MATH 229: Calculus III for Engineers

Takahiro Sakai

Martin Gong / 七海喬介

Jan 8 – ??, 2024

Contents

1 Vector and the Geometry of Space **1**
- 1.1 3-Dimensional Space . . . 1
  - 1.1.1 2D Coordinates . . . 1
  - 1.1.2 3D Coordinates . . . 1
- 1.2 Vectors . . . 1
  - 1.2.1 Vector Operation . . . 2
  - 1.2.2 Components . . . 2
  - 1.2.3 Standard Basis Vectors . . . 2
- 1.3 The Dot Products . . . 2
  - 1.3.1 Law of Cosine . . . 3
  - 1.3.2 Projection . . . 3
  - 1.3.3 Work . . . 4
- 1.4 The Cross Product . . . 4
- 1.5 Lines and Planes . . . 6
- 1.6 Cylinders and Quadric Surfaces . . . 8
  - 1.6.1 Cylinders . . . 8
  - 1.6.2 Quadric Surface . . . 9
- 1.7 Vector Functions . . . 10
- 1.8 Arc Length and Curvature . . . 11

2 Partial Derivatives **12**
- 2.1 Functions of Several Variables . . . 12
- 2.2 Continuity . . . 12
- 2.3 Partial Derivatives . . . 12
- 2.4 Tangent Planes and Linear Derivatives . . . 13
- 2.5 The Chain Rule . . . 13
- 2.6 Maxima and Minima . . . 13
  - 2.6.1 Local Maxima and Minima . . . 13
  - 2.6.2 Absolute Maxima and Minima . . . 13
- 2.7 Lagrange Multiplier . . . 14

3 Vector Calculus **16**
- 3.1 Vector Fields . . . 16
- 3.2 Line Integrals . . . 16
- 3.3 Fundamental Theorem of Line Integrals . . . 16
- 3.4 Green's Theorem . . . 17

Chapter 1

Vector and the Geometry of Space

1.1 3-Dimensional Space

1.1.1 2D Coordinates

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\} \tag{1.1.1}$$

1.1.2 3D Coordinates

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} \tag{1.1.2}$$

Lemma 1.1.1 (Distance Between 2 Points)

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{1.1.3}$$

Proof. Easily proven by using the Pythagorean Theorem twice. ■

Lemma 1.1.2 (Spherical Surface)
Given point $C(a, b, c)$ and $P(x, y, z)$ where P is a point on the spherical surface and r is the radius of the sphere.

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \tag{1.1.4}$$

To define a solid spherical space

$$\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} \leq r \tag{1.1.5}$$

1.2 Vectors

Definition 1.2.1
[Vector] Vector is a quantity that has a **magnitude** and a **direction**.

We say that two vectors $\vec{u}$ and $\vec{v}$ are equal if they have the same length and direction.

1.2.1 Vector Operation

Omitted

1.2.2 Components

In $\mathbb{R}^2$

$$\vec{a} \equiv \langle a_1, a_2 \rangle \tag{1.2.1}$$

In $\mathbb{R}^3$

$$\begin{cases} \vec{a} & \equiv \langle a_1, a_2, a_3 \rangle \\ \vec{0} & \equiv \langle 0, 0, 0, \rangle \end{cases} \tag{1.2.2}$$

Definition 1.2.2

Length of $\vec{a} \equiv \langle a_1, a_2, a_3 \rangle$ is

$$|\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2} \tag{1.2.3}$$

1.2.3 Standard Basis Vectors

$$\begin{cases} \hat{i} & = \langle 1, 0, 0 \rangle \\ \hat{j} & = \langle 0, 1, 0 \rangle \\ \hat{k} & = \langle 0, 0, 1 \rangle \end{cases} \tag{1.2.4}$$

1.3 The Dot Products

Definition 1.3.1

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle \tag{1.3.1}$$

Then, the dot product is

$$\vec{a} \cdot \vec{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.3.2}$$

Properties

1. $\vec{a} \cdot \vec{a} = {a_1}^2 + {a_2}^2 + {a_3}^2 = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

Theorem 1.3.1

$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\theta \tag{1.3.3}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\left|\vec{b}\right|}, 0 \leq \theta \leq \pi \tag{1.3.4}$$

Lemma 1.3.2

- If $\vec{a} \cdot \vec{b} > 0$ then $\cos\theta > 0 \implies \theta < \frac{\pi}{2}$
- If $\vec{a} \cdot \vec{b} < 0$ then $\cos\theta < 0 \implies \theta > \frac{\pi}{2}$
- If $\vec{a} \cdot \vec{b} = 0$, then $\theta = \frac{\pi}{2}, \vec{a} \perp \vec{b}$

