PHYS 161 Lecture Notes
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Contents

Chapter 1

Mathematical Interlude

Definition 1.0.1
Kinematics is the study of motion without regard to its cause.

1.1 Units & Dimensions

In *Classical Mechanics* all quantities are expressed in terms of three dimensions, and we use SI units to define them:

- length – meters, m
- time – seconds, s
- mass – kilograms, kg

How do we measure distance? Sometimes it is easier to use the **point particle** approximation where we think of an object just as a point object with all of its mass concentrated at that point.

1.2 Coordinate System

A **coordinate system** is a collection of coordinate axis & a point called the origin.

A coordinate system is often called a **frame of reference**.

Physics should apply in whatever coordinate system (**covariant**), so scalars, vectors, tensors, ...

1.2.1 Cartesian Coordinates

$$\{(x, y, z)|x, y, z \in \mathbb{R}\} \tag{1.1}$$

1.2.2 Spherical Coordinates

$$\{(r, \theta, \phi)|0 \leq r \leq \infty, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\} \tag{1.2}$$

where ϕ is the angle of the radius deviating from the z-axis and θ is the deviation from the x-axis.

The coordinate converstions are

$$\begin{cases} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan\left({}^{y}\!/_{x}\right) \\ \phi &= \arctan\left({}^{\sqrt{x^2+y^2}}\!/_{z}\right) \end{cases} \tag{1.3}$$

1.2.3 Cylindrical Coordinates

$$\{(s,\theta,z)|0 \leq s \leq \infty, 0 \leq \theta \leq 2\pi, -\infty \leq z \leq \infty\} \tag{1.4}$$

$$\begin{cases} s &= \sqrt{x^2+y^2} \\ \theta &= \arctan\left({}^{y}\!/_{x}\right) \\ z &= z \end{cases} \tag{1.5}$$

1.3 Position Vectors

The position of a particle can be specified by its *unique* coordinates or by a **position vector**, $\vec{r}$.

A vector is just an arrow, an arrow is a vector – a geometric quantity.

Definition 1.3.1 (Vector)
A **vector** is a directed line segment, i.e. an arrow.

A vector has both **magnitude** and **direction**.

1.4 Vector Algebra

Notation

$\vec{A}$ the vector

$A = \left|\vec{A}\right|$ the magnitude

$\hat{A} = {}^{\vec{A}}\!/_{A}$ direction / unit vector

Remark
Technically, magnitude cannot be negative, but notation wise we do that anyways. $-\vec{A} = A(-\hat{A})$

1.4.1 Vector Addition

$$\vec{C} = \vec{A} + \vec{B} \tag{1.6}$$

Note that addition is commutative and associative.

1.4.2 Vector Subtraction

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \tag{1.7}$$

Final - Initial

1.4.3 Vector Multiplication

Dot product

$$\vec{A} \cdot \vec{B} = AB\cos(\theta) \tag{1.8}$$

Facts:

- if $\vec{A} \perp \vec{B} \iff \vec{A} \cdot \vec{B} = 0$
- if $\vec{A} \parallel \vec{B} \iff \vec{A} \cdot \vec{B} = AB$ is maximal
- $$\begin{cases} \vec{A} \cdot \vec{B} > 0 & \Longrightarrow \text{point in similar directions} \\ \vec{A} \cdot \vec{B} < 0 & \Longrightarrow \text{point in opposite directions} \end{cases}$$
- $\vec{A} \cdot \vec{A} = A^2$

Also defined component wise

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i \tag{1.9}$$

> **Example 1.4.1**
> Prove the law of cosines.
> Consider the triangle, ABC where θ is the angle between vectors $\vec{A}$ and $\vec{B}$.
>
> $$c^2 = a^2 + b^2 - 2ab\cos(\theta) \tag{1.10}$$

Proof. Define $\vec{A}, \vec{B}, \vec{C}$ by $A = a, B = b, C = c; \vec{C} = \vec{A} - \vec{B}$

