PHYS 161 Lecture Notes

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Contents

Chapter 1

Mathematical Interlude

Definition 1.0.1
Kinematics is the study of motion without regard to its cause.

1.1 Units & Dimensions

In *Classical Mechanics* all quantities are expressed in terms of three dimensions, and we use SI units to define them:

- length – meters, m
- time – seconds, s
- mass – kilograms, kg

How do we measure distance? Sometimes it is easier to use the **point particle** approximation where we think of an object just as a point object with all of its mass concentrated at that point.

1.2 Coordinate System

A **coordinate system** is a collection of coordinate axis & a point called the origin.

A coordinate system is often called a **frame of reference**.

Physics should apply in whatever coordinate system (**covariant**), so scalars, vectors, tensors, ...

1.2.1 Cartesian Coordinates

$$\{(x, y, z) | x, y, z \in \mathbb{R}\} \tag{1.2.1}$$

1.2.2 Spherical Coordinates

$$\{(r, \theta, \phi) | 0 \leq r \leq \infty, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\} \tag{1.2.2}$$

where ϕ is the angle of the radius deviating from the z-axis and θ is the deviation from the x-axis.

The coordinate converstions are

$$\begin{cases} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arctan\left(y/x\right) \\ \phi &= \arctan\left(\sqrt{x^2+y^2}/z\right) \end{cases} \tag{1.2.3}$$

1.2.3 Cylindrical Coordinates

$$\{(s, \theta, z) | 0 \leq s \leq \infty, 0 \leq \theta \leq 2\pi, -\infty \leq z \leq \infty\} \tag{1.2.4}$$

$$\begin{cases} s &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(y/x\right) \\ z &= z \end{cases} \tag{1.2.5}$$

1.3 Position Vectors

The position of a particle can be specified by its *unique* coordinates or by a **position vector**, $\vec{r}$.

A vector is just an arrow, an arrow is a vector – a geometric quantity.

Definition 1.3.1 (Vector)
A **vector** is a directed line segment, i.e. an arrow.

A vector has both **magnitude** and **direction**.

1.4 Vector Algebra

Notation

$\vec{A}$ the vector

$A = \left|\vec{A}\right|$ the magnitude

$\hat{A} = \vec{A}/A$ direction / unit vector

Remark
Technically, magnitude cannot be negative, but notation wise we do that anyways. $-\vec{A} = A(-\hat{A})$

1.4.1 Vector Addition

$$\vec{C} = \vec{A} + \vec{B} \tag{1.4.1}$$

Note that addition is commutative and associative.

1.4.2 Vector Subtraction

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \tag{1.4.2}$$

Final - Initial

1.4.3 Vector Multiplication

Dot product

$$\vec{A} \cdot \vec{B} = AB\cos(\theta) \tag{1.4.3}$$

Facts:

- if $\vec{A} \perp \vec{B} \iff \vec{A} \cdot \vec{B} = 0$
- if $\vec{A} \parallel \vec{B} \iff \vec{A} \cdot \vec{B} = AB$ is maximal
- $$\begin{cases} \vec{A} \cdot \vec{B} > 0 & \implies \text{point in similar directions} \\ \vec{A} \cdot \vec{B} < 0 & \implies \text{point in opposite directions} \end{cases}$$
- $\vec{A} \cdot \vec{A} = A^2$

Also defined component wise

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i \tag{1.4.4}$$

Example 1.4.1
Prove the law of cosines.
Consider the triangle, ABC where θ is the angle between vectors $\vec{A}$ and $\vec{B}$.

$$c^2 = a^2 + b^2 - 2ab\cos(\theta) \tag{1.4.5}$$

Proof. Define $\vec{A}, \vec{B}, \vec{C}$ by $A = a, B = b, C = c; \vec{C} = \vec{A} - \vec{B}$

Then,

$$\begin{aligned} \vec{C} \cdot \vec{C} = C^2 &= (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) && (1.4.6) \\ &= A^2 - 2\vec{A} \cdot \vec{B} + B^2 && (1.4.7) \\ &= a^2 + b^2 - 2ab\cos(\theta) && (1.4.8) \end{aligned}$$

■

Cross Product

$$\vec{A} \times \vec{B} \equiv AB\sin(\theta)\hat{n} \tag{1.4.9}$$

Facts:

- If $\vec{A} \parallel \vec{B}$ or antiparallel $\implies \vec{A} \times \vec{B} = 0$
- If $\vec{A} \perp \vec{B} \implies \vec{A} \times \vec{B}$ is maximal.
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- $\vec{A} \times \vec{A} = \vec{0}$

Also defined component wise as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.4.10}$$

1.5 Components of Vectors Basis Vectors

Say we have in Cartesian coordinates (x, y)

$$\vec{A} = \vec{A}_x + \vec{A}_y \tag{1.5.1}$$

Then,

$$\begin{cases} A_x &= A\cos\theta \\ A_y &= A\sin\theta \end{cases} \tag{1.5.2}$$

$$\begin{cases} \vec{A}_x &= A\cos\theta\vec{x} \\ \vec{A}_y &= A\sin\theta\vec{y} \end{cases} \tag{1.5.3}$$

$$\begin{cases} \hat{x} &= \langle 1, 0, 0\rangle \\ \hat{y} &= \langle 0, 1, 0\rangle \\ \hat{z} &= \langle 0, 0, 1\rangle \end{cases} \tag{1.5.4}$$

1.6 Vectors in Different Basis

1.6.1 Cartesian Coordinates

This is to say the same vectors but different components represented in different coordinates.