1.3.1 Law of Cosine

$$\left|\vec{a} - \vec{b}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2|\vec{a}|\left|\vec{b}\right| \cos\theta \tag{1.3.5}$$

Proof.

$$\left|\vec{a} - \vec{b}\right|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \tag{1.3.6}$$

$$= |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + \left|\vec{b}\right|^2 \tag{1.3.7}$$

$$= |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2ab\cos(\theta) \tag{1.3.8}$$

■

1.3.2 Projection

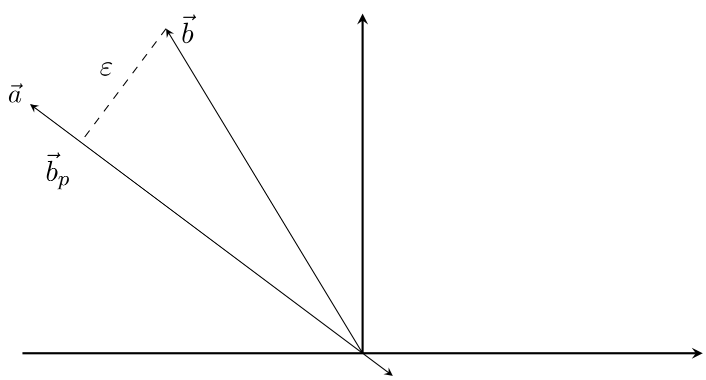

Figure 1.1: Projection

Add to this.

$$\left|\vec{b}\right| \tag{1.3.9}$$

Example 1.3.1

$$\vec{u} = \langle 1, 1, 2 \rangle \qquad \vec{v} = \langle -2, 3, 1 \rangle \tag{1.3.10}$$

Find projection of $\vec{u}$ onto $\vec{v}$

Solution:

$$\text{comp}_{\vec{c}}\vec{u} = \vec{u} \cdot \frac{\vec{v}}{|\vec{v}|} \tag{1.3.11}$$

$$= \frac{-2+3+2}{\sqrt{14}} = \frac{3}{\sqrt{14}} \tag{1.3.12}$$

$$\text{proj}_{\vec{v}}\vec{u} = (\text{comp}_{\vec{v}}\vec{u})\frac{\vec{v}}{|\vec{v}|} = \frac{3}{\sqrt{14}} \cdot \frac{\vec{v}}{\sqrt{v}} = \frac{3}{14}\vec{v} \tag{1.3.13}$$

1.3.3 Work

Move an an object from P to Q with a force $\vec{F}$ forming an angle θ with the displacement vector $\vec{D}$.

$$\text{Work} \equiv \text{Force} \times \text{Dist} \tag{1.3.14}$$

$$W = (\left|\vec{F}\right| \cos\theta)\left|\vec{D}\right| \tag{1.3.15}$$

$$= \left|\vec{F}\right|\left|\vec{D}\right| \cos\theta \tag{1.3.16}$$

$$= \vec{F} \cdot \vec{D} \tag{1.3.17}$$

$$\implies W = \vec{F} \cdot \vec{D} \tag{1.3.18}$$

Example 1.3.2
Move a particle from $P(2,1,0)[m]$ to $Q(4,6,2)$ with a force $\vec{F} = \langle 3,4,5\langle [N]$.
What is the work done by $\vec{F}$?

Solution:

$$W = \vec{F} \cdot \vec{PQ} \tag{1.3.19}$$

$$= \langle 3,4,5 \rangle \cdot \langle 2,5,2 \rangle \tag{1.3.20}$$

$$= 36\,\text{N}\,\text{m} \tag{1.3.21}$$

1.4 The Cross Product

Definition 1.4.1
Given the vectors

$$\vec{a} = \langle a_1, a_2, a_3 \rangle, \vec{b} = \langle b_1, b_2, b_3 \rangle \tag{1.4.1}$$

The cross product is defined as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \tag{1.4.2}$$

Properties of the Dot Product

1. $(\vec{a} \times \vec{b}) \perp \vec{a} \& \vec{b}$ and the direction follows the right-hand rule.
2. $\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \left|\vec{b}\right| \sin\theta, 0 \leq \theta \leq \pi$
3. $\left|\vec{a} \times \vec{b}\right| =$ the area of the parallelogram formed by the two vectors.
4. If $\vec{a} \parallel \vec{b}$, then $\vec{a} \times \vec{b} = \vec{0}$
5. Cross product of basis vectors

$$\begin{cases} \hat{i} \times \hat{j} & = \hat{k} \\ \hat{j} \times \hat{k} & = \hat{i} \\ \hat{k} \times \hat{i} & = \hat{j} \end{cases} \tag{1.4.3}$$

