag{6.6.5}$$

A side note, moment of inertia $I$ can be written as a matrix multiplication with some $\vec{\omega}$.

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \tag{6.6.6}$$

But, $I$ is actually also not a matrix but rather a rank 2 tensor – the **moment of inertia tensor**:

$$I = \sum \left[m_i {r_i}^2 E_3 - m_i \vec{r}_i \otimes \vec{r}_i\right] \tag{6.6.7}$$

where $E_3$ is the 3-by-3 identity matrix.

### 6.6.1 Parallel Axis Theorem

**Theorem 6.6.1**
The moment of inertia $I$ about an axis parallel to an axis through the center of mass, $I_0$, a distance $d$ apart, is given by

$$I = I_0 + Md^2 \tag{6.6.8}$$

## 6.7 Dynamics of Fixed Axis Rigid Body Motion

In this case, $L = I\omega$ with $I = \text{const}$. Therefore, rotational 2nd law may be written as

$$\sum \tau = \frac{\mathrm{d}L}{\mathrm{d}t} = I \frac{\mathrm{d}\omega}{\mathrm{d}t} \implies \sum \tau = I\alpha \tag{6.7.1}$$

[1] Because the second term is zero for fixed axis rotation with $r_i = s_i$

**Remark**
Note that Chasles' Theorem still applies when the center of mass is acceleration. That is to say the motion in the angular aspect is still valid given the net torque exerted on the object.

## 6.8 Examples of Rotational Motion

### 6.8.1 Physical Pendulum

Take positive orientation, we have forces

$$\begin{cases} \tau_T &= 0 \qquad \text{because } F_T \parallel l \\ \tau_g &= \vec{r} \times \vec{F}_g \implies \tau_T = -lmg\sin\theta \end{cases} \tag{6.8.1}$$

Now, we solve for the motion of the pendulum

$$\tau_g = I\alpha \tag{6.8.2}$$

$$-lmg\sin\theta = I\ddot{\theta} \tag{6.8.3}$$

$$-lmg\theta = I\ddot{\theta} \qquad \theta \ll 1 \implies \sin\theta \approx \theta \tag{6.8.4}$$

$$\ddot{\theta} = -\left[\frac{lmg}{I}\right]\theta \tag{6.8.5}$$

so we have that the frequency is

$$\omega = \sqrt{\frac{lmg}{I}} \qquad I = ml^2 \implies \omega = \sqrt{\frac{g}{l}} \tag{6.8.6}$$

Now, let us consider a physical pendulum, defined to be a pendulum with an unrestricted bob and string.

The torque due to gravity comes from gravity at the center of mass, so its basically the same deal as what we solved just now.

Well, we have a difference moment of inertia $I$ for our new axis of rotation, we can find that via the Parallel Axis Theorem:

$$I = I_0 + Ml^2 \tag{6.8.7}$$

So we have

$$\omega = \sqrt{\frac{lMg}{I_0 + Ml^2}} \tag{6.8.8}$$

$$= \sqrt{\frac{lg}{I_0/M + l^2}} \tag{6.8.9}$$

$$= \sqrt{\frac{lg}{k^2 + l^2}} \tag{6.8.10}$$

where $k$ is the **radius of gyration** defined as

$$k = \sqrt{\frac{I_0}{M}} \tag{6.8.11}$$

### 6.8.2 Dynamics of Translation & Rotation

This section talks about the angular momentum of some continuously distributed body about the origin (i.e. the center of mass is in motion WRT the origin).

**TL;DR;**

Basically, angular momentum about the origin is the sum of the angular momentum about the center of mass of an object and the angular momentum of its center of mass about the origin.

Wikipedia covers this pretty well:

https://en.wikipedia.org/wiki/Angular_momentum#Collection_of_particles

### 6.8.3 Examples

**Example 6.8.1**
Ramp mass pulley.

Reference provided lecture notes for this example.

**Example 6.8.2**
A ball of mass $M$ and radius $b$ rolls without slipping.

***Solution:***

$$
\begin{aligned}
L_z &= \pm I_{\text{cm}}\omega + (\vec{R} \times M\vec{V})_z && (6.8.12)\\
&= -I_{\text{cm}}\omega + \left[(x\hat{x} + b\hat{y}) \times M(v\hat{x})\right]_z && (6.8.13)\\
&= -\left(\frac{2}{5}Mb^2\right)\omega + [-Mvb\hat{z}]_z && (6.8.14)\\
&= -\frac{2}{5}Mb^2\omega - M(\omega b)b && (6.8.15)\\
&= -\left(\frac{2}{5} + 1\right)Mb^2\omega && (6.8.16)\\
\end{aligned}
$$

$$\boxed{L_z = -\frac{7}{5}Mb^2\omega} \tag{6.8.17}$$

**Example 6.8.3**
Disk of mass $M$ and radius $b$ pulled by constant force $F$ (at the bottom) slides on ice without friction. What is its motion?