Then,

$$\begin{aligned} \vec{C} \cdot \vec{C} = C^2 &= (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) && (1.11) \\ &= A^2 - 2\vec{A} \cdot \vec{B} + B^2 && (1.12) \\ &= a^2 + b^2 - 2ab\cos(\theta) && (1.13) \end{aligned}$$

■

Cross Product

$$\vec{A} \times \vec{B} \equiv AB\sin(\theta)\hat{n} \tag{1.14}$$

Facts:

- If $\vec{A} \parallel \vec{B}$ or antiparallel $\implies \vec{A} \times \vec{B} = 0$
- If $\vec{A} \perp \vec{B} \implies \vec{A} \times \vec{B}$ is maximal.
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- $\vec{A} \times \vec{A} = \vec{0}$

Also defined component wise as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.15}$$

1.5 Components of Vectors Basis Vectors

Say we have in Cartesian coordinates (x, y)

$$\vec{A} = \vec{A}_x + \vec{A}_y \tag{1.16}$$

Then,

$$\begin{cases} A_x &= A\cos\theta \\ A_y &= A\sin\theta \end{cases} \tag{1.17}$$

$$\begin{cases} \vec{A}_x &= A\cos\theta\vec{x} \\ \vec{A}_y &= A\sin\theta\vec{y} \end{cases} \tag{1.18}$$

$$\begin{cases} \hat{x} &= \langle 1, 0, 0\rangle \\ \hat{y} &= \langle 0, 1, 0\rangle \\ \hat{z} &= \langle 0, 0, 1\rangle \end{cases} \tag{1.19}$$

1.6 Vectors in Different Basis

1.6.1 Cartesian Coordinates

This is to say the same vectors but different components represented in different coordinates.

We can express them in the same way where θ is the original relative angle and θ' is the new relative angle:

$$\begin{cases} \vec{A} &= A\cos\theta\hat{x} + A\sin\theta\hat{y} \\ \vec{A}' &= A\cos\theta'\hat{x} + A\sin\theta'\hat{y} \end{cases} \tag{1.20}$$

Now, say we want to express our components in a different basis that rotates our standard basis by an angle of ϕ in the counterclockwise direction.

$$\begin{bmatrix} A'_x \\ B'_x \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} A_x \\ B_x \end{bmatrix} \tag{1.21}$$

1.6.2 Polar Coordinates

We have two basis vectors defined by the following

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} \tag{1.22}$$

$\hat{r}$ is in the direction

The conversion between the bases of Cartesian and Polar are the following:

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \tag{1.23}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} \tag{1.24}$$

Remark
It can be useful because coordinates will be much easier to express with

$$\vec{A} = A(\theta, r)\hat{r} \tag{1.25}$$

1.7 Calculus with Vectors

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \equiv \lim_{\Delta t \to 0} \frac{\vec{A}(t + \Delta t) - \vec{A}(t)}{\Delta t} \tag{1.26}$$

$\vec{A}(t)$ generally changes in magnitude and direction and this does capture both.

There are two cases:

Case 1: $\vec{A}(t)$ changes in magnitude only

Then $\mathrm{d}\vec{A}$ is parallel to $\vec{A}(t)$ (or antiparallel).

Let $\frac{\mathrm{d}\vec{A}_\parallel}{\mathrm{d}t}$ the component of $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \parallel \vec{A}$.

then here

$$\left\| \frac{\mathrm{d}\vec{A}_{\parallel}}{\mathrm{d}t} \right\| = \frac{\mathrm{d}A}{\mathrm{d}t} \tag{1.27}$$

Case 2: $\vec{A}(t)$ changes in direction only

Then $\mathrm{d}\vec{A}$ is perpendicular to $\vec{A}(t)$ (Almost, if we see the angle as small enough, the $\mathrm{d}\vec{A}$ would be at a right angle).

