Compendium of Mathematics & Physics

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Introduction

Goals

This compendium originated out of the necessity for a compact summary of important theorems and formulas during physics and mathematics classes at university. When the interest in more (and more exotic) subjects grew, this collection lost its compactness and became the chaos it now is. Although there should exist some kind of overall structure, it was not always possible to keep every section self-contained or respect the order of the chapters.

It should definitely not be used as a formal introduction to any subject. It is neither a complete work nor a fact-checked one, so the usefulness and correctness is not guaranteed. However, it can be used as a look-up table for theorems and formulas, and as a guide to the literature. To this end, each chapter begins with a list of useful references. At the same time, only a small number of statements are proven in the text (or appendices). This was done to keep the text as concise as possible (a failed endeavour). However, in some cases the major ideas underlying the proofs are provided.

Structure and conventions

Sections and statements that require more advanced concepts, in particular concepts from later chapters or (higher) category theory, will be labelled by the *clubs* symbol ♣. Some definitions, properties or formulas are given with a proof or an extended explanation whenever I felt like it. These are always contained in a blue frame to make it clear that they are not part of the general compendium. When a section uses notions or results from a different chapter at its core, this will be recalled in a green box at the beginning of the section.

Definitions in the body of the text will be indicated by the use of **bold font**. Notions that have not been defined in this summary but that are relevant or that will be defined further on in the compendium (in which case a reference will be provided) are indicated by *italic text*. Names of authors are also written in *italic*.

Objects from a general category will be denoted by a lower-case letter (depending on the context, upper-case might be used for clarity), functors will be denoted by upper-case letters and the categories themselves will be denoted by symbols in **bold font**. In the later chapters on physics, specific conventions for the different types of vectors will often be adopted. Vectors in Euclidean space will be denoted by a bold font letter with an arrow above, e.g. $\vec{\boldsymbol{a}}$, whereas vectors in Minkowski space (4-vectors) and differential forms will be written without the arrow, e.g. $\boldsymbol{a}$. Matrices and tensors will always be represented by capital letters and, dependent on the context, a specific font will be adopted.

Chapter 1

Category theory

For the general theory of categories, the classical reference is Mac Lane (2013) or the more modern account Riehl (2017). The main reference for (co)end calculus is Loregian (2021), while a thorough introduction to the theory of enrichment is given in Kelly (1982). For the theory of higher categories and its applications to topology and algebra, the reader is referred to the book by Baez and May (2009). A good starting point for bicategories (and more) is the paper by Leinster (1998).

Chapter Contents

1.1 Categories

Definition 1.1.1 (Category). A category $\mathbf{C}$ consists of two collections, the objects $\mathrm{ob}(\mathbf{C})$ and the morphisms $\mathrm{hom}(\mathbf{C})$ or $\mathrm{mor}(\mathbf{C})$, that satisfy the following conditions:

1. **Source and target**: For every morphism $f \in \mathrm{hom}(\mathbf{C})$, there exist two objects $s(f), t(f) \in \mathrm{ob}(\mathbf{C})$, the source and the target. The collection of all morphisms with source x and target y is denoted by $\mathrm{Hom}_{\mathbf{C}}(x, y)$ or $\mathbf{C}(x, y)$.

2. **Composition**: For every two morphisms $f \in \mathbf{C}(y, z)$ and $g \in \mathbf{C}(x, y)$, the composite $f \circ g$ is an element of $\mathbf{C}(x, z)$. Moreover, composition is required to be associative.

3. **Identity**: For every $x \in \mathrm{ob}(\mathbf{C})$, there exists an identity morphism $\mathbb{1}_x \in \mathbf{C}(x, x)$. Identity morphisms are required to satisfy $f \circ \mathbb{1}_x = f = \mathbb{1}_y \circ f$ for all morphisms $f \in \mathbf{C}(x, y)$.

Remark 1.1.2. One technically does not need to consider objects as a separate notion since every object can be identified with its identity morphism (which exists by definition) and, hence, one can work solely with morphisms. It should be noted that for higher categories this remark can be omitted since the objects are always regarded as 0-morphisms in that context.

Definition 1.1.3 (Subcategory). Consider two categories $\mathbf{C}$ and $\mathbf{S}$. $\mathbf{S}$ is called a subcategory of $\mathbf{C}$ if $\mathrm{ob}(\mathbf{S})$ and $\mathrm{hom}(\mathbf{S})$ are subcollections of $\mathrm{ob}(\mathbf{C})$ and $\mathrm{hom}(\mathbf{C})$, respectively.

A subcategory is said to be **full** if for every two objects $x, y \in \mathrm{ob}(\mathbf{S})$:

$$\mathbf{S}(x, y) = \mathbf{C}(x, y)\,. \tag{1.1}$$

A subcategory is said to be **wide** or **lluf** if it contains all objects:

$$\mathrm{ob}(\mathbf{S}) = \mathrm{ob}(\mathbf{C})\,. \tag{1.2}$$

Definition 1.1.4 (Replete subcategory). A subcategory $\mathbf{S} \subseteq \mathbf{C}$ such that if $x \in \mathrm{ob}(\mathbf{S})$ and $f : x \cong y \in \mathrm{hom}(\mathbf{C})$, then also $y \in \mathrm{ob}(\mathbf{S})$ and $f \in \mathrm{hom}(\mathbf{S})$.

Definition 1.1.5 (Small category). A category $\mathbf{C}$ for which both $\mathrm{ob}(\mathbf{C})$ and $\mathrm{hom}(\mathbf{C})$ are sets. A category $\mathbf{C}$ is said to be **locally small** if for every two objects $x, y \in \mathrm{ob}(\mathbf{C})$ the collection of morphisms $\mathbf{C}(x, y)$ is a set. A category *equivalent* (see further down below) to a small category is said to be **essentially small**.

Definition 1.1.6 (Opposite category). Let $\mathbf{C}$ be a category. The opposite category $\mathbf{C}^{\mathrm{op}}$ is constructed by reversing all arrows in $\mathbf{C}$, i.e. a morphism in $\mathbf{C}^{\mathrm{op}}(x,y)$ is a morphism in $\mathbf{C}(y,x)$.

Property 1.1.7 (Involution). From the definition of the opposite category it readily follows that *op* is an involution:

$$(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}\,. \tag{1.3}$$

1.2 Functors

Definition 1.2.1 (Covariant functor). Let $\mathbf{C}, \mathbf{D}$ be categories. A (covariant) functor is an assignment $F : \mathbf{C} \to \mathbf{D}$ satisfying the following conditions:

1. F maps every object $x \in \mathsf{ob}(\mathbf{C})$ to an object $Fx \in \mathsf{ob}(\mathbf{D})$.
2. F maps every morphism $\phi \in \mathbf{C}(x,y)$ to a morphism $F\phi \in \mathbf{D}(Fx, Fy)$.
3. F preserves identities, i.e. $F\mathbb{1}_x = \mathbb{1}_{Fx}$.
4. F preserves compositions, i.e. $F(\phi \circ \psi) = F\phi \circ F\psi$.

Remark 1.2.2 (Category of categories). Small categories, together with (covariant) functors between them, form a category $\mathbf{Cat}$. The restriction to small categories is important since otherwise one would obtain an inconsistency similar to *Russell's paradox*. In certain foundations one can also consider the 'category' $\mathbf{CAT}$ of all categories, but this would not be a large category anymore. It would be something like a 'very large' category.

Definition 1.2.3 (Contravariant functor). Let $\mathbf{C}, \mathbf{D}$ be categories. A contravariant functor is an assignment $F : \mathbf{C} \to \mathbf{D}$ satisfying the following conditions:

1. F maps every object $x \in \mathsf{ob}(\mathbf{C})$ to an object $Fx \in \mathsf{ob}(\mathbf{D})$.
2. F maps every morphism $\phi \in \mathbf{C}(x,y)$ to a morphism $F\phi \in \mathbf{D}(Fy, Fx)$.
3. F preserves identities, i.e. $F\mathbb{1}_x = \mathbb{1}_{Fx}$.
4. F reverses compositions, i.e. $F(\phi \circ \psi) = F\psi \circ F\phi$.

A contravariant functor can also be defined as a covariant functor from the opposite category and, accordingly, from now on the word 'covariant' will be dropped when talking about functors.

Definition 1.2.4 (Endofunctor). A functor of the form $F : \mathbf{C} \to \mathbf{C}$.

Definition 1.2.5 (Presheaf). A functor $G : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. The collection of all presheaves on a (small) category $\mathbf{C}$ forms a category $\mathbf{Psh}(\mathbf{C})$. This is sometimes also denoted by $\widehat{\mathbf{C}}$.

Example 1.2.6 (Hom-functor). Let $\mathbf{C}$ be a locally small category. Every object $x \in \mathrm{ob}(\mathbf{C})$ induces a functor $h^x : \mathbf{C} \to \mathbf{Set}$ defined as follows:

- h^x maps every object $y \in \mathrm{ob}(\mathbf{C})$ to the set $\mathbf{C}(x,y)$.
- For all $y,z \in \mathrm{ob}(\mathbf{C})$, h^x maps every morphism $f \in \mathbf{C}(y,z)$ to the fucntion

$$f \circ - : \mathbf{C}(x,y) \to \mathbf{C}(x,z) : g \mapsto f \circ g\,.$$

Remark 1.2.7. The contravariant hom-functor h_x is defined by replacing $\mathbf{C}(x,-)$ with $\mathbf{C}(-,x)$ and replacing postcomposition with precomposition.

Definition 1.2.8 (Faithful functor). A functor $F : \mathbf{C} \to \mathbf{D}$ for which the map

$$\mathbf{C}(x,y) \to \mathbf{D}(Fx,Fy)$$

is injective for all objects $x,y \in \mathrm{ob}(\mathbf{C})$.

Definition 1.2.9 (Full functor). A functor $F : \mathbf{C} \to \mathbf{D}$ for which the map

$$\mathbf{C}(x,y) \to \mathbf{D}(Fx,Fy)$$

is surjective for all objects $x,y \in \mathrm{ob}(\mathbf{C})$.

Definition 1.2.10 (Embedding). A fully faithful functor.

Definition 1.2.11 (Concrete category). A category equipped with an embedding into **Set**. The objects of such categories can be interpreted as sets with additional structure.

Definition 1.2.12 (Essentially surjective functor). A functor $F : \mathbf{C} \to \mathbf{D}$ such that for every object $y \in \mathrm{ob}(\mathbf{D})$, there exists an object $x \in \mathrm{ob}(\mathbf{C})$ with $Fx \cong y$.

Definition 1.2.13 (Profunctor[1]). A functor of the form $F : \mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$. Such a functor is often denoted by $F : \mathbf{C} \nrightarrow \mathbf{D}$.[2] Elements of the set $F(x,y)$ are called **heteromorphisms** (between x and y).

It should be noted that presheaves on $\mathbf{C}$ are profunctors of the form $1 \nrightarrow \mathbf{C}$.

Definition 1.2.14 (Reflection). A functor $F : \mathbf{C} \to \mathbf{D}$ is said to **reflect** a property if whenever the property holds for Fc, it also holds for $c \in \mathbf{C}$. (Here, c could also be a morphism.)

[1] Sometimes called a **distributor**.

[2] This is the convention by *Borceux*. Some other authors, such as Johnstone (2014), use the opposite convention.

1.2.1 Natural transformations

Definition 1.2.15 (Natural transformation). Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors. A natural transformation $\psi : F \Rightarrow G$[3] consists of a collection of morphisms satisfying the following two conditions:

1. For every object $x \in \mathrm{ob}(\mathbf{C})$, there exists a morphism $\psi_x : Fx \to Gx$ in $\mathrm{hom}(\mathbf{D})$. This morphism is called the **component** of ψ at x. (It is often said that ψ_x **is natural in** x.)
2. For every morphism $f \in \mathbf{C}(x, y)$, the diagram below commutes:

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\psi_x"'] & Fy \arrow[d, "\psi_y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

Definition 1.2.16 (Functor category). Consider two categories $\mathbf{C}$ and $\mathbf{D}$, where $\mathbf{C}$ is small. The functors $F : \mathbf{C} \to \mathbf{D}$ form the objects of a category with the natural transformations as morphisms. This category is denoted by $[\mathbf{C}, \mathbf{D}]$ or $\mathbf{D}^{\mathbf{C}}$ (the latter is a generalization of **??**).

Definition 1.2.17 (Dinatural transformation). Consider two profunctors $F, G : \mathbf{C} \nrightarrow \mathbf{C}$ or, more generally, two functors $F, G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$. A dinatural transformation is a family of morphisms

$$\eta_x : F(x, x) \to G(x, x)$$

that make Diagram 1.1 commute for every morphism $f : y \to x$.

Definition 1.2.18 (Representable functor). Let $\mathbf{C}$ be a locally small category. A functor $F : \mathbf{C} \to \mathbf{Set}$ is said to be representable if there exists an object $x \in \mathrm{ob}(\mathbf{C})$ such that F is naturally isomorphic to h^x. The pair $(x, \psi : F \Rightarrow h^x)$ is called a **representation** of F.

Theorem 1.2.19 (Yoneda lemma). *Let $\mathbf{C}$ be a locally small category and let $F : \mathbf{C} \to \mathbf{Set}$ be a functor. For every object $x \in \mathrm{ob}(\mathbf{C})$, there exists a natural isomorphism*[4]

$$\eta_x : \mathrm{Nat}(h^x, F) \to Fx : \psi \mapsto \psi_x(\mathbb{1}_x)\,. \tag{1.4}$$

Corollary 1.2.20 (Yoneda embedding). When F is another hom-functor h^y, the following result is obtained:

$$\mathrm{Nat}(h^x, h^y) \cong \mathbf{C}(y, x)\,. \tag{1.5}$$

[3]This notation is in analogy with the general notation for *2-morphisms*. See **??** for more information.
[4]Here, the fact that $\mathrm{Nat}(h^-, -)$ can be seen as a functor $\mathbf{Set}^{\mathbf{C}} \times \mathbf{C} \to \mathbf{Set}$ is used.

$$\begin{tikzcd}
& F(x,y) \arrow[dl, "F(\mathbb{1}_x, f)"'] \arrow[dr, "F(f, \mathbb{1}_y)"] & \\
F(x,x) \arrow[d, "\eta_x"'] & & F(y,y) \arrow[d, "\eta_y"] \\
G(x,x) \arrow[dr, "G(f, \mathbb{1}_x)"'] & & G(y,y) \arrow[dl, "G(\mathbb{1}_y, f)"] \\
& G(y,x) &
\end{tikzcd}$$

Figure 1.1: Dinatural transformation.

Note that y appears in the first argument on the right-hand side.

Let $\mathbf{C}(f,-)$ denote the natural transformation corresponding to the morphism $f \in \mathbf{C}(y,x)$. The functor h^-, mapping an object $x \in \mathrm{ob}(\mathbf{C})$ to its hom-functor $\mathbf{C}(x,-)$ and a morphism $f \in \mathbf{C}(y,x)$ to the natural transformation $\mathbf{C}(f,-)$, can also be interpreted as a covariant functor $G : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{C}}$. This way, the Yoneda lemma can be seen to give rise to an embedding h^- of $\mathbf{C}^{\mathrm{op}}$ in the functor category $\mathbf{Set}^{\mathbf{C}}$.

As usual, all of this can be done for contravariant functors. This gives an embedding

$$よ := h_- : \mathbf{C} \hookrightarrow \mathbf{Psh}(\mathbf{C}), \tag{1.6}$$

called the Yoneda embedding.

Definition 1.2.21 (Local object). Consider a collection of morphisms $S \subseteq \mathrm{hom}(\mathbf{C})$. An object $c \in \mathrm{ob}(\mathbf{C})$ is said to be S-local if the Yoneda embedding $よc$ maps morphisms in S to isomorphisms in $\mathbf{Set}$. A morphism $f \in \mathrm{hom}(\mathbf{C})$ is said to be S-local if its image under the Yoneda embedding of every S-local object is an isomorphism in $\mathbf{Set}$.

1.2.2 Equivalences

Definition 1.2.22 (Equivalence of categories). Two categories $\mathbf{C}, \mathbf{D}$ are said to be equivalent if there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors.

A weaker notion is that of a **weak equivalence**. Two categories $\mathbf{C}, \mathbf{D}$ are said to be weakly equivalent if there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ that are fully faithful and essentially surjective. Assuming the axiom of choice, every weak equivalence is also a (strong) equivalence (in fact this statement is equivalent to the axiom of choice).

Definition 1.2.23 (Skeletal category). A category in which every isomorphism is necessarily an identity morphism. The **skeleton** of a category is an equivalent skeletal category (often taken to be a subcategory by choosing a representative from every isomorphism class).

If one does not assume the axiom of choice, the skeleton is merely a weakly equivalent skeletal category.

Definition 1.2.24 (Decategorification). Let $\mathbf{C}$ be an (essentially) small category. The set of isomorphism classes of $\mathbf{C}$ is called the decategorification of $\mathbf{C}$. This amounts to a functor $\mathrm{Decat} : \mathbf{Cat} \to \mathbf{Set}$.

1.2.3 Stuff, structure and property

To classify properties of objects and the *forgetfulness* of functors, it is interesting to make a distinction between stuff, structure and property. Consider for example a group. This is a set (*stuff*) equipped with a number of operations (*structure*) that obey some relations (*properties*).

Using these notions one can classify forgetful functors in the following way:

- A functor forgets nothing if it is an equivalence of categories.
- A functor forgets at most properties if it is fully faithful.
- A functor forgets at most structure if it is faithful.
- A functor forgets at most stuff if it is just a functor.

@@ COMPLETE (see e.g. nLab or the paper "Why surplus structure is not superfluous" by Nicholas Teh et al.) @@

1.2.4 Adjunctions

Definition 1.2.25 (Hom-set adjunction). Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be two functors. These functors form a (hom-set) adjunction $F \dashv G$ if the following isomorphism is natural in both x and y:

$$\Phi_{x,y} : \mathbf{D}(Fx, y) \cong \mathbf{C}(x, Gy)\,. \tag{1.7}$$

The functor F (resp. G) is called the left (resp. right) adjoint and the image of a morphism under either of the natural isomorphisms is called the adjunct of the other morphism.[5].

[5]The terms 'adjunct' and 'adjoint' are sometimes used interchangeably (cf. French versus Latin).

Notation 1.2.26. An adjunction $F \dashv G$ between categories $\mathbf{C}, \mathbf{D}$ is often denoted by

$$\begin{tikzcd} \mathbf{D} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, shift right, "F"'] \end{tikzcd}.$$

Definition 1.2.27 (Unit-counit adjunction). Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be two functors. These functors form a unit-counit adjunction if there exist natural transformations

$$\varepsilon : F \circ G \Rightarrow \mathbb{1}_{\mathbf{D}} \tag{1.8}$$

$$\eta : \mathbb{1}_{\mathbf{C}} \Rightarrow G \circ F \tag{1.9}$$

such that the following compositions are identity morphisms:

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F\,, \tag{1.10}$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G\,. \tag{1.11}$$

These identities are sometimes called the **triangle** or **zig-zag identities** (the latter results from the shape of the associated *string diagram*). The transformations η and ε are called the **unit** and **counit**, respectively.

Property 1.2.28 (Equivalence). Every hom-set adjunctions induces a unit-counit adjunction where the counit ε_y is obtained as the adjunct $\Phi^{-1}_{Gy,y}(\mathbb{1}_{Gy})$ of the identity morphism on $Gy \in \mathrm{ob}(\mathbf{C})$ and the unit η_x is given by the adjunct $\Phi_{c,Fc}(\mathbb{1}_{Fx})$ of the identity morphism at $Fx \in \mathrm{ob}(\mathbf{D})$.

Conversely, every unit-counit adjunction induces a hom-set adjunction. The (right) adjunct of a morphism $f : Fx \to y$ is given by the composition

$$\tilde{f} := Gf \circ \eta_x : x \to (G \circ F)x \to Gy \tag{1.12}$$

and the (left) adjunct of a morphism $\tilde{g} : x \to Gy$ is given by:

$$g := \varepsilon_y \circ F\tilde{g} : Fx \to (F \circ G)y \to y\,. \tag{1.13}$$

Definition 1.2.29 (Reflective subcategory). A full subcategory is said to be reflective (resp. coreflective) if the inclusion functor admits a left (resp. right) adjoint.

Property 1.2.30 (Adjoint equivalence). Any equivalence of categories is part of an adjoint equivalence, i.e. an adjunction for which the unit and counit morphisms are invertible.

Property 1.2.31. Given an adjunction, one obtains an adjoint equivalence by restricting to the full subcategories on which the unit and counit become isomorphisms.

Adjunctions can also be defined through a third alternative. This links the definition to universal properties.

Definition 1.2.32 (Universal morphism). Consider a functor $F : \mathbf{C} \to \mathbf{D}$ and an object $d \in \mathsf{ob}(\mathbf{D})$. A universal morphism from d to F is a pair $(c, f : d \to Fc)$ such that all other morphisms $f' : d \to Fc'$ factor uniquely through f by the image of a morphism in $\mathbf{C}$.

Alternative Definition 1.2.33 (Adjoint functor). A functor $F : \mathbf{C} \to \mathbf{D}$ is a left adjoint if for each object $c \in \mathsf{ob}(\mathbf{C})$ there exists a universal morphism from F to c. A functor $G : \mathbf{C} \to \mathbf{D}$ is a right adjoint if for each object $d \in \mathsf{ob}(\mathbf{D})$ there exists a universal morphism from d to G.

The functor to which these functors are adjoint can be recovered as that mapping objects to the object-part of their universal morphism.

1.3 General constructions

Definition 1.3.1 (Dagger category). A category equipped with a contravariant involutive endofunctor, this functor is often denoted by $\dagger : \mathbf{C} \to \mathbf{C}$, similar to the adjoint operator for Hermitian matrices.

Remark 1.3.2. The concept of a dagger structure allows the usual definition of **unitary** and **self-adjoint** morphisms, i.e. morphism satisfying

$$f^\dagger = f^{-1} \qquad \text{or} \qquad f^\dagger = f\,. \tag{1.14}$$

Definition 1.3.3 (Comma category). Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be three categories and let $F : \mathbf{A} \to \mathbf{C}$ and $G : \mathbf{B} \to \mathbf{C}$ be two functors. The comma category $F \downarrow G$ is defined as follows:

- **Objects**: The triples (x, y, γ) where $x \in \mathsf{ob}(\mathbf{A}), y \in \mathsf{ob}(\mathbf{B})$ and $\gamma : Fx \to Gy$.
- **Morphisms**: The morphisms $(x, y, \gamma) \to (k, l, \sigma)$ are pairs (f, g) with $f : x \to k \in \mathsf{hom}(\mathbf{A})$ and $g : y \to l \in \mathsf{hom}(\mathbf{B})$ such that $\sigma \circ Ff = Gg \circ \gamma$.

Composition of morphisms is defined componentwise.

Definition 1.3.4 (Arrow category). The comma category of the pair of functors $(\mathbb{1}_{\mathbf{C}}, \mathbb{1}_{\mathbf{C}})$. This is equivalently the functor category $[\mathbf{2}, \mathbf{C}]$, where $\mathbf{2}$ is the **interval category/walking arrow** $\{0 \to 1\}$.

Definition 1.3.5 (Functorial factorization). A *section* (see Definition 1.4.1) of the composition functor

$$\circ : [\mathbf{3}, \mathbf{C}] \to [\mathbf{2}, \mathbf{C}]\,,$$

where $\mathbf{3}$ is the poset $\{0 \to 1 \to 2\}$.

If F is the identity functor and $G : \mathbf{1} \to \mathbf{C}$ picks out a single object, the notion of a slice category is obtained (by interchanging these choices, one can also define **coslice categories**).

Definition 1.3.6 (Slice category). Let $\mathbf{C}$ be a category and consider an object $x \in \mathrm{ob}(\mathbf{C})$. The slice category (or **overcategory**) $\mathbf{C}_{/x}$ of $\mathbf{C}$ over x is defined as follows:

- **Objects**: The morphisms in $\mathbf{C}$ with codomain x, and
- **Morphisms**: The morphisms $f \to g$ are morphisms h in $\mathbf{C}$ such that $g \circ h = f$.

By dualizing one obtains the **undercategory** of x.

1.3.1 Fibred categories ♣

Definition 1.3.7 (Fibre category). Let $\Pi : \mathbf{C} \to \mathbf{D}$ be a functor. The fibre category (of Π) over $y \in \mathrm{ob}(\mathbf{D})$ is the subcategory of $\mathbf{C}$ consisting of all objects $x \in \mathrm{ob}(\mathbf{C})$ such that $\Pi x = y$ and all morphisms $f \in \mathrm{hom}(\mathbf{C})$ such that $\Pi f = \mathbb{1}_y$. It will be denoted by $\mathbf{C}_y$.

Morphisms in $\mathbf{C}$ that are mapped to a morphism g in $\mathbf{D}$ are called **g-morphisms** and, in particular (using the identification of objects and their identity morphisms), morphisms in $\mathbf{C}_y$ are called **y-morphisms**. Similarly, **D-categories** are defined as the categories equipped with a (covariant) functor to $\mathbf{D}$. (It is not hard to see that these form a *2-category* under composition of functors that respects the $\mathbf{D}$-category structure.)

Definition 1.3.8 (Cartesian morphism). Consider a $\mathbf{D}$-category $\Pi : \mathbf{C} \to \mathbf{D}$. A morphism f in $\mathbf{C}$ is called Π-Cartesian if every Πf-morphism factors uniquely through a y-morphism, where y is the domain of Πf.

There also exists a stronger notion. A **strongly Cartesian morphism** is a morphism $f \in \mathrm{hom}(\mathbf{C})$ such that for every morphism $\varphi \in \mathrm{hom}(\mathbf{C})$ with the same target and every factorization of $\Pi\varphi$ through Πf, there exists a unique factorization of φ through f that maps to the given factorization of $\Pi\varphi$.

The following diagram, where the triangles commute, should clarify the above (technical) definitions:

$$\begin{tikzcd}
\forall x' \arrow[d, dashed, "\exists! g"'] \arrow[dr, "\forall\varphi"] & & & \Pi x' \arrow[d, "\forall\nu"'] \arrow[dr, "\Pi\varphi"] & \\
x_1 \arrow[r, "f"'] & x_2 & \overset{\Pi}{\mapsto} & \Pi x_1 \arrow[r, "\Pi f"'] & \Pi x_2
\end{tikzcd}$$

The diagram for (weak) Cartesian morphisms is obtained by identifying the objects $\Pi x'$ and Πx_1, i.e. by restricting to the case $\nu = \mathbb{1}_{\Pi x_1}$.

The Cartesian morphisms are said to be **inverse images** of their projections under Π and the object x_1 is called an **inverse image** of x_2 by Πf. The Cartesian morphisms of a fibre category are exactly the isomorphisms of that category.

Definition 1.3.9 (Fibred category). A $\mathbf{D}$-category $\Pi : \mathbf{C} \to \mathbf{D}$ is called a fibred category or **Grothendieck fibration** if the following conditions are satisfied:

1. For each morphism in $\mathbf{D}$, whose codomain lies in the range of Π, and each lift of this codomain to $\mathbf{C}$, there exists at least one inverse image with the given codomain (in the weak sense).

2. The composition of two Cartesian morphisms is again Cartesian (in the weak sense).

If one instead works with strongly Cartesian morphisms, the second condition follows from the first one. However, it should be noted that, in a fibred category, a morphism is weakly Cartesian if and only if it is strongly Cartesian.

Definition 1.3.10 (Cleavage). Given a $\mathbf{D}$-category $\Pi : \mathbf{C} \to \mathbf{D}$, a cleavage is the choice of a Cartesian g-morphism $f : x \to y$ for every $y \in \mathrm{ob}(\mathbf{C})$ and morphism $g : d \to \Pi y$. A $\mathbf{D}$-category equipped with a cleavage is said to be **cloven**.

The existence of cleavage is sufficient for a category to be fibred and, conversely (assuming the axiom of choice), every fibred category admits a cleavage.

The following example can be obtained as a Grothendieck fibration with discrete fibres.

Example 1.3.11 (Discrete fibration). A functor $F : \mathbf{C} \to \mathbf{D}$ such that, for every object $x \in \mathrm{ob}(\mathbf{C})$ and every morphism $f : y \to Fx$ in $\mathbf{D}$, there exists a unique morphism $g : z \to x$ in $\mathbf{C}$ such that $Fg = f$.

Example 1.3.12 (Groupoidal fibration). If every morphism is required to be Cartesian, the notion of a groupoid(al) fibration or a **category fibred in groupoids** is obtained. The reason for this name is that every fibre is a groupoid. An equivalent definition is that the associated *pseudofunctor* (see the construction below) factors through the embedding $\mathbf{Grpd} \hookrightarrow \mathbf{Cat}$.

Property 1.3.13 (Grothendieck construction ♣). Every fibred category $\Pi : \mathbf{C} \to \mathbf{D}$ defines a *pseudofunctor*[6] $F : \mathbf{D}^{\mathrm{op}} \to \mathbf{Cat}$ that sends objects to fibre categories and arrows $f : c \to c'$ to the pullback functor $f^* : \mathbf{C}_{c'} \to \mathbf{C}_c$ constructed from a Cartesian lift of f. This pullback functor acts as follows:

- For every object $x \in \mathbf{C}_{c'}$, f^*x is the domain of the Cartesian lift of f through x.

[6]See **??**.

- For every morphism $(\alpha : x \to y) \in \mathbf{C}_{c'}$ there exists a diagram of the form

$$\begin{tikzcd} f^*x \arrow[r] \arrow[d, dashed, "f^*\alpha"'] & x \arrow[d, "\alpha"] \\ f^*y \arrow[r] & y \end{tikzcd}$$

  Because the horizontal morphism are both projected to f and α is projected to the identity, there exists a unique factorization of the diagram through a morphism $f^*\alpha : f^*x \to f^*y$.

Conversely, every *pseudofunctor* gives rise to a fibred category through the Grothendieck construction $\int : [\mathbf{C}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{Cat}/\mathbf{C}$ as follows. (These two constructions constitute a 2-equivalence of 2-categories.). Consider a *pseudofunctor* $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$. The 'bundle' $\int F$ consists of the following data:

- The objects are pairs (x, y) with $x \in \mathrm{ob}(\mathbf{C})$ and $y \in \mathrm{ob}(Fx)$.
- The morphisms $(x, y) \to (x', y')$ are pairs $\big(f : x \to x', \alpha : y \to Ff(y')\big)$.

Given a cleavage, the morphisms of the Grothendieck construction are exactly the factorizations of f-morphisms through the canonical lifting of f in the cleavage.

Property 1.3.14 (Functors). A *pseudofunctor* is a functor if and only if the cleavage of the associated fibred category is **split(ting)**, i.e. it contains all identities and is closed under composition.

Definition 1.3.15 (Category of elements). Consider a presheaf $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. Its category of elements $\mathsf{El}(F)$ or $\int_{\mathbf{C}} F$ is defined as the comma category $(\mathcal{Y} \downarrow !_F)$, where $!_F : * \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ sends the unique object to F itself. Equivalently, it is the category with objects the pairs $(c, x) \in \mathrm{ob}(\mathbf{C}) \times Fc$ and morphisms $f \in \mathbf{C}(c, c')$ such that $c = Ff(c')$, i.e. it is the Grothendieck construction applied to F.

This category comes equipped with a canonical forgetful functor

$$\mathbf{C}_F : \mathsf{El}(F) \to \mathbf{C} : (c, x) \mapsto c\,. \tag{1.15}$$

Remark 1.3.16. The category of elements is often defined for covariant functors. To obtain that definition, one should take the opposite of the category of elements (and also take the opposite of the forgetful functor).

1.3.2 Monads

Definition 1.3.17 (Monad). A monad is a triple (T, μ, η) where $T : \mathbf{C} \to \mathbf{C}$ is an endofunctor and $\mu : T^2 \to T, \eta : \mathbb{1}_{\mathbf{C}} \to T$ are natural transformations satisfying the following (coherence) conditions:

1. As natural transformations from T^3 to T:

$$\mu \circ T\mu = \mu \circ \mu_T \,. \tag{1.16}$$

2. As natural transformations from T to itself:

$$\mu \circ T\eta = \mu \circ \eta_T = \mathbb{1} \,. \tag{1.17}$$

These conditions say that a monad is a monoid (**??**) in the category $\mathbf{End}_{\mathbf{C}}$ of endofunctors on $\mathbf{C}$. Accordingly, η and μ are often called the **unit** and **multiplication** maps.

Example 1.3.18 (Adjunction). Every adjunction $F \dashv G$, with unit ε and counit η, induces a monad of the form $(GF, G\varepsilon F, \eta)$.

Definition 1.3.19 (Algebra over a monad[7]). Consider a monad (T, μ, η) on a category $\mathbf{C}$. An algebra over T or T-algebra is a couple (x, κ), where $x \in \mathsf{ob}(\mathbf{C})$ and $\kappa : Tx \to x$, such that the following conditions are satisfied:

1. $\kappa \circ T\kappa = \kappa \circ \mu_x$, and
2. $\kappa \circ \eta_x = \mathbb{1}_x$.

Morphisms $(x, \kappa_x) \to (y, \kappa_y)$ of T-algebras are morphisms $f : x \to y$ in $\mathbf{C}$ such that $f \circ \kappa_x = \kappa_y \circ Tf$. An algebra of the form (Tx, μ_x) is said to be **free**. The object x is called the **carrier** of the algebra

Definition 1.3.20 (Eilenberg–Moore category). Given a monad T over a category $\mathbf{C}$, the Eilenberg–Moore category $\mathbf{C}^T$ is defined as the category of T-algebras.

Definition 1.3.21 (Kleisli category). Consider a monad T on a category $\mathbf{C}$. The Kleisli category $\mathbf{C}_T$ is defined as the full subcategory of $\mathbf{C}^T$ on the free T-algebras. This is equivalently the category with objects $\mathsf{ob}(\mathbf{C}_T) := \mathsf{ob}(\mathbf{C})$ and morphisms $\mathbf{C}_T(x, y) := \mathbf{C}(x, Ty)$.

Morphisms in the Kleisly category are composed in the 'obvious way':

$$f \circ_{\mathbf{C}_T} g := \mu_Z \circ Tf \circ g \tag{1.18}$$

for all $f \in \mathbf{C}_T(Y, TZ)$ and $g \in \mathbf{C}_T(X, TY)$.

Definition 1.3.22 (Monadic adjunction). Consider an adjunction $L \dashv R$ between categories $\mathbf{C}$ and $\mathbf{D}$ with the induced monad T. The natural morphism $R\varepsilon : T \circ R \Rightarrow R$ endows R with a T-algebra structure and, hence, induces a functor $\mathbf{C} \to \mathbf{D}^T$ between $\mathbf{C}$ and the Eilenberg-Moore category of T. $L \dashv R$ is said to be monadic if this functor is an equivalence.

[7] A more suitable name would be "module over a monad", since these are modules over a monoid if monads are regarded as monoids in $\mathbf{End}_{\mathbf{C}}$.

Definition 1.3.23 (Monadic functor). A functor is said to be monadic if it admits a left adjoint such that the adjunction is monadic.

The converse of 1.3.18 is also true:

Property 1.3.24. Every monad $T : \mathbf{C} \to \mathbf{C}$ can be obtained from an adjunction. The canonical choice is the adjunction

$$\begin{tikzcd} \mathbf{C} \arrow[r, phantom, "\perp"] & \mathbf{C}^T \arrow[l, bend right, "F_T"'] \end{tikzcd} \xleftarrow{} \; \mathbf{C} \underset{U_T}{\overset{F_T}{\rightleftarrows}} \mathbf{C}^T, \tag{1.19}$$

where F_T is the forgetful functor and U_T sends an object to the free T-algebra on it.

The following theorem characterizes monadic functors (for more information on some of the concepts, see Section 1.4 further below).

Theorem 1.3.25 (Beck's monadicity theorem). *Consider a functor $F : \mathbf{C} \to \mathbf{D}$. This functor is monadic if and only if the following conditions are satisfied:*

- *F admits a left adjoint.*
- *F reflects isomorphisms.*
- *$\mathbf{C}$ has all coequalizers of F-split parallel pairs[8] and F preserves these coequalizers.*

Remark 1.3.26 (Crude monadicity theorem). A sufficient condition for monadicity is obtained by replacing the third condition above by the following weaker statement: "$\mathbf{C}$ has all coequalizers of reflexive pairs and F preserves these coequalizers."

Definition 1.3.27 (Closure operator). Consider a monad $(T : \mathbf{C} \to \mathbf{C}, \eta, \mu)$. This monad is called a closure operator or **modal operator** if the multiplication map is a natural isomorphism, i.e. if the monad is idempotent. Equivalently, it is idempotent if and only if $\eta \circ T$ is a natural isomorphism.

Given a closure operator $T : \mathbf{C} \to \mathbf{C}$, the object Tx is called the closure of $x \in \mathrm{ob}(\mathbf{C})$ and the associated morphism η_x is called the **closing map**. An object $x \in \mathrm{ob}(\mathbf{C})$ itself is said to be T**-closed** exactly if its closing map is an isomorphism.

An object $x \in \mathrm{ob}(\mathbf{C})$ is called a **modal type** if the unit $\eta_x : x \to Tx$ is an isomorphism.

Property 1.3.28. Every (co)reflective subcategory inclusion (Definition 1.2.29) induces a closure operator. Conversely, every closure operator is induced by a (co)reflective subcategory.

[8]These are parallel pairs f, g such that the images Ff, Fg under F admit a split coequalizer.

Remark 1.3.29 (Bicategories ♣). A monad can be defined in any bicategory as a 1-morphism $t : x \to x$ together with two 2-morphisms that satisfy conditions similar to the ones above. The above definition is then just a specific case of this more general definition in **Cat**.

In the general setting one can then also define a **module** over a monad. First of all, one can regard any object $x \in \mathrm{ob}(\mathbf{C})$ as a functor from the terminal category $\mathbf{1}$. By replacing $\mathbf{1}$ by any other category in the ordinary definition one obtains a general algebra (or module). It is this definition that readily generalizes to bicategories, i.e. a module is a 1-morphism $a : x \to y$ together with a 2-morphism that satisfies the same conditions as an algebra over a monad in **Cat**.

1.4 Morphisms and diagrams

1.4.1 Morphisms

Definition 1.4.1 (Section). A section of a morphism $f : x \to y$ is a right inverse, i.e. a morphism $g : y \to x$ such that $f \circ g = \mathbb{1}_y$. f itself is called a **retraction** of g and y is called a **retract** of x.

Definition 1.4.2 (Monomorphism). Let $\mathbf{C}$ be a category. A morphism $\mu \in \mathbf{C}(x,y)$ is called a monomorphism, **mono** or **monic morphism** if for every object $z \in \mathrm{ob}(\mathbf{C})$ and every two morphisms $\alpha_1, \alpha_2 \in \mathbf{C}(z,x)$ such that $\mu \circ \alpha_1 = \mu \circ \alpha_2$, one can conclude that $\alpha_1 = \alpha_2$.

Definition 1.4.3 (Epimorphism). Let $\mathbf{C}$ be a category. A morphism $\varepsilon \in \mathbf{C}(x,y)$ is called an epimorphism, **epi** or **epic morphism** if for every object $z \in \mathrm{ob}(\mathbf{C})$ and every two morphisms $\alpha_1, \alpha_2 \in \mathbf{C}(y,z)$ such that $\alpha_1 \circ \varepsilon = \alpha_2 \circ \varepsilon$, one can conclude that $\alpha_1 = \alpha_2$.

A family of morphisms $\{f_i : x_i \to y\}_{i \in I}$ is called **jointly epimorphic** if

$$\alpha_1 \circ f_i = \alpha_2 \circ f_i \tag{1.20}$$

for all $i \in I$ implies that $\alpha_1 = \alpha_2$.

Definition 1.4.4 (Split monomorphism). A morphism $f : x \to y$ that is a section of some other morphism $g : y \to x$. It can be shown that every split mono is in fact a mono and even an **absolute mono**, i.e. it is preserved by all functors.

The morphism g can be seen to satisfy the dual condition and, hence, is called a **split epimorphism**. It can be shown to be an absolute epi.

Definition 1.4.5 (Balanced category). A category in which every monic epi is an isomorphism.

Definition 1.4.6 (Reflexive pair). Two parallel morphisms $f, g : x \to y$ are said to form a reflexive pair if they have a common section, i.e. if there exists a morphism $\sigma : y \to x$ such that $f \circ \sigma = g \circ \sigma = \mathbb{1}_y$.

Definition 1.4.7 (Subobject). Let $\mathbf{C}$ be a category and let $x \in \mathsf{ob}(\mathbf{C})$ be any object. A subobject y of x is a mono $y \hookrightarrow x$.

In fact, one should work up to isomorphisms and, accordingly, the formal definition goes as follows: a subobject y of x is an isomorphism class of monos $i : y \hookrightarrow x$ in the slice category $\mathbf{C}_{/x}$.

Definition 1.4.8 (Well-powered category). A category $\mathbf{C}$ such that for every object $x \in \mathsf{ob}(\mathbf{C})$ the class of subobjects $\mathsf{Sub}(x)$ is small.

1.4.2 Initial and terminal objects

Definition 1.4.9 (Initial object). An object $\emptyset$ such that for every other object x there exists a unique morphism $\iota_x : \emptyset \to x$. If one drops the uniqueness, the notion of a **weakly initial object** is obtained.

Definition 1.4.10 (Terminal object). An object 1 such that for every other object x there exists a unique morphism $\tau_x : x \to 1$.

Property 1.4.11 (Uniqueness). If an initial (or terminal) object exists, it is unique (up to isomorphisms).

Definition 1.4.12 (Zero object). An object that is both initial and terminal. The zero object is often denoted by 0.

Property 1.4.13 (Zero morphism). From the definition of the zero object it follows that for any two objects x, y there exists a unique morphism $0_{xy} : x \to 0 \to y$.

Definition 1.4.14 (Pointed category). A category containing a zero object.

Definition 1.4.15 (Global element). Let $\mathbf{C}$ be a category with a terminal object 1. A global element of an object $x \in \mathsf{ob}(\mathbf{C})$ is a morphism $1 \to x$.

Property 1.4.16. Every global element is monic.

Definition 1.4.17 (Pointed object). An object x equipped with a global element $1 \to x$. This morphism is sometimes called the **basepoint**.

Remark 1.4.18. In the category $\mathbf{Set}$, the elements of a set S are in one-to-one correspondence with the global elements of S. Furthermore, there is the the important property (*axiom of functional extensionality*) that two functions $f, g : S \to S'$ coincide if their values at every element $s \in S$ coincide or, equivalently, if their precompositions with global elements coincide.

However, this way of checking equality can fail in other categories. Consider for example **Grp**, the category of groups, with its zero object $0 = \{e\}$. The only morphism from this group to any other group G is the one mapping e to the unit in G. It is obvious that precomposition with this morphism says nothing about the equality of other morphisms. To recover the extensionality property from **Set**, the notion of an 'element' should be generalized:

Definition 1.4.19 (Generalized element). Let $\mathbf{C}$ be a category and consider an object $x \in \mathrm{ob}(\mathbf{C})$. For any object $y \in \mathrm{ob}(\mathbf{C})$, a morphism $y \to x$ is called a generalized element of x. These morphisms are also called **y-elements** in x or elements of **shape** y in x.

Definition 1.4.20 (Generator). Let $\mathbf{C}$ be a category. A collection of objects $\mathcal{O} \subset \mathrm{ob}(\mathbf{C})$ is called a collection of generators or **separators** for $\mathbf{C}$ if the generalized elements of shape $\mathcal{O}$ are sufficient to distinguish between all morphisms in $\mathbf{C}$:

$$\forall x, y \in \mathrm{ob}(\mathbf{C}) : \forall f, g \in \mathbf{C}(x, y) : \big(f \neq g \implies \exists o \in \mathcal{O} : \exists h \in \mathbf{C}(o, x) : f \circ h \neq g \circ h\big). \tag{1.21}$$

Definition 1.4.21 (Well-pointed category). A category for which the terminal object is a generator.

Definition 1.4.22 (Free object). Consider a forgetful functor $U : \mathbf{C} \to \mathbf{D}$ (whatever this may mean). An object $c \in \mathrm{ob}(\mathbf{C})$ is said to be free over an object $x \in \mathrm{ob}(\mathbf{D})$ if there exists a universal morphism $\eta_x : x \to Uc$. These are the initial objects of the comma categories $x/\mathbf{U}$.

Property 1.4.23 (Free functor). Note that if the forgetful functor admits a left adjoint $F : \mathbf{D} \to \mathbf{C}$, every object in the image of F is free according to the previous definition. Moreover, if U admits a free object for every $x \in \mathrm{ob}(\mathbf{D})$, it has a left adjoint. For this reason, left adjoints to fortgetful functors are often called free functors.

Definition 1.3.19 can be generalized to endofunctors as follows:

Definition 1.4.24 (Algebra over an endofunctor). Consider an endofunctor $F : \mathbf{C} \to \mathbf{C}$. An algebra over F is a pair $(x, f : Fx \to x)$, where $x \in \mathrm{ob}(\mathbf{C})$ is often called the **carrier**.

Property 1.4.25. The category of algebras over an endofunctor is equivalent to the Eilenberg–Moore category of the **(algebraically-)free monad** it generates (if it exists).