***Solution:***

Compute $\tau_z = \tau_0 + (\vec{R} \times \vec{F})_z$

Here $\tau_{\text{cm}} = +I_{\text{cm}}\alpha = \frac{1}{2}Mb^2\alpha = bF \implies \alpha = \frac{2F}{Mb}$

$$(\vec{R} \times \vec{F})_z = \left[(x\hat{x} + b\hat{y}) \times (F\hat{x})\right] = -bF \tag{6.8.18}$$

Thus, $\tau_z = bF - bF = 0$. So the torque with respect to the origin is 0.

For angular momentum

$$L_z = I_{\text{cm}}\omega + (\vec{R} \times M\vec{V})_z = \frac{1}{2}Mb^2\omega - Mvb \tag{6.8.19}$$

and

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = \frac{1}{2}Mb^2\alpha - Mba = \tau_z = 0 \tag{6.8.20}$$

$$\frac{1}{2}Mb^2\left(\frac{2F}{Mb}\right) - Mba = bF - Mba \implies F = Ma \tag{6.8.21}$$

## 6.9 Collection of Particles

Our rotational 2nd law states that

$$\sum \vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} \tag{6.9.1}$$

Consider a system of $N$ particles, with positions $\vec{r}_i$ subject to forces $\vec{F}_i = \vec{F}_i^{\text{ext}} + \sum_{j \neq i} \vec{F}_{ji}$ Their momenta are $\vec{P}_i$.

Compute $\vec{\tau}_i$:

$$\vec{\tau}_i = \vec{r}_i \times \vec{F}_i = \vec{r}_i \times \vec{F}_i^{\text{ext}} + \vec{r}_i \times \sum_{j \neq i} \vec{F}_{ji} \tag{6.9.2}$$

The total torque is

$$\vec{\tau} = \sum \vec{\tau}_i = \sum \vec{r}_i \times \vec{F}_i^{\text{ext}} + \sum_i \vec{r}_i \times \left( \sum_{j \neq i} \vec{F}_{ji} \right) \tag{6.9.3}$$

For our second term, consider torque on $i$-th particle due to particke $k$

$$\vec{\tau}_{k_i} = \vec{r}_i \times \vec{F}_{k_i} \tag{6.9.4}$$

now consider torque on $k$ due to $i$:

$$\vec{\tau}_{ik} = \vec{r}_k \times \vec{F}_{ik} \tag{6.9.5}$$

Now, with Newton's 3rd law $\vec{F}_{ik} = \vec{F}_{ki}$, so:

$$\vec{\tau}_{ki} + \vec{\tau}_{ik} = \vec{r}_i \times \vec{F}_{ki} + \vec{r}_k \times \vec{F}_{ik} \tag{6.9.6}$$

$$= (\vec{r}_i - \vec{r}_k)\vec{F}_{ki} \tag{6.9.7}$$

This expression is not immediately 0, however, if the force $\vec{F}_{ik}$ is in the direction connecting the two objects, then the torque sums up to 0.

Consequently we have is that

$$\vec{\tau} = \vec{\tau}_{\text{ext}} + \frac{1}{2} \sum_i \sum_j (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} \tag{6.9.8}$$

If all internal forces are central, the second term is zero and (this seems reasonable for a rigid body consisting of $n$ particles, but is not so reasonable for free particles.)

$$\vec{\tau} = \vec{\tau}_{\text{ext}} \tag{6.9.9}$$

If we consider the right hand side of the 2nd law

$$L_i = \vec{r}_i \times \vec{P}_i \tag{6.9.10}$$
$$= \sum_i \vec{r}_i \times \vec{P}_i \tag{6.9.11}$$
$$= \sum \vec{L}_i \tag{6.9.12}$$
$$= \vec{L}_{\text{tot}} \tag{6.9.13}$$

So, if $\vec{\tau}_{\text{ext}} = 0$, we have that

$$\frac{\mathrm{d}\vec{L}_{\text{tot}}}{\mathrm{d}t} = 0 \tag{6.9.14}$$

Thus, $\vec{L}_{\text{tot}} = \text{const}$

**Theorem 6.9.1** (Law of Conservation of Angular Momentum)
If the total external torque on a system is zero, then the total angular momentum of the system is conserved:

$$\Delta\vec{L}_{\text{tot}} = 0 \tag{6.9.15}$$

**Remark**
However, this is assuming all pairwise internal forces are central

Recall $\vec{F} = -\nabla u$. Consider spherical coordinates: $u(r, \theta, \phi)$.

**Example 6.9.1**
Considering the rolling wheel.

***Solution:***

Let's consider the dynamics of a rolling wheel. When the wheel is given a non-zero net force at its center to accelerate it across some frictional surface, the wheel also starts to accelerate rotationally due to the frictional force that is acting on it.

Say positive direction is to the right, we apply a force to the right, friction would act to the left at the contact point of the wheel with the ground, which provides a torque that accelerates the wheel rotationally.