We can express them in the same way where θ is the original relative angle and θ' is the new relative angle:

$$\begin{cases} \vec{A} &= A\cos\theta\hat{x} + A\sin\theta\hat{y} \\ \vec{A}' &= A\cos\theta'\hat{x} + A\sin\theta'\hat{y} \end{cases} \tag{1.6.1}$$

Now, say we want to express our components in a different basis that rotates our standard basis by an angle of ϕ in the counterclockwise direction.

$$\begin{bmatrix} A'_x \\ B'_x \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} A_x \\ B_x \end{bmatrix} \tag{1.6.2}$$

1.6.2 Polar Coordinates

We have two basis vectors defined by the following

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} \tag{1.6.3}$$

$\hat{r}$ is in the direction

The conversion between the bases of Cartesian and Polar are the following:

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \tag{1.6.4}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} \tag{1.6.5}$$

Remark
It can be useful because coordinates will be much easier to express with

$$\vec{A} = A(\theta, r)\hat{r} \tag{1.6.6}$$

1.7 Calculus with Vectors

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \equiv \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t} \tag{1.7.1}$$

$\vec{A}(t)$ generally changes in magnitude and direction and this does capture both.

There are two cases:

Case 1: $\vec{A}(t)$ changes in magnitude only

Then $\mathrm{d}\vec{A}$ is parallel to $\vec{A}(t)$ (or antiparallel).

Let $\frac{\mathrm{d}\vec{A}_\parallel}{\mathrm{d}t}$ the component of $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \parallel \vec{A}$.

then here

$$\left\| \frac{\mathrm{d}\vec{A}_{\|}}{\mathrm{d}t} \right\| = \frac{\mathrm{d}A}{\mathrm{d}t} \tag{1.7.2}$$

Case 2: $\vec{A}(t)$ changes in direction only

Then $\mathrm{d}\vec{A}$ is perpendicular to $\vec{A}(t)$ (Almost, if we see the angle as small enough, the $\mathrm{d}\vec{A}$ would be at a right angle).

Call $\frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t}$ the component of $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \perp \vec{A}(t)$.

then here

$$\left\| \frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t} \right\| = A\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{1.7.3}$$

Generally

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{A}_{\|}}{\mathrm{d}t} + \frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t} \tag{1.7.4}$$

But $\vec{A} = A\hat{A}$ is naively

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\hat{A} + A\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} \tag{1.7.5}$$

and

$$\frac{\mathrm{d}\vec{A}_{\|}}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\hat{A} \qquad \frac{\mathrm{d}\vec{A}_{\perp}}{\mathrm{d}t} = A\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} \tag{1.7.6}$$

1.7.1 With Cartesian Components

Derivative

$$\vec{A}(t) = A_x(t)\hat{x} + A_y(t)\hat{y} \to \frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}A_x}{\mathrm{d}t}\hat{x} + \frac{\mathrm{d}A_y}{\mathrm{d}t}\hat{y} \tag{1.7.7}$$

Notation

$$\dot{f} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} \qquad f' = \frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{space derivative} \tag{1.7.8}$$

Hence

$$\dot{\vec{A}} = \dot{A}_x\hat{x} + \dot{A}_y\hat{y} \tag{1.7.9}$$

Integral

$$\int \vec{A}(t)\,\mathrm{d}t \equiv (\int A_x\,\mathrm{d}t)\hat{x} + (\int A_y\,\mathrm{d}t)\hat{y} \tag{1.7.10}$$

Note that the fundamental theorem of calculus still applies.

1.7.2 With Polar Components

$$\vec{A}(t) = A_r(t)\hat{r}(t) + A_\theta(t)\hat{\theta}(t) \tag{1.7.11}$$

Then

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\mathrm{d}A_r}{\mathrm{d}t}\hat{r} + A_r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} + \frac{\mathrm{d}A_\theta}{\mathrm{d}t}\hat{\theta} + A_\theta\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} \tag{1.7.12}$$

If we derive Eq. (1.6.4), we obtain

$$\begin{cases} \dot{\hat{r}} & = (-\sin\theta)\dot{\theta}\hat{x} + (\cos\theta)\dot{\theta}\hat{y} = \dot{\theta}\hat{\theta} \\ \dot{\hat{\theta}} & = (-\cos\theta)\dot{\theta}\hat{x} + (-\sin\theta)\dot{\theta}\hat{y} = -\dot{\theta}\hat{r} \end{cases} \tag{1.7.13}$$

which means that

$$\dot{\hat{r}} = \dot{\theta}\hat{\theta} \qquad \dot{\hat{\theta}} = -\dot{\theta}\hat{r} \tag{1.7.14}$$

which makes sense if we think about it.

