

PHYS 161 Lecture Notes
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Chapter 1

Mathematical Interlude

Definition 1.0.1

Kinematics is the study of motion without regard to its cause.

1.1 Units & Dimensions

In *Classical Mechanics* all quantities are expressed in terms of three dimensions, and we use SI units to define them:

- length – meters, m
- time – seconds, s
- mass – kilograms, kg

How do we measure distance? Sometimes it is easier to use the **point particle** approximation where we think of an object just as a point object with all of its mass concentrated at that point.

1.2 Coordinate System

A **coordinate system** is a collection of coordinate axis & a point called the origin.

A coordinate system is often called a **frame of reference**.

Physics should apply in whatever coordinate system (**covariant**), so scalars, vectors, tensors, ...

1.2.1 Cartesian Coordinates

$$\{(x, y, z) | x, y, z \in \mathbb{R}\} \quad (1.1)$$

1.2.2 Spherical Coordinates

$$\{(r, \theta, \phi) | 0 \leq r \leq \infty, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\} \quad (1.2)$$

where ϕ is the angle of the radius deviating from the z -axis and θ is the deviation from the x -axis.

The coordinate conversions are

$$\begin{cases} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arctan(y/x) \\ \phi &= \arctan(\sqrt{x^2 + y^2}/z) \end{cases} \quad (1.3)$$

1.2.3 Cylindrical Coordinates

$$\{(s, \theta, z) | 0 \leq s \leq \infty, 0 \leq \theta \leq 2\pi, -\infty \leq z \leq \infty\} \quad (1.4)$$

$$\begin{cases} s &= \sqrt{x^2 + y^2} \\ \theta &= \arctan(y/x) \\ z &= z \end{cases} \quad (1.5)$$

1.3 Position Vectors

The position of a particle can be specified by its *unique* coordinates or by a **position vector**, \vec{r} .

A vector is just an arrow, an arrow is a vector – a geometric quantity.

Definition 1.3.1 (Vector)

A **vector** is a directed line segment, i.e. an arrow.

A vector has both **magnitude** and **direction**.

1.4 Vector Algebra

Notation

\vec{A} the vector

$A = |\vec{A}|$ the magnitude

$\hat{A} = \vec{A}/A$ direction / unit vector

Remark

Technically, magnitude cannot be negative, but notation wise we do that anyways. $-\vec{A} = A(-\hat{A})$

1.4.1 Vector Addition

$$\vec{C} = \vec{A} + \vec{B} \quad (1.6)$$

Note that addition is commutative and associative.

1.4.2 Vector Subtraction

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \quad (1.7)$$

Final - Initial

1.4.3 Vector Multiplication

Dot product

$$\vec{A} \cdot \vec{B} = AB \cos(\theta) \quad (1.8)$$

Facts:

- if $\vec{A} \perp \vec{B} \iff \vec{A} \cdot \vec{B} = 0$
- if $\vec{A} \parallel \vec{B} \iff \vec{A} \cdot \vec{B} = AB$ is maximal
- $$\begin{cases} \vec{A} \cdot \vec{B} > 0 & \implies \text{point in similar directions} \\ \vec{A} \cdot \vec{B} < 0 & \implies \text{point in opposite directions} \end{cases}$$
- $\vec{A} \cdot \vec{A} = A^2$

Also defined component wise

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i \quad (1.9)$$

Example 1.4.1

Prove the law of cosines.

Consider the triangle, ABC where θ is the angle between vectors \vec{A} and \vec{B} .

$$c^2 = a^2 + b^2 - 2ab \cos(\theta) \quad (1.10)$$

Proof. Define $\vec{A}, \vec{B}, \vec{C}$ by $A = a, B = b, C = c; \vec{C} = \vec{A} - \vec{B}$

Then,

$$\vec{C} \cdot \vec{C} = C^2 = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) \quad (1.11)$$

$$= A^2 - 2\vec{A} \cdot \vec{B} + B^2 \quad (1.12)$$

$$= a^2 + b^2 - 2ab \cos(\theta) \quad (1.13)$$

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Cross Product

$$\vec{A} \times \vec{B} \equiv AB \sin(\theta) \hat{n} \quad (1.14)$$

Facts:

- If $\vec{A} \parallel \vec{B}$ or antiparallel $\implies \vec{A} \times \vec{B} = 0$
- If $\vec{A} \perp \vec{B} \implies \vec{A} \times \vec{B}$ is maximal.
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- $\vec{A} \times \vec{A} = \vec{0}$

Also defined component wise as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (1.15)$$

1.5 Components of Vectors Basis Vectors

Say we have in Cartesian coordinates (x, y)

$$\vec{A} = \vec{A}_x + \vec{A}_y \quad (1.16)$$

Then,

$$\begin{cases} A_x &= A \cos \theta \\ A_y &= A \sin \theta \end{cases} \quad (1.17)$$

$$\begin{cases} \vec{A}_x &= A \cos \theta \vec{x} \\ \vec{A}_y &= A \sin \theta \vec{y} \end{cases} \quad (1.18)$$

$$\begin{cases} \hat{x} &= \langle 1, 0, 0 \rangle \\ \hat{y} &= \langle 0, 1, 0 \rangle \\ \hat{z} &= \langle 0, 0, 1 \rangle \end{cases} \quad (1.19)$$

1.6 Vectors in Different Basis**1.6.1 Cartesian Coordinates**

This is to say the same vectors but different components represented in different coordinates.