If the translational acceleration is greater than the maximum providable by friction with our relations, then the wheel will start to slip, which is when

$$a > \frac{RF_f}{I_w} R \tag{6.9.16}$$

Now, if we simply consider a wheel rolling, with no net force acting on it other than friction, we would have linear and angular acceleration either both to be 0, or both trying to reach agreement to the relation $v = \omega r$.

Once they reach this agreement, there is no longer any net force acting on the rolling object since the relative motion between the surface of the wheel and the ground is 0. This means that the wheel in this state would keep rolling on forever, in an ideal situation.

Now, you might be wondering what then causes the wheel to slow down as it rolls? Well, there is rolling resistance which has a nice explanation on Wikipedia. There is also possible air drag.

## 6.10 Rotational Energy

**Definition 6.10.1**
We have our new form of kinetic energy as

$$K = K_{\text{rot}} + K_{\text{trans}} = \frac{1}{2} I_{\text{cm}} \omega^2 + \frac{1}{2} M {V_{\text{cm}}}^2 \tag{6.10.1}$$

and rotational KE

$$K_{\text{rot}} = \frac{1}{2} I \omega^2 \tag{6.10.2}$$

There is also a work-energy equivalent in angular momentum

$$W_{\text{torque}} = \int_{\theta_0}^{\theta} \tau_{\text{cm}} \, \mathrm{d}\theta \tag{6.10.3}$$

# Chapter 7

# Gravitation

## 7.1 Kepler's Laws

**Definition 7.1.1** (Kepler's Laws)
Kepler studied and culminated his study with 3 laws:

1. Planets move in elliptical orbits with the Sun at one focus (out of two possible foci).
2. Equal areas in equal times.
3. The period of revolution $T$ is related to the semi-major axis $A$ by

$$T^2 = kA^3 \tag{7.1.1}$$

where $k$ is some constant for all planets.

## 7.2 Newton's Law of Universal Gravitation

The **gravitational force** $F_g$ between two point masses $m_1$ and $m_2$ is given by

$$\vec{F}_{\mathrm{G}12} = -\mathrm{G}\frac{m_1 m_2}{r^2}\hat{r} \tag{7.2.1}$$

where $r = |\vec{r}_1 - \vec{r}_2|$ is the distance between particles, and $\hat{r}$ is the direction of $\vec{r}_2 - \vec{r}_1$. and G, **Newton's Constant**, is

$$\mathrm{G} = 6.67 \times 10^{-11}\,\mathrm{Nm}^2/\mathrm{kg}^2 \tag{7.2.2}$$

This is also an inverse square law

$$F_{\mathrm{G}} \propto \frac{1}{r^2} \tag{7.2.3}$$

Gravity is incredibly small compared to other fundamental forces.

Often, one mass is significantly larger than the other (i.e. Sun & Planet) in which case a language is used: supposed $m_1 \gg m_2$, write $M \equiv m$, $m \equiv m_2$, then

$$F_G = -G\frac{Mm}{r^2}\hat{r} \tag{7.2.4}$$

is force on $m$ due to $M$.

Call $M$ the "source" mass and $m$ is the "test" mass because studying only motion of $m$.

Law is for point masses, however it holds in same form for spherical masses where $u$ is the distance between their centers.

## 7.3 Connection to Weight / Surface Gravity

Consider a mass $m$ at/near the surface of $M$, let the distance from the center of $m$ to the surface of $M$ be $h$, then we have

$$F_{\mathrm{G}} = -\mathrm{G}\frac{Mm}{r^2}\hat{r} = -\mathrm{G}\frac{Mm}{(R+h)^2}\hat{r} = -\left(\frac{GM}{R^2}\right)m\frac{1}{(1+{}^{h}/_{R})^2} \tag{7.3.1}$$

If $h \ll R$, the ${}^{h}/_{R} \ll 1$. Let $\delta \equiv {}^{h}/_{R}$, then we have $\frac{1}{(1+\delta)^2}$ with $\delta \ll 1$.

Taylor expansion about $\delta = 0$ gives

$$(1+\delta)^{-2} \approx 1 - 2\delta + O(\delta^2) \tag{7.3.2}$$

Then,

$$F_{\mathrm{G}} \approx -\left(\frac{GM}{R^2}\right)m\left[1 - 2\left(\frac{h}{R}\right) + O\left(\left(\frac{h}{R}\right)^2\right)\right]\hat{r} \tag{7.3.3}$$

Since ${}^{h}/_{R} \ll 1$, we can neglect the terms of order ${}^{h}/_{R}$ and higher

$$\vec{F}_{\mathrm{G}} \cong -\left(\frac{GM}{R^2}\right)m\hat{r} \tag{7.3.4}$$

Consider Earth, then we have

$$\frac{\mathrm{G}M_E}{{R_E}^2} = 9.81\,\mathrm{m/s^2} \equiv \mathrm{g} \tag{7.3.5}$$

Hence, $\vec{F}_{\mathrm{G}} = -mg\hat{z}$ is the weight.