And if we put it together

$$\dot{\vec{A}} = \dot{A}_r\hat{r} + A_r\dot{\theta}\hat{\theta} + \dot{A}_\theta\hat{\theta} - A_\theta\dot{\theta}\hat{r} \tag{1.7.15}$$

$$\implies \dot{\vec{A}} = (\dot{A}_r - A_\theta\dot{\theta})\hat{r} + (A_r\dot{\theta} + \dot{A}_\theta)\hat{\theta} \tag{1.7.16}$$

Chapter 2

Kinematics

We have our position vector

$$\vec{r}(t) = (x(t), y(t)) \tag{2.0.1}$$

We use $\vec{r}$ because it seems natural, it is the direction we are pointing in.

Remark
Sometimes when reference to radial $\vec{r}$ is misleading, we use $\vec{x}(t)$.

The change of the vector in space across time sweeps over some **trajectory**.

2.1 Displacement

Definition 2.1.1 (Displacement)
The *displacement vector* $\Delta\vec{r}$ is a measure of where the particle went (which depends on the origin!).

$$\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = \vec{r}(t_f) - \vec{r}(t_i) \tag{2.1.1}$$

1. $\|\Delta\vec{r}\| \neq$ distance travelled in general
   - distance traveled = arc length of trajectory
2. $\Delta\vec{r}$ is coordinate independent.

   Take two coordinate systems S and S'. Let them be defined with the relation $\vec{r} = \vec{r}' + \vec{R}$ where $\vec{r}$ and $\vec{r}'$ are vectors in the respective coordinate systems.

   $$\begin{cases} S: & \Delta\vec{r} = \vec{r}_f - \vec{r}_i \\ S': & \Delta\vec{r}' = \vec{r}_f{}' - \vec{r}_i{}' \end{cases} \tag{2.1.2}$$

   If we plug in the relation, we realize that they are the same, $\Delta\vec{r} = \Delta\vec{r}'$

2.2 Velocity

Definition 2.2.1 (Average Velocity)

$$\vec{v}_{\text{avg}} \equiv \frac{\Delta \vec{r}}{\Delta t} \tag{2.2.1}$$

Let $\mathrm{d}\vec{r}$ be the infinitesimal displacement.

When we consider a smaller interval:

$$\lim_{\Delta t \to 0} \implies \|\mathrm{d}\vec{r}\| = \mathrm{d}r \quad \text{(distance traveled)} \tag{2.2.2}$$

A small change to t results in a small change in $\mathrm{d}S$ (the distance / speed), proportionally

$$\mathrm{d}S \propto \mathrm{d}t \tag{2.2.3}$$

$$\implies \mathrm{d}S = \left(\frac{\mathrm{d}S}{\mathrm{d}t}\right)\mathrm{d}t \tag{2.2.4}$$

Definition 2.2.2 (Velocity)
AKA the *instantaneous velocity*

$$\vec{v}(t) \equiv \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \tag{2.2.5}$$

- $\|\vec{v}\|$ = speed
- $\hat{v}$ = direction of motion

Remark
A note on average velocity:

$$\vec{v}_{\text{avg}} = \frac{1}{\Delta t}\int_{t_i}^{t_f} \vec{v}(t)\,\mathrm{d}t = \frac{1}{\Delta t}\int_{t_i}^{t_f} \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t = \frac{\Delta \vec{r}}{\Delta t} \tag{2.2.6}$$

Note also if we find the magnitude, it would not be the same as the average speed since the norm would go over the integrals instead of what is being integrated.

- $\vec{v}$ a vector, so write $\vec{v}(t) = \dot{x}\hat{x} + \dot{y}\hat{y} = \dot{\vec{r}}$
- Compare to frames of reference, S & S'

  Suppose $\dot{\vec{R}} \neq 0$.

  Then we have

$$\begin{cases} \vec{r} &= \vec{r}' + \vec{R} \\ \vec{v} &= \vec{v}' + \vec{V} \end{cases} \tag{2.2.7}$$

> This is known as the Galilean transformations, which, at higher velocities, "translates" to the Lorentz transformations.

We can also obtain $\vec{r}(t)$ given $\vec{v}(t)$

$$\Delta\vec{r} = \int \mathrm{d}\vec{r} = \int_{t_i}^{t_f} \vec{v}\,\mathrm{d}t \tag{2.2.8}$$

and

$$\vec{r}(t) = \vec{r}_i + \vec{v}_i(t - t_i) \tag{2.2.9}$$

2.3 Acceleration

Definition 2.3.1

$$\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}^2 t} \tag{2.3.1}$$

Similar to what is mentioned in section 1.7, $\vec{a}_{\parallel}$ is change in speed, $\vec{a}_{\perp}$ is change in direction of motion.

