Compendium of Mathematics & Physics

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Introduction

Goals

This compendium originated out of the necessity for a compact summary of important theorems and formulas during physics and mathematics classes at university. When the interest in more (and more exotic) subjects grew, this collection lost its compactness and became the chaos it now is. Although there should exist some kind of overall structure, it was not always possible to keep every section self-contained or respect the order of the chapters.

It should definitely not be used as a formal introduction to any subject. It is neither a complete work nor a fact-checked one, so the usefulness and correctness is not guaranteed. However, it can be used as a look-up table for theorems and formulas, and as a guide to the literature. To this end, each chapter begins with a list of useful references. At the same time, only a small number of statements are proven in the text (or appendices). This was done to keep the text as concise as possible (a failed endeavour). However, in some cases the major ideas underlying the proofs are provided.

Structure and conventions

Sections and statements that require more advanced concepts, in particular concepts from later chapters or (higher) category theory, will be labelled by the *clubs* symbol ♣. Some definitions, properties or formulas are given with a proof or an extended explanation whenever I felt like it. These are always contained in a blue frame to make it clear that they are not part of the general compendium. When a section uses notions or results from a different chapter at its core, this will be recalled in a green box at the beginning of the section.

Definitions in the body of the text will be indicated by the use of **bold font**. Notions that have not been defined in this summary but that are relevant or that will be defined further on in the compendium (in which case a reference will be provided) are indicated by *italic text*. Names of authors are also written in *italic*.

Objects from a general category will be denoted by a lower-case letter (depending on the context, upper-case might be used for clarity), functors will be denoted by upper-case letters and the categories themselves will be denoted by symbols in **bold font**. In the later chapters on physics, specific conventions for the different types of vectors will often be adopted. Vectors in Euclidean space will be denoted by a bold font letter with an arrow above, e.g. $\vec{\mathbf{a}}$, whereas vectors in Minkowski space (4-vectors) and differential forms will be written without the arrow, e.g. $\mathbf{a}$. Matrices and tensors will always be represented by capital letters and, dependent on the context, a specific font will be adopted.

Chapter 1

Calculus

For the section on umbral calculus, see di Bucchianico (1998).

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1.1 General definitions

Definition 1.1.1 (Domain). A connected, open subset of $\mathbb{R}^n$. (Not to be confused with the domain of a function as in **??**.)

Definition 1.1.2 (Factorial).

$$n! := n(n-1)\cdots 1\,, \tag{1.1}$$

where $n \in \mathbb{N}$. The convention is that $0! = 1$. (This, for example, agrees with the combinatorial result that there is a unique way to order zero objects.)

Definition 1.1.3 (Envelope). Consider a set $\mathcal{F}$ of real-valued functions with common domain X. An envelope (function) for $\mathcal{F}$ is any function $F : X \rightarrow \mathbb{R}$ such that

$$\forall f \in \mathcal{F}, x \in X : |f(x)| \leq F(x)\,. \tag{1.2}$$

Theorem 1.1.4 (Binomial theorem).

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{1.3}$$

for all $n \in \mathbb{N}$.

1.2 Continuity

Definition 1.2.1 (Darboux function). A function that has the intermediate value property (**??**).

Theorem 1.2.2 (Darboux). *Every differentiable function defined on a closed interval is Darboux.*

Corollary 1.2.3 (Bolzano). If $f(a) < 0$ and $f(b) > 0$ (or vice versa), there exists at least one point x_0 for which $f(x_0) = 0$.

Theorem 1.2.4 (Weierstrass's extreme value theorem). *Let $I = [a,b]$ be a closed interval and let $f : I \rightarrow \mathbb{R}$ be a continuous function. Then f attains a minimum and maximum at least once on I.*

Definition 1.2.5 (Absolute continuity). A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be absolutely continuous if for every $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that for every finite collection of disjoint intervals $]x_i, y_i[$ satisfying

$$\sum_i (y_i - x_i) < \delta_\varepsilon\,, \tag{1.4}$$

the function f satisfies

$$\sum_i |f(y_i) - f(x_i)| < \varepsilon\,. \tag{1.5}$$

Property 1.2.6. The different types of continuity form the following hierarchy:

$$\text{Lipschitz-continuous} \subset \text{absolutely continuous} \subset \text{uniformly continuous} \subset \text{continuous}\,.$$

Definition 1.2.7 (Function of bounded variation). A function f is said to be of bounded variation on the interval $[a, b]$ if the following quantity is finite:

$$V_{a,b}(f) := \sup_{P\in\mathcal{P}} \sum_{i=0}^{|P|-1} |f(x_{i+1}) - f(x_i)|\,, \tag{1.6}$$

where the supremum is taken over all partitions of $[a, b]$.

Property 1.2.8. Every function of bounded variation can be decomposed as the difference of two monotonically increasing functions.

Example 1.2.9. Every absolutely continuous function is of bounded variation.

1.3 Convergence

Definition 1.3.1 (Pointwise convergence). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions. The sequence is said to converge pointwise to a limit function f if

$$\forall x \in \mathsf{dom}(f_n) : \lim_{n\to\infty} f_n(x) = f(x)\,. \tag{1.7}$$

Definition 1.3.2 (Uniform convergence). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions. The sequence is said to converge uniformly to a limit function f if

$$\lim_{n\to\infty} \sup_{x\in\mathsf{dom}(f_n)} |f_n(x) - f(x)| = 0\,. \tag{1.8}$$

Definition 1.3.3 (Limit superior). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The limit superior is defined as follows:

$$\limsup_{n\to\infty} x_n := \inf_{n\geq 1} \sup_{k\geq n} x_k\,. \tag{1.9}$$

Definition 1.3.4 (Limit inferior). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The limit superior is defined as follows:

$$\liminf_{n\to\infty} x_n := \sup_{n\geq 1} \inf_{k\geq n} x_k\,. \tag{1.10}$$

Property 1.3.5. A sequence $(x_n)_{n\in\mathbb{N}}$ converges pointwise if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n\,. \tag{1.11}$$

1.4 Series

Definition 1.4.1 (Series). Consider a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$. For every index $n \in \mathbb{N}$, consider the **partial sum**

$$S_n := \sum_{k=0}^{n} x_n\,. \tag{1.12}$$

The would-be limit $S_\infty := \sum_{k=0}^{+\infty} x_n$ is called the series associated to $(x_n)_{n\in\mathbb{N}}$. If the induced sequence $(S_n)_{n\in\mathbb{N}}$ converges, the series is said to be **convergent** with limit S_∞. Otherwise, the series is said to be **divergent**.

Definition 1.4.2 (Absolute/conditional convergence). Consider a series

$$S = \sum_{k=0}^{+\infty} x_k \,. \tag{1.13}$$

If the associated series $S' = \sum_{k=1}^{+\infty} |x_k|$ converges, so does $S = \sum_{k=1}^{+\infty} x_k$. In this case, S is said to be absolutely convergent. If S converges but S' does not, S is said to be conditionally convergent.

Property 1.4.3 (Riemann rearrangement theorem). If a series $\sum_{k=0}^{+\infty} x_k$ is conditionally convergent, then for any number $\lambda \in \overline{\mathbb{R}}$, there exist a permutation π of $(x_n)_{n\in\mathbb{N}}$ such that

$$\sum_{k=0}^{+\infty} x_{\pi(k)} = \lambda \,. \tag{1.14}$$

1.4.1 Convergence tests

Property 1.4.4. A necessary condition for the convergence of a series $\sum_{i=1}^{+\infty} a_i$ is that

$$\lim_{n\to\infty} a_n = 0 \,. \tag{1.15}$$

Definition 1.4.5 (Majorizing series). Let $S_a = \sum_{i=1}^{+\infty} a_i$ and $S_b = \sum_{i=1}^{+\infty} b_i$ be two series. The series S_a is said to majorize S_b if for every $k > 0$ the partial sums satisfy $S_{a,k} \geq S_{b,k}$, i.e.

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \tag{1.16}$$

for all $k \in \mathbb{N}$.

Method 1.4.6 (Comparison test). Let S_a, S_b be two series such that S_a majorizes S_b.

- If S_b diverges, then S_a diverges.
- If S_a converges, then S_b converges.
- If S_b converges, nothing can be said about S_a.
- If S_a diverges, nothing can be said about S_b.

Method 1.4.7 (Maclaurin–Cauchy integral test). Let f be a nonnegative, continuous and monotonically decreasing function defined on the interval $[n, +\infty[$ for some $n \in \mathbb{N}$. If $\int_n^{+\infty} f(x)\,dx$ is convergent, so is $\sum_{k=n}^{+\infty} f(k)$. On the other hand, if the integral is divergent, so is the series.

Remark 1.4.8. The function does not have to be nonnegative and decreasing on the complete interval. As long as it does on the interval $[N, +\infty[$ for some $N \geq n$, the statement holds. This can be seen by writing $\sum_{k=n}^{+\infty} f(k) = \sum_{k=n}^{N} f(k) + \sum_{k=N}^{+\infty} f(k)$ and noting that the first term is always finite (and similarly for the integral).

Property 1.4.9. If the integral in the previous theorem converges, the series is bounded in the following way:

$$\int_n^{+\infty} f(x)\,dx \leq \sum_{i=n}^{+\infty} a_i \leq f(n) + \int_n^{+\infty} f(x)\,dx\,. \tag{1.17}$$

Method 1.4.10 (d'Alembert's ratio test). Consider the quantity

$$R := \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|. \tag{1.18}$$

The following cases can be distinguished:

- $R < 1$: the series converges absolutely.
- $R > 1$: the series does not converge.
- $R = 1$: the test is inconclusive.

Method 1.4.11 (Cauchy's root test). Consider the quantity

$$R := \limsup_{n\to\infty} \sqrt[n]{|a_n|}\,. \tag{1.19}$$

The following cases can be distinguished:

- $R < 1$: the series converges absolutely.
- $R > 1$: the series does not converge.
- $R = 1$ and the limit approaches strictly from above: the series diverges.
- $R = 1$: the test is inconclusive.

Definition 1.4.12 (Radius of convergences). The number $1/R$ is called the radius of convergence.

Remark 1.4.13. The root test is stronger than the ratio test. However, if the ratio test can determine the convergence of a series, the radius of convergence of both tests will coincide and, hence, it is a well-defined quantity.

Method 1.4.14 (Gauss's test). If $a_n > 0$ for all $n \in \mathbb{N}$, one can write the ratio of successive terms as follows:

$$\left|\frac{a_n}{a_{n+1}}\right| = 1 + \frac{h}{n} + \frac{B(n)}{n^k}\,, \tag{1.20}$$

where $k > 1$ and $B(n)$ is a bounded function when $n \longrightarrow \infty$. The series converges if $h > 1$ and diverges otherwise.

Definition 1.4.15 (Asymptotic expansion). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A series $\sum_{i=0}^{+\infty} a_n x^n$ is called an asymptotic expansion of f if there exists an $N \in \mathbb{N}$ such that

$$f(x) - \sum_{i=0}^{n} a_i x^i = O(x^{n+1}) \tag{1.21}$$

for all $x \in \mathbb{R}$ and $n \geq N$.

1.4.2 Geometric series and resummation

Definition 1.4.16 (Geometric series). Consider a number $r \in \mathbb{R}$. The geometric series induced by r is given by

$$S_n = \sum_{k=0}^{n} r^k \,. \tag{1.22}$$

The parameter r is often called the **growth rate**.

It can be shown that the series is convergent for $|r| < 1$ with value

$$\sum_{k=0}^{+\infty} r^k = \frac{1}{1-r} \,. \tag{1.23}$$

Example 1.4.17 (Grandi's series). For $r = -1$, the geometric series becomes Grandi's series. This series has alternating partial sums:

$$S_n = \begin{cases} 1 & \text{if } n \in 2\mathbb{Z}, \\ 0 & \text{if } n \in 2\mathbb{Z} + 1 \,. \end{cases} \tag{1.24}$$

As such, this series is divergent.

However, there is a way to assign a value to Grandi's series. Either through an intuitive, nonformal approach, or through a rigourous analytic approach, both leading to the same result. For the former, one uses associativity of addition. For convergent series, associativity still holds, but for divergent series this can only be formally applied. Brackets can be added in two ways[1]:

$$\begin{aligned} S_1 &= 1 + (-1 + 1) + (-1 + 1) + \cdots = 1 \,, \\ S_2 &= (1 - 1) + (1 - 1) + \cdots = 0 \,. \end{aligned}$$

The 'renormalized' value of S is then given by the average of these possibilities:

$$S \doteq \frac{S_1 + S_2}{2} = \frac{1}{2} \,. \tag{1.25}$$

[1] This 'trick' is very similar to the *Eilenberg–Mazur swindle* from algebraic topology.