Construction 1.4.26. This property could actually be interpreted as the definition of the free monad generated by an endofunctor. If it exists, it can be obtained as the monad induced by the free-forgetful adjunction induced by $F\mathbf{Alg} \to \mathbf{C}$.

When the free functor exists, it can be constructed as follows. Consider an endofunctor $F : \mathbf{C} \to \mathbf{C}$. The term introduction of an inductive type corresponds to a morphism

$Fc \to c$, i.e. to an algebra over F. Now, algebras over F should correspond to the algebras over its free monad, the functor $F^* := U \circ \mathcal{F}$, where $U : F\mathbf{Alg} \to \mathbf{C}$ is the forgetful functor and $\mathcal{F} : \mathbf{C} \to F\mathbf{Alg}$ the free functor. The latter sends every object $d \in \mathrm{ob}(\mathbf{C})$ to the initial object of the comma category d/U, i.e. to an object $(c, \alpha : Fc \to c, \beta : d \to c)$. However, as long as $\mathbf{C}$ admits coproducts, such a triple is equivalent to a pair $(c, \gamma : d + Fc \to c)$. Since the latter is an algebra over $\mathbb{1}_{\mathbf{C}} + F$, one finds that algebras over F are equivalent to initial algebras over $\mathbb{1}_{\mathbf{C}} + F$.

Theorem 1.4.27 (Lambek). *If $F : \mathbf{C} \to \mathbf{C}$ has an initial algebra $f : Fx \to x$, then f is an isomorphism.*

1.4.3 Lifts

Definition 1.4.28 (Lifts and extensions). A lift of a morphism $f : x \to y$ along an epi $e : z \to y$ is a morphism $g : x \to z$ satisfying $f = e \circ g$. Dualizing this definition gives the notion of extensions. (The epi/mono condition is often dropped in the literature.)

Definition 1.4.29 (Lifting property). A morphism $f : x \to y$ has the left lifting property with respect to a morphism $g : x' \to y'$ (or g has the right lifting property with respect to f) if for every commutative diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] & x' \arrow[d, "g"] \\ y \arrow[r] \arrow[ur, dashed, "\exists\psi"] & y' \end{tikzcd}$$

there exists a morphism $\psi : y \to x'$ such that the triangles commute. If the morphism ψ is unique, then f and g are said to be **orthogonal**.

Definition 1.4.30 (Injective / projective morphisms). Consider a class of morphisms $I \subseteq \mathrm{hom}(\mathbf{C})$. A morphism $f \in \mathrm{hom}(\mathbf{C})$ is said to be I-injective (resp. I-projective) if it has the right (resp. left) lifting property with respect to all morphisms in I.

Given a set of morphisms I, the sets of I-injective and I-projective morphisms are denoted by $\mathrm{rlp}(I)$ and $\mathrm{llp}(I)$, respectively.

Definition 1.4.31 (Injective and projective objects). If $\mathbf{C}$ has a terminal object 1, an object x is called I-injective if its terminal morphism is I-injective. If $\mathbf{C}$ has an initial object, I-projective objects can be defined dually. (See Fig. 1.2.)

If I is the class of monomorphisms (resp. epimorphisms), the terminology is simplified to **injective** (resp. **projective**) objects. For projective objects, this is also equivalent to requiring that the (covariant) hom-functor preserves epimorphisms.

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "\forall f \in I"'] & i \\ y \arrow[ur, dashed, "\exists\psi"'] & \end{tikzcd}$$

(a) Injective object i.

$$\begin{tikzcd} & x \arrow[d, "\forall f \in I"] \\ p \arrow[r, "g"'] \arrow[ur, dashed, "\exists\psi"] & y \end{tikzcd}$$

(b) Projective object p.

Figure 1.2: Injective and projective objects.

A category **C** is said to **have enough injectives** if, for every object, there exists a monomorphism into an injective object. The category is said to **have enough projectives** if, for every object, there exists an epimorphism from a projective object.

Definition 1.4.32 (Fibrations and cofibrations). Consider a category **C** together with a class $I \subseteq \mathsf{hom}(\mathbf{C})$ of morphisms. A morphism $f \in \mathsf{hom}(\mathbf{C})$ is called an I-fibration (resp. I-cofibration) if it has the right (resp. left) lifting property with respect to all I-projective (resp. I-injective) morphisms.

1.4.4 Limits and colimits

Definition 1.4.33 (Diagram). A diagram in **C** with index category **I** is a (covariant) functor $D : \mathbf{I} \to \mathbf{C}$.

Definition 1.4.34 (Cone). Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram. A cone from $c \in \mathsf{ob}(\mathbf{C})$ to D consists of a family of morphisms $\psi_i : c \to Di$ indexed by **I** such that $\psi_j = Df \circ \psi_i$ for all morphisms $f : i \to j \in \mathsf{hom}(\mathbf{I})$. (This is depicted in Fig. 1.3a.)

$$\begin{tikzcd} & c \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_{f(i)}"] & \\ Di \arrow[rr, "Df"'] & & Df(i) \end{tikzcd}$$

(a) Component of cone over D.

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "\psi_i"'] & & b \arrow[dl, "\phi_i"] \\ & Di & \end{tikzcd}$$

(b) Morphism of cones.

Figure 1.3: Category of cones.

Alternative Definition 1.4.35. The above definition can be reformulated by defining an additional functor $\Delta_x : \mathbf{I} \to \mathbf{C}$ that maps every element $i \in \mathsf{ob}(\mathbf{I})$ to x and every morphism $g \in \mathsf{hom}(\mathbf{I})$ to $\mathbb{1}_x$, i.e. $\Delta : C \to [\mathbf{I}, \mathbf{C}]$ is the **diagonal functor**. The morphisms ψ_i can then be seen to be the components of a natural transformation $\psi : \Delta_x \Rightarrow D$. Hence, a cone (x, ψ) is an element of $[\mathbf{I}, \mathbf{C}](\Delta_x, D)$.

Definition 1.4.36 (Morphism of cones). Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram and let (x, ψ) and (y, ϕ) be two cones over D. A morphism between these cones is a morphism of the apexes $f : x \to y$ such that the diagrams of the form 1.3b commute for all $i \in \mathrm{ob}(\mathbf{I})$. The cones over D together with these morphisms form a category $\mathbf{Cone}(D)$. In fact this can easily be seen to be the comma category $\Delta \downarrow D$.

Definition 1.4.37 (Limit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$. The limit of this diagram, denoted by $\lim D$, is (if it exists) the terminal object of the category $\mathbf{Cone}(D)$.

Remark 1.4.38. In the older literature, the name **projective limit** was sometimes used. The dual notion, a **colimit**, was often called an **inductive limit** in the older literature.

This definition leads to the following universal property.

Universal Property 1.4.39. Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram. For every cone $(x, \psi) \in \mathbf{Cone}(D)$, there exists a unique morphism $f : x \to \lim D$. This defines a bijection

$$[\mathbf{I}, \mathbf{C}](\Delta_x, D) \cong \mathbf{C}(x, \lim D)\,. \tag{1.22}$$

If all (small) limits exist, the limit functor $\lim : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$ can be defined. The universal property of limits then implies that it is right adjoint to the constant functor Δ.

For diagrams in $\mathbf{Set}$, one can use the fully faithfulness of the Yoneda embedding to obtain the following expression:

$$\lim D \cong [\mathbf{I}, \mathbf{Set}](\Delta_*, D)\,. \tag{1.23}$$

Remark 1.4.40. In **??** on enriched category theory, a generalization of the above construction (the so-called *weighted limits*) will be given that is better suited to the enriched setting and allows to express a wide variety of constructions as (weighted) limits.

Example 1.4.41 (Terminal object). The terminal object 1 is the limit of the empty diagram.

Definition 1.4.42 (Finitely complete category). A category is said to be finitely complete if it has all finite limits. If all (small) limits exist, the category is said to be **complete**. The dual notion for colimits is called **(finite) cocompleteness**.

Example 1.4.43 (Presheaf categories). All presheaf categories are both complete and cocomplete.

Definition 1.4.44 (Continuous functor). A functor that preserves all small limits.

A more restricted form is also common in the literature.

Definition 1.4.45 (Exact functor). A functor that preserves all finite limits is said to be **left exact**. Analogously, a functor that preserves all finite colimits is said to be **right exact**.

Example 1.4.46 (Hom-functors). In a locally small category every hom-functor is continuous (in fact these functors even preserve limits that are not necessarily small). This implies for example that

$$\mathbf{C}(x, \lim D) \cong \lim \mathbf{C}(x, D)\,. \tag{1.24}$$

In the case where $\mathbf{C}$ is small, one can characterize the Yoneda embedding through a universal property:

Universal Property 1.4.47 (Free cocompletion). The Yoneda embedding $\mathbf{C} \hookrightarrow \widehat{\mathbf{C}}$ turns the presheaf category $\widehat{\mathbf{C}}$ into the **free cocompletion** of $\mathbf{C}$, i.e. there exists an equivalence of categories between the functor category of cocontinuous functors $[\widehat{\mathbf{C}}, \mathbf{D}]_{\mathrm{cont}}$ and the ordinary functor category $[\mathbf{C}, \mathbf{D}]$.

Definition 1.4.48 (Tiny object). An object in a locally small category for which the covariant hom-functor preserves small colimits. This is sometimes called a **small-projective** object since it is in particular projective[9].

Definition 1.4.49 (Cauchy completion). Let $\mathbf{C}$ be a small category. An important (small and full) subcategory of the free cocompletion of $\mathbf{C}$ is given by the Cauchy completion, i.e. the subcategory of $\widehat{\mathbf{C}}$ on the tiny objects.[10] It can be shown that the free cocompletion of the Cauchy completion coincides with the one on $\mathbf{C}$ (up to equivalence).

A category is said to be **Cauchy-complete** if it is equivalent to its Cauchy completion. It can be shown that a category is Cauchy-complete if and only if it has all small absolute colimits.

Definition 1.4.50 (Filtered category). A category in which every finite diagram admits a cocone. For regular cardinals κ, this notion can be generalized. A category is said to be κ-filtered if every diagram with less than κ arrows admits a cocone. (In this terminology, filtered categories are the same as ω-filtered categories.)

Definition 1.4.51 (Directed limit). Consider a diagram $D : \mathbf{I} \rightarrow \mathbf{C}$. The limit (resp. colimit) of D is said to be (co)directed (resp. directed) if $\mathbf{I}$ is a downward (resp. upward) directed set **??**.

The following definition is a categorification of the previous one.

Definition 1.4.52 (Filtered limit). Consider a diagram $D : \mathbf{I} \rightarrow \mathbf{C}$. The limit (resp. colimit) of D is said to be (co)filtered (resp. filtered) if $\mathbf{I}$ is a cofiltered (resp. filtered) category.

[9] Epimorphisms are characterized by a *pushout* (see Definition 1.4.72 further below).

[10] A generalization in the context of enriched categories is given by the *Karoubi envelope*.

Property 1.4.53. A category has all directed limits if and only if it has all filtered limits. (A dual statement holds for colimits.)

Definition 1.4.54 (Finitary functor). A functor that preserves all filtered colimits.

Definition 1.4.55 (Pro-object). A functor $F : \mathbf{I} \to \mathbf{C}$ from a cofiltered category. By composing these functors with the Yoneda embedding $\mathcal{Y} : \mathbf{C} \to [\mathbf{C}^{op}, \mathbf{Set}]$, pro-objects can also be identified with cofiltered limits of representable presheaves. In conjunction with Remark 1.4.38, this clarifies the terminology.

Universal Property 1.4.56. The **procategory** $\mathbf{Pro}(\mathbf{C})$ is the universal completion of $\mathbf{C}$ under cofiltered limits. $\mathbf{Pro}(\mathbf{C})$ satisfies (cf. Universal Property 1.4.47):

- it admits all cofiltered limits, and
- if $\mathbf{D}$ admits all cofiltered limits, there is an equivalence of functor categories

$$[\mathbf{C}, \mathbf{D}] \cong \mathbf{Fin}\big(\mathbf{Pro}(\mathbf{C}), \mathbf{D}\big)\,, \tag{1.25}$$

  where the category on the right-hand side is the category of finitary functors.

Remark 1.4.57 (Ind-objects). By dualizing the above definitions, i.e. by replacing cofiltered limits by filtered colimits, the category of ind-objects $\mathbf{Ind}(\mathbf{C})$ is obtained.

Definition 1.4.58 (Compact object). An object for which the covariant hom-functor preserves all filtered colimits. These objects are also said to be **finitely presentable**.[11]

Definition 1.4.59 (Product). Let $\mathbf{I}$ be a discrete category. The (co)limit over a diagram $D : \mathbf{I} \to \mathbf{C}$ is called a (co)product in $\mathbf{C}$.

Example 1.4.60 (Equalizer). Consider a diagram of the form

$$x \underset{g}{\overset{f}{\rightrightarrows}} y\,.$$

The limit of this diagram is called the equalizer of f and g. It consists of an object e and a morphism $\varepsilon : e \to x$ such that the following **fork** diagram

$$e \xrightarrow{\varepsilon} x \underset{g}{\overset{f}{\rightrightarrows}} y \tag{1.26}$$

is universal with respect to (e, ε). By dualizing one obtains **cofork** diagrams $x \rightrightarrows y \to z$ and their universal versions, the **coequalizers**.

Example 1.4.61 (Split coequalizer). A cofork diagram

$$x \underset{g}{\overset{f}{\rightrightarrows}} y \xrightarrow{\tau} z$$

together with a section φ of f and a section σ of τ such that $\sigma \circ \tau = g \circ \varphi$.

[11] This name derives from the fact that modules are finitely presented if and only if their covariant hom-functor preserves direct limits (i.e. directed colimits in the context of algebra).

Definition 1.4.62 (Regular morphisms). A mono (resp. epi) is said to be regular if it arises as an equalizer (resp. coequalizer) of two parallel morphisms.

Although not all categories are balanced (Definition 1.4.5), the following property does hold in any category.

Property 1.4.63 (Regular bimorphism). Both monic regular epimorphisms and epic regular monomorphisms are isomorphisms.

Alternative Definition 1.4.64 (Finitely complete category). A category is said to be finitely complete if it has a terminal object and if all binary equalizers and products exist.

Definition 1.4.65 (Span). A span in a category C is a diagram of the form 1.4a. By definition of a diagram, a span in C is equivalent to a functor $S : \boldsymbol{\Lambda} \to \mathbf{C}$, where $\boldsymbol{\Lambda}$ is the category with three objects $\{-1, 0, 1\}$ and two morphisms $i : 0 \to -1$ and $j : 0 \to 1$. For this reason $\boldsymbol{\Lambda}$ is sometimes called the walking or universal span.

$$\begin{tikzcd} & s \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x & & y \end{tikzcd}$$

(a) Span (category theory).

$$\begin{tikzcd} x \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

(b) Cospan.

Figure 1.4: (Co)span diagrams.

Definition 1.4.66 (Pullback). The pullback of two morphisms $f : x \to z$ and $g : y \to z$ is defined as the limit of cospan 1.4b. The full diagram characterizing the pullback, which has the form of a square, is sometimes called a **Cartesian square**.

Notation 1.4.67 (Pullback). The pullback of two morphisms $f : x \to z$ and $g : y \to z$ is often denoted by $x \times_z y$. The associated pullback square is sometimes written as in Fig. 1.5a.

$$\begin{tikzcd} x \times_z y \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & y \arrow[d, "g"] \\ x \arrow[r, "f"'] & z \end{tikzcd}$$

(a) Pullback square.

$$\begin{tikzcd} z \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{po}"] & y \arrow[d] \\ x \arrow[r] & x \sqcup_z y \end{tikzcd}$$

(b) Pushout square.

Figure 1.5: Pullback and pushout diagrams.

Example 1.4.68 (Product). If a terminal object 1 exists, the pullback $x \times_1 y$ is equal to the product $x \times y$.

In fact, pullbacks are sometimes also called **fibred products**. The reason for this terminology is not only that products reduce to a particular case, but also that in the case of **Set** the pullbacks have a fibrewise product structure:

$$x \times_y z \cong \bigsqcup_{a \in x} f^{-1}(a) \times g^{-1}(a)\,. \tag{1.27}$$

Example 1.4.69 (Kernel pair). Consider a morphism $f : x \to y$. Its kernel pair is defined as the pullback of f along itself.

Definition 1.4.70 (Pushout). The dual notion of a pullback, i.e. the colimit of a span. See Fig. 1.5b.

Property 1.4.71. Pullbacks preserve monos and pushouts preserve epis.

Alternative Definition 1.4.72 (Epimorphism). A morphism whose cokernel pair is the identity.

Property 1.4.73 (Pasting law). Consider a diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

If the right square is a pullback diagram, the left square is a pullback diagram if and only if the total diagram is. Dually, if the left square is a pushout diagram, the right square is a pushout diagram if and only if the total diagram is.

Property 1.4.74 (Span category ♣). Consider a category **C** with pullbacks. The category **Span**(**C**) is defined as the category with the same objects as **C** but with spans as morphisms. Composition of spans is given by pullbacks. By including morphisms of spans, **Span**(**C**) can be refined to a bicategory.

Definition 1.4.75 (Wedge). Consider a profunctor $F : \mathbf{C} \nrightarrow \mathbf{C}$. A wedge $e : w \to F$ is an object $w \in \mathrm{ob}(\mathbf{Set})$ together with a collection of morphisms $e_x : w \to F(x,x)$ indexed by **C** such that for every morphism $f : x \to y$ the following diagram commutes:

$$\begin{tikzcd} & w \arrow[dl, "e_x"'] \arrow[dr, "e_y"] & \\ F(x,x) \arrow[dr, "F(\mathbb{1}_x, f)"'] & & F(y,y) \arrow[dl, "F(f, \mathbb{1}_y)"] \\ & F(x,y) & \end{tikzcd}$$

As was the case for cones, this can be reformulated in terms of (di)natural transformations. A wedge (w,e) of a profunctor $F:\mathbf{C}\nrightarrow\mathbf{C}$ is a dinatural transformation from the constant profunctor Δ_w to F.

Definition 1.4.76 (End). The end of a profunctor $F:\mathbf{C}\nrightarrow\mathbf{C}$ is defined as the universal wedge of F. The components of the wedge are called the **projection maps** of the end. This stems from the fact that for a discrete category the end coincides with the product $\prod_{x\in\mathrm{ob}(\mathbf{C})}F(x,x)$.

This is equivalent to a definition in terms of equalizers. Consider the two canonical maps

$$\prod_{x\in\mathrm{ob}(\mathbf{C})}\mathbf{C}(x,x)\rightrightarrows\prod_{f:x\to y}\mathbf{C}(x,y)\,. \tag{1.28}$$

This diagram can be interpreted as the product of all lower halves of the wedge diagrams above. It is not hard to see that its equalizer (universally) satisfies the wedge condition for all $f\in\mathrm{hom}(\mathbf{C})$.

Notation 1.4.77 (End). The end of a profunctor $F:\mathbf{C}\nrightarrow\mathbf{C}$ is often denoted using an integral sign with subscript:

$$\int_{x\in\mathbf{C}}F(x,x)\,.$$

For the dual construction, called a **coend**, an integral sign with superscript is used.

Example 1.4.78 (Natural transformations). Consider two functors $F,G:\mathbf{C}\to\mathbf{D}$. The map $(x,y)\mapsto\mathbf{D}(Fx,Gy)$ gives a profunctor $H:\mathbf{C}\nrightarrow\mathbf{C}$. By looking at the wedge condition for this profunctor, the following equality for all morphisms $f:x\to y$ can be derived:

$$\tau_y\circ Ff=Gf\circ\tau_x\,, \tag{1.29}$$

where τ is the wedge projection. Comparing this equality to Definition 1.2.15 gives

$$\mathrm{Nat}(F,G)=\int_{x\in\mathbf{C}}\mathbf{D}(Fx,Gx)\,. \tag{1.30}$$

Property 1.4.79. Using the continuity of the hom-functor (Definition 1.4.44), one can prove the following equality which can be used to turn ends into coends and vice versa:

$$\mathbf{Set}\left(\int^{x\in\mathbf{C}}F(x,x),y\right)=\int_{x\in\mathbf{C}}\mathbf{Set}\big(F(x,x),y\big)\,. \tag{1.31}$$

Using the above properties and definitions, one obtains the following two statements, called the **Yoneda reduction** and **co-Yoneda lemma**:

Property 1.4.80 (Ninja Yoneda lemma). Let $F : \mathbf{C} \to \mathbf{Set}$ be a covariant functor (similar statements hold for contravariant functors).

$$\int_{x\in\mathbf{C}} \mathbf{Set}\big(\mathbf{C}(-,x), Fx\big) \cong F \tag{1.32}$$

$$\int^{x\in\mathbf{C}} \mathbf{C}(x,-)\times Fx \cong F \tag{1.33}$$

For a generalization to the enriched setting see **??**.

Remark 1.4.81. A common remark at this point is the comparison with the Dirac distribution (**??**):

$$\int \delta(x-y)f(x) = f(y)\,. \tag{1.34}$$

By interpreting the functor F as a function, the representable functors can be seen to behave as Dirac distributions.

Property 1.4.82.

$$\int_{F\in\mathbf{coPsh}(\mathbf{C})} \mathbf{Set}(Fx, Fy) \cong \mathbf{C}(x,y) \tag{1.35}$$

Definition 1.4.83 (Kan extension). Consider two functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{A} \to \mathbf{C}$. The right Kan extension of F along G is given by the universal functor $\mathsf{Ran}_G F : \mathbf{C} \to \mathbf{B}$ and natural transformation $\eta : \mathsf{Ran}_G F \circ G \Rightarrow F$:

$$\begin{tikzcd}
\mathbf{C} \arrow[dr, dashed, "\mathsf{Ran}_G F"] & \\
\mathbf{A} \arrow[u, "G"] \arrow[r, "F"'] \arrow[ur, phantom, "\Downarrow\eta"] & \mathbf{B}
\end{tikzcd}$$

The left Kan extension $\mathsf{Lan}_G F$ is obtained by dualizing this construction.

Property 1.4.84 (Complete categories). Complete (resp. cocomplete) categories admit all right (resp. left) Kan extensions.

Definition 1.4.85 (Preservation of Kan extension). A Kan extension $\mathsf{Lan}_G F$ is said to be **absolute** if every functor with the same codomain as preserves the Kan extension, i.e. a Kan extension is absolute if right whiskering it by another functor defines the Kan extension of the composition. If it is only preserved by all representable functors, the Kan extension is said to be **pointwise** or **strong**. Pointwise Kan extensions can be expressed as (co)limits, an expression is provided in **??** in the *enriched setting* using (co)ends.

Alternative Definition 1.4.86 (Kan extension). The definition above gives a natural isomorphism (here given for left extension):

$$[\mathbf{A},\mathbf{B}](F, G^*-) \cong [\mathbf{C},\mathbf{B}](\mathsf{Lan}_G F, -)\,. \tag{1.36}$$

In the spirit of partial adjoints or partial limits, this construction defines so-called **local Kan extensions**. If local Kan extensions exist for all functors $F \in [\mathbf{A}, \mathbf{B}]$, a right adjoint $\mathrm{Ran}_G : [\mathbf{A}, \mathbf{B}] \to [\mathbf{C}, \mathbf{B}]$ to the pullback functor $G^* : F \mapsto F \circ G$ is obtained. Similarly, left Kan extension can be defined as the left adjoint to the pullback functor.

Remark 1.4.87. Using this equivalence of hom-spaces, Kan extensions can be generalized from **Cat** to any *2-category*.

Example 1.4.88 (Limit). Denote the terminal category by **1**. By choosing the functor G in the definition of a right Kan extension to be the unique functor $!_{\mathbf{C}} : \mathbf{C} \to \mathbf{1}$, one obtain the universal property characterizing limits (Universal Property 1.4.39):

$$\lim F \cong \mathrm{Ran}_{!_C} F\,. \tag{1.37}$$

Similarly, colimits can be obtained as left Kan extensions.

The existence of Kan extensions can also be used to determine the existence of adjoints.

Property 1.4.89 (Adjoint functors). A functor $F : \mathbf{C} \to \mathbf{D}$ admits a left (resp. right) adjoint if and only if the right (resp. left) Kan extension of the identity functor $\mathbb{1}_{\mathbf{C}}$ along F exists. If it exists as an absolute extension, the left adjoint is given exactly by this Kan extension.

Definition 1.4.90 (Codensity monad). Consider a general functor $F : \mathbf{C} \to \mathbf{D}$. If the right Kan extension $\mathrm{Ran}_F F$ exists, it defines a monad. Functors for which this monad is the identity are said to be **codense**.[12] Left Kan extensions give, by duality, rise to *density comonads*.

Property 1.4.91 (Faithfulness). Kan extension along a fully faithful functor is itself a fully faithful functor.

Property 1.4.92 (Representability). (Left) Kan extension of a representable along a functor F is equivalent to the representable of the image of F.

Property 1.4.93 (Adjoint quadruple). Consider an adjunction $F \dashv G$ and consider Kan extension of presheaves taking values in a bicomplete category. In this case precomposition with (the opposite of) one of the adjoints coincides with Kan extension along (the opposite) of the other:

$$(F^{\mathrm{op}})^* \cong \mathrm{Lan}_{G^{\mathrm{op}}}\,, \tag{1.38}$$

$$(G^{\mathrm{op}})^* \cong \mathrm{Ran}_{F^{\mathrm{op}}}\,. \tag{1.39}$$

This implies that every adjunction induces an adjoint quadruple

$$\mathrm{Lan}_{F^{\mathrm{op}}} \dashv \mathrm{Lan}_{G^{\mathrm{op}}} \dashv \mathrm{Ran}_{F^{\mathrm{op}}} \dashv \mathrm{Ran}_{G^{\mathrm{op}}}\,. \tag{1.40}$$

[12] Codense functors are usually defined in a different way, but one can show that this is an equivalent definition (hence the name).

1.5 Internal structures

Property 1.5.1 (Eckmann–Hilton argument). A monoid internal to **Mon**, the category of monoids, is the same as a commutative monoid. (See also **??**.)

Definition 1.5.2 (Internal category). Let $\mathcal{E}$ be a category with pullbacks. A category **C** internal to $\mathcal{E}$ consists of the following data:

- an object $C_0 \in \mathrm{ob}(\mathcal{E})$ of objects;
- an object $C_1 \in \mathrm{ob}(\mathcal{E})$ of morphisms;
- source and target morphisms $s, t \in \mathcal{E}(C_1, C_0)$;
- an 'identity-assigning' morphism $e \in \mathcal{E}(C_0, C_1)$ such that

$$s \circ e = \mathbb{1}_{C_0} \qquad t \circ e = \mathbb{1}_{C_0}; \tag{1.41}$$

  and

- a composition morphism $c : C_1 \times_{C_0} C_1 \to C_1$ such that the following equations hold:

$$\begin{gathered} s \circ c = s \circ \pi_1 \qquad t \circ c = t \circ \pi_2 \\ \pi_1 = c \circ (e \times_{C_0} \mathbb{1}) \qquad c \circ (\mathbb{1} \times_{C_0} e) = \pi_2 \\ c \circ (c \times_{C_0} \mathbb{1}) = c \circ (\mathbb{1} \times_{C_0} c), \end{gathered} \tag{1.42}$$

  where π_1, π_2 are the canonical projections associated with the pullback $C_1 \times_{C_0} C_1$ of (s, t).

Morphisms between these categories, suitably called **internal functors**, are given by a pair of morphisms (in $\mathcal{E}$) between internal objects and morphisms, that preserve composition and identities. Internal natural transformations are defined in a similar way.

Notation 1.5.3. The *(bi)category* of internal categories in $\mathcal{E}$ is denoted by $\mathbf{Cat}(\mathcal{E})$. It should be noted that for $\mathcal{E} = \mathbf{Set}$, the ordinary category of small categories $\mathbf{Cat}(\mathbf{Set}) = \mathbf{Cat}$ is obtained.

Alternative Definition 1.5.4. The above definition can be reformulated in a very elegant way. An internal category in $\mathcal{E}$ is a monad in the bicategory $\mathbf{Span}(\mathcal{E})$ as shown in Fig. 1.6.

Functors between internal categories are not the only relevant morphisms. However, when defining (co)presheaves such as the hom-functor, a problem occurs. In **Cat** there exist, by definition, maps to the ambient category **Set** (ordinary category theory has a

Span gives source and target maps

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 & & C_0 \end{tikzcd}$$

Multiplication gives composition

$$\begin{tikzcd} & C_1 \times_{C_0} C_1 \arrow[dl, "s \circ \pi_0"'] \arrow[dr, "t \circ \pi_1"] \arrow[dd, "c"] & \\ C_0 & & C_0 \\ & C_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

Unit gives identity

$$\begin{tikzcd} & C_0 \arrow[dl, "\mathbb{1}_a"'] \arrow[dr, "\mathbb{1}_a"] \arrow[dd, "e"] & \\ C_0 & & C_0 \\ & C_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

Figure 1.6: Internal category as a monad in $\mathbf{Span}(\mathcal{E})$.

set-theoretic foundation). However, for internal categories there does not necessarily exist a morphism $\mathbf{C} \to \mathcal{E}$. To solve this problem, one can consider a more general structure.

Definition 1.5.5 (Internal diagram). A left module over a monad in $\mathbf{Span}(\mathcal{E})$. The dual notion is better known as an **internal presheaf**. The category of internal diagrams on an internal category $\mathbf{C} \in \mathbf{Cat}(\mathcal{E})$ is denoted by $\mathcal{E}^{\mathbf{C}}$.

This can be spelled out more explicitly. An internal diagram in an internal category $\mathbf{C} \in \mathbf{Cat}(\mathcal{E})$ consists of:

1. a morphism $\gamma_0 : F_0 \to C_0$, and
2. a morphism $\mathsf{ap} : F_0 \times_{C_0} C_1 \to F_0$

satisfying:

$$\begin{aligned} \gamma_0 \circ \mathsf{ap} &= d_1 \circ \pi_{C_1} \\ \mathsf{ap} \circ (\mathbb{1}_{F_0} \times e) &= \pi_{F_0} \\ \mathsf{ap} \circ (\mathsf{ap} \times \mathbb{1}_{C_0}) &= \mathsf{ap} \circ (\mathbb{1}_{F_0} \times c) \end{aligned} \tag{1.43}$$

Alternative Definition 1.5.6 (Internal diagram). An object of the slice category $\mathbf{Cat}(\mathcal{E})_{/\mathbf{C}}$

satisfying the following pullback condition:

$$\begin{tikzcd} F_1 \arrow[r, "d_1"] \arrow[d, "\gamma_1 \equiv \pi_{C_1}"'] \arrow[dr, phantom, "\mathrm{pb}"] & F_0 \arrow[d, "\gamma_0"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd} \tag{1.44}$$

In fact, this is a specific instance of an even more general concept. For more information on the definitions and applications, see Johnstone (2014); Mac Lane (2013).

Definition 1.5.7 (Internal profunctor). A bimodule between monads in $\mathbf{Span}(\mathcal{E})$. Together with the above definitions, this gives rise to an equivalence

$$\mathbf{Mod}(\mathbf{Span}(\mathcal{E})) \cong \mathbf{Prof}(\mathcal{E})\,. \tag{1.45}$$

Construction 1.5.8 (Internal Yoneda profunctor). Consider an internal functor $F : \mathbf{C} \to \mathbf{D}$. This functor induces two internal profunctors $F_* : \mathbf{D} \nrightarrow \mathbf{C}$ and $F^* : \mathbf{C} \nrightarrow \mathbf{D}$. For F_* (the profunctor F^* is defined similarly) the object span is defined as

$$C_0 \xleftarrow{\pi_0} C_0 \times_{D_0} D_1 \xrightarrow{t \circ \pi_1} D_0\,. \tag{1.46}$$

The action of $f \in D_1$ is given by postcomposition with f in the second factor, while the action of $g \in C_1$ is given by precomposition with Fg in the second factor and changing to the domain of g in the first factor.

It can easily be shown that the profunctors induced by an identity functor $\mathbb{1}_{\mathbf{C}}$ have an object span that corresponds to the internal category $\mathbf{C}$ with the actions given by (internal) composition. In the case of $\mathcal{E} = \mathbf{Set}$, this boils down to the hom-functor. The fact that the object span is equivalent to the category $\mathbf{C}$ is essentially the Yoneda embedding. For this reason, this profunctor is in general called the (internal) Yoneda profunctor $\mathcal{Y}(\mathbf{C})$.

1.5.1 Groupoids

Definition 1.5.9 (Groupoid). A (small) groupoid $\mathcal{G}$ is a (small) category in which all morphisms are invertible.

Example 1.5.10 (Action groupoid). Consider a set X with an action of a group G. The action groupoid $X/\!\!/G$ is defined as the following category:

1. **Objects**: X,
2. **Morphisms**: An arrow $x \to y$ for every $g \in G$ such that $g \cdot x = y$.

Example 1.5.11 (Delooping). Consider a group G. Its delooping $\mathbf{B}G$ is defined as the one-object groupoid for which $\mathbf{B}G(*,*) = G$.

Property 1.5.12 (Representations). Consider a group G together with its delooping $\mathbf{B}G$. When considering *representations* as functors $\rho : \mathbf{B}G \to \mathbf{FinVect}$, one can see that the intertwiners (**??**) are exactly the natural transformations. More generally, all G-sets (**??**) can be obtained as functors $\mathbf{B}G \to \mathbf{Set}$.

Definition 1.5.13 (Core). Let $\mathbf{C}$ be a (small) category. The core $\mathrm{Core}(\mathbf{C}) \in \mathbf{Grpd}$ of $\mathbf{C}$ is defined as the maximal subgroupoid of $\mathbf{C}$.

Definition 1.5.14 (Orbit). Let $\mathcal{G}$ be a groupoid with O, M respectively the sets of objects and morphisms. On O one can define an equivalence $x \sim y \iff \exists \phi : x \to y$. The equivalence classes are called orbits and the set of orbits is denoted by O/M.

Definition 1.5.15 (Transitive component). Let $\mathcal{G}$ be a groupoid with O, M respectively the sets of objects and morphisms and let s, t denote the source and target maps on M. Given an orbit $o \in O/M$, the transitive component of M associated to o is defined as $s^{-1}(o)$, or equivalently, as $t^{-1}(o)$.

Property 1.5.16. Every groupoid is a (disjoint) union of its transitive components.

Definition 1.5.17 (Transitive groupoid). A groupoid $\mathcal{G}$ is said to be transitive if for all objects $x \neq y \in \mathrm{ob}(\mathcal{G})$, the set $\mathcal{G}(x,y)$ is not empty.

1.6 Lawvere theories ♣

Definition 1.6.1 (Lawvere theory). Let $\mathbf{F}$ denote the skeleton of $\mathbf{FinSet}$. A Lawvere theory consists of a small category $\mathbf{L}$ and a strict (finite) product-preserving *identity-on-objects* functor $\mathcal{L} : \mathbf{F}^{\mathrm{op}} \to \mathbf{L}$.

Equivalently, a Lawvere theory is a small category $\mathbf{L}$ with a **generic object** c_0 such that every object $c \in \mathrm{ob}(\mathbf{L})$ is a finite power of c_0.

Property 1.6.2. Lawvere theories $(\mathbf{L}, \mathcal{L})$ form a category $\mathbf{Law}$. Morphisms between Lawvere theories are (finite) product-preserving functors.

Definition 1.6.3 (Model). A model or **algebra** over a Lawvere theory $\mathbf{L}$ is a (finite) product-preserving functor $A : \mathbf{L} \to \mathbf{Set}$.

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1.7 Operad theory ♣

1.7.1 Operads

Definition 1.7.1 (Plain operad[13]). Let $\mathcal{O} = \{P(n)\}_{n\in\mathbb{N}}$ be a collection of sets, called ***n*-ary operations** (where n is called the **arity**). The collection $\mathcal{O}$ is called a plain operad if it satisfies following axioms:

1. $P(1)$ contains an identity element $\mathbb{1}$.
2. For all positive integers $n, k_1, \dots, k_n$ there exists a composition map

$$\begin{aligned} \circ : P(n) \times P(k_1) \times \cdots \times P(k_n) &\to P(k_1 + \cdots + k_n) \\ : (\psi, \theta_1, \dots, \theta_n) &\mapsto \psi \circ (\theta_1, \dots, \theta_n) \end{aligned} \tag{1.47}$$

that satisfies two additional axioms:

- **identity**:

$$\theta \circ (\mathbb{1}, \dots, \mathbb{1}) = \mathbb{1} \circ \theta = \theta, \tag{1.48}$$

and

- **associativity**:

$$\begin{aligned} &\psi \circ \Big(\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n})\Big) \\ &\qquad = \Big(\psi \circ (\theta_1, \dots, \theta_n)\Big) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \theta_{2,1}, \dots, \theta_{n,k_n}). \end{aligned} \tag{1.49}$$

If the operad is represented using planar tree diagrams, the associativity obtains a nice intuitive form. When combining planar tree diagrams in three layers, the associativity axiom says that one can either first glue the first two layers together or one can first glue the last two layers together.

Remark 1.7.2. Plain operads can be defined in any monoidal category. In the same way symmetric operad can be defined in any symmetric monoidal category.

Example 1.7.3 (Endomorphism operad). Consider a vector space V. For every $n \in \mathbb{N}$, one can define the endomorphism algebra $\mathsf{End}(V^{\otimes n}, V)$. The endomorphism operad $\mathcal{E}\mathrm{nd}(V)$ is defined as $\{\mathsf{End}(V^{\otimes n}, V)\}_{n\in\mathbb{N}}$.

Definition 1.7.4 (O-algebra). An object X is called an algebra over an operad O if there exist morphisms

$$O(n) \times X^n \to X$$

for every $n \in \mathbb{N}$ satisfying the usual composition and identity laws. Alternatively, this can be rephrased as the existence of a (plain) operad morphism $O(n) \to \mathcal{E}\mathrm{nd}(X)$.

Example 1.7.5 (Categorical O-algebra). An O-algebra in the category **Cat**.

[13] Also called a **nonsymmetric operad** or **non-Σ operad**.

1.7.2 Algebraic topology

Definition 1.7.6 (Stasheff operad). A topological operad $\mathcal{K}$ such that $\mathcal{K}(n)$ is given by the n^{th} *Stasheff polytope/associahedron*. Composition is given by the inclusion of faces.

Definition 1.7.7 (A_∞-space). An algebra over the Stasheff operad. This induces the structure of a multiplication that is associative up to a coherent homotopy.

Definition 1.7.8 (Little k-cubes operad). A topological operad for which every topological space $\mathcal{P}(n)$ consists of all possible configurations of n embedded k-cubes in a (unit) k-cube. Composition is given by the obvious way of inserting one unit k-cube in one of the smaller embedded k-cubes.

Property 1.7.9 (Recognition principle). If a connected topological space X forms an algebra over the little k-cubes operad, it is (weakly) homotopy equivalent to the k-fold loop space $\Omega^k Y$ of another pointed topological space Y. For $k = 1$, one should technically use the Stasheff operad, but it can be shown that this is related to the little interval operad.

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Part I

Higher Set Theory

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Chapter 2

Topos theory

The main reference for this chapter is Caramello (2018, 2019); Johnstone (2014); Mac Lane and Moerdijk (1994). For an introduction to stacks and descent theory, see Vistoli (2004).

Chapter Contents

2.1 Elementary topoi

Definition 2.1.1 (Subobject classifier). Consider a finitely complete category (in fact, the existence of a terminal object suffices). A subobject classifier is a mono[1] $\mathtt{true} : 1 \hookrightarrow \Omega$ from the terminal object such that for every mono $\phi : x \hookrightarrow y$ there exists a unique morphism $\chi : y \to \Omega$ that fits in the following pullback square:

Alternative Definition 2.1.2. Consider a well-powered category $\mathbf{C}$. The assignment of subobjects $\mathsf{Sub}(x)$ to an object $x \in \mathsf{ob}(\mathbf{C})$ defines functor $\mathsf{Sub} : \mathbf{C}^{op} \to \mathbf{Set}$. A subobject

[1] The symbol for this morphism will become clear in Section 2.3.

$$\begin{tikzcd}
x \arrow[r] \arrow[d, hook, "\phi"'] \arrow[dr, phantom, "\mathrm{pb}"] & 1 \arrow[d, hook, "\mathtt{true}"] \\
y \arrow[r, "\exists!\chi"'] & \Omega
\end{tikzcd}$$

Figure 2.1: Subobject classifier.

classifier Ω is a representation of this functor, i.e. the following isomorphism is natural in x:

$$\mathsf{Sub}(x) \cong \mathbf{C}(x, \Omega)\,. \tag{2.1}$$

Example 2.1.3 (Indicator function). The category **Set** has the 2-element set $\{\mathtt{true}, \mathtt{false}\}$ as subobject classifier. The morphism $\chi : S \to \Omega$ is the indicator function

$$\chi_S(x) = \begin{cases} \mathtt{true} & \text{if } x \in S\,, \\ \mathtt{false} & \text{if } x \notin S\,. \end{cases} \tag{2.2}$$

Definition 2.1.4 (Elementary topos). An elementary topos is a finitely complete, Cartesian closed category (**??**) admitting a subobject classifier. Equivalently, one can define an elementary topos as a finitely complete category that has all power objects exist.

The power object Px of $x \in \mathsf{ob}(\mathcal{E})$ is related to the subobject classifier Ω by the following relation:

$$Px = \Omega^x\,. \tag{2.3}$$

Remark 2.1.5 (Finite colimits). The original definition by *Lawvere* also required the existence of finite colimits. However, it can be proven that finite cocompleteness follows from the other axioms.

Theorem 2.1.6 (Fundamental theorem of topos theory). *Let $\mathcal{E}$ be an elementary topos. The slice category $\mathcal{E}_{/x}$ is also a topos for every object $x \in \mathsf{ob}(\mathcal{E})$. The subobject classifier is given by $\pi_2 : \Omega \times x \to x$.*

Property 2.1.7 (Balanced). All monos in a topos are regular. Hence, every mono arises as an equalizer. Since, by Property 1.4.63, every epic equalizer is necessarily an isomorphism, it follows that every topos is balanced (Definition 1.4.5).

Property 2.1.8 (Epi/mono factorization). Every morphism $f : x \to y$ in a topos factorizes uniquely as an epi followed by a mono:

$$x \overset{e}{\twoheadrightarrow} z \overset{m}{\hookrightarrow} y\,. \tag{2.4}$$

The mono is called the **image** of f.

2.2 Morphisms

Definition 2.2.1 (Base change). Consider a category $\mathbf{C}$ with pullbacks. For every morphism $f : x \to y$ one can define a functor $f^* : \mathbf{C}/y \to \mathbf{C}/x$. This functor acts by pullback along f.

Definition 2.2.2 (Dependent sum and product). Consider a base change functor f^*. The dependent sum and product functors are given by the right and left adjoints (if they exist):

$$\sum_f \dashv f^* \dashv \prod_f . \tag{2.5}$$

Remark 2.2.3. The dependent sum can be shown to exist over any category. In fact, as a functor, it is simply given by postcomposition with f. However, the interpretation is much less trivial. Any morphism can be interpreted as a "space" with fibres the preimages. For example, in $\mathbf{Set}$, a morphism $g : Z \to X$ represents Z as follows:

$$Z \cong \bigsqcup_{x\in X} g^{-1}(x) \equiv \sum_{x:X} g^{-1}(x).$$

The dependent sum generalizes this construction in a fibrewise manner, i.e. the fibre over a point $y \in Y$ is given by

$$\sum_f g\Big|_y = \bigsqcup_{x\in f^{-1}(y)} g^{-1}(x) . \tag{2.6}$$

Instead of combining all fibres into a single space, it combines those that lie in a single fibre of f.

Although the dependent sum always exists, the dependent product requires more structure (see e.g. Huang (2022)).

Property 2.2.4 (Locally Cartesian closed categories). In a locally Cartesian closed category (**??**), all dependent products exist. In fact, the existence of the adjoint triple (2.5) is equivalent to $\mathbf{C}$ being locally Cartesian closed. For example, in $\mathbf{Set}$, the dependent product is given by

$$\prod_f g\Big|_y = \prod_{x\in f^{-1}(y)} g^{-1}(x) . \tag{2.7}$$

When f is the terminal morphism, this represents the space of sections $X \to Z \equiv \sum_{x:X} g^{-1}(x)$ of g.

More generally, the functor is obtained through the product-exponential adjunction in slice categories. The pullback is equal to a product in a slice category. Since $\mathbf{C}$ is locally Cartesian, one obtains for all $p : a \to y$ and $q : b \to y$:

$$\mathbf{C}_{/x}(p \times f, q) \cong \mathbf{C}_{/y}(p, q^f) \tag{2.8}$$

for some morphism $q^f : b^f \to y$. Applying this exponential functor to the identity morphism $\mathbb{1}_f$, gives a morphism $s : \mathbb{1}_y \to f^f$ or $s : y \to x^f$ (this simply picks out the identity morphism in the exponential object internally). The dependent product is then defined as the pullback $\prod_f g := s^* g^f$, where g^f is interpreted as the morphism $(f \circ g)^f \to f^f$ in $\mathbf{C}_{/y}$.

The idea behind this pullback construction is that in **Set**, the pullback of $g^X : Z^X \to X^X$ along $s : \mathbb{1}_X \to X^X$ is equal to the set

$$\{h \in Z^X \mid h \circ g = \mathbb{1}_Z\}, \tag{2.9}$$

i.e. it consists of all sections of g. Hence, the pullback construction generalizes Eq. (2.7).