Remark
We do have the *jerk*, but it just seems that it never really matters, and acceleration is fully sufficient.

2.3.1 Cartesian Coordinates

$$\begin{cases} \vec{r}(t) &= x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \\ \vec{v}(t) &= \dot{x}(t)\hat{x} + \dot{y}(t)\hat{y} + \dot{z}(t)\hat{z} \\ \vec{a}(t) &= \ddot{x}(t)\hat{x} + \ddot{y}(t)\hat{y} + \ddot{z}(t)\hat{z} \end{cases} \tag{2.3.2}$$

Example 2.3.1
Suppose particle's position is $\vec{r}(t) = A(e^{\alpha t}\hat{x} + e^{-\alpha t}\hat{y})$ with A and α constants. ($[A] = \mathrm{m}$, $[\alpha] = \mathrm{m}^{-1}$) Find $\vec{v}(t)$ and $\vec{a}(t)$ and sketch trajectory.

Solution:

Velocity

$$\begin{aligned} \vec{v}(t) &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} && (2.3.3) \\ &= A(\alpha e^{\alpha t}\hat{x} - \alpha e^{-\alpha t}\hat{y}) && (2.3.4) \\ &= \alpha A(e^{\alpha t}\hat{x} - e^{-\alpha t}\hat{y}) && (2.3.5) \end{aligned}$$

Acceleration

$$\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{2.3.6}$$
$$= \alpha^2 A(e^{\alpha t}\hat{x} + e^{-\alpha t}\hat{y}) \tag{2.3.7}$$
$$= \alpha^2 \vec{r}(t) \tag{2.3.8}$$

Speed

$$|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} \tag{2.3.9}$$
$$= \sqrt{(\alpha A)^2\left[e^{2\alpha t} + e^{-2\alpha t}\right]} \tag{2.3.10}$$
$$= \alpha A\sqrt{2\cosh(2\alpha t)} \tag{2.3.11}$$

Note that (by definition)

$$\begin{cases} x(t) &= Ae^{\alpha t} \\ y(t) &= Ae^{-\alpha t} \end{cases} \tag{2.3.12}$$

We can try to find $y(x)$ by eliminating t, which is the equation for the trajectory, we obtain:

$$y(x) = \frac{A^2}{x} \qquad y \propto \frac{1}{x} \tag{2.3.13}$$

So although the velocity and acceleration changes at a exponential rate, the trajectory that it produces exhibits the inverse curve.

Example 2.3.2
A particle moves in the plane with trajectory of a circle of radius R. The particle sweeps out the circle at a uniform and constant rate. That is, it undergoes uniform circular motion. Find $\vec{r}(t), \vec{v}(t)$, and $\vec{a}(t)$.

Solution:

We know that the magnitude of the position vector $|\vec{r}| = R$ and that

$$\vec{r} = R\cos\theta(t)\hat{x} + R\sin\theta(t)\hat{y} \tag{2.3.14}$$

Remark
$\vec{v}$ changes direction, but with uniform rate $|\vec{v}| = c$.

From our $\vec{r}(t)$ we have that

$$\vec{v}(t) = -R\sin\theta(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{x} + R\cos\theta(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{y} \tag{2.3.15}$$

$$= R\dot{\theta}\left[-\sin\theta\hat{x} + \cos\theta\hat{y}\right] \tag{2.3.16}$$

We know that v is constant and that $v = R\dot{\theta}$, so $R\dot{\theta}$ must also be constant. Since R is constant, $\dot{\theta}$ is constant.

$$\dot{\theta} \equiv \omega \implies \theta(t) = \omega t \tag{2.3.17}$$

This is assuming $\theta(0) = 0$.

As a result of our derivation, we find

$$\begin{cases} \vec{r}(t) & = R\cos(\omega t)\hat{x} + R\sin(\omega t)\hat{y} \\ \vec{v}(t) & = -\omega R\sin(\omega t)\hat{x} + \omega R\cos(\omega t)\hat{y} \end{cases} \tag{2.3.18}$$

Now, noting the magnitude:

$$\begin{cases} r & = R \\ v & = \omega R \\ a & = \omega^2 R = \frac{v^2}{R} \end{cases} \tag{2.3.19}$$

Acceleration

$$\vec{a}(t) = -\omega^2 R\cos(\omega t)\hat{x} - \omega^2 R\sin(\omega t)\hat{y} \tag{2.3.20}$$

$$= -\omega^2\vec{r}(t) \tag{2.3.21}$$

Remark
Because $\hat{a}$ points towards the origin [$\hat{a} = -\hat{r}$], we call it “centripetal” (← central seeking).