The rigorous approach considers Eq. (1.23) for the geometric series. Taking the limit $r \longrightarrow -1$ gives

$$S \doteqdot \lim_{r \to -1} \frac{1}{1-r} = \frac{1}{2}\,. \tag{1.26}$$

In fact, this latter resolution is an example of *analytic continuation* (see Definition 2.4.5).

A similar issue arises for $r = 2$. This gives the geometric series

$$S = 1 + 2 + 4 + 8 + \cdots\,. \tag{1.27}$$

This series is clearly divergent, but analytic continuation also allows to assign a 'renormalized' value:

$$S \doteqdot \lim_{r \to 2} \frac{1}{1-x} = -1\,. \tag{1.28}$$

Some other approaches exist to assign finite, 'renormalized' values to divergent series.

Definition 1.4.18 (Summation). A summation method is a function $\Sigma : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ that assigns a definite value to a sequence of partial sums.

To be deemed a 'good' summation method, some properties are usually required:

1. **Regularity**: If $S_n \longrightarrow S$, then $\Sigma(S_n)_{n\in\mathbb{N}} = S$.
2. **Linearity**: $\Sigma(\lambda S_n + T_n)_{n\in\mathbb{N}} = \lambda\Sigma(S_n)_{n\in\mathbb{N}} + \Sigma(T_n)_{n\in\mathbb{N}}$ for all $\lambda \in \mathbb{R}$.
3. **Stability**: $\Sigma(S_n)_{n\in\mathbb{N}} = S_0 + \Sigma(S_{n+1} - S_0)_{n\in\mathbb{N}}$. This means that if $(S_n)_{n\in\mathbb{N}}$ are the partial sums of a sequence $(x_n)_{n\in\mathbb{N}}$, one can calculate the series with x_0 omitted and add it afterwards.
4. **Finite reindexability**: If $(S_n)_{n\in\mathbb{N}}, (T_n)_{n\in\mathbb{N}}$ are two sequences such that $T_n = S_{\pi(n)}$ for some permutation $\pi \in \mathrm{Aut}(\mathbb{N})$ that only acts nontrivially on a finite initial segment of $\mathbb{N}$, then $\Sigma(S_n)_{n\in\mathbb{N}} = \Sigma(T_n)_{n\in\mathbb{N}}$.

Method 1.4.19 (Césaro summation). Consider a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and its associated partial sums $(S_n)_{n\in\mathbb{N}}$. The sequence $(x_n)_{n\in\mathbb{N}}$ is said to be Césaro summable if the arithmetic means

$$\overline{S}_n := \frac{1}{n}\sum_{k=0}^{n} S_n \tag{1.29}$$

converge, and its Césaro sum is given by $S \doteqdot \lim_{n\to\infty} \overline{S}_n$. If a series is convergent, it is Césaro summable and the results coincide.

Remark 1.4.20. Grandi's series is Césaro summable, but the geometric series for $r = 2$ is not.

Remark 1.4.21. Césaro summation satisfies all the above regularity conditions, but, for example, *Borel summation* (see Definition 1.6.7 further below) is not stable.

1.5 Differentiation

Formula 1.5.1 (Derivative). Consider a function $f : \mathbb{R} \to \mathbb{R}$. If it exists, the following limit is called the derivative of f at $x \in \mathbb{R}$:

$$\frac{\mathrm{d}f}{\mathrm{d}x} \equiv f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\,. \tag{1.30}$$

If the derivative exists at every point of some interval I, then f is said to be differentiable on I. For multivariate functions $f : \mathbb{R}^n \to \mathbb{R}$, one can similarly define the partial derivatives:

$$\frac{\partial f}{\partial x_i} := \frac{f(x+he_i)-f(x)}{h}\,, \tag{1.31}$$

where e_i is the i^{th} coordinate vector, i.e. the partial derivatives determine the rate of change in the coordinate directions.

Notation 1.5.2. Iterated derivatives are often denoted as follows:

$$f^{(i)}(x) := \frac{\mathrm{d}^i f}{\mathrm{d}x^i}\,. \tag{1.32}$$

Theorem 1.5.3 (Mean value theorem). *Let f be a continuous function defined on the closed interval $[a,b]$ and differentiable on the open interval $]a,b[$. There exists a point $c \in\,]a,b[$ such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}\,. \tag{1.33}$$

Definition 1.5.4 (Differentiablity class). A function $f : \mathbb{R} \to \mathbb{R}$ is said to be of class C^n if it is $n \in \mathbb{N}$ times continuously differentiable, i.e. $f^{(i)}$ exists and is continuous for $i = 1,\dots,n$. Multivariate functions are said to be of class C^n if all of their partial derivatives are of class C^{n-1} or, by recursion, if all mixed partial derivatives up to order n exists and are continuous.

Definition 1.5.5 (Smooth function). A function f is said to be smooth if it is of class C^∞.

Theorem 1.5.6 (Boman). *Consider a function $f : \mathbb{R}^d \to \mathbb{R}$. If, for every smooth function $g : \mathbb{R} \to \mathbb{R}^d$, the composition $f \circ g$ is smooth, the function f is also smooth.*

Property 1.5.7 (Taylor expansion). Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Around every point $x \in \mathbb{R}$, one can express f as the following series:

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2}(y-x)^2 + \cdots = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n\,. \tag{1.34}$$

For the special case $x = 0$, the name **Maclaurin series** is sometimes used. A similar expression exists for multivariate functions, where derivatives are replaced by partial derivatives.

Definition 1.5.8 (Analytic function). A function f is said to be analytic if it is smooth and if its Taylor series expansion around any point x converges to f in some neighbourhood of x. The set of analytic functions defined on V is denoted by $C^\omega(V)$.

Theorem 1.5.9 (Hadamard lemma). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function defined on an open, star-convex set U. One can expand the function as follows:*

$$f(x) = f(x_0) + \sum_{i=1}^{n} (x^i - x_0^i) g_i(x_0)\,, \tag{1.35}$$

where all functions g_i are also smooth on U.

From this expression, one can also see that the functions g_i, evaluated at 0, give the partial derivatives of f. These functions are sometimes called the **Hadamard quotients**.

Remark 1.5.10. This lemma gives a finite-order approximation of the Taylor expansion of f.

Theorem 1.5.11 (Schwarz[2]). *Consider a function $f \in C^2(\mathbb{R}^n, \mathbb{R})$. The mixed partial derivatives of f coincide for all indices $i, j \leq n$:*

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right). \tag{1.36}$$

Formula 1.5.12 (Derivative of $f(x)^{g(x)}$). Consider a function of the form

$$u(x) = f(x)^{g(x)}\,,$$

with $f, g : \mathbb{R} \to \mathbb{R}$ differentiable. After taking the logarithm and applying the standard rules of differentiation, one can obtain the following expression:

$$\left(f(x)^{g(x)}\right)' = f(x)^{g(x)}\left(g'(x)\ln\left[f(x)\right] + \frac{g(x)}{f(x)} f'(x)\right). \tag{1.37}$$

Definition 1.5.13 (Euler operator). On the space $C^{n>1}(\mathbb{R}^n, \mathbb{R})$, the Euler operator $\mathbb{E}$ is defined as follows:

$$\mathbb{E} := \sum_{i=1}^{n} x_i \frac{\partial}{\partial x^i}\,. \tag{1.38}$$

Theorem 1.5.14 (Euler). *Let f be a homogeneous function, i.e.*

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^n f(x_1, \dots, x_n)\,. \tag{1.39}$$

This function satisfies the following equality:

$$\mathbb{E}(f) = n f(x_1, \dots, x_n)\,. \tag{1.40}$$

[2] Also called **Clairaut's theorem**.

1.6 Integration theory

1.6.1 Riemann integral

Definition 1.6.1 (Improper Riemann integral).

$$\int_{-\infty}^{+\infty} f(x)\,dx := \lim_{\substack{a\to-\infty\\ b\to+\infty}} \int_a^b f(x)\,dx \tag{1.41}$$

One-sided improper integrals are defined in a similar fashion.

Theorem 1.6.2 (First fundamental theorem of calculus). *Let f be a continuous function defined on an open interval I and consider any number $c \in I$. Then the integral*

$$F(x) = \int_c^x f(x')\,dx' \tag{1.42}$$

is differentiable (and uniformly continuous) and gives an antiderivative of f:

$$\forall x \in I : F'(x) = f(x)\,. \tag{1.43}$$

Remark 1.6.3. The function F in the previous theorem is called a **primitive (function)** of f. Remark that F is just 'a' primitive function, since adding a constant to F does not change anything because the derivative of a constant is zero (the number $c \in \mathbb{R}$ was arbitrary).

Theorem 1.6.4 (Second fundamental theorem of calculus[3]). *Consider an integrable function $f : [a, b] \to \mathbb{R}$. If $F : [a, b] \to \mathbb{R}$ is an antiderivative of f, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)\,. \tag{1.44}$$

Formula 1.6.5 (Euler–MacLaurin). Consider a function $f \in C^p\big([0, n]\big)$ for some $n \in \mathbb{N}$. The integral of f can be approximated as follows:[4]

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac{f(n)+f(0)}{2} + \sum_{k=1}^{\lfloor p/2\rfloor} \frac{B_{2k}}{(2k)!}\big(f^{(2k-1)}(n) - f^{(2k-1)}(0)\big) + R_p\,, \tag{1.45}$$

where R_p is a remainder term and B_{2k} are the *Bernoulli numbers* (see Definition 1.8.1 further below).

Formula 1.6.6 (Differentiation under the integral sign[5]).

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,y)\,dy = f\big(x, b(x)\big)b'(x) - f\big(x, a(x)\big)a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\,dy \tag{1.46}$$

[3]Sometimes called the **Newton–Leibniz** theorem.

[4]This formula gives a correction to the ordinary trapezium rule.

[5]This is a more general version of the *Leibniz integral rule*.

Definition 1.6.7 (Borel transform). Consider the following function:

$$F(x) := \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n . \tag{1.47}$$

If

$$\int_0^{+\infty} e^{-t} F(xt)\, dt < +\infty \tag{1.48}$$

for all $x \in \mathbb{R}$, then F is called the Borel transform of

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n . \tag{1.49}$$

Furthermore, the integral gives a convergent expression for f.

Proof. The function F is defined as follows:

$$F(x) := \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n .$$

The Borel transform gives:

$$\begin{aligned}
\int_0^{+\infty} F(xt) e^{-t}\, dt &= \sum_{n=0}^{+\infty} \int_0^{+\infty} \frac{a_n}{n!} x^n t^n e^{-t}\, dt \\
&= \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n \int_0^{+\infty} t^n e^{-t}\, dt \\
&= \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n\, \Gamma(n+1) \\
&= \sum_{n=0}^{+\infty} a_n x^n ,
\end{aligned}$$

where the definition of the Gamma function (Formula 1.6.10) was used on line 3, and the relation (1.6.13) between the factorial function and the Gamma function was used on line 4. □

Theorem 1.6.8 (Watson). *The Borel transform F is unique if the function f is holomorphic (see Definition 2.2.1) on the domain* $\{z \in \mathbb{C} \mid |\arg(z)| < \frac{\pi}{2} + \varepsilon\}$.