In general, we can write gravity as the **gravitational field**

$$\vec{g}(r) \equiv -\frac{\mathrm{G}M}{r^2}\hat{r} \tag{7.3.6}$$

sourced by $M$.

## 7.4 Principle of Equivalence

Let's consider $m_{\mathrm{G}} = \vec{F}_{\mathrm{G}}/\vec{g}$ as the **gravitational mass** and $m_I = \vec{F}_{\mathrm{net}}/\vec{a}$ to be the inertial mass.

These have be repeatedly proven to be the same.

Thus, for a particle accelerating via gravity

$$\vec{a} = \vec{g} \tag{7.4.1}$$

**Remark**
But, think of a situation where we are in a box, and we experience our own weight. Do we know if we are on Earth or being accelerated at gravitational acceleration on earth?
Also, when we are experiencing free-fall, we don't experience our own weight, and there doesn't seem to be a force on us.
We don't know. They are the same. So what Einstein figured was that gravity is not a force, but a curvature in spacetime. [a]

[a] If a man falls from the roof of a house, he would not feel his own weight.

## 7.5 Gravitational Potential

Gravity is a central force and hence conservative. The gravitational potential energy $U_{\mathrm{G}}$ is

$$U_{\mathrm{G}}(r) - U_{\mathrm{G}}(\infty) = -\int_{\infty}^{r} \vec{F}_{\mathrm{G}} \cdot \mathrm{d}\vec{l} = \int_{\infty}^{r} \frac{\mathrm{G}Mm}{r^2}\,\mathrm{d}r = -\frac{\mathrm{G}Mm}{r} \tag{7.5.1}$$

So, we have the gravitational potential.

**Definition 7.5.1** (Gravitational Potential Energy)
The gravitational potential energy is defined as

$$U_{\mathrm{G}}(r) = -\frac{\mathrm{G}Mm}{r} \tag{7.5.2}$$

We define the gravitational potential (**not** energy) $\Phi$ by

$$\Phi = \frac{U_{\mathrm{G}}}{m} = -\frac{\mathrm{G}M}{r} \tag{7.5.3}$$

Now, if also find that

$$-\nabla\Phi = -\frac{\partial\Phi}{\partial r}\hat{r} = -\frac{\mathrm{G}M}{r^2}\hat{r} = \vec{g} \tag{7.5.4}$$

Hence, we have that

$$\vec{g} = -\nabla\Phi \tag{7.5.5}$$

and is analogous to

$$\vec{F}_{\mathrm{G}} = -\nabla U_{\mathrm{G}} \tag{7.5.6}$$

**Remark**
Note that in E&M, we have something very similar:

$$\vec{F}_E = -\nabla U_E \quad \vec{E} = \frac{\vec{F}_E}{q} \quad V = \frac{U_E}{q} \quad \vec{E} = -\nabla V \tag{7.5.7}$$

## 7.6 Two-Body Problem

A system of two masses $m_1$ and $m_2$ , that interact solely via a central force.

$$m_1\ddot{\mathbf{r}}_1 = f(r)\hat{r} \tag{7.6.1}$$
$$m_2\ddot{\mathbf{r}}_2 = -f(r)\hat{r} \tag{7.6.2}$$
$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2 \tag{7.6.3}$$

expressed explicitly

$$m_1\ddot{\mathbf{r}}_1 = -\frac{\mathrm{G}m_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2) \tag{7.6.4}$$
$$m_2\ddot{\mathbf{r}}_1 = +\frac{\mathrm{G}m_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2) \tag{7.6.5}$$

instead, we can rewrite it as

$$\ddot{\mathbf{r}}_1 = \frac{1}{m_1}f(r)\hat{\mathbf{r}} \tag{7.6.6}$$
$$\ddot{\mathbf{r}}_1 = -\frac{1}{m_2}f(r)\hat{\mathbf{r}} \tag{7.6.7}$$

we combine these equations to obtain

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \ddot{\mathbf{r}} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{\mathbf{r}} \tag{7.6.8}$$

Where

$$\mu \equiv \frac{m_1m_2}{m_1 + m_2} \tag{7.6.9}$$

is the reduced mass.

Our tw-body problem is equivalent to the one-body problem

$$\mu \ddot{\mathbf{r}} = -f(r)\hat{\mathbf{r}} \tag{7.6.10}$$

If we look at the C.O.M. frame

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{7.6.11}$$

We have

$$\begin{align} \mathbf{r}_1 &= \mathbf{r}_1' + \mathbf{R} \tag{7.6.12} \\ \mathbf{r}_1' &= \mathbf{r}_1 - \mathbf{R} \tag{7.6.13} \\ &= \frac{(m_1 + m_2)\mathbf{r}_1 - m_1\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 + m_2} \tag{7.6.14} \\ &= \frac{m_2(\mathbf{r}_1 - \mathbf{r}_2)}{m_1 + m_2} \tag{7.6.15} \\ &= \frac{\mu}{m_1}\mathbf{r} \tag{7.6.16} \end{align}$$

so we have

$$\mu\mathbf{r} = m_1\mathbf{r}_1' \tag{7.6.17}$$

hence if we know $\mathbf{r}$ we know $\mathbf{r}_1$ and $\mathbf{r}_2$.

