Summary: Mathematics & Physics

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December 12, 2017

Contents

Chapter 1

Introduction

Definitions, properties and formulas marked by a dagger symbol † are explained and/or derived in one of the appendices. This has been done such that the 'summary' itself contains only core notions and theorems.

Definitions of words in the middle of a text will be indicated by **bold text**. Terminology that has been defined in the past but that receives a new meaning/nuance will be indicated by *italic text*. Notions that have not been defined in this 'summary' but that are relevant/crucial are also indicated by *italic text*.

Vectors in Euclidean space will be denoted by a bold font letter with an arrow above: $\vec{\boldsymbol{a}}$. Vectors in Minkowski space (4-vectors) will be written without the arrow: $\mathbf{a}$. Matrices and tensors will always be represented by capital letters and dependent on the context we will use bold font or normal font.

Part I

Set Theory

Chapter 2

Set theory

2.1 Collections

Definition 2.1.1 (Power set). Let S be a set. The power set is defined as the set of all subsets of S and is (often) denoted by $P(S)$ or 2^S. The existence of this set is stated by the *axiom of power set*.

Corollary 2.1.2. $S \subset P(S)$

Definition 2.1.3 (Collection). Let A be a set. A collection of elements in A is a subset of A.

Definition 2.1.4 (Family). Let A be a set and let I be another set, called the **index set**. A family of elements of A is a map $f : I \to A$. A family with index set I is often denoted by $(x_i)_{i\in I}$. In contrast to collections a family can 'contain' multiple copies of a single element.

Definition 2.1.5 (Helly family). A Helly family of order k is a pair (X, F) with $F \subset 2^X$ such that for every finite $G \subset F$:

$$\bigcap_{V\in G} V = \emptyset \implies \exists H \subseteq G : \left(\bigcap_{V\in H} V = \emptyset\right) \wedge \Big(|H| \leq k\Big) \tag{2.1}$$

A Helly family of order 2 is sometimes said to have the **Helly property**.

Definition 2.1.6 (Diagonal). Let S be a set. The diagonal of S is defined as follows:

$$\Delta_S = \{(a, a) \in S \times S : a \in S\} \tag{2.2}$$

Definition 2.1.7 (Partition). A partition of X is a family of disjoint subsets $(A_i)_{i\in I} \subset X$ such that $\bigcup_{i\in I} A_i = X$.

Definition 2.1.8 (Refinement). Let P be a partition of X. A refinement P' of P is a finite collection of subsets such that every $A \in P$ can be written as a disjoint union of elements in P'. Hence P' is also a partition.

Definition 2.1.9 (Cover). A cover of S is a collection of sets $\mathcal{F} \subseteq 2^S$ such that

$$\bigcup_{V\in\mathcal{F}} V = S \tag{2.3}$$

2.2 Set operations

Definition 2.2.1 (Symmetric difference).

$$A\Delta B = (A\backslash B) \cup (B\backslash A) \tag{2.4}$$

Definition 2.2.2 (Complement). Let Ω be the universal set . Let $E \subseteq \Omega$. The complement of E is defined as:

$$E^c = \Omega\backslash E \tag{2.5}$$

Formula 2.2.3 (de Morgan's laws).

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c \tag{2.6}$$

$$\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c \tag{2.7}$$

2.3 Ordered sets

2.3.1 Posets

Definition 2.3.1 (Preordered set). A preordered set is a set equipped with a reflexive and transitive binary relation.

Definition 2.3.2 (Partially ordered set). A set P equipped with a binary relation $\leq$ is called a partially ordered set (**poset**) if the following 3 axioms are fulfilled for all elements $a, b, c \in P$:

1. Reflexivity: $a \leq a$
2. Antisymmetry: $a \leq b \wedge b \leq a \implies a = b$
3. Transitivity: $a \leq b \wedge b \leq c \implies a \leq c$

It is a preordered set for which the binary relation is also anti-symmetric.

Definition 2.3.3 (Totally ordered set). A poset P with the property that for all $a, b \in P : a \leq b$ or $b \leq a$ is called a (non-strict) totally ordered set. This property is called **totality**.

Definition 2.3.4 (Strict total order). A non-strict order $\leq$ has an associated strict order $<$ that satisfies $a < b \iff a \leq b \wedge a \neq b$.

Definition 2.3.5 (Maximal element). An element m of a poset P is maximal if for every $p \in P$, $m \leq p$ implies that $m = p$.

Definition 2.3.6 (Chain). A totally ordered subset of a poset is called a chain.

Theorem 2.3.7 (Zorn's lemma[1]). *Let $(P, \leq)$ be a poset. If every chain in P has an upper bound in P, then P has a maximal element.*

[1] This theorem is equivalent to the *axiom of choice*.

2.3.2 Lattices

Definition 2.3.8 (Semilattice). A poset $(P, \leq)$ for which every 2-element subset has a supremum (also called a **join**) in P is called a join-semillatice. Similarly, a poset $(P, \leq)$ for which every 2-element subset has an infimum (also called a **meet**) in P is called a meet-semilattice.

Notation 2.3.9. The join of $\{a, b\}$ is denoted by $a \wedge b$. The meet of $\{a, b\}$ is denoted by $a \vee b$.

Definition 2.3.10 (Lattice). A poset $(P, \leq)$ is called a lattice if it is both a join- and a meet-semilattice.

Definition 2.3.11 (Directed[2] set). A directed set is a set X equipped with a preorder $\leq$ and with the additional property that every 2-element subset has an upper bound, i.e. for every two elements $a, b \in X$ there exists an element $c \in X$ such that $a \leq c \wedge b \leq c$.

Definition 2.3.12 (Net). A net on a topological space X is a subset of X indexed by a directed set I.

2.3.3 Bounded sets

Definition 2.3.13 (Supremum). The supremum $\sup(X)$ of a set X is the smallest upper bound of X.

Definition 2.3.14 (Infimum). The infimum $\inf(X)$ of a set X is the greatest lower bound of X.

Definition 2.3.15 (Maximum). If $\sup(X) \in X$ the supremum is called the maximum of X. This is denoted by $\max(X)$.

Definition 2.3.16 (Minimum). If $\inf(X) \in X$ the supremum is called the minimum of X. This is denoted by $\min(X)$.

2.3.4 Real numbers

Property 2.3.17 (First axiom). The set of real numbers is an ordered field $(\mathbb{R}, +, \cdot, <)$

Property 2.3.18 (Completeness axiom[3]). Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum.

Property 2.3.19. $\mathbb{Q} \subset \mathbb{R}$

[2]Sometimes called an *upward* directed set. Downward directed sets are analogously defined with a lower bound for every two elements. Directed sets are also sometimes called **filtered sets**.

[3]This form of the completeness axiom is also called the supremum property or the Dedekind completeness.

Remark. There is only one way to extend the field of rational numbers to the field of reals such that it satisfies the two previous axioms. This means that for every possible construction, their exists a bijection (isomorphism) between the two.

Definition 2.3.20 (Extended real line).

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty] \tag{2.8}$$

2.3.5 Filter

Definition 2.3.21 (Filter). Let X be a partially ordered set. A family $\mathcal{F} \subseteq 2^X$ is a filter on X if it satisfies following conditions:

1. $\emptyset \notin \mathcal{F}$
2. $\forall A, B \in \mathcal{F} : A \cap B \in \mathcal{F}$
3. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$

2.4 Algebra of sets

Definition 2.4.1 (Algebra of sets). A collection $\mathcal{F}$ of subsets of X is a called an algebra over X if it is closed under finite unions, finite intersections and complements. The pair $(X, \mathcal{F})$ is also called a **field of sets**.

2.4.1 σ-algebra

Definition 2.4.2 (σ-algebra). A collection of sets Σ is a σ-algebra over a set X if it satisfies the following 3 axioms:

1. $X \in \Sigma$
2. Closed under complements: $\forall E \in \Sigma : E^c \in \Sigma$
3. Closed under countable unions: $\forall \{E_i\}_{i=1}^n \subset \Sigma : \bigcup_{i=1}^n E_i \in \Sigma$

Remark 2.4.3. Axioms (2) and (3) together with de Morgan's laws[4] imply that a σ-algebra is also closed under countable intersections.

Corollary 2.4.4. Every algebra of sets is also a σ-algebra.

Property 2.4.5. The intersection of a family of σ-algebras is again a σ-algebra.

[4]See equations 2.6 and 2.7.

Definition 2.4.6. A σ-algebra $\mathcal{G}$ is said to be generated by a collection of sets $\mathcal{A}$ if

$$\mathcal{G} = \bigcap \{\mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-algebra that contains } \mathcal{A}\} \tag{2.9}$$

It is the smallest σ-algebra containing $\mathcal{A}$.

Notation 2.4.7. The σ-algebra generated by a collection of sets $\mathcal{A}$ is often denoted by $\mathcal{F}_{\mathcal{A}}$ or $\sigma(\mathcal{A})$.

Definition 2.4.8 (Borel set). Let $\mathcal{B}$ be the σ-algebra generated by all open[5] sets $O \subset X$. The elements $B \in \mathcal{B}$ are called Borel sets.

Definition 2.4.9 (Product σ-algebra). The smallest σ-algebra containing the products $A_1 \times A_2$ for all $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$ is called the product σ-algebra of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Notation 2.4.10. The product σ-algebra of $\mathcal{F}_1$ and $\mathcal{F}_2$ is denoted by $\mathcal{F}_1 \times \mathcal{F}_2$.

Alternative Definition 2.4.11. The product σ-algebra $\mathcal{F}$ can also be equivalently defined in the following two ways:

1. $\mathcal{F}$ is generated by the collection
$$\mathcal{C} = \{A_1 \times \Omega_2 : A_1 \in \mathcal{F}_1\} \cup \{\Omega_1 \times A_2 : A_2 \in \mathcal{F}_2\}$$
2. $\mathcal{F}$ is the smallest σ-algebra such that the following projections are measurable (see 10.1.32):
$$\mathrm{Pr}_1 : \Omega \to \Omega_1 : (\omega_1, \omega_2) \mapsto \omega_1$$
$$\mathrm{Pr}_2 : \Omega \to \Omega_2 : (\omega_1, \omega_2) \mapsto \omega_2$$

Remark. Previous definitions can easily be generalized to higher dimensions.

2.4.2 Monotone class

Definition 2.4.12 (Monotone class). Let $\mathcal{A}$ be a collection of sets. $\mathcal{A}$ is called a monotone class if it has the following two properties:

- For every increasing sequence $A_1 \subset A_2 \subset ...$:
$$\bigcup_{i=1}^{+\infty} A_i \in \mathcal{A}$$
- For every decreasing sequence $A_1 \supset A_2 \supset ...$:
$$\bigcap_{i=1}^{+\infty} A_i \in \mathcal{A}$$

Theorem 2.4.13 (Monotone class theorem). *Let $\mathcal{A}$ be an algebra of sets 2.4.1. If $\mathcal{G}_{\mathcal{A}}$ is the smallest monotone class containing $\mathcal{A}$ then it coincides with the σ-algebra generated by $\mathcal{A}$.*

[5]For $X = \mathbb{R}$ we find that open, closed and half-open (both types) intervals generate the same σ-algebra.

2.5 Functions

2.5.1 Domain

Definition 2.5.1 (Domain). Let $f : X \to Y$ be a function. The set X, containing the arguments of f, is called the domain of f.

Notation 2.5.2. The domain of f is denoted by $\mathrm{dom}(f)$.

Definition 2.5.3 (Support). Let $f : X \to \mathbb{R}$ be a function with an arbitrary domain X. The support of f is defined as the set of points where f is non-zero.

Notation 2.5.4. The support of f is denoted by $\mathrm{supp}(f)$

Remark. The support of a function is a subset of its domain.

Notation 2.5.5. Let X, Y be two sets. The set of functions $\{f : X \to Y\}$ is often denoted by X^Y.

2.5.2 Codomain

Definition 2.5.6 (Codomain). Let $f : X \to Y$ be a function. The set Y, containing (at least) all the output values of f, is called the codomain of f.

Definition 2.5.7 (Image). Let $f : X \to Y$ be a function. The following subset of Y is called the image of f:

$$\{y \in Y \mid \exists x \in X : f(x) = y\}$$

It is denoted by $\mathrm{im}(f)$.

Definition 2.5.8 (Level set). Let $f : X \to \mathbb{R}$ be a real-valued function and let $c \in \mathbb{R}$. The following set is called the level set of f:

$$L_c(f) = \{x \in X : f(x) = c\} \tag{2.10}$$

For $X = \mathbb{R}^2$ the level set is called a **level curve** and for $X = \mathbb{R}^3$ it is called the **level surface**.

Chapter 3

Algebra

3.1 Groups

Definition 3.1.1 (Semigroup). Let G be a set equipped with a binary operation $\star$. $(G, \star)$ is a semigroup if it satisfies following axioms:

1. G is closed under $\star$
2. $\star$ is asssociative

Definition 3.1.2 (Monoid). Let M be a set equipped with a binary operation $\star$. $(M, \star)$ is a monoid if it satisfies following axioms:

1. M is closed under $\star$
2. $\star$ is associative
3. M contains an identity element with respect to $\star$

Definition 3.1.3 (Group). Let G be a set equipped with a binary operation $\star$. $(G, \star)$ is a group if it satisfies following axioms:

1. G is closed under $\star$
2. $\star$ is asssociative
3. G has an identity element with respect to $\star$
4. Every element in G has an inverse element with respect to $\star$

Definition 3.1.4 (Commutative group[1]). Let $(G, \star)$ be a group. If $\star$ is commutative, then G is called a commutative group.

[1] Also called an Abelian group.

Construction 3.1.5 (Grothendieck completion). Let the couple $(A, \boxplus)$ be an Abelian monoid. From this monoid one can construct an Abelian group $G(A)$, called the Grothendieck completion of A, as the quotient of $A \times A$ by the equivalence relation

$$(a_1, a_1') \sim (a_2, a_2') \iff \exists c \in A : a_1 + \boxplus a_2' \boxplus + c = a_1' \boxplus + a_2 \boxplus + c \tag{3.1}$$

The identity element is given by the equivalence class of $(0, 0)$, which will be denoted by 0. By the definition of $G(A)$, this class contains all elements $\alpha \in \Delta_A$. From this last remark it follows that $[(a, b)] + [(b, a)] = 0$ which implies that the additive inverse of $[(a, b)]$ is given by $[(b, a)]$.

Example 3.1.6. The Grothendieck completion of the natural numbers $G(\mathbb{N})$ coincides with the additive group of integers $\mathbb{Z}$. The positive integers are then given by the equivalence classes $[(n, 0)]$ and the negative integers are given by the classes $[(0, n)]$.

Property 3.1.7. The Grothendieck completion satisfies the following universal property: For every monoid morphism $m : A \to B$ between an Abelian monoid and an Abelian group, there exists a group morphism $\varphi : G(A) \to B$.

3.1.1 Cosets

Definition 3.1.8 (Coset). Let G be a group and H a subgroup of G. The left coset of H with respect to $g \in G$ is defined as the set

$$gH = \{gh : h \in H\} \tag{3.2}$$

The right coset is analogously defined as Hg. If for all $g \in G$ the left and right cosets coincide then the subgroup H is said to be a **normal subgroup**. The sets of left and right cosets are denoted by G/H and $H\backslash G$ respectively.

Definition 3.1.9 (Quotient group). Let G be a group and N a normal subgroup. The quotient group G/N is defined as the set of cosets of N in G. This set can be turned into a group itself by equipping it with a product such that the product of aN and bN is $(aN)(bN)$. The fact that N is a normal subgroup can be used to rewrite this as $(aN)(bN) = (ab)N$.

Definition 3.1.10 (Center). The center of a group is defined as follows:

$$Z(G) = \{z \in G : \forall g \in G, zg = gz\} \tag{3.3}$$

This set is a normal subgroup of G.

3.1.2 Order

Definition 3.1.11 (Order of a group). The number of elements in the group. It is denoted by $|G|$ or $\mathrm{ord}(G)$.

Definition 3.1.12 (Order of an element). The order of an element $a \in G$ is the smallest integer n such that

$$a^n = e \tag{3.4}$$

where e is the identity element of G.

Definition 3.1.13 (Torsion group). A torsion group is a group for which all element have finite order. The torsion set $\mathrm{Tor}(G)$ of a group G is the set of all elements $a \in G$ that have finite order. For Abelian groups, $\mathrm{Tor}(G)$ is a subgroup.

3.1.3 Symmetric and alternating groups

Definition 3.1.14 (Symmetric group). The symmetric group S_n or Sym_n of the set $V = \{1, 2, ..., n\}$ is defined as the set of all permutations of V. The number n is called the **degree** of the symmetric group. The symmetric group $\mathrm{Sym}(X)$ of a finite set X is analogously defined.

Definition 3.1.15 (Alternating group). The alternating group A_n is the subgroup of S_n containing all even permutations.

Definition 3.1.16 (Cycle). A k-cycle is a permutation of the form $(a_1\ a_2\ ...\ a_k)$ sending a_i to a_{i+1} (and a_k to a_1). A **cycle decomposition** of an arbitrary permutation is the decomposition into a product of disjoint cycles.

Formula 3.1.17. Let τ be a k-cycle. Then τ is k-cyclic (hence the name *cycle*):

$$\tau^k = \mathbb{1}_G \tag{3.5}$$

Example 3.1.18. Consider the set $\{1, 2, 3, 4, 5, 6\}$. The permutation $\sigma : x \mapsto x + 2 \pmod 6$ can be written using the cycle decomposition $\sigma = (1\ 3\ 5)(2\ 4\ 6)$.

Definition 3.1.19 (Transposition). A permutation which exchanges two elements but lets the other ones unchanged.

3.1.4 Direct product

Definition 3.1.20 (Direct product). Let G, H be two groups. The direct product $G \otimes H$ is defined as the set-theoretic Cartesian product $G \times H$ equipped with a binary operation $\cdot$ such that:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2) \tag{3.6}$$

where the operations on the right hand side are the group operations in G and H. The structure $G \otimes H = (G \times H, \cdot)$ forms a group.

Notation 3.1.21. When the groups are Abelian, the direct product is sometimes called the **direct sum** and is denoted by $\oplus$.

Definition 3.1.22 (Inner semidirect product). Let G be a group, H a subgroup of G and N a normal subgroup of G. G is said to be the inner semidirect product of H and N, denoted by $N \rtimes H$, if it satifies the following equivalent statements:

- $G = NH$ where $N \cap H = \{\mathbb{1}\}$.
- For every $g \in G$ there exist unique $n \in N, h \in H$ such that $g = nh$.
- For every $g \in G$ there exist unique $h \in H, n \in N$ such that $g = hn$.
- There exists a group homomorphism $\rho : G \to H$ which satisfies $\rho|_H = \mathbb{1}$ and $\ker(\rho) = N$.
- The composition of the natural embedding $i : H \to G$ and the projection $\pi : G \to G/N$ is an isomorphism between H and G/N.

G is also said to **split** over N.

Definition 3.1.23 (Outer semidirect product). Let G, H be two groups and let $\varphi : H \to \mathrm{Aut}(G)$ be a group homomorphism. The outer semidirect product $G \rtimes_\varphi H$ is defined as the set-theoretic Cartesian product $G \times H$ equipped with a binary relation $\cdot$ such that:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \varphi(h_1)(g_1), h_1 h_2) \tag{3.7}$$

The structure $(G \rtimes_\varphi H, \cdot)$ forms a group.

By noting that the set $N = \{(g, \mathbb{1}_H) | g \in G\}$ is a normal subgroup isomorphic to G and that the set $B = \{(\mathbb{1}_G, h) | h \in H\}$ is a subgroup isomorphic to H we can also construct the outer semidirect product $G \rtimes_\varphi H$ as the inner semidirect product $N \rtimes H$.

Remark 3.1.24. The direct product of groups is a special case of the outer semidirect product where the group homomorphism is given by the trivial map $\varphi : h \mapsto \mathbb{1}_G$.

3.1.5 Free groups

Definition 3.1.25 (Free Abelian group). An abelian group G with generators $\{g_i\}_{i \in I}$ is said to be freely generated if every element $g \in G$ can be uniquely written as a formal linear combination of the generators:

$$G = \left\{ \sum_i a_i g_i \,\middle|\, a_i \in \mathbb{Z} \right\} \tag{3.8}$$

The set of generators $\{g_i\}_{i \in I}$ is then called a **basis**[2] of G. The number of elements in the basis is called the **rank** of G.

Property 3.1.26. Consider a free group G. Let $H \subset G$ be a subgroup. Then H is also free.

[2] In analogy with the basis of a vector space.

Theorem 3.1.27. *Let G be a finitely generated Abelian group of rank n. This group can be constructed in two different ways:*

$$G = F/H \tag{3.9}$$

where both F, H are free and finitely generated Abelian groups. The second decomposition is:

$$G = A \oplus T \qquad \text{where} \qquad T = Z_{h_1} \oplus \cdots \oplus Z_{h_m} \tag{3.10}$$

where A is a free and finitely generated group of rank $n - m$ and all Z_{h_i} are cyclic groups of order h_i. The group T is called the torsion subgroup[3]*.*

Property 3.1.28. The rank $n - m$ and the numbers h_i from previous theorem are unique.

3.1.6 Group presentations

Definition 3.1.29 (Relations). Let G be a group. If the product of a number of elements $g \in G$ is equal to the identity e then this product is called a relation on G.

Definition 3.1.30 (Complete set of relations). Let H be a group generated by a subgroup G. Let R be a set of relations on G. If H is uniquely (up to an isomorphism) determined by G and R then the set of relations is said to be complete.

Definition 3.1.31 (Presentation). Let H be a group generated by a subgroup G and a complete set of relations R on G. The pair (G, R) is called a presentation of H.

It is clear that every group can have many different presentations and that it is (very) difficult to tell if two groups are isomorphic by just looking at their presentations.

3.1.7 Group actions

Definition 3.1.32 (Group homomorphism). A group homomorphism $\Phi : G \to H$ is a map satisfying $\forall g, h \in G$

$$\Phi(gh) = \Phi(g)\Phi(h) \tag{3.11}$$

Definition 3.1.33 (Kernel). The kernel of a group homomorphism $\Phi : G \to H$ is defined as the set

$$K = \{g \in G : \Phi(g) = \mathbb{1}_H\} \tag{3.12}$$

Theorem 3.1.34 (First isomorphism theorem). *Let G, H be a groups and let $\varphi : G \to H$ be a group homomorphism. If φ is surjective than $G/\ker\varphi \cong H$.*

Definition 3.1.35 (Group action). Let G be a group. Let V be a set. A map $\rho : G \times V \to V$ is called an action of G on V if it satisfies the following conditions:

- Identity: $\rho(\mathbb{1}_G, v) = v$

[3] See also definition 3.1.13.

- Compatibility: $\rho(gh, v) = \rho(g, \rho(h, v))$

For all $g, h \in G$ and $v \in V$. The set V is called a (left) **G-space**.

Remark 3.1.36. A group action can alternatively be defined as a group homomorphism from G to $\mathrm{Sym}(V)$. It assigns a permutation of V to every element $g \in G$.

Notation 3.1.37. The action $\rho(g, v)$ is often denoted by $g \cdot v$ or even gv.

Definition 3.1.38 (Orbit). The orbit of an element $x \in X$ with respect to a group G is defined as the set:

$$G \cdot x = \{g \cdot x | g \in G\} \tag{3.13}$$

The relation $p \sim q \iff \exists g \in G : p = g \cdot q$ induces an equivalence relation for which the equivalence classes coincide with the orbits of G. The set of equivalence classes $X/\sim$ (sometimes denoted by X/G) is called the **orbit space**.

Definition 3.1.39 (Stabilizer). The stabilizer group or **isotropy group** of an element $x \in X$ with respect to a group G is defined as the set:

$$G_x = \{g \in G | g \cdot x = x\} \tag{3.14}$$

This is a subgroup of G.

Definition 3.1.40 (Free action). A group action is free if $g \cdot x = x$ implies $g = e$ for every $x \in X$. Equivalently, a group action is free if the stabilizer group of all elements is trivial.

Definition 3.1.41 (Faithful action[4]). A group action is faithful if the homomorphism $G \to \mathrm{Sym}(X)$ is injective. Alternatively, a group action is faithful if for every two group elements $g, h \in G$ there exists an element $x \in X$ such that $g \cdot x \neq h \cdot x$.

Definition 3.1.42 (Transitive action). A group action is transitive if for every two elements $x, y \in X$ there exists a group element $g \in G$ such that $g \cdot x = y$. Equivalently we can say that there is only one orbit.

Definition 3.1.43 (Homogeneous space). If the group action of a group G on a G-space X is transitive, then X is said to be a homogeneous space.

Property 3.1.44 (†). Let X be a set and let G be a group such that the action of G on X is transitive. Then their exists a bijection $X \cong G/G_x$ where G_x is the stabilizer of any element $x \in X$.

Definition 3.1.45 (G-module). Let G be a group. Let M be a commutative group. M equipped with a left group action $\varphi : G \times M \to M$ is a (left) G-module if φ satisfies the following equation (distributivity):

$$g \cdot (a + b) = g \cdot a + g \cdot b \tag{3.15}$$

where $a, b \in M$ and $g \in G$.

Definition 3.1.46 (G-module homomorphism). A G-module homomorphism is a map $f : V \to W$ satisfying

$$g \cdot f(v) = f(g \cdot v) \tag{3.16}$$

where the $\cdot$ symbol represents the group action in W and V respectively. It is sometimes called a **G-map**, a **G-equivariant map** or an **intertwining map**.

[4] A faithful action is also called an **effective** action.

3.2 Rings

Definition 3.2.1 (Ring). Let R be a set equipped with two binary operations $+, \cdot$ (called addition and multiplication). $(R, +, \cdot)$ is a ring if it satisfies the following axioms:

1. $(R, +)$ is a commutative group.
2. $(R, \cdot)$ is a monoid.
3. Multiplication is distributive with respect to addition.

Definition 3.2.2 (Unit). An invertible element of ring $(R, +, \cdot)$. The set of units forms a group under multiplication.

3.2.1 Ideals

Definition 3.2.3 (Ideal). Let $(R, +, \cdot)$ be a ring with $(R, +)$ its additive group. A subset $I \subseteq R$ is called an ideal[5] of R if it satisfies the following conditions:

1. $(I, +)$ is a subgroup of $(R, +)$
2. $\forall n \in I, \forall r \in R : (n \cdot r), (r \cdot n) \in I$

Definition 3.2.4 (Unit ideal). Let $(R, +, \cdot)$ be a ring. R itself is called the unit ideal.

Definition 3.2.5 (Proper ideal). Let $(R, +, \cdot)$ be a ring. A subset $I \subset R$ is said to be a proper ideal if it is an ideal of R and if it is not equal to R.

Definition 3.2.6 (Prime ideal). Let $(R, +, \cdot)$ be a ring. A proper ideal I is a prime ideal if for any $a, b \in R$ the following relation holds:

$$ab \in I \implies \text{a} \in I \vee b \in I \tag{3.17}$$

Definition 3.2.7 (Maximal ideal). Let $(R, +, \cdot)$ be a ring. A proper ideal I is said to be maximal if there exists no other proper ideal T in R such that $I \subset T$.

Definition 3.2.8 (Minimal ideal). A proper ideal is said to be minimal if it contains no other nonzero ideal.

Construction 3.2.9 (Generating set of an ideal). Let R be a ring and let X be a subset of R. The two-sided ideal generated by X is defined as the intersection of all two-sided ideals containing X. An explicit construction is given by:

$$I = \left\{ \sum_{i=1}^{n} l_i x_i r_i \;\middle|\; \forall i \leq n : l_i, r_i \in R \text{ and } x_i \in X \right\} \tag{3.18}$$

Left and right ideals are generated in a similar fashion.

[5] More generally: two-sided ideal

3.2.2 Modules

Definition 3.2.10 (R-Module). Let $(R, +, \cdot)$ be a ring. A set X is an R-module if it satisfies the same axioms as those of a vector space 16.2.1 but where the scalars are only elements of a ring instead of a field.

Property 3.2.11. For a general R-module the existence of a basis is not guaranteed unless R is a division ring. See construction 16.2.8 to see how this basis can be constructed.