6. The cross product is not commutative
7. The cross product is not associative

Example 1.4.1

$$\begin{cases} \hat{i} \times (\hat{i} \times \hat{j}) & = \hat{i} \times \hat{k} = -\hat{j} \\ (\hat{i} \times \hat{i}) \times \hat{j} & = \vec{0} \times \hat{j} = \vec{0} \end{cases} \tag{1.4.4}$$

8. You can find the normal vector to a plane by applying the cross product to two non-parallel vectors on that plane.

Example 1.4.2
Given points

$$P(1, 4, 6), Q(-2, 5, 1), R(1, -1, 1)$$

that lie on a plane

a) Find the vector normal to the plane
b) Find the area of $\triangle PQR$

Solution:

TBA

Definition 1.4.2
[Triple Products]

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) \tag{1.4.5}$$

Eq. (1.4.5) shows the scalar triple product. This is also the volume of the parallelepiped.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{1.4.6}$$

Eq. (1.4.6) shows the vector triple product.

Lemma 1.4.1
If $\vec{a}, \vec{b}$, and $\vec{c}$ are on the same plane (*coplanar*), then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

1.5 Lines and Planes

Definition 1.5.1
[Line] We define a line with a direction vector $\vec{v} = \langle a, b, c \rangle$

$$\vec{r} = \vec{v}_0 + t\vec{v} \tag{1.5.1}$$

Parametric Form

$$\begin{cases} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{cases} \tag{1.5.2}$$

Symmetric Form

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{1.5.3}$$

Notice how the symmetric form does not require parameters, it tells the relationship between the coordinates.

Example 1.5.1
Intersection problem

Definition 1.5.2
[Plane] Given a point $P_0 \equiv \vec{r}_0$ and another point $P \equiv \vec{r}$ on the plane, along with the normal vector $\hat{n} = \langle a, b, c \rangle$.
Now, we see that $\vec{r} - \vec{r}_0$ is always on the plane, so that it follows that

$$\hat{n} \cdot (\vec{r} - \vec{r}_0) = 0 \tag{1.5.4}$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{1.5.5}$$
$$ax + by + cx = d \tag{1.5.6}$$

where $d = ax_0 + by_0 + cz_0$

Example 1.5.2
Given $A(2, 0, 3), B(0, -4, 6), C(-3, 6, 0)$, on a plane, find the equation of the plane.

Solution:

We find that

$$\overrightarrow{AB} \times \overrightarrow{AC} = -8\langle 2, 3, 4\rangle \tag{1.5.7}$$

We take any point and compute d

$$d = 2 \cdot 2 + 0 \cdot 3 + 3 \cdot 4 = 16 \tag{1.5.8}$$

so the equation is

$$2x + 3y + 4z = 16 \tag{1.5.9}$$

What if we want to sketch the plane?

We simply find the x, y, z-intersection of the plane, label them on a skeleton, then connect they for a triangle.

Example 1.5.3
Given two planes

$$\begin{cases} x + y + z &= 1 \\ x - 2y + 3z &= 1 \end{cases} \tag{1.5.10}$$

a) Find the angle between the two planes
b) Find the equation of the intersecting line

Solution:

a) We have the normal vectors

$$\begin{cases} \hat{n}_1 &= \langle 1, 1, 1\rangle \\ \hat{n}_2 &= \langle 1, -2, 3\rangle \end{cases} \tag{1.5.11}$$

We simply find the angle between them using the dot product.

$$\arccos\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right) = \arccos\left(\frac{2}{\sqrt{42}}\right) \tag{1.5.12}$$

b) We need the direction vector and a point on the line.

We can find a point on the line by defining either x, y, or z for the two equations and solve for the other variables. (e.g. A point here on the line is $P(1, 0, 0)$)

For the direction vector, we can cross the normal vectors $\vec{n_1} \times \vec{n_2}$ to find the vector.

Definition 1.5.3
[Distance Between a Point and a Plane] Given some point P and a random point A on the plane, we can have some vector $\overrightarrow{AP}$, which, if we project onto the normal vector $\hat{n}$ of the plane, will give us the component of the vector $\overrightarrow{AP}$ parallel to the normal vector.

$$d = \left|\overrightarrow{AP}\right| |\cos\theta| = \left|\overrightarrow{AP}\cdot\hat{n}\right| \tag{1.5.13}$$

where θ is the angle between the $\overrightarrow{AP}$ and $\hat{n}$

Example 1.5.4
We want to find the distance between two paralle planes.