Definition 2.2.5 (Logical morphism). A functor $f : \mathcal{E} \to \mathcal{F}$ between elementary topoi is said to be logical if it preserves finite limits, exponential objects and the subobject classifier.

Property 2.2.6. If a logical morphism has a left adjoint if and only if it has a right adjoint.

Definition 2.2.7 (Geometric morphism). A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ of elementary topoi consists of an adjunction

$$\begin{tikzcd} \mathcal{E} \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, bend right, "f^*"'] \arrow[l, leftarrow, bend left, "f_*"] \end{tikzcd},$$

where the left adjoint is left exact. The right adjoint f_* is called the **direct image** part of f and the left adjoint f^* is called the **inverse image** part. If f^* itself has a left adjoint, then f is said to be **essential**.

Definition 2.2.8 (Geometric surjection). A geometric morphism for which the inverse image part is faithful or, equivalently, reflects isomorphisms, is said to be **surjective** or is called a geometric surjection.

Definition 2.2.9 (Geometric embedding). A geometric morphism for which the direct image part is fully faithful. If $\mathcal{E} \hookrightarrow \mathcal{F}$ is a geometric embedding, $\mathcal{E}$ is sometimes called a **subtopos** of $\mathcal{F}$. Moreover, if it is an essential geometric morphisms, the subtopos is itself said to be **essential**. Essential subtopoi $\mathcal{E} \hookrightarrow \mathcal{F}$ are also called **levels** of $\mathcal{F}$.

Property 2.2.10 (Characterization of geometric embeddings). Let $f : \mathcal{E} \hookrightarrow \mathcal{F}$ be a geometric embedding and let $W \subset \mathsf{hom}(\mathcal{F})$ be the collection of morphisms that are mapped to isomorphisms under f^*. $\mathcal{E}$ is both equivalent to the full subcategory of $\mathcal{F}$ on W-local objects (Definition 1.2.21) and the *localization* $\mathcal{F}[W^{-1}]$ at W (see **??**).

Property 2.2.11 (Base change). The base change functors on a topos are logical and admit a left adjoint, the postcomposition functor. This implies that these functors can be refined to essential geometric morphisms.

Example 2.2.12 (Topological spaces). Every continuous function $f : X \to Y$ induces a geometric morphism

$$\begin{tikzcd} \mathbf{Sh}(X) \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathbf{Sh}(Y) \arrow[l, bend right, "f^*"'] \end{tikzcd}, \tag{2.10}$$

where the direct image functor f_* is defined as

$$f_*F(U) := F(f^{-1}U) \tag{2.11}$$

for any sheaf $F \in \mathbf{Sh}(X)$ and any open subset $U \in \mathbf{Open}(Y)$. The inverse image functor f^* is defined using the equivalence between sheaves on topological spaces and étalé spaces. Consider a sheaf $E \in \mathbf{Sh}(Y)$ as an étalé space $\pi : E \to Y$. The inverse image of E along a continuous function $f : X \to Y$ is the pullback of π along f.

This example implies that the global elements $* \to X$ of a topological space induce geometric morphisms of the form $\mathbf{Sh}(*) \to \mathbf{Sh}(X)$. By noting that $\mathbf{Sh}(*) = \mathbf{Set}$, one obtains the following generalization.

Definition 2.2.13 (Point). A point of a topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

Notation 2.2.14 (Category of topoi). The category of elementary topoi and geometric morphisms is a 2-category. It is denoted by $\mathbf{Topos}$.

To obtain the structure of a 2-category, one needs to define an appropriate notion of 2-morphism. Because a geometric morphism consists of an adjunction, one can consider two distinct conventions. Either, one can choose the 2-morphisms in $\mathbf{Topos}$ to be the natural transformations $f^* \Rightarrow g^*$ (with associated transformations $g_* \Rightarrow f_*$) or, one can choose them to be the natural transformations $f_* \Rightarrow g_*$ (and associated transformations $g^* \Rightarrow f^*$). This chapter follows Johnstone (2014) and the 'inverse image convention' is used, i.e. a 2-morphism $f \Rightarrow g$ consists of natural transformations $f^* \Rightarrow g^*$ and $g_* \Rightarrow f_*$.

Property 2.2.15 (Balanced). The category $\mathbf{Topos}$ is balanced (Definition 1.4.5), i.e. every geometric morphism that is both a geometric embedding and a geometric surjection is an equivalence.

Theorem 2.2.16 (Factorization). *Every geometric morphisms can be factorized as a geometric surjection followed by a geometric embedding.*

2.3 Internal logic

In this subsection, finitely complete categories that admit a subobject classifier are considered (they do not have to be elementary topoi).

Definition 2.3.1 (Truth value). A global element of the subobject classifier, i.e. a morphism $1 \to \Omega$. The subobject classifier Ω is also sometimes called the **object of truth values**.

Property 2.3.2 (Heyting algebra). For all objects x in an elementary topos, the poset of subobjects $\mathsf{Sub}(x)$ has the structure of an internal meet-semilattice since forming pullbacks of monos gives a natural transformation

$$\cap_x : \mathsf{Hom}(x, \Omega \times \Omega) \to \mathsf{Hom}(x, \Omega)\,. \tag{2.12}$$

By the Yoneda lemma, this induces an internal map

$$\wedge : \Omega \times \Omega \to \Omega\,. \tag{2.13}$$

In fact, it can be shown that this gives the structure of an internal Heyting algebra (**??**) and, in particular, that of an internal locale (**??**). Hence, every topos canonically gives an external Heyting algebra, namely $\mathsf{Sub}(1)$. Furthermore, every power object is an internal Heyting algebra. This in particular includes the subobject classifier $\Omega = P1$.

Definition 2.3.3 (Mitchell–Bénabou language). Let $\mathcal{E}$ be an elementary topos with subobject classifier Ω.

1. **Type**: An object $x \in \mathsf{ob}(\mathcal{E})$.
2. **Variable** (of type x): An identity morphism $\mathbb{1}_x$.
3. **Term** (of type x in variables α_i of type x_i): a morphism $\prod_{i\in I} x_i \to x$.
4. **Formula** or **proposition**: a term of type Ω. Moreover, a formula is deemed true if it factors through $\mathtt{true} : 1 \to \Omega$.

The logical connectives are induced by the internal Heyting structure on Ω. Quantifiers are induced by the (internal) completeness of Ω.

Every type also comes equipped with two binary relations:

1. $=_x$ is obtained from the characteristic morphism of the diagonal inclusion.
2. $\in_x$ is obtained by using the evalution map of the exponential object together with Eq. (2.3).

Since the slices of an elementary topos are themselves elementary topoi, one can also define dependent types. An **indexed type** is an object of a slice category. **(Dependent)**

sums and **products** of type families are goven by the left and right adjoints to the base change functor.

An interpretation of the Mitchell–Bénabou language in an elementary topos $\mathcal{E}$ goes as follows.

Definition 2.3.4 (Kripke–Joyal semantics). The **entailment** $U \vdash \phi(\alpha)$ for a term $\alpha : U \to X$ and a proposition $\phi(x) : X \to \Omega$ is said to hold if α factors through the pullback $\{x \in X \mid \phi(x)\}$.

Property 2.3.5 (Monotonicity). If $V \vdash \phi(\alpha)$ and $f : U \to V$, then $U \vdash \phi(\alpha \circ f)$.

Property 2.3.6 (Locality). If $f : U \to V$ is epic and $V \vdash \phi(\alpha \circ f)$, then $U \vdash \phi(\alpha)$.

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2.4 Grothendieck topoi

2.4.1 Grothendieck topologies

Definition 2.4.1 (Sieve). Let $\mathbf{C}$ be a small category. A sieve S on $\mathbf{C}$ is a fully faithfull discrete fibration $S \hookrightarrow \mathbf{C}$.

A sieve S on an object $x \in \mathbf{C}$ is a sieve in the slice category $\mathbf{C}_{/x}$. This means that S is a subset of $\mathrm{ob}(\mathbf{C}_{/x})$ that is closed under precomposition, i.e. if $y \to x \in S$ and $z \to y \in \mathrm{hom}(\mathbf{C})$, then $z \to y \to x \in S$.

All of this can be summarized by saying that a sieve on an object $x \in \mathrm{ob}(\mathbf{C})$ is a subfunctor of the hom-functor $\mathbf{C}(-, x)$.

Example 2.4.2 (Maximal sieve). Let $\mathbf{C}$ be a category. The maximal sieve on $x \in \mathrm{ob}(\mathbf{C})$ is the collection of all morphisms $\{f \in \mathrm{hom}(\mathbf{C}) \mid \mathrm{cod}(f) = x\}$ or, equivalently, all of $\mathrm{ob}(\mathbf{C}_{/x})$.

Example 2.4.3 (Pullback sieve). Consider a morphism $f : x \to y$. Given a sieve S on y, one can construct the pullback sieve f^*S on x as the sieve of morphisms in S that factor through f:

$$f^*S(x) = \left\{g \mid f \circ g \in S(y)\right\}. \tag{2.14}$$

Property 2.4.4 (Presheaf topos). Consider the presheaf category $\mathbf{Psh}(\mathbf{C})$ on an arbitrary (small) category $\mathbf{C}$. This category is an elementary topos, where the subobject classifier assigns sieves:

$$\underline{\Omega}(x) := \{S \mid S \text{ is a sieve on } x\}. \tag{2.15}$$

The action on a morphism $f : x \to y$ gives the morphism $\underline{\Omega}(f)$ that sends a sieve S to its pullback sieve f^*S.

The morphism $\texttt{true} : \underline{1} \hookrightarrow \underline{\Omega}$ is defined as the natural transformation assigning to every object its maximal sieve. For every subobject $\underline{Y} \hookrightarrow \underline{X}$ the characteristic morphism $\chi_{\underline{Y}}$ is defined as follows. Consider an object $c \in \mathrm{ob}(\mathbf{C})$. The component $\chi_{\underline{Y}}|_c$ is then given by

$$\chi_{\underline{Y}}|_c(x) := \left\{ f \in \mathbf{C}(d, c) \mid \underline{X}(f)(x) \in \underline{Y}(d) \right\}, \tag{2.16}$$

for all $x \in \underline{X}(c)$.

The following definition is due to *Giraud* (for the original definition using the notion of a *cover*, see the end of this section).

Definition 2.4.5 (Grothendieck topology). A Grothendieck topology on a category $\mathbf{C}$ is a map J assigning to every object a collection of sieves satisfying the following conditions:

1. **Identity**[2]: For every object $x \in \mathrm{ob}(\mathbf{C})$, the maximal sieve M_x is an element of $J(x)$ or, equivalently, all sieves generated by isomorphisms are in $J(x)$.
2. **Base change**: If $S \in J(x)$, then $f^*S \in J(y)$ for every morphism $f : y \to x$.
3. **Locality**: Consider a sieve S on $x \in \mathrm{ob}(\mathbf{C})$. If there exists a sieve $R \in J(x)$, such that for every morphism $(f : y \to x) \in R$ the pullback sieve $f^*S \in J(y)$, then $S \in J(x)$.

The sieves in J are called (J-)**covering sieves**. A collection of morphisms with codomain $x \in \mathrm{ob}(\mathbf{C})$ is called a **cover**[3] of x if the sieve generated by these morphisms is a covering sieve on x.

Example 2.4.6 (Topological spaces). These conditions have the following interpretation in the case of topological spaces:

- The collection of all open subsets covers a space U.
- If $\{U_i\}_{i\in I}$ covers U, then $\{U_i \cap V\}_{i\in I}$ covers $U \cap V$.
- If $\{U_i\}_{i\in I}$ covers U and, if for every $i \in I$ the collection $\{U_{ij}\}_{j\in J_i}$ covers U_i, then $\{U_{ij}\}_{i\in I, j\in J_i}$ covers U.

The canonical Grothendieck topology on $\mathbf{Open}(X)$ is given by the sieves $S = \{U_i \hookrightarrow U\}_{i\in I}$, where $\bigcup_{i\in I} U_i = U$. This topology is denoted by $J_{\mathbf{Open}(X)}$.

Definition 2.4.7 (Site). A (small) category equipped with a Grothendieck topology J.

[2] The name itself stems from the fact that the maximal sieve is generated from the identity morphism.

[3] Sometimes this term is also used to denote any collection of morphism with common codomain x, i.e. without reference to a covering sieve.

A slightly weaker notion than that of a (Grothendieck) topology is the following.

Definition 2.4.8 (Coverage). Let $\mathbf{C}$ be a category. A coverage on $\mathbf{C}$ is a map that assigns to every object $x \in \mathsf{ob}(\mathbf{C})$ a collection of families $\{f : y \to x\} \subset \mathsf{hom}(\mathbf{C})$, the **covering families** or **covers**, satisfying the following condition. If $\{f : y \to x\}$ is a covering family on x, then for every morphism $g : x' \to x$ there exists a covering family $\{f' : y' \to x'\}$ on x' such that every composite $g \circ f'$ factors through some f.

Definition 2.4.9 (Matching family). Consider a presheaf $F \in \mathbf{Psh}(\mathbf{C})$ together with a sieve S on $x \in \mathsf{ob}(\mathbf{C})$. A matching family for S with respect to F is a natural transformation $\alpha : S \Rightarrow F$ between S, regarded as a subfunctor of $\mathbf{C}(-, x)$, and F.

More explicitly, it is an assignment of an element $x_f \in Fy$ to every morphism $(f : y \to x) \in S$ such that

$$F(g)(x_f) = x_{f \circ g} \tag{2.17}$$

for all morphisms $g : z \to y$. Equivalently, a matching family for S with respect to F is a set of elements $\{x_f\}_{f \in S}$ such that for all covering morphisms $f : y \to x, g : z \to x \in S$ and all morphisms $f' : c \to y, g' : c \to z$ such that $f \circ f' = g \circ g'$ the following equations holds:

$$F(f')(x_f) = F(g')(x_g)\,. \tag{2.18}$$

Given such a matching family, one calls an element $a \in Fx$ an **amalgamation** if it satisfies

$$F(f)(a) = x_f \tag{2.19}$$

for all morphisms $f \in S(y)$.

The existence of such an element can also be stated in terms of natural transformations. Consider the obvious inclusion ι_S of S into the the hom-functor $\mathbf{C}(-, x)$. Every morphism with codomain x can be obtained from the identity morphism by precomposition and, hence, a natural transformation $\mathbf{C}(-, x) \Rightarrow F$ is determined by its action on the identity morphisms $\mathbb{1}_x$. The existence of an amalgamation is thus equivalent to the existence of an extension of S along ι_S.

Remark 2.4.10. If the base category has all pullbacks, for example if it is a topos on its own, one can restrict the above commuting diagrams to the pullback diagrams of morphisms in the sieve S.

Definition 2.4.11 (Sheaf). Consider a site $(\mathbf{C}, J)$. A presheaf F on $\mathbf{C}$ is called a J-sheaf if every matching family, for every covering sieve in J, admits a unique amalgamation[4] or, equivalently, if all sieves admit a unique extension to representable presheaves.

[4] If there exists at most one amalgamation, the presheaf is said to be **separated**.

The category $\mathbf{Sh}(\mathbf{C}, J)$ of J-sheaves on the site $(\mathbf{C}, J)$ is the full subcategory of $\mathbf{Psh}(\mathbf{C})$ on the presheaves that satisfy the above condition.

This definition can also be restated in terms of local objects (Definition 1.2.21).

Alternative Definition 2.4.12 (Sheaf). By definition every covering sieve admits a morphism into the Yoneda embedding: $\eta : S \hookrightarrow \mathcal{Y}x$. If the collection of all these morphisms is denoted by $\mathcal{S}$, a presheaf is a sheaf if and only if it is $\mathcal{S}$-local, i.e. if the following morphism is an isomorphism for all $\eta \in \mathcal{S}$:

$$Fx \cong \mathbf{Psh}(\mathcal{Y}x, F) \xrightarrow{\mathbf{Psh}(\eta, F)} \mathbf{Psh}(S, F)\,. \tag{2.20}$$

This is also called the **descent condition** for sheaves. In this context the collection of matching families $\mathsf{Match}(S, F) := \mathbf{Psh}(S, F)$ for a sieve S with respect to a presheaf F is often called the **descent object** of S with respect to F.

Example 2.4.13 (Topological spaces). The usual category of sheaves $\mathbf{Sh}(X)$ on a topological space X is obtained as the category of sheaves on the site $(\mathbf{Open}(X), J_{\mathbf{Open}(X)})$. A Grothendieck topos of this form is also called a **spatial topos**. Since the morphisms in the covering sieves are exactly the inclusion maps $U_i \hookrightarrow U$, the pullback of two such morphisms is given by the intersection $U_i \cap U_j$. Hence, the condition for a matching family, as formulated in Definition 2.4.9 above, gives the second part of **??**. The uniqueness of an amalgamation is equivalent to the first part of that definition.

For topological spaces, sheaves are easily represented visually. A matching family assigns to every set U_i of an open cover $\mathcal{U} \equiv \{U_i\}_{i\in I}$ of U an element $x_i \in FU_i$, such that the restrictions coincide on double overlaps, as in step (1) in the figure below.

Cover Matching family Unique amalgamation

$$\begin{tikzcd}[column sep=small]
& U_i \cap U_j \arrow[dl] \arrow[dr] & & & & x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j} \arrow[dl, leftarrow] \arrow[dr, leftarrow] & & & & x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j} \arrow[dl, leftarrow] \arrow[dr, leftarrow] & \\
U_i \arrow[dr] & & U_j \arrow[dl] \arrow[r, "(1)"] & {} & x_i & & x_j \arrow[r, "(2)"] & {} & x_i = x|_{U_i} & & x_j = x|_{U_j} \\
& U & & & & & & & & x \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The descent condition then states that for every such matching family, there exists a unique element x on U, such that the elements of the matching family are restrictions of x as in step (2) of the figure above.

The classical example would be the assignment of the set of continuous functions to open subsets of a topological space. When two functions, defined on two open sets, coincide on the intersection, there exists a unique continuous function defined on the union, such that it restricts to the given functions.

Above, in Definition 2.4.8, the notion of a coverage was introduced. It should be clear that every coverage generates a sieve. Furthermore, although coverages are weaker and easier to handle, they are in fact equivalent for the purpose of sheaf theory.

Property 2.4.14. Consider a covering family C and let S_C be the sieve it generates. A presheaf is a sheaf for C if and only if it is a sheaf for S_C.

Example 2.4.15 (Canonical topology). The canonical topology on a category is the finest Grothendieck topology for which all representable presheaves are sheaves. A **subcanonical topology** is then defined as a subtopology of the canonical one, i.e. any Grothendieck topology for which all representable presheaves are sheaves.

Example 2.4.16 (Minimal and maximal topologies). The minimal Grothendieck topology on a category is the one for which only the maximal sieves are covering sieves. In this topology all presheaves are sheaves. The maximal Grothendieck topology is the one for which all sieves are covering sieves. In this topology only the terminal element of the associated presheaf category is a sheaf.

Definition 2.4.17 (Grothendieck topos). A category equivalent to the category of sheaves on a (small) site. This site is often called the **site of definition** for the given topos.

Property 2.4.18. Every Grothendieck topos is an elementary topos.

Property 2.4.19. For every Grothendieck topos, there exists a (standard) site of definition for which the Grothendieck topology is (sub)canonical.

Property 2.4.20. Let $\mathcal{E}$ be a Grothendieck topos. The canonical topology J_{can} on $\mathcal{E}$ is the topology for which the covering sieves are jointly epimorphic families. Moreover,

$$\mathcal{E} \cong \mathbf{Sh}(\mathcal{E}, J_{\text{can}})\,. \tag{2.21}$$

Construction 2.4.21 (Sheafification). Given a presheaf $\mathcal{F}$, one can construct a sheaf $\overline{\mathcal{F}}$ along the same lines of **??**.

Definition 2.4.22 (Global sections functor). Every Grothendieck topos $\mathcal{E}$ admits a geometric morphism to **Set**, where the right adjoint assigns to an object x its set of global elements:

$$\Gamma : \mathcal{E} \to \mathbf{Set} : x \mapsto \mathcal{E}(1, x)\,. \tag{2.22}$$

When $\mathcal{E}$ is the sheaf topos over a topological space, this is exactly the global sections functor (**??**). The left adjoint assigns to every set S the copower $S \cdot 1 \equiv \coprod_{s\in S} 1$. When $\mathcal{E}$ is a sheaf topos, this adjoint is exactly the constant sheaf functor. It is sometimes denoted by LConst (cf. **??**).

A different approach for defining sheaf topoi is through an embedding of sheaves into presheaves.

Definition 2.4.23 (Local isomorphism). A system of local isomorphisms in $\mathbf{Psh}(\mathbf{C})$ is a class of morphisms in $\mathbf{Psh}(\mathbf{C})$ forming a *system of weak equivalences* (see **??**) closed under pullbacks along morphisms out of representable presheaves.

Property 2.4.24 (Local isomorphisms and Grothendieck topologies). A system of local isos induces a *system of local epis* in the following way. $f : X \to Y$ is a local epi if $\mathrm{im}(f) \to Y$ is a local iso. A Grothendieck topology is defined by declaring a presheaf $F \in \mathbf{Psh}(\mathbf{C})$ to be a covering sieve at $X \in \mathrm{ob}(\mathbf{C})$ if $F \hookrightarrow \mathcal{Y}X$ is a local epi.

Alternative Definition 2.4.25 (Sheaf topos). A category $\mathbf{Sh}(\mathbf{C})$ equipped with a geometric embedding into $\mathbf{Psh}(\mathbf{C})$.

Proof (Proof of equivalence). By Property 2.2.10, such a category is equivalent to the full subcategory on S-local presheaves for some system of local isomorphisms S and, therefore, also to a sheaf topos in the sense of Grothendieck by the property above. □

Remark 2.4.26 (Descent condition). This is essentially a restatement of the descent condition (cf. 2.4.12). Covering sieves, regarded as subfunctors, are in particular local isomorphisms. Stability of sieves under pullback, together with the co-Yoneda lemma 1.4.80, which says that every presheaf is a colimit of representables, generates the full collection of local isomorphisms.

The characterization of geometric embeddings and, hence, of Grothendieck topoi in terms of *localizations* (Property 2.2.10) is equivalent to a definition in terms of reflections (this is due to *Street*).

Corollary 2.4.27. There exists a bijection between the Grothendieck topologies on a small category $\mathbf{C}$ and the equivalence classes of left exact reflective subcategories of $\mathbf{Psh}(\mathbf{C})$.

Definition 2.4.28 (Representable morphism). A natural transformation of presheaves $F \to G$ is said to be representable if, for every representable presheaf h_X and every morphism $h_X \to G$, the pullback $h_X \times_G F$ is representable.

Property 2.4.29 (Diagonals). The diagonal morphism $\Delta_F : F \to F \times F$, for F a presheaf on a category with pullbacks, is representable if and only if any of the following two equivalent properties holds:

1. For every two representable presheaves h_X, h_Y and natural transformations $h_X \to F, h_Y \to F$, the pullback $h_X \times_F h_Y$ is representable.

2. Every natural transformation $h_X \to F$ from a representable presheaf is representable.

2.4.2 Topological sheaves

See **??** for the application of sheaves to topology.

Property 2.4.30 (Presheaf topos). Consider the presheaf category

$$\mathbf{Psh}(X) := \mathbf{Psh}\big(\mathbf{Open}(X)\big) \tag{2.23}$$

over a topological space (X, τ). Unpacking Property 2.4.4 shows that this category is an elementary topos where the subobject classifier is given by

$$\Omega(U) = \{V \in \tau \mid V \subseteq U\}\,. \tag{2.24}$$

Construction 2.4.31 (Sheaves and étale bundles). Let X be a topological space. The functor

$$I : \mathbf{Open}(X) \to \mathbf{Top}_{/X} : U \mapsto (U \hookrightarrow X) \tag{2.25}$$

induces the following adjunction:

$$\begin{tikzcd} \mathbf{Top}_{/X} \arrow[r, bend right, "\Gamma"'] \arrow[r, phantom, "\perp"] & \mathbf{Psh}(X) \arrow[l, bend right, "E"'] \end{tikzcd}\,. \tag{2.26}$$

The slice category on the left-hand side is the category of (topological) *bundles* (see **??**) over X. Both directions of the adjunction have a clear interpretation. The right adjoint assigns to every bundle its sheaf of local sections and the left adjoint assigns to every presheaf its bundle of germs.

By restricting to the subcategories on which this adjunction becomes an adjoint equivalence, one obtains the **étale space** and **sheaf categories** respectively:

$$\mathbf{Et}(X) \cong \mathbf{Sh}(X)\,. \tag{2.27}$$

The category on the right-hand side is the category of sheaves on a topological space X. The category on the left is the full subcategory on local homeomorphisms, i.e. the étale spaces (**??**).

Property 2.4.32 (Associated sheaf). The inclusion functor $\mathbf{Sh}(X) \hookrightarrow \mathbf{Psh}(X)$ admits a left adjoint, the **sheafification functor**, that assigns to every presheaf its associated sheaf. This functor is given by the composition $\Gamma \circ E$, which is simply **??**.

The fact that the counit of the adjunction (Construction 2.4.31) restricts to an isomorphism on the full subcategory $\mathbf{Sh}(X)$ is equivalent to the fact that the sheafification of a sheaf Γ is again Γ.

Definition 2.4.33 (Petit and gros topoi[5]). Consider a topological space X together with its category of opens $\mathbf{Open}(X)$. The petit topos over X is defined as the sheaf topos $\mathbf{Sh}(X) \equiv \mathbf{Sh}\big(\mathbf{Open}(X)\big)$. It represents X as a 'generalized space'. (By Construction 2.4.31, the objects in a petit topos are the étale spaces over the given base space.) Topoi equivalent to such petit topoi are sometimes said to be **spatial**. However, one can also build a topos whose objects are themselves generalized spaces. To this end, choose a site S of 'probes' and call the sheaf topos $\mathbf{Sh}(S)$ a gros topos. (See Section 5.4 for more information.)

Property 2.4.34 (Localic reflection). Mapping a topological space to its sheaf of continuous sections defines a functor $\mathbf{Sh} : \mathbf{Top} \to \mathbf{Topos}$ by Example 2.2.12. When restricted to the full subcategory of sober spaces (**??**), this functor becomes fully faithful. Generalizing to locales even gives a reflective inclusion (Definition 1.2.29).

This property states that no information is lost when regarding (sober) topological spaces as sheaf topoi. This also explains the name 'petit topos'.

Definition 2.4.35 (Localic topos). Multiple equivalent definitions exist:

1. A topos equivalent to a sheaf topos over a locale (**??**) equipped with the topology of jointly surjective morphisms.
2. A topos generated under colimits of subobjects of the terminal object 1.
3. A topos $\mathcal{E}$ for which the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$ is localic, i.e. every object in $\mathcal{E}$ is a subquotient of an object in the inverse image Γ^*

Given a geometric morphism to some base topos $\mathcal{S}$, one can define $\mathcal{S}$-localic topoi by generalizing the third point.

The following property shows that the locale in the first definition has a specific meaning.

Property 2.4.36. By Property 2.3.2, for every topos the poset $\mathsf{Sub}(1)$ is a locale. Every localic topos $\mathcal{E}$ satisfies $\mathcal{E} \cong \mathsf{Sh}\big(\mathsf{Sub}(1)\big)$, where $\mathsf{Sub}(1)$ is equipped with the topology of jointly surjective morphisms.

The equivalence between localic topoi and locales carries over to the notion of $\mathcal{S}$-localic topoi.

Property 2.4.37. The (2-)category of localic topoi over a base topos $\mathcal{S}$ is equivalent to the (2-)category $\mathbf{Loc}(\mathcal{S})$ of locales internal to $\mathcal{S}$.

Property 2.4.38. Given a locale X, the category $\mathbf{Loc}\big(\mathbf{Sh}(X)\big)$ is equivalent to the slice category $\mathbf{Loc}_{/X}$.

[5]For those that do not master French, '*petit*' and '*gros*' mean small and big, respectively.

2.4.3 Lawvere–Tierney topology

This section gives an alternative approach to the construction of sheaf topoi, which is more closely linked to the logical aspect of topos theory.

Definition 2.4.39 (Lawvere–Tierney topology). As noted in Section 2.3 on the internal logic of elementary topoi, the subobject classifier Ω has the structure of an internal Heyting algebra and, in particular, that of a (bounded) meet-semilattice. This internal poset, viewed as an internal category, admits the construction of a closure operator $j:\Omega\to\Omega$ (Definition 1.3.27) satisfying the following condition:

$$j\circ\wedge=\wedge\circ(j\times j)\,. \tag{2.28}$$

This condition states (in a nontrivial way) that j is (internally) order-preserving. More concisely, a Lawvere–Tierney topology is internally a left exact modality on Ω.

Remark 2.4.40. The condition satisfied by the unit morphism in the definition of a closure operator can also be reformulated in this context as follows:

$$j\circ\mathtt{true}=\mathtt{true}\,. \tag{2.29}$$

Example 2.4.41 (Double negation topology). As mentioned above, the subobject classifier in an elementary topos has the structure of an internal Heyting algebra and, hence, admits a negation morphism $\neg:\Omega\to\Omega$. It can be shown that double negation $\neg\neg:\Omega\to\Omega$ gives a Lawvere–Tierney topology.

Property 2.4.42. The Booleanization $\mathcal{E}_{\neg\neg}\hookrightarrow\mathcal{E}$ of an elementary subtopos is the smallest dense subtopos of $\mathcal{E}$ and the largest Boolean subtopos of $\mathcal{E}$, i.e. it is the unique subtopos such that:

1. **Dense**: $\mathcal{E}_{\neg\neg}$ contains the initial object $\perp$.
2. **Boolean**: Ω is a Boolean algebra.

As noted above, Lawvere–Tierney operators induce, internally, *universal closure operators* on all posets $\mathrm{Sub}(x)$ in the topos. Given an object x and a subobject $u\in\mathrm{Sub}(x)$, one defines the closure $j_*(u)\in\mathrm{Sub}(x)$ as the subobject classified by the characteristic morphism $j\circ\chi_u:x\to\Omega$.

Definition 2.4.43 (Dense object). Given a Lawvere–Tierney topology $j:\Omega\to\Omega$, a subobject $u\in\mathrm{Sub}(x)$ is said to be dense (in x) if it satisfies $j_*(u)\cong x$. An object is said to be **closed** if $j_*(u)\cong u$.

Example 2.4.44. On a presheaf topos, objects are dense exactly if they are covering sieves (Definition 2.4.5).

The following definition allows to generalize sheaves from topoi to finitely complete categories.

Alternative Definition 2.4.45 (Sheaf). Given a Lawvere–Tierney topology $j : \Omega \to \Omega$ on a topos $\mathcal{E}$, one calls an object $s \in \mathrm{ob}(\mathcal{E})$ a j-sheaf if it is local with respect to all dense morphisms (Definition 1.2.21), i.e. for all dense morphisms $u \hookrightarrow x$ the induced map

$$\mathcal{E}(x, s) \to \mathcal{E}(u, s)$$

is a bijection. If this map is only a monomorphism, the object is said to be ***j*-separated**.

Property 2.4.46. Lawvere–Tierney topologies on $\mathcal{E}$ are in correspondence with subtopoi of $\mathcal{E}$, given by the sheaves relative to the chosen topology. As before, the left adjoint to the embedding is the sheafification functor.

In the case of (pre)sheaf topoi, Lawvere–Tierney topologies recover Grothendieck topologies.

Corollary 2.4.47. For the presheaf topos on a small category $\mathbf{C}$, the Grothendieck topologies on $\mathbf{C}$ are equivalent to Lawvere–Tierney topologies on $\mathbf{Psh}(\mathbf{C})$.

Proof (Sketch of proof). Since a Grothendieck topology assigns to every object a collection of sieves, Property 2.4.4 implies that $J(x) \subseteq \Omega_{\mathbf{Psh}}(x)$ for all $x \in \mathrm{ob}(\mathbf{C})$. By the base change condition of Grothendieck topologies, this relation is natural in x and, hence, J is a subobject of $\Omega_{\mathbf{Psh}}$. One thus finds a characteristic morphism $j : \Omega_{\mathbf{Psh}} \to \Omega_{\mathbf{Psh}}$ that can be proven (by the other conditions of Grothendieck topologies) to define a Lawvere–Tierney topology on $\mathbf{Psh}(\mathbf{C})$. Conversely, a Lawvere–Tierney topology is a morphism $j : \Omega \to \Omega$ and, hence, determines a unique subobject of $\Omega_{\mathbf{Psh}}$, i.e. a unique collection of sieves for every object $x \in \mathrm{ob}(\mathbf{C})$. From the conditions of Lawvere–Tierney topologies, one can then prove that this collection satisfies the conditions of a Grothendieck topology. $\square$

2.4.4 Diaconescu's theorem

The following concept weakens the notion of left exact functors to categories where not all (finite) limits exist.

Definition 2.4.48 (Flat functor). A functor $F : \mathbf{C} \to \mathbf{Set}$ is said to be flat if the opposite of the category of elements $\mathrm{El}(F)^{\mathrm{op}}$ (Definition 1.3.15) is filtered (Definition 1.4.50). The category of flat functors on $\mathbf{C}$ is denoted by $\mathbf{Flat}(\mathbf{C}, \mathbf{Set})$.

A general functor $F : \mathbf{C} \to \mathbf{D}$ is said to be **(representably) flat** if, for every object $d \in \mathrm{ob}(\mathbf{D})$, the opposite comma category $(d/F)^{\mathrm{op}}$, which is equivalently the opposite category of elements $\mathrm{El}\big(\mathbf{D}(d, -) \circ F\big)^{\mathrm{op}}$, is filtered.

The definitions above can be stated more explicitly as follows (cf. Definition 1.4.64). The functor $F : \mathbf{C} \to \mathbf{D}$ is representably flat if for every $d \in \mathrm{ob}(\mathbf{D})$:

1. There is at least one object $c \in \mathrm{ob}(\mathbf{C})$ and a morphism $f : d \to Fc$. Hence, at least one inhabited set in the image of F.

2. For every two objects $c, c' \in \mathrm{ob}(\mathbf{C})$ and every two morphisms $f \in \mathbf{D}(d, Fc)$ and $g \in \mathbf{D}(d, Fc')$, there exists an object $x \in \mathrm{ob}(\mathbf{C})$ and morphisms $i \in \mathbf{C}(x, c)$, $j \in \mathbf{C}(x, c')$ and $h \in \mathbf{D}(d, Fx)$ such that $Fi \circ h = f$ and $Fj \circ h = g$.

3. For every two parallel morphisms $f, g \in \mathbf{C}(c, c')$ and morphism $h \in \mathbf{D}(d, Fc)$ such that $Ff \circ h = Fg \circ h$, there exists a morphism $i \in \mathbf{C}(x, c)$ and a morphism $j \in \mathbf{D}(d, Fx)$ such that $f \circ i = g \circ i$ and $Fi \circ j = h$.

Remark 2.4.49. Note that flatness and representable flatness are only equivalent for **Set**-valued functors when the domain $\mathbf{C}$ is finitely complete.

Property 2.4.50. Consider the **Yoneda extension** that maps a functor $F \in [\mathbf{C}, \mathbf{Set}]$ to a functor $\tilde{F} : [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ through left Kan extension along the Yoneda embedding. F is flat if and only if $\tilde{F}$ is left exact.[6]

Property 2.4.51. A functor $F : \mathbf{C} \to \mathbf{Set}$ is flat if and only if it is the filtered colimit of representable functors or, equivalently, if F-weighted filtered colimits commute with finite limits. Note that, by Universal Property 1.4.56, this means that

$$\mathbf{Flat}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}) \cong \mathbf{Ind}(\mathbf{C})\,. \tag{2.30}$$

The previous property implies the following statement, which is the **Set**-valued version of *Diaconescu's theorem*, which will be covered in its internal version below.

Theorem 2.4.52 (Diaconescu: Set version). *A functor $F : \mathbf{C} \to \mathbf{Set}$ is flat if and only if the induced adjunction $\mathbf{Set} \leftrightarrows [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is a geometric morphism or, alternatively, if flat functors are equivalent to global points of the presheaf topos $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$.*

Definition 2.4.53 (Morphism of sites). A representably flat functor preserving covers.

Now, for $\mathcal{E}$ a cocomplete topos, a functor $F : \mathbf{C} \to \mathcal{E}$ is said to be **(internally) flat** if it representably flat with respect to the internal logic of $\mathcal{E}$ or, equivalently, if the Yoneda extension $\mathrm{Lan}_y F$ is left exact.

[6] By Definition 1.4.86, Kan extensions admit an expression is terms of adjoints. In this setting, the right adjoint has the form of a Hom-functor and, accordingly, the Yoneda extension is sometimes written as a tensor product $- \otimes_{\mathbf{C}} F$. The terminology 'flat' is then in analogy with **??**. This essentially coincides with the tensor product of functors **??**.

Property 2.4.54. A functor $F : \mathbf{C} \to \mathcal{E}$, with $\mathcal{E}$ a cocomplete topllos, is flat if and only if:

- The family of terminal morphisms in $\mathbf{C}$ is jointly epimorphic.
- For all $c, d \in \mathsf{ob}(\mathbf{C})$, the family of morphisms into $Ac \times Ad$ induced by spans $c \leftarrow b \to d$ is jointly epimorphic.
- For any two parallel morphisms f, g in $\mathbf{C}$, the family of factorizations of morphisms through their equalizer is jointly epimorphic.

These conditions are equivalent to the following ones:

- For every object $x \in \mathcal{E}$, there exists a jointly epimorphic family $\{x_i \to x\}_{i\in I}$ and and a family $\{x_i \to F(c_i) \mid c_i \in \mathsf{ob}(\mathbf{C})\}_{i\in I}$.
- For all $c, d \in \mathsf{ob}(\mathbf{C})$ and generalized elements $\lambda : x \to Fc \times Fd$, there exists an epimorphic family $\{\lambda_i : x_i \to x\}_{i\in I}$ and a family of spans $\{c \xleftarrow{f} b_i \xrightarrow{g} d\}_{i\in I}$ equipped with a family of generalized elements $\{\kappa_i : x_i \to Fb_i\}_{i\in I}$ such that
$$(Ff, Fg) \circ \kappa_i = \lambda \circ \lambda_i\,. \tag{2.31}$$
- For any two parallel morphisms $f, g : c \to d$ in $\mathbf{C}$ and any generalized element $\lambda : x \to Fc$ such that $Ff \circ \lambda = Fg \circ \lambda$, there exists an epimorphic family $\{\lambda_i : x_i \to x\}_{i\in I}$ and families of morphisms $\{f_i : c_i \to c\}$ and $\{\kappa_i : x_i \to Fc_i\}$ such that
$$\begin{aligned} f \circ f_i &= g \circ f_i \\ Ff_i \circ \kappa_i &= \lambda \circ \lambda_i \end{aligned} \tag{2.32}$$
  for all $i \in I$.

Theorem 2.4.55 (Diaconescu: Topos version). *Consider a cocomplete topoi $\mathcal{E}$ and a (small) category $\mathbf{C}$. There exists an equivalence*
$$\mathbf{Topos}\big(\mathcal{E}, [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]\big) \leftrightarrows \mathbf{Flat}(\mathbf{C}, \mathcal{E})\,, \tag{2.33}$$
where

- *flat functors are sent to geometric morphisms whose inverse image part is given by left Kan extension along the Yoneda embedding and direct image part is given by:*
$$F \mapsto \big(x \mapsto \mathcal{E}(F-, x)\big)\,, \tag{2.34}$$
  and
- *geometric morphisms are sent to the precomposition of the inverse image part with the Yoneda embedding.*

To pass to the internal logic of topoi, Definition 1.4.50 needs to be generalized to internal category theory (Section 1.5).

Definition 2.4.56 (Internally filtered category). An internal category $\mathbf{C} \in \mathbf{Cat}(\mathcal{E})$ satisfying:[7]

1. The terminal morphism $C_0 \twoheadrightarrow 1$ is an epi.
2. For any two generalized elements $\lambda_1, \lambda_2 : U \to C_0$, there is an epi $\theta : V \twoheadrightarrow U$ and corresponding V-elements $\gamma_1, \gamma_2 : V \to C_0$ such that
$$\begin{aligned} d_1 \circ \gamma_1 &= d_1 \circ \gamma_2 \\ d_0 \circ \gamma_i &= \lambda_i \circ \theta \end{aligned} \tag{2.35}$$
for $i = 1, 2$.
3. For any two parallel generalized elements $\gamma_1, \gamma_2 : U \to C_1$ such that $d_0 \circ \gamma_1 = d_0 \circ \gamma_2$, there exists an epi $\theta : V \twoheadrightarrow U$ and a V-element $\kappa : V \to C_1$ such that
$$\begin{aligned} d_1 \circ \kappa = d_0 \circ \lambda_1 \circ \theta &= d_0 \circ \lambda_2 \circ \theta \\ c(\kappa, \lambda_1 \circ \theta) &= c(\kappa, \lambda_2 \circ \theta)\,. \end{aligned} \tag{2.36}$$

[7]Some sources require all these morphisms to be regular epis.

Chapter 3

Logic and Type Theory

The main reference for this chapter is The Univalent Foundations Program (2013). For a formal introduction to λ-calculus, see Selinger (2008).

In almost every section of this chapter (at least the ones about type theory), some cross-references to analogous definitions and propositions in other parts of this compendium could have been inserted (Chapter 1 on category theory in particular). However, to reduce the number of references, these relations will only be mentioned and the reader is encouraged to take a look at the relevant chapters whilst or after reading this chapter.

Chapter Contents

3.1 Logic

3.1.1 Languages

Definition 3.1.1 (Language). An **alphabet** is a set of symbols. A **word** in the language is a string of symbols in the alphabet.

Consider an alphabet A. From this alphabet one can construct the free monoid A^* (the multiplication $*$ is sometimes called the **Kleene star**). This monoid represents the set of all words in A and a (formal) language is a subset $L \subseteq A^*$.

Definition 3.1.2 (Signature). Consider an alphabet A and a language L. A signature is a tuple (F, R, ar) that assigns a syntactic meaning to the symbols in A. F and R are respectively the sets of function symbols and relation symbols ($A = F \sqcup R$). The function $\mathsf{ar} : A \to \mathbb{N}$ assigns to every symbol its arity **arity**. Nullary function symbols are also called **constants**.

To give meaning to a language, some extra structure needs to be introduced.

Definition 3.1.3 (L-structure). Consider a (formal) language L. An L-structure consists of the following data:

1. A nonempty set U called the **universe**.
2. For each function symbol f, a function $\mathsf{ap}_f : U^{\mathsf{ar}(f)} \to U$. In particular, for each constant c, an element $u_c \in U$.
3. For each relation symbol $\in$, a set $R_\in \subseteq U^{\mathsf{ar}(\in)}$.

Definition 3.1.4 (L-term). A word in L, possibly containing new symbols (called **variables**), defined recursively as follows:

1. Every variable and every constant is a term.

2. For every n-ary function symbol f and terms $x_1, \dots, x_n$, $f(x_1, \dots, x_n)$ is also a term.

Definition 3.1.5 (L-formula). Consider a (formal) language L. An L-formula is a sentence consisting of terms in L together with parentheses and the following logical symbols (also called **logical connectives**):

- **Equality**: $=$,
- **Negation**: $\neg$,
- **Conjunction**: $\wedge$, and
- **Existential quantification**: $\exists$.

A variable is said to be **free** if it does not first appear next to a quantifier, otherwise it is said to be **bound**.

3.1.2 Propositional logic

Definition 3.1.6 (Proposition). A statement that is either *true* or *false* (not both).

Definition 3.1.7 (Paradox). A statement that cannot (consistently) be assigned a truth value.

Definition 3.1.8 (Contradiction). A statement that is always *false*.

Definition 3.1.9 (Tautology). A statement that is always *true*.

Notation 3.1.10 (Truth values). The truth values *true* and *false* are denoted by $\top$ and $\perp$ respectively.

Definition 3.1.11 (Logical connectives). The following logical operators are used in propositional logic:

- logical 'and' (**conjunction**): $P \wedge Q$,
- logical 'or' (**disjunction**): $P \vee Q$, and
- logical 'then' (**implication**): $P \implies Q$.

Using implication, one can also define the logical 'not' (**negation**): $\neg P :\equiv P \implies \perp$.

The basic inference rule is given by **modus ponens**:

$$\text{If } P \text{ and } P \implies Q, \text{ then } Q. \tag{3.1}$$

One could also use negation as a primitive connective and introduce implication as

$$P \implies Q :\equiv \neg P \wedge Q\,. \tag{3.2}$$

The general deductive system for propositional logic is obtained by combining this rule with the following axioms:

1. If P, then $Q \implies P$.
2. If $P \implies Q \implies R$, then $P \implies Q$ implies $P \implies R$.
3. If $P \wedge Q$, then both P and Q.
4. If P, then $P \vee Q$.
5. If Q, then $P \vee Q$.
6. If P, then Q implies $P \wedge Q$.
7. If $P \implies Q$, then $R \implies Q$ implies $P \vee R \implies Q$.
8. If $\perp$, then P. This principle is often called ***ex falso quodlibet***.

Property 3.1.12 (Boolean algebra). The set of propositions in classical logic admits the structure of a complete Boolean algebra (**??**).