We can express them in the same way where θ is the original relative angle and θ' is the new relative angle:

$$\begin{cases} \vec{A} &= A \cos \theta \hat{x} + A \sin \theta \hat{y} \\ \vec{A}' &= A \cos \theta' \hat{x} + A \sin \theta' \hat{y} \end{cases} \quad (1.20)$$

Now, say we want to express our components in a different basis that rotates our standard basis by an angle of ϕ in the counterclockwise direction.

$$\begin{bmatrix} A'_x \\ B'_x \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} A_x \\ B_x \end{bmatrix} \quad (1.21)$$

1.6.2 Polar Coordinates

We have two basis vectors defined by the following

$$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} \quad (1.22)$$

\hat{r} is in the direction

The conversion between the bases of Cartesian and Polar are the following:

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \quad (1.23)$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} \quad (1.24)$$

Remark

It can be useful because coordinates will be much easier to express with

$$\vec{A} = A(\theta, r) \hat{r} \quad (1.25)$$

1.7 Calculus with Vectors

$$\frac{d\vec{A}}{dt} \equiv \lim_{\Delta t \rightarrow 0} \frac{\vec{A}(t + \Delta t) - \vec{A}(t)}{\Delta t} \quad (1.26)$$

$\vec{A}(t)$ generally changes in magnitude and direction and this does capture both.

There are two cases:

Case 1: $\vec{A}(t)$ changes in magnitude only

Then $d\vec{A}$ is parallel to $\vec{A}(t)$ (or antiparallel).

Let $\frac{d\vec{A}}{dt} \parallel \vec{A}$ the component of $\frac{d\vec{A}}{dt} \parallel \vec{A}$.

then here

$$\left\| \frac{d\vec{A}_{\parallel}}{dt} \right\| = \frac{dA}{dt} \quad (1.27)$$

Case 2: $\vec{A}(t)$ changes in direction only

Then $d\vec{A}$ is perpendicular to $\vec{A}(t)$ (Almost, if we see the angle as small enough, the $d\vec{A}$ would be at a right angle).

Call $\frac{d\vec{A}_{\perp}}{dt}$ the component of $\frac{d\vec{A}}{dt} \perp \vec{A}(t)$.

then here

$$\left\| \frac{d\vec{A}_{\perp}}{dt} \right\| = A \frac{d\theta}{dt} \quad (1.28)$$

Generally

$$\frac{d\vec{A}}{dt} = \frac{d\vec{A}_{\parallel}}{dt} + \frac{d\vec{A}_{\perp}}{dt} \quad (1.29)$$

But $\vec{A} = A\hat{A}$ is naively

$$\frac{d\vec{A}}{dt} = \frac{dA}{dt}\hat{A} + A\frac{d\hat{A}}{dt} \quad (1.30)$$

and

$$\frac{d\vec{A}_{\parallel}}{dt} = \frac{dA}{dt}\hat{A} \quad \frac{d\vec{A}_{\perp}}{dt} = A\frac{d\hat{A}}{dt} \quad (1.31)$$

1.7.1 With Cartesian Components

Derivative

$$\vec{A}(t) = A_x(t)\hat{x} + A_y(t)\hat{y} \rightarrow \frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{x} + \frac{dA_y}{dt}\hat{y} \quad (1.32)$$

Notation

$$\dot{f} \equiv \frac{df}{dt} \quad f' = \frac{df}{dx} \quad \text{space derivative} \quad (1.33)$$

Hence

$$\dot{\vec{A}} = \dot{A}_x\hat{x} + \dot{A}_y\hat{y} \quad (1.34)$$

Integral

$$\int \vec{A}(t) dt \equiv \left(\int A_x dt \right) \hat{x} + \left(\int A_y dt \right) \hat{y} \quad (1.35)$$

Note that the fundamental theorem of calculus still applies.