Call $\frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t}$ the component of $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \perp \vec{A}(t)$.

then here

$$\left\| \frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t} \right\| = A\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{1.28}$$

Generally

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{A}_{\parallel}}{\mathrm{d}t} + \frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t} \tag{1.29}$$

But $\vec{A} = A\hat{A}$ is naively

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\hat{A} + A\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} \tag{1.30}$$

and

$$\frac{\mathrm{d}\vec{A}_{\parallel}}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\hat{A} \qquad \frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t} = A\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} \tag{1.31}$$

1.7.1 With Cartesian Components

Derivative

$$\vec{A}(t) = A_x(t)\hat{x} + A_y(t)\hat{y} \to \frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}A_x}{\mathrm{d}t}\hat{x} + \frac{\mathrm{d}A_y}{\mathrm{d}t}\hat{y} \tag{1.32}$$

Notation

$$\dot{f} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} \qquad f' = \frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{space derivative} \tag{1.33}$$

Hence

$$\dot{\vec{A}} = \dot{A}_x\hat{x} + \dot{A}_y\hat{y} \tag{1.34}$$

Integral

$$\int \vec{A}(t)\,\mathrm{d}t \equiv (\int A_x\,\mathrm{d}t)\hat{x} + (\int A_y\,\mathrm{d}t)\hat{y} \tag{1.35}$$

Note that the fundamental theorem of calculus still applies.

1.7.2 With Polar Components

$$\vec{A}(t) = A_r(t)\hat{r}(t) + A_\theta(t)\hat{\theta}(t) \tag{1.36}$$

Then

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}A_r}{\mathrm{d}t}\hat{r} + A_r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} + \frac{\mathrm{d}A_\theta}{\mathrm{d}t}\hat{\theta} + A_\theta\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} \tag{1.37}$$

If we derive Eq. (1.23), we obtain

$$\begin{cases} \dot{\hat{r}} & = (-\sin\theta)\dot{\theta}\hat{x} + (\cos\theta)\dot{\theta}\hat{y} = \dot{\theta}\hat{\theta} \\ \dot{\hat{\theta}} & = (-\cos\theta)\dot{\theta}\hat{x} + (-\sin\theta)\dot{\theta}\hat{y} = -\dot{\theta}\hat{r} \end{cases} \tag{1.38}$$

which means that

$$\dot{\hat{r}} = \dot{\theta}\hat{\theta} \qquad \dot{\hat{\theta}} = -\dot{\theta}\hat{r} \tag{1.39}$$

which makes sense if we think about it.

And if we put it together

$$\dot{\vec{A}} = \dot{A}_r\hat{r} + A_r\dot{\theta}\hat{\theta} + \dot{A}_\theta\hat{\theta} - A_\theta\dot{\theta}\hat{r} \tag{1.40}$$

$$\implies \dot{\vec{A}} = (\dot{A}_r - A_\theta\dot{\theta})\hat{r} + (A_r\dot{\theta} + \dot{A}_\theta)\hat{\theta} \tag{1.41}$$

Chapter 2

Kinematics

We have our position vector

$$\vec{r}(t) = (x(t), y(t)) \tag{2.1}$$

We use $\vec{r}$ because it seems natural, it is the direction we are pointing in.

Remark
Sometimes when reference to radial $\vec{r}$ is misleading, we use $\vec{x}(t)$.

The change of the vector in space across time sweeps over some **trajectory**.

2.1 Displacement

Definition 2.1.1 (Displacement)
The *displacement vector* $\Delta\vec{r}$ is a measure of where the particle went (which depends on the origin!).

$$\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = \vec{r}(t_f) - \vec{r}(t_i) \tag{2.2}$$

1. $\|\Delta\vec{r}\| \neq$ distance travelled in general
   - distance traveled = arc length of trajectory
2. $\Delta\vec{r}$ is coordinate independent.