Remark 3.1.13 (Intuitionistic logic). The above axioms (together with modus ponens) define a specific type of propositional logic, called intuitionistic or **constructive** (propositional) logic. The main difference with classic logic is that the *law of the excluded middle* or, equivalently, the *double negation elimination* principle was not added. The reason why this makes the logic *constructive* is that to prove a statement it is not sufficient anymore to exclude the possibility of the statement being false. One has to explicitly construct evidence for the truth of the statement.

As was remarked in the chapter on topoi, intuitionistic logic can be defined internal to any elementary topos. All one needs is a Heyting algebra (**??**).

@@ EXPLAIN THIS @@

3.1.3 Sequent calculus

Definition 3.1.14 (Sequent). A general sequent is of the form

$$P_1, \dots, P_m \vdash Q_1, \dots, Q_n \,. \tag{3.3}$$

In such expressions, the commas on the left-hand side indicate conjunction, whereas those on the right-hand side indicate disjunction, i.e. this sequent states "when every P_i holds, then at least one of the Q_j hold as well". The above sequent if (strongly) equivalent to

$$\vdash (P_1 \wedge \cdots \wedge P_m) \implies (Q_1 \vee \cdots \vee Q_n) \,. \tag{3.4}$$

@@ COMPLETE @@

3.1.4 Predicate logic

@@ ADD @@

3.2 Introduction to type theory

In ordinary set theory, the main objects are sets and their elements (and derived concepts such as functions). The framework in which to state and prove propositions is (in general) given by first-order logic. (See **??** for more on this.) In type theory, however, one puts all these notions on the same footing. That is, one considers all concepts such as functions, propositions, sets, etc. as specific instances of the general notion of *type*. A specific function, proof or element can then be seen as an *inhabitant* of a given type.

Definition 3.2.1 (Type judgement). A statement of the form $a : A$, saying that a has the type A, is called a type judgement. Objects having a certain type are in general called **terms** (of that type).

Method 3.2.2 (Type definition). The general method for defining a new type consists of 4 steps/rules:

1. **Formation rule**: This rule says when the new type can be introduced, given a collection of pre-existing types.
2. **Introduction rule**: This rule gives a **constructor** of the new type, i.e. a way to construct a term of the new type[1], in terms of the types required by the formation rule. The pre-existing terms from which a new term can be constructed is often called the **context**.
3. **Elimination rule**: This rule says how the new type can be used.
4. **Computation rule**: This rule says how the elimination and introduction rules interact, i.e. how the elimination rules can actually be applied to a term of the given type.

As in The Univalent Foundations Program (2013), a universe hierarchy à la Russell will be adopted, i.e. a sequence of universes $(\mathcal{U}_n)_{n\in\mathbb{N}}$ will be used where the terms of every universe are types and every universe is cumulative in the sense that $A : \mathcal{U}_n \implies A : \mathcal{U}_{n+1}$. In general, the subscripts will be omitted. However, one should take into account that every well-typed judgement should admit a formulation in which subscripts can be assigned in a consistent way.

In contrast to ordinary set theory, two kinds of equality will be considered. First, there is the **judgemental** or **definitional equality**. This says, as the name implies, that two

[1] As in object-oriented programming languages.

judgements are equal by definition and, as such, its validity lives in the metatheory (it is not a proposition and, hence, cannot be proven). For example, if $f(x)$ is defined as x^2, then $f(5)$ is, by definition, equal to 5^2. Equalities of this sort will be denoted by the $\equiv$ symbol (and, in definitions, $:\equiv$ will be used instead of $:=$). The second equality is the **propositional equality**. This states that two judgements are provably equal. Again, consider the function $f(x) :\equiv x^2$. In this case, the proposition $f(5) = 25$ is not true by definition and can be proven (it would, however, depend on the definition of the natural numbers). This sort of equality will be denoted by an ordinary equals sign =.

3.3 Basic constructions

3.3.1 Functions

Functions can be introduced in two ways. Either through a direct definition, such as in the case of the default example $f(x) :\equiv x^2$, or through *λ-abstraction*. Although the former one is clearly more useful during explicit calculations, the latter will often be used when working with abstract proofs. (For an introduction to λ-calculus, see the next section.)

Definition 3.3.1 (Function type). A general function type is introduced as follows:

- **Formation rule**: Given two types $A, B : \mathcal{U}$, one can form the function type $A \to B : \mathcal{U}$.
- **Introduction rule**: One can either define a function by an explicit definition $f(x) :\equiv \Phi$, where Φ is an expression possibly involving x, or by λ-abstraction $f :\equiv \lambda x.\Phi$.
- **Elimination rule**: If $a : A$ and $\lambda x.\Phi : A \to B$, then $\lambda x.\Phi(a) : B$.
- **Computation rule**[2]: $\lambda x.\Phi(a) :\equiv \Phi(a)$, i.e. function application is equivalent to the substitution of a for the variable x in the expression Φ. (To be completely correct, one should require the substitution to be *capture-avoiding*, i.e. free variables should remain free and distinct variables should not be assigned the same symbol.)

The **uniqueness principle** for function types should also be included in the definition, i.e. $\lambda x.f(x) \equiv f$. This says that every function is uniquely defined by its image.

An important generalization is obtained when the type of the output of a function is allowed to depend on the type of the input.

[2]In λ-calculus, this is often called β-reduction. (See the next section.)

Definition 3.3.2 (Dependent function types). Given a type $A : \mathcal{U}$ and a type family $B : A \to \mathcal{U}$, one can form the dependent function type

$$\prod_{a:A} B(a) : \mathcal{U}\,. \tag{3.5}$$

When B is a constant family, this type reduces to the ordinary function type $A \to B$. All other defining rules remain (formally) the same as in the nondependent setting.

Remark 3.3.3 (Scope). The Π-symbol scopes over all expressions to the right of the symbol unless delimited (similar to λ-calculus), e.g.

$$\prod_{a:A} B(a) \to C(a) \equiv \prod_{a:A} \big(B(a) \to C(a)\big)\,. \tag{3.6}$$

Example 3.3.4 (Polymorphic functions). An interesting example is obtained when the type A in the above definition is taken to be a universe $\mathcal{U}$ (this is a valid choice since universes are themselves types) together with $B(A) :\equiv A$. In this case, one obtains a function that takes a type as input and then acts on this type (or any other type constructed from it), e.g. the **polymorphic identity function**

$$\mathsf{id} : \prod_{A:\mathcal{U}} A \to A \tag{3.7}$$

defined by

$$\mathsf{id} :\equiv \lambda(A : \mathcal{U}).\lambda(a : A).a\,. \tag{3.8}$$

3.3.2 λ-calculus

@@ COMPLETE (e.g. Curry–Howard or even Curry–Howard–Lambek, typed vs. untyped calculus, ...) @@

3.3.3 Products

As in classic set theory, one of the basic notions is that of a product. This construction is ubiquitous throughout all of mathematics (and computer science). However, as opposed to set theory à la ZFC, products are not explicitly constructed as the set of all pairs of elements of its constituents. On the contrary, in type theory, one can prove that all elements necessarily have to be pairs.

Definition 3.3.5 (Product). First, the binary product of types is defined.

- **Formation rule**: Given any two types $A, B : \mathcal{U}$, one can form the product type $A \times B : \mathcal{U}$.
- **Introduction rule**: Given terms $a : A, b : B$, one can construct the term $(a, b) : A \times B$. This is called the **pairing** of the terms a and b.

- **Elimination and computation rules**: Functions out of a product $A \times B$ are defined through currying, i.e. given a function $A \to B \to C$, one can define a function $A \times B \to C$. Instead of giving an explicit definition every time one wants to construct a new function, a universal point of view is adapted: a single function that turns terms $f : A \to B \to C$ into terms $g : A \times B \to C$ is constructed. To this end, consider the **recursor**

$$\mathsf{rec}_{A\times B} : \prod_{C:\mathcal{U}} (A \to B \to C) \to A \times B \to C \tag{3.9}$$

with the constraint

$$\mathsf{rec}_{A\times B}\big(C, f, (a,b)\big) :\equiv f(a)(b)\,. \tag{3.10}$$

Example 3.3.6 (Projections). Analogous to the projection functions associated to the Cartesian product, one should have functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ that act on constructors as

$$\pi_1(a,b) :\equiv a \qquad \text{and} \qquad \pi_2(a,b) :\equiv b\,. \tag{3.11}$$

Using the recursor, one can define these functions by taking $C = A, f = \lambda a.\lambda b.a$ and $C = B, f = \lambda a.\lambda b.b$, respectively.

Definition 3.3.7 (Nullary product). One can also define a nullary product. In this case, it is called the **unit type 1**.

- **Formation rule**: $\mathbf{1} : \mathcal{U}$.
- **Introduction rule**: There is a unique nullary constructor $* : \mathbf{1}$.
- **Elimination and computation rules**: Since the constructor is a nullary operation, one does not expect to have projection maps and, likewise, one also does not expect function definition to be based on binary currying. Instead, the recursor is defined as follows:

$$\mathsf{rec}_{\mathbf{1}} : \prod_{C:\mathcal{U}} C \to \mathbf{1} \to C\,. \tag{3.12}$$

On the constructor $* : \mathbf{1}$, it is required to act trivially:

$$\mathsf{rec}_{\mathbf{1}}(C, c_0, *) :\equiv c_0\,. \tag{3.13}$$

Definition 3.3.8 (Dependent functions). One can easily generalize the above recursors to **inductors**, to allow for the definition of dependent functions out of product types (these functions are then said to be defined by an **induction principle**). In fact, one only has to change the type judgement of $\mathsf{rec}_{A\times B}$. This is accomplished by replacing $C : \mathcal{U}$ by a type family $C : A \times B \to \mathcal{U}$ and by replacing nondependent function types by dependent function types (the form of the computation rules virtually remain the same):

$$\begin{aligned}
\mathsf{ind}_{A\times B} &: \prod_{C:A\times B\to\mathcal{U}} \left(\prod_{a:A,b:B} C(a,b) \to \prod_{x:A\times B} C(x) \right), \\
\mathsf{ind}_{\mathbf{1}} &: \prod_{C:\mathbf{1}\to\mathcal{U}} C(*) \to \prod_{x:\mathbf{1}} C(x) .
\end{aligned} \tag{3.14}$$

Property 3.3.9 (Uniqueness principle). Using the induction principle, one can prove that every term $x : A \times B$ is necessarily of the form (a, b) for some $a : A, b : B$. Furthermore, one can also prove that $* : \mathbf{1}$ is the unique term in $\mathbf{1}$.

One can also generalize products in such a way that the type of the second factor depends on the type of the first one (in classical set theory, this would correspond to an indexed disjoint union).

Definition 3.3.10 (Dependent pair type). As with function types, the definition is not given as explicit as for nondependent types. Suffice it to say that, given a type $A : \mathcal{U}$ and a type family $B : A \to \mathcal{U}$, one can form the dependent pair type

$$\sum_{a:A} B(a) : \mathcal{U} . \tag{3.15}$$

When B is a constant family, the type reduces to the ordinary product type $A \times B$. The recursion and induction functions are defined as in the product case, except for the obvious replacements, such as $A\times B \longrightarrow \sum_{a:A} B(a)$, needed to make everything consistent.

Remark 3.3.11. Dependent pair types are often called **Σ-types** (due to the notation).

Remark 3.3.12 (Scope). Like the Π-symbol, the Σ-symbol scopes over the entire expression to the right unless delimited.

Definition 3.3.13 (Coproduct). Here, a standalone definition is given. The relation with the ordinary product will be mentioned afterwards.

- **Formation rule**: Given two types $A, B : \mathcal{U}$, one can form the coproduct type $A+B : \mathcal{U}$.
- **Introduction rule**: Since in ordinary mathematics (and, in particular, category theory) the coproduct is dual to the product, one expects the projections to be replaced by **injections/inclusions**. In fact, these are taken to be the constructors of coproduct types, i.e. given terms $a : A$ and $b : B$, one can construct the terms $\iota_1(a) : A + B$ and $\iota_2(b) : A + B$.

- **Elimination rules**: Similar to the use of currying for the definition of functions out of a product, functions out of a coproduct are defined in steps. To this intent the recursor and inductor are defined as follows:

$$
\begin{aligned}
&\mathsf{rec}_{A+B} : \prod_{C:\mathcal{U}} (A \to C) \to (B \to C) \to A + B \to C\,, \\
&\mathsf{ind}_{A+B} : \prod_{C:A+B\to\mathcal{U}} \left(\prod_{a:A} C\big(\iota_1(a)\big)\right) \to \left(\prod_{b:B} C\big(\iota_2(b)\big)\right) \to \prod_{x:A+B} C(x)\,.
\end{aligned}
\tag{3.16}
$$

- **Computation rules**: The recursor acts on the constructors as follows (the inductor virtually has the same action):

$$
\begin{aligned}
\mathsf{rec}_{A+B}\big(C, f_1, f_2, \iota_1(a)\big) &:\equiv f_1(a)\,, \\
\mathsf{rec}_{A+B}\big(C, f_1, f_2, \iota_2(b)\big) &:\equiv f_2(b)\,.
\end{aligned}
\tag{3.17}
$$

Definition 3.3.14 (Nullary coproduct). As was the case for products, one can also define a nullary version of the coproduct, the **empty type 0**:

- **Formation rule**: $\mathbf{0} : \mathcal{U}$.
- **Introduction rule**: There is no constructor for $\mathbf{0}$.
- **Elimination and computation rules**: Since there is no constructor for $\mathbf{0}$, one can always trivially 'construct' a function out of $\mathbf{0}$:

$$
\begin{aligned}
&\mathsf{rec}_{\mathbf{0}} : \prod_{C:\mathcal{U}} \mathbf{0} \to C\,, \\
&\mathsf{rec}_{\mathbf{0}} : \prod_{C:\mathbf{0}\to\mathcal{U}} \prod_{x:\mathbf{0}} C(x)\,.
\end{aligned}
\tag{3.18}
$$

  This trivial function corresponds to the logical principle *ex falso quodlibet* as introduced in the section on logic above.

Since coproducts in set theory occur as binary disjoint unions, one could expect that there is a way to express coproducts in terms of dependent pair types.

Construction 3.3.15 (Coproducts as Σ-types). First, introduce the type $\mathbf{2} : \mathcal{U}$ (in set theory, this would be the 2-element set). The introduction rule constructs two terms $0, 1 : \mathbf{2}$. The elimination and computation rules say that one can use this type for binary indexing:

$$
\mathsf{rec}_{\mathbf{2}} : \prod_{C:\mathcal{U}} C \to C \to \mathbf{2} \to C
\tag{3.19}
$$

with

$$
\begin{aligned}
\mathsf{rec}_{\mathbf{2}}(C, c_0, c_1, 0) &:\equiv c_0\,, \\
\mathsf{rec}_{\mathbf{2}}(C, c_0, c_1, 1) &:\equiv c_1\,.
\end{aligned}
\tag{3.20}
$$

Using this type, one can prove that $A + B$ is judgementally equal to $\sum_{x:\mathbf{2}} \mathsf{rec}_{\mathbf{2}}(\mathcal{U}, A, B, x)$. The injections are given by pairing, i.e. $\iota_1(a) \equiv (0, a)$ and $\iota_2(b) \equiv (1, b)$. In a similar way, one can obtain binary products as dependent function types over $\mathbf{2}$.

3.3.4 Propositions as types

To conclude this section, an overview of all the concepts introduced above is given from a propositions-as-types perspective. In intuitionistic logic, this is often called the **Brouwer–Heyting–Kolmogorov interpretation** and, more specifically, it should be seen as an incarnation of the Curry–Howard correspondence.

- Types and their terms correspond to propositions and their proofs, respectively. In a proof-relevant context the fact that a type can have multiple terms makes it clear that, although distinct proofs eventually have the same result, the difference in their content can be important as well.
- Function types correspond to implications. A proof of the proposition $A \to B$ boils down to showing that every proof of A gives a proof of B.
- Π-types correspond to universal quantification, i.e. $\prod_{a:A} B(a)$ can be read as $\forall a \in A : B(a)$. Giving a proof of $\prod_{a:A} B(a)$ is the same as giving for every $a : A$ a proof of $B(a)$. This is indeed compatible with the fact that elements of Π-types are dependent functions, i.e. every element $a : A$ gives rise to a (possibly) distinct type/proposition.
- Σ-types correspond to existential quantification, i.e. $\sum_{a:A} B(a)$ can be read as $\exists a \in A : B(a)$. Giving a proof of $\sum_{a:A} B(a)$ is the same as giving a proof for some $(a, B(a))$. This is compatible with the fact that Σ-types can be identified with disjoint unions and hence every element can be associated with a specific constituent type.
- The logical connectives (conjunction and disjunction) correspond to the product and coproduct types.
- The truth values, *true* and *false*, correspond to the unit and empty types, respectively. Furthermore, if the negation of A is defined as the type $\neg A :\equiv A \to \mathbf{0}$, this indeed corresponds to the logical negation by the statements above.

3.3.5 Identity types

One of the most important, but at the same time most subtle, concepts in type theory (especially when moving on to extensions such as homotopy type theory) is the identity type. Since in predicate (and even propositional) logic the equality of two terms is a proposition, one could expect that to every two terms $a, b : A$ there corresponds an associated equality type $a =_A b : \mathcal{U}$. Note that the type of the terms is assumed to be the same since it does not make any sense to compare terms of different types.

Definition 3.3.16 (Equality type[3]). The type corresponding to a propositional equality is defined by the following rules:

- **Formation rule**: Given terms $a, b : A$, one can form the equality type $a =_A b : \mathcal{U}$. When the type A is clear from the context, this is also often written as $a = b : \mathcal{U}$.
- **Introduction rule**: For every term $a : A$, there is a canonical identity element

  $$\mathsf{refl}_a : a = a\,. \tag{3.21}$$

  The notation points to the fact that this term can be seen as a proof of the reflexivity of equalities.
- **Elimination and computation rules**: Here, the so-called **path induction principle** for equality types is presented, for the equivalent *based path induction principle* see The Univalent Foundations Program (2013).

  Given a type family

  $$C : \prod_{a,b:A} a = b \to \mathcal{U}$$

  and a term

  $$I : \prod_{a:A} C(a, a, \mathsf{refl}_a)\,,$$

  there exists a function

  $$f : \prod_{a,b:A} \prod_{p:a=b} C(a, b, p) \tag{3.22}$$

  such that

  $$f(a, a, \mathsf{refl}_a) :\equiv I(a) \tag{3.23}$$

  for all $a : A$.

Informally this principle says that all terms of the form (a, b, p), with $p : a = b$, are inductively generated by the 'constant' terms (a, a, refl_a). (See the section on homotopy type theory for a more geometric perspective).

Using the notion of identity types one can say when a given type resembles a proposition:

[3]Sometimes called an **identity type**.

Definition 3.3.17 (Mere proposition). A type $A : \mathcal{U}$ for which

$$\mathsf{isProp}(A) :\equiv \prod_{a,b:A} a = b \tag{3.24}$$

is inhabited. This is also called an h**-proposition**.

Given a mere proposition $P : \mathcal{U}$, the related identity type is either uninhabited, if P itself is uninhabited, or has a unique term. Types of this form are said to be **contractible**.

Definition 3.3.18 (Contractible type). A type $A : \mathcal{U}$ for which

$$\mathsf{isContr}(A) :\equiv \sum_{a:A} \prod_{b:A} a = b \tag{3.25}$$

is inhabited.

Definition 3.3.19 (Homotopy level). The homotopy level or h**-level** of a type $A : \mathcal{U}$ is recursively defined as follows:

$$\mathsf{hasHLevel}(0, A) :\equiv \mathsf{isContr}(A) \tag{3.26}$$

$$\mathsf{hasHLevel}(n+1, A) :\equiv \prod_{a,b:A} \mathsf{hasHLevel}(n, a = b)\,. \tag{3.27}$$

A type of homotopy level $n + 2$ is also called a **homotopy** n**-type**.

Example 3.3.20. The lowest h-levels are given by

-2 : Contractible types,

-1 : mere propositions,

0 : sets,

1 : groupoids, ...

Because of the inductive nature of identity types, any homotopy n-type can be truncated to a homotopy k-type for $k < n$. In the case of a (-1)-truncation, the following notion is obtained.

Definition 3.3.21 (Bracket type). Consider a type $A : \mathcal{U}$. The bracket $[A] : \mathcal{U}$ or $[A] : \mathcal{U}$ is inhabited (and uniquely so) if and only if A is inhabited. It is defined by the following rules:

$$\mathsf{isInhab} : A \mapsto [A]\,, \tag{3.28}$$

$$\mathsf{propTrunc} : \prod_{a,b:A} \mathsf{isInhab}(a) = \mathsf{isInhab}(b)\,. \tag{3.29}$$

The term $\mathsf{propTrunc}$ exactly says that $[A]$ is a mere proposition.

Definition 3.3.22 (Homotopy fibre). Consider a function $f : A \to B$. For every term $b : B$, the homotopy fibre of f is defined as follows:

$$\mathsf{hfibre}(b, f) :\equiv \sum_{a:A} f(a) = b\,. \tag{3.30}$$

Definition 3.3.23 (Equivalence). A function $f : A \to B$ such that

$$\mathsf{isEquiv}(f) :\equiv \prod_{b:B} \mathsf{isContr}\big(\mathsf{hfibre}(b, f)\big) \tag{3.31}$$

is inhabited.

Slightly different notions exist. A **homotopy equivalence** is a function $f : A \to B$ such that there exists a function $g : B \to A$ and homotopies

$$p : \prod_{a:A} g\big(f(a)\big) = a \qquad \text{and} \qquad q : \prod_{b:B} f\big(g(b)\big) = b. \tag{3.32}$$

It is called an **adjoint equivalence** if it is a homotopy equivalence equipped with higher homotopies representing triangle identities for f and g.

Property 3.3.24. The types $\mathsf{isEquiv}(f)$, $\mathsf{isHomEquiv}(f)$ and $\mathsf{isAdjEquiv}(f)$ are co-inhabited.

3.4 Categorical semantics

3.4.1 Inductive types

Inductive types also admit semantics in category theory. The right concept for these is that of algebras over endofunctors (Definition 1.4.24). The recursion principle and accompanying computation rule of inductive types exactly state for every other F-algebra A there exists a morphism $T \to A$ and that this is a unique algebra morphism, i.e. inductive types are initial algebras over endofunctors. An induction principle assigns to every algebra morphism $B \to T$, where B should be interpreted as the total space $\sum_{x:T} B(t)$, a section $T \to B$. The computation rule then again says that this section is an algebra morphism.

Property 3.4.1. When working in sets, the recursion and induction principles are equivalent.

Remark 3.4.2. When passing from extensional to intensional type theory, one has to replace initial algebras by weakly initial algebras.

3.4.2 Polynomial functors

Definition 3.4.3 (Polynomial). A diagram of the form

$$W \xleftarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z\,. \tag{3.33}$$

In terms of ordinary polynomials, Y gives the index set of the terms, X as a subset of $\mathbb{N}$ gives the degrees of the terms and W gives the coefficients. If g is an identity morphism, one often speaks of **linear functors**.

Given a polynomial in a locally Cartesian closed category $\mathbf{C}$, the induced polynomial functor is given by the composition

$$\mathbf{C}/W \xrightarrow{f^*} \mathbf{C}/X \xrightarrow{\prod_g} \mathbf{C}/Y \xrightarrow{\sum_h} \mathbf{C}/Z\,, \tag{3.34}$$

where $\prod_g$ and $\sum_h$ are dependent product and sum functors (Definition 2.2.2).

Definition 3.4.4 (**W-type**)**.** Consider a type A and a type family $B : A \to \mathcal{U}$. The W-type $W_{a:A}B(a)$ is obtained as the initial algebra for the polynomial functor

@@ COMPLETE @@

3.5 Homotopy type theory

3.5.1 Introduction

This section gives a reformulation or extension of the concept introduced before using the language of homotopy theory (and, more generally, algebraic topology). The relevant concepts can be found in **??** and Section 1.5.1. The resulting theory is called homotopy type theory or **HoTT**.

The general idea is to associate types with topological spaces and terms with points in those spaces. The main novelty is given by the identification of (propositional) equalities with paths between points. Since everything happens in a proof-relevant context, two equalities $p, q : a =_A b$ are not necessarily equal themselves and, hence, one can consider equalities between equalities (and so on). In the topological picture this gives rise to homotopies between paths. By going all the way and working out all coherence laws, one obtains the structure of a (weak) ∞-groupoid.[4]

It is also this interpretation that explains the name 'path induction' for the induction principle of equality types. Namely, what this induction principle says is that the free path space ΩA is *inductively generated* by constant loops (ranging over all possible points). This principle, however, sounds quite crazy. How can one build a path between two distinct points from (constant) loops? Here it is important to remind that everything only has to be equal up to homotopy and any path is indeed homotopy-equivalent to a constant loop if one retracts one of the endpoints along the path. It

[4]This characterization is strongly related to the homotopy hypothesis (or theorem when using the right model for ∞-categories).

is thus important that one does not require the homotopies to act rel endpoints (as is often done in classical homotopy theory).

Definition 3.5.1 (Pointed type). A type $A : \mathcal{U}$ together with a distinguished term $a : A$, called the **base point**. Pointed types are often denoted by a pair (A, a). It should be clear that the type of pointed types $\mathcal{U}_\bullet$ is equal to $\sum_{A:\mathcal{U}} A$.

Definition 3.5.2 (Loop space). The loop space $\Omega(A, a)$ of a pointed type (A, a) is the pointed type $(a =_A a, \mathsf{refl}_a)$.

Now, the important aspect of HoTT is that the ∞-groupoid structure of a type can be derived solely from the (path) induction principle of the equality types. Some examples are given.

Property 3.5.3 (Inversion). For every type $A : \mathcal{U}$ and terms $a, b : A$, there exists a function

$$p \mapsto p^{-1} : (a = b) \to (b = a) \tag{3.35}$$

such that $\mathsf{refl}_a^{-1} :\equiv \mathsf{refl}_a$ for all $a : A$.

Property 3.5.4 (Concatenation). For every type $A : \mathcal{U}$ and terms $a, b, c, d : A$, there exists a function

$$p \mapsto q \mapsto p \cdot q : (a = b) \to (b = c) \to (c = d) \tag{3.36}$$

such that $\mathsf{refl}_a \cdot \mathsf{refl}_a :\equiv \mathsf{refl}_a$ for all $a : A$. (Note that the composition does not follow the usual convention of right-to-left. This is why the symbol $\cdot$ and not $\circ$ was used.)

Property 3.5.5. The above operations satisfy the group relations (up to higher equalities):

- $p \cdot \mathsf{refl}_b = p$ and $\mathsf{refl}_a \cdot p = p$ for all $p : a = b$.
- $p \cdot p^{-1} = \mathsf{refl}_a$ and $p^{-1} \cdot p = \mathsf{refl}_b$ for all $p : a = b$.
- $(p^{-1})^{-1} = p$ for all $p : a = b$.
- $p \cdot (q \cdot r) = (p \cdot q) \cdot r$ for all $p : a = b, q : b = c, r : c = d$.

3.5.2 Transport

The relation with homotopy theory and category theory becomes even stronger when looking at function types:

Property 3.5.6. Given a function $f : A \to B$, there exists an **application function**

$$\mathsf{ap}_f : (a =_A b) \to \big(f(a) =_B f(b)\big) \tag{3.37}$$

such that $\mathsf{ap}_f(\mathsf{refl}_a) :\equiv \mathsf{refl}_{f(a)}$ for all $a, b : A$. Furthermore, this function behaves functorially in that it preserves concatenation, inverses and identities (again this should be interpreted in the full weak ∞-sense). From the topological perspective this can be interpreted as if all functions are 'continuous'.

Notation 3.5.7. Because functors in category theory are generally given the same notation when acting on objects or morphisms, the application function ap_f is also often denoted by f.

For dependent functions one can obtain a similar result. However, for this generalization, one needs some kind of 'parallel transport' since for two terms with $a = b$, it does not necessarily hold that $f(a)$ and $f(b)$ have the same type.

Property 3.5.8 (Transport). Given a type family $P : A \to \mathcal{U}$ and an equality $p : a =_A b$, there exists a **transport function**

$$p_* : P(a) \to P(b) \tag{3.38}$$

such that $(\mathsf{refl}_a)_* :\equiv \mathsf{id}(a)$ for all $a : A$. The pushforward notation is used since p_* can be (informally) interpreted as the pushforward of p along P.

From a topological perspective, this transport function allows to regard type families as fibrations (**??**). For every type family $P : A \to \mathcal{U}$, term $\alpha : P(a)$ and equality $p : a = b$, there exists a **lift**

$$\mathsf{lift}(p, \alpha) : (a, \alpha) = \big(b, p_*(\alpha)\big) \tag{3.39}$$

such that

$$\pi_1\big(\mathsf{lift}(p, u)\big) = p\,. \tag{3.40}$$

The equality $\mathsf{lift}(p, u)$ acts between terms of the Σ-type $\sum_{a:A} P(a)$, which can be interpreted as the total space of a **fibration** $\pi_1 : \sum_{a:A} P(a) \to A$. To take this terminology even further, one can could call functions $\sigma : \prod_{a:A} P(a)$ **sections** (of π_1).

Now, as mentioned before, for dependent functions one cannot just compare $f(a)$ and $f(b)$ if $a \not\equiv b$. However, the function $\mathsf{lift}(p, \cdot)$ gives a canonical path from one fibre to the other and every path between these fibres should factor through this canonical path essentially uniquely. Hence, one can define a path between α and β in the total space $\sum_{a:A} P(a)$, lying over $p : a = b$, to be a path $p_*(\alpha) = \beta$ (up to equivalence).

Property 3.5.9. Given a dependent function $f : \prod_{a:A} P(a)$, there exists a function

$$\mathsf{apd}_f : \prod_{p:a=b} p_*\big(f(a)\big) =_{P(b)} f(b)\,. \tag{3.41}$$

Again, with some abuse of notation, this function is also denoted by f

Since an ordinary function is a specific instance of a Π-type, one might expect that the application functions ap_f and apd_f are related in this case. The following property shows that this intuition is not unreasonable.

Property 3.5.10. Consider two types $A, B : \mathcal{U}$ and a function $f : A \to B$. For every equality $p : a =_A b$ and term $\alpha : P(b)$, there exists an equality $\tilde{p} : p_*(\alpha) =_{P(b)} \alpha$. Using this equality one can relate the application functions as follows:

$$\mathsf{apd}_f(p) = \tilde{p}\big(f(a)\big) \cdot \mathsf{ap}_f(p)\,. \tag{3.42}$$

3.5.3 Equivalences

In this paragraph the notions of equivalences and isomorphisms are considered in more detail. As is known from the chapter on category theory, the distinction between the various notions of similarity (or equality) is important yet subtle.

Lead by the intuition from topology a **homotopy** between functions is defined.

Definition 3.5.11 (Homotopy). Consider two sections $f, g : \prod_{a:A} P(a)$. A homotopy between f and g is a term of the type

$$f \sim g :\equiv \prod_{a:A} f(a) = g(a)\,. \tag{3.43}$$

It can be shown that homotopies induce equivalence relations on function types.

It has already been noted that functions can be regarded as functors between ∞-groupoids. Since homotopies act between functions, one might expect that these can be regarded as (weak) natural transformations between the (∞-)functors.

Property 3.5.12. Consider two sections $f, g : \prod_{a:A} P(a)$ and an equality $p : a = b$. If H is a homotopy between f and g, then

$$H(a) \cdot g(p) = f(p) \cdot H(b)\,. \tag{3.44}$$

Using the notion of homotopy one can introduce a first kind of 'equivalence'.

Definition 3.5.13 (Quasi-inverse). Given a function $f : A \to B$, a quasi-inverse of f is a triple (g, α, β), where $g : B \to A$ and

$$\alpha : f \circ g \sim \mathsf{id}_B \qquad\qquad \beta : g \circ f \sim \mathsf{id}_B\,. \tag{3.45}$$

From a homotopy theoretical perspective one would call the pair (f, g) a homotopy equivalence. The corresponding type is given by

$$\mathsf{qInv}(f) :\equiv \sum_{g:B\to A} (f \circ g \sim \mathsf{id}_B) \times (g \circ f \sim \mathsf{id}_A)\,. \tag{3.46}$$

Now, although this type may seem to give the right notion of equivalence, it is better to generalize it since it is in general not very well-behaved. (This is similar to the fact that adjoint equivalences between categories are better behaved than ordinary equivalences.)

In general an equivalence should satisfy three requirements:

1. For every function $f : A \to B$, there exists a function $\mathsf{qInv}(f) \to \mathsf{isEquiv}(f)$.
2. For every function $f : A \to B$, there also exists a function $\mathsf{isEquiv}(f) \to \mathsf{qInv}(f)$.
3. For every two terms $eq_1, eq_2 : \mathsf{isEquiv}(f)$, there exists an equality $eq_1 = eq_2$.

So, inducing an equivalence is logically equivalent to admitting a quasi-inverse and as such finding a quasi-inverse is sufficient to show that a function induces an equivalence.

3.5.4 Equality types: revisited

In the section on (intensional) type theory, equality types were introduced in a general and uniform way. The defining rules did not assume any specific structure on the underlying types. Although this made the technique of path induction widely applicable, it has the downside that one cannot leverage the internal structure of specific types to get more useful characterizations.

First, consider binary products (and by extension Σ-types). Can one express the equality of two elements $x, y : A \times B$ in terms of their projections? The answer is yes: there exists an equivalence

$$(x =_{A\times B} y) \simeq \big(\pi_1(x) =_A \pi_1(y)\big) \times \big(\pi_2(a) =_B \pi_2(y)\big)\,. \tag{3.47}$$

However, one should bear in mind that this is merely an equivalence. A term (resp. proof) of one side gives a term (resp. proof) of the other side, but it is not a judgemental equality (it is not even a propositional one). One could see this as a problem or defect of the theory and to resolve this kind of (apparent) issue the univalence axiom will be introduced at the end of this section. Still, one can leverage this equivalence to give a practical alternative[5] for the defining rules of the equality type in the case of product types:

Remark 3.5.14. The function $\big(\pi_1(a) = \pi_1(b)\big) \times \big(\pi_2(a) = \pi_2(b)\big) \to (a = b)$ associated to the above equivalence can be interpreted as an introduction rule of the equality type for binary products. At the same time one can take the application functions induced by the projections on $A \times B$ as elimination rules for the equality type. The homotopies associated to the equivalence in their turn induce the propositional computation rules and uniqueness principle.

One can also express the transport of properties along an equality $p : x =_{A\times B} y$ in terms of transport in the individual spaces:

Property 3.5.15. Consider two types $A, B : \mathcal{U}$ together with type families $P : A \to \mathcal{U}$ and $Q : B \to \mathcal{U}$. For every term α of the product family $(P\times Q)(x) :\equiv P\big(\pi_1(x)\big)\times Q\big(\pi_2(x)\big)$, the following equality type is inhabited:

$$p_*(\alpha) = \big(p_*(\pi_1(\alpha)), p_*(\pi_2(\alpha))\big)\,. \tag{3.48}$$

Note that all three occurrences of the pushforward p_* denote a different operation or, more precisely, the same operation but applied to different types.

One would intuitively expect that given two functions $f, g : A \to B$ that take the same value at all points, i.e. $f(a) = g(a)$ for all $a : A$, there exists an equality $f =_{A\to B} g$. However, this cannot be proven within the frame work of intensional type theory. This issue should also not come as a shock, since two functions that are defined differently might still take the same value at all points. To resolve this apparent gap in the theory, the following axiom is introduced.

Axiom 3.1 (Function extensionality). Given two functions $f, g : \prod_{a:A} P(a)$, there exists an equivalence $(f = g) \to \prod_{a:A} f(a) = g(a)$ that sends refl_f to $f(\mathsf{refl}_x)$.

Axiom 3.2 (Univalence axiom). Given two types $A, B : \mathcal{U}$, there exists an equivalence $(A =_{\mathcal{U}} B) \to (A \simeq B)$ that takes refl_A to id_A. A universe in which the univalence axiom holds is said to be univalent.

@@ COMPLETE @@

3.6 Modal logic

3.6.1 Modalities

The two most important or most well-known modalities are 'necessity' and 'possibility'. For any proposition p,

[5]Note that this is not a judgementally equal alternative. It is merely a convenient interpretation.

- $\Box p$ means that p is necessarily true, and
- $\Diamond p$ means that p is possibly true.

To formalize these modalities, a few axioms can be introduced.

Axiom 3.3 (K). $\Box$ preserves implication:

$$\Box(p \to q) \to (\Box p \to \Box q)\,. \tag{3.49}$$

In intuitionistic logic, a similar axiom has to be introduced for $\Diamond$:

$$\Box(p \to q) \to (\Diamond p \to \Diamond q)\,. \tag{3.50}$$

Axiom 3.4 (Necessitation). If p is true, then $\Box p$ is true or, equivalently:

$$\mathtt{true} \to \Box\mathtt{true}\,. \tag{3.51}$$

Axiom 3.5 (T).

$$\Box p \to p \tag{3.52}$$

As for Axiom K, a separate axiom has to be introduced for $\Diamond$ in intuitionistic logic:

$$p \to \Diamond p\,. \tag{3.53}$$

Axiom 3.6 (S4).

$$\Box p \to \Box\Box p \tag{3.54}$$

and

$$\Diamond\Diamond p \to \Diamond p \tag{3.55}$$

The categorical semantics for intuitionistic (propositional) logic is given by the internal language of Cartesian closed categories (**??**). When passing to (S4-)modal logic, one has to specialize these categories. The three axioms above imply that $\Box$ is a monoidal comonad. The analogous axioms for $\Diamond$ imply that it is a $\Box$-strong monad, i.e. it is a monad equipped with a natural transformation

$$\eta_{x,y} : \Box x \otimes \Diamond y \to \Diamond(\Box x \otimes y)\,. \tag{3.56}$$

3.6.2 Type theory

Definition 3.6.1 (Modality). Consider an $(\infty, 1)$-topos $\mathbf{H}$. A modality or **modal operator** on $\mathbf{H}$ is an idempotent[6] (co)monad on $\mathbf{H}$ (see also Definition 1.3.27). For clarity, idempotent comonads will be called **comodalities** in this document (unless specifically mentioned, all statements can be dualized to pertain to comodalities).

[6]Sometimes, idempotency is not required.

Definition 3.6.2 (Modal type). A type $X \in \mathbf{H}$ is said to be modal for a modality $\Diamond$ if the unit $\eta_X : X \to \Diamond X$ is an equivalence.

Property 3.6.3. The modal types of a modality $\Diamond$ on $\mathbf{H}$ are exactly the $\Diamond$-algebras or, equivalently, the objects of the Eilenberg–Moore category $\mathbf{H}^{\Diamond}$ (Definition 1.3.20). This property allows to define modal types for monads that are not idempotent as their algebras. It also means that one can recover the monad objectwise as the composition of the unit and the inclusion.

Definition 3.6.4 (Opposite). An **adjoint cylinder** is an adjoint triple $\iota_! \dashv \iota^* \dashv \iota_*$ where the left and, equivalently, the right adjoint are fully faithful:

$$\mathbf{H} \begin{smallmatrix} \hookrightarrow \\ \leftarrow \\ \hookrightarrow \end{smallmatrix} \mathbf{H}' \tag{3.57}$$

In the case where these functors are themselves modalities, these can be seen as full inclusions of modal types, turning $\mathbf{H}$ into an essential subtopos of $\mathbf{H}'$ (Definition 2.2.9), and such an adjoint cylinder represents **opposite** modalities. Such a situation also gives rise to a new pair of adjoint modalities $\iota_!\iota^* \dashv \iota_*\iota^*$, with the left adjoint being a monad (the other adjunction is trivial because of the fully faithfulness).

A dual situation arises for an adjoint triple of the form

$$\mathbf{H} \begin{smallmatrix} \leftarrow \\ \hookrightarrow \\ \leftarrow \end{smallmatrix} \mathbf{H}', \tag{3.58}$$

exhibiting $\mathbf{H}$ as a bireflective subcategory of $\mathbf{H}'$ (Definition 1.2.29). Here, the resulting adjoint modalities have as left adjoint a comonad.

Given an opposition of modalities (either of the two situations), one has a modality $\Diamond$ and a comodality $\Box$ and, hence, a unit and counit morphism:

$$\Box x \to x \to \Diamond x\,. \tag{3.59}$$

The composite map is said to represent the **unity** of the opposition in the Hegelian interpretation of *Lawvere*.

Remark 3.6.5 (Interpretation). Note that the adjoint triples in this definition have a slightly different interpretation due to there only being one inclusion arrow in the second case. If one interprets a modality as projecting out some kind of property of objects, the first opposition embodies the case where two related properties are considered, while the second case involves only one property but projected out in two different ways.

Remark 3.6.6 (Adjoint sequences). In some situations, longer sequences of adjoint functors arise (see, for example, Section 5.3). In the case of adjoint quadruples one gets induced adjunctions of the form

$$\mathsf{modality} \dashv \mathsf{comodality} \dashv \mathsf{modality}$$

and

$$\text{comodality} \dashv \text{modality} \dashv \text{comodality}.$$

These are sometimes called **Yin** and **Yang triples**, respectively.

Notation 3.6.7. One can show that adjoint triples $F \dashv G \dashv H$ are equivalent to adjoint pairs in the 2-category having adjunctions as morphisms, hence to adjunctions of adjunctions. This inspires the following notations:

$$\begin{array}{ccc} F & \dashv & G \\ \perp & & \perp \\ G & \dashv & H \end{array}$$

Definition 3.6.8 (Negative). Consider a comodality $\bigcirc : \mathbf{H} \to \mathbf{H}$. The negative of a modal type X is obtained by removing its 'pure $\bigcirc$-part', i.e. its projection under $\bigcirc$. Categorically, this means that one takes the cofibre of the counit:

$$\overline{\bigcirc}X := \mathrm{cofib}(\bigcirc X \to X). \tag{3.60}$$

Definition 3.6.9 (Aufhebung). Consider an inclusion of adjoint modalities (or, equivalently, an inclusion of essential subtopoi[7]):

$$\begin{array}{ccc} \Diamond_2 & \dashv & \Box_2 \\ \vee & & \vee \\ \Diamond_1 & \dashv & \Box_1 \end{array} \tag{3.61}$$

Level 2 is said to **resolve** level 1 if $\Diamond_1 < \Box_2$. The smallest resolution (if it exists) is called the (right) Aufhebung (of opposites).

Example 3.6.10 (*Dasein*). Consider the adjoint modality of '(pure) being' and 'nothing' (or non-being):

$$\emptyset \dashv *, \tag{3.62}$$

where the former is the constant comonad on the initial object and the latter the constant monad on the terminal object. The associated unity of opposites

$$\emptyset \cong \emptyset X \to X \to *X \cong *, \tag{3.63}$$

corresponds to *Hegel*'s "*there is nothing which is not an intermediate state between being and nothing*".

[7]One can show that these form a (complete) lattice and, hence, inclusion is well defined.

Part II

Linear Algebra

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Chapter 4

Operator Theory

The main reference for this chapter is Blackadar (2013).

Chapter Contents

4.1 Operators

4.1.1 Operator topologies

Definition 4.1.1 (Weak operator topology). The topology generated by the seminorms $\left\{T \mapsto |\lambda(Tv)| \mid v \in V, \lambda \in V^*\right\}$. A net of linear operators $(T_\alpha)_{\alpha\in I}$ on a space V converges to a linear operator T in the weak (operator) topology if $T_\alpha v \longrightarrow Tv$ for all $v \in V$ in the weak topology.

In the case of Hilbert spaces, one can simplify the above definition using Riesz's representation theorem (**??**). The weak operator topology on a Hilbert space is generated by the seminorms $\left\{T \mapsto |\langle Tv \mid w\rangle| \mid v, w \in \mathcal{H}\right\}$.

Definition 4.1.2 (Strong operator topology). The topology generated by the seminorms $\left\{T \to \|Tv\| \mid v \in V\right\}$. A net of linear operators $(T_n)_{n\in\mathbb{N}}$ on a space V converges to a linear operator T in the strong (operator) topology if $T_\alpha v \longrightarrow Tv$ for all $v \in V$ in the norm (or strong) topology.

Definition 4.1.3 (Operator norm). The operator norm of L is defined as follows:

$$\|T\|_{\text{op}} = \inf\left\{\lambda \in \mathbb{R} \mid \forall v \in V : \|Tv\|_W \leq \lambda\|v\|_V\right\}. \tag{4.1}$$

Equivalent definitions of the operator norm are:

$$\|T\|_{\text{op}} = \sup_{\|v\|\leq 1} \|Tv\| = \sup_{\|v\|=1} \|Tv\| = \sup_{v\neq 0} \frac{\|Tv\|}{\|v\|}. \tag{4.2}$$

Definition 4.1.4 (Norm topology[1]). A sequence of linear operators $(T_n)_{n\in\mathbb{N}}$ on a space V converges to a linear operator T in the norm topology if the sequence $(\|T_n - T\|)_{n\in\mathbb{N}}$ converges to 0. (Sequences suffice since the norm topology is metrizable and, therefore, sequential by **??**.)

4.1.2 Bounded operators

Definition 4.1.5 (Bounded operator). Let $T : V \to W$ be a linear operator between two normed spaces. The linear operator is said to be bounded if it satisfies

$$\|T\|_{\text{op}} < +\infty. \tag{4.3}$$

Notation 4.1.6. The space of bounded linear operators from V to W is denoted by $\mathcal{B}(V, W)$.