Now, we should consider the **conservation of angular momentum**.

$$\begin{align} |\mathbf{L}| &= |\mathbf{r} \times \mu\dot{\mathbf{r}}| = \left|\mathbf{r} \times \mu(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta})\right| \tag{7.6.18} \\ &= \mu r^2\dot{\theta} \tag{7.6.19} \end{align}$$

This is basically Kepler's 2nd Law!

Conservation of Energy or the frist integral of motion:

We have total energy $E$, a constant, as:

$$E = \frac{1}{2}\mu\dot{\mathbf{r}}^2 + U(r) \tag{7.6.20}$$

where $f(r)\hat{\mathbf{r}} = -\nabla U$.

In polar coordinates:

$$\dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \implies \dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{7.6.21}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r) \tag{7.6.22}$$

Since $L$ is constant, we write $\dot{\theta} = \frac{L}{\mu r^2}$ and substitute:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\left(\frac{L}{\mu r^2}\right)^2 + U(r) \tag{7.6.23}$$

$$= \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) \tag{7.6.24}$$

where

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + U(r) \tag{7.6.25}$$

which is the effective potential.

For gravity $U(r) = -{}^{C}\!/_{r}$ where $C = \mathrm{G}m_1m_2$

The effective potential is:

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} - \frac{C}{r} \tag{7.6.26}$$

Now, we return to solving for the motion of the bodies; we want the trajectory $r = r(\theta)$.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}r}\dot{r} \tag{7.6.27}$$

From energy we have

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{C}{r} \tag{7.6.28}$$

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E + \frac{C}{r} - \frac{L^2}{2\mu r^2}\right)} \tag{7.6.29}$$

From angular momentum, we have

$$\dot{\theta} = \frac{L}{\mu r^2} \tag{7.6.30}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{L}{\mu r^2} \tag{7.6.31}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{L}{\mu r^2} \frac{1}{\sqrt{\frac{2}{\mu}\left(E + \frac{C}{r} - \frac{L^2}{2\mu r^2}\right)}} \tag{7.6.32}$$

$$= \frac{L}{r\sqrt{2\mu\left(Er^2 + Cr - {}^{L^2}/{}_{2\mu}\right)}} \tag{7.6.33}$$

$$\boxed{\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{L}{r\sqrt{2\mu E r^2 + 2\mu C r - L^2}}} \tag{7.6.34}$$

Then we want to find

$$\int_{\theta_0}^{\theta} \mathrm{d}\theta = \theta - \theta_0 = \int_{r_0}^{r} \frac{L\,\mathrm{d}r}{r\sqrt{2\mu E r^2 + 2\mu C r - L^2}} \tag{7.6.35}$$

The result is (process not shown):

$$\theta - \theta_0 = -\arcsin\left[\left(-\frac{\mu C}{L^2}\right)\sqrt{\frac{L^4}{2\mu E L^2 + (\mu C)^2}}\right] \tag{7.6.36}$$

We invert and rearrange to obtain

$$r = \frac{r_0}{1 - \epsilon\cos\theta} \tag{7.6.37}$$

where

$$r_0 \equiv \frac{L^2}{\mu C} \tag{7.6.38}$$

$$\epsilon \equiv \sqrt{1 + \frac{2EL^2}{\mu C^2}} \tag{7.6.39}$$

**Remark**
$r_0$ is minimum of $U_{\text{eff}}(r)$ if $E = U_{\min}$ then $\epsilon = 0 \implies r = r_0$
If $E = U_{\min}$ the orbits are then circles.
If $E < 0$, then $0 \leq \epsilon \leq 1$

$$\sqrt{x^2+y^2} = \frac{r_0}{1-\frac{\epsilon x}{r}} \tag{7.6.40}$$
$$\sqrt{x^2+y^2}\left(1-\frac{\epsilon x}{r}\right) = r_0 \tag{7.6.41}$$
$$\sqrt{x^2+y^2} - \epsilon x = r_0 \tag{7.6.42}$$
$$x^2+y^2 = {r_0}^2 + \epsilon^2 x^2 + 2\epsilon r_0 x \tag{7.6.43}$$
$$(1-\epsilon^2)x^2 + y^2 - 2\epsilon r_0 x = {r_0}^2 \tag{7.6.44}$$

Notice that this is the equation of an ellipse!
$\epsilon$ is **eccentricity** of the ellipse, which is basically Kepler's 1st Law!