Corollary 3.2.12. As every field is in particular a division ring, the existence of a basis follows from the above property for R-modules.

Definition 3.2.13 (Free module). A module is said to be free if it admits a basis.

Definition 3.2.14 (Projective module). A module P is said to be projective if:

$$P \oplus M = F \tag{3.19}$$

where M is a module and F is a free module.

3.2.3 Graded rings

Definition 3.2.15 (Graded ring). Let R be a ring that can be written as the direct sum of Abelian groups A_k:

$$R = \bigoplus_{k \in \mathbb{N}} A_k \tag{3.20}$$

If R has the property that for every $i, j \in \mathbb{N} : A_i \star A_j \subseteq A_{i+j}$, where $\star$ is the ring multiplication, then R is said to be a graded ring. The elements of the space A_k are said to be **homogeneous of degree** k.

Formula 3.2.16 (Graded commutativity). Let $m = \deg v$ and let $n = \deg w$. If

$$vw = (-1)^{mn} wv \tag{3.21}$$

for all elements v, w of the graded ring then it is said to be a graded-commutative ring.

3.3 Other algebraic structures

Definition 3.3.1 (Chain complex). Let $(A_k)_{k \in \mathbb{N}}$ be a sequence of algebraic structures together with a sequence $\{\partial_k : A_k \to A_{k-1}\}_{k \in \mathbb{N}}$ of morphisms, called the **boundary operators** or **differentials**, such that for all k:

$$\partial_k \circ \partial_{k+1} = 0 \tag{3.22}$$

This structure is called a chain complex[6]. Elements in $\mathrm{im}(\partial_k)$ are called **boundaries** and elements in $\ker(\partial_k)$ are called **cycles**.

[6] A **cochain complex** is constructed similarly. For this structure we consider an ascending order, i.e.: $\partial_k : A_k \to A_{k+1}$.

3.3.1 Direct systems

Definition 3.3.2 (Direct system). Let $(I, \leq)$ be a directed set[7]. Let $\{A_i\}_{i\in I}$ be a family of algebraic objects (groups, rings, ...) and let $\{f_{ij} : A_i \to A_j\}_{i,j\in I}$ be a set of homomorphisms with the following properties:

- For every $i \in I$: $f_{ii} = e_i$, where e_i is the identity in A_i.
- For every $i \leq j \leq k \in I$: $f_{ik} = f_{jk} \circ f_{ij}$.

The pair (A_i, f_{ij}) is called a direct system over I.

Definition 3.3.3 (Direct limit). Consider a direct system (A_i, f_{ij}) over a (directed) set I. The direct limit A of these direct systems is defined as follows:

$$\varinjlim A_i = \bigsqcup_{i\in I} A_i \Big/ \sim \tag{3.23}$$

where the equivalence relation is given by $x \in A_i \sim y \in A_j \iff \exists k \in I : f_{ik}(x) = f_{jk}(y)$. Informally put: two elements are equivalent if they eventually become the same.

The algebraic operations on A are defined such that the inclusion maps $\phi_i : A_i \to A$ are morphisms.

3.3.2 Exact sequences

Definition 3.3.4 (Exact sequence). Consider a sequence (finite or infinite) of algebraic structures and their corresponding homomorphisms:

$$A_0 \xrightarrow{\Phi_1} A_1 \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_n} A_n \tag{3.24}$$

The sequence is exact if for every $k \in \mathbb{N} : \mathrm{im}(\Phi_k) = \ker(\Phi_{k+1})$. This implies that $\Phi_{k+1} \circ \Phi_k = 0$ for all $h \in \mathbb{N}$.

Definition 3.3.5 (Short exact sequence). A short exact sequence is an exact sequence of the form:

$$0 \to A_0 \xrightarrow{\Phi_1} A_1 \xrightarrow{\Phi_2} A_3 \to 0 \tag{3.25}$$

A long exact sequence is an infinite exact sequence.

Property 3.3.6. Looking at some small examples we can derive some important constraints for certain exact sequences and especially for short exact sequences. Consider the sequence

$$0 \to A \xrightarrow{\Phi} B$$

This sequence can only be exact if Φ is an injective homomorphism (**monomorphism**). This follows from the fact that the only element in the image of the map $0 \to A$ is 0 because the map is a homomorphism. The kernel of Φ is thus trivial which implies that Φ is injective.

[7] See definition 2.3.11.

Analogously, the sequence

$$A \xrightarrow{\Psi} B \to 0$$

is exact if Ψ is a surjective homomorphism (**epimorphism**). This follows from the fact that the kernel of the map $B \to 0$ and thus the image of Ψ is all of B which implies that Ψ is surjective.

It follows that the sequence

$$0 \to A \xrightarrow{\Sigma} B \to 0$$

is exact if Σ is a **bimorphism** (which is often an isomorphism).

3.4 Integers

3.4.1 Partition

Definition 3.4.1 (Composition). Let $n \in \mathbb{N}$. A k-composition of n is a k-tuple $(t_1, ..., t_k)$ such that $\sum_{i=1}^{k} t_k = n$.

Definition 3.4.2 (Partition). Let $n \in \mathbb{N}$. A partition of n is an ordered composition of n.

Definition 3.4.3 (Young diagram[8]). A Young diagram is a visual representation of the partition of an integer n. It is a left justified system of boxes, where every row corresponds to a part of the partition.

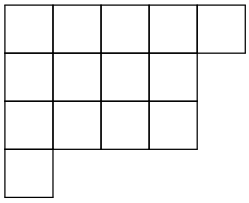


Figure 3.1: A Young diagram representing the partition $(5, 4, 4, 1)$ of 14.

Definition 3.4.4 (Conjugate partition). Let λ be a partition of n with Young diagram $\mathcal{D}$. The conjugate partition λ' is obtained by reflecting $\mathcal{D}$ across its main diagonal.

Example 3.4.5. Using the diagram 3.1 we obtain the conjugate partition $(4, 3, 3, 3, 1)$ represented by

[8]Sometimes called a *Ferrers* diagram.

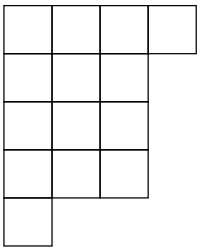


Figure 3.2: A Young diagram representing the partition $(4, 3, 3, 3, 1)$ of 14.

3.4.2 Superpartition

Definition 3.4.6 (Superpartition). Let $n \in \mathbb{N}$. A superpartition in the m-fermion sector is a sequence of integers of the following form:

$$\Lambda = (\Lambda_1, ..., \Lambda_m; \Lambda_{m+1}, ..., \Lambda_N) \tag{3.26}$$

where the first m numbers are strictly ordered, i.e. $\Lambda_i > \Lambda_{i+1}$ for all $i < m$, and the last $N - m$ numbers form a normal partition.

Both sequences, separated by a semicolon, form in fact distinct partitions themself. The first one represents the antisymmetric fermionic sector (this explains the strict order) and the second one represents the symmetric bosonic sector. This amounts to the following notation:

$$\Lambda = (\lambda^a; \lambda^s)$$

The degree of the superpartition is given by $n \equiv |\Lambda| = \sum_{i=1}^{N}$.

Notation 3.4.7. A superpartition of degree n in the m-fermion sector is said to be a superpartition of $(n|m)$. To every superpartition Λ we can also associate a unique partition Λ^* by removing the semicolon and reordering the numbers such that they form a partition of n. The superpartition Λ can then be represented by the Young diagram belonging to Λ^* where the rows belonging to the fermionic sector are ended by a circle.

Chapter 4

Categories, Operads and Topoi

4.1 Category theory

4.2 Operad theory

4.2.1 Operads

Definition 4.2.1 (Plain operad[1]). Let $\mathcal{O} = \{P(n)\}_{n\in\mathbb{N}}$ be a sequence of sets, called n**-ary operations** (n is the **arity**). The set $\mathcal{O}$ is called a plain operad if it satisfies following axioms:

1. $P(1)$ contains an identity element $\mathbb{1}$.
2. For all positive integers $n, k_1, ..., k_n$ there exists a composition

$$\circ : P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots k_n) : (\psi, \theta_1, ..., \theta_n) \mapsto \psi \circ (\theta_1, ..., \theta_n) \quad (4.1)$$

that satisfies two additional axioms:

- Identity:

$$\theta \circ (\mathbb{1}, ..., \mathbb{1}) = \mathbb{1} \circ \theta = \theta \quad (4.2)$$

- Associativity:

$$\psi \circ \Big(\theta_1 \circ (\theta_{1,1}, ..., \theta_{1,k_1}), ..., \theta_n \circ (\theta_{n,1}, ..., \theta_{n,k_n})\Big) = \Big(\psi \circ (\theta_1, ..., \theta_n)\Big) \circ (\theta_{1,1}, ..., \theta_{1,k_1}, \theta_{2,1}, ..., \theta_{n,k_n}) \quad (4.3)$$

4.3 Topos theory

[1] Also called a **non-symmetric operad** or **non-Σ operad**.

Part II

Topology

Chapter 5

General topology

5.1 Topological spaces

Definition 5.1.1 (Topology). Let Ω be a set. Let $\tau \subseteq 2^{\Omega}$. The set τ is a topology on Ω if it satisfies following axioms:

1. $\emptyset \in \tau$ and $\Omega \in \tau$
2. $\forall\ \mathcal{F} \subseteq \tau : \bigcup_{V \in \mathcal{F}} V \in \tau$
3. $\forall\ U, V \in \tau : U \cap V \in \tau$

Furthermore we call the elements of τ open sets and the couple (Ω, τ) a topological space.

Remark. On topological spaces the open sets are thus defined by axioms.

Property 5.1.2. Consider a topological space (X, τ). Let $U \subseteq V \in \tau$. The inclusion maps $U \hookrightarrow V$ are morphisms. The set of these morphisms together with the topology τ form a (small) category $\mathrm{Open}(X)$.

Definition 5.1.3 (Relative topology). Let (X, τ_X) be a topological space and Y a subset of X. We can turn Y into a topological space by equipping it with the following topology, called the relative topology:

$$\tau_{\mathrm{rel}} = \{U_i \cap Y : U_i \in \tau_X\} \tag{5.1}$$

Definition 5.1.4 (Disjoint union). Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Now consider the disjoint union

$$X = \bigsqcup_{i \in I} X_i \tag{5.2}$$

together with the canonical inclusion maps $\phi_i : X_i \to X : x_i \mapsto (x_i, i)$. We can turn X into a topological space by equipping it with the following topology:

$$\tau_X = \{U \subseteq X | \forall i \in I : \phi_i^{-1}(U) \text{ is open in } X_i\} \tag{5.3}$$

Definition 5.1.5 (Quotient space). Let X be a topological space and let $\sim$ be an equivalence relation defined on X. The set $X/_\sim$ can be turned into a topological space by equipping it with the following topology:

$$\tau_\sim = \{U \subseteq X/_\sim | \pi^{-1}(U) \text{ is open in } X\} \tag{5.4}$$

where π is the canonical surjective map from X to $X/_\sim$.

Example 5.1.6 (Discrete topology). The discrete topology is the topology such that every subset is open (and thus also closed).

Example 5.1.7 (Product topology). First consider the case where the index set I is finite. The product space $X = \prod_{i \in I} X_i$ can be turned into a topological space by equipping it with the topology generated by the following basis:

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i \;\middle|\; U_i \in \tau_i \right\} \tag{5.5}$$

For general cases (countably infinite and uncountable index sets) the topology can be defined using the canonical projections $\pi_i : X \to X_i$. The general product topology (**Tychonoff topology**) is the coarsest (finest) topology such that all projections π_i are continuous.

Definition 5.1.8 (Topological group). A topological group is a group G equipped with a topology such that both the multiplication and inversion map are continuous.

Definition 5.1.9 (Pointed topological space). Let $x_0 \in X$. The triple (X, τ, x_0) is called a pointed topological space with base point x_0.

Definition 5.1.10 (Suspension). Let X be a topological space. The suspension of X is defined as the following quotient space:

$$SX = (X \times [0,1])/\{(x,0) \sim (y,0) \text{ and } (x,1) \sim (y,1) | x, y \in X\} \tag{5.6}$$

5.2 Neighbourhoods

5.2.1 Neighbourhoods

Definition 5.2.1 (Neighbourhood). A set $V \subseteq \Omega$ is a neighbourhood of a point $a \in \Omega$ if there exists an open set $U \in \tau$ such that $a \in U \subseteq V$.

Definition 5.2.2 (Basis). Let $\mathcal{B} \subseteq \tau$ be a family of open sets. The family $\mathcal{B}$ is a basis for the topological space (Ω, τ) if every $U \in \tau$ can be written as:

$$U = \bigcup_{V \in \mathcal{F}} V \tag{5.7}$$

where $\mathcal{F} \subseteq \mathcal{B}$.

Definition 5.2.3 (Local basis). Let $\mathcal{B}_x$ be a family of open neighbourhoods of a point $x \in \Omega$. $\mathcal{B}_x$ is a local basis of x if every neighbourhood of x contains at least one element in $\mathcal{B}_x$.

Definition 5.2.4 (First-countability). A topological space (Ω, τ) is first-countable if for every point $x \in \Omega$ there exists a countable local basis.

Property 5.2.5 (Decreasing basis). Let $x \in \Omega$. If there exists a countable local basis for x then there also exists a countable decreasing local basis for x.

Definition 5.2.6 (Second-countability). A topological space (Ω, τ) is second-countable if there exists a countable global basis.

Property 5.2.7. Let X be a topological space. The closure of a subset V is given by:

$$\overline{V} = \{x \in X | \exists \text{ a net } (x_\lambda)_{\lambda \in I} \text{ in } X : x_\lambda \to x\} \tag{5.8}$$

This implies that the topology on X is completely determined by the convergence of nets[1].

Corollary 5.2.8. In first-countable spaces we only have to consider the convergence of sequences.

Definition 5.2.9 (Germ). Let X be a topological space and let Y be a set. Consider two functions $f, g : X \to Y$. If there exists a neighbourhood U of a point $x \in X$ such that

$$f(u) = g(u) \qquad \forall u \in U$$

then this property defines an equivalence relation denoted by $f \sim_x g$ and the equivalence classes are called **germs**.

Property 5.2.10. Let the set Y in the previous definition be the set of reals $\mathbb{R}$. Then the germs at a point $p \in X$ satisfy following closure/linearity relations:

- $[f] + [g] = [f + g]$
- $\lambda[f] = [\lambda f]$
- $[f][g] = [fg]$

where $[f], [g]$ are two germs at p and $\lambda \in \mathbb{R}$ is a scalar.

5.2.2 Separation axioms

Definition 5.2.11 (T_0 axiom[2]). A topological space is T_0 if for every two distinct points x, y at least one of them has a neighbourhood not containing the other. The points are said to be topologically distinguishable.

[1] See definition 2.3.12.

[2] T_1 spaces are also said to carry the **Kolmogorov topology**.

Definition 5.2.12 (T_1 axiom[3]). A topological space is T_1 if for every two distinct points x, y there exists a neighbourhood U of x such that $y \notin U$. The points are said to be separated.

Definition 5.2.13 (Hausdorff space). A topological space is a Hausdorff space or T_2 space if it satisfies the following axiom:

$$(\forall x, y \in \Omega)(\exists \text{ neighbourhoods } U, V)(x \in U, y \in V, U \cap V = \emptyset) \tag{5.9}$$

This axiom is called the **Hausdorff separation axiom** or T_2 axiom. The points are said to be separated by neighbourhoods.

Property 5.2.14. Every singleton (and thus also every finite set) is closed in a Hausdorff space.

Definition 5.2.15 (Urysohn space[4]). A topological space is an Urysohn space if every two distcinct points are separated by closed neighbourhoods.

Definition 5.2.16 (Regular space). A topological space is said to be regular if for every closed subset F and every point $x \notin F$ there exist disjoint open subsets U, V such that $x \in U$ and $F \subset V$.

Definition 5.2.17 (T_3 axiom). A space that is both regular and T_0 is T_3.

Definition 5.2.18 (Normal space). A topological space is said to be normal if every two closed subsets have disjoint neighbourhoods.

Definition 5.2.19 (T_4 axiom). A space that is both normal and T_1 is T_4.

Property 5.2.20. A space satisfying the separation axiom T_k also satisfies all separation axioms $T_{i \leq k}$.

5.3 Convergence and continuity

5.3.1 Convergence

Definition 5.3.1 (Convergence). A sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to converge to a point $a \in X$ if:

$$(\forall \text{ neighbourhoods } V \text{ of } a)(\exists\, N > 0)(\forall\, n > N)(x_n \in V) \tag{5.10}$$

Property 5.3.2. Every subsequence of a converging sequence converges to the same point[5].

Property 5.3.3. Let X be a Hausdorff space. The limit of a converging sequence in X is unique.

[3] T_1 spaces are also said to carry the **Fréchet topology**.

[4] Sometimes called a $T_{2\frac{1}{2}}$ space.

[5] This limit does not have to be unique. See the next property for more information.

5.3.2 Continuity

Definition 5.3.4 (Continuity). A function $f : X \to Y$ is continuous if the inverse image $f^{-1}(U)$ of every open set U is also open.

Theorem 5.3.5. *Let X be a first-countable space. Consider a function $f : X \to Y$. The following statements are equivalent:*

- *f is continuous*
- *The sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(a) \in Y$ whenever the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $a \in X$.*

Corollary 5.3.6. If the space Y in the previous theorem is Hausdorff then the limit $f(a)$ does not need to be known because the limit is unique (see 5.3.3).

Remark 5.3.7. If the space X is not first-countable, we have to consider the convergence of nets 2.3.12.

Theorem 5.3.8 (Urysohn's lemma). *A topological space X is normal*[6] *if and only if every two closed disjoint subsets $A, B \subset X$ can be separated by a continuous function $f : X \to [0, 1]$ i.e.*

$$f(a) = 0, \forall a \in A \qquad\qquad f(b) = 1, \forall b \in B \tag{5.11}$$

Theorem 5.3.9 (Tietze extension theorem). *Let X be a normal space and let $A \subset X$ be a closed subset. Consider a continuous function $f : A \to \mathbb{R}$. There exists a continuous function $F : X \to \mathbb{R}$ such that $\forall a \in A : F(a) = f(a)$. Furthermore, if the function f is bounded then F can be chosen to be bounded by the same number.*

Remark. The Tietze extension theorem is equivalent to Urysohn's lemma.

5.3.3 Homeomorphisms

Definition 5.3.10 (Homeomorphism). A function f is called a homeomorphism if both f and f^{-1} are continuous and bijective.

Definition 5.3.11 (Embedding). A function is an embedding if it is homeomorphic onto its image.

Definition 5.3.12 (Mapping cylinder). Let $f : X \to Y$ be a continuous function. The mapping cylinder M_f is defined as follows:

$$M_f = \left([0, 1] \times X \bigsqcup Y\right) / \sim_f \tag{5.12}$$

where the equivalence relation $\sim_f$ is generated by the relations $(0, x) \sim f(x)$. From this definition it follows that the "top" of the cylinder is homeomorphic to X and the "base" is homeomorphic to $f(X) \subseteq Y$.

[6] See definition 5.2.18.

Definition 5.3.13 (Covering space). Consider two topological spaces X, C and a continuous surjective map $\phi : C \to X$, called the **covering map**. C is said to be a covering space of X if for all points $x \in X$ there exists a neighbourhood U of x such that $\phi^{-1}(U)$ can be written as a disjoint union $\bigsqcup_i C_i$ of open sets in C such that every set C_i is mapped homeomorphically onto U. The neighbourhoods U are said to be **evenly covered**.

Definition 5.3.14 (Universal covering space). A covering space C is said to be universal if it is simply-connected[7].

5.4 Connectedness

Definition 5.4.1 (Connected space). A topological space X is connected if it cannot be written as the disjoint union of two non-empty open sets. Equivalently, X is connected if the only clopen sets are X and $\emptyset$.

Property 5.4.2. Let X be a connected space. Let f be a function on X. If f is locally constant, i.e. for every $x \in X$ there exists a neighbourhood U on which f is constant, then f is constant on all of X.

Theorem 5.4.3 (Intermediate value theorem). *Let X be a connected space. Let $f : X \to \mathbb{R}$ be a continuous function. If $a, b \in f(X)$ then for every $c \in]a, b[$ we have that $c \in f(X)$.*

Definition 5.4.4 (Path-connected space[8]). Let X be a topological space. If for every two points $x, y \in X$ there exists a continuous function $\varphi : [0, 1] \to X$ (i.e. a **path**) such that $\varphi(0) = x$ and $\varphi(1) = y$ then the space is said to be path-connected.

Property 5.4.5. Every path-connected space is connected.

The converse does not hold. There exists however the following (stronger) relation:

Property 5.4.6. A connected and locally path-connected space is path-connected.

Remark 5.4.7. The notions of connectedness and path-connectedness define equivalence relations on the space X. The equivalence classes are closed in X and form a cover of X.

5.5 Compact spaces

5.5.1 Compactness

Definition 5.5.1 (Sequentially compact). A topological space is sequentially compact if every sequence[9] has a convergent subsequence.

[7] See definition 5.6.9.

[8] A similar notion is that of **arcwise-connectedness** where the function φ is required to be a homeomorphism.

[9] The sequence itself does not have to converge.

Definition 5.5.2 (Finite intersection property). A family $\mathcal{F} \subseteq 2^X$ of subsets has the finite intersection property[10] if every finite subfamily has a non-zero intersection:

$$\bigcap_{i \in I} V_i \neq \emptyset \tag{5.13}$$

for all finite index sets I.

Definition 5.5.3 (Locally finite cover). An open cover of a topological space X is said to be locally finite if every $x \in X$ has a neighbourhood that intersects only finitely many sets in the cover of X.

Property 5.5.4. A first-countable space is sequentially compact if and only if every countable open cover has a finite subcover.

Definition 5.5.5 (Lindelöf space). A space for which every open cover has a countable subcover.

Property 5.5.6. Every second-countable space is also a Lindelöf space.

Definition 5.5.7 (Compact space). A topological space X is compact if every open cover of X has a finite subcover.

Theorem 5.5.8 (Heine-Borel[11]). *If a topological space X is sequentially compact and second-countable then every open cover has a finite subcover. This implies that X is compact.*

Theorem 5.5.9 (Heine-Borel on $\mathbb{R}^n$). *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 5.5.10 (Tychonoff's theorem). *Any product[12] of compact topological spaces is again compact when equipped with the (Tychonoff) product topology 5.1.7.*

Definition 5.5.11 (Relatively compact). A topological space is called relatively compact if its closure is compact.

Definition 5.5.12 (Locally compact). A topological space is locally compact if every point $x \in X$ has a compact neighbourhood.

Theorem 5.5.13 (Dini's theorem). *Let (X, τ) be a compact space. Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence of continuous functions $f_n : X \to \mathbb{R}$. If $(f_n)_n \to f$ pointwise to a continuous function f then the convergence is uniform.*

Definition 5.5.14 (Paracompact space). A topological space is paracompact if every open cover has a locally finite open refinement.

Property 5.5.15 (ω-boundedness). Let X be a topological space. X is said to be ω-bounded if the closure of every countable subset is compact.

[10] The family is then called a FIP-family.

[11] Also Borel-Lebesgue.

[12] Finite, countably infinite or even uncountably infinite.

Definition 5.5.16 (Partition of unity). Let $\{\varphi_i : X \to [0,1]\}_i$ be a collection of continuous functions such that for every $x \in X$:

- For every neighbourhood U of x, the set $\{f_i : \text{supp} f_i \cap U \neq \emptyset\}$ is finite.
- $\sum_i f_i = 1$

Definition 5.5.17. Consider an open cover $\{V_i\}_{i\in I}$ of X, indexed by a set I. If there exists a partition of unity, also indexed by I, such that $\text{supp}(\varphi_i) \subseteq U_i$, then this partition of unity is said to be **subordinate** to the open cover.

5.5.2 Compactifications

Definition 5.5.18 (Dense). A subset $V \subseteq X$ is dense in a topological space X if $\overline{V} = X$.

Definition 5.5.19 (Separable space). A topological space is separable if it contains a countable dense subset.

Property 5.5.20. Every second-countable space is separable.

Definition 5.5.21 (Compactification). A compact topological space (X', τ') is a compactification of a topological space (X, τ) if X is a dense subspace of X'.

Example 5.5.22. Standard examples of compactifications are the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$ and the extended complex plane $\mathbb{C} \cup \{\infty\}$ for the real line and the complex plane respectively.

Remark. It is important to note that compactifications are not unique.

Definition 5.5.23 (One-point compactification). Let X be a Hausdorff space. A one-point compactification or **Alexandrov compactification** is a compactification X' such that $X' \setminus X$ is a singleton.

5.6 Homotopy theory

5.6.1 Homotopy

Definition 5.6.1 (Homotopy). Let $f, g \in \mathcal{C}(X, Y)$ where X, Y are topological spaces. If there exists a continuous function $H : X \times [0,1] \to Y$ such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ then f and g are said to be homotopic. This relation also induces an equivalence relation on $\mathcal{C}(X, Y)$.

Definition 5.6.2 (Homotopy type). Let X, Y be two topological spaces. X and Y are said to be homotopy equivalent, or of the same homotopy type, if there exist functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to $\mathbb{1}_Y$ and $g \circ f$ is homotopic to $\mathbb{1}_X$. The maps f, g are called **homotopy equivalences**.

Property 5.6.3. Every homeomorphism is a homotopy equivalence.

Definition 5.6.4 (Null-homotopic). A continuous function is null-homotopic if it is homotopic to a constant function.

Definition 5.6.5 (Contractible space). A topological space X is said to be contractible if the identity map $\mathbb{1}_X$ is null-homotopic. Equivalently, the space is homotopy-equivalent to a point.

5.6.2 Fundamental group

In this subsection we will always assume to be working with pointed spaces 5.1.9. The base point will be denoted by x_0.

Definition 5.6.6 (Loop space). The set of all **loops** in X, i.e. all continuous functions $\delta : [0,1] \to X$ for which $\delta(0) = \delta(1)$. It is denoted by ΩX. This set can be equipped with a multiplication operation corresponding to the concatenation of loops[13].

Definition 5.6.7 (Fundamental group). The fundamental group $\pi_1(X, x_0)$ based at $x_0 \in X$ is defined as the loop space (with base x_0) modulo homotopy. As the name implies the fundamental group can be given the structure of a multiplicative group where the operation is inherited from that of the loop space.

Property 5.6.8. In general, as the notation implies, the fundamental group depends on the base point x_0. However when the space X is path-connected, the fundamental groups belonging to different base points are isomorphic. It follows that we can speak of "the" fundamental group in the case of path-connected spaces.

Definition 5.6.9 (Simply-connected space). A topological space is said to be simply-connected if it is path-connected and if the fundamental group is trivial.

The definition of a fundamental group can be generalized to arbitrary dimensions in the following way[14]:

Definition 5.6.10 (Homotopy group). The homotopy group $\pi_n(X, x_0)$ is defined as the set of homotopy classes of continuous maps $f : S^n \to X$ based at $x_0 \in X$. The set $\pi_0(X, x_0)$ is defined as the set of path-connected components of X.

Property 5.6.11. For $n \geq 1$ the sets $\pi_n(X, x_0)$ are groups.

Property 5.6.12. For $n \geq 2$ the homotopy groups $\pi_n(X, x_0)$ are abelian.

Property 5.6.13. If X is path-connected, then the homotopy groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic for all $x_0, x_1 \in X$ and all $n \in \mathbb{N}$.

[13]It should be noted that the speed at which the concatenated loops are traversed is doubled because the parameter t should remain an element of $[0,1]$.

[14]Note however that we replace the interval $[0,1]$ by the sphere S^1. This is nonrestrictive as we can construct S^n by mapping (identifying) the boundary of $[0,1]^n$ to the basepoint x_0.

Property 5.6.14. Homeomorphic spaces have the same homotopy groups π_n.

Formula 5.6.15. Let (X, x_0) and (Y, y_0) be pointed topological spaces with homotopy groups $\pi_n(X, x_0)$ and $\pi_n(Y, y_0)$. The homotopy groups of their product is given by:

$$\pi_n(X \times Y, (x_0, y_0)) = \pi_n(X, x_0) \otimes \pi_n(Y, y_0) \tag{5.14}$$

where $\otimes$ denotes the direct product of groups 3.1.20.