Property 4.1.7. If V is a Banach space, then $\mathcal{B}(V)$ is also a Banach space.

[1] Also called the **uniform (operator) topology**.

The following property reduces the problem of continuity to that of boundedness (or vice versa).

Property 4.1.8. Consider a linear operator $T \in \mathcal{L}(V, W)$. The following statements are equivalent:

- T is bounded.
- T is continuous at 0.
- T is continuous on V.
- T is uniformly continuous on V.
- T maps bounded sets to bounded sets.

Property 4.1.9 (Bounded eigenvalues). The eigenvalues of a bounded operator are bounded by its operator norm. Furthermore, every bounded linear operator on a Banach space has at least one eigenvalue.

Property 4.1.10 (BLT theorem[2]). Consider a bounded linear operator $f : X \to W$, where X is a dense subset of a normed space V and W is a Banach space. There exists a unique extension $F : V \to W$ such that $\|f\|_{\text{op}} = \|F\|_{\text{op}}$.

Definition 4.1.11 (Schatten class operator). Consider the space of bounded linear operators on a Hilbert space $\mathcal{H}$. The **Schatten p-norm**, for $p \in [1, +\infty[$, is defined as

$$\|T\|_p := \mathrm{tr}\left(\sqrt{T^*T}^{\,p}\right)^{1/p}. \tag{4.4}$$

Linear operators for which this norm is finite constitute the p^{th} Schatten class $\mathcal{I}_p$.

Property 4.1.12. The Schatten classes are Banach spaces with respect to the associated Schatten norms.

Example 4.1.13 (Trace-class operator). The space of trace-class operators on a Hilbert space $\mathcal{H}$ is defined as follows:

$$\mathcal{B}_1(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) \mid \mathrm{tr}(|T|) < +\infty\}, \tag{4.5}$$

where the trace functional was defined in **??** and $|T| := \sqrt{T^*T}$.

The following theorem can be seen as the analogue of Riesz's theorem for trace-class operators.

[2] BLT stands for 'bounded linear transformation'.

Property 4.1.14. For every bounded linear functional ρ on the space of trace-class operators $\mathcal{B}_1(\mathcal{H})$, there exists a unique bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ such that

$$\rho(S) = \mathrm{tr}(ST) \tag{4.6}$$

for all $S \in \mathcal{B}_1(\mathcal{H})$. This implies that $\mathcal{B}_1^*(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ are isometrically equivalent.

The previous property allows for the following definition.

Definition 4.1.15 (Weak-* operator topology). The weak-* (also **ultraweak** or **σ-weak**) topology on $\mathcal{B}(\mathcal{H})$ with respect to the trace-class operators $\mathcal{B}_1(\mathcal{H})$. This is also called the σ-weak topology on $\mathcal{B}(\mathcal{H})$.

Example 4.1.16 (Hilbert–Schmidt operator). Consider the Hilbert–Schmidt norm $\|\cdot\|_2$ from **??**. A linear operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a Hilbert–Schmidt operator if it satisfies

$$\|T\|_2 < +\infty\,. \tag{4.7}$$

This space is closed under taking adjoints.

A more general, but still well-behaved, class of linear operators is the space of closed operators.

Definition 4.1.17 (Closed operator). A linear operator $T : V \to W$ such that for every sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathrm{dom}(T)$ converging to $v \in V$, where $f(v_n)$ converges to $w \in W$, one finds that $v \in \mathrm{dom}(T)$ and $Tv = w$.

Equivalently, one can define a closed linear operator as a linear operator for which its graph is a closed subset in the direct sum $V \oplus W$.

Definition 4.1.18 (Closure). Let $T : V \to W$ be a linear operator. Its closure (if it exists) is the closed linear operator $\overline{T}$ such that the graph of $\overline{T}$ is the closure of the graph of T in $V \oplus W$.

Theorem 4.1.19 (Closed graph theorem). *A linear operator on a Banach space is closed if and only if it is bounded.*

4.1.3 Self-adjoint operators

There is a multitude of different notions available in the literature that try to indicate in what sense a linear operator is related to its adjoint (not everyone agrees on the definitions). Here, an overview is given in the case of Hilbert spaces where linear operators are allowed to be unbounded.

?? for finite-dimensional spaces can be generalized as follows.

Definition 4.1.20 (Adjoint). Let T be a linear operator on a Hilbert space $\mathcal{H}$. A linear operator $T^\dagger$ is said to be the (Hermitian) adjoint of T if the following conditions are satisfied:

1. $\langle v \mid Tw\rangle = \langle T^\dagger v \mid w\rangle$ for all $v \in \mathrm{dom}(T^\dagger)$ and $w \in \mathrm{dom}(T)$.
2. Every other linear operator satisfying this property is a restriction of $T^\dagger$ (i.e. the domain of T^* is maximal with respect to the above property).

Property 4.1.21. Let T be a bounded linear operator. Its adjoint is also bounded and $\|T\|_{\mathrm{op}} = \|T^\dagger\|_{\mathrm{op}}$.

Definition 4.1.22 (Symmetric operator). A linear operator T on a Hilbert space $\mathcal{H}$ such that $\mathrm{dom}(T) \subseteq \mathrm{dom}(T^\dagger)$ and $T = T^\dagger|_{\mathrm{dom}(T)}$.

Definition 4.1.23 (Self-adjoint operator). A linear operator T on a Hilbert space $\mathcal{H}$ such that $\mathrm{dom}(T)$ is dense in $\mathcal{H}$ and $T = T^\dagger$.

The notion of Hermitian operator is the one where almost nobody agrees upon its definition. Here, the definition from Schreiber (2008) is chosen.

Definition 4.1.24 (Hermitian operator). A bounded, symmetric operator.

Theorem 4.1.25 (Hellinger–Toeplitz). *A self-adjoint operator on a Hilbert space $\mathcal{H}$ is bounded if and only if its domain is all of $\mathcal{H}$.*

Theorem 4.1.26 (Stone). *Consider a strongly continuous unitary one-parameter group, i.e. a family of unitary operators $U : \mathbb{R} \to \mathrm{U}(\mathcal{H})$ such that*

- *U is continuous in the strong operator topology:*
$$\lim_{t\to t_0} U(t)v = U(t_0)v \tag{4.8}$$
for all $t_0 \in \mathbb{R}, v \in \mathcal{H}$.
- *U is a one-parameter group in the sense of* **??**.

There exists a self-adjoint operator T such that $U(t) = e^{itT}$. Furthermore, the linear operator T is bounded if and only if U is continuous in the norm topology.

Definition 4.1.27 (Generator). The linear operator T in the preceding theorem is called the (infinitesimal) generator of the family U. It can be obtained through a formal derivative:
$$T = \left.\frac{\mathrm{d}U(t)}{\mathrm{d}t}\right|_{t=0}. \tag{4.9}$$

4.1.4 Compact operators

Definition 4.1.28 (Compact operator). Let V, W be Banach spaces. A linear operator $T : V \to W$ is said to be compact if the image of any bounded set in V is relatively compact (**??**).

Alternative Definition 4.1.29 (Compact operator). Let V, W be Banach spaces. A linear operator $T : V \to W$ is compact if, for every bounded sequence $(v_n)_{n\in\mathbb{N}}$ in V, the sequence $(Tv_n)_{n\in\mathbb{N}} \subset W$ has a convergent subsequence.

Notation 4.1.30. The space of compact, bounded linear operators between Banach spaces V, W is denoted by $\mathcal{B}_0(V, W)$. If $V = W$, this is abbreviated as $\mathcal{B}_0(V)$ as usual.

Property 4.1.31. $\mathcal{B}_0(V)$ is a two-sided ideal in the (Banach) algebra $\mathcal{B}(V)$. Moreover, $\mathcal{B}_0(V) = \mathcal{B}_1^*(V)$.

Property 4.1.32 (Finite-rank operators). All finite-rank operators are compact. In fact, the space of compact operators is the norm closure of that of finite-rank operators.

Property 4.1.33. Every compact operator is bounded.

Corollary 4.1.34. Every linear map between finite-dimensional Banach spaces is bounded.

Property 4.1.35. If T is a compact self-adjoint operator on a Hilbert space, then $-\|T\|$ or $\|T\|$ are an eigenvalue of T. Furthermore, the set of nonzero eigenvalues is either finite or converges to 0.

Definition 4.1.36 (Calkin algebra). Consider the algebra $\mathcal{B}(V)$ of bounded linear operators on V together with its two-sided ideal $\mathcal{B}_0(V)$ of compact operators. The quotient algebra $\mathcal{Q}(V) = \mathcal{B}(V)/\mathcal{B}_0(V)$ is called the Calkin algebra of V.

Definition 4.1.37 (Fredholm operator). A bounded linear operator for which the kernel and cokernel are finite-dimensional. The space of Fredholm operators on a space V is denoted by $\mathfrak{F}(V)$.

By the following theorem, one can characterize Fredholm operators using the Calkin algebra.

Property 4.1.38 (Atkinson). A bounded linear operator $T : V \to W$ is a Fredholm operator if and only if it is invertible in the Calkin algebra, i.e. there exists a bounded linear operator $S : W \to V$ and compact operators C_1, C_2 such that $\mathbb{1}_W - TS = C_1$ and $\mathbb{1}_V - ST = C_2$. S is called the **parametrix** of T.

Definition 4.1.39 (Fredholm index). The index of a Fredholm operator F is defined as follows:

$$\mathrm{ind}(F) := \dim\ker(F) - \dim\mathrm{coker}(F)\,. \tag{4.10}$$

Property 4.1.40. The induced function

$$\mathrm{ind}: \pi_0\big(\mathfrak{F}(\mathcal{H})\big) \rightarrow \mathbb{Z} \tag{4.11}$$

is a group isomorphism:

- $\mathrm{ind}(F^*) = -\mathrm{ind}(F)$, and
- $\mathrm{ind}(FG) = \mathrm{ind}(F) + \mathrm{ind}(G)$.

This theorem is generalized in *K-theory* (see **??**) by the *Atiyah–Jänich theorem* **??**.

4.1.5 Spectrum

Definition 4.1.41 (Resolvent operator). Consider an operator $T \in \mathcal{B}(V)$ on a normed space V. The resolvent operator T_λ for some $\lambda \in \mathbb{C}$ is defined as the linear operator $(T - \lambda\mathbb{1}_V)^{-1}$.

Definition 4.1.42 (Resolvent set). The resolvent set $\rho(T)$ consists of all scalars $\lambda \in \mathbb{C}$ for which the resolvent operator of A is a bounded linear operator on a dense subset of V. These scalars λ are called **regular values** of T.

Definition 4.1.43 (Spectrum). The set of scalars $\mu \in \mathbb{C}\backslash\rho(T)$ is called the spectrum $\sigma(T)$.

From **??**, it is clear that every eigenvalue of T belongs to the spectrum of T. The converse, however, is not true. This is remedied by introducing the following concepts.

Definition 4.1.44 (Point spectrum). The set of scalars $\mu \in \mathbb{C}$ for which $T - \mu\mathbb{1}_V$ fails to be injective is called the point spectrum $\sigma_p(T)$. This set coincides with the set of eigenvalues of T.

Definition 4.1.45 (Continuous spectrum). The set of scalars $\mu \in \mathbb{C}$ for which $T - \mu\mathbb{1}_V$ is injective with dense image but fails to be surjective is called the continuous spectrum of T.

Definition 4.1.46 (Residual spectrum). The set of scalars $\mu \in \mathbb{C}$ for which $T - \mu\mathbb{1}_V$ is injective but fails to have a dense image is called the residual spectrum $\sigma(T)$.

Definition 4.1.47 (Essential spectrum). The set of scalars $\mu \in \mathbb{C}$ for which $T - \mu\mathbb{1}_V$ is not a Fredholm operator is called the essential spectrum $\sigma_{\mathrm{ess}}(T)$.

From Atkinson's theorem 4.1.38, one can derive the following result.[3]

[3] In fact, one could (equivalently) define the essential spectrum in terms of the Calkin algebra using Atkinson's theorem. Then, this property would be an obvious consequence.

Property 4.1.48. Let T be a bounded linear operator and let C be a compact operator. The essential spectra of T and $T + C$ coincide.

Property 4.1.49 (Bounded spectrum). A self-adjoint operator is bounded if and only if its spectrum is bounded. Furthermore, it is positive if and only if its spectrum lies in $\mathbb{R}^+$.

4.1.6 Spectral theorem & functional calculus

This section focuses on the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a (complex) Hilbert space $\mathcal{H}$.

Property 4.1.50 (Closed subspaces). There exists a bijection between the set of closed subspaces of $\mathcal{H}$ and the set of orthogonal projections in $\mathcal{B}(\mathcal{H})$. Furthermore, if the projection p corresponds to a subspace $\mathcal{H}_p$, the projection $\mathbb{1}_{\mathcal{H}} - p$ corresponds to the orthogonal complement $\mathcal{H}_p^\perp$.

Definition 4.1.51 (Projection-valued measure). Consider a measurable space (X, Σ). A projection-valued measure (PVM) or **spectral measure**[4] on X is a map $P : \Sigma \to \mathcal{B}(\mathcal{H})$ satisfying the following conditions:[5]

1. P_E is a projection for all $E \in \Sigma$,
2. $P_X = \mathbb{1}_{\mathcal{H}}$,
3. $P_A P_B = P_{A\cap B}$, and
4. for all disjoint $(E_n)_{n\in\mathbb{N}} \subset \Sigma$:

$$\sum_{n=0}^{+\infty} P_{E_n} = P_{\cup_{n\in\mathbb{N}} E_n} . \tag{4.12}$$

Property 4.1.52. Let P be a spectral measure on (X, Σ). For every two elements $v, w \in \mathcal{H}$, the map

$$E \mapsto \mu^P_{v,w}(E) := \langle v \mid P_E w\rangle \tag{4.13}$$

defines a (complex) measure $\mu^P_{v,w}$ on X. The square of the norm of an element $v \in \mathcal{H}$ is then simply given by $\mu^P_{v,v}(X)$ due to the second condition above.

Property 4.1.53. Let $f : X \to \mathbb{C}$ be a measurable function on a measurable space (X, Σ). Given a spectral measure P, one defines Δ_f to be the set of all $v \in \mathcal{H}$ for which $f \in L^2(X, \mu^P_{v,v})$. This set is dense in $\mathcal{H}$. Moreover, the map

$$\int_X f(\lambda)\, dP(\lambda) : \Delta_f \to \mathcal{H} \tag{4.14}$$

[4] Sometimes also called a **resolution of the identity**.

[5] The third property can in fact be shown to follow from the others.

defined by

$$\left\langle v \,\middle|\, \int_X f(\lambda)\, dP(\lambda) w \right\rangle := \int_X f(\lambda)\, d\mu^P_{v,w}(\lambda) \tag{4.15}$$

is closed and normal. It also satisfies the following two equalities:

$$\left(\int_X f(\lambda)\, dP(\lambda)\right)^\dagger = \int_X \overline{f(\lambda)}\, dP(\lambda)\,, \tag{4.16}$$

$$\left\|\int_X f(\lambda)\, dP(\lambda) v\right\|^2 = \int_X |f(\lambda)|^2\, d\mu^P_{v,v}(\lambda)\,. \tag{4.17}$$

If f is bounded, the above operator is bounded by the supremum norm of f:

$$\left\|\int_X f(\lambda)\, dP(\lambda)\right\| \leq \|f\|_\infty\,. \tag{4.18}$$

Theorem 4.1.54 (Spectral theorem). *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. There exists a unique spectral measure $P_A : \Sigma_{\mathbb{R}} \to \mathcal{B}(\mathcal{H})$ on the Borel σ-algebra of the real line such that*

$$A = \int_{\mathbb{R}} \lambda\, dP_A(\lambda)\,. \tag{4.19}$$

If $\mathbb{R}$ is replaced by $\mathbb{C}$, this theorem also holds for normal operators. If A is compact, the measure becomes purely atomic, i.e. there exists a countable orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A such that $\lambda_i \longrightarrow 0$:

$$A = \sum_{i=1}^{+\infty} \lambda_i e_i^* \otimes e_i\,. \tag{4.20}$$

Property 4.1.55 (Spectrum and support). The spectrum of a self-adjoint operator A coincides with the support of its associated spectral measure P_A. A number $\lambda \in \mathbb{R}$ belongs to the point spectrum of A if and only if P_A does not vanish on $\{\lambda\}$. A number $\lambda \in \mathbb{R}$ belongs to the continuous spectrum of A if P_A vanishes on $\{\lambda\}$ but is nonvanishing on any open set containing λ.

Definition 4.1.56 (Singular value). Let A be a compact operator. The singular values of A are given by the square roots of the eigenvalues of the self-adjoint (and compact) operator A^*A. If A is self-adjoint, its singular values and eigenvalues coincide.

Definition 4.1.57 (Measurable functional calculus). Let (X, Σ) be a measurable space and let $\mathcal{H}$ be a Hilbert space. A measurable functional calculus $(\Phi, \mathcal{H})$ on (X, Σ) is an assignment

$$\Phi : \mathsf{Meas}(X, \mathbb{C}) \to \mathcal{B}_0(\mathcal{H}) \tag{4.21}$$

satisfying the following conditions:

1. **Unitality**: $\Phi(1) = \mathbb{1}_{\mathcal{H}}$.
2. **Sublinearity**: $\Phi(\lambda f + g) \subseteq \lambda\Phi(f) + \Phi(g)$ and the equality holds if either operator is bounded.
3. **Submultiplicativity**: $\Phi(f)\Phi(g) \subseteq \Phi(fg)$ and the equality holds if either operator is bounded. Moreover, the product is commutative if either operator is bounded.
4. **Density**:[6] $\Phi(f)$ is densely defined.
5. **Involutivity**: $\Phi\left(\overline{f}\right) = \overline{\Phi(f)}$.
6. **Boundedness**: $\Phi(f)$ is bounded if f is bounded.
7. **Convergence**: If the bounded sequence $f_n \longrightarrow f$ pointwise, then $\Phi(f_n) \longrightarrow \Phi(f)$ strongly.

Property 4.1.58. If $(\Phi, \mathcal{H})$ is a measurable functional calculus on (X, Σ), then

$$P_\Phi : \Sigma \to \mathcal{B}(\mathcal{H}) : E \mapsto \Phi(\mathbb{1}_E) \tag{4.22}$$

is a projection-valued measure. Conversely, every projection-valued measure P gives rise to a measurable functional calculus Φ such that $P = P_\Phi$.

Definition 4.1.59 (Borel functional calculus). A functional calculus on (X, Σ) where X is a topological space and Σ is its Borel σ-algebra. If $X \subseteq \mathbb{C}$ and the identity function $\mathbb{1}_{\mathbb{C}} : z \mapsto z$ is measurable, $(\Phi, \mathcal{H})$ is called a Borel functional calculus for the operator A if $\Phi(\mathbb{1}_{\mathbb{C}}) = A$.

The above properties allow to compose self-adjoint operators with (measurable) functions similar to how one can compute $f(X)$ for finite-dimensional operators by applying f to the eigenvalues of X.

Formula 4.1.60 (Borel functional calculus). Consider a bounded operator $A \in \mathcal{B}(\mathcal{H})$. Let $f : \sigma(A) \to \mathbb{C}$ be a measurable function (with respect to the restriction of the Borel algebra on $\mathbb{R}$) and let $g : \mathbb{R} \to \mathbb{C}$ be any other measurable function that coincides with f on $\sigma(A)$.

$$f(A) := \int_{\sigma(A)} f(\lambda)\, dP_A(\lambda) = \int_{\mathbb{R}} g(\lambda)\, dP_A(\lambda) =: g(A)\,. \tag{4.23}$$

The Spectral Theorem 4.1.54 can also be stated in terms of functional calculus.

Theorem 4.1.61 (Spectral theorem). *A normal operator $A \in \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ induces a unique Borel functional calculus on $\sigma(A)$.*

[6] This condition is actually redundant.

4.1.7 Dixmier trace

Definition 4.1.62 (Dixmier ideal). For every compact operator, one can sort the singular values in decreasing order. The Dixmier ideal is defined as follows:

$$\mathfrak{D}(\mathcal{H}) := \left\{A \in \mathrm{End}_0(\mathcal{H}) \mid \sigma_n(A) = O(\ln(n))\right\}, \tag{4.24}$$

where

$$\sigma_n(A) := \sum_{i=1}^{n} s_i(A)\,. \tag{4.25}$$

The following functional gives an alternative for the ordinary trace on operators that are not trace-class.

Definition 4.1.63 (Dixmier trace). For every element in $A \in \mathfrak{D}(\mathcal{H})$, the Dixmier trace is defined as follows:

$$\mathrm{tr}_{\mathfrak{D}}(A) := \omega\left(\left\{\frac{\sigma_n(A)}{1+\ln(n)}\,\middle|\, n \in \mathbb{N}\right\}\right), \tag{4.26}$$

where $\omega : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ is a linear functional satisfying the following conditions:

1. **Positivity**: $\omega(s) \geq 0$ if $s_n \geq 0$ for all $n \in \mathbb{N}$.
2. **Convergence**: $\omega(s) = \lim_{n\to\infty} s_n$ if s is convergent.
3. **Dilation-invariant**: Let $D : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ be the following operator:

   $$D : (s_1, s_2, \ldots) \mapsto (s_1, s_1, s_2, s_2, \ldots). \tag{4.27}$$

   The state ω is dilation-invariant if $\omega(s) = \omega\big(D(s)\big)$ for all $s \in \ell^\infty(\mathbb{N})$.

If for an operator $A \in \mathfrak{D}(\mathcal{H})$ the Dixmier trace is independent of the chosen state, it is said to be **measurable**.

Property 4.1.64. The Dixmier trace has the following properties:

- $\mathrm{tr}_{\mathfrak{D}}(A+B) = \mathrm{tr}_{\mathfrak{D}}(A) + \mathrm{tr}_{\mathfrak{D}}(B)$ for all $A, B \in \mathfrak{D}(\mathcal{H})$.
- If A is bounded and B is an element of the Dixmier ideal, then $\mathrm{tr}_{\mathfrak{D}}(AB) = \mathrm{tr}_{\mathfrak{D}}(BA)$.
- The Dixmier ideal vanishes on trace-class operators (in particular, it is **singular**, i.e. it vanishes on finite-rank operators):

  $$A \in \mathcal{B}_1(\mathcal{H}) \implies \mathrm{tr}_{\mathfrak{D}}(A) = 0\,. \tag{4.28}$$

4.2 C^*-algebras

4.2.1 Involutive algebras

Definition 4.2.1 (Involutive algebra). An involutive algebra is an associative algebra A over a commutative involutive ring $(R, \overline{\cdot})$ together with an algebra involution $\cdot^* : A \to A$ such that:

1. $(a+b)^* = a^* + b^*$,
2. $(ab)^* = b^*a^*$, and
3. $(\lambda a)^* = \overline{\lambda} a^*$,

for all $a, b \in A$ and $\lambda \in R$. These algebras are also sometimes called **$*$-algebras**.

Definition 4.2.2 (Isometry). An element a of a $*$-algebra A satisfying

$$a^*a = 1\,, \tag{4.29}$$

where 1 is the unit of A.

Definition 4.2.3 (Normal element). An element of a $*$-algebra that commutes with its adjoint:

$$a^*a = aa^*\,. \tag{4.30}$$

Property 4.2.4. Every element in a $*$-algebra can be decomposed as the sum of two normal elements:

$$a = \frac{1}{2}\big((a + a^*) + (a - a^*)\big)\,. \tag{4.31}$$

This implies that a linear operator defined on normal elements extends uniquely to the whole algebra.

Definition 4.2.5 (Projection). An element p of a $*$-algebra such that

$$p = p^2 = p^*\,. \tag{4.32}$$

This terminology reflects the property that in an algebra of bounded operators on a Hilbert space, the projections are exactly the operators associated to a orthogonal projections (cf. **??**).

Definition 4.2.6 (C^*-algebra). A C^*-algebra is an involutive Banach algebra A (**??**) such that the **C^*-identity**

$$\|a^*a\| = \|a\|\,\|a^*\| \tag{4.33}$$

is satisfied for all $a \in A$.

The Artin–Wedderburn theorem **??** implies the following decomposition.

Theorem 4.2.7. *Let A be a finite-dimensional C^*-algebra over the field $\mathfrak{K}$. There exist unique integers N and $d_1, \dots, d_N$ such that*

$$A \cong \bigoplus_{i=1}^{N} M_{d_i}(\mathfrak{K})\,. \tag{4.34}$$

This implies that every C^*-algebra can be represented using block matrices.

Example 4.2.8 (Bounded operators). Let $\mathcal{H}$ be a finite-dimensional Hilbert space. The space of bounded operators $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

Property 4.2.9. Every norm-closed $*$-subalgebra of a C^*-algebra is a C^*-algebra.

Definition 4.2.10 (H^*-algebra). A Hilbert space $\mathcal{H}$ equipped with a unital $*$-algebra structure such that $*$-adjoint and multiplicative adjoints coincide:

1. $\langle ab \mid c\rangle = \langle b \mid a^*c\rangle$, and
2. $\langle ab \mid c\rangle = \langle a \mid cb^*\rangle$,

for all $a, b, c \in \mathcal{H}$.

Example 4.2.11 (Linear operators). The canonical example of H^*-algebras is given by the algebra of linear operators on a Hilbert space $\mathcal{H}$, where the involution is given by taking adjoints and the inner product is the Hilbert–Schmidt inner product induced by the norm (**??**) (up to a factor $k > 0$):

$$\langle f \mid g\rangle_{\mathrm{HS}} := k\ \mathrm{tr}(f^*g)\,. \tag{4.35}$$

The resulting space is denoted by $L^2(\mathcal{H}, k)$. A result, analogous to the Artin–Wedderburn theorem **??**, states that every H^*-algebra can be decomposed as an orthogonal direct sum of finitely many algebras of the form $L^2(\mathcal{H}_i, k_i)$.

4.2.2 Positive maps

Definition 4.2.12 (Positive element). A self-adjoint element of a C^*-algebra whose spectrum is contained in $[0, +\infty[$. The cone of all positive elements in A is often denoted by A^+.

Property 4.2.13 (Positive decomposition). An element $a \in A$ is positive if and only if $a = b^*b$ for some element $b \in A$. A normal element is positive if and only if it is self-adjoint.

Definition 4.2.14 (Positive map). A morphism $T : A \to B$ of C^*-algebras such that $T(A^+) \subseteq B^+$.

Definition 4.2.15 (Completely positive map). A morphism $T : A \to B$ of C^*-algebras such that the following map is positive for all $k \in \mathbb{N}$:

$$\mathbb{1}_k \otimes T : \mathbb{C}^{k\times k} \otimes A \to \mathbb{C}^{k\times k} \otimes B\,. \tag{4.36}$$

If T satisfies this condition only up to an integer $n \in \mathbb{N}$, it is said to be n**-positive**.

Definition 4.2.16 (State). A positive linear functional of unit norm on a C^*-algebra.

Property 4.2.17 (Convexity). The set of states is a convex set. The extreme points of this set are called **pure states**, all other elements are called **mixed states**.

Definition 4.2.18 (Positivity-improving map). A positive map T that satisfies

$$a \geq 0 \implies T(a) > 0\,. \tag{4.37}$$

Definition 4.2.19 (Ergodic map). A positive map T that satisfies

$$\forall a > 0 : \exists t_a \in \mathbb{R}_0 : \exp(t_a T)\,a > 0\,. \tag{4.38}$$

The following theorem can be seen as a Jordan algebra-theoretic analogue of the Gel'fand–Naimark theorem 4.2.42.

Theorem 4.2.20 (Alfsen–Shultz). *The state spaces of two C^*-algebras are isomorphic if and only if the algebras are isomorphic as (special) Jordan algebras (**??**).*

Definition 4.2.21 (Cuntz algebra). The n^{th} Cuntz algebra $\mathcal{O}_n$ is defined as the (universal) unital C^*-algebra generated by $n \in \mathbb{N}$ isometries s_i under the additional relation

$$\sum_{i=1}^{n} s_i s_i^* = 1\,, \tag{4.39}$$

where 1 is the unit element.

4.2.3 Traces

Definition 4.2.22 (Trace). Let A be a C^*-algebra. A trace on A is a linear functional that satisfies the following conditions:

1. **Positivity**: $\mathrm{tr}(A^+) \geq 0$, and
2. **Tracial**: $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in A$.

Remark 4.2.23. The tracial property could have been replaced by the following equivalent, but seemingly weaker, condition: $\mathrm{tr}(a^*a) = \mathrm{tr}(aa^*)$ for all $a \in A$.

Definition 4.2.24 (Adjoint map). Assume that a trace functional tr is given on a C^*-algebra and consider a continuous linear operator T defined on the Schatten class $\mathcal{I}_p$ (Definition 4.1.11). One can define the adjoint map T^* on $\mathcal{I}_q$ whenever p, q are Hölder conjugate (**??**). This adjoint is given by the following equation:

$$\mathrm{tr}\big((T^*a)b\big) = \mathrm{tr}\big(a^*Tb\big)\,, \tag{4.40}$$

where $a \in \mathcal{I}_q$ and $b \in \mathcal{I}_p$.

Definition 4.2.25 (Trace-preserving map). A linear operator $T : A \to B$ is said to be trace-preserving if it satisfies

$$\mathrm{tr}_B(Ta) = \mathrm{tr}_A(a) \tag{4.41}$$

for all trace-class elements $a \in A$. Using the above definition, it is easy to see that, on a unital C^*-algebra, this is equivalent to

$$T^*1 = 1\,, \tag{4.42}$$

i.e. to its adjoint being unital.

Property 4.2.26. A completely positive, trace-preserving map T satisfies:

$$\|T\|_1 = 1\,, \tag{4.43}$$

where the subscript 1 indicates that this operator is defined on trace-class elements.

Property 4.2.27 (States). Whenever a C^*-algebra is commutative, every state defines a trace and, after a suitable normalization, every trace defines a state. However, for noncommutative C^*-algebras, only the latter implication holds.

4.2.4 Representations

Definition 4.2.28 (C^*-representation). A representation of a C^*-algebra A on a Hilbert space $\mathcal{H}$ is a unital $*$-morphism $\pi : A \to \mathcal{B}(\mathcal{H})$.

Definition 4.2.29 (Normal state). Consider a Hilbert space $\mathcal{H}$ and a C^*-representation $\pi : A \to \mathcal{B}(\mathcal{H})$. A normal state ω on A is a state such that there exists a trace-class operator $\rho \in \mathcal{B}_1(\mathcal{H})$ with the following property:

$$\omega(a) = \frac{\mathrm{tr}\big(\rho\pi(a)\big)}{\mathrm{tr}(\rho)}\,. \tag{4.44}$$

When $A = \mathcal{B}(\mathcal{H})$, the normal states are exactly the σ-weakly continuous states (Definition 4.1.15).

Definition 4.2.30 (Folium). Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a C^*-representation. The space of normal states of this representation is called its folium.

Definition 4.2.31 (Cyclic vector). A cyclic vector for a C^*-algebra representation $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ is a vector $\xi \in \mathcal{H}$ such that $\{\pi(a)\xi \mid a \in A\}$ is (norm-)dense in $\mathcal{H}$.

The injective tensor product of Banach spaces admits an equivalent construction in the case of C^*-algebras.

Definition 4.2.32 (Spatial tensor product). Let A, B be two C^*-algebras with faithful representations $\pi_A : A \rightarrow \mathcal{B}(\mathcal{H}_1)$ and $\pi_B : B \rightarrow \mathcal{B}(\mathcal{H}_2)$. The spatial tensor product $A \otimes_{\text{sp}} B$ is the norm closure of the algebraic tensor product $A \otimes B$. It can be shown that this definition does not depend on the choice of representation and, moreover, coincides with the injective tensor product of the underlying Banach spaces as induced by **??**.

4.2.5 Gel'fand duality

Definition 4.2.33 (Gel'fand spectrum). Consider a commutative C^*-algebra A (in fact any involutive algebra suffices). Its set of **characters**, i.e. the algebra morphisms $A \rightarrow \mathbb{C}$, can be equipped with a locally compact Hausdorff topology: the weak-* topology (**??**).

Property 4.2.34. The Gel'fand spectrum of a C^*-algebra is compact if and only if the algebra is unital.

Definition 4.2.35 (Gel'fand representation). Consider a C^*-algebra A and let Φ_A denote its Gel'fand spectrum. The **Gel'fand transformation** of an element $a \in A$ is defined as the morphism $\hat{a} : \Phi_A \rightarrow \mathbb{C}$ given by the following formula:

$$\hat{a}(\lambda) = \langle \lambda, a \rangle \,, \tag{4.45}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between A and Φ_A. By definition of the Gel'fand spectrum, the functional $\hat{a}$ is continuous for all $a \in A$. The mapping $a \mapsto \hat{a}$ is called the Gel'fand representation of A.

Theorem 4.2.36 (Gel'fand–Naimark: Commutative case). *Let A be a commutative C^*-algebra. The Gel'fand representation gives an isometric $*$-isomorphism between A and the set $C_0(\Phi_A)$ of continuous complex-valued functions that vanish at infinity (**??**) on its Gel'fand spectrum.*

Remark 4.2.37. In fact, the Gel'fand–Naimark theorem gives an equivalence between the category of commutative nonunital C^*-algebras and the category of locally compact Hausdorff spaces (with continuous functions that vanish at infinity).

Property 4.2.38 (Connectedness). A compact Hausdorff space is connected if and only if its algebra of continuous functions has no nontrivial projections.

Property 4.2.39 (Metrizability). A commutative C^*-algebra is separable if and only if its Gel'fand spectrum is metrizable.

Formula 4.2.40 (Spectrum). Recall Definition 4.1.43 of the spectrum of an operator. This is related to the spectrum of a unital, commutative C^*-algebra as follows:

$$\sigma(a) = \{a(x) \mid x \in \Phi_A\}, \tag{4.46}$$

for all $a \in A$.

Construction 4.2.41 (Gel'fand–Naimark–Segal). Let A be a C^*-algebra. Given a state ω on A, there exists a C^*-representation $\pi : A \to \mathcal{B}(D)$, where $D \subset \mathcal{H}$ is a dense subspace of a Hilbert space $\mathcal{H}$, such that the following conditions are satisfied:

- There exists a cyclic (unit) vector $\xi \in D$.
- For all elements $a \in A$, the following equality holds:

$$\omega(a) = \langle \pi(a)\xi \mid \xi \rangle. \tag{4.47}$$

@@ COMPLETE CONSTRUCTION @@

Theorem 4.2.42 (Gel'fand–Naimark: General case). *Every C^*-algebra is isometrically $*$-isomorphic to a norm closed (C^*-)algebra of bounded operators on a Hilbert space $\mathcal{H}$.*

The GNS construction can be generalized as follows.

Theorem 4.2.43 (Stinespring[7]). *Consider a linear operator $T : A \to \mathcal{B}(\mathcal{H})$ between C^*-algebras. It is completely positive if and only if there exists a C^*-representation $\pi : A \to \mathcal{B}(\mathcal{K})$ and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ such that*

$$Ta = V^\dagger \pi(a) V \tag{4.48}$$

or, equivalently,

$$T^\dagger b = \pi^\dagger(VbV^\dagger). \tag{4.49}$$

Moreover, V is an isometry if and only if T is unital and, hence, if $T^\dagger$ is trace-preserving by Definition 4.2.25. The triple $(\mathcal{K}, \pi, V)$ is often called a ***Stinespring representation*** *of T.*

Remark 4.2.44. Because the adjoint of a completely positive map is again completely positive, the above two characterizations can be used interchangeably. The latter is mostly used by mathematicians, whereas the former is better known in the physics literature.

[7] Sometimes called the **Stinespring dilation theorem** or **Stinespring factorization theorem**.

The most common situation is where $\mathcal{H} := \mathcal{H}_1 = \mathcal{H}_2$ and

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) : a \mapsto a \otimes b \tag{4.50}$$

for some density operator $b \in \mathcal{B}(\mathcal{K})$. The adjoint to tensoring by a density operator is taking the partial trace $\mathrm{tr}_{\mathcal{K}}$. This way, one obtains the following expressions:

$$Ta = V^\dagger(a \otimes b)V \tag{4.51}$$

and, equivalently,

$$T^\dagger b = \mathrm{tr}_{\mathcal{K}}(VbV^\dagger)\,. \tag{4.52}$$

Corollary 4.2.45 (GNS construction). The GNS construction follows from the Stinespring theorem by taking T to be a state (hence $\mathcal{H} = \mathbb{C}$).

Corollary 4.2.46 (Choi). Let $\mathcal{H}_1, \mathcal{H}_2$ be two finite-dimensional Hilbert spaces of dimensions $m, n \in \mathbb{N}$, respectively. A linear map $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ is completely positive if and only if it can be expressed as

$$\Phi(a) = \sum_{i=1}^{mn} V_i^\dagger a V_i \tag{4.53}$$

for some bounded operators $V_i : \mathcal{H}_2 \to \mathcal{H}_1$. Furthermore, it is trace-preserving if and only if

$$\sum_{i=1}^{mn} A_i^\dagger A_i = \mathbb{1}\,. \tag{4.54}$$

This decomposition is also often called an **operator-sum decomposition** and the operators $\{V_i\}_{i\leq mn}$ are called **Kraus operators** of Φ.

4.2.6 Hilbert modules ♣

Definition 4.2.47 (Hilbert C^*-module). Let A be a C^*-algebra and H a vector space. H is a (right) pre-Hilbert A-module if there exists a map

$$\langle\cdot \mid \cdot\rangle_A : H \times H \to A\,, \tag{4.55}$$

sometimes called the A**-inner product**, satisfying the following conditions:

1. **Conjugate linearity**: For all $x, y, z \in H$ and $\lambda \in \mathbb{C}$:

$$\langle x \mid \lambda y + z\rangle_A = \langle x \mid y\rangle_A + \overline{\lambda}\langle x \mid z\rangle_A\,, \tag{4.56}$$

$$\langle x \mid y\rangle_A = \langle y \mid x\rangle_A^*\,. \tag{4.57}$$

2. **Nondegeneracy**: For all $x \in H$:

$$\langle x \mid x \rangle_A \geq 0\,, \tag{4.58}$$

$$\langle x \mid x \rangle_A = 0 \iff x = 0\,. \tag{4.59}$$

3. **Equivariance**: For all $x, y \in H$ and $a \in A$:

$$\langle x \mid y \rangle_A\, a = \langle x \mid y \cdot a \rangle_A\,. \tag{4.60}$$

A left module is obtained by requiring linearity and (left-)equivariance in the first argument. The completion of a pre-Hilbert module with respect to the norm $x \mapsto \sqrt{\|\langle x|x\rangle_A\|}$ is called a **Hilbert A-module**.

Definition 4.2.48 (Hilbert C^*-bimodule). Let A, B be C^*-algebras and consider a Hilbert C^*-module H over B. H is called a Hilbert (A, B)-bimodule if it admits a left $*$-representation of A such that

$$\langle a^* \cdot x \mid y \rangle_B = \langle x \mid a \cdot y \rangle_A \tag{4.61}$$

for all $a \in A$, i.e. adjoints in A correspond to adjoints with respect to the B-inner product.

Definition 4.2.49 (Fredholm module). Let $\pi : A \to \mathsf{End}(\mathcal{H})$ be a $*$-representation of a unital C^*-algebra on a Hilbert space $\mathcal{H}$. When equipped with an operator $F \in \mathsf{End}(\mathcal{H})$, this gives an (**odd**) Fredholm module if

$$F = F^* \qquad \text{and} \qquad F^2 = \mathbb{1}_{\mathcal{H}} \qquad \text{and} \qquad [F, \pi(A)] \subseteq \mathcal{B}_0(\mathcal{H})\,. \tag{4.62}$$

The Fredholm operator is said to be **even** if $\mathcal{H}$ is a super-Hilbert space with grading $\Gamma \in \mathsf{End}(\mathcal{H})$ satisyfing

$$\Gamma^* = \Gamma \qquad \text{and} \qquad \Gamma^2 = \mathbb{1}_{\mathcal{H}} \qquad \text{and} \qquad [\Gamma, \pi(A)] = 0 \qquad \text{and} \qquad \{\Gamma, F\}_+ = 0\,. \tag{4.63}$$

F has the form

$$\begin{pmatrix} 0 & F_+ \\ F_- & 0 \end{pmatrix}, \tag{4.64}$$

where F_+ is Fredholm.

Example 4.2.50. Consider $A = \mathbb{C}$ and let $\pi : \mathbb{C} \to \mathsf{End}(\mathcal{H})$ be the unique unital representation. Fredholm modules with this data correspond to (essentially self-adjoint) Fredholm operators on $\mathcal{H}$.

Remark 4.2.51 (Kasparov K-theory). If A is not unital, one should relax the first two relations up to compact operators, i.e. $F = F^*$ and $F^2 = \mathbb{1}_{\mathcal{H}}$ in the Calkin algebra. This gives a clear relation to *Kasparov's K-theory* (see below), where the homology complex $KK_\bullet$ is defined in terms of such generalized Fredholm modules. However, it can be shown that these induce the same KK-classes.

Definition 4.2.52 (Kasparov bimodule). Let A, B be two C^*-algebras and let H be a super-Hilbert (A, B)-bimodule. H is called a Kasparov (A, B)-bimodule if there exists an odd adjointable operator $F \in \mathcal{B}(H)$ satisfying the following properties for all $a \in A$:

1. $(F^2 - \mathbb{1}_H)\pi_A(a)$ is compact,
2. $(F - F^*)\pi_A(a)$ is compact, and
3. $[F, \pi_A(a)]$ is compact.

If A is unital (and the representation respects units), this implies that F is a projection in the Calkin algebra. If F is a projection on the nose, the bimodule is said to be **normalized**.

Definition 4.2.53 (Kasparov K-theory). The set, in fact Abelian group, of homotopy classes of Kasparov bimodules $KK(A, B)$, where a homotopy between two Kasparov (A, B)-bimodules is a Kasparov $(A, C([0, 1], B))$-bimodule that restricts to the given bimodules on the boundaries of the interval.

Property 4.2.54. Every Kasparov bimodule is homotopic to a normalized bimodule. Hence, for Kasparov K-theory, one can restrict to normalized bimodules.

4.3 von Neumann algebras

Definition 4.3.1 (von Neumann algebra). A $*$-subalgebra of a C^*-algebra equal to its double commutant (**??**): $M = M''$.

Definition 4.3.2 (Concrete von Neumann algebra). A weakly closed unital $*$-algebra of bounded operators on some Hilbert space.

Theorem 4.3.3 (Double Commutant theorem[8]). *The above definitions are equivalent.*

Property 4.3.4 (Projections). Every von Neumann algebra is generated by its projections.

Property 4.3.5 (Orthomodular lattices). The set of projections in a von Neumann algebra or, equivalently, the set of closed subspaces of a Hilbert space forms a complete orthomodular lattice (**??**).

Given two closed subspaces, the join and meet are constructed as follows:

[8]Often called **von Neumann's double commutant theorem**.

- Meet: $A \wedge B = A \cap B$, and
- Join: $A \vee B =$ smallest closed subspace containing $A \cup B$.

Definition 4.3.6 (Murray–von Neumann equivalence). Closed subspaces of a Hilbert space are said to be Murray–von Neumann equivalent if they are isomorphic through a partial isometry. In terms of projections, this means that $p \sim q$ if and only if there exists a partial isometry u such that $p = uu^*$ and $q = u^*u$.

Property 4.3.7 (Partial order). The inclusion of subspaces induces a partial order on the set of projections. A projection p is said to be finite if there exists no smaller projection q that is equivalent to p.

4.3.1 Factors

Definition 4.3.8 (Factor). Consider a von Neumann algebra M. A $*$-subalgebra $A \subseteq M$ is called a factor of M if its center $Z(A)$ is given by the scalar multiples of the identity.

Definition 4.3.9 (Type-I factor). A factor that contains a minimal projection.

Property 4.3.10 (Type-I_n factors). Any type I-factor is isomorphic to the algebra of all bounded operators on a Hilbert space. To indicate the dimension $n \in \overline{\mathbb{N}}$ of this Hilbert space (which may be $+\infty$), one sometimes uses the subclassification of type I_n-factors.

Definition 4.3.11 (Powers index). Consider a Hilbert space $\mathcal{H}$ together with its von Neumann algebra of bounded operators $\mathcal{B}(\mathcal{H})$. A unital $*$-endomorphism α is said to have Powers index $n \in \mathbb{N}$ if the space $\alpha(\mathcal{B}(\mathcal{H}))$ is isomorphic to a type I_n-factor.

Definition 4.3.12 (Type-II factor). A factor that contains nonzero finite projections but no minimal ones. If the identity is finite, the factor is sometimes said to be of type II_1, otherwise it is said to be of type II_∞.

Definition 4.3.13 (Type-III factor). A factor that does not contain any nonzero finite projections.

Part III

Differential Geometry

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Chapter 5

Higher-dimensional Geometry ♣

In this chapter, certain constructions and theorems introduced in the previous chapters are generalized to the setting of supergeometries and higher categories. As such, it can been as an analogue to **??** for (differential) geometry.

The main references are Baez and Schreiber (2005); Schreiber (2005). The section on spectral geometry is based on Sanders (2012), whereas that on smooth spaces is inspired by Baez and Hoffnung (2011). For an introduction to (higher) category theory, see Chapter 1. **??** gives a different approach to the higher-dimensional analogues of Lie algebras. For an introduction to supergeometry, see Cattaneo and Schätz (2011).