1.6.2 Euler integrals

Formula 1.6.9 (Beta function). The beta function (also known as the **Euler integral of the first kind**) is defined as follows:

$$B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1}\, dt . \tag{1.50}$$

Formula 1.6.10 (Gamma function). The gamma function (also known as the **Euler integral of the second kind**) is defined as follows:

$$\Gamma(x) := \int_0^{+\infty} t^{x-1}e^{-t}\,dt\,. \tag{1.51}$$

Formula 1.6.11. The following formula relates the beta and gamma functions:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,. \tag{1.52}$$

Property 1.6.12 (Recursion). The gamma function satisfies the following recursion relation for all points x in its domain:

$$\Gamma(x+1) = z\Gamma(x)\,. \tag{1.53}$$

Formula 1.6.13 (Factorial). For integers $n \in \mathbb{N}$, the gamma function can be expressed in terms of the factorial (Definition 1.1.2):

$$\Gamma(n) = (n-1)!\,. \tag{1.54}$$

Formula 1.6.14 (Stirling). This formula (originally stated for the factorial of natural numbers) gives an asymptotic expansion of the gamma function:

$$\ln\Gamma(z) \approx z\ln z - z + \frac{1}{2}\ln\left(\frac{2\pi}{z}\right). \tag{1.55}$$

Property 1.6.15 (Euler's reflection formula). When $x \notin \mathbb{Z}$, the following formula holds:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}\,. \tag{1.56}$$

1.6.3 Gaussian integrals

Formula 1.6.16 (n-dimensional Gaussian integral). An integral of the form

$$I\left(A,\vec{\mathbf{b}}\right) := \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\vec{\mathbf{x}}\cdot A\vec{\mathbf{x}} + \vec{\mathbf{b}}\cdot\vec{\mathbf{x}}\right) d^n x\,, \tag{1.57}$$

where A is a real symmetric matrix. By performing the transformation $\vec{\mathbf{x}} \to A^{-1}\vec{\mathbf{b}} - \vec{\mathbf{x}}$ and diagonalizing A, one can obtain the following expression:

$$I\left(A,\vec{\mathbf{b}}\right) = \sqrt{\frac{(2\pi)^n}{\det(A)}}\exp\left(\frac{1}{2}\vec{\mathbf{b}}\cdot A^{-1}\vec{\mathbf{b}}\right). \tag{1.58}$$

More generally, one has the following result:

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\vec{\mathbf{x}}\cdot A\vec{\mathbf{x}}\right) f(\vec{\mathbf{x}})\,d^n x = \sqrt{\frac{(2\pi)^n}{\det(A)}}\exp\left(\frac{1}{2}\sum_{i,j=1}^n A^{-1}_{ij}\partial_i\partial_j\right) f(\vec{\mathbf{x}})\Bigg|_{\vec{\mathbf{x}}=0}\,. \tag{1.59}$$

This result is sometimes called **Wick's lemma**.

Corollary 1.6.17. A functional generalization is given by:

$$\begin{aligned} I(iA, iJ) &= \int \exp\left(-i\int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi(x)A(x,y)\varphi(y)\, d^n x\, d^n y + i\int_{\mathbb{R}^n} \varphi(x)J(x)\, d^n x\right)[d\varphi] \\ &= C \det(A)^{-1/2} \exp\left(\frac{i}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} J(x)A^{-1}(x,y)J(y)\, d^n x\, d^n y\right), \end{aligned} \tag{1.60}$$

where the analytic continuation $I(iA, iJ)$ of Eq. (1.58) was used. One should pay attention to the normalization factor C which is infinite in general.

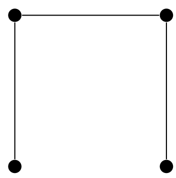
Method 1.6.18 (Feynman diagrams). The expansion of the exponential in the general expression for Gaussian integrals admits a diagrammatic expression. Let $f(\vec{\mathbf{x}})$ be a polynomial function of the coordinates.

If the number of factors in a monomial is odd, the resulting integral will vanish (since the integral of an odd function over an even domain is zero). For an even number of factors, one gets the following expression:

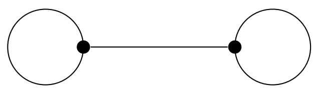
$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\vec{\mathbf{x}}\cdot A\vec{\mathbf{x}}\right) x^{i_1}\cdots x^{i_k}\, d^n x = \sqrt{\frac{(2\pi)^n}{\det(A)}} \sum_{\sigma\in S_k} A^{-1}_{\sigma(i_1)\sigma(i_2)}\cdots A^{-1}_{\sigma(i_{k-1})\sigma(i_k)}. \tag{1.61}$$

To every coordinate dimension, one can assign a vertex in the plane, i.e. the object x_i can be interpreted as a real-valued function on the set of $k\in\mathbb{N}$ elements. The sum on the right-hand side above can then be expressed as a 'sum' over all possible diagrams, where a factor A^{-1}_{ij} is represented by a line connecting the vertices i and j.

Example 1.6.19 (Feynman diagrams). Some simple examples are given:

$A^{-1}_{13}A^{-1}_{12}A^{-1}_{24} =$ 

Higher powers of a given coordinate would then, for example, give rise to diagrams with loops at a given vertex:

$A^{-1}_{11}A^{-1}_{12}A^{-1}_{22} =$ 

Remark 1.6.20 (Normalization). In practice, one often divides all Gaussian integrals by the quantity $I(A, 0)$ to cancel the normalization factor. In the functional setting, this is even imperative since, as mentioned above, the normalization factor diverges for infinite-dimensional spaces.

1.6.4 Generalizations

[6] Also called the **Perron**, **Lusin**, **(narrow) Denjoy** or **gauge** integral.

Definition 1.6.21 (Henstock–Kurzweil integral[6]). Consider the usual definition of the (proper) Riemann integral, where tagged partitions P of $[a,b]$ are chosen and the integral is obtained as the limit of the Riemann sums

$$I = \sum_P f(x_i)(t_i - t_{i-1}) \tag{1.62}$$

as the mesh size of the partitions goes to zero.

Now, to obtain the generalized integral, consider a strictly positive function $\delta : [a,b]\mathbb{R}^{>0}$, the **gauge function**. Given such a gauge, a tagged partition P is said to be **δ-fine** if

$$[t_{i-1}, t_i] \subset [x_i - \delta(x_i), x_i + \delta(x_i)] \tag{1.63}$$

for subintervals in the partition.[7]

If the integral exists, it is given by the number $I \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to \mathbb{R}^{>0}$ such that, if P is δ-fine, then

$$\left| I - \sum_P f(x_i)(t_i - t_{i-1}) \right| < \varepsilon\,. \tag{1.64}$$

Remark 1.6.22 (Riemann integral). If the gauge functions are chosen to be constant, the classical (ε,δ)-definition of ordinary Riemann integrals is obtained.

The following statement can be seen as a refinement of **??**. Moreover, it is also sometimes known as the **Borel–Lebesgue theorem**.

Property 1.6.23 (Cousin). For every gauge $\delta : [a,b] \to \mathbb{R}^{>0}$, there exists a δ-fine partition.

Property 1.6.24 (Integrability). If $f : [a,b] \to \mathbb{R}$ is bounded, then the following are equivalent:

- f is Henstock–Kurzweil integrable, and
- f is *Lebesgue integrable* (see **?? ??**).

More generally, a function $f : [a,b] \to \mathbb{R}$ is Henstock–Kurzweil integrable if and only if both f and $|f|$ are *Lebesgue integrable*.

The following property shows that 'improper' Henstock–Kurzweil integrals are only truly improper for unbounded domains.

Property 1.6.25 (Hake).

$$\int_a^b f\,dx = \lim_{c\nearrow b} \int_a^c f\,dx\,, \tag{1.65}$$

whenever either side exists.

[7]If the condition $x_i \in [t_{i-1}, t_i]$ in the definition of tagged partitions is dropped, the **McShane integral** is obtained. This can be shown to be equivalent to the *Lebesgue integral* (see **?? ??**).

One of the most important arguments for using the Henstock–Kurzweil integral is its refinement of the Second Fundamental Theorem of Calculus 1.6.4. Note that the theorem for the Riemann integral required that the derivative was integrable. The gauge integral relaxes this condition.

Theorem 1.6.26 (Second fundamental theorem of calculus). *Let $f : [a,b] \to \mathbb{R}$ be differentiable, then*

$$\int_a^x f'(x')\,dx' = f(x) - f(a) \;\; a.e. \tag{1.66}$$

1.7 Convexity

Definition 1.7.1 (Convex set). A subset of X of a vector space V (**??**) is said to be convex if $x,y \in X$ implies that $\{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\} \subset X$, i.e. if all straight lines connecting elements of the set are completely contained in that set. The **convex hull** of a subset X is defined as the smallest convex subset containing X.

Definition 1.7.2 (Extreme point). Consider a convex set X. The extreme points of X are the points $p \in X$ such that, if

$$p = \lambda p_1 + (1-\lambda)p_2 \tag{1.67}$$

for some $p_1, p_2 \in X$ and $\lambda \in [0,1]$, then $p_1 = p_2 = p$.

Definition 1.7.3 (Convex function). Let X be a convex set. A function $f : X \to \mathbb{R}$ is said to be convex if for all $x,y \in X$ and $\lambda \in [0,1]$:

$$f\big(\lambda x + (1-\lambda)y\big) \leq t\lambda(x) + (1-\lambda)f(y)\,. \tag{1.68}$$

For the definition of a **concave** function, the inequality has to be turned around.

Definition 1.7.4 (Linear map). A function $f : X \to \mathbb{R}$ is linear if and only if it is both convex and concave.

Theorem 1.7.5 (Karamata's inequality). *Consider an interval $I \subset \mathbb{R}$ and let $f : I \to \mathbb{R}$ be a convex function. If $(x_1, \dots, x_n)$ is a tuple that majorizes $(y_1, \dots, y_n)$, i.e.*

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i \tag{1.69}$$

and

$$x_{(1)} + \cdots + x_{(k)} \geq y_{(1)} + \cdots + y_{(k)} \tag{1.70}$$

for all $k \leq n$, where $x_{(i)}$ denotes the i^{th} largest element of $(x_1, \dots, x_n)$, then

$$\sum_{i=1}^n f(x_i) \geq \sum_{i=1}^n f(y_i)\,. \tag{1.71}$$

The following inequality can be derived directly from the definition of convexity by induction.

Theorem 1.7.6 (Jensen's inequality). *Let f be a convex function and consider a point $\{a_i\}_{i\leq n}$ in the probability simplex Δ^n (**??**).*

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} a_i f(x_i)\,. \tag{1.72}$$

Definition 1.7.7 (Legendre transformation). Consider a function $f:\mathbb{R}\to\mathbb{R}$. In certain cases (especially in physics) it is sometimes useful to replace the argument x by the slope of f at x, i.e. to perform the transformation

$$x \longrightarrow f'(x)\,. \tag{1.73}$$

However, it should be clear that this transformation is not always well-defined and, even if it is, it does not always preserve all the information contained in f.

These conditions are satisfied exactly if f is convex (or concave). In this case, the Legendre transform of f is defined as

$$f^*(x^*) := \sup_x \big(x^* x - f(x)\big)\,. \tag{1.74}$$

Now, consider the case where f is differentiable. The above supremum can then be obtained by differentiating the right-hand side and equating it to zero. This results in $x^* = f'(x)$, which is exactly the transformation that was required. By expressing everything in terms of the Legendre tranformed quantity x^*, one can also find the derivative of f^*:

$$\frac{\mathrm{d}f^*}{\mathrm{d}x^*}(x^*) = x(x^*)\,. \tag{1.75}$$

Property 1.7.8 (Alternative characterization). In fact, up to an additive constant, the condition

$$(f^*)' = (f')^{-1} \tag{1.76}$$

uniquely determines the Legendre transformation.

Remark 1.7.9. These definitions can easily be extended to higher dimensions ($n \geq 2$).

1.8 Umbral calculus

1.8.1 Special functions

Definition 1.8.1 (Bernoulli polynomial). The Bernoulli polynomials $B_n(x)$, for $n\in\mathbb{N}$, are generated as follows:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{+\infty} B_n(x)\frac{t^n}{n!}\,. \tag{1.77}$$

The **Bernoulli numbers** are defined as

$$B_n := B_n(0)\,. \tag{1.78}$$

Property 1.8.2 (Recursion).

$$B_n(x) = \sum_{k=0}^{+\infty} \binom{n}{k} B_k x^{n-k} \tag{1.79}$$

or, more generally,

$$B_n(x+y) = \sum_{k=0}^{+\infty} \binom{n}{k} B_k(x) y^{n-k} \tag{1.80}$$

Formula 1.8.3 (Differential formula).

$$B_n(x) = \frac{D}{e^D - 1} x^n\,, \tag{1.81}$$

where $D \equiv \frac{\mathrm{d}}{\mathrm{d}x}$.

The recursion formula shows that Bernoulli polynomials are a concrete example of a more general class of polynomial sequences.

Definition 1.8.4 (Appell sequence). A sequence of polynomials $(p_n)_{n\in\mathbb{N}} \subset \mathbb{R}[x]$ that satisfies

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) y^{n-k}\,. \tag{1.82}$$

This is equivalent to the following recursion relation

$$\frac{\mathrm{d}p_n}{\mathrm{d}x} = np_{n-1}\,. \tag{1.83}$$

The second recursion relation for Appell sequences can be generalized to arbitrary linear operators. For so-called *delta operators*, i.e. generalized derivative operators such as the *forward differencing operator*, another important class of polynomials is recovered.