5.7 Homology

5.7.1 simplices

Definition 5.7.1 (Simplex). A k-simplex σ^k is defined as the following set:

$$\sigma^k = \left\{ \sum_{i=0}^{k} \lambda_i t_i \,\middle|\, \sum_{i=0}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \right\} \tag{5.15}$$

where the points (**vertices**) $t_i \in \mathbb{R}^n$ are linearly independent, i.e. the vectors $t_i - t_0$ are linearly independent. Equivalently it is the convex hull of the $k+1$ vertices $\{t_0, ..., t_k\}$.

Remark 5.7.2 (Barycentric coordinates). The coordinates λ_i from previous definition are called barycentric coordinates. This follows from the fact that the point $\sum_{i=0}^{k} \lambda_i t_i$ represents the barycenter of a gravitational system consisting of masses λ_i placed at the points t_i.

Definition 5.7.3 (Simplicial complex). A simplicial complex $\mathcal{K}$ is a set of simplices satisfying following conditions:

- If σ is a simplex in $\mathcal{K}$ then so are its faces.
- If $\sigma_1, \sigma_2 \in \mathcal{K}$ then either $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

A simplicial k-complex is a simplicial complex where every simplex has dimension at most k.

Definition 5.7.4 (Path-connectedness). Let $\mathcal{K}$ be a simplicial complex. $\mathcal{K}$ is said to be path-connected if every two vertices in $\mathcal{K}$ are connected by edges in $\mathcal{K}$.

Definition 5.7.5 (Polyhedron). Let $\mathcal{K}$ be a simplicial complex. The polyhedron associated with $\mathcal{K}$ is the topological space constructed by equipping $\mathcal{K}$ with the Euclidean subspace topology.

Definition 5.7.6 (Triangulable spaces). Let X be a topological space and let $\mathcal{K}$ be a polyhedron. If there exists a homeomorphism $\varphi : \mathcal{K} \to X$ then we say that X is triangulable and we call $\mathcal{K}$ a **triangulation** of X.

Theorem 5.7.7. *Let $\mathcal{K}$ be a path-connected polyhedron with basepoint a_0. Let $\mathcal{C} \subset \mathcal{K}$ be a contractible 1-dimensional subpolyhedron containing all vertices of $\mathcal{K}$. Let G be the free group generated by the elements g_{ij} corresponding to the ordered 1-simplices $[v_i, v_j] \in \mathcal{C}$.*

The group G is isomorphic to the fundamental group $\pi_1(\mathcal{K}, a_0)$ if the generators g_{ij} satisfy following two relations:

- *$g_{ij}g_{jk} = g_{ik}$ for every ordered 2-simplex $[v_i, v_j, v_k] \in \mathcal{K}\backslash\mathcal{C}$*
- *$g_{ij} = e$ if $[v_i, v_j] \in \mathcal{C}$.*

Corollary 5.7.8. From the theorem that homeomorphic spaces have the same homotopy groups it follows that the fundamental group of a triangulable space can be computed by looking at its triangulations.

5.7.2 Simplicial homology

Definition 5.7.9 (Chain group). Let $\mathcal{K}$ be a simplicial n-complex. The k^{th} chain group $C_k(\mathcal{K})$ is defined as the free Abelian group generated by the k-simplices in $\mathcal{K}$:

$$C_k(\mathcal{K}) = \left\{ \sum_i a_i \sigma_i \;\middle|\; \sigma_i \text{ is a } k\text{-simplex in } \mathcal{K} \text{ and } a_i \in \mathbb{Z} \right\} \tag{5.16}$$

For $k > n$ we define $C_k(\mathcal{K})$ to be $\{0\}$.

Definition 5.7.10 (Boundary operator). The boundary operator $\partial_k : C_k(\mathcal{K}) \to C_{k-1}(\mathcal{K})$ is the group morphism defined by following properties:

- Linearity:
$$\partial_k \left(\sum_i a_i \sigma_i \right) = \sum_i a_i \partial_k \sigma_i \tag{5.17}$$
- For every oriented k-simplex $[v_0, ..., v_k]$:
$$\partial_k [v_0, ..., v_k] = \sum_{i=0}^{k} (-1)^i [v_0, ..., \hat{v}_i, ..., v_k] \tag{5.18}$$
where $[v_0, ..., \hat{v}_i, ..., v_k]$ denotes the $(k-1)$-simplex obtained by removing the vertex v_i.
- The boundary of every 0-chain is the identity 0.

Property 5.7.11. The boundary operators satisfy following relation:

$$\partial_k \circ \partial_{k+1} = 0 \tag{5.19}$$

This property turns the system (C_k, ∂_k) into a chain complex[15].

[15] See definition 3.3.1.

Definition 5.7.12 (Cycle group). The k^{th} cycle group $Z_k(\mathcal{K})$ is defined as the set of k-chains σ_k such that $\partial_k \sigma_k = 0$. These chains are called *cycles*.

Definition 5.7.13 (Boundary group). The k^{th} boundary group $B_k(\mathcal{K})$ is defined as the set of k-chains σ_k for which there exists a $(k+1)$-chain N such that $\partial_{k+1} N = \sigma_k$. These chains are called *boundaries*.

Definition 5.7.14 (Homology group). From property 5.19 it follows that $B_k(\mathcal{K})$ is a subgroup of $Z_k(\mathcal{K})$. We can thus define the k^{th} homology group $H_k(\mathcal{K})$ as the following quotient group:

$$H_k(\mathcal{K}) = Z_k(\mathcal{K})/B_k(\mathcal{K}) \tag{5.20}$$

Theorem 3.1.27 tells us that we can write $H_k(\mathcal{K})$ as $G_k \oplus T_k$. Both of these groups tell us something about $\mathcal{K}$. The rank of G_k, denoted by $R_k(\mathcal{K})$, is equal to the number of $(k+1)$-dimensional holes in $\mathcal{K}$. The torsion subgroup T_k tells us how the space $\mathcal{K}$ is twisted.

Property 5.7.15. If two topological spaces have the same homotopy type then they have isomorphic homology groups. It follows that homeomorphic spaces have isomorphic homology groups.

Corollary 5.7.16. As was the case for the fundamental group, tt follows from the definition of a triangulation that we can construct the homology groups for a given triangulable space by looking at one of its triangulations.

Definition 5.7.17 (Betti numbers). The numbers $R_k(\mathcal{K})$ from the definition of homology groups are called the Betti numbers of $\mathcal{K}$.

Formula 5.7.18 (Euler characteristic). The Euler characteristic of a triangulable space X is defined as follows[16]:

$$\boxed{\chi(X) = \sum_i (-1)^i R_i(X)} \tag{5.21}$$

Construction 5.7.19. The definition of homology groups can be generalized by letting the (formal) linear combinations used in the definition of the chain group (see 5.7.9) be of the following form:

$$c^k = \sum_i g_i \sigma_i^k \tag{5.22}$$

where $G = \{g_i\}$ is an Abelian group and σ_i^k are k-simplices. The k^{th} homology group of X with coefficients in G is denoted by $H_k(X; G)$.

Property 5.7.20. When G is a field, such as $\mathbb{Q}$ or $\mathbb{R}$, the torsion subgroups T_k vanish. The relation between integral homology and homology with coefficients in a group is given by the *Universal coefficient theorem*.

Formula 5.7.21 (Künneth formula). Let X, Y be two triangulable spaces. The homology groups of the Cartesian product $X \times Y$ with coefficients in a field F is given by:

$$H_k(X \times Y; F) = \bigoplus_{k=i+j} H_i(X; F) \otimes H_j(Y; F) \tag{5.23}$$

Remark. When the requirement of F being a field is relaxed to it merely being a group, the torsion subgroups have to be taken into account. This will not be done here.

[16] This formula is sometimes called the *Poincaré* or *Euler-Poincaré* formula.

5.7.3 Relative homology

In this section we use a simplicial complex K and a subcomplex L.

Definition 5.7.22 (Relative chain group). The k-chain group of K modulo L is defined as the following quotient group:

$$C_k(K, L) = C_k(K)/C_k(L) \tag{5.24}$$

Definition 5.7.23 (Relative boundary operator). The relative boundary operator $\overline{\partial}_k$ is defined as follows:

$$\overline{\partial}_k(c_k + C_k(L)) = \partial_k c_k + C_{k-1}(L) \tag{5.25}$$

where $c_k \in C_k(K)$. This operator is a group morphism, just like the ordinary boundary operator ∂_k.

Definition 5.7.24 (Relative homology groups). The relative cycle and relative boundary groups are defined analogous to their ordinary counterparts. The relative homology groups are then defined as follows:

$$H_k(K, L) = \frac{\ker \overline{\partial}_k}{\text{im } \overline{\partial}_{k+1}} \tag{5.26}$$

Elements $h_k \in H_k(K, L)$ can thus be written as $h_k = z_k + C_k(L)$ where z_k does not have to be a relative k-cycle but merely a chain in $C_{k-1}(L)$.

Definition 5.7.25 (Homology sequence). Using the relative homology groups we obtain following (long) exact sequence:

$$\cdots \to H_k(L) \xrightarrow{i_*} H_k(K) \xrightarrow{j_*} H_k(K, L) \xrightarrow{\partial_k} H_{k-1}(L) \to \cdots \tag{5.27}$$

where i_* and j_* are the homology morphisms induced by the inclusions $i : L \to K$ and $j : K \to (K, L)$.

Theorem 5.7.26 (Excision theorem). *Let U, V and X be triangulable spaces such that $U \subset V \subset X$. If the closure $\overline{U}$ is contained in the interior V° then:*

$$H_k(X, V) = H_k(X \backslash U, V \backslash U) \tag{5.28}$$

5.7.4 Singular homology

Definition 5.7.27 (Singular simplex). Consider the **standard** k-simplex Δ^k:

$$\Delta^k = \left\{ (x_0, ..., x_k) \in \mathbb{R}^{k+1} : \sum_i x_i = 1 \text{ and } x_i \geq 0 \right\} \tag{5.29}$$

A singular k-simplex in a topological space X is defined as a continuous map $\sigma^k : \Delta^k \to X$. The name singular comes from the fact that the maps σ^k need not be invertible.

Definition 5.7.28 (Singular chain group). The singular chain group $S_k(X)$ with coefficients in a group G is defined as the set of formal linear combinations $\sum_i g_i \sigma_i^k$. The basis of this free group is in most cases infinite as there are multiple ways to map Δ^k to X.

Definition 5.7.29 (Singular boundary operator). The singular boundary operator ∂ (we use the same notation as for simplicial boundary operators) is defined by its linear action on the singular chain group $S_k(X)$. It follows that we only have to know the action on the singular simplices σ^k.

We first introduce the notation $[e_0, ..., e_k] = \Delta_k$ where e_i is the i^{th} vertex of the standard simplex Δ^k. The action on the singular simplex σ^k is then given by:

$$\partial\sigma^k = \sum_i^k (-1)^k \sigma^k \Big([e_0, ..., \hat{e}_i, ..., e_k] \Big) \tag{5.30}$$

The singular boundary operators satisfy the same relation as in the simplicial case:

$$\partial_k \circ \partial_{k-1} = 0 \tag{5.31}$$

Definition 5.7.30 (Singular homology group). The singular homology groups are defined as follows:

$$H_k(X; G) = \frac{\ker \partial_k}{\text{im } \partial_{k+1}} \tag{5.32}$$

Property 5.7.31. Let X be a triangulable space. The k^{th} singular homology group of X is isomorphic to the k^{th} simplicial homology group of X.

Remark 5.7.32. When X is not triangulable the previous theorem is not valid. The singular approach to homology is thus a more general construction, but it is often more difficult to compute the homology groups (even in the case of triangulable spaces).

5.7.5 Examples

Example 5.7.33. Let X be a contractible space. We find that:

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \\ \{0\} & k > 0 \end{cases} \tag{5.33}$$

Example 5.7.34. Let P be a connected polyhedron. We find that:

$$H_0(P) = \mathbb{Z} \tag{5.34}$$

Example 5.7.35. The homology groups of the n-sphere S^n are given by:

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \\ \{0\} & \text{otherwise} \end{cases} \tag{5.35}$$

5.7.6 Axiomatic approach

Definition 5.7.36 (Eilenberg-Steenrod axioms). All homology theories have a set of properties in common. By treating these properties as axioms we can construct homology theories as a sequence of functors H_k. The axioms are as follows:

1. **Homotopy**: If f, g are homotopic maps then their induced homology maps are the same.
2. **Excision**[17]: If $U \subset V \subset X$ and $\overline{U} \subset V^\circ$ then $H_k(X, V) \cong H_k(X \backslash U, V \backslash U)$
3. **Dimension**: If X is a singleton then $H_k(X) = \{0\}$ for all $k \geq 1$.
4. **Additivity**: If $X = \bigsqcup_i X_i$ then $H_k(X) \cong \bigoplus_i H_k(X_i)$
5. **Exactness**: Each pair (X, A), where $A \subset X$, induces a long exact sequence

$$\cdots \to H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X, A) \xrightarrow{\partial_k} H_{k-1}(A) \to \cdots \tag{5.36}$$

   where i_* and j_* are the homology morphisms induced by the inclusions $i : A \to X$ and $j : X \to (X, A)$.

Let X be a singleton. The group $H_0(X)$ is called the **coefficient group** and gives the coefficients used in the construction of the free Abelian chain groups C_k.

Remark 5.7.37. If the dimension axiom is removed from the set of axioms, then we obtain a so-called *extraordinary homology theory*.

5.8 Sheaf theory

5.8.1 Presheafes

Definition 5.8.1 (Presheaf). Let (X, τ) be a topological space. A presheaf over X consists of an algebraic structure S_U for every open set $U \in \tau$ and a morphism $\Phi_V^U : S_U \to S_V$ for every two open sets $U, V \in \tau$ with $V \subseteq U$ such that the following conditions are satisfied:

- $\Phi_U^U = \mathrm{Id}_{U,V}$
- If $W \subseteq V \subseteq U$ then $\Phi_W^U = \Phi_W^V \circ \Phi_V^U$.

The set S_U is called the set of **sections** over U and the morphisms Φ_V^U are called the **restriction maps**.

Definition 5.8.2 (Morphism of presheaves). Let S, S' be two presheaves of a space X. A morphism $S \to S'$ is a set of morphisms $\Psi_U : S_U \to S'_U$ that commute with the restriction maps Φ_V^U.

[17] See also theorem 5.7.26.

Alternative Definition 5.8.3 (Category theory). Using the language of category theory one can more easily introduce presheaves: Let $\mathbf{C}$ be a category and let X be a topological space. A $\mathbf{C}$-valued presheaf on X is a contravariant functor $\mathrm{Open}(X) \to \mathbf{C}$.

5.8.2 Sheafs

Definition 5.8.4 (Sheaf). Let X be a topological space. A sheaf over X is a tuple (S, X, π), where S is a topological space and $\pi : S \to X$ a continuous surjection, such that the following two conditions are satisfied:

- For every point $s \in S$ there exists a neighbourhood U such that $\pi|_U$ is a homeomorphism onto some open neighbourhood of $\pi(s) \in X$. This map induces the discrete topology on S
- For every $x \in X$, the set $\pi^{-1}(x)$ is an algebraic structure such that the corresponding algebraic operation is continuous.

Definition 5.8.5 (Stalk). The preimage $\pi^{-1}(x)$ is called the stalk over x and is often denoted by S_x.

Definition 5.8.6 (Morphism of sheaves). Let S, S' be two sheaves over the same space with projections π and π'. A morphism of sheaves is a map Φ satisfying the following conditions:

- $\Phi : S \to S'$ is continuous.
- $\pi = \pi' \circ \Phi$, i.e. Φ maps stalks in S to corresponding stalks in S'.
- For each $x \in X$, the restriction $\Phi|_x : S_x \to S'_x$ is a morphism of the algebraic structures corresponding to the stalks.

Construction 5.8.7. For every presheaf over X we can construct a sheaf (S, X, π). For every $x \in X$ we set the stalk S_x to be the direct limit 3.23 of the direct system (S_U, Φ_V^U). The set S is then defined as the union of all sets S_x and π maps every element of $S_x \subset S$ to x.

The topology on S is defined by means of the following basis. For every $U \in X$ and every element $f \in S_U$ we construct a subset $f_U \subset S$ given by $\{f_x \in S_x : x \in U\}$ where f_x is called the **germ**[18] of f at x. The basis for our topology is then given by the set $\{f_U : U \subset X, f \in S_U\}$.

5.8.3 Sections

Definition 5.8.8 (Section). A section of a sheaf (S, X, π) over an open set U is a continuous map $s : U \to S$ such that $\pi \circ s = \mathbb{1}_U$. The set of all sections carries the same algebraic structure as S.

Remark. A **global** section is a section $s : X \to S$.

[18] This is a generalization of definition 5.2.9.

Chapter 6

Metric spaces

6.1 General definitions

Definition 6.1.1 (Metric). A metric (or distance) on a set M is a map $d : M \times M \to \mathbb{R}^+$ that satisfies the following properties:

- Non-degeneracy: $d(x, y) = 0 \iff x = y$
- Symmetry: $d(x, y) = d(y, x)$
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z) \quad , \forall x, y, z \in M$

Definition 6.1.2 (Metric space). A set M equipped with a metric d is called a metric space and is denoted by (M, d).

Definition 6.1.3 (Diameter). The diamater of a subset $U \subset M$ is defined as

$$\text{diam}(U) = \sup_{x,y \in U} d(x, y) \tag{6.1}$$

Definition 6.1.4 (Bounded). A subset $U \subseteq M$ is bounded if $\text{diam}(U) < +\infty$.

Property 6.1.5. Every metric space is a topological space[1].

Multiple topological notions can be reformulated in terms of a metric. The most important of them are given below:

Definition 6.1.6 (Open ball). An open ball centered on a point $x_0 \in M$ with radius $R > 0$ is defined as the set:

$$\boxed{B(x_0, R) = \{x \in M : d(x, x_0) < R\}} \tag{6.2}$$

Definition 6.1.7 (Closed ball). The closed ball $\overline{B}(x_0, R)$ is defined as the union of the open ball $B(x_0, R)$ and its boundary, i.e. $\overline{B}(x_0, R) = \{x \in M : d(x, x_0) \leq R\}$.

[1] See next chapter.

Definition 6.1.8 (Interior point/neighbourhood). Let N be a subset of M. A point $x \in N$ is said to be an interior point of N if there exists an $R > 0$ such that $B(x, R) \subset M$. Furthermore, N is said to be a neighbourhood of x.

Definition 6.1.9 (Open set). A subset $N \subset M$ is said to be open if every point $x \in N$ is an interior point of N.

Definition 6.1.10 (Closed set). A subset $V \subset M$ is said to be closed if its complement is open.

Definition 6.1.11 (Limit point). Let S be a subset of X. A point $x \in X$ is called a limit point of S if every neighbourhood of x contains at least one point of S different from x.

Definition 6.1.12 (Accumulation point). Let $x \in X$ be a limit point of S. Then x is an accumulation point of S if every open neighbourhood of x contains infinitely many points of S.

Definition 6.1.13 (Convergence). A sequence $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \to M$ in a metric space (M, d) is said to be convergent to a point $a \in M$ if:

$$\forall \varepsilon > 0 : \exists N_0 \in \mathbb{N} : \forall n \geq N_0 : d(x_n, a) < \varepsilon \tag{6.3}$$

Definition 6.1.14 (Continuity). Let (M, d) and (M', d') be two metric spaces. A function $f : M \to M'$ is said to be continuous at a point $a \in \mathrm{dom}(f)$ if:

$$\forall \varepsilon > 0 : \exists \delta_\varepsilon : \forall x \in \mathrm{dom}(f) : d(a, x) < \delta_\varepsilon \implies d'(f(a), f(x)) < \varepsilon \tag{6.4}$$

Property 6.1.15. Let (M, d) be a metric space. The distance function $d : M \times M \to \mathbb{R}$ is a continuous function.

Definition 6.1.16 (Uniform continuity). Let (M, d) and (M', d') be two metric spaces. A function $f : M \to M'$ is said to be uniformly continuous if:

$$\forall \varepsilon > 0 : \exists \delta_\varepsilon : \forall x, y \in \mathrm{dom}(f) : d(x, y) < \delta_\varepsilon \implies d'(f(x), f(y)) < \varepsilon \tag{6.5}$$

This is clearly a stronger notion than that of continuity as the number ε is equal for all points $y \in \mathrm{dom}(f)$.

6.2 Examples of metrics

Definition 6.2.1 (Product space). Consider the cartesian product

$$M = M_1 \times M_2 \times ... \times M_n$$

with $\forall n$: (M_n, d_n) a metric space. If equipped with the distance function $d(x, y) = \max_{1\leq i\leq n} d_i(x_i, y_i)$ this product is also a metric space. It is called the product metric space.

Property 6.2.2. The projection associated with the set M_j is defined as:

$$\mathrm{pr}_j : M \to M_j : (a_1, ..., a_n) \mapsto a_j \tag{6.6}$$

A sequence in a product metric space M converges if and only if every component $(\mathrm{pr}_j(x_m))_{m\in\mathbb{N}}$ converges in (M_j, d_j).

Example 6.2.3 (Supremum distance). Let $K \subset \mathbb{R}^n$ be a compact set. Denote the set of continuous functions $f : K \to \mathbb{C}$ by $\mathcal{C}(K, \mathbb{C})$. The following map defines a metric on $\mathcal{C}(K, \mathbb{C})$:

$$d_\infty(f, g) = \sup_{x\in K} |f(x) - g(x)| \tag{6.7}$$

Example 6.2.4 (p-metric). We can define following set of metrics on $\mathbb{R}^n$:

$$\boxed{d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}} \tag{6.8}$$

Example 6.2.5 (Chebyshev distance).

$$d_\infty(x, y) = \max_{1\le i\le n} |x_i - y_i| \tag{6.9}$$

It is also called the **maximum metric** or L_∞ metric.

Remark 6.2.6. This metric is also an example of a product metric defined on the Euclidean product space $\mathbb{R}^n$. The notation d_∞, which is also used for the supremum distance, can be justified if the space $\mathbb{R}^n$ is identified with the set of maps $\{1, ..., n\} \to \mathbb{R}$ equipped with the supremum distance. Another justification is the following relation:

$$d_\infty(x, y) = \lim_{p\to\infty} d_p(x, y) \tag{6.10}$$

which is also the origin of the name L_∞ metric.

6.3 Metrizable spaces

Definition 6.3.1 (Metrizable space). A topological space X is metrizable if it is homeomorphic to a metric space M or equivalently if there exists a metric function $d : X \times X \to \mathbb{R}$ such that it induces the topology on X.

Theorem 6.3.2 (Urysohn's metrization theorem). *Every second-countable T_3 space is metrizable.*

6.4 Compactness in metric spaces

Theorem 6.4.1 (Stone). *Every metric space is paracompact.*

Definition 6.4.2 (Totally bounded). A metric space M is said to be totally bounded if it satisfies the following equivalent statements:

- For every $\varepsilon > 0$ there exists a finite cover $\mathcal{F}$ of M with $\forall F \in \mathcal{F} : \operatorname{diam}(F) \leq \varepsilon$.
- For every $\varepsilon > 0$ there exists a finite subset $E \subset M$ such that $M \subseteq \bigcup_{x \in E} B(x, \varepsilon)$.

Property 6.4.3. Every totally bounded set is bounded and every subset of a totally bounded set is also totally bounded. Furthermore, every totally bounded space is second-countable.

The following theorem is a generalization of the statement "*a set is compact if and only if it is closed and bounded*" known from Euclidean space $\mathbb{R}^n$.

Theorem 6.4.4. *For a metric space M the following statements are equivalent:*

- *M is compact.*
- *M is sequentially compact.*
- *M is complete and totally bounded.*

Theorem 6.4.5 (Heine-Cantor). *Let M, M' be two metric spaces with M being compact. Every continuous function $f : M \to M'$ is also uniformly continuous.*

Definition 6.4.6 (Equicontinuity). Let X be a topological space and let M be a metric space. A collection $\mathcal{F}$ of maps $X \to M$ is equicontinuous in $a \in X$ if for all neighbourhoods U of a:

$$(\forall f \in \mathcal{F})(\forall x \in U)(d(f(x), f(a)) \leq \varepsilon) \tag{6.11}$$

for all $\varepsilon \geq 0$.

Property 6.4.7. Let $I \subseteq \mathbb{R}$ be an open interval. Let $\mathcal{F}$ be a collection of differentiable functions such that $\{f'(t) : f \in \mathcal{F}, t \in I\}$ is bounded. Then $\mathcal{F}$ is equicontinuous.

Theorem 6.4.8 (Arzelà-Ascoli). *Let K be a compact topological space and let M be a complete metric space. The following statements are equivalent for any collection $\mathcal{F} \subseteq C(K, M)$:*

- *$\mathcal{F}$ is compact with respect to the supremum distance[2].*
- *$\mathcal{F}$ is equicontinuous, closed under uniform convergence and $\{f(x) : f \in \mathcal{F}\}$ is totally bounded for every $x \in K$.*

[2]See formula 6.7.

6.5 Complete metric spaces

Definition 6.5.1 (Cauchy sequence). A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (M, d) is Cauchy (or has the Cauchy property) if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)(d(x_m, x_n) < \varepsilon) \tag{6.12}$$

Property 6.5.2.

- Every closed subset of a complete metric space is complete.
- Every complete subset of a metric space is closed.

Property 6.5.3 (Cauchy criterion). A metric space (M, d) satisfies the Cauchy criterion if a sequence converges to a point $a \in M$ if and only if it is Cauchy.

Definition 6.5.4 (Completeness). A metric space is complete if it satisfies the Cauchy criterion.

6.6 Injective metric spaces

Definition 6.6.1 (Metric retraction). Let (M, d) be a metric space. A function $f : X \to X$ is said to be a retraction of metric spaces if:

- f is idempotent
- f is non-expansive, i.e. the following relation holds for all $x, y \in M$:

$$d(f(x), f(y)) \leq d(x, y) \tag{6.13}$$

The image of f is called a (metric) retract of M.

Definition 6.6.2 (Injective metric space). A metric space M is said to be injective if whenever M is isometric to a subspace Y of a metric space X then Y is a retract of X.

Property 6.6.3. Every injective metric space is complete.

6.7 Convex spaces

Definition 6.7.1 (Convex space). A metric space (M, d) is said to be convex if for every two points $x, y \in M$ there exists a third point $z \in M$ such that:

$$d(x, z) = d(x, y) + d(y, z) \tag{6.14}$$

Definition 6.7.2 (Hyperconvex space). A convex space for which the set of closed balls has the Helly property[3] is called a hyperconvex space.

[3] See definition 2.1.5.

Theorem 6.7.3 (Aronszajn & Panitchpakdi). *A metric space is injective if and only if it is hyperconvex.*

Part III

Calculus

Chapter 7

Calculus

7.1 Sequences

Definition 7.1.1 (Limit superior). Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of real numbers. The limit superior is defined as follows:

$$\limsup_{i\to+\infty} x_i = \inf_{i\geq 1}\left\{\sup_{k\geq i} x_k\right\} \tag{7.1}$$

Definition 7.1.2 (Limit inferior). Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of real numbers. The limit superior is defined as follows:

$$\liminf_{i\to+\infty} x_i = \sup_{i\geq 1}\left\{\inf_{k\geq i} x_k\right\} \tag{7.2}$$

Theorem 7.1.3. *A sequence $(x_i)_{i\in\mathbb{N}}$ converges pointwise if and only if* $\limsup_{i\to+\infty} x_i = \liminf_{i\to+\infty} x_i$.

7.2 Continuity

Definition 7.2.1 (Lipschitz continuity). A function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous if there exists a constant $C > 0$ such that

$$|f(x) - f(x')| \leq C|x - x'| \tag{7.3}$$

for all $x, x' \in \mathbb{R}$.

Theorem 7.2.2 (Darboux's theorem). *Let f be a differentiable function on a closed interval I. Then f' has the intermediate value property*[1].