@@ CITE MORE (e.g. BARTELS, ...) @@

Chapter Contents

5.1 Graded manifolds

In this section, some of the notions from Part III will be generalized to supermanifolds and general graded manifolds. The general notation (x^i) will be used for the collection of both even and odd coordinates.

Example 5.1.1 (Supermanifold). A super smooth set in the form of a locally ringed space $(M,\mathcal{A})$ that is locally isomorphic to a super Euclidean space, i.e. $\mathcal{A}$ is locally given by $C^\infty(U)\otimes\Lambda^\bullet\mathbb{R}^n$ for some $n\in\mathbb{N}$ and open $U\subseteq M$. More generally, a **graded manifold** is a locally ringed space modelled on $\big(\mathbb{R}^m, C^\infty(\mathbb{R}^m)\otimes\mathsf{Sym}(V^*)\big)$ for a graded vector space V. (A supermanifold can be recovered by taking $V=\mathbb{R}^{0|n}$.)

Theorem 5.1.2 (Batchelor). *Let $(M,\mathcal{A})$ be an $\mathbb{N}$-graded manifold. There exists a vector bundle $E\rightarrow M$ such that $\mathcal{A}$ is isomorphic to the structure sheaf $\mathcal{O}(\Lambda^\bullet E)$, i.e. $\mathcal{A}$ is locally given by $\mathsf{Sym}^\bullet(\Lambda^\bullet E^*)$. If $(M,\mathcal{A})$ is a supermanifold, there exists a vector bundle $E\rightarrow M$ such that $\mathcal{A}$ is locally given by $\Lambda^\bullet E^*$.*

Example 5.1.3 (Odd tangent bundle). Consider a smooth manifold M. By the associated bundle construction and functoriality, any endofunctor on **Set** (or any concrete category) can turn an existing bundle into a new one. Starting from the tangent bundle TM and applying the parity functor (**??**) gives rise to the odd tangent bundle ΠTM. As a locally ringed space, it is given by the de Rham sheaf $\Omega^\bullet$.

More generally, for any vector bundle $E\rightarrow B$, parity reversal gives a supermanifold with structure sheaf $\mathcal{O}_{\Pi E}:=\Lambda^\bullet_{C^\infty(B)}\Gamma_{E^*}$.

Definition 5.1.4 (Vector field). A graded vector field of degree $k\in\mathbb{N}$ is a degree-k derivation on $C^\infty(M)$. The integer k is called the **degree**.

Definition 5.1.5 (Cohomological vector field). A graded vector field X of degree 1 that satisfies $[X,X]=0$. Every degree-1 graded vector field satisfies

$$[X,X]=2X\circ X\,. \tag{5.1}$$

This implies that every cohomological vector field defines a coboundary operator on $C^\infty(M)$. A graded manifold equipped with a cohomological vector field is called a **differential-graded manifold** (dg-manifold).

Example 5.1.6 (de Rham differential). Consider the de Rham complex $\Omega^\bullet(M)$ with differential d. This differential corresponds to a cohomological vector field Q on ΠTM, locally given by

$$Q := \sum_{i=1}^{n} \mathrm{d}x^i \partial_i \,. \tag{5.2}$$

Note that the differentials $\mathrm{d}x^i$ are here to be regarded as coordinate functions on ΠTM.

Definition 5.1.7 (Degree). Let M be a graded manifold. The degree of a homogeneous element of $\Omega^\bullet(M)$ is defined as the difference of its graded degree and its form degree.

Property 5.1.8 (Euler vector field). Consider the graded vector field

$$E := \sum_{i=1}^{n} \mathrm{deg}(x^i) x^i \partial_i \,. \tag{5.3}$$

The Lie derivative $\mathcal{L}_E$, defined through the Cartan formula

$$\mathcal{L}_E := \iota_E \mathrm{d} + (-1)^{\mathrm{deg}(E)} \mathrm{d}\iota_E \,, \tag{5.4}$$

acts on homogeneous forms by multiplication by their degree.

Definition 5.1.9 (Poisson manifold). Consider a degree-k symplectic form ω. This form induces a Poisson structure on the algebra $C^\infty(M)$ as follows:

$$\{f, g\} := (\partial_i^R f)\omega^{ij}(\partial_j^L g) \,. \tag{5.5}$$

It is not hard to check that this operation is graded commutative. As in **??**, given a symplectic form, a Hamiltonian vector field can be defined for any smooth function $H \in C^\infty(M)$:

$$\omega(X_H, \cdot) = -\mathrm{d}H(\cdot) \,. \tag{5.6}$$

Property 5.1.10. Every closed differential form of degree $k \neq 0$ is exact. More generally, the de Rham cohomology of a graded manifold is isomorphic to the de Rham cohomology of its body. This, for example, implies that a degree-l symplectic vector field X is Hamiltonian with respect to a degree-k symplectic form if $k + l \neq 0$.

Corollary 5.1.11 (dg-symplectic manifold). Consider a Hamiltonian cohomological vector field X. There exists a Hamiltonian function H such that

$$Xf = \{H, f\} \tag{5.7}$$

for all $f \in C^\infty(M)$. If the symplectic form has degree $k \in \mathbb{N}$, the function H can be chosen to be of degree $k+1$ and, accordingly, $\{H, H\}$ will be of degree $k+2$. Now, the identity $[X, X] = 0$ also implies that $\{H, H\}$ is a constant and, since all constants are of degree 0, it follows that

$$\{H, H\} = 0 \tag{5.8}$$

whenever $k \neq -2$. This equation is often called the **classical master equation**. A graded manifold equipped with both a symplectic form and a symplectic cohomological vector field is called a **differential-graded symplectic manifold**.

If ω is of degree 1, it was shown by *Schwarz* that (M, ω) is symplectomorphic to ΠT^*M such that the Poisson bracket is mapped to the Schouten–Nijenhuis bracket (**??**) and the Hamiltonian is mapped to a Poisson bivector field exactly if it satisfies the master equation.

5.2 Infinite-dimensional geometry

In many situations, when considering function spaces, the objects under consideration do not form a finite-dimensional manifold. However, with some care, one can omit this size condition. In **??**, it was shown how calculus could be extended from $\mathbb{R}^n$ to infinite-dimensional vector spaces. Here, geometry is extended to that setting.

The first approach uses a locally convex TVS as local model space.

Definition 5.2.1 (E-manifold). Let E be a locally convex TVS. A Hausdorff space is called an E-manifold if there exists an atlas of charts (U, φ), where $\varphi : U \to \varphi(U) \subseteq E$ is a homeomorphism and the transition maps are Gateaux-smooth.

Definition 5.2.2 (Kinematic tangent bundle). Let M be an E-manifold with a smooth atlas $\{(U_i, \varphi_i)\}_{i\in I}$. The kinematic tangent bundle of M is defined as the quotient of

$$\bigsqcup_{i\in I} U_i \times E \tag{5.9}$$

by the equivalence relations[1] $(x, v) \sim \big(x, \mathrm{d}\psi_{ji}(\varphi_i(x); v)\big)$, where, as in the finite-dimensional case, $\psi_{ji} : U_i \cap U_j \to \mathsf{Aut}(E)$ are the transition functions.

Remark 5.2.3. For infinite-dimensional E, this tangent bundle is not isomorphic to the definition in terms of derivations. The above construction is 'kinematical' because the pair (x, v) represent a vector tangent to a curve at the point $x \in M$.

Since infinite-dimensional vector spaces are, in general, not reflexive, simply defining the cotangent bundle to be the fibrewise dual of the kinematic tangent bundle would lead to even more size issues. *Kriegl* and *Michor* have shown that one can cook up to 12 sensible definitions of a cotangent bundle (this also includes 'operational' definitions using derivations). However, only one of these definitions is well behaved with respect to Lie derivatives, exterior derivatives and pullbacks. Luckily, this is also the most widely used definition in the finite-dimensional setting.

[1]Here, d denotes the Gateaux differential and not the de Rham differential.

Definition 5.2.4 (Kinematic cotangent bundle). Let M be an E-manifold. Consider the set of bounded, alternating linear maps $E^{\times k} \to \mathbb{R}$. This lifts to a vector bundle $L^k_{\text{alt}}(TM, M \times \mathbb{R})$.

5.3 Cohesion

In this section, the terminology (Grothendieck) topos **over** a topos $\mathcal{S}$ (or $\mathcal{S}$-topos) will mean a topos equipped with a geometric morphism to $\mathcal{S}$. Moreover, everything will be stated in terms of ∞-topoi, unless stated otherwise. All notions such as adjoints, limits, etc. should be understood in this ∞-sense. The language of Section 3.6.2 is adopted.

Definition 5.3.1 (Local topos). Consider an $\mathcal{S}$-topos $\mathbf{H}$. $\mathbf{H}$ is said to be ($\mathcal{S}$-)local if the geometric morphism $(f^* \dashv f_*) : \mathbf{H} \leftrightarrows \mathcal{S}$ admits a right adjoint $f^!$ such that one of the following equivalent statements holds:

- $f^!$ is fully faithful,
- f^* is fully faithful,

* $f^!$ is an $\mathcal{S}$-indexed functor (**??**), or

* $f^!$ is Cartesian closed (**??**),

where the starred items hold in the 1-categorical setting. If $\mathcal{S} = \mathbf{Set}$ or $\mathcal{S} = \infty\mathbf{Grpd}$, the conditions are automatically satisfied. The triple of functors $f^* \dashv f_* \dashv f^!$ is also called a **local geometric morphism**.

The right adjoint is sometimes called the **codiscrete object functor** coDisc (in fact, this terminology is applied more generally when $\mathbf{H}$ is just any category). If this functor exists, $\mathbf{H}$ is said to have **codiscrete objects**.

Property 5.3.2. A topos is local if and only if $*$ is tiny (Definition 1.4.48).

Definition 5.3.3 (Locally connected topos). An object in a category is said to be **connected** if its representable functor preserves finite coproducts. A topos is said to be **locally connected** if all objects can be written as coproducts of connected objects. This defines a functor

$$\Pi_0 : \mathbf{H} \to \mathbf{Set} : \bigsqcup_{i \in I} X_i \mapsto I\,, \tag{5.10}$$

left-adjoint to the **discrete object functor** Disc, the inverse image part of the global section geometric morphism. This functor is suitably called the **connected components functor**.

Property 5.3.4. A topos is locally connected if and only if its global section geometric morphism is essential. More generally, an $\mathcal{S}$-topos is said to be **locally connected** if its associated geometric morphism is essential and the left adjoint is $\mathcal{S}$-indexed. In the case of $(\infty, 1)$-topoi, the image of the functor Π_0 is called the **fundamental ∞-groupoid**.

Definition 5.3.5 (Connected topos). A topos is said to be **connected** if the inverse image part of the associated geometric morphism is fully faithful. For sheaf topoi over a topological space X, this is exactly equivalent to X being connected.

For locally connected topoi, this amounts to the property that the left adjoint in its adjoint triple preserves the terminal object. Furthermore, a locally connected topos is said to be **strongly connected** if the left adjoint in its adjoint triple preserves finite products (in particular, turning it into a connected topos).

Property 5.3.6. Every local topos is connected.

Definition 5.3.7 (Cohesive topos). A local, strongly connected topos. This implies the existence of an adjoint quadruple $\Pi_0 \dashv \mathsf{Disc} \dashv \Gamma \dashv \mathsf{coDisc}$, where both Disc and coDisc are fully faithful:

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift left=6, "\Pi_0"] & \infty\mathbf{Grpd} \arrow[l, hook, shift left=2, "\mathsf{Disc}"'] \arrow[l, phantom] \\ \end{tikzcd} \quad \mathbf{H} \xrightarrow{\Pi_0} \infty\mathbf{Grpd},\ \mathbf{H} \xhookleftarrow{\mathsf{Disc}} \infty\mathbf{Grpd},\ \mathbf{H} \xrightarrow{\Gamma} \infty\mathbf{Grpd},\ \mathbf{H} \xhookleftarrow{\mathsf{coDisc}} \infty\mathbf{Grpd} \tag{5.11}$$

Property 5.3.8 (Cohesive modalities). The adjoint quadruple of a cohesive topos induces an adjoint triple of modalities (Definition 1.3.27):

$$(\textstyle\int \dashv \flat \dashv \sharp) := (\mathsf{Disc} \circ \Pi_0 \dashv \mathsf{Disc} \circ \Gamma \dashv \mathsf{coDisc} \circ \Gamma)\,. \tag{5.12}$$

These are respectively called the **shape**, **flat** and **sharp** modalities. The modal types of the flat and sharp modalities are called the **discrete** and **codiscrete objects**, respectively.

Remark 5.3.9 (Unity of opposites). Recall Example 3.6.10. A similar remark holds for the adjunction $\flat \dashv \sharp$. The induced unity of opposites

$$\mathsf{quantity}_X : \flat X \to X \to \sharp X \tag{5.13}$$

represents *Cantor*'s '*Kardinalen*' or *Hegel*'s '*Quantität*', which is the unity of continuity and discreteness.

Definition 5.3.10 (Concrete object). Consider a cohesive topos $\mathbf{H}$. An object $X \in \mathsf{ob}(\mathbf{H})$ is said to be concrete if $\eta^\sharp_X$ is monic.

Property 5.3.11 (Nullstellensatz). Consider a cohesive topos $\mathbf{H}$. The adjoint quadruple induces the following unity of opposites:

$$\mathsf{ptp} : \flat \Rightarrow \int := \varepsilon^{\int} \circ \eta^{\flat} .$$

This is called the **points-to-pieces** transform, representing the unity between 'repulsion' and 'attraction' (or cohesion). Sometimes, the points-to-pieces transform is also considered to be the adjunct $\Gamma \Rightarrow \Pi_0$. If this transformation is an epi for all objects $X \in \mathrm{ob}(\mathbf{H})$, the Nullstellensatz is said to hold (or pieces are said to have points).

Property 5.3.12. In a cohesive topos $\mathbf{H}$, the following statements are equivalent:

- The Nullstellensatz holds.
- Discrete objects are concrete, i.e. $\eta^{\sharp}_{\flat X}$ is a mono for all $X \in \mathrm{ob}(\mathbf{H})$.
- $\flat \dashv \sharp$ exhibits Aufhebung (Definition 3.6.9) with respect to $\emptyset \dashv *$, i.e. the initial object is codiscrete.

Moreover, if $\mathbf{H}$ is the sheaf topos over a *cohesive site* $\mathbf{C}$, these statements hold as soon as every object in $\mathbf{C}$ has at least one global element.

Definition 5.3.13 (Differential cohesion). Consider a cohesive $(\infty, 1)$-topos $\mathbf{H}$. An **infinitesimal cohesive neighbourhood** of $\mathbf{H}$ is a cohesive $(\infty, 1)$-topos $\mathbf{H}_{\mathrm{th}}$ equipped with a strongly connected adjoint quadruple $(\iota_{\mathrm{inf}} \dashv \Pi_{\mathrm{inf}} \dashv \mathsf{Disc}_{\mathrm{inf}} \dashv \Gamma_{\mathrm{inf}}) : \mathbf{H} \to \mathbf{H}_{\mathrm{th}}$ such that ι_{inf} is also fully faithful (and, hence, so is $\mathsf{Disc}_{\mathrm{inf}}$). If such a neighbourhood exists, $\mathbf{H}_{\mathrm{th}}$ is said to be differentially cohesive (or **elastic**) over $\mathbf{H}$.[2]

Remark 5.3.14. The interpretation of the infinitesimal adjoint quadruple is slightly different from that in the definition of cohesion. Here, $\Pi_0 \dashv \mathsf{Disc} \dashv \Gamma$ corresponds to the triple $\Pi_{\mathrm{inf}} \dashv \mathsf{Disc}_{\mathrm{inf}} \dashv \Gamma_{\mathrm{inf}}$. Moreover, the adjoint triple $\Pi_{\mathrm{th}} \dashv \mathsf{Disc}_{\mathrm{th}} \dashv \Gamma_{\mathrm{th}}$ characterizing $\mathbf{H}_{\mathrm{th}}$ as a cohesive topos factors exactly through these two triples by uniqueness of adjoint and global sections (see Eq. (5.16) for the full structure):

$$\begin{array}{ccc} \mathbf{H}_{\mathrm{th}} \overset{\xrightarrow{\Pi_{\mathrm{inf}}}}{\underset{\xrightarrow[\Gamma_{\mathrm{inf}}]{}}{\xleftarrow{\mathsf{Disc}_{\mathrm{inf}}}}} \mathbf{H} \overset{\xrightarrow{\Pi_0}}{\underset{\xrightarrow[\Gamma]{}}{\xleftarrow{\mathsf{Disc}}}} \infty\mathbf{Grpd} & = & \mathbf{H}_{\mathrm{th}} \overset{\xrightarrow{\Pi_{\mathrm{th}}}}{\underset{\xrightarrow[\Gamma_{\mathrm{th}}]{}}{\xleftarrow{\mathsf{Disc}_{\mathrm{th}}}}} \infty\mathbf{Grpd} \end{array} \tag{5.14}$$

Property 5.3.15 (Differential modalities). Consider a differentially cohesive $(\infty, 1)$-topos $\mathbf{H}$. The adjoint quadruple to its infinitesimal thickening induces an adjoint cylinder of modalities:

$$(\Re \dashv \Im \dashv \&) := (\iota_{\mathrm{inf}}\Pi_{\mathrm{inf}} \dashv \mathsf{Disc}_{\mathrm{inf}}\Pi_{\mathrm{inf}} \dashv \mathsf{Disc}_{\mathrm{inf}}\Gamma_{\mathrm{inf}}) .$$

[2] The subscript 'th' indicates that $\mathbf{H}_{\mathrm{th}}$ can be seen as an 'infinitesimal thickening' of $\mathbf{H}$. (See Definition 5.4.31 for more information.)

There are called the **reduction**, **infinitesimal shape** and **infinitesimal flat** modalities, respectively. The modal types of $\mathfrak{R}$ and $\mathfrak{I}$ are said to be **reduced** and **coreduced**, respectively.

The modalities coming from being cohesive and differentially cohesive fit into the following diagram, where $\vee$ indicates inclusion of modal types:

$$\begin{array}{ccccccc}
\mathfrak{R} & \dashv & \mathfrak{I} & \dashv & \& & & \\
 & & \vee & & \vee & & \\
 & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & \vee & & \vee \\
 & & & & \emptyset & \dashv & *
\end{array} \tag{5.15}$$

Definition 5.3.16 (Jet comonad). Consider a differentially cohesive topos $\mathbf{H}$. The jet bundle $J^\infty(E)$ of a bundle $E \in \mathsf{ob}(\mathbf{H}/B)$ over some base space $B \in \mathsf{ob}(\mathbf{H})$ is given by the base-change comonad (Definition 2.2.1) along the unit $\eta^{\mathfrak{I}}_B : B \to \mathfrak{I}B$:[3]

$$J^\infty E := (\eta^{\mathfrak{I}}_B)^*(\eta^{\mathfrak{I}}_B)_* E\,.$$

Analogously, the **infinitesimal (or formal) disk bundle** is defined as the adjoint monad induced by the base-change triple:

$$T^\infty E := (\eta^{\mathfrak{I}}_B)^*(\eta^{\mathfrak{I}}_B)_! E\,.$$

By definition of the base-change triple and the pasting laws (Property 1.4.73), this is equivalent to $T^\infty B \times_B E$, where the infinitesimal disk bundle $T^\infty B$ of an object is defined as the pullback of the counit $\eta^{\mathfrak{I}}_B$ along itself.

If $\mathfrak{I}$ is interpreted as identifying all infinitesimally close points, the fibre over each (global) point of the infinitesimal disk bundle can indeed be interpreted as the infinitesimal disk around that point (again by the pasting laws).

In analogy to **??** and **??**, one can also define local diffeomorphisms/étale morphisms between formal smooth sets.

Definition 5.3.17 (Formally étale morphism[4]). A morphism $f : X \to Y$ in a differentially cohesive topos $\mathbf{H}$ such that the naturality square of the shape modality is a

[3] Sometimes, the object $(\eta^{\mathfrak{I}}_B)_* E \to \mathfrak{I}B$ is called the jet bundle of $E \to B$.
[4] Also called a **local diffeomorphism**.

pullback square:

$$
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{pb}"] & \Im X \arrow[d, "\Im f"] \\
Y \arrow[r, "\eta_Y"'] & \Im Y\,.
\end{tikzcd}
$$

Definition 5.3.18 (Super-differential cohesion). Consider a differentially cohesive $(\infty,1)$-topos $\mathbf{H}_{\mathrm{bos}} \to \mathbf{H}_{\mathrm{red}}$. A super-differentially cohesive (or **solid**[5]) topos over $\mathbf{H}_{\mathrm{bos}}$ is a differentially cohesive $(\infty,1)$-topos $\mathbf{H} \to \mathbf{H}_{\mathrm{red}}$ equipped with an adjoint quadruple $(\mathsf{even} \dashv \iota_{\mathrm{sup}} \dashv \Pi_{\mathrm{sup}} \dashv \mathsf{Disc}_{\mathrm{sup}}) : \mathbf{H} \to \mathbf{H}_{\mathrm{bos}}$ such that ι_{sup} (and, hence, also $\mathsf{Disc}_{\mathrm{sup}}$) is fully faithful.

These structures give the following diagram of $(\infty,1)$-functors (again by uniqueness of adjoints and global sections):

$$
\begin{tikzcd}[column sep=huge]
\mathbf{H}
\arrow[r, shift left=12, "\mathsf{even}"]
\arrow[r, hookleftarrow, shift left=8, "\iota_{\mathrm{sup}}"]
\arrow[r, shift left=4, "\Pi_{\mathrm{sup}}"]
\arrow[r, hookleftarrow, "\mathsf{Disc}_{\mathrm{sup}}"]
\arrow[r, shift right=4, "\Gamma_{\mathrm{sup}}"]
& \mathbf{H}_{\mathrm{bos}}
\arrow[r, hookleftarrow, shift left=8, "\iota_{\mathrm{bos}}"]
\arrow[r, shift left=4, "\Pi_{\mathrm{bos}}"]
\arrow[r, hookleftarrow, "\mathsf{Disc}_{\mathrm{bos}}"]
\arrow[r, shift right=4, "\Gamma_{\mathrm{bos}}"]
& \mathbf{H}_{\mathrm{red}}
\arrow[r, shift left=4, "\Pi_{\mathrm{red}}"]
\arrow[r, hookleftarrow, "\mathsf{Disc}_{\mathrm{red}}"]
\arrow[r, shift right=4, "\Gamma_{\mathrm{red}}"]
& \infty\mathbf{Grpd}
\arrow[lll, hook', bend left, "\mathsf{coDisc}"]
\end{tikzcd}
\tag{5.16}
$$

Property 5.3.19 (Solid modalities). As for cohesive and differentially cohesive topoi, the adjoint functors induce a set of (adjoint) modalities:

$$
(\rightrightarrows \dashv \rightsquigarrow \dashv \mathsf{Rh}) := (\iota_{\mathrm{sup}} \circ \mathsf{even} \dashv \iota_{\mathrm{sup}} \circ \Pi_{\mathrm{sup}} \dashv \mathsf{Disc}_{\mathrm{sup}} \circ \Pi_{\mathrm{sup}})\,. \tag{5.17}
$$

These are called the **fermionic**, **bosonic** and **rheonomic** modalities, respectively. Note that, by Diagram 5.16, a super-differentially cohesive topos also inherits the cohesive and differential modalities.

@@ COMPLETE (e.g. work by Schreiber) @@

[5] This terminology stems from the fact that supergeometry is necessary for treating fermions, which are in turn responsible for condensed matter physics and, hence, the study of solid matter.

5.4 Smooth spaces

In this section, some generalizations of spaces that are better behaved when considering their properties as a whole are introduced. Before moving to the smooth setting, a bit of history will be given, starting from the ordinary topological setting.

The first problem in the study of the global properties of spaces arose in algebraic topology. When consider mapping spaces, it is sometimes useful to use the currying operation

$$C(X \times Y, Z) \to C\big(X, C(Y, Z)\big) . \tag{5.18}$$

However, in general, this is not a homeomorphism, i.e. currying does not define an adjunction and, therefore, **Top** is not Cartesian closed (**??**). This problem was treated by *Steenrod* and others, and the solution was simply to restrict to a smaller class of better behaved spaces: the compactly generated Hausdorff spaces.[6]

Whilst studying varieties in algebraic geometry, people experienced similar problems. For this reason, *Grothendieck* invented schemes (see **??** and **??** in particular). The main takeaway of this approach was that "it's better to work with a nice category containing some nasty objects, than a nasty category containing only nice objects" as *Baez* phrases it succinctly.

The category **Diff** of finite-dimensional smooth manifolds suffers the same problems, namely the space of smooth functions $C^\infty(X, Y)$ is, in general, some kind of infinite-dimensional manifold and, hence, cannot be defined internally. It becomes even worse if one studies the mapping spaces between those. *Kriegl* and *Michor* have introduced a framework in which one can work safely, but the main problem with their solution is that not all spaces of interest are included. Certain other operations such as quotients and (co)limits are also not guaranteed to exist within that category.

5.4.1 Frölicher spaces

Definition 5.4.1 (Frölicher space). A triple (X, C_X, F_X) consisting of the following data:

- a set X,
- a set of curves $C_X \subseteq \mathbf{Set}(\mathbb{R}, X)$, and
- a set of functionals $F_X \subseteq \mathbf{Set}(X, \mathbb{R})$.

These are required to satisfy the following closure and consistency conditions:

[6] In practice, this is not a problem since all interesting spaces, such as CW complexes, belong to this class.

1. For all $c \in C_X$ and $f \in F_X$, the composite is smooth: $f \circ c \in C^\infty(\mathbb{R})$.
2. If, given $c \in \mathbf{Set}(\mathbb{R}, X)$, the composite $f \circ c \in C^\infty(\mathbb{R})$ for all functionals $f \in F_X$, then $c \in C_X$.
3. If, given $f \in \mathbf{Set}(X, \mathbb{R})$, the composite $f \circ c \in C^\infty(\mathbb{R})$ for all curves $c \in C_X$, then $f \in F_X$.

Morphisms of Frölicher spaces are those functions that preserve curves, functionals and smoothness under composition.

Property 5.4.2. The category of Frölicher spaces is bicomplete and Cartesian closed.

Definition 5.4.3 (Topology). Consider a Frölicher space (X, C_X, F_X). The **curvaceous topology** on X is the final topology with respect to the curves C_X. The **functional topology** on X is the initial topology with respect to the functionals F_X.

@@ COMPLETE (e.g. Isbell enveloppe / duality) @@

5.4.2 Smooth sets

Definition 5.4.4 (Concrete site). Consider a site $(\mathbf{C}, J)$. It is said to be concrete if

1. the functor $\mathbf{C}(*, -) : \mathbf{C} \to \mathbf{Set}$ is faithful, and
2. for every covering family $\{f_i : U_i \to U\}_{i \in I} \in J(U)$, the morphism

$$\bigsqcup_{i \in I} \mathbf{C}(*, f_i) : \bigsqcup_{i \in I} \mathbf{C}(*, U_i) \to \mathbf{C}(*, U) \tag{5.19}$$

   is surjective.

Definition 5.4.5 (Concrete presheaf). Consider a concrete site $(\mathbf{C}, J)$. For every presheaf $X \in \mathbf{Psh}(\mathbf{C})$ and object $U \in \mathrm{ob}(\mathbf{C})$, denote by

$$X_U : X(U) \to \mathbf{Set}\big(\mathbf{C}(*, U), X(*)\big) \tag{5.20}$$

the adjunct of the restriction morphism

$$X(U) \times \mathbf{C}(*, U) \to X(*)\,. \tag{5.21}$$

The presheaf X is said to be concrete if, for each $U \in \mathrm{ob}(\mathbf{C})$, the map X_U is injective. This says that concrete presheaves are subobjects of presheaves of the form

$$U \mapsto \mathbf{Set}\big(\mathbf{C}(*, U), X(*)\big)\,. \tag{5.22}$$

This definition also agrees with Definition 5.3.10.

Property 5.4.6. The category of sheaves on a concrete site is a local topos (Definition 5.3.1). The right adjoint $\mathsf{coDisc} : \mathbf{Set} \to \mathbf{H}$ sends a set A to the functor $U \mapsto \mathbf{Set}\big(\mathbf{C}(*, U), A\big)$.

This property allows to define concrete objects in any local topos in an alternative way (by slightly modifying Property 2.2.10 to allow for separation instead of locality).

Definition 5.4.7 (Concrete object). Consider a local geometric morphism $\Gamma : \mathbf{H} \to \mathcal{S}$. Consider the class V of local isomorphisms with respect to Γ. An object in $\mathbf{H}$ is said to be concrete if it is V-separated, i.e. its Yoneda embedding maps morphisms in V to monos.

Remark 5.4.8. Note that concrete sheaves carry two kinds of sheaf-theoretic information. On the one hand they are actual sheaves with respect to the Grothendieck topology of the underlying site and, on the other hand, they are separated presheaves with respect to the topology generated by global elements $* \to U$.

Property 5.4.9 (Concretification). Consider a local 1-topos $\mathbf{H}$.[7] Every topos has an epi/mono factorization giving rise to the image $\mathrm{im}(f)$ of a morphism $f \in \mathrm{Hom}(\mathbf{H})$. The concretification of an object $X \in \mathrm{ob}(\mathbf{H})$ is given by the image $\mathrm{im}(\eta^\sharp_X)$.

In the case of the site of Cartesian spaces, the following notion is obtained.

Definition 5.4.10 (Diffeological space). A diffeology on a set X is a concrete sheaf $\mathcal{D}_X$ on $\mathbf{CartSp}_{\mathrm{diff}}$ such that $\Gamma\mathcal{D}_X = X$.

Alternative Definition 5.4.11 (Diffeological space). Let X be a set. A diffeology $\mathcal{D}_X$ on X is defined as a collection of functions $f : U \subseteq \mathbb{R}^n \to X$, called **plots**, satisfying the following conditions (where U, V and W are open sets):

1. $\mathcal{D}_X$ contains all constant functions.
2. If $\{U_i\}_{i\in I}$ is an open cover of U and if $f|_{U_i} \in \mathcal{D}_X$ for all $i \in I$, then $f \in \mathcal{D}_X$.
3. If $f \in \mathcal{D}_X$ and $g : W \subseteq \mathbb{R}^m \to \mathrm{dom}(f)$ is smooth, then $f \circ g \in \mathcal{D}_X$.

The set X can be turned into a topological space by equipping it with the **$\mathcal{D}_X$-topology**, the final topology with respect to $\mathcal{D}_X$.

Remark 5.4.12. Note that, in contrast to ordinary manifolds, the plots in a diffeology can have domains of different dimensions.

Definition 5.4.13 (Smooth map). Let $\mathcal{D}_X$ and $\mathcal{D}_Y$ be diffeological spaces. A map $g : X \to Y$ is said to be smooth if $g \circ f \in \mathcal{D}_Y$ for all $f \in \mathcal{D}_X$. The diffeological spaces together with their differentiable morphisms form a category $\mathbf{DiffSp}$.

[7] The restriction to ordinary topoi ensures that ordinary image factorization suffices. For higher topoi, this would not be the case and higher image factorizations would be required.

Definition 5.4.14 (Chen space). If the open sets in the definition of a diffeological space are replaced by convex sets, the notion of smooth spaces due to *Chen* is obtained.

Alternative Definition 5.4.15 (Manifold). A diffeological space that is locally diffeomorphic to a Euclidean space. A map between manifolds is smooth in the diffeological sense if and only if it smooth in the sense of **??**.

There exist two trivial smooth structures.

Example 5.4.16 (Discrete structure). The minimal smooth structure, i.e. the one obtained by taking the plots to be the constant functions.

Example 5.4.17 (Indiscrete structure). The maximal smooth structure, i.e. the one obtained by taking all functions to be plots.

The notion of diffeological spaces can be further generalized by passing to the full sheaf topos on Cartesian spaces.

Definition 5.4.18 (Smooth set).

$$\mathbf{SmoothSet} := \mathbf{Sh}(\mathbf{CartSp}_{\mathrm{diff}}) \tag{5.23}$$

The topology on this site is generated by the coverage of differentiably good covers (**??**). In fact, this topology coincides with the usual one consisting of open covers. The category of smooth sets is also denoted by $\mathbf{C}^\infty$.

To phrase this in the sense of Section 5.3, smooth sets form a cohesive topos over **Set**, where Disc and coDisc assign the discrete and codiscrete structures from the preceding examples, respectively. Moreover, this cohesive topos satisfies the *Nullstellensatz* 5.3.12.

Property 5.4.19. There exists an adjunction

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift right, "\mathsf{diff}"'] \arrow[r, phantom, "\perp"] & \mathbf{C}^\infty \arrow[l, shift right, "\mathsf{top}"'] \end{tikzcd}. \tag{5.24}$$

The right adjoint endows a topological space X with the smooth structure for which every continuous map $U \to X$ is a plot. Its left adjoint sends a smooth space to the topological space equipped with the finest topology for which all plots become continuous maps.

Property 5.4.20. The adjunction $\Pi_0 \dashv \mathrm{Disc}$ exhibits the discrete smooth sets, i.e. the constant sheaves, as the reflective subcategory on $\mathbb{R}$-local objects (see also **??**).

As noted above, the subcategory of diffeological spaces exactly consists of the concrete smooth sets, i.e. those smooth sets that have an underlying set of points. A common example of smooth sets that do not have such an underlying set is given by de Rham spaces.

Example 5.4.21 (Differential forms). Consider the k^{th} de Rham functor Ω^k on the category **Diff**. This functor assigns to every smooth manifold its space of differential k-forms. Locally defined forms can be glued together if they agree on intersections, i.e. they satisfy the sheaf condition. This shows that Ω^k defines a smooth set, albeit one that is far from an ordinary smooth manifold. It is called the **(universal) moduli space of differential k-forms**. Note, however, that for a given smooth set $X \in \mathbf{C}^\infty$, the plots of $[X, \Omega^k]$ correspond to differential k-forms on the product $\mathbb{R}^n \times X$ and not to $\mathbb{R}^n$-parametrized forms on X. For the latter, one should consider the concretification $\mathsf{conc}[X, \Omega^k]$.

One can go even further and characterize specific geometric structures in terms of this functor. Consider for example the subfunctor Ω^2_{cl} that assigns closed two-forms to a manifold. This also defines a smooth space and, hence, one can consider the slice category $\mathbf{C}^\infty/\Omega^2_{\text{cl}}$. It is not hard to show that the category **SpMfd** of symplectic manifolds admits an embedding into this slice category.

Property 5.4.22 (Universal differential forms). Consider the differential n-form

$$\omega_{\text{univ}} \in \Omega^n(\Omega^n) \tag{5.25}$$

modulated by the identity morphism on Ω^n, the universal differential n-form. Every differential n-form, on any smooth set, can be obtained as a pullback of ω_{univ}.

Example 5.4.23 (Path space). Since every topos is Cartesian closed, all internal homs exist. The path space of a smooth set X is defined as

$$\mathbf{P}X := [\mathbb{R}, X]\,. \tag{5.26}$$

Its underlying set is given by the set of smooth plots (trajectories): $\Gamma[\mathbb{R}, X] \cong \mathbf{C}^\infty(\mathbb{R}, X)$.

Analogously, the internal hom out of X gives the moduli space of smooth functions on X:

$$\mathbf{C}^\infty(X) := [X, \mathbb{R}]\,. \tag{5.27}$$

Definition 5.4.24 (Transgression). Consider a smooth set $X \in \mathsf{ob}(\mathbf{C}^\infty)$. Transgression of differential forms on X along a compact k-manifold Σ is defined as the composite:

$$\tau_\Sigma\omega := \int_\Sigma [\Sigma, \omega] : [\Sigma, X] \to [\Sigma, \Omega^n] \xrightarrow{\int_\Sigma} \Omega^{n-k}\,. \tag{5.28}$$

An alternative approach uses a pullback along the evaluation morphism $\mathsf{ev}_\Sigma : \Sigma \times [\Sigma, X] \to X$.

Property 5.4.25. If the compact k-manifold Σ has a boundary, transgression of closed differential forms along Σ and $\partial\Sigma$ is related by the de Rham differential:

$$\tau_{\partial\Sigma}(\omega) = (-1)^{k+1}\mathrm{d}\tau_\Sigma\omega \tag{5.29}$$

for all $\omega : X \to \Omega^n_{\text{cl}}$.

5.4.3 Smooth algebras

Definition 5.4.26 (Smooth algebra). For any smooth manifold M, the algebra of smooth functions can be obtained as a hom-object:

$$C^\infty(M) := C^\infty(M,\mathbb{R}) = \mathbf{Diff}(M,\mathbb{R})\,. \tag{5.30}$$

Since hom-functors are (finite) product-preserving, the multiplication $C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ can be seen to be induced by the multiplication on $\mathbb{R}$:

$$C^\infty(M,\mathbb{R}\times\mathbb{R}) \cong C^\infty(M)\times C^\infty(M)\,. \tag{5.31}$$

Furthermore, the hom-functor is covariant in the second argument and, hence, defines a copresheaf on the category $\mathbf{CartSp}_{\mathrm{diff}}$. Generalizing this situation, smooth algebras are defined as finite product-preserving copresheaves on $\mathbf{CartSp}_{\mathrm{diff}}$. This (functor) category is denoted by $\mathbf{C^\infty Alg}$.

Definition 5.4.27 (Underlying algebra). Given a smooth algebra $R \in \mathbf{C^\infty Alg}$, its underlying algebra $U(R)$ is defined as the set $R(\mathbb{R})$ equipped with the canonically induced ring operations.

Definition 5.4.28 (Finitely generated smooth algebra). Since ordinary R-algebras are finitely generated if and only if they are of the form $R[x_1,\dots,x_k]/I$ for some integer $k\in\mathbb{N}$ and some ideal I, a smooth algebra is said to be finitely generated if it is of the form $C^\infty(\mathbb{R}^n)/I$ for some $n\in\mathbb{N}$ and some ideal I in the underlying algebra.

Definition 5.4.29 (Smooth locus). Let $\mathbf{C^\infty Alg^{fin}}$ denote the category of finitely generated smooth algebras. The category of **smooth loci** is defined as $(\mathbf{C^\infty Alg^{fin}})^{\mathrm{op}}$. The smooth locus corresponding to a smooth algebra R is often denoted by ℓR.

5.4.4 Supergeometry

In this section, the definition of smooth spaces (and sets) is generalized to the odd (fermionic) sector, i.e. 'super smooth sets' will be defined. This gives an explicit example of a super-differentially cohesive $(\infty,1)$-topos.

Definition 5.4.30 (Superscheme). The category of affine superschemes is defined as the opposite of the category of supercommutative superalgebras, the commutative monoids internal to **sVect**:

$$\mathbf{Aff}(\mathbf{sVect}) := \mathbf{sCAlg}^{\mathrm{op}} := \mathbf{CMon}(\mathbf{sVect})\,. \tag{5.32}$$

More generally, one can define affine schemes internal to any symmetric monoidal category.

Definition 5.4.31 (Infinitesimally thickened space). First, consider a point $\mathbb{R}^0$. Its infinitesimal thickening should be a space such that every function that vanishes at the origin is actually nilpotent (this is essentially a version of the Kock–Lawvere axiom **??**):

$$\mathbb{D} := \mathsf{Spec}(A)\,, \tag{5.33}$$

where $A := \mathbb{R} \oplus V$ for V a finite-dimensional nilpotent ideal.[8] A Euclidean space can be infinitesimally thickened by taking the product with $\mathbb{D}$ or, at the algebraic level, by taking the tensor product with A. A morphism of such spaces is defined by an $\mathbb{R}$-algebra morphism between their associated algebras. These form the category $\mathbf{FormalCartSp}_{\mathrm{diff}}$.

Definition 5.4.32 (Superpoint). A space of the form $\mathsf{Spec}(\Lambda^\bullet\mathbb{R}^n)$. This is often denoted by $\mathbb{R}^{0|n}$. The **super-Euclidean space** $\mathbb{R}^{m|n}$ is obtained as the product of an ordinary Euclidean space $\mathbb{R}^m$ and the superpoint $\mathbb{R}^{0|n}$, i.e. its algebra of smooth functions is $C^\infty(\mathbb{R}^m) \otimes \Lambda^\bullet\mathbb{R}^n$.[9]

Example 5.4.33 (First-order neighbourhood). For $A = \mathbb{R}[\varepsilon]/\varepsilon^2$, the definition of an infinitesimal thickening recovers the first-order infinitesimal neighbourhood of **??**:

$$\mathbb{D}^1 := \mathsf{Spec}(\mathbb{R}[\varepsilon]/\varepsilon^2)\,. \tag{5.34}$$

The morphism dual to the mapping implied by the Kock–Lawvere axiom **??** gives an inclusion map $\mathbb{D}^1 \hookrightarrow \mathbb{R}^1$. (This example can easily be generalized to k^{th}-order neighbourhoods.) This algebra is the algebra of functions on the even part of the superplane:

$$\mathcal{O}\left(\mathsf{even}\,\mathbb{R}^{0|2}\right) \cong \mathbb{R}[\varepsilon]/\varepsilon^2\,. \tag{5.35}$$

Property 5.4.34 (Morphisms). First, consider the morphisms from a Euclidean space into an infinitesimal neighbourhood $\mathbb{D}^k$. Since such morphisms are dual to algebra morphisms, one should consider morphisms of the form $\mathbb{R}[\varepsilon]/\varepsilon^{k+1} \to C^\infty(\mathbb{R}^n)$. However, being an algebra morphism implies that $f(1) = 1$ and that nilpotents are mapped to nilpotents. The algebra of smooth functions on a Euclidean space does not contain nilpotents and, hence, there exists a unique function into an infinitesimal neighbourhood (the one that factorizes through the one-point set).

For morphisms out of (first-order) infinitesimal neighbourhoods, one obtains the property, known from synthetic geometry, that morphisms of the form $\mathbb{R}^n \times \mathbb{D}^1 \to \mathbb{R}^n$ correspond to vector fields on $\mathbb{R}^n$ exactly when the postcomposition with the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \mathbb{D}^1$ is the identity.

Definition 5.4.35 (Formal smooth set). A sheaf on the site of infinitesimally thickened Euclidean spaces, where covers are of the form $\{U_i \times \mathsf{Spec}(A) \mid U_i \subseteq \mathbb{R}^n\}$, for A a commutative (hence, even) algebra. The category of formal smooth sets or, equivalently,

[8] Algebras of this form are also called **Weil algebras** or **local Artin algebras**.

[9] Compare this to the definition of supermanifolds (Example 5.1.1).

the sheaf topos on $\mathbf{FormalCartSp_{diff}}$ is also called the **Cahiers topos**. The sets in the image of a formal smooth set X are called the sets of **plots** of X and can be interpreted as sets of functions into X (in analogy with the definition of smooth sets).

Definition 5.4.36 (Super smooth set). By combining infinitesimal thickenings and superpoints, the category $\mathbf{SuperFormalCartSp_{diff}}$ is obtained. A sheaf on this category is called a super smooth set. Since *Dubuc* introduced the Cahiers topos, the category of super smooth sets is sometimes called the **super-Dubuc topos**.

Property 5.4.37 (Cohesion). The full subcategory inclusion $\mathbf{CAlg} \hookrightarrow \mathbf{sCAlg}$ is part of an adjoint cylinder, where the left and right adjoints are given by projection onto the even part and quotienting out the odd part:

$$\mathrm{even} \dashv \iota \dashv -/(-)_{\mathrm{odd}}\,, \tag{5.36}$$

where $(-)_{\mathrm{odd}}$ indicates that the ideal generated by odd elements is quotiented out, i.e. every term containing at least one odd factor is set to zero. These define canonical functors, even and Π_{sup}, from the category of affine superschemes to affine schemes (and, hence, also between the larger categories of superschemes and schemes). Moreover, the inclusion of affine schemes into infinitesimally thickened spaces also admits a right adjoint given by reduction: $\Pi_{\mathrm{inf}}(\mathbb{R}^n \times \mathbb{D}) := \mathbb{R}^n$.

These, in turn, induce a system of ajunctions between the sheaf topos of smooth sets, the Cahiers topos and the super-Dubuc topos, characterizing them as a cohesive, elastic and solid topoi, respectively:

$$\begin{tikzcd}[column sep=huge]
\begin{array}{c}\textbf{Super}\\ \textbf{Smooth}\\ \textbf{Set}\end{array}
\arrow[r, shift left=9, "\mathrm{even}"]
\arrow[r, leftarrow, hook', shift left=6, "\iota_{\mathrm{sup}}"]
\arrow[r, shift left=3, "\Pi_{\mathrm{sup}}"]
\arrow[r, leftarrow, hook', "\mathrm{Disc}_{\mathrm{sup}}"]
\arrow[r, shift right=3, "\Gamma_{\mathrm{sup}}"']
\arrow[rrr, leftarrow, hook', shift right=7, "\mathrm{coDisc}"']
& \begin{array}{c}\textbf{Formal}\\ \textbf{Smooth}\\ \textbf{Set}\end{array}
\arrow[r, leftarrow, hook', shift left=6, "\iota_{\mathrm{inf}}"]
\arrow[r, shift left=3, "\Pi_{\mathrm{inf}}"]
\arrow[r, leftarrow, hook', "\mathrm{Disc}_{\mathrm{inf}}"]
\arrow[r, shift right=3, "\Gamma_{\mathrm{inf}}"']
& \mathbf{C}^\infty
\arrow[r, shift left=3, "\Pi_{\mathrm{red}}"]
\arrow[r, leftarrow, hook', "\mathrm{Disc}_{\mathrm{red}}"]
\arrow[r, shift right=3, "\Gamma_{\mathrm{red}}"']
& \mathbf{Set}
\end{tikzcd} \tag{5.37}$$

The top four functors in every column are obtained by Kan extension from the functors between affine spaces (Property 1.4.93). The others follow from uniqueness and composition properties of adjoints and the fact that the coverages on $\mathbf{SuperSmoothSet}$ and $\mathbf{FormalSmoothSet}$ are trivial along infinitesimal thickenings and odd dimensions.