2.4 Formal Solution of Kinematic Equations

We want to obtain $\vec{v}(t)$ and $\vec{r}(t)$ given $\vec{a}(t)$.

2.4.1 $\vec{v}$ from $\vec{a}$

$$\int_0^t \vec{a}(t)\,\mathrm{d}t' = \int_{\vec{v}_0}^{\vec{v}} \frac{\mathrm{d}\vec{v}}{\mathrm{d}t'}\,\mathrm{d}t' \tag{2.4.1}$$

$$= \vec{v}(t) - \vec{v}_0 \tag{2.4.2}$$

$$\vec{v}(t) = \boxed{\vec{v}_0 + \int_0^t \vec{a}(t')\,\mathrm{d}t'} \tag{2.4.3}$$

2.4.2 $\vec{r}$ from $\vec{v}$ (from $\vec{a}$)

$$\int_0^t \vec{v}(t')\,\mathrm{d}t' = \int_{\vec{r}_0}^{\vec{r}} \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t \tag{2.4.4}$$

$$= \vec{r}(t) - \vec{r}_0 \tag{2.4.5}$$

$$\vec{r}(t) = \boxed{\vec{r}_0 + \int_0^t \vec{v}(t')\,\mathrm{d}t'} \tag{2.4.6}$$

$$= \vec{r}_0 + \int_0^t \left[\vec{v}_0 + \int_0^{t'} \vec{a}(t'')\,\mathrm{d}t''\right]\mathrm{d}t' \tag{2.4.7}$$

$$\vec{r}(t) = \boxed{\vec{r}_0 + \vec{v}_0 t + \int_0^t \int_0^{t'} \vec{a}(t'')\,\mathrm{d}t''\,\mathrm{d}t'} \tag{2.4.8}$$

Remark
We need to know $\vec{r}_0$.

To find $\vec{r}(t)$ given $\vec{a}(t)$ we need also to know the initial conditions, $\vec{r}_0$ and $\vec{v}_0$.

2.5 Constant Acceleration Motion

Theorem 2.5.1 (Kinematic Equations with Constant $\vec{a}$)
There are many cases of constant $\vec{a}$ motion. With our previous analysis, the cases of when $\vec{a} = \text{const}$ gives:

$$\begin{cases} \vec{r}(t) & = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \\ \vec{v}(t) & = \vec{v}_0 + \int_0^t \vec{a}\,\mathrm{d}t' = \vec{v}_0 + \vec{a}t \\ v^2 & = {v_0}^2 + 2\vec{a}\cdot\Delta\vec{r} \end{cases} \tag{2.5.1}$$

Remark
if $t_0 \neq 0$, the $t \to \Delta t$ in formulas.

Let's eliminate t from these equations:

From $\vec{v} = \vec{v}_0 + \vec{a}t$, compute $v^2 = \vec{v}\cdot\vec{v}$

$$v^2 = {v_0}^2 + 2\vec{v}_0 \cdot \vec{a}t + a^2t^2 \tag{2.5.2}$$

$$\frac{1}{2}v^2 = \frac{1}{2}{v_0}^2 + \vec{v}_0 \cdot \vec{a}t + \frac{1}{2}a^2t^2 \tag{2.5.3}$$

Now, from $\vec{r}$ compute

$$\vec{a} \cdot \vec{r} = \vec{a} \cdot \vec{r}_0 + \vec{a} \cdot \vec{v}_0 t + \frac{1}{2}a^2t^2 \tag{2.5.4}$$

Then, we take the difference, we have

$$\frac{1}{2}v^2 - \vec{a} \cdot \vec{r} = \frac{1}{2}{v_0}^2 - \vec{a} \cdot \vec{r}_0 \tag{2.5.5}$$

$$\frac{1}{2}v^2 = \frac{1}{2}{v_0}^2 + \vec{a} \cdot (\vec{r} - \vec{r}_0) \tag{2.5.6}$$

$$\boxed{v^2 = {v_0}^2 + 2\vec{a} \cdot \Delta\vec{r}} \tag{2.5.7}$$

2.5.1 Components of the Equations

Remark
These laws are also applicable in components.

2.6 Two-Dimensional Motion

2.6.1 Free Fall

All objects regardless of mass, shape, composition, etc., fall downward towards earth with same motion – *free fall.*

Free fall is vertical motion subject *only* to earth's gravity, which is constant acceleration motion.

The acceleration due to gravity, g, is

$$g = 9.8\,\mathrm{m/s^2} \qquad \vec{a} = -g\hat{z}^{1} \tag{2.6.1}$$

2.6.2 Projectile Motion

Projectile motion is motion subject only to gravity, that is, motion for which $\vec{a} = -g\hat{z}$.

Remark
Projectile motion lies in the plane formed by $\vec{v}_0$ and $\vec{a}$. This implies 2D motion.