Definition 1.8.5 (Sheffer sequence). Let $Q : \mathbb{R}[x] \to \mathbb{R}[x]$ be a **delta operator**, i.e. a shift-equivariant operator on polynomial functions that lowers the degree by 1:

$$Q\big(p(x+\lambda)\big) = Qp(x+\lambda) \tag{1.84}$$

and

$$\mathrm{deg}(Qp) = \mathrm{deg}(p) - 1 \tag{1.85}$$

for all $\lambda \in \mathbb{R}$ and $p \in \mathbb{R}[x]$. A polynomial sequence $(p_n)_{n\in\mathbb{N}} \subset \mathbb{R}[x]$ is said to be a Sheffer sequence (or a **poweroid**) if

$$Qp_n = np_{n-1} \tag{1.86}$$

for some delta operator Q.

Example 1.8.6. Many common polynomials form Sheffer sequences, a few examples are Abel, Bernoulli, Euler, Hermite and Laguerre polynomials.

Property 1.8.7. Since $\deg(p_n) = n$ for Sheffer sequences, every Sheffer sequence constitutes a basis for $\mathbb{R}[x]$.

Property 1.8.8 (Umbral composition). The Sheffer sequences form a group under the following composition law:

$$(p \circ q)_n(x) := \sum_{k=0}^{n} a_{n,k} q_k(x) = \sum_{0 \leq k \leq l \leq n} a_{n,k} b_{k,l} x^l \tag{1.87}$$

where $p_n = \sum_{k=0}^{n} b_{n,k} x^k$ and $q_n = \sum_{k=0}^{n} a_{n,k} x^k$. The identity element of the umbral group is given by the standard monomials $p_n(x) = x^n$.

Definition 1.8.9 (Binomial type). A polynomial sequence $(p_n)_{n\in\mathbb{N}} \subset \mathbb{R}[x]$ is said to be of binomial type if[8]

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \,. \tag{1.88}$$

A Sheffer sequence is a polynomial of binomial type if and only if

- $p_0(x) = 1$, and
- $p_n(0) = 0$ for all $n \in \mathbb{N}_0$.

Property 1.8.10 (Semidirect product). The umbral group is a semidirect product (**??**) of the (Abelian) subgroup of Appell sequences and the subgroup of binomial type.

This gives rise to a general recursion relation. Let $(s_n)_{n\in\mathbb{N}}$ be a Sheffer sequence and consider the unique sequence $(p_n)_{n\in\mathbb{N}}$ of binomial-type polynomials in the same coset as s.

$$s_n(x+y) = \sum_{k=0}^{n} p_k(x) s_{n-k}(y) \tag{1.89}$$

1.8.2 Umbrae

Now, what about the term 'umbral calculus'? In the 19th century, some mathematicians noticed the apparent similarity between the binomial theorem 1.1.4 for monomials and the recursion relation 1.8.2 for Bernoulli polynomials. (This is, of course, explained by the fact that they are both Appell sequences.) These relations allowed to treat exponents as indices and were, in turn, used to 'prove' various statements about special polynomials. Passage from one sequence to another happens through a linear operator, as shown by *Rota*:

$$L_{p,q} : \mathbb{R}[x] \to \mathbb{R}[x] : p_n \mapsto q_n \,. \tag{1.90}$$

[8]By rescaling the polynomials p_n by $n!$, an ordinary convolution formula is obtained.

Definition 1.8.11 (Umbral operator). A linear operator $L : \mathbb{R}[x] \to \mathbb{R}[x]$ is said to be umbral if there exist Sheffer sequences $(p_n)_{n\in\mathbb{N}}, (q_n)_{n\in\mathbb{N}}$ such that

$$Lp_n = q_n \tag{1.91}$$

for all $n \in \mathbb{N}$.

For the relation between monomials and Bernoulli polynomials, the operator is

$$L_B : x^n \mapsto B_n\,. \tag{1.92}$$

Property 1.8.12. Polynomial sequences of binomial type are closed under the action of umbral operators. This implies that, if the binomial-type sequence $(p_n)_{n\in\mathbb{N}}$ is associated to the delta operator Δ, the sequence $(Lp_n)_{n\in\mathbb{N}}$ is associated to the delta operator $L\Delta L^{-1}$. Accordingly, if Δ admits the series expansion[9]

$$\Delta = \sum_{k=0}^{+\infty} \frac{(\Delta x^k)(0)}{k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k} \equiv q\left(\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{1.93}$$

then

$$L\Delta L^{-1} = q\left(L\frac{\mathrm{d}}{\mathrm{d}x}L^{-1}\right). \tag{1.94}$$

The umbral composition rule is then simply given by the composition 1.8.8 of two such linear operators. More generally, the umbral composition can be extended to the entire $\mathbb{R}$-vector space of polynomial functions, sometimes called the **umbral algebra** in this setting:

$$(L_A \circ L_B)x^n := \sum_{k=0}^{n} \binom{n}{k}(L_A x^k)(L_B x^{n-k})\,. \tag{1.95}$$

Example 1.8.13 (Probability theory). Umbral calculus also has a relation to *probability theory* (see **??**), where the linear operator L is given by the *expectation value* E (see **??**). Here, the random variable X is the *umbra* for the sequence of moments $m_n := \mathsf{E}[X^k]$ and, hence, for the *probability distribution* P_X.

@@ NEEDS EXPANDING @@

[9]Such an expansion always exists.

Chapter 2

Complex Analysis

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2.1 Complex algebra

The set of complex numbers $\mathbb{C}$ forms a 2-dimensional *vector space* over the field of real numbers (see **??** **??**). At the same time, the operations of complex addition and complex multiplication also turn the complex numbers into a field.

Definition 2.1.1 (Complex conjugate). Complex conjugation

$$\overline{a+bi} := a - bi \tag{2.1}$$

is an involution (**??**). It is sometimes denoted by z^* instead of $\bar{z}$, but, unless this would cause confusion, the former notation will be used throughout this compendium.

Formula 2.1.2 (Real/imaginary part). A complex number can also be written as

$$\Re(z) + i\Im(z)\,, \tag{2.2}$$

where

$$\Re(z) := \frac{z + \bar{z}}{2}, \tag{2.3}$$

$$\Im(z) := \frac{z - \bar{z}}{2i}. \tag{2.4}$$

Note that the notations $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are also frequently used.

Definition 2.1.3 (Argument and modulus). Since $\mathbb{C} \cong \mathbb{R}^2$, every complex number can be represented as a point in the plane. Now, consider this **polar form**of a complex number:

$$z = re^{i\theta} \tag{2.5}$$

The number $\theta \in [0, 2\pi[$ is called the argument of z and it is denoted by $\arg(z)$. The number $r \in \mathbb{R}^+$ is called the modulus of Z and is denoted by $|z|$.

Definition 2.1.4 (Riemann sphere). Consider the one-point compactification $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (**??**). This set is called the Riemann sphere or **extended complex plane**. The standard operations on $\mathbb{C}$ can be generalized to $\overline{\mathbb{C}}$ for all nonzero $z \in \mathbb{C}$ in the following way:

$$\begin{aligned} z + \infty &:= \infty \\ z \cdot \infty &:= \infty \\ \frac{z}{\infty} &:= 0\,. \end{aligned} \tag{2.6}$$

Since there exists no multiplicative inverse for ∞, the Riemann sphere is not a field.

Definition 2.1.5 (Poincaré plane). The Poincaré upper half plane is the subset of $\mathbb{C}$ with positive imaginary part:

$$\mathfrak{h} := \{z \in \mathbb{C} \mid \Im(z) > 0\}\,. \tag{2.7}$$

2.2 Holomorphic functions

Definition 2.2.1 (Holomorphic function). A function f on an open set $U \subseteq \mathbb{C}$ that is complex differentiable at every point $z_0 \in U$, i.e. for every point $z_0 \in U$ the following limit exists:

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}\,. \tag{2.8}$$

Definition 2.2.2 (Biholomorphic function). A holomorphic function admitting a holomorphic inverse.

Definition 2.2.3 (Entire). A function that is holomorphic on all of $\mathbb{C}$.

Property 2.2.4 (Cauchy–Riemann conditions). A holomorphic function f satisfies the following conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,. \tag{2.9}$$

These conditions can be combined into one equation using the so-called **Wirtinger derivative**:

$$\frac{\partial f}{\partial \bar{z}} = 0\,. \tag{2.10}$$

Theorem 2.2.5 (Looman–Menchoff[1]). *Let f be a continuous function defined on a subset $U \in \mathbb{C}$. If the partial derivatives of the real and imaginary part exist and if f satisfies the Cauchy–Riemann conditions, then f is holomorphic on U.*

Property 2.2.6 (Laplace equation). The functions u, v satisfying the Cauchy–Riemann conditions are harmonic functions, i.e. they satisfy the *Laplace equation* (see **??**).

Property 2.2.7 (Level sets). The functions u, v satisfying the Cauchy–Riemann conditions have orthogonal level curves (**??**).

Property 2.2.8 (Real functions). Consider a real-valued function f defined on the complex plane. If it is holomorphic, the Cauchy–Riemann conditions imply that f is a constant.

Theorem 2.2.9 (Identity theorem). *If two holomorphic functions on a domain D coincide on a set containing an accumulation point of D, they coincide on all of D.*

2.3 Contour integrals

Remark. Whenever contours are considered for integration purposes, they have been chosen to be evaluated counter-clockwise (by convention). To obtain results concerning clockwise evaluation, most of the time adding a minus sign is sufficient.

Definition 2.3.1 (Contour integral). The contour integral of a complex function

$$f(z) = u(z) + iv(z) \tag{2.11}$$

is defined as the following line integral:

$$\int_{z_1}^{z_2} f(z)\,dz = \int_{(x_1,y_1)}^{(x_2,y_2)} \big(u(x,y) + iv(x,y)\big)(dx + idy)\,. \tag{2.12}$$

[1]This is the most general theorem on the holomorphy of continuous functions. It generalizes the original results by *Riemann* and *Cauchy–Goursat*.

[2]Also called the **Cauchy–Goursat theorem**.

Theorem 2.3.2 (Cauchy's integral theorem[2]). *Let Ω be a simply connected subset of $\mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on Ω. The contour integral around every closed, rectifiable (i.e. of finite length) contour C in Ω vanishes:*

$$\oint_C f(z)\,dz = 0\,. \tag{2.13}$$

Corollary 2.3.3 (Freedom of contour). The contour integral of a holomorphic function depends only on the limits of integration and not on the contour connecting them.

Formula 2.3.4 (Cauchy's integral formula). Let Ω be a connected subset of $\mathbb{C}$ and let C be a closed contour in Ω. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on Ω, then, at every point z_0 inside C, one can express f as follows:

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz\,. \tag{2.14}$$

Corollary 2.3.5 (Analytic function). Let Ω be a connected subset of $\mathbb{C}$ and let C be a closed contour in Ω. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on Ω, then f is analytic (Definition 1.5.8) on Ω and

$$f^{(n)}(z_0) = \frac{1}{2\pi i}\oint_C f(z)\frac{n!}{(z - z_0)^{n+1}}\,dz\,. \tag{2.15}$$

Furthermore, the derivatives are also holomorphic on Ω.

Theorem 2.3.6 (Morera). *If $f : \mathbb{C} \to \mathbb{C}$ is continuous on a connected open set Ω and*

$$\oint_C f(z)\,dz = 0 \tag{2.16}$$

for every closed contour C in Ω, then f is holomorphic on Ω.

Theorem 2.3.7 (Liouville). *Every bounded, entire function is constant.*

2.4 Laurent series

Definition 2.4.1 (Laurent series). If $f : \mathbb{C} \to \mathbb{C}$ is an analytic function defined on an **annulus**, i.e. a ring-shaped region, then f can be expanded as the following series:

$$f(z) = \sum_{n=-N}^{+\infty} a_n(z - z_0)^n \qquad \text{with} \qquad a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z - z_0)^{n+1}}\,dz\,, \tag{2.17}$$

The subseries containing all terms of negative degree is called the **principal part** of the Laurent series.

Notation 2.4.2. The ring of Laurent series in the indeterminate z is given by the ring $\mathbb{C}[[z, z^{-1}]]$.

Remark 2.4.3 (Multiplication). This definition can easily be generalized from $\mathbb{C}$ to general rings R. However, to be able to multiply two Laurent series, one has to restrict to series with a finite number of negative powers (unless R is topological such that convergence can be defined). This gives rise to the ring of **formal Laurent series** $R((z))$. This ring can be obtained as the fraction field of $R[[z]]$ (**??**).

Property 2.4.4 (Convergence). The Laurent series of an analytic function $f : \mathbb{C} \to \mathbb{C}$ converges uniformly to f on the annulus $R_1 < |z-z_0| < R_2$, with R_1 and R_2 the distances from z_0 to the two closest *poles* (see Definition 2.5.1 further below).