[1]This means that the function satisfies the conclusion of the intermediate value theorem 5.4.3.

Remark 7.2.3 (Darboux function). Functions that have the intermediate value property are called Darboux functions.

Corollary 7.2.4 (Bolzano's theorem). If $f(a) < 0$ and $f(b) > 0$ (or vice versa) then there exists at least one point x_0 where $f(x_0) = 0$.

Corollary 7.2.5. The image of a compact set is also a compact set.

Theorem 7.2.6 (Weierstrass' extreme value theorem). *Let $I = [a, b] \subset \mathbb{R}$ be a compact interval. Let f be a continuous function defined on I. Then f attains a minimum and maximum at least once on I.*

7.3 Convergence

Definition 7.3.1 (Pointwise convergence). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions. The sequence is said to converge pointwise to a limit function $f(x)$ if

$$\forall x \in \mathrm{dom}(f_n) : \lim_{n\to+\infty} f_n(x) = f(x) \tag{7.4}$$

Definition 7.3.2 (Uniform convergence). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions. The sequence is said to converge uniformly to a limit function $f(x)$ if

$$\sup_{x\in\mathrm{dom}(f_n)} \left\{ |\lim_{n\to+\infty} f_n(x) - f(x)| \right\} = 0 \tag{7.5}$$

7.4 Derivative

7.4.1 Single variable

Formula 7.4.1 (Derivative).

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}} \tag{7.6}$$

Theorem 7.4.2 (Mean value theorem). *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$. Then there exists a point $c \in]a, b[$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{7.7}$$

Definition 7.4.3 (Differentiablity class). Let I be a set. Let f be a function defined on I. If f is n times continuously differentiable on I (i.e. $f^{(i)}$ exists and is continuous for $i = 1, \ldots, n$) then f is said to be of class $\mathbf{C}^n(I)$.

Definition 7.4.4 (Smooth function). A function f is said to be smooth if it is of class $\mathbf{C}^\infty$.

Definition 7.4.5 (Analytic function). A function f is said to be analytic if it is smooth and if its Taylor series expansion around any point x_0 converges to f in some neighbourhood of x_0. The class of analytic functions defined on I is denoted by $\mathbf{C}^\omega(I)$.

Theorem 7.4.6 (Schwarz's theorem). *Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$, then:*

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) \tag{7.8}$$

for all indices $i, j \leq n$.

Method 7.4.7 (Derivative of $f(x)^{g(x)}$). Let us consider a function of the form $u(x) = f(x)^{g(x)}$. To find the derivative of this function we can use the derivative of the natural logarithm:

First we take a look at the natural logarithm of the function:

$$\ln[u(x)] = g(x)\ln[f(x)]$$

Then we look at the derivative of the natural logarithm:

$$\frac{d\ln[u(x)]}{dx} = \frac{1}{u(x)}\frac{du(x)}{dx} \implies \frac{du(x)}{dx} = u(x)\frac{d\ln[u(x)]}{dx}$$

But according to the first equation we also have:

$$\frac{d\ln[u(x)]}{dx} = \frac{d}{dx}g(x)\ln[f(x)] = \frac{dg(x)}{dx}\ln[f(x)] + \frac{g(x)}{f(x)}\frac{df(x)}{dx}$$

Combining these two equations gives:

$$\boxed{\frac{d}{dx}\left[f(x)^{g(x)}\right] = f(x)^{g(x)}\left[\frac{dg}{dx}(x)\ln[f(x)] + \frac{g(x)}{f(x)}\frac{df}{dx}(x)\right]} \tag{7.9}$$

Theorem 7.4.8 (Euler's homogeneous function theorem). *Let f be a homogeneous function, i.e. $f(ax_1, ..., ax_n) = a^n f(x_1, ..., x_n)$. Then f satisfies following equality:*

$$\sum_k x_k \frac{\partial f}{\partial x_k} = nf(x_1, ..., x_n) \tag{7.10}$$

7.5 Riemann integral

Definition 7.5.1 (Improper Riemann integral).

$$\boxed{\int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b f(x)dx} \tag{7.11}$$

7.6 Fundamental theorems

Theorem 7.6.1 (First fundamental theorem of calculus). *Let $f(x)$ be a continuous function defined on the open interval I. Let $c \in I$. The following theorem establishes a link between integration and differentiation:*

$$\boxed{\exists F(x) = \int_c^x f(x')dx' : F'(x) = f(x)} \tag{7.12}$$

Furthermore this function $F(x)$ is uniformly continuous on I.

Remark 7.6.2. The function $F(x)$ in the previous theorem is called a **primitive function** of $f(x)$. Remark that $F(x)$ is just 'a' primitive function as adding a constant to $F(x)$ does not change anything because the derivative of a constant is zero.

Theorem 7.6.3 (Second fundamental theorem of calculus). *Let $f(x)$ be a function defined on the interval $[a, b]$. Furthermore, let $f(x) \in C^1[a, b]$. We then find the following important theorem:*

$$\boxed{\int_a^b f'(x)dx = f(b) - f(a)} \tag{7.13}$$

Theorem 7.6.4 (Differentiation under the integral sign[2]).

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)dy = f(x,b(x)) \cdot b'(x) - f(x,a(x)) \cdot a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}dy} \tag{7.14}$$

7.7 Taylor expansion

Formula 7.7.1 (Exponential function).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{7.15}$$

7.8 Euler integrals

7.8.1 Euler integral of the first kind

Formula 7.8.1 (Beta function).

$$\boxed{B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt} \tag{7.16}$$

[2]This is a more general version of the so called 'Leibnitz integral rule'.

7.8.2 Euler integral of the second kind

Formula 7.8.2 (Gamma function).

$$\boxed{\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt} \tag{7.17}$$

Formula 7.8.3. B(x, y) = $\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Formula 7.8.4 (Recursion formula). Let $n!$ denote the factorial for integer numbers.

$$\Gamma(n) = (n-1)! \tag{7.18}$$

Chapter 8

Series

8.1 Convergence tests

Theorem 8.1.1. *A series* $\sum_{i=1}^{+\infty} a_i$ *can only converge if* $\lim_{i\to+\infty} a_i = 0$.

Property 8.1.2 (Absolute/conditional convergence). If $S' = \sum_{i=1}^{+\infty} |a_i|$ converges then so does the series $S = \sum_{i=1}^{+\infty} a_i$ and S is said to be absolutely convergent. If S converges but S' does not, then S is said to be conditionally convergent.

Definition 8.1.3 (Majorizing series). Let $S_a = \sum_{i=1}^{+\infty} a_i$ and $S_b = \sum_{i=1}^{+\infty} b_i$ be two series. The series S_a is said to majorize S_b if for every $k > 0$ the partial sum $S_{a,k} \geq S_{b,k}$.

Method 8.1.4 (Comparison test). Let S_a, S_b be two series such that S_a majorizes S_b. We have the following cases:

- If S_b diverges, then S_a diverges.
- If S_a converges, then S_b converges.
- If S_b converges, nothing can be said about S_a.
- If S_a diverges, nothing can be said about S_b.

Method 8.1.5 (MacLaurin-Cauchy integral test). Let f be a continuous non-negative monotone decreasing function on the interval $[n, +\infty[$. If $\int_n^{+\infty} f(x)dx$ is convergent then so is $\sum_{k=n}^{+\infty} f(k)$. On the other hand, if the integral is divergent, so is the series.

Remark 8.1.6. The function does not have to be non-negative and decreasing on the complete interval. As long as it does on the interval $[N, +\infty[$ for some $N \geq n$. This can be seen by writing $\sum_{k=n}^{+\infty} f(k) = \sum_{k=n}^{N} f(k) + \sum_{k=N}^{+\infty} f(k)$ and noting that the first term is always finite (the same argument applies for the integral).

Property 8.1.7. If the integral in the previous theorem converges, then the series has following lower and upper bounds:

$$\int_n^{+\infty} f(x)dx \leq \sum_{i=n}^{+\infty} a_i \leq f(n) + \int_n^{+\infty} f(x)dx \tag{8.1}$$

Method 8.1.8 (d'Alembert's ratio test).

$$R = \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{8.2}$$

Following cases arise:

- $R < 1$: the series converges absolutely
- $R > 1$: the series does not converge
- $R = 1$: the test is inconclusive

Method 8.1.9 (Cauchy's root test).

$$R = \limsup_{n \to +\infty} \sqrt[n]{|a_n|} \tag{8.3}$$

We have the following cases:

- $R < 1$: the series converges absolutely
- $R > 1$: the series does not converge
- $R = 1$ and the limit approaches strictly from above: the series diverges
- $R = 1$: the test is inconclusive

Definition 8.1.10 (Radius of convergences). The number $\frac{1}{R}$ is called the radius of convergence.

Remark 8.1.11. The root test is stronger than the ratio test. Whenever the ratio test determines the convergence/divergence of a series, the radius of convergence of both tests will coincide.

Method 8.1.12 (Gauss's test). If $u_n > 0$ for all n then we can write the ratio of successive terms as follows:

$$\left| \frac{u_n}{u_{n+1}} \right| = 1 + \frac{h}{n} + \frac{B(n)}{n^k} \tag{8.4}$$

where $k > 1$ and $B(n)$ is a bounded function when $n \to \infty$. The series converges if $h > 1$ and diverges otherwise.

8.2 Asymptotic expansions

Definition 8.2.1 (Asysmptotic expansion). Let $f(x)$ be a continuous function. A series expansion of order N is called an asymptotic expansion of $f(x)$ if it satisfies:

$$f(x) - \sum_{n=0}^{N} = O(x^{N+1}) \tag{8.5}$$

Method 8.2.2 (Borel transform†). Define the function $F(x) = \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n$. If the integral

$$\int_0^{+\infty} e^{-t} F(xt) dt < +\infty \tag{8.6}$$

for all $x \in \mathbb{R}$ then $F(x)$ is called the Borel transform of $f(x)$. Furthermore the integral will give a convergent expression for $f(x)$.

Theorem 8.2.3 (Watson). *The uniqueness of the function $F(x)$ is guaranteed if the function $f(x)$ is holomorphic on the domain $\{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2} + \varepsilon\}$.*

Chapter 9

Complex calculus

9.1 Complex algebra

The set of complex numbers $\mathbb{C}$ forms a 2-dimensional vector space over the field of real numbers. Furthermore the operations of complex addition and complex multiplication also turn the complex numbers into a field.

Definition 9.1.1 (Complex conjugate). The complex conjugate $\overline{z} : a + bi \mapsto a - bi$ is an involution, i.e. $\overline{\overline{z}} = z$. It is sometimes denoted by z^* instead of $\overline{z}$.

Formula 9.1.2 (Real/imaginary part). A complex number z can also be written as $\mathrm{Re}(z) + i\mathrm{Im}(z)$ where

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \tag{9.1}$$

$$\mathrm{Im}(z) = \frac{z - \overline{z}}{2i} \tag{9.2}$$

Definition 9.1.3 (Argument). Let z be a complex number parametrized as $z = re^{i\theta}$. The number θ is called the argument of z and it is denoted by $\arg(z)$.

Definition 9.1.4 (Riemann sphere). Consider the one-point compactification[1] $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This set is called the Riemann sphere or extended complex plane. The standard operations on $\mathbb{C}$ can be generalized to $\overline{\mathbb{C}}$ in the following way:

$$\begin{aligned} z + \infty &= \infty \\ z * \infty &= \infty \\ \frac{z}{\infty} &= 0 \end{aligned} \tag{9.3}$$

for all non-zero $z \neq \infty$. As there exists no multiplicative inverse for ∞ the Riemann sphere does not form a field.

[1] See definition 5.5.23.

9.2 Holomorphic functions

Definition 9.2.1 (Holomorphic). A function f is holomorphic on an open set U if it is complex differentiable at every point $z_0 \in U$.

Definition 9.2.2 (Biholomorphic). A complex function f is said to be biholomorphic if both f and f^{-1} are holomorphic.

Property 9.2.3 (Cauchy-Riemann conditions). A holomorphic function $f(z)$ satisfies the following conditions:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \tag{9.4}$$

or equivalently:

$$\boxed{\frac{\partial f}{\partial \overline{z}} = 0} \tag{9.5}$$

Theorem 9.2.4 (Looman-Menchoff[2]). *Let $f(z)$ be a continuous complex-valued function defined on a subset $U \in \mathbb{C}$. If the partial derivatives of the real and imaginary part exist and if f satisfies the Cauchy-Riemann conditions then f is holomorphic on U.*

Property 9.2.5. Functions u, v satisfying the CR-conditions are harmonic functions, i.e. they satisfy Laplace's equation.

Property 9.2.6. Functions u, v satisfying the CR-conditions have orthogonal level curves 2.10.

9.3 Complex integrals

In this and further sections, all contours have been chosen to be evaluated counterclockwise (by convention). To obtain results concerning clockwise evaluation, most of the time adding a minus sign is sufficient.

Definition 9.3.1 (Contour). A contour is a curve $z(t)$ that can be parametrized by

$$\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\} \rightarrow z(t) = z = x + iy \tag{9.6}$$

Formula 9.3.2 (Complex contour integral). The complex contour integral of a function $f(z) = u(z) + iv(z)$ is defined as the following line integral:

$$\int_{z_1}^{z_2} f(z)dz = \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y) + iv(x,y)](dx + idy) \tag{9.7}$$

[2]This is the strongest (most general) theorem on the holomorphy of continuous functions as it generalizes the original results by Riemann and Cauchy-Goursat.

Theorem 9.3.3 (Cauchy's Integral Theorem[3]). *Let Ω be a simply-connected subset of $\mathbb{C}$ and let f be a holomorphic function on Ω. Then for every closed rectifiable contour C in Ω:*

$$\boxed{\oint_C f(z)dz = 0} \tag{9.8}$$

Corollary 9.3.4. The contour integral of a holomorphic function depends only on the limits of integration and not on the contour connecting them.

Formula 9.3.5 (Cauchy's Integral Formula). Let Ω be a connected subset of $\mathbb{C}$ and let f be a holomorphic function on Ω. Let C be a contour in Ω. For every point z_0 inside C we find:

$$\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz} \tag{9.9}$$

Corollary 9.3.6 (Analytic function). Let Ω be a connected subset of $\mathbb{C}$ and C a closed contour in Ω. If f is holomorphic on Ω then f is analytic[4] on Ω and:

$$\boxed{f^{(n)}(z_0) = \frac{1}{2\pi i}\oint_C f(z)\frac{n!}{(z - z_0)^{n+1}}dz} \tag{9.10}$$

Furthermore, the derivatives are also holomorphic on Ω.

Theorem 9.3.7 (Morera's Theorem). *If f is continuous on a connected open set Ω and $\oint_C f(z)dz = 0$ for every closed contour C in Ω, then f is holomorphic on Ω.*

Definition 9.3.8 (Meromorphic). A function f is called meromorphic when it is analytic on the whole complex plane with exception of isolated poles and removable singularities.

Theorem 9.3.9 (Sokhotski-Plemelj[5]). *Let $f(x)$ be a continuous complex-valued function defined on the real line and let $a < 0 < b$.*

$$\lim_{\varepsilon \to 0^+} \int_a^b \frac{f(x)}{x \pm i\varepsilon}dx = \mp i\pi f(0) + \mathcal{P}\int_a^b \frac{f(x)}{x}dx \tag{9.11}$$

where $\mathcal{P}$ denotes the Cauchy principal value.

9.4 Laurent series

Definition 9.4.1 (Laurent series). If f is function, analytic on an annulus A, then f can be expanded as the following series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \qquad \text{with} \qquad a_n = \frac{1}{2\pi i}\oint \frac{f(z')}{(z' - z_0)^{n+1}}dz' \tag{9.12}$$

[3] Also called the *Cauchy-Goursat theorem*.

[4] See definition 7.4.5.

[5] See for example [17], page 104.

Remark 9.4.2. The Laurent series of an analytic function f converges uniformly to f in the ring shaped region ('*annulus*') $R_1 < |z - z_0| < R_2$, with R_1 and R_2 the distances from z_0 to the two closest poles.

Definition 9.4.3 (Principal part). The principal part of a Laurent series is defined as the sum:

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n \tag{9.13}$$

9.5 Singularities

9.5.1 Poles

Definition 9.5.1 (Pole). A function $f(z)$ has a pole of order $m > 0$ at a point z_0 if its Laurent series at z_0 satisfies $\forall n < -m : a_n = 0$ and $a_{-m} \neq 0$.

Definition 9.5.2 (Essential singularity). A function $f(z)$ has an essential singularity at a point z_0 if its Laurent series at z_0 satisfies $\forall n \in \mathbb{N} : a_{-n} \neq 0$, i.e. its Laurent series has infinitely many negative degree terms.

Theorem 9.5.3 (Picard's great theorem). *Let $f(z)$ be an analytic function with an essential singularity at z_0. On every punctured neighbourhood of z_0, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely many times.*

Method 9.5.4 (Frobenius transformation). To study the behaviour of a function $f(z)$ at $z \to \infty$, one should apply the Frobenius transformation $h = 1/z$ and study the limit $\lim_{h \to 0} f(h)$.

9.5.2 Branch cuts

Formula 9.5.5 (Roots). Let $z \in \mathbb{C}$. The n^{th} roots[6] of $z = re^{i\theta}$ are given by:

$$z^{1/n} = \sqrt[n]{r} \exp\left(i \frac{\theta + 2\pi k}{n}\right) \tag{9.14}$$

where $k \in \{0, 1, ..., n\}$.

Formula 9.5.6 (Complex logarithm). We parametrize z as $z = re^{i\theta}$.

$$\mathrm{LN}(z) = \ln(r) + i(\theta + 2\pi k) \tag{9.15}$$

[6] Also see the *fundamental theorem of algebra* 16.1.3.

Definition 9.5.7 (Branch). From these two formulas it is clear that the complex roots and logarithms are multi-valued functions. To get an unambiguous image it is necessary to fix a value of the parameter k. By doing so there will arise curves in the complex plane where the function is discontinuous. These are the branch cuts. A **branch** is then defined as a particular choice of the parameter k. For the logarithm the choice for $\arg(\mathrm{LN}) \in]\alpha, \alpha + 2\pi]$ is often denoted by LN_α or $\log_\alpha$.

Definition 9.5.8 (Branch point). Let $f(z)$ be a complex valued function. A point z_0 such that there exists no neighbourhood $|z - z_0| < \varepsilon$ where $f(z)$ is single valued is called a branch point.

Definition 9.5.9 (Branch cut). A line connecting exactly two branch points is called a branch cut. One of the branch points can be at infinity. In case of multiple branch cuts, they do not cross.

Example 9.5.10. Consider the complex function

$$f(z) = \frac{1}{\sqrt{(z - z_1)...(z - z_n)}}$$

This function has singularities at $z_1, ..., z_n$. If n is even, this function will have n (finite) branch points. This implies that the points can be grouped in pairs connected by non-intersecting branch cuts. If n is odd, this function will have n (finite) branch points and one branch point at infinity. The finite branch points will be grouped in pairs connected by non-intersecting branch cuts and the remaining branch point will be joined to infinity by a branch cut which does not intersect the others.(See [6] for the proof.)

Definition 9.5.11 (Principal value). The principal value of a multi-valued complex function is defined as the choice of branch such that $\arg(f) \in]-\pi, \pi]$.

9.5.3 Residue theorem

Definition 9.5.12 (Residue). By applying formula 9.7 to a polynomial function we find:

$$\int_C (z - z_0)^n dz = 2\pi i \delta_{n,-1} \tag{9.16}$$

where C is a circular contour around the pole $z = z_0$. This means that integrating a Laurent series around a pole isolates the coefficient a_{-1}. This coefficient is therefore called the residue of the function at the given pole.

Notation 9.5.13. The residue of a complex function $f(z)$ at a pole z_0 is denoted by $\mathrm{Res}[f(z)]_{z=z_0}$.

Formula 9.5.14. For a pole of order m, the residue is calculated as follows:

$$\mathrm{Res}\left[f(z)\right]_{z=z_j} = a_{-1} = \lim_{z \to z_0} \frac{1}{(m-1)!} \left(\frac{\partial}{\partial z}\right)^{m-1} (f(z)(z - z_0)) \tag{9.17}$$

For essential singularities the residue can be found by writing out the Laurent series explicitly.

Theorem 9.5.15 (Residue theorem). *If $f(z)$ is a meromorphic function in Ω and if C is a closed contour in Ω which contains the poles z_j of $f(z)$, then:*

$$\boxed{\oint_C f(z)dz = 2\pi i \sum_j \text{Res}\left[f(z)\right]_{z=z_j}} \tag{9.18}$$

Remark 9.5.16. For poles on the contour C, only half of the residue contributes to the integral.

Formula 9.5.17 (Argument principle). Let $f(z)$ be a meromorphic function. Let Z_f, P_f be respectively the number of zeroes and poles of $f(z)$ inside the contour C. From the residue theorem we can derive the following formula:

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{f'(z)}dz = Z_f - P_f \tag{9.19}$$

Formula 9.5.18 (Winding number). Let $f(z)$ be a meromorphic function and let C be a simple closed contour. For all $a \notin f(C)$ the winding number or **index** of a with respect to the function f is defined as:

$$\text{Ind}_f(a) = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)-a}dz \tag{9.20}$$

This number will always be an integer.

9.6 Limit theorems

Theorem 9.6.1 (Small limit theorem). *Let f be a function that is holomorphic almost every where on $\mathbb{C}$. Let the contour C be a circular segment with radius ε and central angle α. If z is parametrized as $z = \varepsilon e^{i\theta}$ then*

$$\int_C f(z)dz = i\alpha A$$

with

$$A = \lim_{\varepsilon \to 0} f(z)$$

Theorem 9.6.2 (Great limit theorem). *Let f be a function that is holomorphic almost every where on $\mathbb{C}$. Let the contour C be a circular segment with radius R and central angle α. If z is parametrized as $z = Re^{i\theta}$ then*

$$\int_C f(z)dz = i\alpha B$$

with

$$B = \lim_{R \to +\infty} f(z)$$

Theorem 9.6.3 (Jordan's lemma). *Let g be a continuous function with $g(z) = f(z)e^{bz}$. Let the contour C be a semicircle lying in the half-plane bounded by the real axis and oriented away of the point $\overline{b}i$. If z is parametrized as $z = Re^{i\theta}$ and*

$$\lim_{R\to\infty} f(z) = 0$$

then

$$\int_C g(z)dz = 0$$

9.7 Analytic continuation

Theorem 9.7.1 (Schwarz' reflection principle). *Let $f(z)$ be analytic on the upper half plane. If $f(z)$ is real when z is real then*

$$f(\overline{z}) = \overline{f(z)} \tag{9.21}$$

Chapter 10

Measure theory and Lebesgue integration

10.1 Measure

10.1.1 General definitions

Definition 10.1.1 (Measure). Let X be a set. Let Σ be a σ-algebra over X. A function $\mu : \Sigma \to \overline{\mathbb{R}}$ is called a measure if it satisfies the following conditions:

1. Non-negativity: $\forall E \in \Sigma : \mu(E) \geq 0$
2. Null empty set: $\mu(\emptyset) = 0$
3. Countable-additivity[1] : $\forall i \neq j : E_i \cap E_j = \emptyset \implies \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$

Definition 10.1.2 (Measure space). The pair (X, Σ) is called a measurable space. The elements $E \in \Sigma$ are called measurable sets. The triplet (X, Σ, μ) is called a measure space.

Definition 10.1.3 (Almost everywhere[2]). Let (X, Σ, μ) be a measure space. A property P is said to hold on X almost everywhere (a.e.) if it satisfies the following equation:

$$\mu\left(\{x \in X : \neg P(x)\}\right) = 0 \tag{10.1}$$

Definition 10.1.4 (Complete measure space). The measure space (X, Σ, μ) is said to be complete if for every $E \in \Sigma$ with $\mu(E) = 0$ the following property holds for all $A \subset E$:

$$A \in \Sigma \quad \text{and} \quad \mu(A) = 0$$

Definition 10.1.5 (Completion). Let $\mathcal{F}, \mathcal{G}$ be σ-algebras over a set X. $\mathcal{G}$ is said to be the completion of $\mathcal{F}$ if it is the smallest σ-algebra such that the measure space $(X, \mathcal{G}, \mu)$ is complete.

[1] also called σ-additivity

[2] In probability theory this is foten often called **almost surely**.

Definition 10.1.6 (Regular Borel measure). Let μ be a non-negative countably additive set function defined on $\mathcal{B}$. μ is called a regular Borel measure if it satisifes following equations for every Borel set B:

$$\begin{aligned} \mu(B) &= \inf\{\mu(O) : O \text{ open}, O \supset B\} \\ \mu(B) &= \sup\{\mu(F) : F \text{ closed}, F \subset B\} \end{aligned} \tag{10.2}$$

Definition 10.1.7 (σ-finite measure). Let $(\Omega, \mathcal{F}, P)$ be a measure space. The measure P is said to be σ-finite if there exists a sequence $(A_i)_{i\in\mathbb{N}}$ of measurable sets such that $\bigcup_{i=1}^{+\infty} A_i = \Omega$ with $\forall A_i : P(A_i) < +\infty$.

Method 10.1.8. To show that two measures coincide on a σ-algebra, it suffices to show that they coincide on the generating sets and apply the monotone class theorem 2.4.13.

10.1.2 Lebesgue measure

Formula 10.1.9 (Length of an interval). The length of an open interval $I = (a, b)$ is defined as:

$$l(I) = b - a \tag{10.3}$$

Definition 10.1.10 (Null set). A set $A \subset \mathbb{R}$ is called a null set if it can be covered by a sequence of intervals of arbitrarily small length: $\forall \varepsilon > 0$ there exists a sequence $(I_n)_{n\in\mathbb{N}}$ such that

$$A \subseteq \bigcup_{n=1}^{+\infty} I_n \tag{10.4}$$

with

$$\sum_{i=1}^{+\infty} l(I_n) < \varepsilon \tag{10.5}$$

Theorem 10.1.11. *Let $(E_i)_{i\in\mathbb{N}}$ be a sequence of null sets. The union $\bigcup_{i=1}^{+\infty} E_i$ is also null.*

Corollary 10.1.12. Any countable set is null.

Definition 10.1.13 (Outer measure). Let $X \subseteq \mathbb{R}$ be an open set. The (Lebesgue) outer measure is defined as:

$$\boxed{m^*(X) = \inf\left\{\sum_{i=1}^{+\infty} l(I_i) \text{ with } (I_i)_{i\in\mathbb{N}} \text{ a sequence of open intervals that covers } X\right\}} \tag{10.6}$$

Property 10.1.14. Let I be an interval. The outer measure equals the length: $m^*(I) = l(I)$.

Property 10.1.15. The outer measure is translation invariant: $m^*(A+t) = m^*(A) \quad , \forall A, t$

Property 10.1.16. $m^*(A) = 0$ if and only if A is null.

Property 10.1.17. If $A \subset B$ then $m^*(A) \leq m^*(B)$.

Property 10.1.18 (Countable subadditivity). For every sequence of sets $(E_i)_{i\in\mathbb{N}}$ the following inequality holds:

$$m^*\left(\bigcup_{i=1}^{+\infty} E_i\right) \leq \sum_{i=1}^{+\infty} m^*(E_i) \tag{10.7}$$

Theorem 10.1.19 (Carathéodory's criterion / Lebesgue measure). *Let X be a set. If X satisfies the following equation, it is said to be Lebesgue measurable:*

$$\forall E \subseteq \mathbb{R} : m^*(E) = m^*(E \cap X) + m^*(E \cap X^c) \tag{10.8}$$

This is denoted by $X \in \mathcal{M}$ and the outer measure $m^*(X)$ is called the Lebesgue measure of X denoted by $m(X)$.

Property 10.1.20. All null sets and intervals are measurable.

Property 10.1.21 (Countable additivity). For every sequence $(E_i)_{i\in\mathbb{N}}$ with $E_i \in \mathcal{M}$ satisfying $i \neq j : E_i \cap E_j = \emptyset$ the following equation holds:

$$\boxed{m\left(\bigcup_{i=1}^{+\infty} E_i\right) = \sum_{i=1}^{+\infty} m(E_i)} \tag{10.9}$$

Remark. Previous property, together with the properties of the outer measure, implies that the Lebesgue measure is indeed a proper measure as defined in 10.1.1.