[1] True near earth's surface

Now, the equations:

But a lot of times what we do is to consider the two components in Cartesian.

$$x \text{ component} \implies x(t) = x_0 + v_{0x}t \tag{2.6.2}$$

$$y \text{ component} \implies 0 \tag{2.6.3}$$

$$z \text{ component} \implies \begin{cases} z(t) & = z_0 + v_{0z}t - \frac{1}{2}gt^2 \\ v_z(t) & = v_{0z} - gt \\ {v_z}^2 & = {v_{0z}}^2 - 2g\Delta z \end{cases} \tag{2.6.4}$$

Example 2.6.1
Consider a projectile launched with initial velocity $\vec{v}_0$ that makes angle θ with the horizontal. Choose coordinates s.t. $(x_0, y_0, z_0) = (0, 0, h)$ with the plane of motion the xz-plane.
Find:

a) the trajectory of the projectile, $z = z(x)$
b) the maximum height and horizontal distance (i.e. range) of the projectile
c) the velocity of the projectile when it hits the ground
d) the launch angle, ϕ, that maximizes the range. Here, let $h = 0$.

Solution:

a) Equations for the motion are:

$$\begin{cases} z(t) & = h + v_0 \sin\theta t + \frac{1}{2}gt^2 \\ v_z(t) & = v_0 \sin\theta + gt \\ {v_z}^2 & = {v_0}^2 \sin^2\theta - 2g(z - h) \\ x(t) & = v_0 \cos\theta t \end{cases} \tag{2.6.5}$$

We simply have to find z in terms of x, notice how $z(x) = z(t(x))$. We just need $t(x)$.

We find that

$$x = v_0 \cos\theta t \tag{2.6.6}$$

$$t = \frac{x}{v_0 \cos\theta} \tag{2.6.7}$$

Now we substitute

$$z(t) = h + v_0 \sin\theta t + \frac{1}{2}gt^2 \tag{2.6.8}$$

$$z(t) = h + v_0 \sin\theta \left(\frac{x}{v_0\cos\theta}\right) + \frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 \tag{2.6.9}$$

$$= \boxed{h + x\tan\theta + \frac{gx^2}{2{v_0}^2\cos^2\theta}} \tag{2.6.10}$$

b) **Maximum Heigh** z_{max}

Obtained when $v_z = 0$

$$\implies 0 = v_0\sin\theta - gt_{\text{max}} \tag{2.6.11}$$

$$t_{\text{max}} = \frac{v_0\sin\theta}{g} \tag{2.6.12}$$

Sub into z-equation

$$z_{\text{max}} = h + v_0\sin\theta\left(\frac{v_0\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{v_0\sin\theta}{g}\right)^2 \tag{2.6.13}$$

$$= \boxed{h + \frac{{v_0}^2\sin^2\theta}{2g}} \tag{2.6.14}$$

Alternatively,

$$0 = {v_0}^2\sin^2\theta - 2g(z_{\text{max}} - h) \tag{2.6.15}$$

$$z_{\text{max}} = \boxed{h + \frac{{v_0}^2\sin^2\theta}{2g}} \tag{2.6.16}$$

Range x_{max}

Occurs when $z = 0$

$$0 = h + v_0\sin\theta t_f - \frac{1}{2}g{t_f}^2 \tag{2.6.17}$$

$$t_f = \frac{-v_0\sin\theta \pm \sqrt{{v_0}^2\sin^2\theta + 2gh}}{-g} \tag{2.6.18}$$

$$= \frac{v_0\sin\theta}{g} \mp \sqrt{\left(\frac{v_0\sin\theta}{g}\right)^2 + \frac{2h}{g}} \tag{2.6.19}$$

Remark
We have to chose the positive of the $\mp$ because larger time.

We notice that if $h = 0$ (more generally, $\Delta z = z_f - z_0 = 0$)

$$t_f = \frac{2v_0 \sin\theta}{g} = 2t_{\max} \implies \text{symmetry of } z(t) \text{ parabola} \tag{2.6.20}$$

From x-equation:

$$x_{\max} = \frac{{v_0}^2 \sin\theta\cos\theta}{g} + v_0\cos\theta\sqrt{\left(\frac{v_0\sin\theta}{g}\right)^2 + \frac{2h}{g}} \tag{2.6.21}$$

Use the identity $2\sin\theta\cos\theta = \sin(2\theta)$

Which gives us

$$\begin{aligned} x_{\max} &= \boxed{\frac{{v_0}^2\sin(2\theta)}{2g} + \sqrt{\left(\frac{{v_0}^2\sin(2\theta)}{2g}\right)^2 + \frac{2h{v_0}^2\cos^2\theta}{g}}} && (2.6.22)\\ &= \frac{{v_0}^2\sin(2\theta)}{2g}\left[1 + \sqrt{1 + \frac{2gh}{{v_0}^2\sin^2\theta}}\right] && (2.6.23)\end{aligned}$$