Definition 2.4.5 (Analytic continuation). Consider an analytic function $f : \mathbb{C} \to \mathbb{C}$ defined on an open subset $U \subset \mathbb{C}$. If $V \subset \mathbb{C}$ is an open subset containing U and if there exists an analytic function F on V such that $F(z) = f(z)$ for all $z \in U$, then F is called the analytic continuation of f to V. Using the identity theorem for holomorphic functions, one can prove that analytic continuations are unique (on connected domains).

Theorem 2.4.6 (Schwarz's reflection principle). *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on the upper half plane. If $z \in \mathbb{R} \implies f(z) \in \mathbb{R}$, then*

$$f\left(\overline{z}\right) = \overline{f(z)}\,. \tag{2.18}$$

2.5 Singularities

2.5.1 Poles

Definition 2.5.1 (Pole). A function $f : \mathbb{C} \to \mathbb{C}$ has a pole of order $m \in \mathbb{N}_0$ at a point $z_0 \in \mathbb{C}$ if its Laurent series at z_0 satisfies $\forall n < -m : a_n = 0$ and $a_{-m} \neq 0$.

Definition 2.5.2 (Removable singularity). Let $f : U\backslash\{a\} \to \mathbb{C}$ be a holomorphic function on a punctured open set. The point a is called a removable singularity of f if there exists a holomorphic function $g : U \to \mathbb{C}$ such that $f(z) = g(z)$ for all $z \in U\backslash\{a\}$.

Theorem 2.5.3 (Riemann). *A point $a \in \mathbb{C}$ is a removable singularity for a holomorphic function $f : U\backslash\{a\} \to \mathbb{C}$, with $U \subseteq \mathbb{C}$ an open set, if and only if there exists a neighbourhood of a on which f is bounded.*

Definition 2.5.4 (Meromorphic). A function $f : \mathbb{C} \to \mathbb{C}$ is said to be meromorphic if it is analytic on the whole complex plane with exception of isolated poles and removable singularities. Every meromorphic function can be written as a fraction of two holomorphic functions, where the poles coincide with the zeros of the denominator.

Definition 2.5.5 (Essential singularity). A function f has an essential singularity at a point z_0 if its Laurent series at z_0 satisfies $\forall n \in \mathbb{N} : a_{-n} \neq 0$, i.e. if its Laurent series has infinitely many negative degree terms.

Method 2.5.6 (Frobenius transformation). To study the behaviour of a function f at

$z \longrightarrow \infty$, one can apply the Frobenius transformation $h = 1/z$ and study the limit $\lim_{h\to 0} f(h)$. For example, a singularity at ∞ is defined as a singularity of $f(1/z)$ at 0.

Property 2.5.7 (Polynomials). An entire function $f : \mathbb{C} \to \mathbb{C}$ is polynomial if and only if it has a pole at ∞.

Theorem 2.5.8 (Casorati–Weierstrass). *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic on the punctured open set $U\backslash\{z_0\}$ with an essential singularity at z_0. For every neighbourhood V of z_0 contained in U, the image $f(V\backslash\{z_0\})$ is dense in $\mathbb{C}$.*

Corollary 2.5.9. If $f : \mathbb{C} \to \mathbb{C}$ is a nonpolynomial, entire function, then, for every $c \in \mathbb{C}$, there exists a sequence $z_n \longrightarrow \infty$ such that $f(z_n) \longrightarrow c$.[3]

Theorem 2.5.10 (Picard's little theorem). *The range of a nonconstant, entire function is the complex plane with at most a single exception.*

Theorem 2.5.11 (Picard's great theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function with an essential singularity at z_0. On every punctured neighbourhood of z_0, f takes on all possible values, with at most a single exception, infinitely many times.*

Definition 2.5.12 (Cauchy principal value). If $f : \mathbb{C} \to \mathbb{C}$ has a pole z_0 on the contour C, then

$$\mathcal{P}\int_C f(z)\,dz := \lim_{\varepsilon\downarrow 0}\int_{C_\varepsilon} f(z)\,dz\,, \tag{2.19}$$

where $C_\varepsilon := C\backslash B(z_0,\varepsilon)$, assuming f is integrable on all C_ε.

Theorem 2.5.13 (Sokhotski–Plemelj). *Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and consider $a < 0 < b$, then*

$$\lim_{\varepsilon\to 0^+}\int_a^b \frac{f(x)}{x \pm i\varepsilon}\,dx = \mp i\pi f(0) + \mathcal{P}\int_a^b \frac{f(x)}{x}\,dx\,. \tag{2.20}$$

2.5.2 Branch cuts

Formula 2.5.14 (Roots). Let $z \in \mathbb{C}$. The n^{th} roots[4] of $z = re^{i\theta}$ are given by

$$\left\{\sqrt[n]{r}\exp\left(\frac{\theta + 2\pi k}{n}i\right)\,\middle|\, k \in \{0,1,\dots,n\}\right\}. \tag{2.21}$$

Formula 2.5.15 (Complex logarithm). The natural logarithm can be continued to the complex plane (as a multivalued function) as follows:

$$\mathrm{LN}(z) := \left\{\ln(r) + i(\theta + 2\pi k) \mid k \in \mathbb{Z}\right\}. \tag{2.22}$$

Definition 2.5.16 (Branch). The problem with the previous two formulas is that they represent multivalued functions. To get an unambiguous image it is necessary to fix

[3] Polynomials are excluded due to the property above.
[4] Also see the fundamental theorem of algebra (**??**).

a value of the parameter k. By doing so there will arise curves, called **branch cuts**, in the complex plane where the function becomes discontinuous. A **branch** is defined as a particular choice of the parameter k.

For the logarithm, the choice for $\mathrm{arg}(\mathrm{LN}) \in]\alpha, \alpha + 2\pi]$ is often denoted by LN_α or $\log_\alpha$.

Definition 2.5.17 (Principal value). The principal value[5] of a multivalued complex function is defined as the value associated with a choice of branch for which $\mathrm{arg}(f) \in]-\pi, \pi]$.

Definition 2.5.18 (Branch point). Let f be a complex-valued function. A point z_0 for which there exists no neighbourhood on which f is single-valued is called a branch point.

Definition 2.5.19 (Branch cut). A line connecting exactly two branch points, one possibly being ∞, is called a branch cut. In case there exist multiple branch cuts, they are required to never cross.

Example 2.5.20. Consider the complex function

$$f(z) = \frac{1}{\sqrt{(z - z_1) \cdots (z - z_n)}} \,. \tag{2.23}$$

This function has singularities at $z_1, \dots, z_n$. If $n \in \mathbb{N}$ is even, this function will have n (finite) branch points. This implies that the points can be grouped in pairs connected by non-intersecting branch cuts. If n is odd, this function will have n (finite) branch points and one branch point at infinity. The finite branch points will be grouped in pairs connected by non-intersecting branch cuts and the remaining branch point will be joined to infinity by a branch cut that does not intersect the others.

2.5.3 Residue theorem

Definition 2.5.21 (Residue). By applying Definition 2.3.1 to a polynomial function, one finds

$$\oint_C (z - z_0)^n \, dz = 2\pi i \delta_{n,-1} \,, \tag{2.24}$$

where C is a contour around the pole $z = z_0$. This means that integrating a Laurent series around a pole isolates the coefficient a_{-1}. This coefficient is, therefore, called the residue of the function at the given pole.

Notation 2.5.22. The residue of a complex function $f : \mathbb{C} \to \mathbb{C}$ at a pole z_0 is denoted by

$$\mathrm{Res}[f(z)]_{z=z_0} \,. \tag{2.25}$$

[5] Not to be confused with the Cauchy principal value.

Formula 2.5.23. For a pole of order $m \in \mathbb{N}_0$, the residue is calculated as follows:

$$\mathrm{Res}\big[f(z)\big]_{z=z_j} = \lim_{z\to z_0} \frac{1}{(m-1)!}\left(\frac{\partial}{\partial z}\right)^{m-1} \big(f(z)(z-z_0)\big)\,. \tag{2.26}$$

For essential singularities, the residue can be found by writing out the Laurent series explicitly.

Theorem 2.5.24 (Residue theorem). *If $f : \mathbb{C} \to \mathbb{C}$ is meromorphic on $\Omega \subseteq \mathbb{C}$ and if C is a closed contour in Ω that contains the poles z_j of f, then*

$$\oint_C f(z)\,dz = 2\pi i \sum_j \mathrm{Res}\big[f(z)\big]_{z=z_j}\,. \tag{2.27}$$

For poles on the contour C, only half of the residue contributes to the integral.

Formula 2.5.25 (Argument principle). Let $f : \mathbb{C} \to \mathbb{C}$ be meromorphic and denote the number of zeros and poles of f inside the contour C by $Z_f(C)$ and $P_f(C)$, respectively. From the residue theorem, one can derive the following formula:

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{f'(z)}\,dz = Z_f(C) - P_f(C)\,. \tag{2.28}$$

Definition 2.5.26 (Winding number). Let $f : \mathbb{C} \to \mathbb{C}$ be meromorphic and let C be a simple closed contour. For all $a \notin f(C)$, the winding number, also called the **index**, of a with respect to the function f is defined as follows:

$$\mathrm{Ind}_f(a) := \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)-a}\,dz\,. \tag{2.29}$$

This number is always an integer.

2.6 Limit theorems

Theorem 2.6.1 (Small limit theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be a function that is holomorphic almost everywhere and let the contour C be a circular segment with radius ε and central angle α. If z is parametrized as $z = \varepsilon e^{i\theta}$, then*

$$\oint_C f(z)\,dz = i\alpha A \tag{2.30}$$

with

$$A = \lim_{\varepsilon\to 0} f(z)\,. \tag{2.31}$$

Theorem 2.6.2 (Great limit theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be a function that is holomorphic almost everywhere and let the contour C be a circular segment with radius R and central angle α. If z is parametrized as $z = Re^{i\theta}$, then*

$$\oint_C f(z)\,dz = i\alpha B \tag{2.32}$$

with

$$B = \lim_{R\to\infty} f(z)\,. \tag{2.33}$$

Theorem 2.6.3 (Jordan's lemma). *Let $g : \mathbb{C} \to \mathbb{C}$ be a continuous function that can be written as $g(z) = f(z)e^{bz}$ and let the contour C be a semicircle lying in the half-plane bounded by the real axis and oriented away of the point $i\bar{b}$. If z is parametrized as $z = Re^{i\theta}$ and*

$$\lim_{R\to\infty} f(z) = 0\,, \tag{2.34}$$

then

$$\oint_C g(z)\,dz = 0\,. \tag{2.35}$$

Chapter 3

Number Theory

The section on modular forms and L-functions is based on Mustaţă (2011); Sutherland (2017). A general overview is given in Waldschmidt, Moussa, Luck, and Itzykson (1992)

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3.1 Algebraic number theory

Definition 3.1.1 (Diophantine equation). A polynomial with integer coefficients. In general, only integer solutions are of interests.

Theorem 3.1.2 (Prime factorization). *Every integer $n \geq 2$ can be written in a unique way (up to permutation of the factors) as the product of prime numbers.*[1]

Theorem 3.1.3 (Lagrange). *Every integer $n \in \mathbb{N}_0$ is the sum of four squares.*

Lagrange's theorem can also be restated in terms of power series. Consider the **Jacobi θ-function**

$$\theta(q) := \sum_{k\in\mathbb{Z}} q^{k^2} = 1 + 2\sum_{n\in\mathbb{N}_0} q^{n^2}\,. \tag{3.1}$$

This function satisfies the following property:

$$\theta^k(q) = \sum_{n=0}^{+\infty} r_k(n) q^n\,, \tag{3.2}$$

where $r_k(n)$ counts the number of (integer) solutions to the Diophantine equation

$$x_1^2 + \cdots + x_k^2 = n\,. \tag{3.3}$$

This Jacobi θ-function is often generalized.

Definition 3.1.4 (Jacobi θ-function). The series (3.1) is convergent for $|q| < 1$ and, accordingly, q can be reparametrized as

$$q = \exp(i\pi\tau) \tag{3.4}$$

for some $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$. The parameter q is often called the **nome**.

The general Jacobi θ-function is given by

$$\Theta(z;\tau) := \sum_{k\in\mathbb{Z}} e^{i\pi(k^2\tau+2kz)} = \sum_{k\in\mathbb{Z}} q^{k^2}u^k\,, \tag{3.5}$$

where $z \in \mathbb{C}$.