By the general theory of cohesive topoi, the diagram of adjunctions induce a another diagram of adjoint modalities. These are listed below.

Definition 5.4.38 (Fermionic modalities). The adjoint cylinder induced by the inclusion **CAlg** $\hookrightarrow$ **sCAlg** in turn gives rise to the adjoint modalities

$$\overset{\rightrightarrows}{(-)} \dashv \overset{\rightsquigarrow}{(-)}\,. \tag{5.38}$$

The left adjoint sends a super smooth set to its 'bifermionic' space, where only paired fermions occur. The right adjoint sends a super smooth set to its underlying bosonic space and might, therefore, be called the **bosonic modality**.

Moreover, since these modalities come from an adjoint quadruple, a further right adjoint exists. The rheonomy modality corresponds to localization at the superpoint:

$$\mathsf{Rh} \cong L_{\mathbb{R}^{0|1}}\,, \tag{5.39}$$

akin to Property 5.4.20.

The following definition is dual to **??**.

Definition 5.4.39 (Reduction). By Property 5.3.15, **SuperSmoothSet** admits differential modalities to both **FormalSmoothSet** and **SmoothSet**. The reduction modality $\Re$ simply drops all infinitesimal directions, both even- and odd-graded:

$$\Re(\mathbb{R}^{m|n} \times \mathbb{D}) = \mathbb{R}^m\,. \tag{5.40}$$

The **infinitesimal neighbourhood** (to arbitrary order) of a formal smooth subset $Y \hookrightarrow X$ is defined by taking its plots to be those plots of X for which the reductions factorize through plots of Y.

Definition 5.4.40 (Shape modality). The **(infinitesimal) shape** or **de Rham shape** $\Im X$ of a super smooth set X is given, throught duality/adjointness, by the super smooth set obtained by reducing the plots of X:

$$\Im X(U) := X(\Re(U))\,. \tag{5.41}$$

This construction identifies all infinitesimally close points.

Definition 5.4.41 (Tangent bundle). Consider a super smooth set X. Its tangent bundle is given by the internal hom out of the infinitesimal disk:

$$TX := [\mathbb{D}^1, X]\,. \tag{5.42}$$

Precomposition by the unique global element of $\mathbb{D}^1$ gives the bundle projection $TX \to X$. Analogously, the internal hom out of the superpoint gives the odd tangent bundle (see also Definition 5.1.3):

$$\Pi TX := [\mathbb{R}^{0|1}, X]\,. \tag{5.43}$$

Definition 5.4.42 (Local diffeomorphism[10]). Recall Definition 5.3.17. In the setting of super smooth sets, this is equivalent to a morphism $f : X \to Y$ such that the thickened plots of X can be identified with those of Y whose reduction comes from a Euclidean plot of X. Equivalently, by taking internal homs, a morphism of super smooth sets is formally étale if and only if

$$\begin{tikzcd}
{[\mathbb{D}, X]} \arrow[r] \arrow[d, "{[\mathbb{D}, f]}"'] \arrow[dr, phantom, "\mathrm{pb}"] & X \arrow[d, "f"] \\
{[\mathbb{D}, Y]} \arrow[r] & Y
\end{tikzcd}$$

is a pullback square for all infinitesimally thickened (super)points. By taking $\mathbb{D} = \mathbb{D}^1$, the Inverse Function Theorem **??** can be recovered.

This can also be related to the definition in commutative algebra (and dually, that in algebraic geometry) as follows. The shape modality is right adjoint to the reduction modality. Sending a ring (extension) to its representable presheaf and using the Yoneda lemma gives the diagram

$$\begin{tikzcd}
\mathbf{CRing}(A, B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & \mathbf{CRing}(A, B/I) \arrow[d] \\
\mathbf{CRing}(R, B) \arrow[r] & \mathbf{CRing}(R, B/I)
\end{tikzcd}$$

This square being a pullback exactly corresponds to the lifting condition in **??**.

Property 5.4.43. Local diffeomorphisms are preserved by the bosonic modality. If f is formally étale, so is $\overset{\rightsquigarrow}{f}$.

Alternative Definition 5.4.44 (Smooth manifold). A diffeological space (in its incarnation as a formal smooth set) equipped with a family of local diffeomorphisms from Euclidean spaces (also regarded as formal smooth sets) such that every point of the space lies in the image of at least one such morphism (and such that the final topology induced by the plots of the smooth set is paracompact Hausdorff).

Definition 5.4.45 (V-manifold). Consider a group object V in **SuperSmoothSet**. A V-manifold is a super smooth set X equipped with a span

$$\begin{tikzcd}
 & U \arrow[dl, "\text{ét}"'] \arrow[dr, "\text{ét, epi}"] & \\
V & & X
\end{tikzcd}$$

Definition 5.4.46 (Supermanifold). A supermanifold of **(super)dimension** $(m \mid n)$ is a super smooth set X equipped with a formally étale epimorphism $\sqcup_{i \in I} \mathbb{R}^{m|n} \to X$ for some index set I.

Using the notion of supermanifold, one can also define super fibre bundles (and other common notions from differential geometry). Given such a bundle $E \to M$, its (super)space of sections is the fibre product

$$\Gamma(E) := [M, E] \times_{[M,M]} \mathcal{Y}\{\mathbb{1}_M\}\,.$$

Property 5.4.47. By Property 5.4.43 above, the bosonic space underlying a V-manifold is a $\overset{\leadsto}{V}$-manifold.

5.5 Higher Lie theory

In this section, some notions about groups, Lie groups and groupoids (**??** **??**, **??** and 1.5.1) are extended to the setting of higher category theory.

5.6 Gauge theory ♣

Recall the notions of Chapter 2, in particular the notions of stacks and higher topoi. The $(\infty, 1)$-category of smooth ∞-stacks can be described in terms of the (left Bousfield) localization of a suitable presheaf category by Lurie's theorem **??**.

The first possibility is the category of ∞-presheaves on **Diff** with the localization at open covers. The second possibility is the dense subsite $\mathbf{CartSp}_{\mathrm{diff}}$ with localization at good open covers. Both will result in a Čech model structure (**??**). However, the specific properties will differ.

Example 5.6.1 (Classifying stacks). Consider the example of a Lie group G and its classifying stack $\mathbf{B}G$. In the first model structure, the mapping space $\mathbf{H}(M, \mathbf{B}G)$, for M a smooth manifold, is simply presented[11] by $\mathrm{Hom}(M, \mathbf{B}G)$, since M is cofibrant as a representable presheaf and $\mathbf{B}G$ is fibrant by gluing over covers. So, mapping spaces $\mathbf{H}(M, \mathbf{B}G)$ are just given by groupoids of G-bundles over M.

On the subsite $\mathbf{CartSp}_{\mathrm{diff}}$, the presheaves represented by manifolds are not cofibrant anymore. However, Čech nerves of open covers give a cofibrant replacement. On the other hand, over Cartesian spaces, the stacks are trivial and can be presented as action groupoids $*/\!\!/G$ (the ordinary deloopings). A fibrant replacement is given by the

[11] This means the homotopy-invariant hom-object in the underlying presheaf category, where the domain is replaced by a cofibrant object and the codomain by a fibrant object.

presheaf

$$U \mapsto N_\Delta\big(*/\!\!/C^\infty(U,G)\big)\,. \tag{5.44}$$

This presheaf is also equivalent to the groupoid of G-bundles (over U). The derived mapping space in this situation is given by (normalized) G-valued Čech cocycles.

5.6.1 Lie groupoids

Definition 5.6.2 (Lie groupoid[12]). A groupoid internal to **Diff**.

Remark 5.6.3. Note that Definition 1.5.2 requires the existence of pullbacks. In the category **Diff**, this is equivalent to assuming that the source and target morphisms are (surjective) submersions.

In the Ehresmannian approach, the manifold of composable morphisms $D_1 \times_{D_0} D_1$ is given as part of the data. Hence, no further assumptions have to be made about the source and target morphisms.

Definition 5.6.4 (Lie algebroid). A vector bundle $\pi : E \to M$ together with a morphism $\rho : E \to TM$, called the **anchor map**, and a Lie bracket on $\Gamma(E)$ such that the following Leibniz-type property is satisfied:

$$[X, fY] = f[X,Y] + \rho(X)(f)Y\,. \tag{5.45}$$

This property also implies that ρ preserves the Lie bracket:

$$\rho([X,Y]) = [\rho(X), \rho(Y)]\,. \tag{5.46}$$

In local coordinates x^i and for a local basis of sections s_α, the bracket and anchor can be expressed in terms of structure functions:

$$\begin{aligned}\rho(s_\alpha) &= R^i_\alpha \partial_i\,,\\ [s_\alpha, s_\beta] &= C^\gamma_{\alpha\beta}(x)s_\gamma\,.\end{aligned} \tag{5.47}$$

The Lie algebroid properties then imply the following conditions on these structure functions:

$$R^j_\alpha \frac{\partial R^i_\beta}{\partial x^j} - R^j_\beta \frac{\partial R^i_\alpha}{\partial x^j} = R^i_\gamma C^\gamma_{\alpha\beta} \tag{5.48}$$

and

$$R^i_\alpha \frac{\partial C^\kappa_{\beta\gamma}}{\partial x^i} + C^\kappa_{\alpha\mu}C^\mu_{\beta\gamma} + (\alpha \leftrightarrow \beta \leftrightarrow \gamma) = 0\,. \tag{5.49}$$

[12] In a similar way one could define *topological groupoids*, *étalé groupoids*, ...

Example 5.6.5 (Tangent Lie algebroid). The tangent bundle over a smooth manifold is a Lie algebroid with $\rho = \mathbb{1}_{TM}$.

Consider the **pair groupoid** or **codiscrete groupoid** $\mathsf{M} \times \mathsf{M}$:

- **Objects**: M, and
- **Morphisms**: $M \times M$, i.e. between every two points there exists a unique morphism.

Both the fundamental groupoid $\boldsymbol{\Pi}_1(M)$ of **??** and the pair groupoid $\mathsf{M} \times \mathsf{M}$ *integrate* the tangent Lie algebroid.

@@ ADD LIE INTEGRATION OF ALGEBROIDS @@

One can generalize the dual construction of L_∞-algebras even further.

Definition 5.6.6 (L_∞-algebroid). Consider **??** of the Chevalley–Eilenberg algebra for an L_∞-algebra. By replacing the base field by a smooth algebra $C^\infty(M)$ for some smooth manifold M and the (graded) vector space V by a module of sections $\Gamma(E)$ of a (graded) vector bundle $E \to M$, one obtains the notion of a L_∞-algebroid.

Property 5.6.7. L_∞-algebras can be recovered by considering the special case $M = \{*\}$.

Example 5.6.8 (de Rham complex). Consider the tangent algebroid of a smooth manifold M. The associated Chevalley–Eilenberg complex is equivalent to the de Rham complex $\Omega^\bullet(M)$.

5.7 Weak groups

Definition 5.7.1 (Weak 2-group). Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a monoidal category. This category is called a weak 2-group or **categorical group** if it satisfies the following conditions:

1. All morphisms are invertible.
2. Every object is weakly invertible with respect to the monoidal structure.

By **??**, one can equivalently define a weak 2-group as a 2-category with a single object, weakly invertible 1-morphisms and invertible 2-morphisms.

The definition of a weak 2-group can be strengthened to that of a **coherent 2-group** (sometimes called a **gr-category**), where the isomorphisms $x \otimes x^{-1} \to \mathbf{1}$ and $x^{-1} \otimes x \to \mathbf{1}$ glue together to form an adjoint equivalence.

Example 5.7.2 (Automorphism 2-group). Consider a 2-category $\mathbf{C}$. For every object $c \in \mathrm{ob}(\mathbf{C})$, the automorphism category $\mathbf{Aut}(c)$ of autoequivalences of c and invertible 2-morphisms between them forms a coherent 2-group.[13]

Definition 5.7.3 (2-groupoid). A 2-groupoid is a 2-category in which all 1-morphisms are invertible and every 2-morphisms has a 'vertical' inverse. (The 'horizontal' inverse can be constructed from the other ones.)

Definition 5.7.4 (Strict 2-group). A (strict) 2-group is defined as a (strict) 2-groupoid with only one object. From this it follows that the set of 1-morphisms forms a group and so does the set of 2-morphisms under horizontal composition. However, the 2-morphisms do not form a group under vertical composition because the sources/targets may not match.

This definition is equivalent to the following internal version. A (strict) 2-group is a group object in $\mathbf{Cat}$ or an internal category in $\mathbf{Grp}$. If $\mathbf{Grp}$ is replaced by $\mathbf{Lie}$, the notion of a (strict) Lie 2-group is obtained.

Definition 5.7.5 (∞-groupoid). A ∞-category in which all morphisms are invertible. This is equivalent to a $(\infty, 0)$-category in the language of (n, r)-categories.

Property 5.7.6 (Lie crossed modules). The 2-category of (strict) 2-groups is biequivalent to the 2-category of (Lie) crossed modules (**??**). Given a 2-group $\mathcal{G}$, a crossed module is obtained as follows:

- $G := \mathrm{ob}(\mathcal{G})$,
- $H := \{h \in \mathrm{hom}(\mathcal{G}) \mid \mathfrak{s}(f) = e\}$,
- $t(h) := \mathfrak{t}(h)$, and
- $\alpha(g)h := \mathbb{1}_g h \mathbb{1}_g^{-1}$,

where $\mathfrak{s}, \mathfrak{t}$ are the source and target morphisms in $\mathcal{G}$.

To every Lie crossed module one can also assign a **differential crossed module**. This consists of the following data:

- two Lie algebras $\mathfrak{g}, \mathfrak{h}$,
- a Lie algebra morphism $\partial : \mathfrak{h} \to \mathfrak{g}$, and
- a Lie algebra morphism $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$.

The equivariance and Peiffer conditions induce similar conditions for the above data:

- $\partial(\rho(h)g) = [h, \partial g]$, and

[13] Relaxing the objects to weakly invertible 1-morphisms gives a weak 2-group.

- $\rho(\partial h)(h') = [h, h']$,

where $g \in \mathfrak{g}$ and $h, h' \in \mathfrak{h}$. The biequivalence of crossed modules and strict 2-groups induces a biequivalence of differential crossed modules and strict Lie 2-algebras.

Example 5.7.7 (Automorphism 2-group). Given a Lie group H, one can construct a crossed module with $G := \mathrm{Aut}(H)$, t assigning inner automorphisms (conjugations) and α the obvious map. The associated 2-group $\mathbf{Aut}(H)$ is the automorphism 2-group of H of the delooping $\mathbf{B}H$.

Definition 5.7.8 (Exponentiable group). A smooth group for which every smooth function $f : [0,1] \to \mathfrak{g}$ corresponds to a smooth function $g : [0,1] \to G$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = f(t) g(t) \tag{5.50}$$

with $g(0) = e$. A smooth 2-group is said to be exponentiable if both of its component groups are exponentiable. Since all Lie groups are exponentiable, all Lie 2-groups are also exponentiable

Remark 5.7.9 (Lie's third theorem). In ordinary Lie theory, Lie's third theorem states that every (finite-dimensional) Lie algebra can be obtained as the infinitesimal version of a Lie group. However, this does not carry over to the 2-group setting. Consider, for example, the Lie 2-algebras $\mathfrak{g}_\lambda$ constructed in **??**. As shown by Baez and Lauda (2003), only $\mathfrak{g}_0$ gives rise to a Lie 2-group (or even a topological 2-group). This remark also extends to the setting of Lie algebroids.

5.7.1 Spaces

To overcome the problem encountered in Definition 5.6.2 (see the subsequent remark), one should pass from **Diff** to $\mathbf{C}^\infty$. It can be shown that this category admits all pullbacks, quotients, path spaces, etc.

Definition 5.7.10 (Smooth 2-space). A category internal to $\mathbf{C}^\infty$.

In the remainder of this, chapter all spaces will be assumed to be smooth in this generalized sense. The notions of 2-groups as introduced in the previous section are easily generalized to this setting.

Definition 5.7.11 (2-group action). Consider a smooth 2-group G and a smooth 2-space E. A strict action of G on E is a smooth morphism $G \to \mathbf{Aut}(E)$, i.e. a smooth (ana)functor preserving products and inverses.

Definition 5.7.12 (Thin homotopy). Let M be a smooth manifold. A smooth homotopy $H : [0,1]^2 \to M$ is said to be thin if

$$H(s,t) = F(s) \tag{5.51}$$

for some smooth F near $t = 0, 1$ and if it pulls back every two-form to 0:

$$\forall \omega \in \Omega^2(M) : H^*\omega = 0 . \tag{5.52}$$

Definition 5.7.13 (Lazy path). Let M be a smooth manifold. A path $f : [0,1] \to M$ is said to be lazy or **to have sitting instants** if it is locally constant on some neighbourhoods of 0 and 1.

Definition 5.7.14 (Path groupoid). Let M be a smooth space. The path groupoid $\mathcal{P}_1(M)$ is the smooth groupoid consting of the following data:

- **Objects**: M, and
- **Morphisms**: thin homotopy classes of lazy paths with fixed endpoints on M.

The laziness combined with the first condition of thin homotopies implies that the morphisms of this groupoid are (locally) constant near the boundary of their domain.

In fact, by suitably generalizing the smoothness properties of the homotopies and paths, one can extend this definition to surfaces, volumes and so on. This results in the n-path n-groupoid $\mathcal{P}_n(M)$.

Remark 5.7.15. The restriction to lazy paths is required to ensure the smoothness of composite paths. The quotient by thin homotopies is required to ensure the validity of the associativity and invertibility properties.

The following definition generalizes **??**.

Definition 5.7.16 (Algebraic stack). A stack $X \in \mathbf{Sh}_{(2,1)}(\mathbf{Sch}_{\text{fppf}})$ on the big fppf-site such that

1. The diagonal $\Delta_X : X \to X \times X$ is representable by an algebraic space (**??**).
2. There exists a scheme $S \in \mathbf{Sch}$ and a morphism $h_S \to X$ that is surjective and smooth.

If the covering morphism is étale and not just smooth, the notion of **Deligne–Mumford stacks** is obtained. To distinguish these cases, algebraic stacks are sometimes called **Artin stack**.

@@ COMPLETE @@

5.8 2-Bundles

A first step is the generalization of the categorical definition of a general bundle (**??**), i.e. as an object of a slice category.

Definition 5.8.1 (Smooth 2-bundle). A triple (E, B, π) where both E and B are smooth 2-spaces and π is a smooth map.

Definition 5.8.2 (Locally trivial 2-bundle). A locally trivial 2-bundle with typical fibre F over a smooth 2-space B is defined as a 2-bundle (E, B, π) with an open cover $\{U_i\}_{i\in I}$ of B such that for every $i \in I$ there exists an equivalence $\varphi_i : E|_{U_i} \cong U_i \times F$ that makes the diagram below commute:

$$\begin{tikzcd} E|_{U_i} \arrow[rr, "\varphi_i"] \arrow[dr, "\pi"'] & & U_i \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd}$$

It should be noted that the existence of such a cover is not a trivial matter. The general definition becomes quite involved when allowing for arbitrary smooth 2-spaces B. For convenience, it will always be assumed that B is an ordinary smooth space regarded as a 2-space with only trivial morphisms.

As was the case in **??**, one can also characterize locally trivial 2-bundles by their transition data. Since the trivilizations φ_i are equivalences, they admit an inverse (up to an invertible 2-map) and one can thus construct transition maps $\varphi_i \circ \varphi_j^{-1} = U_{ij} \times F \cong U_{ij} \times F$ as usual. By the commutative diagram above, these transition maps only act on the fibre F. Because $\varphi_i \circ \varphi_j^{-1}$ is itself an (auto)equivalence, the action on F is given by a functor $g_{ij} : U_{ij} \to \mathbf{Aut}(F)$, where the 2-space $\mathbf{Aut}(F)$ is the (coherent) automorphism 2-group of F.

The interesting (and important) part is how the cocycle conditions (**??** and **??**) for the maps g_{ij} are modified. Since the equivalences g_{ij} are only invertible up to 2-morphisms, one cannot expect these conditions to hold as equations. Instead, two higher transition maps (i.e. natural isomorphisms) $h_{ijk} : g_{ij} \circ g_{jk} \Rightarrow g_{ik}$ and $k_i : g_{ii} \Rightarrow \mathrm{id}_F$ are obtained. These higher data should in turn satisfy the necessary conditions coming from associativity and unitality constraints (similar to the coherence conditions from **??**).

Definition 5.8.3 (G-bundle). A locally trivial 2-bundle with typical fibre F is said to have the 2-group G as its structure (2-)group if the transition data factor through an action $G \to \mathbf{Aut}(F)$. If $F = G$, the 2-bundle is called a **principal G-2-bundle**.

Remark 5.8.4 (Gerbes). If the transition maps k_i are chosen to be trivial and G is chosen to be, respectively, the trivial Lie 2-group associated to an Abelian Lie group G or the automorphism 2-group of a Lie group H, one obtains Abelian and non-Abelian *gerbes*. In fact, it can be shown that the 2-category of principal 2-bundles is equivalent to the 2-category of gerbes for every Lie 2-group of the aforementioned type.

By categorifying **??** of principal connections, one can define connections for principal n-bundles.

Definition 5.8.5 (n-connection). Let M be a smooth space and let G be a Lie n-groupoid. Given a locally trivial principal n-bundle P over M, an n-connection with n-holonomy is defined by the following data:

- for every coordinate chart $U_i \subseteq M$ a local holonomy n-functor

$$\mathsf{hol}_i : \mathcal{P}_n(U_i) \to G\,, \tag{5.53}$$

- for every double intersection U_{ij} a 1-transfor (**??**)

$$g_{ij} : \mathsf{hol}_i \Rightarrow \mathsf{hol}_j\,, \tag{5.54}$$

- for every triple intersection U_{ijk} a 2-transfor

$$f_{ijk} : g_{ij} \circ g_{jk} \Rrightarrow g_{ik}\,, \tag{5.55}$$

- and so on ...

This is equivalently given by a global n-functor

$$\mathsf{hol} : \mathcal{P}_n(M) \to \mathbf{Trans}_n(P)\,. \tag{5.56}$$

@@ ADD GERBES (e.g. BRYLINSKI) @@

5.9 Space and quantity

In this section, the general notions of spaces and observables are reconsidered. From the start, everything will be formulated in an enriched setting, where $\mathcal{V}$ is a cosmos (**??**). The categories of interest will also be assumed to be small.

In Section 5.4, spaces modelled on a base space or, more generally, on a category of spaces were presented as (concrete) sheaves on a suitable site. Here, this notion is relaxed as much as possible.

Definition 5.9.1 (Space). A (generalized) space modelled on a category $\mathbf{C}$ is a presheaf on $\mathbf{C}$.

As before, the object $X(C)$ can be interpreted as the collection of 'probes' from C to X. The Yoneda lemma assures that ordinary test spaces in $\mathbf{C}$ can be viewed as spaces modelled on $\mathbf{C}$ and that their probes are indeed the ordinary maps in $\mathbf{C}$.

In a similar vein, one can define observables as maps out of a space.

Definition 5.9.2 (Quantity). A (generalized[14]) quantity on a category $\mathbf{C}$ is a copresheaf on $\mathbf{C}$.

Property 5.9.3 (Isbell duality). Given a space X, one can look at the quantities that live on it (in ordinary geometry this would have been its algebra of functions). This defines a functor:

$$\mathcal{O} : \mathbf{Psh}(\mathbf{C}) \to \mathbf{coPsh}^{\mathrm{op}}(\mathbf{C}) : X \mapsto \mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(X, \mathcal{Y}-)\,. \tag{5.57}$$

Similarly, given a quantity Q one can ask on which space it behaves as the algebra of functions. This also defines as functor:

$$\mathrm{Spec} : \mathbf{coPsh}^{\mathrm{op}}(\mathbf{C}) \to \mathbf{Psh}(\mathbf{C}) : Q \mapsto \mathrm{Hom}_{\mathbf{coPsh}(\mathbf{C})}(\mathcal{Y}^{\mathrm{op}}-, Q)\,, \tag{5.58}$$

where $\mathcal{Y}^{\mathrm{op}}$ denotes the co-Yoneda embedding $\mathbf{C} \to [\mathbf{C}, \mathcal{V}]^{\mathrm{op}} : c \mapsto \mathbf{C}(c, -)$.

The incredible result is now that $(\mathcal{O} \dashv \mathrm{Spec})$ is an adjunction, called the **Isbell adjunction**. Objects that are preserved (up to isomorphism) under the associated (co)monad are said to be **Isbell selfdual**.

Example 5.9.4 (Cartesian spaces). When working over the site **CartSp** (with its usual topology) and restricting to coherent sheaves and product-preserving presheaves, the Isbell adjunction maps spaces to smooth algebras.

[14]It is generalized because it 'measures' a category instead of a single object.

Part IV

Quantum Theory

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Chapter 6

Quantum Mechanics

The main reference for this chapter is Bransden and Joachain (2000). In the first two sections, the two basic formalisms of quantum mechanics are introduced: wave and matrix mechanics. The main reference for the mathematically rigorous treatment of quantum mechanics, in particular in the infinite-dimensional setting, is Moretti (2016). The main reference for the generalization to curved backgrounds is Schuller (2016). The section on the WKB approximation is based on Bates and Weinstein (1997). Relevant chapters in this compendium are, amongst others, **??**, **??** and 4.

Chapter Contents

6.1 Introduction

This section will give both an introduction and formal treatment of the objects and notions used in quantum mechanics.

6.1.1 Dirac–von Neumann postulates

Axiom 6.1 (States). The states of a (closed) system are represented by vectors in a (complex) Hilbert space $\mathcal{H}$. In the infinite-dimensional setting, one often further restricts to separable spaces, i.e. the spaces are required to admit a countable Hilbert basis.

Notation 6.1.1 (Dirac notation). State vectors $|\psi\rangle$ are called **ket**'s and their duals $\langle\psi|$ are called **bra**'s. The inner product of a state $|\phi\rangle$ and a state $|\psi\rangle$ is denoted by $\langle\phi\,|\,\psi\rangle$. This notation is often called the **braket notation** (or Dirac notation).

Axiom 6.2 (Observables). Every physical property is represented by a bounded, self-adjoint operator. In the finite-dimensional case, this is equivalent to an operator that admits a complete set of eigenfunctions.

Definition 6.1.2 (Compatible observables). Two observables are said to be compatible if they share a complete set of eigenvectors.

Formula 6.1.3 (Closure relation). For a complete set of eigenvectors, the closure relation (also called the **resolution of the identity**) is given by (see also **??**)

$$\sum_n |\psi_n\rangle\langle\psi_n| + \int_X |x\rangle\langle x|\,dx = \mathbb{1}\,, \tag{6.1}$$

where the sum ranges over the discrete spectrum and the integral over the continuous spectrum. For simplicity, the summation will also be used for the continuous part.

Axiom 6.3 (Born rule). Let $\mathcal{H}$ be the Hilbert space of a physical system and consider an observable $\widehat{O}$. If $|\psi\rangle$ is a state vector and $\widehat{P}_\phi$ is the projection onto an eigenvector $|\phi\rangle$ of $\widehat{O}$, the probability of observing the state $|\phi\rangle$ is given by:

$$\frac{\langle\psi \mid \widehat{P}_\phi \mid \psi\rangle}{\langle\psi\mid\psi\rangle} = \frac{|\langle\psi\mid\phi\rangle|^2}{\langle\psi\mid\psi\rangle}\,. \tag{6.2}$$

Property 6.1.4 (Projectivization). In light of the Born rule, the dynamics of a system does not depend on the global phase or normalization, i.e. states are represented by rays in a projective Hilbert space $\mathcal{H}\mathbb{P}$ (**??**).

Combining Born's rule with **??**, gives the following definition.

Definition 6.1.5 (Expectation value). The expectation value of an observable $\widehat{A}$ in a (normalized) state $|\psi\rangle$ is defined as follows:

$$\langle\widehat{A}\rangle_\psi := \langle\psi|\widehat{A}|\psi\rangle\,. \tag{6.3}$$

The subscript ψ is often left implicit. As in ordinary statistics (**??**), the uncertainty or variance is defined as follows:

$$\Delta A := \langle\widehat{A}^2\rangle - \langle\widehat{A}\rangle^2\,. \tag{6.4}$$

Formula 6.1.6 (Uncertainty relation). Let $\widehat{A}, \hat{B}$ be two observables and let $\Delta A, \Delta B$ be the corresponding uncertainties. The (**Robertson**) uncertainty relation reads as follows:

$$\Delta A \Delta B \geq \frac{1}{4}\left|\left\langle\left[\widehat{A}, \hat{B}\right]\right\rangle\right|^2. \tag{6.5}$$

Axiom 6.4 (Projection[1]). Let $\mathcal{H}$ be the Hilbert space of a physical system and consider an observable $\widehat{O}$ with eigenvalues $\{o_i\}_{i\in I}$. After measuring the observable $\widehat{O}$ in the state $|\psi\rangle$, the outcome will be one of the eigenvalues o_i and system will 'collapse' to, i.e. get projected onto, the eigenstate $\widehat{P}_{o_i}|\psi\rangle \equiv |o_i\rangle$.

Axiom 6.5 (Unitary evolution). The evolution of a closed system is unitary, i.e. there exists a unitary operator $\widehat{U}(t,t') \in \mathsf{Aut}(\mathcal{H})$, for all times $t \leq t'$, such that

$$|\psi(t')\rangle = \widehat{U}(t,t')|\psi(t)\rangle\,. \tag{6.6}$$

[1] Also called the **measurement postulate**.

6.2 Schrödinger picture

Since the energy is of paramount importance in physics, the associated eigenvalue equation deserves its own name.

Formula 6.2.1 (Time-independent Schrödinger equation).

$$\widehat{H}|\psi\rangle = E|\psi\rangle \tag{6.7}$$

The operator $\widehat{H}$ is called the **Hamiltonian** of the system. The wave function ψ is an element of the vector space $L^2(\mathbb{R},\mathbb{C})\otimes\mathcal{H}$ with $\mathcal{H}$ the internal Hilbert space (describing, for example, the spin or charge of a particle). This is an eigenvalue equation for the energy levels of the system.

@@ INTRODUCE POSITION/CONFIGURATION REPRESENTATION @@

The time evolution of a wave function was governed by Axiom 6.5. By passing to generators, the following equation is obtained.

Formula 6.2.2 (Time-dependent Schrödinger equation).

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle\,. \tag{6.8}$$

In case $\widehat{H}$ is time independent, the TISE can be obtained from this equation by separation of variables (see below).

Proof (Derivation of TISE from TDSE). Starting from the one-dimensional TDSE in position space with a time-independent Hamiltonian, one can perform a separation of variables and assert a solution of the form $\psi(x,t) = X(x)T(t)$. Inserting this in the previous equation gives

$$i\hbar X(x)T'(t) = \big(\widehat{H}X(x)\big)T(t)\,.$$

Dividing both sides by $X(x)T(t)$ and rearranging the terms gives

$$i\hbar\frac{T'(t)}{T(t)} = \frac{\widehat{H}X(x)}{X(x)}\,.$$

Because the left side only depends on t and the right side only depends on x, one can conclude that they both have to equal a constant $E\in\mathbb{C}$. This leads to the following system of differential equations:

$$\begin{cases} i\hbar T'(t) = ET(t)\,,\\ \widehat{H}X(x) = EX(x)\,. \end{cases}$$

The first equation immediately gives a solution for T:

$$T(t) = C\exp\left(-\frac{iE}{\hbar}t\right). \tag{6.9}$$

The second equation is exactly the TISE (Formula 6.2.1). □

Example 6.2.3 (Massive particle in a stationary potential).

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right)\psi(x,t) \tag{6.10}$$

In this case, the TISE reads as follows:

$$\psi''(x) = -\frac{2m}{\hbar^2}\big(E - V(x)\big)\psi(x). \tag{6.11}$$

Formula 6.2.4 (General solution). A general solution of the TDSE (for time-independent Hamiltonians) is given by the following formula (cf. **??**):

$$\psi(x,t) = \sum_E c_E\psi_E(x)e^{-\frac{i}{\hbar}Et}, \tag{6.12}$$

where the functions $\psi_E(x)$ are the eigenfunctions of the TISE. The coefficients c_E can be found using the orthogonality relations

$$c_E = \left(\int_{\mathbb{R}}\overline{\psi_E}(x)\psi(x,t_0)\,dx\right)e^{\frac{i}{\hbar}Et_0}. \tag{6.13}$$

6.3 Heisenberg–Born–Jordan picture

In the previous section, the central object was the wave function. It was this object that evolved in time and the operators acting on the Hilbert space of physical states were assumed to be fixed. However, it is also possible to transfer this dependence on time to the operators.

Formula 6.3.1 (Time-dependent observables).

$$\widehat{O}_H(t) := e^{\frac{i}{\hbar}\widehat{H}t}\widehat{O}_S(t)e^{-\frac{i}{\hbar}\widehat{H}t} \tag{6.14}$$

The equivalence between the Schödinger and Heisenberg pictures essentially come from the fact that the time-evolving expectation values of operators are given by the following formula:

$$\langle\widehat{O}(t)\rangle = \langle\psi|e^{\frac{i}{\hbar}\widehat{H}t}\widehat{O}(t)e^{-\frac{i}{\hbar}\widehat{H}t}|\psi\rangle. \tag{6.15}$$

The difference between the pictures is simply the choice of whether to include the evolution operator in the states or in the operators.

Using the above transformation, the Schödinger equation (Formula 6.2.2) can also be reexpressed.

Formula 6.3.2 (Time-dependent Schrödinger equation).

$$\frac{\partial \widehat{O}_H}{\partial t}(t) = \frac{i}{\hbar}\left[\widehat{H}_H(t), \widehat{O}_H(t)\right] + \left(\frac{\partial \widehat{O}}{\partial t}(t)\right)_H \tag{6.16}$$

Taking this expression for the Schrödinger equation and taking expectation values (using the linearity of the equation), gives the following (interaction-independent) result.

Theorem 6.3.3 (Ehrenfest). *Let $\widehat{H}$ be the Hamiltonian and consider an observable $\widehat{O}$. The expectation value of this operator evolves as follows:*

$$\frac{\mathrm{d}\langle\widehat{O}\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\langle[\widehat{O}, \widehat{H}]\rangle + \left\langle\frac{\partial \widehat{O}}{\partial t}\right\rangle. \tag{6.17}$$

Remark 6.3.4 (Equivalence). It is important to note that the Schrödinger equation could be replaced by Ehrenfest's theorem. They are entirely equivalent.

But, given the abstract state vectors $|\psi\rangle$ from Section 6.1.1, how does one recover the position (configuration) representation $\psi(x)$? This is simply the projection of the state vector $|\psi\rangle$ on the 'basis function' $\delta(x)$, i.e. $\psi(x)$ represents an expansion coefficient in terms of a 'basis' for the physical Hilbert space. In the same way, one can obtain the momentum representation $\psi(p)$ by projecting onto the plane waves e^{ipx}.

Remark 6.3.5. It should be noted that neither the 'basis states' $\delta(x)$, nor the plane waves e^{ipx} are square integrable and, hence, they are not elements of the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$. This issue can be resolved through the concept of *rigged Hilbert spaces*.

@@ COMPLETE @@

6.3.1 Hydrogen atom

Consider the hydrogen atom, i.e. a single proton (the nucleus) orbited by a single electron with only the electrostatic Coulomb force acting between them (gravity can safely be neglected):

$$\widehat{H} := \frac{\widehat{p}_p^2}{2m_p} + \frac{\widehat{p}_e^2}{2m_e} - \frac{e^2}{4\pi\varepsilon r^2}. \tag{6.18}$$

It is not hard to see that this is the quantum mechanical version of the Kepler problem (**??**). The special property of the Kepler problem was that it contained a 'hidden' symmetry that gave rise to the conserved Laplace–Runge–Lenz vector (**??**). As is the case for all conserved charges in quantum mechanics, this symmetry induces a degeneracy of the energy eigenvalues. Degeneracy of the magnetic quantum number $m \in \mathbb{N}$ follows from rotational symmetry, but the energy levels of the hydrogen atom only depend on the principal quantum number $n \in \mathbb{N}$. It is the degeneracy of the total angular quantum number $l \in \mathbb{N}$ that is due to this 'hidden' $SO(4)$-symmetry. It is often called an 'accidental degeneracy' for this reason.

@@ COMPLETE @@

6.3.2 Molecular dynamics

Consider the Hamiltonian of two interacting atoms:

$$\widehat{H} = \frac{\widehat{P}_1^2}{2M_1} + \frac{\widehat{P}_2^2}{2M_2} + \frac{\hat{q}_1\hat{q}_2}{4\pi\varepsilon R^2} + \sum_i \frac{\hat{p}_i^2}{2m} - \frac{e\hat{q}_1}{4\pi\varepsilon r_{i1}^2} - \frac{e\hat{q}_2}{4\pi\varepsilon r_{i2}^2} + \sum_{i\neq j} \frac{e^2}{4\pi\varepsilon r_{ij}^2}, \tag{6.19}$$

where the indices i, j indicate the electrons and uppercase symbols denote operators associated to the nuclei.

Except for the most simple situations, solving the Schrödinger equation for this Hamiltonian becomes intractable (both analytically and numerically). However, in general, one can approximate the situation. The masses of nuclei are much larger than those of the electrons and this influences their motion, they move much slower than the electrons. In essence, the nuclei and electrons live on different time scales and this allows to decouple their dynamics:

$$\widehat{H}_{\text{nucl}} = \frac{\widehat{P}_1^2}{2M_1} + \frac{\widehat{P}_2^2}{2M_2} + \frac{Q_1Q_2}{4\pi\varepsilon R^2} + V_{\text{eff}}(R_1, R_2). \tag{6.20}$$

The electrons generate an effective potential for the nuclei and the Schrödinger equation decouples as follows:

$$\begin{aligned} \widehat{H}_{\text{nucl}}(R)\psi(R) &= E\psi(R), \\ \widehat{H}_{\text{el}}(r, R)\phi(r, R) &= E_{\text{el}}\phi(r, R). \end{aligned} \tag{6.21}$$

This is the so-called **Born–Oppenheimer approximation**. From a more modern physical perspective, this approximation can also be seen to be a specific instance of renormalization theory, where the short time-scale (or, equivalently, the high energy-scale) degrees of freedom are integrated out of the theory.

6.4 Mathematical formalism

6.4.1 Weyl systems

Definition 6.4.1 (Canonical commutation relations). Two observables $\widehat{A}, \widehat{B}$ are said to obey a canonical commutation relation (CCR) if they satisfy (up to a constant factor

$\hbar$)

$$[\widehat{A}, \widehat{B}] = i\,. \tag{6.22}$$

The prime examples are the position and momentum operators $\hat{x}, \hat{p}$. Through functional calculus, one can also define the exponential operators $e^{is\widehat{A}}$ and $e^{it\widehat{B}}$. The above relation then induces the so-called **Weyl form** of the CCR:

$$e^{is\widehat{A}}e^{it\widehat{B}} = e^{ist}e^{it\widehat{B}}e^{is\widehat{A}}\,. \tag{6.23}$$

Theorem 6.4.2 (Stone–von Neumann). *All pairs of irreducible, unitary one-parameter subgroups satisfying the Weyl form of the CCRs are unitarily equivalent.*

Corollary 6.4.3. The Schrödinger and Heisenberg pictures are unitarily equivalent.

In fact, one can generalize the Weyl form of the CCRs.

Definition 6.4.4 (Weyl system). Let (L, ω) be a symplectic vector space and let K be a complex vector space. Consider a map W from L to the space of unitary operators on K. The pair (K, W) is called a Weyl system over (L, ω) if it satisfies

$$W(z)W(z') = e^{i/2\omega(z,z')}W(z + z') \tag{6.24}$$

for all $z, z' \in L$, i.e. W is a projective representation of the Abelian group L and ω is, up to rescaling, the group cocycle inducing it (**??**). The relation itself is called a **Weyl relation**.

Definition 6.4.5 (Heisenberg system). Let W be a Weyl system. The selfadjoint generators $\phi(z)$, which exist by Stone's theorem 4.1.26, of the maps $t \mapsto W(tz)$ are said to form a Heisenberg system. These operators satisfy the following properties:

1. **Positive homogeneity**: $\lambda\phi(z) = \phi(\lambda z)$ for all $\lambda > 0$,
2. **Commutator**: $[\phi(z), \phi(z')] = -i\omega(z, z')$, and
3. **Weak additivity**: $\phi(z + z')$ is the closure (Definition 4.1.18) of $\phi(z) + \phi(z')$.

Remark 6.4.6. It should be noted that the Weyl relations are more fundamental than their infinitesimal counterparts. Only the Weyl relations are well defined on more general spaces and when passing to a relativistic setting.

Recall **??**, where the framework of measure theory and distributions was generalized to the noncommutative context.

Property 6.4.7 (Schrödinger representation). Consider a distribution d on a (real) TVS V. There exists a unique unitary representation U of the additive group V^* on $L^2(V,d)$ such that

$$U(\lambda)f = e^{id(\lambda)}f \tag{6.25}$$

for all bounded tame functions f and such that 1 is cyclic for U in $L^2(V,d)$. Moreover, this representation is continuous with respect to the finest locally convex topology on V (the one generated by all seminorms on V)[2].

@@ EXPLAIN RELEVANCE e.g. Baez, Segal, and Zhou (2014) @@

6.4.2 Dirac–von Neumann postulates: revisited

Section 6.1.1 presented the axioms of quantum mechanics in terms of Hilbert spaces and the operators thereon. However, the incredible insight of *von Neumann* was that one can do away with the Hilbert space. By Example 4.2.8, the observables of a quantum-mechanical system form a C^*-algebra. Consequently, the idea is to rephrase the axioms in purely C^*-algebraic terms (Section 4.2). By Theorem 4.2.42, these two approaches are equivalent.

Axiom 6.6 (Observables). A physical system is characterized by a C^*-algebra, with the observables corresponding to the self-adjoint elements.

Axiom 6.7 (States). A state of a quantum-mechanical system is given by a state of the associated C^*-algebra (Definition 4.2.16).

Axiom 6.8 (Born rule). The expectation value of an observable a in a state ω is given by the evaluation $\omega(a)$.

Remark 6.4.8. **??** will link this axiom to traces and operator theory through Property 4.2.27 and Definition 4.2.29.

Axiom 6.9 (Projection).

Axiom 6.10 (Unitary evolution).

@@ CORRECT ALL AXIOMS @@

[2]This topology is also known as the **algebraic topology**

6.4.3 Symmetries

Property 6.4.9 (States). By the postulates of quantum mechanics, states are represented by rays in the projective Hilbert space $\mathcal{H}\mathbb{P}$. The probabilities, given by the Born rule (Axiom 6.3), can be expressed in terms of the *Fubini–Study metric* on $\mathcal{H}\mathbb{P}$ as follows:

$$\mathcal{P}(\psi,\phi) := \cos^2\big(d_{\mathrm{FS}}(\psi,\phi)\big) = \frac{|\langle\psi\,|\,\phi\rangle|^2}{\langle\psi\,|\,\psi\rangle\langle\phi\,|\,\phi\rangle}, \tag{6.26}$$

where $|\psi\rangle, |\phi\rangle$ are representatives of the states ψ, ϕ in $\mathcal{H}\mathbb{P}$.

Definition 6.4.10 (Symmetry). A quantum symmetry (or **quantum automorphism**) is an isometric automorphism of $\mathcal{H}\mathbb{P}$. The group of these symmetries is denoted by $\mathrm{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P})$.

The following theorem due to *Wigner* gives a (linear) characterization of quantum symmetries.[3]

Theorem 6.4.11 (Wigner). *Every quantum automorphism of $\mathcal{H}\mathbb{P}$ is induced by a unitary or anti-unitary operator on $\mathcal{H}$.*

This is equivalent to saying that the group morphism

$$\pi : \mathrm{Aut}(\mathcal{H},\mathcal{P}) := \mathrm{U}(\mathcal{H}) \times \mathrm{AU}(\mathcal{H}) \to \mathrm{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \tag{6.27}$$

is surjective. Together with the kernel $\mathrm{U}(1)$, given by phase shifts, this forms a short exact sequence:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathrm{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \longrightarrow 1\,. \tag{6.28}$$

In the case of symmetry breaking (e.g. lattice systems), the full symmetry group is reduced to a subgroup $G \subset \mathrm{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P})$. The group of operators acting on $\mathcal{H}$ is then given by the pullback $\widetilde{G}$ of the diagram

$$\mathrm{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathrm{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \longleftarrow G\,. \tag{6.29}$$

It should also be noted that the kernel of the homomorphism $\widetilde{G} \to G$ is again $\mathrm{U}(1)$. This leads to the property that $\widetilde{G}$ is a $\mathbb{Z}_2$-twisted (hence noncentral) $\mathrm{U}(1)$-extension of G, where the twist is induced by the homomorphism $\phi : \mathrm{Aut}(\mathcal{H},\mathcal{P}) \to \mathbb{Z}_2$ that says whether an operator is implemented unitarily or anti-unitarily.