Now, if we solve for the case wehre $h = 0$, we get

$$x_{\max} = \frac{{v_0}^2\sin(2\theta)}{g} \tag{2.6.24}$$

c) We want $\vec{v}_f$, which is $\vec{v}_f = \vec{v}_0 - gt_f\hat{z}$

$$\vec{v}_f = v_0\cos\theta\hat{x} - \left(v_0\sin\theta\sqrt{1 + \frac{2gh}{{v_0}^2\sin^2\theta}}\right)\hat{z} \tag{2.6.25}$$

We can also write it in terms of magnitude and angle:

First to find the magnitude

$$\begin{aligned} {v_f}^2 &= {v_0}^2\cos^2\theta + {v_0}^2\sin^2\theta\left(1 + \frac{2gh}{{v_0}^2\sin^2\theta}\right) && (2.6.26)\\ &= {v_0}^2 + 2gh && (2.6.27)\\ \implies v_f &= \sqrt{{v_0}^2 + 2gh} && (2.6.28)\end{aligned}$$

Now, for the angle of the projectile when it hits the ground

$$\tan\theta_f = \left|\frac{v_{fz}}{v_{fx}}\right| = \tan\theta\sqrt{1+\frac{2gh}{{v_0}^2\sin^2\theta}} \tag{2.6.29}$$

$$\theta_f = \arctan\left[\tan\theta\sqrt{1+\frac{2gh}{{v_0}^2\sin^2\theta}}\right] \tag{2.6.30}$$

Remark
Notice now when $h = 0$, $\theta_f = \theta$.

d) Since $h = 0$, the range is $x_{\max} = \frac{{v_0}^2\sin(2\theta)}{g}$

We want to maximize, so we can think that $x_{\max} = x_{\max}(\theta)$ and find $\theta = \phi$ s.t. $\left.\frac{\mathrm{d}x_{\max}}{\mathrm{d}\theta}\right|_{\phi} = 0$

$$\left.\frac{2{v_0}^2\cos(2\theta)}{g}\right|_{\phi} = \frac{2{v_0}^2}{g}\cos(2\phi) = 0 \tag{2.6.31}$$

$$\implies \cos(2\phi) = 0 \tag{2.6.32}$$

$$\phi = \frac{\pi}{4} = 45\,\mathrm{deg} \tag{2.6.33}$$

Example 2.6.2
A hunter is trying to hunt a bear on a tree with heigh h distance d away. The moment the hunter shoots, the bear is scared and drops from the tree. What angle relative to the bear should the hunter aim at to hit the bear?

Solution:

We can consider the vertical component, which must match for the hunter's arrow to hit

$$y + 0 + v_{0_y}t - \frac{1}{2}gt^2 \tag{2.6.34}$$

$$v_0\sin\theta t - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2 \tag{2.6.35}$$

$$v_0\sin\theta t = h \tag{2.6.36}$$

$$t = \frac{h}{v_0\sin\theta} \tag{2.6.37}$$

then we plug the vertical to horizontal

$$\frac{h}{v_0 \sin\theta} \cos\theta = d \tag{2.6.38}$$

$$d = h \cot\theta \tag{2.6.39}$$

$$\theta = \boxed{\operatorname{arccot}\left(\frac{d}{h}\right)} \tag{2.6.40}$$

We notice that θ then is essentially directly at the bear.

Remark
Another way of thinking about it, is if we consider $g = 0$, then consider the problem, we would come to the conclusion that we should aim at the bear too. Adding g to both bodies shouldn't change that fact.

Since we also want the hunder to hit the bear before it hits the ground, we can find that

$$h = \frac{1}{2} g t^2 \tag{2.6.41}$$

$$t = \sqrt{\frac{2h}{g}} \tag{2.6.42}$$

$$t < \sqrt{\frac{2h}{g}} \tag{2.6.43}$$

Consequently

$$\sqrt{\frac{2h}{g}} v_0 \cos\theta = \sqrt{\frac{2h}{g}} \frac{d}{d^2 + h^2} v_0 \tag{2.6.44}$$

$$v_0 > \boxed{\sqrt{\frac{g(d^2 + h^2)}{2d}}} \tag{2.6.45}$$

Remark
The **Frenet-Serret Formulas** gives a way of finding motion only based on the particle's current motion relative to itself.

2.7 Kinematics in Plane Polar Coordinates

Definition 2.7.1

$$\vec{r}(t) = r\hat{r} = r(t)\hat{r}(t) \tag{2.7.1}$$
$$\dot{\vec{r}}(t) = \dot{r}\hat{r} + r\dot{\hat{r}} \tag{2.7.2}$$
$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{2.7.3}$$
$$= \dot{r}\hat{r} + r\omega\hat{\theta} \tag{2.7.4}$$
$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} \tag{2.7.5}$$
$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \tag{2.7.6}$$

Remark
If we have the trajectory being a circle, we have velocity $\vec{v} = r\omega\hat{\theta}$.