1.7.2 With Polar Components

$$\vec{A}(t) = A_r(t)\hat{r}(t) + A_\theta(t)\hat{\theta}(t) \quad (1.36)$$

Then

$$\frac{d\vec{A}}{dt} = \frac{dA_r}{dt}\hat{r} + A_r \frac{d\hat{r}}{dt} + \frac{dA_\theta}{dt}\hat{\theta} + A_\theta \frac{d\hat{\theta}}{dt} \quad (1.37)$$

If we derive Eq. (1.23), we obtain

$$\begin{cases} \dot{\hat{r}} &= (-\sin \theta)\dot{\theta}\hat{x} + (\cos \theta)\dot{\theta}\hat{y} = \dot{\theta}\hat{\theta} \\ \dot{\hat{\theta}} &= (-\cos \theta)\dot{\theta}\hat{x} + (-\sin \theta)\dot{\theta}\hat{y} = -\dot{\theta}\hat{r} \end{cases} \quad (1.38)$$

which means that

$$\dot{\hat{r}} = \dot{\theta}\hat{\theta} \quad \dot{\hat{\theta}} = -\dot{\theta}\hat{r} \quad (1.39)$$

which makes sense if we think about it.

And if we put it together

$$\dot{\vec{A}} = \dot{A}_r\hat{r} + A_r\dot{\hat{r}} + \dot{A}_\theta\hat{\theta} + A_\theta\dot{\hat{\theta}} \quad (1.40)$$

$$\implies \dot{\vec{A}} = (\dot{A}_r - A_\theta\dot{\theta})\hat{r} + (A_r\dot{\theta} + \dot{A}_\theta)\hat{\theta} \quad (1.41)$$

Chapter 2

Kinematics

We have our position vector

$$\vec{r}(t) = (x(t), y(t)) \quad (2.1)$$

We use \vec{r} because it seems natural, it is the direction we are pointing in.

Remark

Sometimes when reference to radial \vec{r} is misleading, we use $\vec{x}(t)$.

The change of the vector in space across time sweeps over some **trajectory**.

2.1 Displacement

Definition 2.1.1 (Displacement)

The *displacement vector* $\Delta\vec{r}$ is a measure of where the particle went (which depends on the origin!).

$$\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = \vec{r}(t_f) - \vec{r}(t_i) \quad (2.2)$$

1. $\|\Delta\vec{r}\| \neq$ distance travelled in general
 - distance traveled = arc length of trajectory
2. $\Delta\vec{r}$ is coordinate independent.

Take two coordinate systems S and S' . Let them be defined with the relation $\vec{r} = \vec{r}' + \vec{R}$ where \vec{r} and \vec{r}' are vectors in the respective coordinate systems.

$$\begin{cases} S : & \Delta\vec{r} = \vec{r}_f - \vec{r}_i \\ S' : & \Delta\vec{r}' = \vec{r}'_f - \vec{r}'_i \end{cases} \quad (2.3)$$

If we plug in the relation, we realize that they are the same, $\Delta\vec{r} = \Delta\vec{r}'$

2.2 Velocity

Definition 2.2.1 (Average Velocity)

$$\vec{v}_{\text{avg}} \equiv \frac{\Delta \vec{r}}{\Delta t} \quad (2.4)$$

Let $d\vec{r}$ be the infinitesimal displacement.

When we consider a smaller interval:

$$\lim_{\Delta t \rightarrow 0} \implies \|d\vec{r}\| = dr \quad (\text{distance traveled}) \quad (2.5)$$

A small change to t results in a small change in dS (the distance / speed), proportionally

$$dS \propto dt \quad (2.6)$$

$$\implies dS = \left(\frac{dS}{dt} \right) dt \quad (2.7)$$

Definition 2.2.2 (Velocity)

AKA the *instantaneous velocity*

$$\vec{v}(t) \equiv \frac{d\vec{r}}{dt} \quad (2.8)$$

- $\|\vec{v}\|$ = speed
- \hat{v} = direction of motion

Remark

A note on average velocity:

$$\vec{v}_{\text{avg}} = \frac{1}{\Delta t} \int_{t_i}^{t_f} \vec{v}(t) dt = \frac{1}{\Delta t} \int_{t_i}^{t_f} \frac{d\vec{r}}{dt} dt = \frac{\Delta \vec{r}}{\Delta t} \quad (2.9)$$

Note also if we find the magnitude, it would not be the same as the average speed since the norm would go over the integrals instead of what is being integrated.

- \vec{v} a vector, so write $\vec{v}(t) = \dot{x}\hat{x} + \dot{y}\hat{y} = \dot{\vec{r}}$
- Compare to frames of reference, S & S'

Suppose $\dot{\vec{R}} \neq 0$.

Then we have

$$\begin{cases} \vec{r} &= \vec{r}' + \vec{R} \\ \vec{v} &= \vec{v}' + \vec{V} \end{cases} \quad (2.10)$$

This is known as the Galilean transformations, which, at higher velocities, “translates” to the Lorentz transformations.

We can also obtain $\vec{r}(t)$ given $\vec{v}(t)$

$$\Delta\vec{r} = \int d\vec{r} = \int_{t_i}^{t_f} \vec{v} dt \quad (2.11)$$

and

$$\vec{r}(t) = \vec{r}_i + \vec{v}_i(t - t_i) \quad (2.12)$$