Property 3.1.5 (Jacobi identity). The Jacobi θ-function satisfies the following transformation formulas for *modular transformations* (see Definition 3.4.1 further below):

$$\Theta\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,e^{i\pi z^2/\tau}\Theta(z;\tau)\,. \tag{3.6}$$

Theorem 3.1.6 (Gauss). *Every integer of the form $8n+3$, $n \in \mathbb{N}$, is the sum of three odd squares.*

Theorem 3.1.7 (Fermat). *An integer $n \in \mathbb{N}_0$ is the sum of two squares if and only if every prime divisor of n that is congruent to $3 \bmod 4$ appears an even number of times in this decomposition.*

[1] 1 is excluded since it is not a prime number.

3.2 Adic numbers

This section will make use of the content of **??**.

3.2.1 Finite fields

Definition 3.2.1 (Finite field). The finite field (or **Galois field**) $\mathbb{F}_p$ is the field with a finite number of elements $p \in \mathbb{N}$. The number p is called the **order** of $\mathbb{F}_p$.

Property 3.2.2 (Prime order). If $p \in \mathbb{N}$ is prime, then $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$.

3.2.2 Rational numbers

Definition 3.2.3 (p-adic numbers). Consider a prime number $p \in \mathbb{N}$. For every $m, n \in \mathbb{N}$, one has that $\mathbb{Z}/p^n\mathbb{Z} \subseteq \mathbb{Z}/p^m\mathbb{Z}$ whenever $m \leq n$. The group $\mathbb{Z}_p$ of **p-adic integers** is given by the profinite group (**??**) obtained from this system of inclusions. The p-adic integers are of the form

$$\sum_{n=0}^{+\infty} a_n p^n\,, \tag{3.7}$$

with integers $0 \leq a_n < p$ for all $n \in \mathbb{N}$.

Just like $\mathbb{Q}$ is the field of fractions (**??**) of $\mathbb{Z}$ as explained in **??**, the p-adic numbers $\mathbb{Q}_p$ can be obtained as the field of fractions of $\mathbb{Z}_p$. The p-adic numbers are of the form

$$\sum_{n=-N}^{+\infty} a_n p^n\,, \tag{3.8}$$

with $N \in \mathbb{N}$ and integers $0 \leq a_n < p$ for all $n \in \mathbb{N}$.

An alternative definition makes use of the notion of valuations (**??**).

Alternative Definition 3.2.4 (p-adic numbers). Consider a prime number $p \in \mathbb{N}$. The p-adic valuation of an integer $z \in \mathbb{Z}$ is defined as follows:

$$\nu_p(k) := \begin{cases} \max\{n \in \mathbb{N} : p^n \mid k\} & \text{if } k \neq 0\,, \\ +\infty & \text{if } k = 0\,. \end{cases} \tag{3.9}$$

This valuation extends to the rational numbers by setting

$$\nu_p\left(\frac{a}{b}\right) := \nu_p(a) - \nu_p(b)\,. \tag{3.10}$$

In turn, the p-adic valuation induces an absolute value on $\mathbb{Z}$ (and on $\mathbb{Q}$) as in **??**:

$$|z|_p := p^{-\nu_p(z)}\,. \tag{3.11}$$

The metric completions of $\mathbb{Z}$ and $\mathbb{Q}$ by these absolute values are the rings of p-adic integers and numbers, respectively.

Remark 3.2.5. The p-adic valuation of a rational number $q \in \mathbb{Q}$ can also be defined as the (unique) integer $k \in \mathbb{Z}$ such that

$$q = p^k \frac{m}{n} \tag{3.12}$$

for some integers $m, n \in \mathbb{Z}$ such that p^k, m and n are all coprime.

Property 3.2.6. If $b \bmod p \neq 0$, then $\frac{a}{b} \in \mathbb{Z}_p$ for all $a, b \in \mathbb{Z}$ and $p \in \mathbb{N}$.

The construction of the p-adic representation of a rational number $q \in \mathbb{Q}$ proceeds similar to the construction of the usual decimal representation, i.e. through long division:

1. Find the unique coprime integers satisfying $q = \frac{c}{d}p^k$ and, hence, $\nu_p(q) = k$.
2. $k - 1$ zeros have to be added to the p-adic representation.
3. Then, find the unique integers satisfying $q = ap^k + r$ with $0 \leq a < p$ and $\nu_p(r) > k$.
4. Then the next coefficients is a and the algorithm can be restarted with $q := r$.

Theorem 3.2.7 (Ostrowski). *The only nontrivial absolute values on $\mathbb{Q}$ are either the ordinary absolute value $|\cdot|$ or the p-adic absolute values $|\cdot|_p$.*

More generally, consider a complete field $\mathfrak{K}$ with respect to an absolute value. Then, either the absolute value is Archimedean and $\mathfrak{K} \cong \mathbb{R}, \mathbb{C}$, or the absolute value is non-Archimedean.

Definition 3.2.8 (Profinite integers). Similar to the construction of the p-adic integers, one can also construct the profinite completion (**??**) of $\mathbb{Z}$. Instead of taking the inverse limit over the integers modulo a prime power, one simply takes the inverse limit over all finite cyclic groups:

$$\widehat{\mathbb{Z}} := \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}\,. \tag{3.13}$$

It can be shown that this is equivalent to taking the product of all p-adic integers:

$$\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p\,. \tag{3.14}$$

Definition 3.2.9 (Adèles). The ring of integral adèles is defined as the product

$$\mathbb{A}_{\mathbb{R}} := \mathbb{R} \times \widehat{\mathbb{Z}}\,. \tag{3.15}$$

To obtain the proper ring of adèles, the above ring is rationalized:

$$\mathbb{A}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{A}_{\mathbb{R}}\,. \tag{3.16}$$

Definition 3.2.10 (Idèles). The group of units $\mathbb{I}_{\mathbb{Q}} := \mathbb{A}_{\mathbb{Q}}^{\times}$.

3.2.3 Number and global fields

The product expression from the previous section can be reinterpreted by noting that the rationalization of the integral adèles can be written as a **restricted product**. Recall **??** of products. Given an indexing set S and a collection of morphisms $\{\iota_s : X_s \to Y_s\}_{s\in S}$, the restricted product $\prod'_{s\in S} Y_s$ is given by the subset of the ordinary product $\prod_{s\in S} Y_s$ where all but a finite number of elements lie in the image of the morphisms ι_s.

Now, by the prime factorization theorem 3.1.2 and Property 3.2.6, every element of the form

$$\frac{a}{b} \otimes c_p \tag{3.17}$$

with $a, b \in \mathbb{Z}$ and $c_p \in \mathbb{Z}_p$ will either be an element of $\mathbb{Z}_p$ if $b \bmod p \neq 0$ or an element of $\mathbb{Q}_p$ if $b \bmod p = 0$. Since there are only a finite number of prime factors of b, this shows that

$$\mathbb{A}_{\mathbb{Q}} \cong \mathbb{R} \times \prod_{p \text{ prime}}{}^{'} \mathbb{Q}_p\,. \tag{3.18}$$

Moreover, taking $\mathbb{Q}_\infty := \mathbb{R}$ and recalling Ostrowski's theorem, one has

$$\mathbb{A}_{\mathbb{Q}} \cong \prod_{p\in\mathrm{places}(\mathbb{Z})}{}^{'} \mathbb{Q}_p\,. \tag{3.19}$$

Now, consider a number field $\mathfrak{K}$ (**??**). Consider the following notations:

- $\mathcal{O}$ denotes the ring of algebraic integers (**??**) of $\mathfrak{K}$.
- P denotes the set of places of $\mathfrak{K}$ with $S \subset P$ the Archimedean (or infinite) places.
- For every place $v \in P$, $\mathfrak{K}_v$ denotes the completion at v.

By Ostrowski's theorem, the completions at the infinite place are isomorphic to $\mathbb{R}$ or $\mathbb{C}$. For each (*local*) field $\mathfrak{K}_v$, one can also construct the **valuation ring** as the subring $\mathcal{O}_v$ of elements of norm at most 1.

The **integral adèles** are defined as follows, in analogy to Eq. (3.15):

$$\mathbb{A}_{\mathcal{O}} := \widehat{\mathcal{O}} \times \prod_{v\in S} \mathfrak{K}_v \tag{3.20}$$

and the **adèle ring** is defined as its 'rationalization':

$$\mathbb{A}_{\mathfrak{K}} := \mathfrak{K} \otimes_{\mathcal{O}} \mathbb{A}_{\mathcal{O}}\,. \tag{3.21}$$

Note that the profinite completion $\widehat{\mathcal{O}}$ again admits a factorization over non-Archimedean places:

$$\widehat{\mathcal{O}} \cong \prod_{v\in P\setminus S} \mathcal{O}_v\,. \tag{3.22}$$

Combining these facts, one obtains the following expression:

$$\mathbb{A}_{\mathfrak{K}} \cong \prod_{\nu\in S} \mathfrak{K}_\nu \times \prod_{\nu\in P\setminus S}{}' \mathfrak{K}_\nu \cong \prod_{\nu\in P}{}' \mathfrak{K}_\nu\,. \tag{3.23}$$

This definition can now be generalized to global fields as is.

Definition 3.2.11 (Global field). Either a number field or a function field (**??**) over a finite field. Equivalently, a finite extension of either the rational numbers $\mathbb{Q}$ or the rational functions with coefficients in a finite field.

3.3 Elliptic functions

Definition 3.3.1 (Elliptic curve). Consider a lattice of the form $\Gamma_\tau := \mathbb{Z} + \tau\mathbb{Z}$ for some $\tau \in \mathfrak{h}$, where $\mathfrak{h}$ is the Poincaré upper half plane (Definition 2.1.5). An elliptic curve is a quotient $\mathbb{C}/\Gamma_\tau$.

Property 3.3.2. All elliptic curves are compact *Riemann surfaces* (see **??**).

Definition 3.3.3 (Elliptic function). A meromorphic function on $\mathbb{C}$ that is periodic with respect to a lattice Γ_τ.

3.4 Algebraic geometry ♣

This section gives a relation between number theory and (algebraic) geometry (**??**). The content of **?? ??** and Complex Analysis will also be used throughout this section.

3.4.1 Modular forms

Definition 3.4.1 (Modular group). In the setting of number theory, the projective special linear group $\mathrm{PSL}(2,\mathbb{Z})$ is often called the modular group.[2] The modular group acts on the complex plane by **Möbius transformations**:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}\,. \tag{3.24}$$

For this reason, $\mathrm{PSL}(2,\mathbb{C})$ is sometimes also called the **Möbius group**. The full modular group $\mathrm{SL}(2,\mathbb{Z})$ will be denoted by Γ.

Definition 3.4.2 (Modular form). A modular form of weight $k \in \mathbb{R}$ is a holomorphic function on the upper-half plane $f : \mathcal{H} \to \mathbb{C}$ satisfying the following two conditions:

[2] This name is also used for the special linear group $\mathrm{SL}(2,\mathbb{Z})$ since Möbius transformations are invariant under rescaling.

1. **Automorphicity**: For $g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, one has
$$f(g \cdot z) = (cz+d)^k f(z)\,, \tag{3.25}$$
and
2. **Bounded growth**: $f(z)$ is bounded for $z \longrightarrow i\infty$.

If the modular form satisfies the stronger condition $f(z) \longrightarrow 0$ when $z \longrightarrow i\infty$, it is said to be **cuspidal** or it is simply called a **cusp form**. The spaces of modular forms and cusp forms of weight $k \in \mathbb{R}$ are, respectively, denoted by $M_k(\Gamma)$ and $S_k(\Gamma)$.

@@ MAYBE ADD automorphicity in full generality @@

Remark 3.4.3 (Arithmetic group). Modular forms can also be defined for subgroups of $\mathrm{SL}(2,\mathbb{Z})$ of finite index, such as the *congruence groups* (or their generalizations, the *arithmetic groups*).

Property 3.4.4. The generators of the modular group are given by
$$z \mapsto -\frac{1}{z} \qquad \text{and} \qquad z \mapsto z+1\,. \tag{3.26}$$
Invariance under the second generator shows that modular forms are, in particular, periodic and, hence, admit a *Fourier expansion* (see **??**). Cusp forms are exactly those modular forms with vanishing constant Fourier coefficient.