Now, with the period of orbit

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{2}{\mu}(E-U_{\text{eff}}) \tag{7.6.45}$$
$$t_a - t_b = \int_{r_a}^{r_b} \frac{\mathrm{d}r}{\sqrt{\frac{2}{\mu}(E-U_{\text{eff}})}} \tag{7.6.46}$$

If we compute the integral and take $r_a = r_b \implies t_a - t_b = T$ period.

$$T = -\frac{\mu C \pi}{E}\frac{1}{\sqrt{-2\mu E}} \tag{7.6.47}$$
$$T^2 = \frac{\mu\pi^2}{2C}\left(-\frac{C^3}{E^3}\right) \tag{7.6.48}$$

Since $-1 < \cos\theta < 1 \implies r_{\min} < r < r_{\max}$ where $r_{\max} = \frac{r_0}{1-\epsilon}$ and $r_{\min} = \frac{r_0}{1+\epsilon}$

$$T^2 = \frac{\pi^2}{2G(m_s+m_p)}A^3 \tag{7.6.49}$$

# Chapter 8

# Thermodynamics

## 8.1 Ideal Gas Law

**Definition 8.1.1** (Ideal Gass Law)
The ideal gas law describes the relationship between pressure, volume, and temperature.

$$PV = nRT \tag{8.1.1}$$

or

$$PV = Nk_BT \tag{8.1.2}$$

where

- $P$ = pressure (Pa)
- $V$ = volume ($\mathrm{m}^3$)
- $T$ = temperature (K)
- $n$ = number of moles
- $N$ = number of molecules

where

$$N = nN_A \qquad N_A = 6.022 \times 10^{23} \tag{8.1.3}$$

We also have the *universal gas constant* $R = 8.314\,\mathrm{J/molK}$ and the *Boltzmann's Constant* $k_B = 1.381 \times 10^{-23}\,\mathrm{J/K}$.

**Ideal** means the gas is law-density, non-interacting, and above equation of state holds.

## 8.2 Kinetic Theory of Gases

Assumptions:

- non-interacting & low-density
- monatomic gas
- collision of gas with container are elastic

Let there be a piston forming a section of length $L$ with a container.

Volume of the cylinder is $V = LA$

Average pressure particle exerts on the piston:

$$\overline{P} = \frac{\overline{F}_{x,\text{ piston}}}{A} = -\frac{\overline{F}_{x,\text{ particle}}}{A} = -\frac{\Delta P_x/\Delta t}{A} \tag{8.2.1}$$

The momentum change

$$\Delta P_x = mv_{fx} - mvix = -mv_{ix} - mv_{ix} = -2mv_{ix} = -2mv_x \tag{8.2.2}$$

The time between collisions would be

$$\Delta t = \frac{2L}{v_x} \tag{8.2.3}$$

Hence,

$$\overline{P} = \frac{2m{v_x}^2}{2LA} = \frac{m{v_x}^2}{V} \implies \overline{P}V = m{v_x}^2 \tag{8.2.4}$$

When we consider $N$ particles:

$$\overline{P}V = \sum_{i=1}^{N} m{v_{ix}}^2 = N\overline{m{v_x}^2} \tag{8.2.5}$$

For $N$ that is very large, $P$ is approximately continuous so $\overline{P} \to P$ and

$$PV = Nm\overline{{v_x}^2} \implies Nk_BT = Nm\overline{{v_x}^2} \tag{8.2.6}$$

We then obtain the aoverage $x$-direction kinetic energy

$$\frac{1}{2}k_BT = \overline{\frac{1}{2}m{v_x}^2} = \overline{K_x} \tag{8.2.7}$$

where we can apply the same argument to all three walls

$$\overline{K} = \overline{K_x} + \overline{K_y} + \overline{K_z} = \frac{3}{2}k_BT \tag{8.2.8}$$

We should mention a **quadratic degree of freedom** is any energy that is quadratic in a variable

$$\text{For example: } \frac{1}{2}mv^2, \frac{1}{2}I\omega^2, \frac{1}{2}kx^2 \tag{8.2.9}$$

**Theorem 8.2.1** (Equipartition Theorem)
At temperature $T$, the average energy per quadratic degree of freedom is $\frac{1}{2}k_BT$.

Monatomic Ideal Gas: $f = 3 \to$ translation K, so $\frac{3}{2}k_BT$.

Diatomic Ideal Gas: $f = \text{trans} + \text{rot} = 3 + 2 = 5$, so $\frac{5}{2}k_BT$

Let's define a new form of energy, the **thermal** or **internal** energy,

$$U_{\text{th}} = N\frac{f}{2}k_BT \tag{8.2.10}$$

Monatomic Ideal Gas Energy: $f = 3 \to$ translation K, so $\frac{3}{2}Nk_BT$.