Property 10.1.22. $\mathcal{M}$ is a σ-algebra[3] over $\mathbb{R}$.

Theorem 10.1.23. *For every $A \subset \mathbb{R}$ there exists a sequence $(O_i)_{i\in\mathbb{N}}$ of open sets such that:*

$$A \subset \bigcap_i O_i \qquad \textit{and} \qquad m\left(\bigcap_i O_i\right) = m^*(A) \tag{10.10}$$

Theorem 10.1.24. *For every $E \in \mathcal{M}$ there exists a sequence $(F_i)_{i\in\mathbb{N}}$ of closed sets such that:*

$$\bigcup_i F_i \subset E \qquad \textit{and} \qquad m\left(\bigcup_i F_i\right) = m(E) \tag{10.11}$$

Remark. The previous 2 theorems imply that the Lebesgue measure is a regular Borel measure 10.2.

Theorem 10.1.25. *Let $E \subset \mathbb{R}$. $E \in \mathcal{M}$ if and only if for every $\varepsilon > 0$ there exist an open set $O \supset E$ and a closed set $F \subset E$ such that $m^*(O \backslash E) < \varepsilon$ and $m^*(E \backslash F) < \varepsilon$.*

[3] See definition 2.4.2.

Property 10.1.26. Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of sets with $\forall i : A_i \in \mathcal{M}$. The following two properties apply:

$$\forall i : A_i \subseteq A_{i+1} \implies m\left(\bigcup_{i=1}^{+\infty} A_i\right) = \lim_{i\to+\infty} m(A_i) \tag{10.12}$$

$$\forall i : A_i \supseteq A_{i+1} \wedge m(A_1) < +\infty \implies m\left(\bigcap_{i=1}^{+\infty} A_i\right) = \lim_{i\to+\infty} m(A_i) \tag{10.13}$$

Remark 10.1.27. This property is not only valid for the Lebesgue measure but for every countably additive set function.

Property 10.1.28. The Lebesgue measure $m(X)$ is continuous at $\emptyset$, i.e. if $(A_i)_{i\in\mathbb{N}} \to \emptyset$ then $\lim_{i\to+\infty} m(A_i) = 0$.

Theorem 10.1.29. *$\mathcal{M}$ is the completion of $\mathcal{B}$.*

Corollary 10.1.30. $\mathcal{B} \subset \mathcal{M} \subset \mathcal{F}_{\mathbb{R}}$

Definition 10.1.31 (Restricted Lebesgue measure). Let $B \subset \mathbb{R}$ be a measurable set with measure $m(B) > 0$. The restriction of the Lebesgue measure to the set B is defined as follows:

$$\mathcal{M}_B = \{A \cap B : A \in \mathcal{M}\} \qquad \text{and} \qquad \forall E \in \mathcal{M}_B : m_B(E) = m(E) \tag{10.14}$$

Furthermore, the measure space $(B, \mathcal{M}_B, m_B)$ is complete.

10.1.3 Measurable functions

Definition 10.1.32 (Measurable function). A function f is (Lebesgue) measurable if for every interval $I \subset \mathbb{R} : f^{-1}(I) \in \mathcal{M}$.

Definition 10.1.33 (Borel measurable function). A function f is called Borel measurable[4] if for every interval $I \subset \mathbb{R} : f^{-1}(I) \in \mathcal{B}$.

Remark 10.1.34. Inclusion 10.1.30 implies that every Borel function is also Lebesgue measurable.

Theorem 10.1.35. *The class of Lebesgue measurable[5] functions defined on $E \in \mathcal{M}$ is closed under multiplication and it forms a vector space.*

Property 10.1.36. Following types of functions are measurable:

- monotone functions

[4]These functions are often simply called 'Borel functions'.

[5]This property is also valid for Borel functions.

- continuous functions
- indicator functions

Corollary 10.1.37. Let f, g be measurable functions. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. The composition $F(f(x), g(x))$ is also measurable.

Property 10.1.38. Let f be a measurable function. The set[6] $\{x : f(x) = a\}$ is also measurable for all $a \in \mathbb{R}$.

Theorem 10.1.39. *Define following functions, which are measurable if f is measurable as a result of previous properties:*

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases} = \max(f, 0) \tag{10.15}$$

$$f^-(x) = \begin{cases} 0 & \text{if } f(x) > 0 \\ -f(x) & \text{if } f(x) \leq 0 \end{cases} = \max(-f, 0) \tag{10.16}$$

The function $f : E \to \mathbb{R}$ is measurable if and only if both f^+ and f^- are measurable. Furthermore f is measurable if $|f|$ is measurable, the converse is false.

10.1.4 Limit operations

Property 10.1.40. Let $(f_i)_{i \in \mathbb{N}}$ be a sequence of measurable[7] functions. The following operations are measurable:

- $\min\limits_{i \leq k} f_i$ and $\max\limits_{i \leq k} f_i$
- $\inf\limits_{i \in \mathbb{N}} f_i$ and $\sup\limits_{i \in \mathbb{N}} f_i$
- $\liminf\limits_{i \to +\infty} f_i$ and $\limsup\limits_{i \to +\infty} f_i$

Remark. The measurability of the limit inferior and limit superior follows from their definitions and from the measurability of the inf / sup and min / max.

Property 10.1.41. Let f be a measurable function. Let g be a function such that $f = g$ almost everywhere. The function g is measurable.

Corollary 10.1.42. A result of the previous two properties is the following: if a sequence of measurable functions converges pointwise a.e. then the limit is also a measurable function.

Definition 10.1.43 (Essential supremum).

$$\text{ess sup} f = \sup\{z : f \geq z \text{ a.e.}\} \tag{10.17}$$

[6] This set is called the 'level set' of f.

[7] This property is also valid for Borel functions.

Definition 10.1.44 (Essential infimum).

$$\text{ess inf} f = \inf\{z : f \leq z \text{ a.e.}\} \tag{10.18}$$

Property 10.1.45. Let f be a measurable function. $f \leq \text{ess sup} f$ a.e. and $f \geq \text{ess inf} f$ a.e. We also have that: $\text{ess sup} f \leq \sup f$ and $\text{ess inf} f \geq \inf f$, furthermore this last pair of inequalities becomes a pair of equalities if f is continuous.

Property 10.1.46. Let f, g be measurable functions. $\text{ess sup}(f+g) \leq \text{ess sup} f + \text{ess sup} g$. An analogous inequality holds for the essential infimum.

10.2 Lebesgue integral

10.2.1 Simple functions

Definition 10.2.1 (Indicator function). An important function when working with sets is the following one:

$$\boxed{\mathbb{1}_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}} \tag{10.19}$$

Definition 10.2.2 (Simple function). Let f be a function that takes on a finite number of non-negative values $\{a_i\}$ with for every $i \neq j : f^{-1}(a_i) \cap f^{-1}(a_j) = \emptyset$. f is called a simple function if it can be expanded in the following way:

$$f(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(x) \tag{10.20}$$

with $A_i = f^{-1}(a_i) \in \mathcal{M}$

Remark 10.2.3 (Step function). If the sets A_i are intervals, the simple function is often called a 'step function'.

Formula 10.2.4 (Lebesgue integral of simple functions). Let φ be a simple function as defined in equation 10.20. Let $\mu : \mathcal{M} \to \mathbb{R}$ be a Lebesgue measure and let E be a measurable set. The Lebesgue integral of φ over a E with respect to μ is given by:

$$\int_E \varphi d\mu = \sum_{i=1}^{n} a_i \mu(E \cap A_i) \tag{10.21}$$

Example 10.2.5. Let $\mathbb{1}_{\mathbb{Q}}$ be the indicator function of the set of rational numbers. This function is clearly a simple function. Previous formula makes it possible to integrate the rational indicator function over the real line, which is not possible in the sense of Riemann:

$$\int_{\mathbb{R}} \mathbb{1}_{\mathbb{Q}} dm = 1 \times m(\mathbb{Q}) + 0 \times m(\mathbb{R} \backslash \mathbb{Q}) = 0 \tag{10.22}$$

where the measure of the rational numbers is 0 because it is a countable set (see corollary 10.1.12.

10.2.2 Measurable functions

Formula 10.2.6 (Lebesgue integral). Let f be a non-negative measurable function. Let A be measurable set. The Lebesgue integral of f over E is defined as:

$$\int_E f dm = \sup\left\{\int_E \varphi dm : \varphi \text{ a simple function such that } \varphi \leq f\right\} \tag{10.23}$$

Property 10.2.7. The Lebesgue integral $\int_E f dm$ of a measurable function f is always non-negative.

Notation 10.2.8. The following notation is frequently used (both in the sense of Riemann and Lebesgue):

$$\int f dm = \int_{\mathbb{R}} f dm \tag{10.24}$$

Formula 10.2.9. The following equality is easily proved as for every set $A \subseteq \mathbb{R} : A \cup A^c = \mathbb{R}$.

$$\int_A f dm = \int f \mathbb{1}_A dm \tag{10.25}$$

Theorem 10.2.10. *Let f be a non-negative measurable function. Then $f = 0$ a.e. if and only if $\int_{\mathbb{R}} f dm = 0$.*

Property 10.2.11. The Lebesgue integral over a null set is 0.

Property 10.2.12. Let f, g me measurable functions. The Lebesgue integral has the following properties:

- $f \leq g$ a.e. implies $\int f dm \leq \int g dm$.
- Let A be a measurable set. Let $B \subset A$. Then $\int_B f dm \leq \int_A f dm$.
- The Lebesgue integral is linear.
- For every two disjoint measurable sets A and B we have that $\int_{A\cup B} f dm = \int_A f dm + \int_B f dm$.
- **Mean value theorem**: If $a \leq f(x) \leq b$, then $am(A) \leq \int_A f dm \leq bm(A)$.

Theorem 10.2.13. *Let f be a non-negative measurable function. There exists an increasing sequence $(\varphi_i)_{i\in\mathbb{N}}$ of simple functions such that $\varphi_i \nearrow f$.*

Theorem 10.2.14. *Let f be a bounded measurable function defined on the interval $[a, b]$. For every $\varepsilon > 0$ there exists a step function[8] h such that $\int_a^b |f - h| dm < \varepsilon$.*

[8]See remark 10.2.3.

10.2.3 Integrable functions

Definition 10.2.15 (Integrable function). Let $E \in \mathcal{M}$. A measurable function f is said to be integrable over E if both $\int_E f^+ dm$ and $\int_E f^- dm$ are finite. The Lebesgue integral of f over E is defined as:

$$\int_E f dm = \int_E f^+ dm - \int_E f^- dm \tag{10.26}$$

Remark. The difference between the integral 10.23 and the integral of an integrable function is that with the latter f does not have to be non-negative.

Theorem 10.2.16. *f is integrable if and only if $|f|$ is integrable. Furthermore, $\int_E |f| dm = \int_E f^+ dm + \int_E f^- dm$.*

Property 10.2.17. Let f, g be integrable functions. The following important properties apply:

- $f + g$ is also integrable.
- $\forall E \in \mathcal{M}, \int_E f dm \leq \int_E g dm \implies f \leq g$ a.e.
- Let $c \in \mathbb{R}$. $\int_E (cf) dm = c \int_E f dm$.
- f is finite a.e.
- $|\int f dm| \leq \int |f| dm$
- $f \geq 0 \wedge \int f dm = 0 \implies f = 0$ a.e.

Theorem 10.2.18. *The set of functions integrable over a set $E \in \mathcal{M}$ forms a vector space. It is denoted by $\mathcal{L}^1(E)$.*

Property 10.2.19. Let $f \in \mathcal{L}^1$ and $\varepsilon > 0$. There exists a continuous function g, vanishing outside some finite interval, such that $\int |f - g| dm < \varepsilon$.

Property 10.2.20. Let $f \geq 0$. The mapping $E \mapsto \int_E f dm$ is a measure on E (if it exists, hence if f is integrable). Furthermore, this measure is said to be **absolutely continuous**.

Remark. See section 10.6 for further information.

10.2.4 Convergence theorems

Theorem 10.2.21 (Fatou's lemma). *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of non-negative measurable functions.*

$$\int_E \left(\liminf_{n \to \infty} f_n \right) dm \leq \liminf_{n \to \infty} \int_E f_n dm \tag{10.27}$$

Theorem 10.2.22 (Monotone convergence theorem). *Let $E \in \mathcal{M}$. Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence of non-negative measurable functions such that $f_n \nearrow f$ pointwise a.e. We have the following powerful equality:*

$$\boxed{\int_E f dm = \lim_{n \to \infty} \int_E f_n(x) dm} \tag{10.28}$$

Method 10.2.23. To prove 'linear' results concerning integrable functions in spaces such as $\mathcal{L}^1(E)$ we proceed according to the following steps:

1. Verify that the property holds for indicator functions. (This often follows by definition.)
2. Use the linearity to extend the property to simple functions.
3. Apply the monotone convergence theorem to show that the property holds for all non-negative measurable functions.
4. Extend the property to all integrable functions by writing $f = f^+ - f^-$ and applying the linearity again.

Theorem 10.2.24 (Dominated convergence theorem). *Let $E \in \mathcal{M}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions with $\forall n : |f_n| \leq g$ a.e. for a function $g \in \mathcal{L}^1(E)$. If $f_n \to f$ pointwise a.e. then f is integrable over E and*

$$\int_E f dm = \lim_{n\to\infty} \int_E f_n(x) dm \tag{10.29}$$

Property 10.2.25. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions. The following equality applies:

$$\int \sum_{n=1}^{+\infty} f_n(x) dm = \sum_{n=1}^{+\infty} \int f_n(x) dm \tag{10.30}$$

We cannot conclude that the right-hand side is finite a.e., so the series on the left-hand side need not be integrable.

Theorem 10.2.26 (Beppo-Levi). *Suppose that*

$$\sum_{i=1}^{\infty} \int |f_n|(x) dm \text{ is finite.}$$

The series $\sum_{i=1}^{\infty} f_n(x)$ converges a.e. Furthermore, the series is integrable and

$$\int \sum_{i=1}^{\infty} f_n(x) dm = \sum_{i=1}^{\infty} \int f_n(x) dm \tag{10.31}$$

Theorem 10.2.27 (Riemann-Lebesgue lemma). *Let $f \in \mathcal{L}^1$. The sequences*

$$s_k = \int_{-\infty}^{+\infty} f(x) \sin(kx) dx$$

and

$$c_k = \int_{-\infty}^{+\infty} f(x) \cos(kx) dx$$

both converge to 0.

Remark. This theorem is useful in Fourier analysis.

10.2.5 Relation to the Riemann integral

Theorem 10.2.28 (Fundamental theorem of calculus). *If $f : [a, b] \to \mathbb{R}$ is continuous then f is integrable and the function $F : x \mapsto \int_a^x f dm$ is differentiable for $x \in]a, b[$ such that $F' = f$.*

Theorem 10.2.29. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

- *f is Riemann-integrable if and only if f is continuous a.e. with respect to the Lebesgue measure on $[a, b]$.*
- *Riemann-integrable functions on $[a, b]$ are integrable with respect to the Lebesgue measure on $[a, b]$ and the integrals coincide.*

Theorem 10.2.30. *If $f \geq 0$ and the improper Riemann integral 7.11 exists, then the Lebesgue integral $\int f dm$ exists and the two integrals coincide.*

10.3 Examples

Definition 10.3.1 (Dirac measure[9]). We define the Dirac measure as follows:

$$\delta_a(X) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{if } a \notin X \end{cases} \tag{10.32}$$

The integration with respect to the Dirac measure has the following nice property[10]:

$$\int g(x) d\delta_a = g(a) \tag{10.33}$$

Example 10.3.2. Let $\mu = \delta_2, X = (-4; 1)$ and $Y = (-2; 17)$. The following two integrals are easily computed:

$$\int_X d\mu = 0$$
$$\int_Y d\mu = 1$$

10.4 Space of integrable functions

10.4.1 Distance

To define a distance between functions, we first have to define some notion of length of a function. Normally this would not be a problem, because we now do know how to integrate

[9] Compare to 12.6.

[10] This equality can be proved by applying formula 32.14 with $X \equiv a$.

integrable functions, however the fact that two functions differing on a null set have the same integral carries problems with it, i.e. a non-zero function could have a zero length. Therefore we will define the 'length' on a different vector space:

Define the following set of equivalence classes $L^1(E) = \mathcal{L}^1(E)_{/\equiv}$ by introducing the equivalence relation: $f \equiv g$ if and only if $f = g$ a.e.

Property 10.4.1. $L^1(E)$ is a Banach space[11].

Formula 10.4.2. A norm on $L^1(E)$ is given by:

$$||f||_1 = \int_E |f| dm \tag{10.34}$$

10.4.2 Hilbert space L^2

Property 10.4.3. L^2 is a Hilbert space[12].

Formula 10.4.4. A norm on $L^2(E)$ is given by:

$$||f||_2 = \left(\int_E |f|^2 dm \right)^{\frac{1}{2}} \tag{10.35}$$

This norm is induced by the following inner product:

$$\boxed{\langle f|g\rangle = \int_E f\overline{g} dm} \tag{10.36}$$

Now instead of deriving L^2 from $\mathcal{L}^2$ we do the opposite. We define $\mathcal{L}^2$ as the set of measurable functions for which equation 10.35 is finite.

Definition 10.4.5 (Orthogonality). As L^2 is a Hilbert space and thus has an inner product $\langle \cdot | \cdot \rangle$, it is possible to introduce the concept of orthogonality of functions in the following way:

$$\langle f|g\rangle = 0 \implies \text{f and g are orthogonal} \tag{10.37}$$

Furthermore it is also possible to introduce the angle between functions in the same way as equation 16.44.

Formula 10.4.6 (Cauchy-Schwarz inequality). Let $f, g \in L^2(E, \mathbb{C})$. We have that $fg \in L^1(E\mathbb{C})$ and:

$$\boxed{\left| \int_E f\overline{g} dm \right| \leq ||fg||_1 \leq ||f||_2 ||g||_2} \tag{10.38}$$

Remark. This follows immediately from formula 10.40.

Property 10.4.7. If E has finite Lebesgue measure then $L^2(E) \subset L^1(E)$.

[11] See definition 18.1.4.
[12] See definition 18.2.1.

10.4.3 L^p spaces

Generalizing the previous two Lebesgue function classes leads us to the notion of L^p spaces with the following norm:

Property 10.4.8. For all $1 \leq p \leq +\infty$ $L^p(E)$ is a Banach space with a norm given by:

$$||f||_p = \left(\int_E |f|^p \, dm \right)^{\frac{1}{p}} \tag{10.39}$$

Remark 10.4.9. Note that L^2 is the only L^p space that is also a Hilbert space. The other L^p spaces do not have a norm induced by an inner product.

Formula 10.4.10 (Hölder's inequality). Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p \geq 1$. For every $f \in L^p(E)$ and $g \in L^q(E)$ we have that $fg \in L^1(E)$ and:

$$||fg||_1 \leq ||f||_p ||g||_q \tag{10.40}$$

Formula 10.4.11 (Minkowski's inequality). For every $p \geq 1$ and $f, g \in L^p(E)$ we have

$$||f + g||_p \leq ||f||_p + ||g||_p \tag{10.41}$$

Property 10.4.12. If E has finite Lebesgue measure then $L^q(E) \subset L^p(E)$ when $1 \leq p \leq q < +\infty$.

10.4.4 L^∞ space of essentially bounded measurable functions

Definition 10.4.13 (Essentially bounded function). Let f be a measurable function satisfying ess $\sup|f| < +\infty$. The function f is said to be essentially bounded and the set of all such functions is denoted by $L^\infty(E)$.

Formula 10.4.14. A norm on L^∞ is given by:

$$||f||_\infty = \text{ess } \sup|f| \tag{10.42}$$

This norm is called the **supremum norm** and it induces the supremum metric 6.7.

Property 10.4.15. L^∞ is a Banach space.

10.5 Product measures

10.5.1 Real hyperspace $\mathbb{R}^n$

The notions of intervals and lengths from the one dimensional case can be generalized to more dimensions in the following way:

Definition 10.5.1 (Hypercube). Let $I_1, ..., I_n$ be a sequence of intervals.

$$\mathbf{I} = I_1 \times ... \times I_n \tag{10.43}$$

Definition 10.5.2 (Generalized length). Let $\mathbf{I}$ be a hypercube induced by the sequence of intervals $I_1, ..., I_n$. The length of $\mathbf{I}$ is given by:

$$l(\mathbf{I}) = \prod_{i=1}^{n} l(I_i) \tag{10.44}$$

10.5.2 Construction of the product measure

Property 10.5.3 (General condition). The general condition for multi-dimensional Lebesgue measures is given by following equation which should hold for all $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$:

$$\boxed{P(A_1 \times A_2) = P_1(A_1)P_2(A_2)} \tag{10.45}$$

Definition 10.5.4 (Section). Let $A = A_1 \times A_2$. The following two sets are called sections:

$$A_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\} \subset \Omega_2$$

$$A_{\omega_2} = \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in A\} \subset \Omega_1$$

Property 10.5.5. Let $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. If $A \in \mathcal{F}$ then for each ω_1, $A_{\omega_1} \in \mathcal{F}_2$ and for each ω_2, $A_{\omega_2} \in \mathcal{F}_1$. Equivalently the sets $\mathcal{G}_1 = \{A \in \mathcal{F} : \forall \omega_1, A_{\omega_1} \in \mathcal{F}_2\}$ and $\mathcal{G}_2 = \{A \in \mathcal{F} : \forall \omega_2, A_{\omega_2} \in \mathcal{F}_1\}$ coincide with the product σ-algebra $\mathcal{F}$.

Property 10.5.6. The function $A_{\omega_2} \mapsto P(A_{\omega_2})$ is a step function:

$$P(A_{\omega_2}) = \begin{cases} P_1(A_1) & \text{if} \quad \omega_2 \in A_2 \\ 0 & \text{if} \quad \omega_2 \notin A_2 \end{cases}$$

Formula 10.5.7 (Product measure). From previous property it follows that we can write the product measure $P(A)$ in the following way:

$$\boxed{P(A) = \int_{\Omega_2} P_1(A_{\omega_2}) dP_2(\omega_2)} \tag{10.46}$$

Property 10.5.8. Let P_1, P_2 be finite. If $A \in \mathcal{F}$ then the functions

$$\omega_1 \mapsto P_2(A_{\omega_1}) \qquad\qquad \omega_2 \mapsto P_1(A_{\omega_2})$$

are measurable with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively and

$$\boxed{\int_{\Omega_2} P_1(A_{\omega_2}) dP_2(\omega_2) = \int_{\Omega_1} P_2(A_{\omega_1}) dP_1(\omega_1)} \tag{10.47}$$

Furthermore the set function P is countably additive and if any other product measure coincides with P on all rectangles, it is equal to P on the whole product σ-algebra.

10.5.3 Fubini's theorem

Property 10.5.9. Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negtaive function. If f is measurable with respect to $\mathcal{F}_1 \times \mathcal{F}_2$ then for each $\omega_1 \in \Omega_1$ the function $\omega_2 \mapsto f(\omega_1, \omega_2)$ is measurable with respect to $\mathcal{F}_2$ (and vice versa). There integrals with respect to P_1 and P_2 respectively are also measurable.

Definition 10.5.10 (Section of a function). The functions $\omega_1 \mapsto f(\omega_1, \omega_2)$ and $\omega_2 \mapsto f(\omega_1, \omega_2)$ are called sections of f.

Theorem 10.5.11 (Tonelli's theorem). *Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative function. The following equalities apply:*

$$\begin{aligned}\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d(P_1 \times P_2)(\omega_1, \omega_2) &= \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) dP_2(\omega_2) \right) dP_1(\omega_1) \\ &= \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) dP_1(\omega_1) \right) dP_2(\omega_2)\end{aligned} \tag{10.48}$$

Corollary 10.5.12 (Fubini's theorem). Let $f \in L^1(\Omega_1 \times \Omega_2)$. The sections are integrable in the appropriate spaces. Furthermore the functions $\omega_1 \mapsto \int_{\Omega_2} f dP_2$ and $\omega_2 \mapsto \int_{\Omega_1} f dP_1$ are in $L^1(\Omega_1)$ and $L^1(\Omega_2)$ respectively and equality 10.48 holds.

Remark 10.5.13. The previous construction and theorems also apply for higher dimensional product spaces. These thereoms provide a way to construct higher-dimensional Lebesgue measures m_n by defining them as the completion of the product of n one-dimensional Lebesgue measures.

10.6 Radon-Nikodym theorem

Definition 10.6.1. Let $(\Omega, \mathcal{F})$ be a measurable space. Let μ, ν be two measures defined on this space. ν is said to be **absolutely continuous with respect to** μ if

$$\forall A \in \mathcal{F} : \mu(A) = 0 \implies \nu(A) = 0 \tag{10.49}$$

Notation 10.6.2. This relation is denoted by $\nu \ll \mu$.

Theorem 10.6.3 (Absolute continuity). *Let μ, ν be finite measures on a measurable space $(\Omega, \mathcal{F})$. Then $\nu \ll \mu$ if and only if*

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall A \in \mathcal{F} : \mu(A) < \delta \implies \nu(A) < \varepsilon \tag{10.50}$$

Property 10.2.20 can be generalized to arbitrary measure spaces as follows:

Property 10.6.4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $f : \Omega \to \mathbb{R}$ be a measurable function such that $\int f d\mu$ exists. Then $\nu(f) = \int_F f d\mu$ defines a measure $\nu \ll \mu$.

Definition 10.6.5 (Dominated measure). Let μ, ν be two measures. μ is said to **dominate** ν if $0 \leq \nu(F) \leq \mu(F)$ for every $F \in \mathcal{F}$.

Theorem 10.6.6 (Radon-Nikodym theorem for dominated measures). *Let μ be a measure such that $\mu(\Omega) = 1$. Let ν be a measure dominated by μ. There exists a non-negative $\mathcal{F}$-measurable function h such that $\nu(F) = \int_F h d\mu$ for all $F \in \mathcal{F}$.*

Remark. The assumption $\mu(\Omega) = 1$ is non-restrictive as every other finite measure ϕ can be normalized by putting $\mu = \frac{\phi}{\phi(\Omega)}$.

Definition 10.6.7 (Radon-Nikodym derivative). The function h as defined in previous theorem is called the Radon-Nikodym derivative of ν with respect to μ and we denote it by $\frac{d\nu}{d\mu}$.

Theorem 10.6.8 (Radon-Nikodym theorem). *Let $(\Omega, \mathcal{F})$ be a measurable space. Let μ, ν be two σ-finite measures defined on this space such that $\nu \ll \mu$. There exists a non-negative measurable function $g : \Omega \to \mathbb{R}$ such that $\nu(F) = \int_F g d\mu$ for all $F \in \mathcal{F}$.*

Remark 10.6.9. The function g in the previous theorem is unique up to a μ-null (or ν-null) set.

Property 10.6.10. Let μ, ν be finite measures such that μ dominates ν. Let $h_\nu = \frac{d\nu}{d\mu}$ be the associated Radon-Nikodym derivative. For every non-negative $\mathcal{F}$-measurable function f we have

$$\int_\Omega f d\nu = \int_\Omega f h_\nu d\mu \tag{10.51}$$

Remark 10.6.11. This property also holds for all functions $f \in L^1(\mu)$.

Property 10.6.12. Let λ, ν, μ be σ-finite measures. If $\lambda \ll \mu$ and $\nu \ll \mu$ then we have:

- $\frac{d(\lambda + \nu)}{d\mu} = \frac{d\lambda}{d\mu} + \frac{d\lambda}{d\mu}$ a.e.
- Chain rule: if $\lambda \ll \nu$ then $\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}$ a.e.