Example 3.4.5 (Eisenstein series). Consider a complex number $\tau \in \mathcal{H}$. The Eisenstein series of weight $2k \in 2\mathbb{Z}$, with $k > 2$, is defined as follows:
$$G_{2k}(\tau) := \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m+n\tau)^{2k}}\,. \tag{3.27}$$

Remark 3.4.6 (Fourier expansion). Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is an element of $\mathrm{PSL}(2,\mathbb{Z})$, every modular form is periodic and, hence, admits a *Fourier expansion* (see **??**):
$$f(z) = \sum_{n=-\infty}^{+\infty} a_n e^{2\pi i z}\,. \tag{3.28}$$
f can then be interpreted as a function on the punctured disk by making the transformation $z \longrightarrow q := e^{2\pi i z}$. By Riemann's theorem on removable singularities (Theorem 2.5.3), the growth condition on modular forms is equivalent to $f(q)$ admitting a holomorphic continuation to the origin. Moreover, it is a cusp form exactly if $a_0 = 0$. The growth condition also implies that the Fourier series can be truncated to positive integers:
$$f(z) = \sum_{n=0}^{+\infty} a_n e^{2\pi i z}\,. \tag{3.29}$$

Definition 3.4.7 (Hecke operator). For every $n \in \mathbb{N}_0$, let Γ_n denote the subgroup of Γ on matrices with determinant n. On $M_k(\Gamma)$, one can define the following averaging operators:

$$T_n f(z) := n^{k-1} \sum_{g \in \Gamma \backslash \Gamma_n} (cz+d)^{-k} f(g \cdot z)\,, \tag{3.30}$$

where $g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. One can easily see that $[T_m, T_n] = 0$.

The Hecke operators also have an important relation to cusp forms. The subspace $S_k(\Gamma)$ is preserved under the action of the Hecke operators and, moreover the joint eigenforms of the Hecke operators satisfy the following relations on their Fourier coefficients:

$$a_n = \lambda_n a_1\,, \tag{3.31}$$

where λ_n is the n^{th} eigenvalue. $S_k(\Gamma)$ admits an inner product, the **Petersson inner product**, defined as follows:

$$\langle f \,|\, g \rangle := \int_{\mathcal{F}} f(z)\overline{g}(z) y^{k-2}\, dx\, dy\,, \tag{3.32}$$

where $\mathcal{F} \subset \mathcal{H}$ is any fundamental domain for Γ. With respect to this inner product, there exists a normalized eigenbasis of the Hecke operators for $S_k(\Gamma)$, where normalized means that the first Fourier coefficient is equal to 1 (and, hence, the Fourier coefficients correspond to the Hecke eigenvalues).

3.4.2 Algebraic functions

?? can be generalized to the functional setting.

Definition 3.4.8 (Algebraic function). Let R be a commutative ring. A function $f : R^n \to R$ is said to be algebraic if it is the solution of a polynomial equation with coefficients in $R[x_1, \ldots, x_n]$.[3] If f is not algebraic, it is said to be **transcendental**.

3.5 L-functions

3.5.1 Dirichlet series

Definition 3.5.1 (Dirichlet series). A series of the form

$$\sum_{n=1}^{+\infty} a_n n^s\,, \tag{3.33}$$

where $s \in \mathbb{C}$ and $(a_n)_{n \in \mathbb{N}} \subset \mathbb{C}$. If the coefficients satisfy the growth bound $|a_n| = O(n^\sigma)$, the series converges (absolutely) to a holomorphic function whenever $\Re(s) > \sigma$.

[3] Often, the polynomial is required to be irreducible.

The Dirichlet series for unit coefficients is of major interest in mathematics.

Definition 3.5.2 (Riemann ζ-function).

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{3.34}$$

for all $s \in \mathbb{C}$ for which $\Re(s) > 1$. For general s, an *analytic continuation* is used (see Definition 2.4.5).

Formula 3.5.3 (Euler product).

$$\zeta(s) = \prod_{p\ \text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{3.35}$$

for all $s \in \mathbb{C}$ for which $\Re(s) > 1$. This is an example of an Euler product, a product over all prime numbers.

By perturbing the Riemann ζ-function, a more general series is obtained.

Definition 3.5.4 (Hurwitz ζ-function). The function $\zeta_H : \mathbb{C} \times \mathbb{R}_0^+ \to \mathbb{C}$ defined as

$$\zeta_H(s, v) := \sum_{n=0}^{+\infty} (n + v)^{-s} . \tag{3.36}$$

As for ζ, the radius of convergence is 1, with $\zeta(s) = \zeta_H(s, 1)$.

Definition 3.5.5 (Selberg ζ-function). A function $\zeta : \mathbb{C} \to \mathbb{C}$ satisfying the following conditions:

1. There exists a $\eta \in \mathbb{R}^+$ (usually taken to be 1) such that ζ admits a convergent series expansion of the form

$$\zeta(s) = \sum_{n=1}^{+\infty} \lambda^{-s} , \tag{3.37}$$

for some sequence of real numbers $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, or admits a Euler product expansion.

2. ζ analytic continuation (Definition 2.4.5) of the above series on all of $\mathbb{C}$.

3. There exists a 'completion' $\hat{\zeta}$ of ζ that admits a functional relation of the form

$$\hat{\zeta}(1 - s) = \hat{\zeta}(s) . \tag{3.38}$$

4*. *Ramanajuan conjecture*: $a_1 = 1$ and $a_n = O(n^\varepsilon)$ for any $\varepsilon > 0$.

@@ CHECK AND COMPLETE @@

Example 3.5.6 (Riemann ζ). The Riemann ζ-function from Definition 3.5.2 induces a Selberg ζ-function through the following rescaling

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) . \tag{3.39}$$

Example 3.5.7 (Differential operators). Consider a self-adjoint, *elliptic differential operator* (see **??**) D whose spectrum lies in the positive real numbers $\mathbb{R}^+_0$. The associated ζ-function is defined as follows:

$$\zeta_D(s) := \mathrm{tr}\big(D^{-s}\big)\,. \tag{3.40}$$

It can be shown that this is equal to the *Mellin transform* (see **??**) of the *partition function* of D:

$$\zeta_D(s) = \int_0^{+\infty} t^{s-1}\left(\sum_{n=1}^{+\infty}\exp(-t\lambda_n)\right)dt\,. \tag{3.41}$$

Remark 3.5.8 (Bost–Connes system). The *Bost–Connes system*, a *C^*-algebra* (see **??**) of the form $C^*\big(\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}^\times\big)$, admits a Hamiltonian whose associated ζ-function is the Riemann ζ-function. A physical realization of this system would be able to shed light on the *Riemann hypothesis*.

3.5.2 Dirichlet characters

Definition 3.5.9 (Dirichlet character). Let $p \in \mathbb{N}_0$ be an integer. A Dirichlet character with **conductor** p is a function $\chi : \mathbb{Z} \to \mathbb{R}$ satisfying

1. **Periodicity**: $\chi(n+p) = \chi(n)$ for all $n \in \mathbb{Z}$.
2. **Multiplicativity**: $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$ such that $p \nmid m, n$.
3. **Degeneracy**: $\chi(n) = 0$ for all $n \in \mathbb{Z}$ such that $\gcd(p, n) > 1$.

Example 3.5.10 (Principal character). The principal character of conductor $p \in \mathbb{N}_0$ is given by

$$\varepsilon_p(n) := \begin{cases} 1 & \text{if } p \nmid n, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.42}$$

Its mean value is given by

$$\overline{\varepsilon_p} = \frac{\varphi(p)}{p}\,, \tag{3.43}$$

where φ is **Euler's totient function**:

$$\varphi(n) := \big|\{m \in \{1, \ldots, n\} \mid m \nmid n\}\big|\,. \tag{3.44}$$

Definition 3.5.11 (Dirichlet L-series). Consider a periodic function $\chi : \mathbb{Z} \to \mathbb{R}$. The associated L-series is defined as follows:

$$L_\chi(s) := \sum_{n=1}^{+\infty} \chi(n) n^{-s}\,. \tag{3.45}$$

Since χ is bounded (since it is periodic), this series converges for $\Re(s) > 1$. Note that $\zeta(s)$ is simply the case

Formula 3.5.12 (Hurwitz ζ-function). Every Dirichlet L-series can be rewritten using the Hurwitz ζ-function:

$$L_\chi(s) = p^{-s}\sum_{k=1}^{p}\chi(k)\zeta_H\left(s,\frac{k}{p}\right). \tag{3.46}$$

Property 3.5.13 (Holomorphic extension). If the mean value of χ is 0, the associated L-series can be extended to an entire function.

As for the Riemann ζ-function (cf. Formula 3.5.3), every L-series admits an Euler product formula.

Formula 3.5.14 (Euler product). Let χ be a Dirichlet character of conductor $p \in \mathbb{N}_0$.

$$L_\chi(s) = \prod_{n\text{ prime},\, n|p}\left(1-\frac{\chi(n)}{n^s}\right)^{-1} \tag{3.47}$$

Recall the θ-function from Eq. (3.2). This function can be altered to give a Dirichlet series:

$$Z_k(s) := \sum_{n=1}^{+\infty} r_k(n)n^{-s}. \tag{3.48}$$

For $k=2$, one obtains the following factorization:

$$Z_2(s) = 4\zeta(s)L(s) \tag{3.49}$$

with

$$L(s) = \prod_{p\text{ prime},\, p\equiv 1 \bmod 4}\frac{1}{1-p^{-s}}\prod_{p'\text{ prime},\, p'\equiv 3 \bmod 4}\frac{1}{1+p'}. \tag{3.50}$$

The above L-series can be obtained as the one associated to the following character of conductor 4:

$$\chi_4(n) := \begin{cases} 0 & \text{if } n \in 2\mathbb{Z}, \\ (-1)^{(n-1)/2} & \text{if } n \in 2\mathbb{Z}+1. \end{cases} \tag{3.51}$$

Formula 3.5.15 (Induced ζ-function). Every Dirichlet character χ of conductor $p \in \mathbb{N}_0$ induces a Selberg-type ζ-function by mimicking Example 3.5.6:

$$\xi(\chi,s) := \left(\frac{p}{\pi}\right)^{(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L_\chi(s). \tag{3.52}$$

This function satisfies the following functional equation:

$$\xi(\widehat{\chi},1-s) = \xi(\chi,s), \tag{3.53}$$

where $\widehat{\chi}$ is the finite Fourier transform of χ:

$$\widehat{\chi}(n) = \frac{1}{\sqrt{p}}\sum_{k=0}^{p-1}\chi(k)\exp(2\pi ink/p). \tag{3.54}$$

3.5.3 Algebraic geometry

Definition 3.5.16 (Hasse–Weil ζ-function[4]). Let V be a nonsingular projective variety over the finite field $\mathbb{F}_p$ and, for all $k \in \mathbb{N}$, let $N_K := \left|\mathbb{F}_{p^k}(V)\right|$ be the number of rational points over $\mathbb{F}_{p^k}$.

$$Z(V,t) := \exp\left(\sum_{k=1}^{+\infty} N_k \frac{t^k}{k}\right) \in \mathbb{Q}[[t]]\,. \tag{3.55}$$

Remark 3.5.17 (Weil conjectures). The Weil conjectures are three related statements of the Hasse–Weil function of profound importance. The conjectures were:

1. $Z(V,t)$ is a rational function of t.
2. $Z(V,t)$ admits a functional relation as in Eq. (3.38).
3. Discrete *Riemann hypothesis*: The zeroes of $P_k(t)$, for all $k \in \mathbb{N}$, lie on the critical line $\Re(z) = k/2$, where the $P_k(t)$ are the integral polynomials in the rational form of $Z(V,t)$:
$$Z(V,t) = \frac{P_1(t)\cdots P_{2\dim(V)-1}(t)}{P_0(t)\cdots P_{2\dim(V)}(t)} \tag{3.56}$$
with $P_0(t) = 1 - t$ and $P_{2\dim(V)}(t) = 1 - p^n t$.
4. If X is a *good reduction* of nonsingular projective variety over a number field in the complex numbers, then $\deg(P_k)$ equals the k^{th} Betti number (**??**) of the set of complex points of V.

Items 1, 2 and 4 were proven by *Grothendieck* using *étale cohomology*, whereas the discrete Riemann hypothesis was proven by *Deligne* using *l-adic cohomology*.

Definition 3.5.18 (L-function of varieties). Consider a nonsingular projective variety V over $\mathbb{Q}$. For almost all primes p, one obtains a good reduction V_p. The Dirichlet series

$$Z_{\mathbb{Q},V}(s) := \prod_{p\text{ good prime}} Z(V_p, p^{-s}) \tag{3.57}$$

is called the L-function of V (or sometimes also the Hasse–Weil function of V).