$\ddot{r}$ is radial acceleration, and $-r\dot{\theta}^2 = -r\omega^2$ is the centripetal acceleration.
$r\ddot{\theta}$ is angular acceleration ($\alpha \equiv \ddot{\theta}, r\alpha$), and $2\dot{r}\dot{\theta}$ is the Coriolis Acceleration[a]

[a] This is related to the Coriolis Effect in non-inertial frames.

Once again, we can see for circular motion

$$\dot{r} = \ddot{r} = 0 \tag{2.7.7}$$
$$\implies \vec{a} = (-r\dot{\theta}^2)\hat{r} + (r\ddot{\theta})\hat{\theta} = -r\omega^2\hat{r} + r\alpha\hat{\theta} \tag{2.7.8}$$

Note the above is from constant speed only

Remark
Three examples from the notes are not included.

Chapter 3

Newton's Laws

3.1 Dynamics

Newton's Laws provide the framework for the dynamics of classical particle motion.

Question of Classical Mechanics:

Given $\vec{r}_0$ and $\vec{v}_0$ of the particle, with mass m, determine its subsequent motion, $\vec{r}(t)$, for all time t.

3.1.1 Within Context

Newton originally formulated the laws to solve the question of gravity – along the way, he formulated concepts like forces and momentum.

3.2 Newton's Laws

Definition 3.2.1
The three laws of motion:

Law of Invertia A particle remains at rest or moving with constant velocity unless influenced by a force.

$\mathbf{F} = \mathbf{ma}$ The change in a particle's motion (i.e. its acceleration) is proportional to the force impressed, as vectors.

Action / Reaction Forces come in pairs: to every action by one particle on another, there is an equal and opposite force in return.

3.2.1 First Law

There exist *intertial frames of reference*, that is, a frame in which a *free particle*[1] has constant velocity.

Remark
Essentially, a frame at rest and a frame with constant velocity are the same.

[1] particle subject to absolutely no influences

Mathematically, this is expressed as

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}^2t} = 0 \tag{3.2.1}$$

3.2.2 Second Law

Denote the force by $\vec{F}$

Two different particels subject to the same force (e.g. a spring). After the unfluence of the force (e.g. left spring), particle 1 has speed v_1 and particle 2 has speed v_2.

Consider the ratio

$$\frac{v_1}{v_2} \equiv \frac{m_2}{m_1} \tag{3.2.2}$$

where m_i is an intrinsic property of the i-th particle we call its mass [unit: kg].

Assumption: m is independent of $\vec{F}$ and $\vec{v}$.

So we can write a relation:

$$m_1 v_1 = m_2 v_2 \tag{3.2.3}$$

Assume we start from rest and apply some force for some duration, then we have

$$m_1 \Delta v_1 = m_2 \Delta v_2 = F \Delta t \tag{3.2.4}$$

And thus we have

$$F\Delta t = m\Delta v \implies F = \frac{m\Delta v}{\Delta t} = \frac{\Delta(mv)}{\Delta t} \tag{3.2.5}$$

Definition 3.2.2
Define the (physical) **momentum** of a particle to be

$$\vec{p} = m\vec{v} \tag{3.2.6}$$

As so we have with Eq. (3.2.5) the following

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} \tag{3.2.7}$$

1. As $\Delta t \to 0$, we have that

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \tag{3.2.8}$$

2. Forces (empirical) obey the *principle of superposition.*

$$\vec{F}_{\text{net}} = \sum_i \vec{F}_i \tag{3.2.9}$$

Altogether we have that

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \tag{3.2.10}$$

If m is constant, then we have

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = m\vec{a} \implies \boxed{\vec{F}_{\text{net}} = m\vec{a}} \tag{3.2.11}$$

mass is a measure of an object's inertia – *tendency to persist in its state of motion.*

3.2.3 Third Law

Definition 3.2.3
A force is a directed influence between pairs of particles.

If force of 1 on 2 is $\vec{F}_{12}$,

then force of 2 on 1 is $\vec{F}_{21} = -\vec{F}_{12}$.

IMPORTANT: Forces always come in pairs! (e.g. When we are sitting on our seats, its us pushing on the seat, and the seat pushing on us. The force of us pushing on the seat comes from gravity.)

Example 3.2.1
Given $\vec{r}_0$, $\vec{v}_0$, and m, fint $\vec{r}(t)$
Newton's laws:

1. go to an intertial frame: $\vec{r}(t)$
2. Identify forces acting on particle: $\vec{F}$
3. Then we just solve the differential equation.

$$\vec{F}_{\text{net}} = m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} \tag{3.2.12}$$

Our initial conditions are the two givens.

3.3 Scenarios of Newton's Laws

3.3.1 Constant Forces

3.3.2 Variable Forces with Time

3.3.3 Variable Forces with Position

3.3.4 Variable Forces with Velocity