10.7 Lebesgue-Stieltjes measure

Chapter 11

Integral transforms

11.1 Fourier series

Definition 11.1.1 (Dirichlet kernel). The Dirichlet kernel is the collection of functions of the form

$$D_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx} \tag{11.1}$$

Formula 11.1.2 (Sieve property). If $f \in C^1[-\pi, \pi]$ then

$$\lim_{n\to+\infty} \int_{-\pi}^{\pi} f(x) D_n(x) dx = 0 \tag{11.2}$$

Formula 11.1.3. For 2π-periodic functions, the n-th degree Fourier approximation is given by following convolution:

$$s_n(x) = \sum_{k=-n}^{n} \widetilde{f}(k) e^{ikx} = (D_n * f)(x) \tag{11.3}$$

Theorem 11.1.4 (Convergence of the Fourier series). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with period 2π. If $f(x)$ is piecewise C^1 on $[-\pi, \pi]$ the the limit $\lim_{n\to+\infty}(D_n * f)(x)$ converges to $\frac{f(x+)+f(x-)}{2}$ for all $x \in \mathbb{R}$.*

Formula 11.1.5 (Generalized Fourier series). Let $f(x) \in \mathcal{L}^2[-l, l]$ be a $2l$-periodic function. This function can be approximated by the following series:

$$\boxed{f(x) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2l} \int_{-l}^{l} e^{-i\frac{n\pi x'}{l}} f(x') dx' \right) e^{i\frac{n\pi x}{l}}} \tag{11.4}$$

Formula 11.1.6 (Fourier coefficients). As seen in the general formula, the Fourier coefficient $\widetilde{f}(n)$ can be calculated by taking the inner product 18.5 of $f(x)$ and the n-th eigenfunction e_n:

$$\widetilde{f}(n) = \langle e_n | f \rangle = \int_{-l}^{l} e_n^*(x) f(x) dx \qquad \text{with} \qquad e_n = \sqrt{\frac{1}{2l}} e^{i\frac{n\pi x}{l}} \tag{11.5}$$

Definition 11.1.7 (Periodic extension). Let $f(x)$ be piecewise C^1 on $[-L, L]$. The periodic extension $f^L(x)$ is defined by repeating the restriction of $f(x)$ to $[-L, L]$ every $2L$. The **normalized periodic extension** is defined as

$$f^{L,\nu}(x) = \frac{f^L(x+) + f^L(x-)}{2} \tag{11.6}$$

Theorem 11.1.8. *If $f : \mathbb{R} \to \mathbb{R}$ is piecewise C^1 on $[-L, L]$ then the Fourier series approximation of $f(x)$ converges to $f^{L,\nu}(x)$ for all $x \in \mathbb{R}$.*

11.2 Fourier transform

The Fourier series can be used to expand a $2l$-periodic function as an infinite series of exponentials. For expanding a non-periodic function we need the Fourier integral:

$$\boxed{\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt} \tag{11.7}$$

$$f(t) = \mathcal{F}^{-1}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{i\omega t} d\omega \tag{11.8}$$

Equation 11.7 is called the (forward) Fourier transform of $f(t)$ and equation 11.8 is called the inverse Fourier transform.

Notation 11.2.1. The Fourier transform of a function $f(t)$, as seen in equation 11.7, is often denoted by $\widetilde{f}(\omega)$.

Theorem 11.2.2 (Convergence of the Fourier integral). *If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous (see 7.3) and if $\int_{-\infty}^{+\infty} |f(x)| dx$ is convergent then the Fourier integral converges to $f(x)$ for all $x \in \mathbb{R}$.*

Theorem 11.2.3 (Fourier inversion theorem). *If both $f(t), \mathcal{F}(\omega) \in \mathcal{L}^1(\mathbb{R})$ are continuous then the Cauchy principal value in 11.8 can be replaced by a normal integral.*

Remark 11.2.4. Schwartz functions (see 12.1) are continuous elements of $\mathcal{L}^1(\mathbb{R})$ and as such the Fourier inversion theorem also holds for these functions. This is interesting because checking the conditions for Schwartz functions is often easier then checking the more general conditions of the theorem.

Property 11.2.5. From the Riemann-Lebesgue lemma 10.2.27 it follows that

$$\mathcal{F}(\omega) \to 0 \qquad \text{if} \qquad |\omega| \to 0 \tag{11.9}$$

Property 11.2.6 (Parceval's theorem). Let $(f, \widetilde{f})$ and $(g, \widetilde{g})$ be two Fourier transform pairs.

$$\int_{-\infty}^{+\infty} f(x) g(x) dx = \int_{-\infty}^{+\infty} \widetilde{f}(k) \widetilde{g}(k) dk \tag{11.10}$$

Corollary 11.2.7 (Plancherel theorem). The integral of the square (of the modulus) of a Fourier transform is equal to the integral of the square (of the modulus) of the original function:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\widetilde{f}(k)|^2 dk \tag{11.11}$$

11.2.1 Convolution

Formula 11.2.8 (Convolution).

$$\boxed{(f * g)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau} \tag{11.12}$$

Property 11.2.9 (Commutativity).

$$f * g = g * f \tag{11.13}$$

Theorem 11.2.10 (Convolution Theorem).

$$\widetilde{f * g} = \widetilde{g}\widetilde{f} \tag{11.14}$$

11.3 Laplace transform

Formula 11.3.1 (Laplace transform).

$$\mathcal{L}\{F(t)\}_{(s)} = \int_0^{\infty} f(t) e^{-st} dt \tag{11.15}$$

Formula 11.3.2 (Bromwich integral).

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{L}\{F(t)\}_{(s)} e^{st} ds \tag{11.16}$$

Notation 11.3.3. The Laplace transform as defined in equation 11.15 is sometimes denoted by $f(s)$.

11.4 Mellin transform

Formula 11.4.1 (Mellin transform).

$$\mathcal{M}\{f(x)\}(s) = \int_0^{+\infty} x^{s-1} f(x) dx \tag{11.17}$$

Formula 11.4.2 (Inverse Mellin transform).

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{M}\{f(x)\}_{(s)} x^{-s} ds \tag{11.18}$$

11.5 Integral representations

Formula 11.5.1 (Heaviside step function).

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x - i\varepsilon} dk \tag{11.19}$$

Formula 11.5.2 (Dirac delta function).

$$\delta^{(n)}(\vec{\boldsymbol{x}}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}} d^n k \tag{11.20}$$

Chapter 12

Distributions

12.1 Generalized function

Definition 12.1.1 (Schwartz space). The Schwartz space or **space of rapidly decreasing functions**[1] $S(\mathbb{R})$ is defined as:

$$S(\mathbb{R}) = \left\{ f(x) \in C^\infty(\mathbb{R}) : \forall i, j \in \mathbb{N} : \forall x \in \mathbb{R} : |x^i f^{(j)}(x)| < +\infty \right\} \tag{12.1}$$

Remark 12.1.2. This definition can be generalized to functions of the class $C^\infty(\mathbb{R}^n)$ or functions $f : \mathbb{R} \to \mathbb{C}$. The Schwartz space is then denoted by $S(\mathbb{R}, \mathbb{C})$.

Definition 12.1.3 (Functions of slow growth). The set of functions of slow growth $N(\mathbb{R})$ is defined as:

$$N(\mathbb{R}) = \{ f(x) \in C^\infty(\mathbb{R}) : \forall i \in \mathbb{N}, \exists M_i > 0 : |f^{(i)}(x)| = O(|x|^i) \text{ for } |x| \to +\infty \} \tag{12.2}$$

Remark. It is clear that all polynomials belong to $N(\mathbb{R})$ but not to $S(\mathbb{R})$.

Property 12.1.4. If $f(x) \in S(\mathbb{R})$ and $a(x) \in N(\mathbb{R})$ then $a(x)f(x) \in S(\mathbb{R})$.

Definition 12.1.5 (Generalized function). Let $g(x) \in S(\mathbb{R})$ be a test function. Let $\{f_n(x) \in S(\mathbb{R})\}, \{h_n(x) \in S(\mathbb{R})\}$ be sequences such that

$$\lim_{n \to +\infty} \langle f_n(x) | g(x) \rangle = \lim_{n \to +\infty} \int_{-\infty}^{+\infty} f_n(x) g(x) dx$$

and similarly for h_n. Define the equivalence relation $\{f_n(x) \in S(\mathbb{R})\} \sim \{h_n(x) \in S(\mathbb{R})\}$ by saying that the two sequences, satisfying the previous condition, are equivalent if and only if

$$\lim_{n \to +\infty} \langle f_n(x) | g(x) \rangle = \lim_{n \to +\infty} \langle h_n(x) | g(x) \rangle$$

A generalized function is defined as a complete equivalence class under previous relation.

[1] These functions are said to be rapidly decreasing because every derivative $f^{(j)}(x)$ decays faster than any polynomial x^i for $x \to +\infty$.

Notation 12.1.6. Let ψ be a generalized function. Let $f \in S(\mathbb{R})$. The inner product 10.36 is generalized by following functional:

$$\langle \psi | f \rangle = \lim_{n \to +\infty} \int_{-\infty}^{+\infty} \psi_n(x) f(x) dx \tag{12.3}$$

Property 12.1.7. Let ψ be a generalized function. Let $f(x) \in S(\mathbb{R})$. The previous functional has following properties:

- $\forall i \in \mathbb{N} : \langle \psi^{(i)} | f \rangle = (-1)^i \langle \psi | f^{(i)} \rangle$
- $\forall a, b \in \mathbb{R}, a \neq 0 : \langle \psi(ax+b) | f(x) \rangle = |a|^{-1} \langle \psi(x) | f(x - b/a) \rangle$
- $\forall a(x) \in N(\mathbb{R}) : \langle a\psi | f \rangle = \langle \psi | af \rangle$

Property 12.1.8 (Ordinary function as generalized function). Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that $\exists M \geq 0 : (1+x^2)^{-M} |f(x)| \in L(\mathbb{R}, \mathbb{C})$[2]. There exists a generalized function $\psi \sim \{f_n(x) \in S(\mathbb{R}, \mathbb{C})\}$ such that for every $g(x) \in S(\mathbb{R}, \mathbb{C})$:

$$\langle \psi | g \rangle = \langle f | g \rangle$$

Furthermore if $f(x)$ is continuous on an interval, then $\lim_{n \to +\infty} f_n(x) = f(x)$ converges pointwise on that interval.

12.2 Dirac Delta distribution

Definition 12.2.1 (Heaviside function). Define the generalized function $H \sim \{H_n(x) \in S(\mathbb{R})\}$ as:

$$H(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases} \tag{12.4}$$

From this definition it follows that for every $f \in S(\mathbb{R})$:

$$\langle H | f \rangle = \int_0^{+\infty} f(x) dx \tag{12.5}$$

Remark 12.2.2. For the above integral to exist, $f(x)$ does not need to be an element of $S(\mathbb{R})$. It is a sufficient condition, but not a necessary one.

Definition 12.2.3 (Generalized delta function). The Dirac delta function is defined as a representant of the equivalence class of generalized functions $\{H_n'(x) \in S(\mathbb{R})\}$. By equations 12.1.7 and 12.5 we have for every $f \in S(\mathbb{R})$:

$$\begin{aligned} \langle \delta | f \rangle &= \langle H' | f \rangle \\ &= -\langle H | h \rangle \\ &= -\int_0^{+\infty} f'(x) dx \\ &= f(0) \end{aligned} \tag{12.6}$$

[2]The space of Lebesgue integrable functions 10.2.18.

Property 12.2.4 (Sampling property). The result from previous definition can be generalized in the following way:

$$\boxed{f(x_0) = \int_{\mathbb{R}} f(x)\delta(x - x_0)dx} \tag{12.7}$$

Example 12.2.5 (Dirac comb).

$$III_b(x) = \sum_n \delta(x - nb) \tag{12.8}$$

Property 12.2.6. Let $f(x) \in C^1(\mathbb{R})$ be a function with roots at $x_1, x_2, ..., x_n$ such that $f'(x_i) \neq 0$. The Dirac delta distribution has the following property:

$$\delta[f(x)] = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|}\delta(x - x_i) \tag{12.9}$$

Property 12.2.7 (Convolution with delta function). Let $f(x) \in S(\mathbb{R})$. Let $\otimes$ denote the convolution.

$$\delta(x) \otimes f(x) = \int_{-\infty}^{+\infty} \delta(x - \alpha)f(\alpha)d\alpha = f(x) \tag{12.10}$$

Formula 12.2.8 (Differentiation across discontinuities). Let $f(x)$ be a piecewise continuous function with discontinuities at $x_1, ..., x_n$. Let f satisfy the conditions to be a generalized function. Define $\sigma_i = f^+(x_i) - f^-(x_i)$ which are the jumps of f at its discontinuities. Next, define the function

$$f_c(x) = f(x) - \sum_{i=1}^{n} \sigma_i H(x - x_i)$$

which is a continuous function. Differentiation gives

$$f'(x) = f_c'(x) + \sum_{i=1}^{n} \sigma_i \delta(x - x_i)$$

It follows that the derivative in a generalized sense of a piecewise continuous function equals the derivative in the classical sense plus a summation of delta functions at every jump discontinuity.

12.3 Fourier transform

Theorem 12.3.1. *Let $f(x), F(k)$ be a Fourier transform pair. If $f(x) \in S(\mathbb{R}, \mathbb{C})$, then $F(k) \in S(\mathbb{R}, \mathbb{C})$. It follows that for a sequence $\{f_n(x) \in S(\mathbb{R}, \mathbb{C})\}$ the sequence of Fourier*

transformed functions $\{F_n(x) \in S(\mathbb{R}, \mathbb{C})\}$ *is also a subset of the Schwartz space. Furthermore Parceval's theorem 11.10 gives*

$$\int_{-\infty}^{+\infty} f_n(x)g(x)dx = \int_{-\infty}^{+\infty} F_n(x)G(x)dx \in \mathbb{R}$$

where $g(x) \in S(\mathbb{R}, \mathbb{C})$*. From these two properties it follows that the Fourier transform of a generalized functions is also a generalized functions.*

Property 12.3.2. Let ψ be a generalized function with Fourier transform Ψ. Let $f(x) \in S(\mathbb{R}, \mathbb{C})$ with Fourier transform $F(k)$. We have the following equality:

$$\langle \psi | F \rangle = \langle \Psi | f \rangle \tag{12.11}$$

Formula 12.3.3 (Fourier representation of delta function).

$$\boxed{\delta(x-a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-a)} dk} \tag{12.12}$$

Chapter 13

Ordinary differential equations

13.1 Boundary conditions

Unigue solutions of a differential equation are obtained by supplying additional conditions. These are called boundary conditions.

13.1.1 Periodic boundary conditions

Periodic boundary conditions are conditions of the following form:

$$y(x) = y(x + \varphi) \tag{13.1}$$

By induction it follows that for every n:

$$y(x) = y(x + n\varphi) \tag{13.2}$$

13.1.2 Dirichlet boundary conditions

Dirichlet boundary conditions are conditions of the following form:

$$y(x) = f(x) \qquad , \qquad x \in \partial\Omega \tag{13.3}$$

where Ω is the domain of the problem.

Remark 13.1.1. When y is a function of multiple variables, α can be a function as well. For example (in spherical coordinates: ρ, ϕ, θ):

$$y(x, \phi, \theta) = \alpha(\phi, \theta) \tag{13.4}$$

13.1.3 Neumann boundary conditions

Neumann boundary conditions are conditions of the following form:

$$y'(a) = \alpha \tag{13.5}$$

Remark 13.1.2. When y is a function of multiple variables, we obtain the following form (where S is the boundary of the domain and $\hat{n}$ a normal vector to this boundary):

$$\frac{\partial y}{\partial \hat{n}}(\vec{x}) = f(\vec{x}) \qquad , \qquad \vec{x} \in S \tag{13.6}$$

13.2 First order ODE's

Formula 13.2.1 (First order ODE).

$$\boxed{y'(t) + a(t)y(t) = R(t)} \tag{13.7}$$

If the function $R(t)$ is identically zero, then the ODE is said to be **homogenous**.

Theorem 13.2.2. *Let $U \subseteq \mathbb{R}$ be an open set. Let the functions $a(t), R(t) : U \to \mathbb{R}$ be continuous. The solutions $\varphi(t) : U \to \mathbb{R}$ of equation 13.7 are given by:*

$$\boxed{\varphi(t) = e^{-\int a(t)dt}\left(c + \int R(t)e^{\int a(t)dt}dt\right)} \tag{13.8}$$

where c is a constant.

13.3 Second order ODE's

Formula 13.3.1 (Second order ODE).

$$\boxed{y''(t) + a(t)y'(t) + b(t)y(t) = R(t)} \tag{13.9}$$

Formula 13.3.2 (Homogeneous second order ODE).

$$y''(t) + a(t)y'(t) + b(t)y(t) = 0 \tag{13.10}$$

13.3.1 General solution

Formula 13.3.3. Let $\varphi : U \to \mathbb{R}$ be a nowhere zero solution of the homogeneous equation 13.10. The general solution of equation 13.9 is then given by:

$$\boxed{y(t) = c_1\ \varphi + c_2\ \varphi \int \frac{e^{-\int a}}{\varphi^2} + \psi_0} \tag{13.11}$$

where ψ_0 is a particular solution of equation 13.9.

Theorem 13.3.4. *Let ψ_0 be a solution of equation 13.9. The set of all solutions is given by the affine space:*

$$\{\psi_0 + \chi : \chi \text{ is a solution of the homogeneous equation 13.10}\} \tag{13.12}$$

Theorem 13.3.5. *Two solutions of the homogeneous equation 13.10 are independent if the wronskian is nonzero:*

$$W(\varphi_1(x), \varphi_2(x)) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} \neq 0 \tag{13.13}$$

Formula 13.3.6 (Abel's identity). An explicit formula for the wronskian is given by:

$$\boxed{W(x) = W(x_0) \exp\left(-\int_{x_0}^{x} a(x')dx'\right)} \tag{13.14}$$

13.3.2 Constant coefficients

Theorem 13.3.7. *A map $\varphi : U \to \mathbb{C}$ is a complex solution of equation 13.10 if and only if $Re\{\varphi\}$ and $Im\{\varphi\}$ are real solutions of equation 13.10.*

Formula 13.3.8 (Characteristic equation). When having an ODE of the form[1] :

$$y''(t) + py'(t) + qy(t) = 0 \tag{13.15}$$

where p and q are constants, we define the characteristic equation as follows:

$$\lambda^2 + p\lambda + q = 0 \tag{13.16}$$

This polynomial equation generally[2] has two distinct (complex) roots λ_1 and λ_2. From these roots we can derive the solutions of equation 13.15 using the following rules (c_1 and c_2 are constants):

- $\lambda_1 \neq \lambda_2$, λ_1 and $\lambda_2 \in \mathbb{R}$: $y(t) = c_1\ e^{\lambda_1 t} + c_2\ e^{\lambda_2 t}$
- $\lambda_1 = \lambda_2$: $y(t) = c_1\ e^{\lambda t} + c_2\ te^{\lambda t}$
- $\lambda_1 = \lambda_2^*$, where $\lambda_1 = a + ib$: $y(t) = c_1\ e^{at}\cos(bt) + c_2\ e^{at}\sin(bt)$

[1] Any other form of homogeneous second order ODE's with constant coefficients can be rewritten in this form.

[2] See theorem 16.1.3 ("Fundamental theorem of algebra").

13.3.3 Method of Frobenius

Formula 13.3.9 (Method of Frobenius). To find a solution of the homogeneous equation 13.10 we assert a solution of the form:

$$\boxed{y(x) = \sum_{i=0}^{\infty} a_i (x - x_0)^{i+k}} \tag{13.17}$$

where k is a constant.

Definition 13.3.10 (Indicial equation). After inserting the solution 13.17 into the homogeneous equation 13.10 we obtain[3] an equation of the form $\sum_{i=n}^{\infty} H_i(k) x^i = 0$ where $n \in \mathbb{R}$ and $H_i(k)$ is a polynomial in k. This means that for every i we obtain an equation of the form $H_i(k) = 0$, due to the independence of polynomial terms. The equation for the lowest power will be quadratic in k and it is called the indicial equation.

Theorem 13.3.11. *The indicial equation generally has two roots k_1, k_2. The following possibilities arise:*

- *$k_1 = k_2$: Only one solution will be found with the method of Frobenius (another one can be found as in the second term of equation 13.11)*
- *$k_1 - k_2 \in \mathbb{Z}$: A second independent solution might be obtained using this method. If not, then a second solution can be found as mentioned in the previous case.*
- *$k_1 - k_2 \notin \mathbb{Z}$: Two independent solutions can be found using this method.*

Theorem 13.3.12 (Fuch's theorem). *If $a(z)$ and $b(z)$ are analytic at $z = z_0$ then the general solution $y(z)$ can be expressed as a Frobenius' series.*

13.4 Sturm-Liouville theory

Definition 13.4.1 (Sturm-Liouville boundary value problem). The following ODE, subject to mixed boundary conditions, is called a Sturm-Liouville boundary value problem:

$$\boxed{\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [g(x) + \lambda r(x)]\, y(x) = 0} \tag{13.18}$$

where $p(x), q(x)$ and $r(x)$ are continuous on $a \leq x \leq b$. $p(x) \in C^1(a, b)$ with $p(x) < 0$ or $p(x) > 0$ for $a \leq x \leq b$. $r(x) \geq 0$ or $r(x) \leq 0$ for $a \leq x \leq b$ and $r(x)$ is not identically zero on any subinterval.

The boundary conditions are given by

$$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{cases} \tag{13.19}$$

where at least one of the constants $\alpha_1, \alpha_2, \beta_1$ or β_2 is non-zero.

[3] It is important to 'sync' the power of all terms in order to obtain one 'large' coefficient.

Formula 13.4.2. The solutions are of the form

$$y(x) = c_1 u_1(\lambda; x) + c_2 u_2(\lambda; x)$$

Only for certain values of λ will these solutions (u_1, u_2) be non-trivial. The values of λ for which the solutions are non-trivial are called **eigenvalues** and the associated solutions are called **eigenfunctions**. Substituting this form in the boundary conditions gives the following determinant condition for non-trivial solutions, which is also the defining equation of the eigenvalues λ:

$$\begin{vmatrix} \alpha_1 u_1(a;\lambda) + \beta_1 u_1'(a;\lambda) & \alpha_1 u_2(a;\lambda) + \beta_1 u_2'(a;\lambda) \\ \alpha_1 u_1(b;\lambda) + \beta_1 u_1'(b;\lambda) & \alpha_1 u_2(b;\lambda) + \beta_1 u_2'(b;\lambda) \end{vmatrix} = 0 \tag{13.20}$$

The independent eigenfunctions can be found by substituting the found eigenvalues in the ODE 13.18.

Definition 13.4.3 (Self-adjoint form). The SL-problem can be rewritten as

$$\left[\hat{\mathcal{L}} + \lambda r(x)\right] y(x) = 0$$

The operator $\hat{\mathcal{L}} = \frac{d}{dx}\left[p(x)\frac{d}{dx} + g(x)\right]$ is called the self-adjoint form. Now consider the general linear ODE

$$\left[a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)\right] y(x) = 0 \tag{13.21}$$

This equation can be rewritten in a self-adjoint form by setting:

$$p(x) = e^{\int \frac{a_1}{a_2} dx} \qquad \text{and} \qquad g(x) = \frac{a_0}{a_2} e^{\int \frac{a_1}{a_2} dx} \tag{13.22}$$

Property 13.4.4. The eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $r(x)$.

Theorem 13.4.5 (Oscillation theorem). *Let f_n be the n^{th} eigenfunction of a Sturm-Liouville boundary condition problem. Then f_n has precisely $n-1$ roots.*

Chapter 14

Partial differential equations

14.1 General linear equations

Formula 14.1.1 (Cramer's rule). Let $Ax = b$ be a system of linear equations where the matrix A has a nonzero determinant. Then Cramer's rule gives a unique solution where the unknowns are given by;

$$x_i = \frac{\det(A_i)}{\det(A)} \tag{14.1}$$

where A_i is the matrix obtained by replacing the i^{th} column of A by the column matrix b.

Definition 14.1.2 (Characteristic curve). Curve along which the highest order partial derivatives are not uniquely defined.

14.2 First order PDE

Formula 14.2.1 (First order quasilinear PDE).

$$\boxed{P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z)} \tag{14.2}$$

Formula 14.2.2 (Characteristic curve). The PDE will have no unique solution if

$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix} = 0 \tag{14.3}$$

and will have a non-unique solution if

$$\begin{vmatrix} P & R \\ dx & dz \end{vmatrix} = 0 \tag{14.4}$$

The characteristic curves are thus defined by $\frac{dx}{P} = \frac{dy}{Q}$ and along thse curves the condition $\frac{dx}{P} = \frac{dz}{R}$ should hold to ensure a solution.

Theorem 14.2.3. *The general solution of 14.2 is implicitly given by $F(\xi, \eta) = 0$ with $F(\xi, \eta)$ an arbitrary differentiable function where $\xi(x, y, z) = c_1$ and $\eta(x, y, z) = c_2$ are solutions of the equation*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{14.5}$$

where c_1, c_2 are constants which are fixed by boundary conditions.

Remark 14.2.4. Looking at the defining equations of the characteristic curve, it is clear that these fix the general solution of the PDE.

14.3 Characteristics

Formula 14.3.1 (Second order quasilinear PDE). Consider the following pseudolinear differential equation for the function $u(x, y)$:

$$R(x, y)u_{xx} + S(x, y)u_{xy} + T(x, y)u_{yy} = W(x, y, u, p, q) \tag{14.6}$$

where $p = u_x$ and $q = u_y$.

Formula 14.3.2 (Equation of characteristics). Consider the following two differential equations

$$\begin{cases} u_{xx}dx + u_{xy}dy = dp \\ u_{xy}dx + u_{yy}dy = dq \end{cases} \tag{14.7}$$

According to Cramer's rule 14.1 these equations, together with the PDE 14.6, give the following condition for the characteristic curves:

$$\begin{vmatrix} R(x, y) & S(x, y) & T(x, y) \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0 \tag{14.8}$$

which is equivalent to following equation:

$$\boxed{R\left(\frac{dy}{dx}\right)^2 - S\left(\frac{dy}{dx}\right) + T = 0} \tag{14.9}$$

Definition 14.3.3 (Types of characteristics). Equation 14.9 is quadratic in $\frac{dy}{dx}$. If this equation has two distinct real roots then the PDE is said to be **hyperbolic**. If the equation has only one root, the PDE is said to be **parabolic**. In the remaining case, where the equation has two distinct complex roots, the PDE is said to be **elliptic**.

Formula 14.3.4 (Canonical form). Consider the general change of variables $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ and $z = \zeta$. With this change, the PDE 14.6 becomes:

$$A(\xi_x, \xi_y)\frac{\partial^2\zeta}{\partial\xi^2} + 2B(\xi_x, \xi_y, \eta_x, \eta_y)\frac{\partial^2\zeta}{\partial\xi\partial\eta} + A(\eta_x, \eta_y)\frac{\partial^2\zeta}{\partial\eta^2} = F(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta) \tag{14.10}$$

where $A(a, b) = Ra^2 + Sab + Tb^2$ and $B = R\xi_x\eta_x + \frac{1}{2}S(\xi_x\xi_y + \eta_x\eta_y) + Tbd$. Solving the quadratic equation 14.9 will lead to the following three canonical forms:

- **hyperbolic PDE**: With the solutions $\lambda_1(x,y)$ and $\lambda_2(x,y)$ the defining equation can be separated into two ODE's

$$\left(\frac{dy}{dx} + \lambda_1(x,y)\right)\left(\frac{dy}{dx} + \lambda_2(x,y)\right) = 0$$

It is clear that the solutions of these ODE's are also roots of the $A(a,b)$ coefficients such that the change of variables $\xi = f_1(x,y)$ and $\eta = f_2(x,y)$ gives the canonical hyperbolic form

$$\boxed{\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = H(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta)} \tag{14.11}$$

where $H = \frac{F}{2B}$.

- **parabolic PDE**: As in the hyperbolic case we perform the change of variable $\xi = f(x,y)$, however there is only one root of the defining equation so the second variable can be chosen randomly, yet indepedent of $f_1(x,y)$. From the condition $S^2 + 4RT = 0$ it is alos possible to derive the condition that $B(\xi_x, \xi_y \eta_x \eta_y) = 0$ and $A(\eta_x, \eta_y) \neq 0$. This gives the parabolic canonical form

$$\boxed{\frac{\partial^2 \zeta}{\partial \eta^2} = G(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta)} \tag{14.12}$$

where $G = \frac{F}{A(\eta_x, \eta_y)}$.