Example 3.5.19 (Riemann ζ-function). Consider $V = \operatorname{Spec}\mathbb{Q}$ with $V_p \cong \operatorname{Spec}(\mathbb{F}_p)$ for prime $p \in \mathbb{N}$. In this case,

$$Z(V_p,t) = \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n}\right) = (1-t)^{-1} \tag{3.58}$$

and, hence,

$$Z_{\mathbb{Q},V}(s) = \prod_{p\text{ prime}} (1-p^{-s})^{-1} = \zeta(s) \tag{3.59}$$

by Formula 3.5.3.

[4] Also called the **local Hasse–Weil function** or **local L-function**.

Recall the cusp forms of Section 3.4.1.

Definition 3.5.20 (L-function of modular forms). Let $f : \mathcal{H} \to \mathbb{C}$ be a cusp form of weight $k \in \mathbb{R}$ and consider its Fourier series

$$f(z) = \sum_{n=1}^{+\infty} a_n e^{2\pi i z} . \tag{3.60}$$

The associated L-function is given by the following Dirichlet series:

$$L_f(s) := \sum_{n=1}^{+\infty} \frac{a_n}{n^s} . \tag{3.61}$$

The domain of convergence is bounded by $\Re(s) > 1 + \frac{k}{2}$.

Theorem 3.5.21 (Modularity theorem[5]). *Every Hasse–Weil L-function of an elliptic curve can be obtained as the L-function of a modular form.*

[5] Also called the **Taniyama–Shimura(–Weil) theorem/conjecture** (or any of its permutations).

List of Symbols

The following abbreviations and symbols are used throughout the compendium.

Abbreviations

AIC	Akaike information criterion
ARMA	autoregressive moving-average model
BCH	Baker–Campbell–Hausdorff
BPS	Bogomol'nyi–Prasad–Sommerfield
BPST	Belavin–Polyakov–Schwarz–Tyupkin
BRST	Becchi–Rouet–Stora–Tyutin
CCR	canonical commutation relation
CDF	cumulative distribution function
CFT	conformal field theory
CIS	completely integrable system
CP	completely positive
CPTP	completely positive, trace-preserving
CR	Cauchy–Riemann
dga	differential graded algebra
dgca	differential graded-commutative algebra
EMM	equivalent martingale measure
EPR	Einstein–Podolsky–Rosen
ESM	equivalent separating measure
ETCS	Elementary Theory of the Category of Sets
FIP	finite intersection property
FWHM	full width at half maximum
GA	geometric algebra
GHZ	Greenberger–Horne–Zeilinger

GNS	Gel'fand–Naimark–Segal
HJE	Hamilton–Jacobi equation
HoTT	Homotopy Type Theory
KKT	Karush–Kuhn–Tucker
LIVF	left-invariant vector field
MCG	mapping class group
MPO	matrix-product operator
MPS	matrix-product state
MTC	modular tensor category
NDR	neighbourhood deformation retract
OPE	operator product expansion
OTC	over the counter
OZI	Okubo–Zweig–Iizuka
PAC	probably approximately correct
PDF	probability density function
PID	principal ideal domain
PL	piecewise-linear
PMF	probability mass function
POVM	positive operator-valued measure
PRP	predictable representation property
PVM	projection-valued measure
RKHS	reproducing kernel Hilbert space
SVM	support-vector machine
TDSE	time-dependent Schrödinger equation
TISE	time-independent Schrödinger equation
TQFT	topological quantum field theory
TVS	topological vector space
UFD	unique factorization domain
VC	Vapnik–Chervonenkis
VIF	variance inflation factor
VOA	vertex operator algebra
WKB	Wentzel–Kramers–Brillouin

ZFC	Zermelo–Frenkel set theory with the axiom of choice

Operations

Ad_g	adjoint representation of a Lie group G
ad_X	adjoint representation of a Lie algebra $\mathfrak{g}$
arg	argument of a complex number
$\Box$	d'Alembert operator
$\mathsf{deg}(f)$	degree of a polynomial f
e	identity element of a group
$\delta_\xi \phi$	field variation (equal to $\mathcal{L}_\xi \phi$)
$\Gamma(E)$	set of global sections of a fibre bundle E
Im, $\mathfrak{I}$	imaginary part of a complex number
$\mathsf{Ind}_f(z)$	index of a point $z \in \mathbb{C}$ with respect to a function f
$\hookrightarrow$	injective function
$\cong$	is isomorphic to
$A \multimap B$	linear implication
$N \lhd G$	N is a normal subgroup of G
Par_t^γ	parallel transport map along a curve γ
Re, $\mathfrak{R}$	real part of a complex number
Res	residue of a complex function
$\twoheadrightarrow$	surjective function
$\{\cdot,\cdot\}$	Poisson bracket
$X \pitchfork Y$	transversally intersecting manifolds X, Y
∂X	boundary of a topological space X
$\overline{X}$	closure of a topological space X
X°, $\mathring{X}$	interior of a topological space X
$\sphericalangle(\cdot,\cdot)$	angle between two vectors
$X \times Y$	cartesian product of two sets X, Y
$X + Y$	sum of two vector spaces X, Y
$X \oplus Y$	direct sum of two vector spaces X, Y
$V \otimes W$	tensor product of two vector spaces V, W
$\mathbb{1}_X$	identity morphism on an object X

$\approx$	is approximately equal to
$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto

Objects

Ab	category of Abelian groups
$\mathsf{Aut}(X)$	automorphism group of an object X
$\mathcal{B}_0(V, W)$	space of compact bounded operators between two Banach spaces V, W
$\mathcal{B}_1(\mathcal{H})$	space of trace-class operators on a Hilbert space
$\mathcal{B}(V, W)$	space of bounded linear maps between two vector spaces V, W
CartSp	category of Euclidean spaces and 'suitable' morphisms (e.g. linear maps, smooth maps, ...)
$C(X, Y)$	set of continuous functions between two topological spaces X, Y
S'	centralizer of a subset (of a ring)
$C_\bullet$	chain complex
$\mathbf{Ch}(\mathbf{A})$	category of chain complexes with objects in an additive category $\mathbf{A}$
$\mathbf{C}^\infty$, **SmoothSet**	category of smooth sets
$C_p^\infty(M)$	ring of smooth functions $f : M \to \mathbb{R}$ on a neighbourhood of $p \in M$
$C\ell(A, Q)$	Clifford algebra over an algebra A induced by a quadratic form Q
$C^\omega(V)$	set of all analytic functions defined on a set V
$\mathsf{Conf}(M)$	conformal group of a (pseudo-)Riemannian manifold M
$\mathbf{C}^\infty\mathbf{Ring}$, $\mathbf{C}^\infty\mathbf{Alg}$	category of smooth algebras
$S_k(\Gamma)$	space of cusp forms of weight $k \in \mathbb{R}$
Δ_X	diagonal of a set X
Diff	category of smooth manifolds
DiffSp	category of diffeological spaces and smooth maps
$\mathcal{D}_M$	sheaf of differential operators
D^n	standard n-disk
$\mathsf{dom}(f)$	domain of a function f
$\mathsf{End}(X)$	endomorphism monoid of an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad

$\mathbf{FormalCartSp_{diff}}$ category of infinitesimally thickened Euclidean spaces

$\mathsf{Frac}(I)$ field of fractions of an integral domain I

$\mathfrak{F}(V)$ space of Fredholm operators on a Banach space V

$\mathbb{G}_a$ additive group (scheme)

$\mathsf{GL}(V)$ general linear group: group of automorphisms of a vector space V

$\mathsf{GL}(n, \mathfrak{K})$ general linear group: group of invertible $n \times n$-matrices over a field $\mathfrak{K}$

Grp category of groups and group homomorphisms

Grpd category of groupoids

$\mathsf{Hol}_p(\omega)$ holonomy group at a point p with respect to a principal connection ω

$\mathsf{Hom}_{\mathbf{C}}(V, W)$, $\mathbf{C}(V, W)$ collection of morphisms between two objects V, W in a category $\mathbf{C}$

hTop homotopy category

$I(S)$ vanishing ideal on an algebraic set S

$I(x)$ rational fractions over an integral domain I

$\mathsf{im}(f)$ image of a function f

$K^0(X)$ K-theory over a (compact Hausdorff) space X

Kan category of Kan complexes

$K(A)$ Grothendieck completion of a monoid A

$\mathcal{K}_n(A, v)$ Krylov subspace of dimension n generated by a matrix A and a vector v

L^1 space of integrable functions

Law category of Lawvere theories

$\mathfrak{Lie}$ category of Lie algebras

Lie category of Lie groups

$\mathfrak{X}^L$ space of left-invariant vector fields on a Lie group

$\mathsf{llp}(I)$ set of morphisms having the left lifting property with respect to I

LX free loop space on a topological space X

$\mathbf{Man}^p$ category of C^p-manifolds

Meas
- category of measurable spaces and measurable functions, or
- category of measure spaces and measure-preserving functions

M^4 four-dimensional Minkowski space

$M_k(\Gamma)$ space of modular forms of weight $k \in \mathbb{R}$

$\mathbb{F}^X$ natural filtration of a stochastic process $(X_t)_{t \in T}$

$N\mathbf{C}$	simplicial nerve of a small category $\mathbf{C}$
$\mathsf{O}(n, \mathfrak{K})$	group of $n \times n$ orthogonal matrices over a field $\mathfrak{K}$
$\mathbf{Open}(X)$	category of open subsets of a topological space X
$P(X), 2^X$	power set of a set X
$\mathsf{Pin}(V)$	pin group of the Clifford algebra $C\ell(V, Q)$
$\mathbf{Psh}(\mathbf{C}), \widehat{\mathbf{C}}$	category of presheaves on a (small) category $\mathbf{C}$
$R((x))$	ring of (formal) Laurent series in x with coefficients in R
$\mathsf{rlp}(I)$	set of morphisms having the right lifting property with respect to I
$R[[x]]$	ring of (formal) power series in x with coefficients in R
S^n	standard n-sphere
$S^n(V)$	space of symmetric rank n tensors over a vector space V
$\mathbf{Sh}(X)$	category of sheaves on a topological space X
$\mathbf{Sh}(\mathbf{C}, J)$	category of J-sheaves on a site $(\mathbf{C}, J)$
$\mathbf{\Delta}$	simplex category
$\mathsf{sing\ supp}(\phi)$	singular support of a distribution ϕ
$\mathsf{SL}_n(\mathfrak{K})$	special linear group: group of all $n \times n$-matrices with unit determinant over a field $\mathfrak{K}$
$W^{m,p}(U)$	Sobolov space in L^p of order m
$\mathbf{Span}(\mathbf{C})$	span category over a category $\mathbf{C}$
$\mathsf{Spec}(R)$	spectrum of a commutative ring R
$\mathbf{sSet}_{\text{Quillen}}$	Quillen's model structure on simplicial sets
$\mathsf{supp}(f)$	support of a function f
$\mathsf{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
$\mathsf{Sym}(X)$	symmetric group of a set X
S_n	symmetric group of degree n
$\mathsf{Sym}(X)$	symmetric group on a set X
$\mathsf{Sp}(n, \mathfrak{K})$	group of matrices preserving a canonical symplectic form over a field $\mathfrak{K}$
$\mathsf{Sp}(n)$	compact symplectic group
$\mathbb{T}^n$	standard n-torus (n-fold Cartesian product of S^1)
$T_{\leq t}$	set of all elements smaller than (or equal to) $t \in T$ for a partial order T
$\mathsf{TL}_n(\delta)$	Temperley–Lieb algebra with $n - 1$ generators and parameter δ
$\mathbf{Top}$	category of topological spaces and continuous functions

Topos	(2-)category of (elementary) topoi and geometric morphisms
$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\mathsf{U}(n, \mathfrak{K})$	group of $n \times n$ unitary matrices over a field $\mathfrak{K}$
$V(I)$	algebraic set corresponding to an ideal I
$\mathbf{Vect}(X)$	category of vector bundles over a manifold X
$\mathbf{Vect}_{\mathfrak{K}}$	category of vector spaces and linear maps over a field $\mathfrak{K}$
Y^X	set of functions between two sets X, Y
$\mathbb{Z}_p$	group of p-adic integers
$\emptyset$	empty set
$\pi_n(X, x_0)$	n^{th} homotopy space over X with basepoint x_0
$[a, b]$	closed interval
$]a, b[$	open interval
$\Lambda^n(V)$	space of antisymmetric rank-n tensors over a vector space V
ΩX	(based) loop space on a topological space X
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on a manifold M
$\rho(A)$	resolvent set of a bounded linear operator A
$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on a manifold M

Units

C	Coulomb
T	Tesla

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