Diatomic Ideal Gas Energy: $\frac{5}{2}Nk_BT$

## 8.3 First Law of Thermodynamics

**Definition 8.3.1** (First Law of Thermodynamics)

$$\Delta U_{\text{th}} = W + Q \tag{8.3.1}$$

where

1. $W$ is work: Here we consider only compression-expansion work (e.g. of the piston in the example) $\vec{F} \cdot \mathrm{d}\vec{r} \to PA\,\mathrm{d}l = P\,\mathrm{d}V$

   $$W = -\int_{V_0}^{V} P\,\mathrm{d}V \tag{8.3.2}$$

   When we have decrease in volume, we should have increase in temperature, which is increase in energy, so we need the negative sign.

   **Remark**
   In some context (i.e. engineering), the first law has $-W$ instead, and there is no negative sign in the work. This works out ultimately the same, but we think more of the work done on our gas, not the gas on the environment, so we stick with this convention/definition.

2. $Q$ is heat: spontaneous energy exchange due to a temperature difference.

## 8.4 Heat

**Definition 8.4.1** (Heat)
Let **heat capacity**, $C$, is

$$C = \frac{Q}{\Delta T} \tag{8.4.1}$$

and **specific heat**, $c$, is

$$c = \frac{Q}{m\Delta T} = \frac{C}{m} \tag{8.4.2}$$

where there is also a mole variant **molar specific heat**

$$c = -\frac{C}{n} = \frac{Q}{n\Delta T} \tag{8.4.3}$$

Specific heat at constant volume: $Q = mc_V \Delta T$ (implies $W = 0$)
Specific heat at constant pressure: $Q = mc_P \Delta T$ (note $W = -P\Delta V$)

## 8.5 Phase Transition

During a phase transition, $\Delta T = 0$. Between solid, liquid, and gas.

There is the **latent heat** of **fusion** (solid $\leftrightarrow$ liquid)

$$Q = \pm mL_f \tag{8.5.1}$$

and of **vaporization** (liquid $\leftrightarrow$ gas)

$$Q = \pm mL_v \tag{8.5.2}$$

## 8.6 Thermodynamic Processes

Our system is characterized by state variables $(P, V, T, N, S)$, where $S$ is entropy and $N$ is fixed, $U$ is always a function of state $U = U(P, V)$

Generally, only need three variables and an equation of state. (e.g. the ideal gas law)

We can represent our system on a diagram, such as a PV-diagram.

Since we have 3 variables, we can use any point to compute all the state variables of the system at some time. (so we don't necessarily have to use P and V)

The process which state changes occur is a sequence of infinitesimal changes, so a continuous curve on the diagram.

We will now talk about four such processes for monatomic ideal gas.

### 8.6.1 Isobaric

An **isobaric** process occurs at constant pressure. ($P =$ const)

$$W = -P\Delta V \tag{8.6.1}$$

$$Q = nC_P\Delta T \tag{8.6.2}$$

$$\Delta U_{\text{th}} = \frac{f}{2} nR\Delta T \tag{8.6.3}$$

We can combine these expressions together to obtain

$$\Delta U_{\text{th}} = Q + W \tag{8.6.4}$$
$$\frac{f}{2}nR\Delta T = nC_P\Delta T - P\Delta V \tag{8.6.5}$$
$$= nC_P\Delta T - nR\Delta T \tag{8.6.6}$$
$$\frac{f}{2}nR\Delta T + nR\Delta T = nC_p\Delta T \tag{8.6.7}$$
$$C_P = \frac{f}{2}R + R = \frac{f+2}{2}R \tag{8.6.8}$$

### 8.6.2 Isochoric

An **isochoric** process is when we ahve constant volume.

We have the following condition for the process.

$$W = 0 \tag{8.6.9}$$
$$Q = nC_V\Delta T \tag{8.6.10}$$
$$\Delta U_{\text{th}} = \frac{f}{2}nR\Delta T \tag{8.6.11}$$

We can solve for these to obtain the relation:

$$\Delta U_{\text{th}} = Q \tag{8.6.12}$$
$$\frac{f}{2}nR\Delta T = nC_V\Delta T \tag{8.6.13}$$
$$C_V = \frac{f}{2}R \tag{8.6.14}$$

Thus, note that $C_P = C_V + R$

### 8.6.3 Isothermal

An **isothermal** process is when temperature is held constant, as a result, PV is constant.

We basically have a curve that is $P \propto \frac{1}{V}$.

$$W = ... \tag{8.6.15}$$

### 8.6.4 Adiabatic

An **adiabatic** process is one where there is no heat exchange.

From our later calcualtions, we would know that the path for such a cycle is between two isothermal processes of a certain temperature, and present itself as a steeper slope than $P \propto \frac{1}{V}$ as $P \propto \frac{1}{V^\gamma}, \gamma > 1$.