   Take two coordinate systems S and S'. Let them be defined with the relation $\vec{r} = \vec{r}' + \vec{R}$ where $\vec{r}$ and $\vec{r}'$ are vectors in the respective coordinate systems.

$$\begin{cases} S: & \Delta\vec{r} = \vec{r}_f - \vec{r}_i \\ S': & \Delta\vec{r}' = \vec{r}_f{}' - \vec{r}_i{}' \end{cases} \tag{2.3}$$

   If we plug in the relation, we realize that they are the same, $\Delta\vec{r} = \Delta\vec{r}'$

2.2 Velocity

Definition 2.2.1 (Average Velocity)

$$\vec{v}_{\text{avg}} \equiv \frac{\Delta \vec{r}}{\Delta t} \tag{2.4}$$

Let $\mathrm{d}\vec{r}$ be the infinitesimal displacement.

When we consider a smaller interval:

$$\lim_{\Delta t \to 0} \implies \|\mathrm{d}\vec{r}\| = \mathrm{d}r \quad \text{(distance traveled)} \tag{2.5}$$

A small change to t results in a small change in $\mathrm{d}S$ (the distance / speed), proportionally

$$\mathrm{d}S \propto \mathrm{d}t \tag{2.6}$$

$$\implies \mathrm{d}S = \left(\frac{\mathrm{d}S}{\mathrm{d}t}\right) \mathrm{d}t \tag{2.7}$$

Definition 2.2.2 (Velocity)
AKA the *instantaneous velocity*

$$\vec{v}(t) \equiv \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \tag{2.8}$$

- $\|\vec{v}\| =$ speed
- $\hat{v} =$ direction of motion

Remark
A note on average velocity:

$$\vec{v}_{\text{avg}} = \frac{1}{\Delta t}\int_{t_i}^{t_f} \vec{v}(t)\,\mathrm{d}t = \frac{1}{\Delta t}\int_{t_i}^{t_f} \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t = \frac{\Delta \vec{r}}{\Delta t} \tag{2.9}$$

Note also if we find the magnitude, it would not be the same as the average speed since the norm would go over the integrals instead of what is being integrated.

- $\vec{v}$ a vector, so write $\vec{v}(t) = \dot{x}\hat{x} + \dot{y}\hat{y} = \dot{\vec{r}}$
- Compare to frames of reference, S & S'

  Suppose $\dot{\vec{R}} \neq 0$.

  Then we have

$$\begin{cases} \vec{r} &= \vec{r}' + \vec{R} \\ \vec{v} &= \vec{v}' + \vec{V} \end{cases} \tag{2.10}$$

This is known as the Galilean transformations, which, at higher velocities, "translates" to the Lorentz transformations.

We can also obtain $\vec{r}(t)$ given $\vec{v}(t)$

$$\Delta\vec{r} = \int \mathrm{d}\vec{r} = \int_{t_i}^{t_f} \vec{v}\,\mathrm{d}t \tag{2.11}$$

and

$$\vec{r}(t) = \vec{r}_i + \vec{v}_i(t - t_i) \tag{2.12}$$

2.3 Acceleration

Definition 2.3.1

$$\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}^2 t} \tag{2.13}$$

Similar to what is mentioned in section 1.7, $\vec{a}_{\parallel}$ is change in speed, $\vec{a}_{\perp}$ is change in direction of motion.

Remark
We do have the *jerk*, but it just seems that it never really matters, and acceleration is fully sufficient.

2.3.1 Cartesian Coordinates

$$\begin{cases} \vec{r}(t) & = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \\ \vec{v}(t) & = \dot{x}(t)\hat{x} + \dot{y}(t)\hat{y} + \dot{z}(t)\hat{z} \\ \vec{a}(t) & = \ddot{x}(t)\hat{x} + \ddot{y}(t)\hat{y} + \ddot{z}(t)\hat{z} \end{cases} \tag{2.14}$$

Example 2.3.1
Suppose particle's position is $\vec{r}(t) = A(e^{\alpha t}\hat{x} + e^{-\alpha t}\hat{y})$ with A and α constants. ($[A] = \mathrm{m}$, $[\alpha] = \mathrm{m}^{-1}$) Find $\vec{v}(t)$ and $\vec{a}(t)$ and sketch trajectory.