- **elliptic PDE**: Again there are two (complex) roots, so the A coefficients will dissappear. Writing $\xi = \alpha + i\beta$ and $\eta = \alpha - i\beta$ gives the following (real) equation

$$\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = \frac{1}{4}\left(\frac{\partial^2 \zeta}{\partial \alpha^2} + \frac{\partial^2 \zeta}{\partial \beta^2}\right)$$

Substituting this in the hyperbolic case results in the following elliptic canonical form

$$\boxed{\frac{\partial^2 \zeta}{\partial \alpha^2} + \frac{\partial^2 \zeta}{\partial \beta^2} = K(\alpha, \beta, \zeta, \zeta_\alpha, \zeta_\beta)} \tag{14.13}$$

Theorem 14.3.5 (Maximum principle). *Consider a PDE of the parabolic or elliptic type. The maximum of the solution on a domain is to be found on the boundary of that domain.*

14.3.1 D'Alemberts method

Consider the wave equation

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t) \tag{14.14}$$

By applying the method from previous subsection, it is clear that the characteristics are given by

$$\xi = x + ct \qquad \text{and} \qquad \eta = x - ct \tag{14.15}$$

Furthermore, it follows that the wave equation is a hyperbolic equation which can be rewritten in the canonical form:

$$\frac{\partial^2 u}{\partial \xi \partial \eta}(\xi, \eta) = 0 \tag{14.16}$$

Integration with respect to ξ and η and rewriting the solution in terms of x and t gives

$$u(x,t) = f(x+ct) + g(x-ct) \tag{14.17}$$

where f, g are arbitrary functions. This solution represents a superposition of a left-moving wave and a right-moving wave.

Now consider the wave equation subject to the general conditions

$$u(x,0) = v(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = q(x) \tag{14.18}$$

By applying these conditions to the general solution 14.17 it can be shown that the general solution subject to the given boundary conditions is given by:

$$\boxed{u(x,t) = \frac{1}{2}\left[v(x+ct) + v(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} q(z)dz} \tag{14.19}$$

Remark 14.3.6. Because x is not bounded, this solution is only valid for infinite strings.

14.4 Separation of variables

Remark. We begin this section with the remark that solutions obtained by this method are generalized Fourier series, which tend to converge rather slowly. For numerical purposes, other techniques are recommended. However, the series solutions often give a good insight in the properties of the obtained solutions.

14.4.1 Cartesian coordinates

Method 14.4.1 (Separation of variables). Let $\hat{\mathcal{L}}$ be the operator associated with a partial diferential equation such that $\hat{\mathcal{L}}u(\vec{x}) = 0$ where $\vec{x} = (x_1, ..., x_n)$ is the set of variables. A useful method is to propose a solution of the form

$$u(\vec{x}) = \prod_{i=1}^{n} u_i(x_i)$$

By substituting this form in the PDE and using (basic) algebra it is sometimes (!!) possible to reduce the partial differential equation to a system of n ordinary differential equations.

Example 14.4.2. Consider following PDE:

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = 0 \tag{14.20}$$

Substituting a solution of the form $u(x,t) = X(x)T(t)$ gives

$$X(x)\frac{dT(t)}{dt} - aT(t)\frac{d^2X(x)}{dx^2} = 0$$

which can be rewritten as (the arguments are dropped for convenience)

$$\frac{1}{aT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

As both sides are independent, it is clear that they are equal to a constant, say λ. This results in the following system of ordinary differential equations:

$$\begin{cases} X''(x) &= \lambda X(x) \\ T'(t) &= a\lambda T(t) \end{cases}$$

14.4.2 Dirichlet problem

The (interior) Dirichlet problem[1] is the problem of finding a solution to a PDE in a finite region, given the value of the function on the boundary of the region. The uniqueness of this solution can be proven with the maximum principle 14.3.5 if the PDE is of the elliptic kind (!!) such as the Laplace equation[2].

Proof. Let ϕ, ψ be two solutions of the interior Dirichlet problem. Due to the linearity both $\psi - \phi$ and $\phi - \psi$ are solutions too (without applying the boundary conditions). According to the maximum principle, these solutions achieve their maximum on the boundary of the domain. Furthermore, due to the Dirichlet boundary conditions, $\phi(x) = \psi(x)$ for all $x \in \partial\Omega$. Combining these two facts gives $\max(\psi - \phi) = \max(\phi - \psi) = 0$ or alternatively $\psi \leq \phi$ and $\phi \leq \psi$ in the complete domain. Which means that $\phi = \psi$ in the complete domain. □

There is also an exterior Dirichlet problem, where one has to find the solution of the PDE, given the boundary conditions, outside of the boundary.

14.5 Non-homogeneous boundary conditions

Formula 14.5.1 (Non-homogeneous boundary condition).

$$\alpha u(a,t) + \beta\frac{\partial u}{\partial x}(a,t) = h(t) \tag{14.21}$$

When $h(t)$ is identically zero, the boundary condition becomes homogeneous.

[2]Think of the Dirichlet boundary condition 13.3.

[2]The Dirichlet boundary problem originated with the Laplace equation.

Method 14.5.2 (Steady-state solution). Assume that the function $h(t)$ is constant. In this case it is useful to rewite the solution as

$$u(x,t) = v(x) + w(x,t)$$

The 'time'-independent function is called the steady-state solution and the function $w(x,t)$ represents the deviation of this steady-state scenario.
As the PDE is linear, we require the partial solutions $v(x)$ and $w(x,t)$ to individually satisfy the equation. Furthermore we require the function $v(x)$ to also satisfy the given non-homogeneous boundary conditions. This results in $w(x,t)$ being the solution of a homogeneous PDE with homogeneous boundary conditions. This can be seen in the following proof:

Proof. Assume a boundary condition of the form $\alpha u(a,t) + \beta\frac{\partial u}{\partial x}(a,t) = u_0$. Due to the requirements, we also have $\alpha v(a) + \beta\frac{\partial v}{\partial x}(a) = u_0$. Combining these two conditions gives

$$\alpha\left[v(a) + w(a,t)\right] + \beta\left[\frac{\partial v}{\partial x}(a) + \frac{\partial w}{\partial x}(a,t)\right] = \alpha v(a) + \beta\frac{\partial v}{\partial x}(a)$$

which can be reduced to

$$\alpha w(a,t) + \beta\frac{\partial w}{\partial x}(a,t) = 0$$

The steady-state deviation $w(x,t)$ thus satisfies homogeneous boundary conditions. □

Method 14.5.3. If the function $h(t)$ is not a constant, we use a different method. Rewrite the solution as $u(x,t) = v(x,t) + w(x,t)$ where we only require $v(x,t)$ to be some function that satisfies the boundary conditions (and not the PDE)[3]. This will lead to $w(x,t)$ satisfying the homogeneous boundary conditions as in the previous method. After substituting the function $v(x,t)$ in the PDE, we obtain a differential equation for $w(x,t)$ but it can be non-homogeneous.

Method 14.5.4. A third, sometimes useful, method is the following. If the problem consists of 3 homogeneous and 1 non-homogeneous boundary condition then the problem can be solved by first applying the homogenous conditions to restrict the values of the separation constant and obtain a series expansion. Afterwards the obtained series can be fitted to the non-homogeneous condition to obtain the final remaining coefficients.

If there is more than 1 non-homogeneous boundary condition, the method can be extended. Let there be j boundary conditions. Rewrite the general solution as $u(x,t) = \sum_{i=1}^{j} v_j(x,t)$ where $v_j(x,t)$ satisfies the j^{th} non-homogeneous condition and the homogeneous versions of the other conditions. This way the general solution still satisfies all conditions and the first part of the method can be applied to all functions $v_j(x,t)$ to obtain a series expansion.

Method 14.5.5 (Non-homogeneous PDE). A possible way to solve non-homogeneous second order partial differential equations of the form

$$\hat{\mathcal{L}}u(x,t) = f(x,t)$$

[3]As there are infinitely many possible functions that satisfy the boundary conditions, the best choice for $v(x,t)$ is the one that makes the equation for $w(x,t)$ as simple as possible.

given a set of homogeneous boundary conditions and inital value conditions $w(x,0) = \psi(x)$, is the following, where we assume all involved functions to be expandable as a generalized Fourier series:

1. Solve the homogeneous version of the PDE, which will result in a series expansion $\sum_n w_n(t)e_n(x)$, where $e_n(x)$ are a complete set of eigenfunctions in the variable x. This solution should satisfy the (homogeneous[4]) boundary conditions.
2. Expand the function $f(x,t)$ in the same way as $u(x,t)$. The coefficients f_n can be found by using the orthogonality realtions of the functions $e_n(x)$.
3. Inserting these expansions in the original PDE and rewriting the equation will lead to a summation of the form:

$$\sum_n \left[\left(\hat{D} w_n(t) \right) e_n(x) \right] = 0$$

   where $\hat{D}$ is a linear first order differential operator. As all terms are independent, this gives n first order ODE's to obtain the functions $w_n(t)$. These can be generally solved by using formula 13.8.
4. Initial value conditions for the functions $w_n(t)$ are applied by setting $t = 0$ in the series expansion of $u(x,t)$ and equating it with the series expansion of $\psi(x)$. This results in $w_n() = \Psi_n$.
5. The obtained ODE's together with the found boundary conditions $w_n(0) = \Psi_n$ will give the solutions of $w_n(t)$.
6. Entering these solutions in the series expansion of $u(x,t)$ will give the general solution of the non-homogeneuous PDE.

Remark 14.5.6. It is clear that the requirement that all involved functions are expandable as a generalized Fourier series is restricting. Not all non-homogeneous PDE's are solvable with this method.

[4]Non-homogeneous boundary conditions can be turned homogeneous by the previous two methods.

Chapter 15

Bessel functions

15.1 Bessel's differential equation (BDE)

A Bessel's differential equation is an ordinary differential equation of the following form:

$$\boxed{z^2y'' + zy' + (z^2 - n^2)y = 0} \tag{15.1}$$

The solutions of this ODE are the Bessel functions of the first and second kind (also called respectivelly Bessel and Neumann functions).

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left(\frac{z}{2}\right)^{2m+n} \tag{15.2}$$

$$N_n(z) = \lim_{\nu \to n} \frac{cos(\nu\pi)J_N(z) - J_{-n}(z)}{sin(\nu\pi)} \tag{15.3}$$

Remark. Solution 15.2 can be found by using Frobenius' method.

Property 15.1.1. For $n \notin \mathbb{N}$ the solutions $J_n(z)$ and $J_{-n}(z)$ are independent.

Remark 15.1.2. For $n \notin \mathbb{N}$ the limit operation in function 15.3 is not necessary as $\sin(n\pi)$ will never become 0 in this case.

15.2 Generating function

Define the following function:

$$g(x,t) = exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] \tag{15.4}$$

If we expand this function as a Laurent series, we obtain the following formula:

$$g(x,t) = \sum_{n=-\infty}^{+\infty} J_n(x)t^n \tag{15.5}$$

By applying the residue theorem 9.5.15, we can express the functions $J_n(x)$ as follows:

$$J_n(x) = \frac{1}{2\pi i}\oint_C \frac{g(x,t)}{t^{n+1}}dt \tag{15.6}$$

The function $g(x,t)$ is called the generating function of the Bessel functions.

15.3 Applications

15.3.1 Laplace equation

When solving the Laplace equation in cylindrical coordinates we obtain a BDE with integer n, which has the **cylindrical Bessel functions** 15.2 and 15.3 as solutions.

15.3.2 Helmholtz equation

When solving the Helmholtz equation in spherical coordinates we obtain a variant of the BDE for the radial part:

$$z^2y'' + 2zy' + [z^2 - n(n+1)]y = 0 \tag{15.7}$$

where n is an integer. The solutions, called **spherical Bessel functions**, are related to the cylindrical Bessel functions in the following way:

$$j_n(r) = \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(r) \tag{15.8}$$

and similarly for the Neumann functions.

Part IV

Linear Algebra

Chapter 16

Linear Algebra

16.1 General

16.1.1 Polynomes

Definition 16.1.1 (Degree). The exponent of the highest order power in x. It is often denoted by $\deg(f)$.

Definition 16.1.2 (Monic polynome). A polynome of which the highest order term has coefficient 1.

Theorem 16.1.3 (Fundamental theorem of algebra). *Let $f(x) \in K[x]$ with $\deg(f) \geq 1$. Then $f(x)$ has at least 1 root in $\mathbb{C}$.*

Theorem 16.1.4. *If $f(x) \in \mathbb{C}[x]$ is a monic polynome with $\deg(f) \geq 1$, we can write:*

$$f(x) = \prod_{i=1}^{k} (x - a_i)^{n_i}$$

Where $a_1, ..., a_k \in \mathbb{C}$ and $n_1, ..., n_k \in \mathbb{N}$.

16.2 Vector spaces

In this and coming sections all vector spaces can be finite- or infinite-dimensional. If necessary, the dimension will be specified.

Definition 16.2.1 (K-vector space). Let K be a field. A K-vector space V is a set equipped with two operations, vector addition $V \times V \to V$ and scalar multiplication $K \times V \to V$, that satisfy the following 8 axioms:

1. V is an Abelian group under vector addition.

2. $a(b\vec{v}) = (ab)\vec{v}$
3. $1_K\vec{v} = \vec{v}$ where 1_K is the identity element of the field K
4. Distributivity of scalar multiplication with respect to vector addition: $a(\vec{v}+\vec{w}) = a\vec{v}+a\vec{w}$

Definition 16.2.2 (Linear combination). The vector w is a linear combination of elements in the set $\{v_n\}$ if it can be written as:

$$w = \sum_n \lambda_n v_n \tag{16.1}$$

for some subset $\{\lambda_n\}$ of the field K.

Definition 16.2.3 (Linear independence). A set finite $\{v_n\}_{n\leq N}$ is said to be linearly independent if the following relation holds:

$$\sum_{n=0}^{N} \lambda_n v_n = 0 \iff \forall n : \lambda_n = 0 \tag{16.2}$$

A general set $\{w_i\}_{i\in I}$ is linearly independent if every finite subset of it is linearly independent.

Definition 16.2.4 (Span). A set of vectors $\{v_n\}$ is said to span V if every vector $v \in V$ can be written as a linear combination of $\{v_n\}$.

Definition 16.2.5 (Basis). A set $\{v_n\}$ is said to be a basis of V if $\{v_n\}$ is linearly independent and if $\{v_n\}$ spans V.

Corollary 16.2.6. Every set T that spans V contains a basis of V.

Remark 16.2.7. Here it is time for a little side note. In the previous definition we implicitly used the concept of a *Hamel* basis, which is based on two conditions:

- The basis is linearly independent.
- Every element in the vector space can be written as a linear combination of a finite subset of the basis.

It follows that for finite-dimensional spaces we do not have to worry. In infinite-dimensional spaces however we have to keep this in mind. An alternative, which allows infinitely many elements is given by the concept of a *Schauder* basis.

We continue by constructing this peculiar type of basis:

Construction 16.2.8 (Hamel basis). Consider the set of all linearly independent subsets of V. Under the relation of inclusion this set becomes a partially ordered set[1]. From Zorn's lemma 2.3.7 it follows that there exists at least one maximal linearly independent set.

Now we have to show that this maximal subset S is also a generating set of V. For this let us choose a vector $v \in V$ that is not already in S. From the maximality of S it follows that

[1]See definition 2.3.2.

$S \cup v$ is linearly dependent and hence there exists a finite sequence of numbers $(a^1, ..., a^n, b)$ in K and a finite sequence of elements $(e_1, ..., e_n)$ in S such that:

$$\sum_{i=0}^{n} a^i e_i + bv = 0 \tag{16.3}$$

where not all scalars are zero. This then implies that $b \neq 0$ because else the set $\{e_i\}_{i \leq n}$ and hence S would be linearly dependent. It follows that we can write v as[2]:

$$v = -\frac{1}{b} \sum_{i=0}^{n} a^i e_i \tag{16.4}$$

Because v was randomly chosen we conclude that S is a generating set for V. This set is called a Hamel basis of V.

Remark. This construction clearly assumes the ZFC axioms of set theory, only ZF does not suffice. It can even be shown that the existence of a Hamel basis for every vector space is equivalent to the axiom of choice (and thus to Zorn's lemma).

16.2.1 Subspaces

Definition 16.2.9 (Subspace). Let V be a K-vector space. A subset W of V is a subspace if W itself is a K-vector space under the operations of V. Alternatively we can write this as:

$$W \leq V \iff \forall w_1, w_2 \in W : \forall \lambda, \mu \in K : \lambda w_1 + \mu w_2 \in W \tag{16.5}$$

Definition 16.2.10 (Grassmannian). Let V be a K-vector space. The set consisting of all k-dimensional subspaces of V is denoted by $\mathrm{Gr}(k, V)$.

Property 16.2.11. $GL(V)$ acts transitively[3] on all k-dimensional subspaces of V. From property 3.1.44 it follows that the coset space $GL(V)/H_W$ for any stabilizer H_W of some $W \in \mathrm{Gr}(k, V)$ is isomorphic (as a set) to $\mathrm{Gr}(k, V)$.

16.2.2 Algebra

Definition 16.2.12 (Algebra). Let V be a K-vector space. Let V be equipped with the binary operation $\star : V \times V \to V$. $(V, \star)$ is called an algebra over K if it satisfies the following conditions[4]:

1. Right distributivity: $(\vec{x} + \vec{y}) \star \vec{z} = \vec{x} \star \vec{z} + \vec{y} \star \vec{z}$
2. Left distributivity: $\vec{x} \star (\vec{y} + \vec{z}) = \vec{x} \star \vec{y} + \vec{x} \star \vec{z}$
3. Compatibility with scalars: $(a\vec{x}) \star (b\vec{y}) = (ab)(\vec{x} \star \vec{y})$

These conditions turn the binary operation into a bilinear operation.

[2] It is this step that requires R to be a division ring in property 3.2.11 because else we would not generally be able to divide by $b \in R$.

[3] See definition 3.1.42

[4] These conditions imply that the binary operation is a bilinear map.

Definition 16.2.13 (Unital algebra). An algebra V is said to be unital if it contains an identity element with respect to the bilinear map $\star$.

Definition 16.2.14 (Clifford algebra). Let V be a unital associative algebra over the field K. If the bilinear map is a quadratic form then V is called a Clifford algebra.

Notation 16.2.15. Let V be an algebra and Q a quadratic form. The Clifford algebra is denoted by $C\ell(V, Q)$.

16.2.3 Sum and direct sum

Definition 16.2.16 (Sum). Let V be a K-vector space. Let $W_1, W_2, ..., W_k$ be subspaces of V. The 'sum' of the subspaces $W_1, ..., W_k$ is defined as follows:

$$W_1 + ... + W_k := \left\{ \sum_{i=1}^{k} w_i : w_i \in W_i \right\} \tag{16.6}$$

Definition 16.2.17 (Direct sum). If every element v of the sum as defined in definition 16.2.16 can be written as a unique linear combination, then the sum is called a direct sum.

Notation 16.2.18 (Direct sum).

$$W_1 \oplus ... \oplus W_k = \bigoplus_{i=1}^{k} W_i$$

Theorem 16.2.19. *Let V be a K-vector space. Let W, W_1, W_2 be three subspaces of V such that $W = W_1 \oplus W_2$. We have the following properties:*

- *If $\mathcal{B}_1$ is a basis of W_1 and if $\mathcal{B}_2$ is a basis of W_2, $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of W.*
- $\dim(W) = \dim(W_1) + \dim(W_2)$

Theorem 16.2.20. *Let V be a finite-dimensional K-vector space. Let W_1, W_2 be two subspaces of V. Then the following relation holds:*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \tag{16.7}$$

The second item in previous property is a direct consequence of this property.

Definition 16.2.21 (Complement). Let V be a K-vector space. Let W be a subspace of V. A subspace W' of V is called a complement of W if $V = W \oplus W'$.

Theorem 16.2.22. *Let V be a K-vector space. Let U, W be two subspaces of V. If $V = U + W$, then there exists a subspace $Y \leq U$ such that $V = W \oplus Y$. Furthermore every subset W of V has a complement in V.*

16.2.4 Graded vector space

Similar to definition 3.2.15 we can define the following:

Definition 16.2.23 (Graded vector space). Let V_n be a vector space for all $n \in \mathbb{N}$. The vector space

$$V = \bigoplus_{n \in \mathbb{N}} V_n \tag{16.8}$$

is called a graded vector space.

Definition 16.2.24 (Graded algebra). Let V be a graded vector space with the additional structure of an algebra given by the multiplication $\star$. Then V is a graded algebra if $\star$ maps $V^k \times V^l$ to V^{k+l}.

Example 16.2.25 (Superalgebra). Let A be a $\mathbb{Z}_2$-graded algebra, i.e.:

$$A = A_0 \oplus A_1 \tag{16.9}$$

such that for all $i, j \mod 2$:

$$A_i \star A_j \subseteq A_{i+j} \tag{16.10}$$

16.3 Linear maps[5]

Definition 16.3.1 (Zero map). Let $f : A \to B$ be a (linear) map. The map f is called a zero map if:

$$\forall a \in A : f(a) = 0 \tag{16.11}$$

Definition 16.3.2 (Restriction). Let $f : A \to B$ be a (linear) map. Let $C \subset A$. The (linear) map $f|_C : C \to B : c \to f(c)$ is called the restriction of f to C.

Definition 16.3.3 (Injective). A map $f : A \to B$ is called injective if the following condition is satisfied:

$$\forall a, a' \in A : f(a) = f(a') \implies a = a' \tag{16.12}$$

Notation 16.3.4 (Injective map).

$$f : A \hookrightarrow B$$

Definition 16.3.5 (Surjective). A map $f : A \to B$ is called surjective if the following condition is satisfied:

$$\forall b \in B, \exists a \in A : f(a) = b \tag{16.13}$$

Notation 16.3.6 (Surjective map).

$$f : A \twoheadrightarrow B$$

[5]Other names are **linear mapping** and **linear transformation**.

Definition 16.3.7 (Bijective). A map is called bijective if it is both injective and surjective.

Notation 16.3.8 (Bijective map).

$$f : A \xrightarrow{\sim} B$$

Definition 16.3.9 (Isomorphism). A linear bijective map f between two K-vector spaces is called an isomorphism.

Notation 16.3.10 (Isomorphic). If two K-vector spaces V, W are isomorphic we denote it as following:

$$V \cong W$$

Definition 16.3.11 (Automorphism). An isomorphism from V to V is called an automorphism. The set of all automorphisms on V, which is in fact a group, is denoted by $\mathrm{Aut}(V)$.

Definition 16.3.12 (C^r-diffeomorphism). An isomorphism of class $C^r(K)$ with an inverse that is also of class $C^r(K)$ is called a C^r-diffeomorphism.

Theorem 16.3.13. *Let $f : A \to B$ be a map. The following statements are equivalent:*

1. *There exists a map[6] $g : B \to A$ such that $f \circ g = \mathbf{1}_B$ and $g \circ f = \mathbf{1}_A$.*
2. *f is bijective.*

Corollary 16.3.14. From theorem 16.3.13 and the definition of isomorphisms we can conclude that isomorphisms are precisely those maps that are invertible.

Definition 16.3.15 (General linear group[7]). The set of all automorphisms $f : V \to V$ is called the general linear group $GL_K(V)$ of $GL(V)$.

Definition 16.3.16 (Rank). The dimension of the image of a linear map is called the rank.

Definition 16.3.17 (Kernel). The kernel of a linear map $f : V \to W$ is the following subset of V:

$$\ker(f) = \{v \in V \mid f(v) = 0\} \tag{16.14}$$

Definition 16.3.18 (Nullity). The dimension of the kernel is called the nullity.

Theorem 16.3.19. *A linear map $f : V \to W$ is injective if and only if $ker(f) = \{0\}$.*

Property 16.3.20. Let $f : V \to W$ be a linear map. Let $U \leq V$. We have the following two properties of the restriction $f|_U$ of f to U:

- $\ker(f|_U) = \ker(f) \cap U$
- $\mathrm{im}(f|_U) \leq \mathrm{im}(f)$

[6] The map g is called the **inverse** of f.

[7] This group is isomorphic to the general linear group of invertable matrices, hence the similar name and notation. (See definition 16.5.7)

16.3.1 Linear operator

Definition 16.3.21 (Linear operator). A linear automorphism $f : V \to V$ is called a linear operator. It is also more generally known as an **endomorphism** on V.

Property 16.3.22. Let $\lambda, \mu \in K$. An operator $f : V \to V$ is called linear if it satisfies the following condition:

$$f(\lambda v_1 + \mu v_2) = \lambda f(v_1) + \mu f(v_2) \tag{16.15}$$

Theorem 16.3.23. *Let V be finite-dimensional K-vector space. Let $f : V \to V$ be a linear operator. The following statements are equivalent:*

- *f is injective*
- *f is surjective*
- *f is bijective*

16.3.2 Dimension

Definition 16.3.24 (Dimension). Let V be a finite-dimensional K-vector space. Let $\{v_n\}$ be a basis for V that contains n elements. We then define the dimension of V as following:

$$\boxed{\dim(V) = n} \tag{16.16}$$

Property 16.3.25. Let V be a finite-dimensional K-vector space. Every basis of V has the same number of elements.[8]

Theorem 16.3.26 (Dimension theorem[9]). *Let $f : V \to W$ be a linear map.*

$$\dim(\text{im}(f)) + \dim(\ker(f)) = \dim(\text{V}) \tag{16.17}$$

Theorem 16.3.27. *Two K-vector spaces are isomorphic if and only if they have the same dimension.*

16.3.3 Homomorphisms

Definition 16.3.28 (Homomorphism space). Let V, W be two K-vector spaces. The set of all linear maps between V and W is called the homomorphism space of V to W, or shorter: the 'hom-space' of V to W.

$$\text{Hom}_K(V, W) = \{f : V \to W \mid \text{f is a linear map}\} \tag{16.18}$$

[8] This theorem can be generalized to infinite-dimensional spaces by stating that all bases have the same *cardinality*.

[9] Also called the **rank-nullity theorem**.

Theorem 16.3.29. *If V, W are two finite-dimensional K-vector spaces we have:*

$$\dim\left(Hom_K(V,W)\right) = \dim(V) \cdot \dim(W) \tag{16.19}$$

Definition 16.3.30 (Endomorphism ring). The space $\mathrm{Hom}_K(V,V)$ with the composition as multiplication forms a ring, the endomorphism ring. It is denoted as $\mathrm{End}_K(V)$ or $\mathrm{End}(V)$.

Property 16.3.31. The endomormphism ring $\mathrm{End}(V)$ forms a Lie algebra[10] when equipped with the commutator $[A,B] = A \circ B - B \circ A$.

Property 16.3.32 (Jordan-Chevalley decomposition). Every endomorphism A can be decomposed as follows:

$$A = A_{ss} + A_n \tag{16.20}$$

where

- A_{ss} is **semisimple**, i.e. for every the invariant subspace of A_{ss} there exists a invariant complementary subspace.
- A_n is **nilpotent**, i.e. $\exists k \in \mathbb{N} : A_n^k = 0$.

Furthermore, this decomposition is unique and the endomorphisms A_{ss}, A_n can be written as polynomials in A.

Definition 16.3.33 (Minimal polynomial). Let $f \in \mathrm{End}(V)$ and V a finite-dimensional K-vector space. The monic polynomial $\mu_f(x)$ of the lowest order such that $\mu_f(f) = 0$ is called the minimal polynomial of f.

Property 16.3.34. Let $f \in \mathrm{End}(V)$. Let $\mu_f(x)$ be the minimal polynomial of f. Let $\varphi(x) \in K[x]$. If $\varphi(f) = 0$, then the minimal polynomial $\mu_f(x)$ divides $\varphi(x)$.

16.3.4 Dual space

Definition 16.3.35 (Dual space). Let V be a K-vector space. The dual space V^* of V is the following vector space:

$$V^* := \mathrm{Hom}_K(V,K) = \{f : V \to K \ : \ f \text{ is a linear map}\} \tag{16.21}$$

Definition 16.3.36 (Linear form). The elements of V^* are called *linear forms*.