Solution:

Simply find a vector that "connects" the two planes, let that vector be $\vec{v}$. Then, calculate $|\vec{v}\cdot\hat{n}|$ where $\hat{n}$ is the normal vector of the plane.

1.6 Cylinders and Quadric Surfaces

1.6.1 Cylinders

The perimeter of different cross sections of a cylinder are called **traces**, and the lines parallel to the cylindrical axis are called **???**.

Example 1.6.1

$$x = z^2 \tag{1.6.1}$$

This is a parabolic cylinder; we see a parabolic curve on the xz-plane.

Example 1.6.2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.6.2}$$

This is a cylindrical surface with intersect a and b on x and y respectively.

1.6.2 Quadric Surface

Remark
Spherical surfaces are a type of quadric surface.

Example 1.6.3
[Ellipsoid]

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 \tag{1.6.3}$$

Solution:

Let $z = k$ some constant.

Then,

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \tag{1.6.4}$$

We observe that only $-2 \leq k \leq 2$ do we see a surface. So the z intersects are -2 and 2. Now we can solve for $x|_{y=0}$ and $y|_{x=0}$ and we find $\pm 1, \pm 3$

We can keep finding the range by turning the equation into the form $c_1{x_1}^2 + c_2{x_2}^2 = c_3$.

When $x = k$ we get

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \tag{1.6.5}$$

When $y = k$ we get

$$x^2 + \frac{z^2}{4} = 1 - \frac{y^2}{9} \tag{1.6.6}$$

This is called an ellipsoid.

Example 1.6.4
[Elliptic Parabola]

$$z = 4x^2 + y^2 \tag{1.6.7}$$

Solution:

We notice that this surface only exists when $z \geq 0$.

Then, we can just sketch traces and connect.

Let $z = k$

$$4x^2 + y^2 = k \tag{1.6.8}$$

We see an ellipse cross section that grows as k increases.

Let $x = 0$ or $y = 0$

We see a parabolic cross section.

This is an **elliptic parabola**.

Example 1.6.5
[Hyperbolic Parabola]

$$z = y^2 - x^2 \tag{1.6.9}$$

Example 1.6.6
[Elliptic Cone]

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{1.6.10}$$

1.7 Vector Functions

Definition 1.7.1
[Vector Function]

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \tag{1.7.1}$$

Example 1.7.1
The vector function of a line.

$$\vec{r}_0 + \vec{v}t = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle \tag{1.7.2}$$

Definition 1.7.2
[Derivative of Vector Functions]

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \equiv \vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h)\vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle \tag{1.7.3}$$

Definition 1.7.3
[Rules of Differentiation]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[a\vec{u}(t) + b\vec{v}(t)\right] = a\vec{u}'(t) + b\vec{v}'(t) \tag{1.7.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \tag{1.7.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t) \tag{1.7.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t) \tag{1.7.7}$$

1.8 Arc Length and Curvature

Chapter 2

Partial Derivatives

2.1 Functions of Several Variables

Definition 2.1.1
[Function of 2 Variables]

$$z = f(x, y) \tag{2.1.1}$$

Where z is the dependent variable and x, y are independent variables.
Defines a point $(x, y, f(x, y))$ in $\mathbb{R}^3$
We have the domain of f in $\mathbb{R}^2$ and its range in $\mathbb{R}$

2.2 Continuity

Skipped

2.3 Partial Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x, y) \implies \frac{\partial f}{\partial x} \equiv f_x(x, y) \tag{2.3.1}$$

We simply hold all other variables as constant and derive with respect to x.

Definition 2.3.1
[Partial Derivative]

$$\frac{\partial f}{\partial x} \equiv f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \tag{2.3.2}$$

2.4 Tangent Planes and Linear Derivatives

2.5 The Chain Rule

Definition 2.5.1
If

$$y = f(x_1, x_2, x_3, \cdots, x_n) \qquad x_i = g_i(t_1, t_2, t_3, \cdots, t_n), i = [\![1, n]\!] \tag{2.5.1}$$

then

$$\frac{\partial u}{\partial t_k} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_k} \tag{2.5.2}$$

Theorem 2.5.1
[Implicit Function]
Case 1 Single Variable Case
Case 2 Multivariable Case
We have

$$F(x, y, z) = C, z = z(x, y) \tag{2.5.3}$$

We want $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$.

$$F(x, y, z(x, y)) = C \qquad \frac{\partial}{\partial x} \tag{2.5.4}$$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \tag{2.5.5}$$

We obtain

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}} \tag{2.5.6}$$

2.6 Maxima and Minima

2.6.1 Local Maxima and Minima

2.6.2 Absolute Maxima and Minima

In single variable Caluclus, we find all the critical points and the endpoints of interval, then compare the values.