Solution:

Velocity

$$\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \tag{2.15}$$

$$= A(\alpha e^{\alpha t}\hat{x} - \alpha e^{-\alpha t}\hat{y}) \tag{2.16}$$

$$= \alpha A(e^{\alpha t}\hat{x} - e^{-\alpha t}\hat{y}) \tag{2.17}$$

Acceleration

$$\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{2.18}$$
$$= \alpha^2 A(e^{\alpha t}\hat{x} + e^{-\alpha t}\hat{y}) \tag{2.19}$$
$$= \alpha^2 \vec{r}(t) \tag{2.20}$$

Speed

$$|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} \tag{2.21}$$
$$= \sqrt{(\alpha A)^2\left[e^{2\alpha t} + e^{-2\alpha t}\right]} \tag{2.22}$$
$$= \alpha A\sqrt{2\cosh(2\alpha t)} \tag{2.23}$$

Note that (by definition)

$$\begin{cases} x(t) & = Ae^{\alpha t} \\ y(t) & = Ae^{-\alpha t} \end{cases} \tag{2.24}$$

We can try to find $y(x)$ by eliminating t, which is the equation for the trajectory, we obtain:

$$y(x) = \frac{A^2}{x} \qquad y \propto \frac{1}{x} \tag{2.25}$$

So although the velocity and acceleration changes at a exponential rate, the trajectory that it produces exhibits the inverse curve.

Example 2.3.2
A particle moves in the plane with trajectory of a circle of radius R. The particle sweeps out the circle at a uniform and constant rate. That is, it undergoes uniform circular motion. Find $\vec{r}(t), \vec{v}(t)$, and $\vec{a}(t)$.

Solution:

We know that the magnitude of the position vector $|\vec{r}| = R$ and that

$$\vec{r} = R\cos\theta(t)\hat{x} + R\sin\theta(t)\hat{y} \tag{2.26}$$

Remark
$\vec{v}$ changes direction, but with uniform rate $|\vec{v}| = c$.

From our $\vec{r}(t)$ we have that

$$\vec{v}(t) = -R\sin\theta(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{x} + R\cos\theta(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{y} \tag{2.27}$$

$$= R\dot{\theta}\left[-\sin\theta\hat{x} + \cos\theta\hat{y}\right] \tag{2.28}$$

We know that v is constant and that $v = R\dot{\theta}$, so $R\dot{\theta}$ must also be constant. Since R is constant, $\dot{\theta}$ is constant.

$$\dot{\theta} \equiv \omega \implies \theta(t) = \omega t \tag{2.29}$$

This is assuming $\theta(0) = 0$.

As a result of our derivation, we find

$$\begin{cases} \vec{r}(t) &= R\cos(\omega t)\hat{x} + R\sin(\omega t)\hat{y} \\ \vec{v}(t) &= -\omega R\sin(\omega t)\hat{x} + \omega R\cos(\omega t)\hat{y} \end{cases} \tag{2.30}$$

Now, noting the magnitude:

$$\begin{cases} r &= R \\ v &= \omega R \\ a &= \omega^2 R = \frac{v^2}{R} \end{cases} \tag{2.31}$$

Acceleration

$$\vec{a}(t) = -\omega^2 R\cos(\omega t)\hat{x} - \omega^2 R\sin(\omega t)\hat{y} \tag{2.32}$$

$$= -\omega^2\vec{r}(t) \tag{2.33}$$

Remark
Because $\hat{a}$ points towards the origin $[\hat{a} = -\hat{r}]$, we call it "centripetal" ($\leftarrow$ central seeking).