Our conditions are

$$Q = 0 \tag{8.6.16}$$
$$W = -\int_{V_0}^{V} P \, \mathrm{d}V \tag{8.6.17}$$
$$\Delta U_{\text{th}} = \frac{f}{1} nR\Delta T \tag{8.6.18}$$
$$PV = nRT \tag{8.6.19}$$

Now we can setup the infinitesimal first law:

$$\mathrm{d}U_{\text{th}} = \mathrm{d}W + \mathrm{d}Q \tag{8.6.20}$$
$$= \mathrm{d}W \tag{8.6.21}$$
$$\frac{f}{2} nR \, \mathrm{d}T = -P \, \mathrm{d}V \tag{8.6.22}$$

Note that we consider an ideal gas

$$P \, \mathrm{d}V + V \, \mathrm{d}P = nR \, \mathrm{d}T \tag{8.6.23}$$

So we can substitute and obtain the differential equation which solves to

$$\frac{f}{2} P \, \mathrm{d}V + \frac{f}{2} V \, \mathrm{d}P = -P \, \mathrm{d}V \tag{8.6.24}$$
$$\ln \left[ \frac{{V_f}^{1+\frac{2}{f}} P_f}{{V_i}^{1+\frac{2}{f}} P_f} \right] = 0 \tag{8.6.25}$$

So we know that $PV^\gamma = \text{const}$.

Hence,

$$\gamma = 1 + \frac{2}{f} = \frac{2+f}{f} = \frac{C_P}{C_V} \tag{8.6.26}$$

which is called the **adiabatic ratio**.

Let $PV^\gamma = A$ a constant, then, $P = AV^{-\gamma}$ so that

$$W = -\int_{V_0}^{V} P \,\mathrm{d}V \tag{8.6.27}$$

$$= -\int_{V_0}^{V} AV^{-\gamma} \,\mathrm{d}V \tag{8.6.28}$$

$$= \frac{A}{\gamma - 1}({V_f}^{1-\gamma} - V_i^{1-\gamma}) \tag{8.6.29}$$

$$= \frac{P_i V_i^\gamma}{\gamma - 1}\left({V_f}^{1-\gamma} - V_i^{1-\gamma}\right) \tag{8.6.30}$$

Note that $\Delta U_{\text{th}} = W = \frac{f}{2} nR\Delta T$

## 8.7 Cycle

A **cycle** is a sequence of processes that take system from an initial state back to that same initial state.

Since $U_{\text{th}}$ is a function of state, $\Delta U_{\text{th}}$ for whole cycle is zero.

## 8.8 Heal Engines and Refrigerators

A heat engine is a device that takes a working substance (e.g. ideal gas) through a cycle such that it converts a portion of input heat into work done by system on its surroundings.

The idea is that the heat comes from a heat reservoir $T_H$ and it goes into the engine through heat $Q_H$, but then there must be some heat dissipated $Q_C$ to a cold reservoir $T_C$ while doing work $W$.

So, there must be some efficiency, written as

$$\eta = \frac{W_{\text{out}}}{Q_H} = 1 - \frac{|Q_C|}{|Q_H|} \tag{8.8.1}$$

Now let's look at it through the first law:

$$0 = W + |Q_H| - |Q_C| \tag{8.8.2}$$

$$-W = |Q_H| - |Q_C| \tag{8.8.3}$$

$$W_{\text{out}} = |Q_H| - |Q_C| \tag{8.8.4}$$

A refrigerator or heat pump, is a heat engine run in reverse.

We have coefficient of performance C.O.P. which is

$$\text{C.O.P.} = \frac{Q_C}{W_{\text{in}}} = \frac{1}{|Q_H| / |Q_C| - 1} \tag{8.8.5}$$

## 8.9 Most Efficient Heat Engine

This is the **Carnot Cycle**.

This consists of two isotherm process and adiabatic process.

The efficiency can be proven to be

$$\eta_C = 1 - \frac{T_C}{T_H} \geq \eta \tag{8.9.1}$$

## 8.10 Entropy

The things that increases as heat $Q$ is added to system at temperature $T$ is $Q/T$. Defined the entropy, $S$, for a process involving **no** work to be,

$$\Delta S = \int \frac{Q}{T} \implies \mathrm{d}S = \frac{\mathrm{d}Q}{T} \tag{8.10.1}$$

**Definition 8.10.1** (Second Law of Thermodynamics)
Entropy is non-decreasing (globally).

The **multiplicity**, $\Omega$, of a state is the number of microscopic configurations that have some macroscopic state variables (if we think about it, this number is really really big).

Because $\Omega$ tends to be a **very** large number, work with $\ln \Omega$ define

$$S \equiv k_B \ln \Omega \tag{8.10.2}$$

The thing that is the same between two systems A & B in equilibrium is

$$\left(\frac{\partial S}{\partial U}\right)_A = \left(\frac{\partial S}{\partial U}\right)_B \implies \frac{1}{T} = \left.\frac{\partial S}{\partial U}\right|_{\mu,V} \tag{8.10.3}$$