Example 2.6.1
$f(x, y) = xy - x - 2y + 8$
Find absolute max/min in D. Let D be the region in the first quadrant bounded by the curve $y = -x + 4$.

Solution:

$$\begin{cases} f_x &= y - 1 = 0 \\ f_y &= x - 2 = 0 \end{cases} \tag{2.6.1}$$

Re interpret the critical point as $(2, 1)$

$f(2, 1) = 6$

Then we have to look at different segments of the boundaries:

1. $x = 0, 0 \leq y \leq 4$

   $f(0, y) = -2y + 8$

   $f(0, 0) = 8$

   $f(0, 4) = 0$

2. $y = 0, 0 \leq x \leq 4$

   $f(x, 0) = -x + 8$

   $f(4, 0) = 4$

3. $y = 4 - x, 0 \leq x \leq 4$

   $f|_{y=4-x} = -x^2 + 5x$

   Now we can find the critical points of this function to find the extrema on the edge.

2.7 Lagrange Multiplier

Example 2.7.1
Find the extrema of $f(x, y)$ subject to constraint $g(x, y) = k$.

This is the general from of the problem we are trying to solve – of course this could be of three variables.

The condition for optimization is

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \tag{2.7.1}$$

where λ is called the **Lagrangian Multiplier**.

Remark
This statement means that the gradient of the constraint function is parallel to the gradient of the function we are trying to optimize. Why is the point at which this is true the point where the function f is maximized on the curve?

We can agree that when the gradient of level curve/surface of constraint $g(x, y, z) = k$ is parallel to the gradient of the function at some point, the tangent (plane) to that level curve/surface is perpendicular to the gradient of function at that point. At an infinitesimal scale, some increment at that point along the tangent (plane) of constraint will not result in an increase in the value of the function since, given the definition of the gradient, we are currently perpendicular to steepest rate of change, which also mean we are moving on an infinitesimal level segment. In other words, the tangent (planes) of the constraint and the function are parallel (this statement is equivalent to what we started with, but is another way to understand it). Again, any movement along the tangent of the the constraint curve would result in no change in value of the function, thus we must be at an extrema.

Now, continuing with how to utilize the lagrangian multiplier, we can obtain n equations for n dimensions, and an additional equation from our constraint. So we have $n + 1$ unknowns (with the addition of λ) and $n + 1$ equations.

We then have to solve by guessing possible values while thinking about whether certain variable values make sense.

Chapter 3

Vector Calculus

3.1 Vector Fields

3.2 Line Integrals

3.3 Fundamental Theorem of Line Integrals

Given a conservative vector field $\vec{F} = \nabla f$, how do we evaluate the following?

$$\begin{aligned}
\int_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, \mathrm{d}t && (3.3.1) \\
&= \int_a^b \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} \right) \mathrm{d}t && (3.3.2) \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t), z(t)) \, \mathrm{d}t && (3.3.3) \\
&= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a)) && (3.3.4)
\end{aligned}$$

Theorem 3.3.1
[Fundamental Theorem of Line Integrals]

$$\int_C \nabla f \cdot \mathrm{d}\vec{r} = f(B) - f(A) \tag{3.3.5}$$

Remark
If $\vec{F} = \nabla f$, then $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ does not depend on the shape of path C.

For a conservative field

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = 0 \tag{3.3.6}$$

So when is a conservative field conservative?

Lemma 3.3.2
$\mathbb{R}^2$ test: Given $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$

$\vec{F}$ is conservative if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in its domain.

Proof. If $\vec{F}$ is conservative, then $\nabla f = \vec{F}$ i.e. $f_x = P$ and $f_y = Q$.

Then we have that $P_y = (f_x)_y$ and $Q_x = (f_y)_x$, which should be equal. ∎

3.4 Green's Theorem

Given some vector function

$$\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath} \tag{3.4.1}$$

and consider

$$\oint_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} + \oint_{C_2} \vec{F} \cdot \mathrm{d}\vec{r} = \oint_{C_{1+2}} \vec{F} \cdot \mathrm{d}\vec{r} \implies \sum_n \oint_{C_n} \vec{F} \cdot \mathrm{d}\vec{r} = \oint_{C} \vec{F} \cdot \mathrm{d}\vec{r} \tag{3.4.2}$$

Green's Theorem looks at the circulation of a closed area in the positive orientation.