2.4 Formal Solution of Kinematic Equations

We want to obtain $\vec{v}(t)$ and $\vec{r}(t)$ given $\vec{a}(t)$.

2.4.1 $\vec{v}$ from $\vec{a}$

$$\int_0^t \vec{a}(t)\,\mathrm{d}t' = \int_{\vec{v}_0}^{\vec{v}} \frac{\mathrm{d}\vec{v}}{\mathrm{d}t'}\,\mathrm{d}t' \tag{2.34}$$

$$= \vec{v}(t) - \vec{v}_0 \tag{2.35}$$

$$\vec{v}(t) = \boxed{\vec{v}_0 + \int_0^t \vec{a}(t')\,\mathrm{d}t'} \tag{2.36}$$

2.4.2 $\vec{r}$ from $\vec{v}$ (from $\vec{a}$)

$$\int_0^t \vec{v}(t')\,\mathrm{d}t' = \int_{\vec{r}_0}^{\vec{r}} \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t \tag{2.37}$$

$$= \vec{r}(t) - \vec{r}_0 \tag{2.38}$$

$$\vec{r}(t) = \boxed{\vec{r}_0 + \int_0^t \vec{v}(t')\,\mathrm{d}t'} \tag{2.39}$$

$$= \vec{r}_0 + \int_0^t \left[\vec{v}_0 + \int_0^{t'} \vec{a}(t'')\,\mathrm{d}t''\right]\,\mathrm{d}t' \tag{2.40}$$

$$\vec{r}(t) = \boxed{\vec{r}_0 + \vec{v}_0 t + \int_0^t \int_0^{t'} \vec{a}(t'')\,\mathrm{d}t''\,\mathrm{d}t'} \tag{2.41}$$

Remark
We need to know $\vec{r}_0$.

To find $\vec{r}(t)$ given $\vec{a}(t)$ we need also to know the initial conditions, $\vec{r}_0$ and $\vec{v}_0$.

2.5 Constant Acceleration Motion

There are many cases of constant $\vec{a}$ motion. With our previous analysis, the cases of when $\vec{a} =$ const gives:

$$\begin{cases} \vec{v}(t) & = \vec{v}_0 + \int_0^t \vec{a}\,\mathrm{d}t' = \vec{v}_0 + \vec{a}t \\ \vec{r}(t) & = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \\ v^2 & = {v_0}^2 + 2\vec{a}\cdot\Delta\vec{r} \end{cases} \tag{2.42}$$

Remark
if $t_0 \neq 0$, the $t \to \Delta t$ in formulas.

Let's eliminate t from these equations:

From $\vec{v} = \vec{v}_0 + \vec{a}t$, compute $v^2 = \vec{v}\cdot\vec{v}$

$$v^2 = {v_0}^2 + 2\vec{v}_0\cdot\vec{a}t + a^2t^2 \tag{2.43}$$

$$\frac{1}{2}v^2 = \frac{1}{2}{v_0}^2 + \vec{v}_0\cdot\vec{a}t + \frac{1}{2}a^2t^2 \tag{2.44}$$

Now, from $\vec{r}$ compute

$$\vec{a}\cdot\vec{r} = \vec{a}\cdot\vec{r}_0 + \vec{a}\cdot\vec{v}_0 t + \frac{1}{2}a^2t^2 \tag{2.45}$$

Then, we take the difference, we have

$$\frac{1}{2}v^2 - \vec{a}\cdot\vec{r} = \frac{1}{2}{v_0}^2 - \vec{a}\cdot\vec{r}_0 \tag{2.46}$$

$$\frac{1}{2}v^2 = \frac{1}{2}{v_0}^2 + \vec{a}\cdot(\vec{r}-\vec{r}_0) \tag{2.47}$$

$$\boxed{v^2 = {v_0}^2 + 2\vec{a}\cdot\Delta\vec{r}} \tag{2.48}$$

2.5.1 Components of the Equations