@@ COMPLETE @@

[3]It is a particular case of a more general theorem in projective geometry.

6.4.4 Symmetric states

Axiom 6.11 (Symmetrization postulate). Let $\mathcal{H}$ be the single-particle Hilbert space. A system of $n \in \mathbb{N}$ identical particles is described by a state $|\Psi\rangle$ belonging to either $S^n\mathcal{H}$ or $\Lambda^n\mathcal{H}$. These **bosonic** and **fermionic** states are, respectively, of the form

$$|\Psi_B\rangle = \sum_{\sigma\in S_n} \left|\psi_{\sigma(1)}\right\rangle \cdots \left|\psi_{\sigma(n)}\right\rangle \tag{6.30}$$

and

$$|\Psi_F\rangle = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\left|\psi_{\sigma(1)}\right\rangle \cdots \left|\psi_{\sigma(n)}\right\rangle, \tag{6.31}$$

where the $|\psi_i\rangle$ are single-particle states and S_n is the permutation group on n elements.

Remark 6.4.12. In ordinary quantum mechanics, this is a postulate, but in quantum field theory, this is a consequence of the *spin-statistics theorem*. @@ ADD THIS THEOREM TO [QFT] @@

Definition 6.4.13 (Slater determinant). Let $\{\phi_i(\vec{q})\}_{i\leq n}$ be a set of wave functions, called **spin orbitals**, describing a system of n identical fermions. The totally antisymmetric wave function of the system is given by

$$\psi(\vec{q}_1,\dots,\vec{q}_n) = \frac{1}{\sqrt{n!}}\det\begin{pmatrix}\phi_1(\vec{q}_1) & \cdots & \phi_n(\vec{q}_1)\\ \vdots & \ddots & \vdots\\ \phi_1(\vec{q}_n) & \cdots & \phi_n(\vec{q}_n)\end{pmatrix}. \tag{6.32}$$

A similar function can be defined for bosonic systems using the concept of *permanents*.

6.5 Foundations ♣

6.5.1 Measurement problem

If one looks at the Schrödinger equation (Formula 6.2.2) or Ehrenfest's theorem (Theorem 6.3.3), it is easy to see that time evolution is entirely linear and deterministic. Superpositions are preserved under Hamiltonian flow (a crucial ingredient of quantum mechanics) and, given an initial state, time evolution will always lead to the same final state. However, the Born rule (Axiom 6.3), which governs 'measurements' is very nonlinear and nondeterministic. It is probabilistic and, once a 'measurement' has been performed, the state has 'collapsed' onto an eigenstate of the observable under consideration.

The issue of what constitutes a 'measurement' — Is it a conscious human doing an experiment? Is it a mouse interfering with an experiment? Is it two particles interacting? ...[4] — and why exactly the Born rule holds and what it entails, i.e. how probabilities arise, is known as the measurement problem. On a historical note, it should be noted that, after an initial surge of interest shortly after the 5th Solvay Conference (1927), where quantum mechanics was formally established, the study of the foundations of quantum mechanics (the measurement problem specifically) became an infamous topic due to the pragmatic mentality of nuclear physics during the 20th century.

@@ ADD (dynamical collapse, epistemic) @@

6.5.2 Copenhagen interpretation

The Copenhagen interpretation[5] takes the foundations of quantum mechanics as presented above very literally.

@@ COMPLETE (e.g. collapse) @@

6.5.3 Many-worlds interpretation

This interpretation, originating with *Everett*, posits a different idea, which does away with the need of the explicit Born rule axiom. In this interpretation, there is a kind of 'universal wave function', which governs both the observer and the experiment. A 'measurement' is then simply an entanglement-inducing interaction between these two subsystems.

The main implication of such an interpretation is, however, that the universal wave function branches every time such an interaction occurs. More precisely, assume that 'we', the observers, perform a measurement on some system (for simplicity, assume that the measurement has a binary outcome). The measurement process is then described as follows:

$$|\mathrm{in}\rangle_{\mathrm{obs}}|\mathrm{in}\rangle_{\mathrm{exp}} \longrightarrow \lambda_0|0\rangle_{\mathrm{obs}}|0\rangle_{\mathrm{exp}} + \lambda_1|1\rangle_{\mathrm{obs}}|1\rangle_{\mathrm{exp}}\,. \tag{6.33}$$

Taking this superposition as a physical reality, this means that if we had measured the state 0, a copy of us living on the other branch will have measured 1 (and the other way around).

@@ COMPLETE (e.g. origin of probabilities) @@

[4] This (perhaps artificial) boundary between classical and quantum is sometimes called the **Heisenberg cut**.

[5] This name stems from the fact that its initial proponents were from the group of physicists centered around *Bohr*.

6.5.4 Relational quantum mechanics

An important notion in classical physics is that of a *reference frame*, i.e. a choice of axes and scales. Usually, this corresponds to choosing an observer, relative to which one expresses the motion of all other objects. In relativity, the relative treatment of physics was the grand breakthrough by Einstein. However, although this notion had been left aside for a long time in the treatment of quantum mechanics and a specific choice of reference frame was silently assumed, this assumption was not as innocuous as it appears. Superposition and complementarity make a definite choice of absolute reference frame impossible.

To understand the relevance of a relational approach to quantum mechanics, consider the following thought experiment.

Definition 6.5.1 (Wigner's friend). Consider two observers, Wigner and his friend, performing an experiment as shown diagrammatically in Fig. 6.1. One envisions Wigner standing outside the laboratory, having no way to observe what happens inside the lab, and his friend who performs an experiment inside the lab. The paradox arises from the two ways one can describe the sequence of the friend performing a measurement and Wigner checking up on the results in the classical (Copenhagen) interpretation.

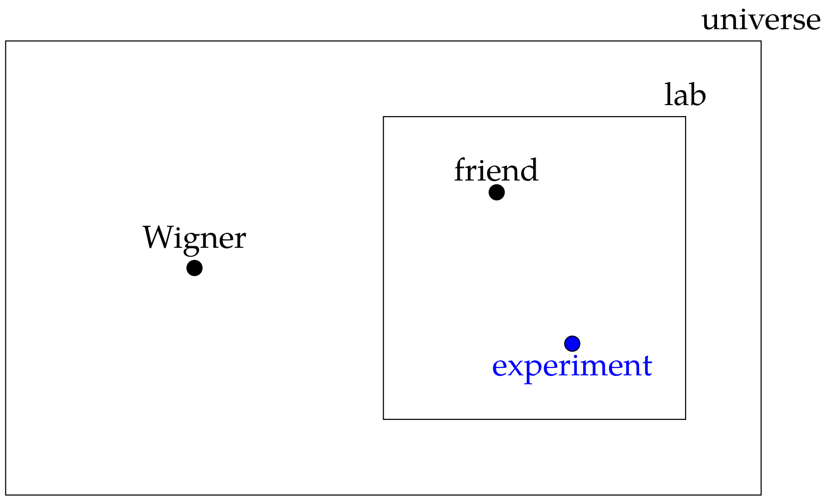

Figure 6.1: Wigner's friend thought experiment.

From the point of view of the friend, at the moment of measurement, the projection/collapse axiom states that the wave function describing friend + experiment 'collapses' to:

$$|\psi\rangle = |\uparrow\rangle_{\text{friend}}|\uparrow\rangle_{\text{exp}}\,. \tag{6.34}$$

However, from the point of view of Wigner, who has not observed the measurement, the state is described by

$$|\psi\rangle' = \alpha|\uparrow\rangle_{\text{friend}}|\uparrow\rangle_{\text{exp}} + \beta|\downarrow\rangle_{\text{friend}}|\downarrow\rangle_{\text{exp}}\,. \tag{6.35}$$

Whereas in the many-world approach one would simply take the branching approach, which is fully unitary and resolves this issue by avoiding collapse, the relational approach takes collapse at face value, but states that observations are relative, i.e. always with respect to some fixed observer (be it a person, a classical object or another quantum-mechanical system). From this point of view, textbook Copenhagen QM is simply quantum mechanics with respect to some god-given observer, and collapse and unitary evolution do not have to be reconciled.

The more general idea is that information and, hence, the values of observables are a relative notion, i.e. variables only attain their values when considered with respect to a certain observer. As such, RQM is an epistemic interpretation of quantum mechanics in that the wave function only captures 'our' information about the system (or universe) and not the 'true' physical state. When applied to the notion of instantaneity (or velocity), this line of thinking will give rise to (special) relativity as in **??** (and **??**).

@@ COMPLETE @@

For example, consider three observers: Alice, Bob and Charlie. Assume that each observer has a spin-$\frac{1}{2}$ particle and that, relative to Alice, the joint state is given by

$$|\psi\rangle^A_{ABC} = |\uparrow\rangle^A_A\Big(|\uparrow\rangle^A_B + |\downarrow\rangle^A_B\Big)|\downarrow\rangle^A_C\,. \tag{6.36}$$

Note that this state is separable. Now, what would the state be relative to Bob? If one supposes that changes of reference frame are *coherent* (to be formalized below), the joint state will be

$$|\psi\rangle^B_{ABC} = |\uparrow\rangle^B_B\Big(|\uparrow\rangle^B_A|\downarrow\rangle^B_C + |\downarrow\rangle^B_A|\uparrow\rangle^B_C\Big)\,. \tag{6.37}$$

A mere change of reference frame, an operation that would classically leave the physics invariant, has transformed a product state into an entangled state.

Axiom 6.12 (Relational physics). Given $n \in \mathbb{N}$ systems[6], any state is described relative to one of these systems. Given a choice of 'observing system', let it be system i, the state of system i is given by a fiducial state $|0\rangle^i_i$.

Axiom 6.13 (Coherent change). Consider a change of reference frame $0 \longrightarrow i$ such that

$$\begin{cases} |\psi\rangle^0 \longrightarrow |\psi\rangle^i \\ |\phi\rangle^0 \longrightarrow |\phi\rangle^i\,. \end{cases} \tag{6.38}$$

Then

$$\alpha|\psi\rangle^0 + \beta|\phi\rangle^0 \longrightarrow \alpha|\psi\rangle^i + \beta|\phi\rangle^i \tag{6.39}$$

for all $\alpha, \beta \in \mathbb{C}$.

[6] An abstraction of the notion of observer.

Abstractly, a (classical) reference frame is defined as follows in the spirit of Section 5.4 and Section 5.9.

Definition 6.5.2 (Reference frame). Let X be an object of interest. Whereas a coordinate chart on X, modeled on an object Y, is given by a morphism $Y \rightarrow X$, a **coordinate system** on X is given by an isomorphism $Y \cong X$, i.e. a global coordinate chart. A reference frame is coordinate system for which Y corresponds the a physical system.

Let the system of interest X admit a group action that is both free and transitive, turning it into a G-torsor (**??**). At the level of sets, one has $X \cong G$ and a choice of origin, i.e. a specific choice of isomorphism, corresponds to a choice of reference frame (the identity element corresponding to the fiducial state above). A change of reference frames $s^0 \longrightarrow s^i$, frome system 0 to system i, is given by the right regular action of the relative coordinate of i on all relative coordinates:

$$\phi^{0\rightarrow i}(e, g_1^0, \ldots, g_n^0) \mapsto (g_0^i, g_1^0 g_0^i, \ldots, e, \ldots, g_n^0 g_0^i)\,, \tag{6.40}$$

where the relation $g_i^0 = (g_0^i)^{-1}$ was used. It should be noted that this boils down to a *passive transformation*. When passing to the quantization of these systems, one should assume that G is locally compact and comes equipped with the canonical Haar measure (**??**). In this case, a quantization is given by the space of square-integrable functions $L^2(G)$, where basis states are labeled by group elements.

@@ VERIFY THIS STATEMENT @@

The change-of-reference-frame operator is given as follows:

$$\widehat{U}^{0\rightarrow i} := \mathsf{SWAP}_{0,i} \circ \int_G \mathbb{1}_{L^2(G)} \otimes \widehat{U}_R(g_i^0)^{\otimes i-2} \otimes |g_0^i\rangle\langle g_i^0| \otimes \widehat{U}_R(g_i^0)^{\otimes n-i-2}\, dg_i^0\,, \tag{6.41}$$

where

$$\widehat{U}_R(g) : |x\rangle \mapsto |xg^{-1}\rangle \tag{6.42}$$

is the unitary implementation of the right regular action and dg denotes integration with respect to the Haar measure on G. It can be shown that $\widehat{U}^{0\rightarrow i}$ is unitary, its inverse being given by $\widehat{U}^{i\rightarrow 0}$ and composition is transitive. It can be shown that this procedure can be extended to any one-particle Hilbert space $\mathcal{H}$ as long as the inclusion $G \rightarrow \mathcal{H}$ is injective and maps G to an orthonormal basis of (a subset of) $\mathcal{H}$.

6.6 Angular Momentum

6.6.1 Angular momentum operator

Property 6.6.1 (Lie algebra). The angular momentum operators generate a Lie algebra (**??**). The Lie bracket is defined by the following commutation relation:

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk}\hat{J}_k\,. \tag{6.43}$$

Since rotations correspond to actions of the orthogonal group SO(3), it should not come as a surprise that the above relation is exactly the defining relation of the Lie algebra $\mathfrak{so}(3)$ from **??**.

Property 6.6.2. The mutual eigenbasis of $\hat{J}^2$ and $\hat{J}_z$ is defined by the following two eigenvalue equations:

$$\hat{J}^2|j,m\rangle = j(j+1)\hbar^2\,|j,m\rangle\,, \tag{6.44}$$

$$\hat{J}_z|j,m\rangle = m\hbar\,|j,m\rangle\,. \tag{6.45}$$

Definition 6.6.3 (Ladder operators[7]). The raising and lowering operators $\hat{J}_+$ and $\hat{J}_-$ are defined as follows:

$$\hat{J}_+ := \hat{J}_x + i\hat{J}_y \qquad \text{and} \qquad \hat{J}_- := \hat{J}_x - i\hat{J}_y\,. \tag{6.46}$$

These operators only change the quantum number $m_z \in \mathbb{N}$, not the total angular momentum.

Corollary 6.6.4. From the commutation relations of the angular momentum operators, one can derive the commutation relations of the ladder operators:

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z\,. \tag{6.47}$$

Formula 6.6.5. The total angular momentum operator $\hat{J}^2$ can now be expressed in terms of $\hat{J}_z$ and the ladder operators using the commutation relation (6.43):

$$\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z\,. \tag{6.48}$$

Remark 6.6.6 (Casimir operator). From the definition of $\hat{J}^2$, it follows that this operator is a Casimir invariant (**??**) of $\mathfrak{so}(3)$.

6.6.2 Rotations

Formula 6.6.7. An infinitesimal rotation $\widehat{R}(\delta\vec{\varphi})$ is given by the following formula:

$$\widehat{R}(\delta\vec{\varphi}) = \mathbb{1} - \frac{i}{\hbar}\vec{J}\cdot\delta\vec{\varphi}\,. \tag{6.49}$$

[7] Also called the **creation** and **annihilation** operators (especially in quantum field theory).

A finite rotation can be generated by applying this infinitesimal rotation repeatedly:

$$\widehat{R}(\vec{\boldsymbol{\varphi}}) = \left(\mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}} \cdot \frac{\vec{\boldsymbol{\varphi}}}{n}\right)^n = \exp\left(-\frac{i}{\hbar}\vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{\varphi}}\right). \tag{6.50}$$

Formula 6.6.8 (Matrix elements). Applying a rotation over an angle φ about the z-axis to a state $|j,m\rangle$ gives

$$\widehat{R}(\varphi\vec{\boldsymbol{e}}_z)|j,m\rangle = \exp\left(-\frac{i}{\hbar}\hat{J}_z\varphi\right)|j,m\rangle = \exp\left(-\frac{i}{\hbar}m\varphi\right)|j,m\rangle\,. \tag{6.51}$$

Multiplying these states with a bra $\langle j',m'|$ and using the orthonormality of the eigenstates, gives the matrix elements of the rotation operator:

$$\widehat{R}_{ij}(\varphi\vec{\boldsymbol{e}}_z) = \exp\left(-\frac{i}{\hbar}m\varphi\right)\delta_{jj'}\delta_{mm'}\,. \tag{6.52}$$

From the expression of the angular momentum operators and the rotation operator, it is clear that a general rotation has no effect on the total angular momentum number $j \in \mathbb{N}$. This means that the rotation matrix will be block diagonal with respect to j. This amounts to the following reduction of the representation of the rotation group:

$$\langle j,m'|\widehat{R}(\varphi\vec{\boldsymbol{n}})|j,m\rangle = \mathcal{D}^{(j)}_{m,m'}(\widehat{R})\,, \tag{6.53}$$

where the functions $\mathcal{D}^{(j)}_{m,m'}(\widehat{R})$ are called the **Wigner *D*-functions**. For every value of j, there are $(2j+1)$ values for m. This implies that the matrix $\mathcal{D}^{(j)}(\widehat{R})$ is a $(2j+1)\times(2j+1)$-matrix.

6.6.3 Spinor representation

Definition 6.6.9 (Pauli matrices).

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.54}$$

From this definition, it is clear that the Pauli matrices are Hermitian and unitary. Together with the 2×2 identity matrix, they form a basis for the space of 2×2 Hermitian matrices. For this reason, the identity matrix is often denoted by σ_0 (especially in the context of relativistic QM).

Formula 6.6.10. In the spinor representation ($J = \frac{1}{2}$), the Wigner-D matrix reads as follows:

$$\mathcal{D}^{(1/2)}(\varphi\vec{\boldsymbol{e}}_z) = \begin{pmatrix} e^{-i/2\varphi} & 0 \\ 0 & e^{i/2\varphi} \end{pmatrix}. \tag{6.55}$$

6.6.4 Coupling of angular momenta

Due to the tensor product structure of a coupled Hilbert space, the angular momentum operator $\hat{J}_i$ should now be interpreted as $\mathbb{1} \otimes \cdots \otimes \hat{J}_i \otimes \cdots \otimes \mathbb{1}$ (cf. **??**). Because the angular momentum operators $\hat{J}_{k\neq i}$ do not act on the space $\mathcal{H}_i$, one can pull these operators through the tensor product:

$$\hat{J}_i|j_1\rangle \otimes \cdots \otimes |j_n\rangle = |j_1\rangle \otimes \cdots \otimes \hat{J}_i|j_i\rangle \otimes \cdots \otimes |j_n\rangle . \tag{6.56}$$

The basis used above is called the **uncoupled basis**.

For simplicity, the total Hilbert space is, from here on, assumed to be that of a two-particle system. Let $\hat{J}$ denote the total angular momentum:

$$\hat{J} = \hat{J}_1 + \hat{J}_2 . \tag{6.57}$$

With this operator, one can define a **coupled** state $|J, M\rangle$, where M is the total magnetic quantum number which ranges from $-J$ to J.

Formula 6.6.11 (Clebsch–Gordan coefficients). Because both bases (coupled and uncoupled) span the total Hilbert space $\mathcal{H}$, there exists an invertible transformation between them. The transformation coefficients can be found by using the resolution of the identity:

$$|J, M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2\,|\,J, M\rangle . \tag{6.58}$$

These coefficients are called the Clebsch–Gordan coefficients.

Property 6.6.12. By acting with the operator $\hat{J}_z$ on both sides of Formula 6.6.11, it is possible to prove that the Clebsch–Gordan coefficients are nonzero if and only if $M = m_1 + m_2$.

6.7 Approximation methods

6.7.1 WKB approximation

The Wentzel–Kramers–Brillouin (WKB) approximation[8] starts from the ansatz

$$\psi(\vec{q}) := \exp\bigl(iS(\vec{q})/\hbar\bigr) , \tag{6.59}$$

[8] This approach to solving second-order ODEs was essentially introduced a century earlier by *Green* and *Liouville*.

with $S : \mathbb{R}^n \to \mathbb{R}$ a phase function that is to be determined. Inserting this in the TDSE (in configuration representation) gives:

$$\left[\frac{\|\vec{\nabla}S(\vec{q})\|^2}{2m} + \big(V(\vec{q}) - E\big) - \frac{i\hbar\Delta S(\vec{q})}{2m}\right]\exp\big(iS(\vec{q})/\hbar\big) = 0\,. \tag{6.60}$$

To first order, i.e. for slowly varying potentials, the last term can be ignored. In this case, the phase function satisfies the Hamilton–Jacobi equation (**??**):

$$H\big(\vec{q}, S'(\vec{q})\big) = \frac{\|\vec{\nabla}S(\vec{q})\|^2}{2m} + \big(V(x) - E\big) = 0\,. \tag{6.61}$$

In physics, the Hamilton–Jacobi equation without time derivative is often called the **eikonal equation**[9]. This leads to the following result.

Property 6.7.1. A function $S : \mathbb{R}^n \to \mathbb{R}$ is a phase function for a first-order solution to the Schrödinger equation if its differential lies in a level set of the classical Hamiltonian $H : T^*\mathbb{R}^n \to \mathbb{R}$. These solutions are said to be **admissible**.

To obtain higher-order approximations, the solution has to be generalized beyond a pure phase function:

$$\psi(\vec{q}) = a(\vec{q})\exp\big(iS(\vec{q})/\hbar\big)\,. \tag{6.62}$$

Assuming S is admissible, the factor $a : \mathbb{R}^n \to \mathbb{R}$ satisfies the **homogeneous transport equation**:

$$a\Delta S + 2\vec{\nabla}a \cdot \vec{\nabla}S = 0\,. \tag{6.63}$$

If a satisfies this equation, ψ is called a **semiclassical state**. Note that this equation is equivalent to $a^2\vec{\nabla}S$ being divergence free or, equivalently:

$$\mathcal{L}_{\pi_* X^H}(a^2\,\mathrm{Vol}) = 0\,. \tag{6.64}$$

Since Lie derivatives pull back under diffeomorphisms (**??**) and the image $\mathrm{im}(\mathrm{d}S)$ gives a trivial subbundle of $T^*\mathbb{R}^n$, this is also equivalent to

$$\mathcal{L}_{X^H}(a^2\pi^*\,\mathrm{Vol}) = 0\,. \tag{6.65}$$

This quadratic behaviour in a leads to the idea that the correct object for representing quantum states is a half-density (see also **??**). This leads to the following statement:

> A second-order solution to the Schrödinger equation is given by a pair (S, a), where S is an admissible phase function and $a \in \Omega^{1/2}(\mathrm{im}(\mathrm{d}S))$ is a half-form that is invariant under the (classical) Hamiltonian flow.

The generalization to curved spaces, i.e. replacing $\mathbb{R}^{2n}$ by a symplectic manifold M, will be covered in Section 6.8.2.

[9] This name stems from optics.

6.8 Curved backgrounds ♣

Using the tools of distribution theory and differential geometry (**?? ??**, **??** and onwards), one can introduce quantum mechanics on curved backgrounds (in the sense of 'space', not 'spacetime').

6.8.1 Extending quantum mechanics

Remark 6.8.1 (Rigged Hilbert spaces). A first important remark to be made is that the classical definition of the wave function as an element of $L^2(\mathbb{R}^d, \mathbb{C})$ is not sufficient, even in flat Cartesian space. A complete description requires the introduction of so-called *Gel'fand triples* or *rigged Hilbert spaces*, where the space of square-integrable functions is replaced by the Schwartz space (**??**) of rapidly decreasing functions. The linear functionals on this space are then given by the tempered distributions.

When working on curved spaces or even in non-Cartesian coordinates on flat space, one can encounter problems with the definition of the self-adjoint operators $\widehat{q}^i$ and $\widehat{p}_i$. The naive definition $\widehat{q}^i = q^i, \widehat{p}_i = -i\partial_i$ gives rise to extra terms that break the canonical commutation relations and the selfadjointness of the operators (e.g. the angular position operator $\widehat{\varphi}$ on the circle together with its conjugate $\widehat{L}$) when calculating inner products.

An elegant solution to this problem is obtained by giving up the definition of the wave function as a well-defined function $\psi : \mathbb{R}^d \to \mathbb{C}$. Assume that the physical space has the structure of a Riemannian manifold (M, g) and that the 'naive' wave functions take values in a vector space V. Then, construct a vector bundle E with typical fibre V over M. By **??**, an invariant description of the 'true' wave function is a map $\Psi : F(E) \to V$ or, locally, the pullback $\psi := \varphi^*\Psi$ for some local section $\varphi : U \subseteq M \to F(E)$. The Levi-Civita connection on M also induces a covariant derivative ∇ on E that can be used to define differential operators.

Now, a general inner product can be introduced:

$$\langle \psi, \phi \rangle := \int_M \overline{\psi(x)}\phi(x) \,\mathrm{Vol}_M \,. \tag{6.66}$$

Because the factor $\sqrt{\det(g)}$ transforms in the inverse manner of the measure dx, the integrand is invariant under coordinate transforms (something that is generally required of physical laws). Using this new inner product, one can for example check the selfadjointness of the momentum operator $\widehat{P}_i := -i\nabla_i$:

$$\begin{aligned}
\langle\psi,\widehat{P}_i\phi\rangle &= \int_M \overline{\psi(x)}(-i\nabla_i)\phi(x)\sqrt{\det(g)}\,dx \\
&\overset{??}{=} \int_M \overline{\psi(x)}(-i\partial_i - i\omega_i)\phi(x)\sqrt{\det(g)}\,dx \\
&= \int_M \overline{(-i\partial_i\psi)(x)}\phi(x)\sqrt{\det(g)}\,dx + i\int_M \overline{\psi(x)}\phi(x)\Big(\partial_i\sqrt{\det(g)}\Big)\,dx \\
&\qquad - i\int_M \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx \\
&= \langle\widehat{P}_i\psi,\phi\rangle - i\int_M \overline{\psi(x)}\overline{\omega_i}\phi(x)\sqrt{\det(g)}\,dx \\
&\qquad + i\int_M \overline{\psi(x)}\phi(x)\Big(\partial_i\sqrt{\det(g)}\Big)\,dx \\
&\qquad - i\int_M \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx\,.
\end{aligned}$$

Selfadjointness then requires that

$$\sqrt{\det(g)}(\omega_i + \overline{\omega_i}) = \partial_i\sqrt{\det(g)} \tag{6.67}$$

or

$$2\mathrm{Re}(\omega_i) = \partial_i \ln\Big(\sqrt{\det(g)}\Big)\,. \tag{6.68}$$

@@ COMPLETE (rewrite in global terms) @@

6.8.2 WKB approximation

Property 6.7.1 is generalized quite trivially after replacing $\mathbb{R}^n$ by a configuration manifold Q. A further step is provided by also generalizing **??**.

Property 6.8.2. A Lagrangian submanifold $\iota : L \hookrightarrow T^*Q$ will be called an admissible phase function for a first-order solution to the Schrödinger equation if it satisfies the classical Hamilton–Jacobi equation, i.e. lies in a level set of the classical Hamiltonian $H : T^*Q \to \mathbb{R}$, for a regular value.

To obtain a second-order solution, one also needs prefactor for the semiclassical states. The homogeneous transport equation (6.63) is generalized as follows:

$$a\Delta S + 2\mathcal{L}_{\nabla S}a = 0\,, \tag{6.69}$$

where Δ is the Laplace–Beltrami operator on Q. As before, a general second-order solution, assuming S is admissible, is given by a half-form $a \in \Omega^{1/2}(L)$ satisfying

$$\mathcal{L}_Y a = 0\,, \tag{6.70}$$

where Y is the (nonsingular) vector field on L induced by X^H. This then gives a second-order solution on Q by pulling back along the inverse $(\pi \circ \iota)^{-1}$, which is a diffeomorphism since L is projectable. Moreover, if L is exact (**??**), then S is induced by a primitive of the induced Liouville form $\iota^*\alpha$. If both the exactness and projectability conditions are dropped, the notion of a **geometric solution** are obtained.

To pass to this more general situation, some more structure is needed. If L is not exact, the Liouville form does not admit a global primitive. However, L does admit a (good) cover $\{U_k\}_{k\in I}$ such that on every patch, a second-order solution can be found, and then the problem becomes how to glue these together. The gluing condition is the following integrality condition:

$$\phi_k(x) - \phi_l(x) \in 2\pi\hbar\mathbb{Z}\,, \tag{6.71}$$

where ϕ_k is the phase function on U_k, for all $x \in U_k \cap U_l$. Note that this condition can only be satisfied for all $\hbar \in \mathbb{R}^+$ if $[\alpha] = 0$. However, this is exactly the condition that should be relaxed. Luckily, $\hbar$ should be a fixed value.

Definition 6.8.3 (Quantizable Lagrangian). A projectable Lagrangian submanifold $L \subset T^*M$ is said to be quantizable if there exists an $\hbar \in \mathbb{R}^+$ such that the restriction of the Liouville class to L is $\hbar$-integral, i.e. the integrality condition (6.71) is satisfied. All values $\hbar$ for which the integrality condition is satisfied, are said to be **admissible**.

Remark 6.8.4. Note that the admissible values for $\hbar$ will form a decreasing sequence of the form

$$\hbar_0, \frac{\hbar_0}{2}, \cdots, \tag{6.72}$$

where $\hbar_0$ is the greatest admissible value.

For the weakening of the projectability condition, see Bates and Weinstein (1997). However, even without weakening that condition, there is still a remaining issue to the quantization of classical solutions. This will involve Maslov indices (**??**) and Morse theory (**??**).

6.9 Topos theory ♣

Definition 6.9.1 (Bohr topos). Consider a C^*-algebra A and denote by $\mathsf{ComSub}(A)$ the poset (**??**) of commutative C^*-subalgebras. This set can be equipped with the **Alexandrov topology**[10], i.e. the topology for which the open sets are the upward closed subsets. The topological space $(\mathsf{ComSub}(A), \tau_{\text{Alex}})$ is called the Bohr site of A.

[10] There exist an equivalences **Pre** $\cong$ **AlexTop** and **Pos** $\cong$ $\mathbf{AlexTop}_{T_0}$.

The sheaf topos over the Bohr site is called the Bohr topos $\mathbf{Bohr}(A)$. It can be turned into a ringed topos, where the ring object (which is even an internal commutative C^*-algebra) is given by the tautological functor

$$\underline{A} : \mathsf{ComSub}(A) \rightarrow \mathbf{Set} : C \mapsto C\,. \tag{6.73}$$

Property 6.9.2. A morphism in $\mathbf{C^*Alg}$ is commutativity reflecting if and only if the induced morphism on posets admits a right adjoint. Moreover, there exists a bijection between the following two classes of morphisms:

- Geometric morphisms $f : \mathbf{Bohr}(B) \rightarrow \mathbf{Bohr}(A)$ admitting a right adjoint together with epimorphisms of internal algebras $\underline{A} \rightarrow f^*\underline{B}$.
- Commutativity-reflecting functions $f : A \rightarrow B$ that restrict to algebra morphisms on all commutative subalgebras.

Definition 6.9.3 (Spectral presheaf). The presheaf on a Bohr site assigning to every commutative subalgebra its Gel'fand spectrum.

The idea behind the Bohr topos is that, given a general C^*-algebra A, the Bohr topos $\mathbf{Bohr}(A)$ is interpreted as its quantum phase space. This is similar to Section 5.4, where smooth spaces are also reinterpreted in terms of sheaf topoi.

Theorem 6.9.4 (Kochen–Specker). *If $A = \mathcal{B}(\mathcal{H})$ with $\dim(\mathcal{H}) > 2$, the spectral presheaf has no global element.*

Property 6.9.5 (Gleason's theorem). There exists a natural bijection between the quantum states of a C^*-algebra A and the classical states of $\underline{A}$ internal to $\mathbf{Bohr}(A)$.

Definition 6.9.6 (Bohrification). Consider a C^*-algebra A together with its Bohr topos $\mathbf{Bohr}(A)$. To its internal C^*-algebra $\underline{A}$, one can assign an internal locale $\underline{\Sigma}_A$ by (internal) Gel'fand duality (Definition 4.2.33). Under the equivalence 2.4.38, one then obtains a locale Σ_A. The functor

$$\Sigma : \mathbf{C^*Alg} \rightarrow \mathbf{Loc} : A \mapsto \Sigma_A \tag{6.74}$$

is called Bohrification. This locale can be constructed as the disjoint union

$$\Sigma_A = \bigsqcup_{C \in \mathsf{ComSub}(A)} \Phi_C\,, \tag{6.75}$$

the étale locale corresponding to the spectral presheaf. Its open sets are given by those subsets whose restrictions to commutative subalgebras are open in such a way that these restrictions are compatible with subalgebra inclusions.

Example 6.9.7 (Gel'fand spectrum). If A is a commutative C^*-algebra, its Bohrification is not isomorphic to its ordinary Gel'fand spectrum Φ_A. However, after replacing the topology on $\mathbf{Bohr}(A)$ by the double negation topology (Example 2.4.41) and repeating the above construction, one obtains

$$\Phi_A \cong \Sigma_A^{\neg\neg}. \tag{6.76}$$

This locale can also be obtained in another way. Double negation $\neg\neg$ defines an (internal) **nucleus** on the (internal) locale Σ_A, i.e. a left-exact monad. $\Sigma_A^{\neg\neg}$ is then given by the fixed points of $\neg\neg : \Sigma_A \to \Sigma_A$.

By Property 6.9.2 above, the following relation is obtained.

Property 6.9.8 (Observables). Morphisms $\mathbf{Bohr}(A) \to \mathbf{Bohr}(C(\mathbb{R})_0)$ admitting a right adjoint together with an epimorphism $\underline{C_0(\mathbb{R})} \to f^*\underline{A}$ correspond to observables on A.

The topological bundle $\Sigma_A \to \mathsf{Alex}(\mathsf{ComSub}(A))$ also admits a topos-theoretic incarnation. There exists a (canonical) morphism of ringed topoi

$$\pi : \mathbf{Bohr}(A) \to \left(\mathbf{Sh}\big(\mathsf{Alex}(\mathsf{ComSub}(A))\big), \underline{\mathbb{R}}\right), \tag{6.77}$$

whose underlying geometric morphism is simply the identity.

Property 6.9.9 (States). A positive and normalized section of the morphism $\pi : \mathbf{Bohr}(A) \to \left(\mathbf{Sh}\big(\mathsf{Alex}(\mathsf{ComSub}(A))\big), \underline{\mathbb{R}}\right)$ in the category of $\underline{\mathbb{R}}$*-module topoi*.

List of Symbols

The following abbreviations and symbols are used throughout the compendium.

Abbreviations

AIC	Akaike information criterion
ARMA	autoregressive moving-average model
BCH	Baker–Campbell–Hausdorff
BPS	Bogomol'nyi–Prasad–Sommerfield
BPST	Belavin–Polyakov–Schwarz–Tyupkin
BRST	Becchi–Rouet–Stora–Tyutin
CCR	canonical commutation relation
CDF	cumulative distribution function
CFT	conformal field theory
CIS	completely integrable system
CP	completely positive
CPTP	completely positive, trace-preserving
CR	Cauchy–Riemann
dga	differential graded algebra
dgca	differential graded-commutative algebra
EMM	equivalent martingale measure
EPR	Einstein–Podolsky–Rosen
ESM	equivalent separating measure
ETCS	Elementary Theory of the Category of Sets
FIP	finite intersection property
FWHM	full width at half maximum
GA	geometric algebra
GHZ	Greenberger–Horne–Zeilinger

GNS	Gel'fand–Naimark–Segal
HJE	Hamilton–Jacobi equation
HoTT	Homotopy Type Theory
KKT	Karush–Kuhn–Tucker
LIVF	left-invariant vector field
MCG	mapping class group
MPO	matrix-product operator
MPS	matrix-product state
MTC	modular tensor category
NDR	neighbourhood deformation retract
OPE	operator product expansion
OTC	over the counter
OZI	Okubo–Zweig–Iizuka
PAC	probably approximately correct
PDF	probability density function
PID	principal ideal domain
PL	piecewise-linear
PMF	probability mass function
POVM	positive operator-valued measure
PRP	predictable representation property
PVM	projection-valued measure
RKHS	reproducing kernel Hilbert space
SVM	support-vector machine
TDSE	time-dependent Schrödinger equation
TISE	time-independent Schrödinger equation
TQFT	topological quantum field theory
TVS	topological vector space
UFD	unique factorization domain
VC	Vapnik–Chervonenkis
VIF	variance inflation factor
VOA	vertex operator algebra
WKB	Wentzel–Kramers–Brillouin

ZFC	Zermelo–Frenkel set theory with the axiom of choice

Operations

Ad_g	adjoint representation of a Lie group G
ad_X	adjoint representation of a Lie algebra $\mathfrak{g}$
arg	argument of a complex number
$\Box$	d'Alembert operator
$\mathsf{deg}(f)$	degree of a polynomial f
e	identity element of a group
$\Gamma(E)$	set of global sections of a fibre bundle E
Im, $\mathfrak{I}$	imaginary part of a complex number
$\mathsf{Ind}_f(z)$	index of a point $z \in \mathbb{C}$ with respect to a function f
$\hookrightarrow$	injective function
$\cong$	is isomorphic to
$A \multimap B$	linear implication
$N \lhd G$	N is a normal subgroup of G
Par_t^γ	parallel transport map along a curve γ
Re, $\mathfrak{R}$	real part of a complex number
Res	residue of a complex function
$\twoheadrightarrow$	surjective function
$\{\cdot,\cdot\}$	Poisson bracket
$X \pitchfork Y$	transversally intersecting manifolds X, Y
∂X	boundary of a topological space X
$\overline{X}$	closure of a topological space X
X°, $\mathring{X}$	interior of a topological space X
$\sphericalangle(\cdot,\cdot)$	angle between two vectors
$X \times Y$	cartesian product of two sets X, Y
$X + Y$	sum of two vector spaces X, Y
$X \oplus Y$	direct sum of two vector spaces X, Y
$V \otimes W$	tensor product of two vector spaces V, W
$\mathbb{1}_X$	identity morphism on an object X
$\approx$	is approximately equal to

$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto

Objects

Ab	category of Abelian groups
$\mathsf{Aut}(X)$	automorphism group of an object X
$\mathcal{B}_0(V,W)$	space of compact bounded operators between two Banach spaces V,W
$\mathcal{B}_1(\mathcal{H})$	space of trace-class operators on a Hilbert space
$\mathcal{B}(V,W)$	space of bounded linear maps between two vector spaces V,W
CartSp	category of Euclidean spaces and 'suitable' morphisms (e.g. linear maps, smooth maps, ...)
$C(X,Y)$	set of continuous functions between two topological spaces X,Y
S'	centralizer of a subset (of a ring)
$C_\bullet$	chain complex
$\mathbf{Ch}(\mathbf{A})$	category of chain complexes with objects in an additive category $\mathbf{A}$
$\mathbf{C}^\infty$, **SmoothSet**	category of smooth sets
$C_p^\infty(M)$	ring of smooth functions $f : M \to \mathbb{R}$ on a neighbourhood of $p \in M$
$C\ell(A,Q)$	Clifford algebra over an algebra A induced by a quadratic form Q
$C^\omega(V)$	set of all analytic functions defined on a set V
$\mathsf{Conf}(M)$	conformal group of a (pseudo-)Riemannian manifold M
$\mathbf{C}^\infty\mathbf{Ring}$, $\mathbf{C}^\infty\mathbf{Alg}$	category of smooth algebras
$S_k(\Gamma)$	space of cusp forms of weight $k \in \mathbb{R}$
Δ_X	diagonal of a set X
Diff	category of smooth manifolds
DiffSp	category of diffeological spaces and smooth maps
$\mathcal{D}_M$	sheaf of differential operators
D^n	standard n-disk
$\mathsf{dom}(f)$	domain of a function f
$\mathsf{End}(X)$	endomorphism monoid of an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad
$\mathbf{FormalCartSp}_{\mathbf{diff}}$	category of infinitesimally thickened Euclidean spaces

$\mathsf{Frac}(I)$	field of fractions of an integral domain I
$\mathfrak{F}(V)$	space of Fredholm operators on a Banach space V
$\mathbb{G}_a$	additive group (scheme)
$\mathsf{GL}(V)$	general linear group: group of automorphisms of a vector space V
$\mathsf{GL}(n, \mathfrak{K})$	general linear group: group of invertible $n \times n$-matrices over a field $\mathfrak{K}$
Grp	category of groups and group homomorphisms
Grpd	category of groupoids
$\mathsf{Hol}_p(\omega)$	holonomy group at a point p with respect to a principal connection ω
$\mathsf{Hom}_{\mathbf{C}}(V, W)$, $\mathbf{C}(V, W)$	collection of morphisms between two objects V, W in a category $\mathbf{C}$
hTop	homotopy category
$I(S)$	vanishing ideal on an algebraic set S
$I(x)$	rational fractions over an integral domain I
$\mathsf{im}(f)$	image of a function f
$K^0(X)$	K-theory over a (compact Hausdorff) space X
Kan	category of Kan complexes
$K(A)$	Grothendieck completion of a monoid A
$\mathcal{K}_n(A, v)$	Krylov subspace of dimension n generated by a matrix A and a vector v
L^1	space of integrable functions
Law	category of Lawvere theories
Lie	category of Lie groups
$\mathfrak{Lie}$	category of Lie algebras
$\mathfrak{X}^L$	space of left-invariant vector fields on a Lie group
$\mathsf{llp}(I)$	set of morphisms having the left lifting property with respect to I
LX	free loop space on a topological space X
$\mathbf{Man}^p$	category of C^p-manifolds
Meas	• category of measurable spaces and measurable functions, or • category of measure spaces and measure-preserving functions
M^4	four-dimensional Minkowski space
$M_k(\Gamma)$	space of modular forms of weight $k \in \mathbb{R}$
$\mathbb{F}^X$	natural filtration of a stochastic process $(X_t)_{t\in T}$
$N\mathbf{C}$	simplicial nerve of a small category $\mathbf{C}$

$\mathsf{O}(n,\mathfrak{K})$	group of $n \times n$ orthogonal matrices over a field $\mathfrak{K}$
$\mathbf{Open}(X)$	category of open subsets of a topological space X
$P(X), 2^X$	power set of a set X
$\mathsf{Pin}(V)$	pin group of the Clifford algebra $C\ell(V,Q)$
$\mathbf{Psh}(\mathbf{C}), \widehat{\mathbf{C}}$	category of presheaves on a (small) category $\mathbf{C}$
$R((x))$	ring of (formal) Laurent series in x with coefficients in R
$\mathsf{rlp}(I)$	set of morphisms having the right lifting property with respect to I
$R[[x]]$	ring of (formal) power series in x with coefficients in R
S^n	standard n-sphere
$S^n(V)$	space of symmetric rank n tensors over a vector space V
$\mathbf{Sh}(X)$	category of sheaves on a topological space X
$\mathbf{Sh}(\mathbf{C},J)$	category of J-sheaves on a site $(\mathbf{C},J)$
$\boldsymbol{\Delta}$	simplex category
$\mathsf{sing\ supp}(\phi)$	singular support of a distribution ϕ
$\mathsf{SL}_n(\mathfrak{K})$	special linear group: group of all $n \times n$-matrices with unit determinant over a field $\mathfrak{K}$
$W^{m,p}(U)$	Sobolov space in L^p of order m
$\mathbf{Span}(\mathbf{C})$	span category over a category $\mathbf{C}$
$\mathsf{Spec}(R)$	spectrum of a commutative ring R
$\mathbf{sSet}_{\mathrm{Quillen}}$	Quillen's model structure on simplicial sets
$\mathsf{supp}(f)$	support of a function f
$\mathsf{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
$\mathsf{Sym}(X)$	symmetric group of a set X
S_n	symmetric group of degree n
$\mathsf{Sym}(X)$	symmetric group on a set X
$\mathsf{Sp}(n,\mathfrak{K})$	group of matrices preserving a canonical symplectic form over a field $\mathfrak{K}$
$\mathsf{Sp}(n)$	compact symplectic group
$\mathbb{T}^n$	standard n-torus (n-fold Cartesian product of S^1)
$T_{\leq t}$	set of all elements smaller than (or equal to) $t \in T$ for a partial order T
$\mathsf{TL}_n(\delta)$	Temperley–Lieb algebra with $n-1$ generators and parameter δ
$\mathbf{Top}$	category of topological spaces and continuous functions
$\mathbf{Topos}$	(2-)category of (elementary) topoi and geometric morphisms

$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\mathsf{U}(n,\mathfrak{K})$	group of $n \times n$ unitary matrices over a field $\mathfrak{K}$
$V(I)$	algebraic set corresponding to an ideal I
$\mathbf{Vect}(X)$	category of vector bundles over a manifold X
$\mathbf{Vect}_{\mathfrak{K}}$	category of vector spaces and linear maps over a field $\mathfrak{K}$
Y^X	set of functions between two sets X, Y
$\mathbb{Z}_p$	group of p-adic integers
$\emptyset$	empty set
$\pi_n(X, x_0)$	n^{th} homotopy space over X with basepoint x_0
$[a, b]$	closed interval
$]a, b[$	open interval
$\Lambda^n(V)$	space of antisymmetric rank-n tensors over a vector space V
ΩX	(based) loop space on a topological space X
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on a manifold M
$\rho(A)$	resolvent set of a bounded linear operator A
$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on a manifold M

Units

C	Coulomb
T	Tesla

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