Property 16.3.37. From theorem 16.3.29 it follows that $\dim(V^*) = \dim(V)$.

Remark 16.3.38. If V is infinite-dimensional, theorem 16.3.37 is not valid. In the infinite-dimensional case we **always** have $|V^*| > |V|$ (where we now use the cardinality instead of the dimension).

[10] See also 25.2.20.

Definition 16.3.39 (Dual basis). Let $\mathcal{B} = \{e_1, e_2, ..., e_n\}$ be a basis for a finite-dimensional K-vector space V. We can define a basis $\mathcal{B}^* = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ for V^*, called the dual basis of $\mathcal{B}$, as follows:

$$\boxed{\varepsilon_i : V \to K : \sum_{j=1}^{n} a_i e_i \mapsto a_i} \tag{16.22}$$

The relation between the basis and dual basis can also be written as:

$$\varepsilon^i(e_j) = \delta^i_j \tag{16.23}$$

Definition 16.3.40 (Dual map). Let $f : V \to W$ be a linear map. The linear map $f^* : W^* \to V^* : \varphi \to \varphi \circ f$ is called the dual map or **transpose** of f.

Notation 16.3.41 (Transpose). When $V = W$ the dual map f^* is often denoted by f^T.

Definition 16.3.42 (Natural pairing). The natural pairing of V and its dual V^* is defined as the following bilinear map:

$$\langle v, v^* \rangle = v^*(v) \tag{16.24}$$

16.3.5 Convex functions

Definition 16.3.43 (Convex function). Let X be a convex subset of V. A function $f : X \to \mathbb{R}$ is convex if for all $x, y \in X$ and $t \in [0, 1]$:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{16.25}$$

Remark 16.3.44. For a concave function we have to turn the inequality around.

Corollary 16.3.45. A linear map $f : X \to \mathbb{R}$ is both convex and concave.

Theorem 16.3.46 (Karamata's inequality). *Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a convex function. If $(x_1, ..., x_n)$ is a tuple that majorizes $(y_1, ..., y_n)$, i.e. $\forall k \leq n$*

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \tag{16.26}$$

$$x_{(1)} + ... + x_{(k)} \geq y_{(1)} + ... + y_{(k)} \tag{16.27}$$

where $x_{(i)}$ denotes the ordering[11] *of the tuple $(x_1, ..., x_n)$. Then*

$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i) \tag{16.28}$$

[11] In decreasing order: $x_{(1)} \geq ... \geq x_{(n)}$.

16.4 Inner product

In the following section all vector spaces V will be $\mathbb{R}$- or $\mathbb{C}$-vector spaces.

Notation 16.4.1 (Inner product). Let v, w be two vectors in V. The map $\langle\cdot|\cdot\rangle : V \times V \to K$ is called an inner product on V if it satisfies the following 3 properties:

1. **Conjugate symmetry:** $\langle v|w\rangle = \langle w|v\rangle^*$
2. **Linearity in the first argument:** $\langle \lambda u + v|w\rangle = \lambda\langle u|w\rangle + \langle v|w\rangle$
3. **Non-degeneracy:** $\langle v|v\rangle = 0 \iff v = 0$
4. **Positive-definiteness** $\langle v|v\rangle \geq 0$

Remark 16.4.2. Inner products are special cases of **non-degenerate Hermitian forms** which do not possses the positive-definiteness property.

Corollary 16.4.3. The first two properties have the result of conjugate linearity in the second argument:

$$\langle f|\lambda g + \mu h\rangle = \overline{\lambda}\langle f|g\rangle + \overline{\mu}\langle f|h\rangle \tag{16.29}$$

16.4.1 Inner product space

Definition 16.4.4 (Inner product space[12]). A vector space equipped with an inner product $\langle\cdot|\cdot\rangle$ is called an inner product space.

Definition 16.4.5 (Metric dual[13]). Using the inner product (or any other non-degenerate Hermitian form) one can define the metric dual of a vector v by the following map:

$$L : V \to V^* : v \mapsto \langle v|\cdot\rangle \tag{16.30}$$

Definition 16.4.6 (Adjoint operator). Let A be a linear operator on V. Let v, w be two vectors in V. The *Hermitian* adjoint of A is defined as the linear operator $A^\dagger$ that satisfies:

$$\langle A^\dagger v, w\rangle = \langle v, Aw\rangle \tag{16.31}$$

Alternatively one can define the adjoint using the metric dual $L(\cdot)$ as follows:

$$\boxed{A^\dagger = L^{-1} \circ A^T \circ L} \tag{16.32}$$

If $A = A^\dagger$ then A is said to be **Hermitian** or **self-adjoint**.

Corollary 16.4.7. The Hermitian adjoint of a complex matrix $A \in \mathbb{C}^{m\times n}$ is given by:

$$A^\dagger = \overline{A}^T \tag{16.33}$$

where $\overline{A}$ denotes the complex conjugate of A and A^T the transpose of A.

[12] Sometimes called a **prehilbert space**.
[13] See also definition 28.1.

The definition of an adjoint operator 16.4.6 can be generalized to the case where $A^\dagger$ is not unique (for example when A is not globally defined) in the following way:

Definition 16.4.8 (Conjugate operators). Two operators B and C are said to be conjugate if:

$$\langle Bx, y\rangle = \langle x, Cy\rangle \tag{16.34}$$

Example 16.4.9. The Lie algebra associated with the group of isometries $\mathrm{Isom}(V)$ of a non-degenerate Hermitian form satisfies following condition:

$$\langle Xv, w\rangle = -\langle v, Xw\rangle \tag{16.35}$$

for all Lie algebra elements X. It follows that the Lie algebra consists of all anti-hermitian operators.

16.4.2 Orthogonality

Definition 16.4.10 (Orthogonal). Let $v, w \in V$. The vectors v and w are said to be orthogonal, denoted by $v \perp w$, if they obey the following relation:

$$\langle v|w\rangle = 0 \tag{16.36}$$

An orthogonal **system** is a set of vectors, none of them the null vector, that are mutually orthogonal.

Property 16.4.11. Orthogonal systems are linearly independent.

Definition 16.4.12 (Orthonormal). A set of vectors $\{v_n\}$ is said to be orthonormal if it is orthogonal and if all the elements v_n obey the following relation:

$$\langle v|v\rangle = 1 \tag{16.37}$$

Definition 16.4.13 (Orthogonal complement[14]). Let W be a subspace of V. The following subspace is called the orthogonal complement of W:

$$W^\perp = \{v \in V \mid \forall w \in W : \langle v|w\rangle = 0\} \tag{16.38}$$

Property 16.4.14. The inner-product is invariant under transformations between orthonormal bases.

Property 16.4.15.

$$W \cap W^\perp = \{0\} \tag{16.39}$$

Property 16.4.16. Let V be a finite-dimensional K-vector space. The orthogonal complement $W^\perp$ is a complementary subspace[15] to W, i.e. $W \leq V$: $W \oplus W^\perp = V$.

[14] $W^\perp$ is pronunciated as 'W-perp'.

Corollary 16.4.17. Let $W \leq V$ where V is a finite-dimensional K-vector space. We have the following relation:

$$(W^{\perp})^{\perp} = W \tag{16.40}$$

Definition 16.4.18 (Orthogonal projection). Let V be a finite-dimensional K-vector space. Let $W \leq V$. Let $w \in W$ and let $\{w_1, ..., w_k\}$ be an orthonormal basis of W. We define the projection of $v \in V$ on W and $w \in W$ as follows:

$$\text{proj}_W(v) = \sum_{i=1}^{k} \langle v|w_i\rangle w_i \tag{16.41}$$

$$\text{proj}_w(v) = \frac{\langle v|w\rangle}{\langle w|w\rangle} w \tag{16.42}$$

Property 16.4.19.

1. $\forall w \in W : \text{proj}_W(w) = w$
2. $\forall u \in W^{\perp} : \text{proj}_W(u) = 0$

Method 16.4.20 (Gram-Schmidt orthonormalisation). Let $\{u_n\}$ be a set of linearly independent vectors. We can construct an orthonormal set $\{e_n\}$ out of $\{u_n\}$ in the following way:

$$\begin{aligned} w_1 &= u_1 & e_1 &= \frac{w_1}{||w_1||} \\ w_2 &= u_2 - \frac{\langle u_2|w_1\rangle}{||u_2||^2} w_1 & e_2 &= \frac{w_2}{||w_2||} \\ &\vdots & &\vdots \\ w_n &= u_n - \sum_{k=1}^{n-1} \frac{\langle u_n|w_k\rangle}{||u_n||^2} w_k & e_n &= \frac{w_n}{||w_n||} \end{aligned} \tag{16.43}$$

16.4.3 Angle

Definition 16.4.21 (Angle). Let v, w be elements of an inner product space. The angle θ between v and w is defined as:

$$\boxed{\cos\theta = \frac{\langle v|w\rangle}{||v||||w||}} \tag{16.44}$$

16.5 Matrices

Notation 16.5.1. The set of all $m \times n$-matrices defined over the field K is denoted as $M_{m,n}(K)$ or $\text{Mat}_{m,n}(K)$. If $m = n$, the set is denoted as $M_n(K)$ or $Mat_n(K)$.

[15] hence the name

Property 16.5.2 (Dimension). The dimension of $M_{m,n}(K)$ is mn.

Definition 16.5.3 (Trace). Let $A = (a_{ij}) \in M_n(K)$. We define the trace of A as follows:

$$\boxed{\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}} \tag{16.45}$$

Property 16.5.4. Let $A, B \in M_n(K)$. We have the following properties of the trace:

1. $\mathrm{tr} : M_n(K) \to K$ is a linear map
2. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
3. $\mathrm{tr}(AB) \neq \mathrm{tr}(A)\mathrm{tr}(B)$
4. $\mathrm{tr}(A^T) = \mathrm{tr}(A)$

Formula 16.5.5 (Hilbert-Schmidt norm). Also called the **Frobenius norm**. This matrix norm is given by following formula

$$||A||_{HS}^2 = \sum_{i,j} |A_{ij}|^2 = \mathrm{tr}(A^\dagger A) \tag{16.46}$$

If one identifies $M_n(\mathbb{C})$ with $\mathbb{C}^{2n}$ then this norm equals the standard Hermitian norm.

Formula 16.5.6 (Hadamard product). Let A, B be two matrices in $M_{m\times n}(K)$. The Hadamard product is defined as the entry-wise product:

$$(A \circ B)_{ij} = A_{ij}B_{ij} \tag{16.47}$$

Definition 16.5.7 (General linear group). The set of invertable matrices is called the general linear group and is denoted by $GL_n(K)$.

Property 16.5.8. For all $A \in GL_n(K)$ we have:

- $A^T \in GL_n(K)$
- $\left(A^T\right)^{-1} = (A^{-1})^T$

Property 16.5.9. Let $A \in M_{m,n}(K)$. Denote the set of columns of A as $\{A_1, A_2, ..., A_n\}$ and the set of rows of A as $\{R_1, R_2, ..., R_m\}$. The set of columns is a subspace of K^m and the set of rows is a subspace of K^n. Furthermore we have:

$$\dim(\mathrm{span}(A_1, ..., A_n)) = \dim(\mathrm{span}(R_1, ..., R_m))$$

Definition 16.5.10 (Rank of a matrix). We can define the rank of matrix $A \in M_{m,n}(K)$ as follows:

$$\mathrm{rk}(A) := \dim(\mathrm{span}(A_1, ..., A_n)) \overset{16.5.9}{=} \dim(\mathrm{span}(R_1, ..., R_m)) \tag{16.48}$$

Property 16.5.11. The rank of a matrix has the folowing properties:

1. Let $A \in M_{m,n}(K)$ and $B \in M_{n,r}(K)$. We have $\text{rk}(AB) \leq \text{rk}(A)$ and $\text{rk}(AB) \leq \text{rk}(A)$.
2. Let $A \in GL_n(K)$ and $B \in M_{n,r}(K)$. We have $\text{rk}(AB) = \text{rk}(B)$.
3. Let $A \in GL_n(K)$ and $B \in M_{r,n}(K)$. We have $\text{rk}(BA) = \text{rk}(B)$.

Property 16.5.12. Let $A \in M_{m,n}(K)$. First define the following linear map:

$$\boxed{L_A : K^n \to K^m : v \mapsto Av} \tag{16.49}$$

This map has the following properties:

1. $\text{im}(L_A) = \text{span}(A_1, ..., A_n)$
2. $\dim(\text{im}(L_A)) = \text{rk}(A)$

Remark. The second property is a direct consequence of the first one and definition 16.48.

16.5.1 System of equations

Theorem 16.5.13. *Let $AX = w$ with $A \in M_{m,n}(K)$, $w \in K^m$ and $X \in K^n$ be a system of m equations in n variables. Let L_A be the linear map as defined in equation 16.49. We then have the following properties:*

1. *The system is false if and only if $w \notin im(L_A)$.*
2. *If the system is not false, the solution set is an affine space. If $v_0 \in K^n$ is a solution, then the solution set is given by: $L_A^{-1}(w) = v_0 + ker(L_A)$.*
3. *If the system is homogeneous ($AX = 0$), then the solution set is equal to $ker(L_A)$.*

Theorem 16.5.14 (Uniqueness). *Let $AX = w$ with $A \in M_n(K)$ be a system of n equations in n variables. If $rk(A) = n$, then the system has a unique solution.*

16.5.2 Coordinates and matrix representations

Definition 16.5.15 (Coordinate vector). Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis of V. Let $v \in V$ such that $v = \sum_{i=1}^{n} \lambda_i b_i$. We define the coordinate vector of v with respect to $\mathcal{B}$ as $(\lambda_1, ..., \lambda_n)^T$. The λ_i's are called the **coordinates** of v with respect to $\mathcal{B}$.

Definition 16.5.16 (Coordinate isomorphism). With the previous definition in mind we can define the coordinate isomorphism of v with respect to $\mathcal{B}$ as follows:

$$\boxed{\beta : V \to K^n : \sum_{i=1}^{n} \lambda_i b_i \mapsto (\lambda_1, ..., \lambda_n)^T} \tag{16.50}$$

Definition 16.5.17 (Matrix representation). Let V be an n-dimensional K-vector space and W an m-dimensional K-vector space. Let $f : V \to W$ be a linear map. Let $\mathcal{B} = \{b_1, ..., b_n\}, \mathcal{C} = \{c_1, ..., c_m\}$ be a basis for V, respectively W. The matrix representation of f with respect to $\mathcal{B}$ and $\mathcal{C}$ can be derived as follows: For every $j \in \{1, ..., n\}$ we can write $f(b_j) = \sum_{i=1}^{m} a_{ij}c_i$, so with this in mind we can define the matrix $(a_{ij}) \in M_{m,n}(K)$ as the matrix represenation of f.

Notation 16.5.18 (Matrix representation of a linear map). The matrix representation of f with respect to $\mathcal{B}$ and $\mathcal{C}$ is denoted by $A_{f,\mathcal{B},\mathcal{C}}$.

Method 16.5.19 (Construction of a matrix representation). From definition 16.5.17 we can see that j-th column of $A_{f,\mathcal{B},\mathcal{C}}$ coincides with the coordinate vector of $f(b_j)$ with respect to $\mathcal{C}$. We use this relation to construct $A_{f,\mathcal{B},\mathcal{C}}$ by writing for every $j \in \{1, ..., n\}$ the coordinate vector of $f(b_j)$ in the j-th column.

Theorem 16.5.20. *Let $(\lambda_1, ..., \lambda_n)^T$ be the coordinate vector of $v \in V$ with respect to $\mathcal{B}$. Let $(\mu_1, ..., \mu_m)^T$ be the coordinate vector of $f(v)$ with respect to $\mathcal{C}$. Then the following relation holds:*

$$\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} = A_{f,\mathcal{B},\mathcal{C}} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \tag{16.51}$$

Theorem 16.5.21. *For every matrix $A \in M_{m,n}(K)$ there exists a linear map $f : V \to W$ such that $A_{f,\mathcal{B},\mathcal{C}} = A$.*

On the other hand we also have the following theorem:

Theorem 16.5.22. *Let $f : K^n \to K^m$ be a linear map. There exists a matrix $A \in M_{m,n}(K)$ such that $f = L_A$.*

Theorem 16.5.23. *Let β and γ be the coordinate isomorphisms with respect to respectively $\mathcal{B}$ and $\mathcal{C}$. From theorem 16.5.20 it follows that:*

$$\gamma(f(v)) = A_f \cdot \beta(v) \tag{16.52}$$

or alternatively

$$\gamma \circ f = L_{A_f} \circ \beta \tag{16.53}$$

Theorem 16.5.24. *The map $Hom_K(V, W) \to M_{m,n}(K) : f \mapsto A_f$ is an isomorphism and for every $f \in Hom_K(V, W)$ and $g \in Hom_K(W, U)$ we have:*

$$A_{g \circ f} = A_g A_f \tag{16.54}$$

Theorem 16.5.25. *The map $End_K(V) \to M_n(K) : f \mapsto A_{f,\mathcal{B},\mathcal{B}}$ is an isomorphism and for every $f, g \in End_K(V)$ we have:*

$$A_{g \circ f} = A_g A_f \tag{16.55}$$

Theorem 16.5.26. *Let $f \in End_K(V)$. Let A_f be the corresponding matrix representation. The linear map f is invertible if and only if A_f is invertible. Furthermore, if A_f is invertible, we have that*

$$(A_f)^{-1} = A_{f^{-1}}$$

In other words, the following map is an isomorphism[16]*:*

$$GL_K(V) \to GL_n(K) : f \mapsto A_f \tag{16.56}$$

Remark 16.5.27. The sets $GL_K(V)$ and $GL_N(K)$ are groups. So the previous theorem states that the map $f \mapsto A_f$ is a group isomorphism.

Theorem 16.5.28. *Let $V = K^n$. Let $f \in V^*$. From construction 16.5.19 it follows that $A_f = (f(e_1), ..., f(e_n)) \in M_{1,n}(K)$ with respect to the standard basis of V. This combined with theorem 16.5.20 gives:*

$$f(\lambda_1, ..., \lambda_n)^T = (f(e_1), ..., f(e_n))(\lambda_1, ..., \lambda_n)^T = \sum_{i=1}^{n} f(e_i)\lambda_i \tag{16.57}$$

or alternatively with $\{\varepsilon_1, ..., \varepsilon_n\}$ the dual basis to the standard basis of V:

$$\boxed{f = \sum_{i=1}^{n} f(e_i)\varepsilon_i} \tag{16.58}$$

Theorem 16.5.29. *Let $f : V \to W$ be a linear map. Let $f^* : W^* \to V^*$ be the corresponding dual map. If A_f is the matrix representation of f with respect to $\mathcal{B}$ and $\mathcal{C}$, then the transpose A_f^T is the matrix representation of f^* with respect to the dual basis of $\mathcal{C}$ and the dual basis of $\mathcal{B}$.*

16.5.3 Coordinate transforms

Definition 16.5.30 (Transition matrix). Let $\mathcal{B} = \{b_1, ..., b_n\}$ and $\mathcal{B}' = \{b'_1, ..., b'_n\}$ be two bases of V. Every element of $\mathcal{B}'$ can be written as a linear combination of elements in $\mathcal{B}$:

$$b'_j = q_{1j}b_1 + ... + q_{nj}b_n \tag{16.59}$$

The matrix $Q = (q_{ij}) \in M_n(K)$ is called the transition matrix from the 'old' basis $\mathcal{B}$ to the 'new' basis $\mathcal{B}'$.

Theorem 16.5.31. *Let $\mathcal{B}, \mathcal{B}'$ be two basis of V. Let Q be the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$. We find the following statements:*

[16] Follows from theorem 16.5.25.

1. *Let $\mathcal{C}$ be an arbitrary basis of V with γ the corresponding coordinate isomorphism. Define the following matrices:*

$$B = (\gamma(b_1), ..., \gamma(b_n)) \quad \textit{and} \quad B' = (\gamma(b'_1), ..., \gamma(b'_n))$$

Then $BQ = B'$.

2. *$Q \in GL_n(K)$ and Q^{-1} is the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$.*

3. *Let $v \in V$ with $(\lambda_1, ..., \lambda_n)^T$ the coordinate vector with respect to $\mathcal{B}$ and $(\lambda'_1, ..., \lambda'_n)^T$ the coordinate vector with respect to $\mathcal{B}'$. Then:*

$$Q \begin{pmatrix} \lambda'_1 \\ \vdots \\ \lambda'_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \lambda'_1 \\ \vdots \\ \lambda'_n \end{pmatrix} = Q^{-1} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

Theorem 16.5.32. *Let V, W be two finite-dimensional K-vector spaces. Let $\mathcal{B}, \mathcal{B}'$ be two bases of V and $\mathcal{C}, \mathcal{C}'$ two bases of W. Let Q, P be the transition matrices from $\mathcal{B}$ to $\mathcal{B}'$ and from $\mathcal{C}$ to $\mathcal{C}'$ respectively. Let $A = A_{f,\mathcal{B},\mathcal{C}}$ and $A' = A_{f,\mathcal{B}',\mathcal{C}'}$. Then:*

$$A' = P^{-1}AQ \tag{16.60}$$

Corollary 16.5.33. Let $f \in \mathrm{End}_K(V)$ and let Q be the transition matrix. From theorem 16.5.32 it follows that:

$$A' = Q^{-1}AQ \tag{16.61}$$

Definition 16.5.34 (Matrix conjugation). Let $A \in M_n(K)$. The set

$$\{Q^{-1}AQ \mid Q \in GL_n(K)\} \tag{16.62}$$

is called the conjugacy class[17] of A. Another name for conjugation is **similarity transformation**.

Remark 16.5.35. If A is a matrix representation of a linear operator f, then the conjugacy class of A consists out of every possible matrix representation of f.

Property 16.5.36. From property 16.5.4 it follows that the trace of a matrix is invariant under similarity transformations:

$$\boxed{\mathrm{tr}(Q^{-1}AQ) = \mathrm{tr}(A)} \tag{16.63}$$

Definition 16.5.37 (Matrix congruence). Let $A, B \in M_n(K)$. If there exists a matrix P such that

$$A = P^T BP \tag{16.64}$$

then the matrices are said to be congruent.

Property 16.5.38. Every matrix congruent to a symmetric matrix is also symmetric.

Theorem 16.5.39. *Let $(V, \langle .|. \rangle)$ be an inner-product space defined over $\mathbb{R}$ (or $\mathbb{C}$). Let $\mathcal{B}, \mathcal{B}'$ be two orthonormal bases of V and let Q be the transition matrix. We can find the following result:*

$$Q^T Q = \mathbb{1}_n$$

[17] This is the general definition of conjugacy classes for groups. Furthermore, these classes induce a partitioning of the group.

16.5.4 Determinant

Definition 16.5.40 (Minor). The (i, j)-th minor of A is defined as:

$$\det(A_{ij})$$

where $A_{ij} \in M_{n-1}(K)$ is the matrix obtained by removing the i-th row and the j-th column from A.

Definition 16.5.41 (Cofactor). The cofactor α_{ij} of the matrix element a_{ij} is equal to:

$$(-1)^{i+j} \det(A_{ij})$$

where $\det(A_{ij})$ is the minor as previously defined.

Definition 16.5.42 (Adjugate matrix). The adjugate matrix of $A \in M_n(K)$ is defined as follows:

$$\operatorname{adj}(A) := \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{pmatrix} \tag{16.65}$$

or shorter: $\operatorname{adj}(A) = (\alpha_{ij})^T$.

Remark. It is important to notice that we have to transpose the matrix after the elements have been replaced by their cofactor.

Property 16.5.43. Let $A, B \in M_n(K)$. Denote the columns of A as $A_1, \ldots, A_n$. We have the following properties of the determinant:

1. $\det(A^T) = \det(A)$
2. $\det(AB) = \det(BA) = \det(A) \det(B)$
3. $\det(A_1, \ldots, A_i + \lambda A_i', \ldots, A_n) = \det(A_1, \ldots, A_i, \ldots, A_n) + \lambda \det(A_1, \ldots, A_i', \ldots, A_n)$ for all $A_i, A_i' \in M_{n,1}(K)$.
4. If two columns of A are equal then $\det(A) = 0$.
5. $\det(A_{i_1}, \ldots, A_{i_n}) = \operatorname{sgn}(i_1, \ldots, i_n) \det(A_1, \ldots, A_n)$
6. $\det(A_1, \ldots, A_n) = \det(A_1, \ldots, A_i + \lambda A_k, \ldots, A_n)$ where A_i, A_k are columns of A.
7. The determinant can be evaluated as follows:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det(A_{ik}) \tag{16.66}$$

Theorem 16.5.44. *Let $A \in M_n(K)$, the following statements are equivalent:*

1. $\det(A) \neq 0$

2. $rk(A) = n$
3. $A \in GL_n(K)$

Theorem 16.5.45. *For all $A \in M_n(K)$ we find $A\,adj(A) = adj(A)A = \det(A)I_n$.*

Theorem 16.5.46. *For all $A \in GL_n(K)$ we find $A^{-1} = \det(A)^{-1}\ adj(A)$.*

An alternative definition of a $k \times k$-minor is:

Definition 16.5.47. Let $A \in M_{m,n}(K)$ and $k \leq \min(m,n)$. A $k \times k$-minor of A is the determinant of a $k \times k$-partial matrix obtained by removing $m-k$ rows and $n-k$ columns from A.

Theorem 16.5.48. *Let $A \in M_{m,n}(K)$ and $k \leq \min(m,n)$. We find that $rk(A) \geq k$ if and only if A contains a $k \times k$-minor different from 0.*

Theorem 16.5.49. *Let $f \in End_K(V)$. The determinant of the matrix representation of f is invariant under basis transformations.*

Definition 16.5.50 (Determinant of a linear operator). The previous theorem allows us to unambiguously define the determinant of f as follows:

$$\det(f) := \det(A)$$

where A is some matrix representation of f.

16.5.5 Characteristic polynomial

Definition 16.5.51 (Characteristic polynomial[18]). Let V be a finite-dimensional K-vector space. Let $f \in \mathrm{End}_K(V)$ be a linear operator with the matrix representation A (with respect to some arbitrary basis). We then find:

$$\chi_f(x) := \det(x\mathbb{1}_n - A) \in K[x] \tag{16.67}$$

is a monic polynomial of degree n in the variable x and the polynomial does not depend on the choice of basis.

Definition 16.5.52 (Characteristic equation[19]). The following equation is called the characteristic equation of f:

$$\boxed{\chi_f(x) = 0} \tag{16.68}$$

[18] This polynomial can also be used directly for a matrix A as theorem 16.5.21 matches every matrix A with some linear operator f.

[19] This equation is sometimes called the **secular equation**.

Formula 16.5.53. Let $A = (a_{ij}) \in M_n(K)$ with characteristic polynomial:

$$\chi_A(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$$

We then have the following result:

$$\begin{cases} c_0 = (-1)^n \det(A) \\ c_{n-1} = -\mathrm{tr}(A) \end{cases}$$

Theorem 16.5.54 (Cayley-Hamilton).

1. *Let $A \in M_n(K)$ with characteristic polynomial $\chi_A(x)$. We find the following relation:*

$$\chi_A(A) = A^n + \sum_{i=1}^{n-1} c_i A^i = 0 \tag{16.69}$$

2. *Let $f \in End_K(V)$ with characteristic polynomial $\chi_f(x)$. We find that*

$$\chi_f(f) = f^n + \sum_{i=1}^{n-1} c_i f^i = 0 \tag{16.70}$$

Corollary 16.5.55. From theorem 16.3.34 and the Cayley-Hamilton theorem it follows that the minimal polynomial $\mu_f(x)$ is a divisor of the characteristic polynomial $\chi_f(x)$.

16.5.6 Linear groups

Definition 16.5.56 (Elementary matrix). An elementary matrix is a matrix of the following form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & c_{ij} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & c_{ij} & \ddots & \vdots \\ 0 & 0 & \vdots & 1 \end{pmatrix}, \ldots$$

i.e. equal to the sum of an identity matrix and a multiple of a matrix unit $U_{ij},\ i \neq j$.

Notation 16.5.57 (Elementary matrix). $E_{ij}(c)$ is the elementary matrix with element c on the i,j-th position.

Property 16.5.58. We have the following property:

$$\det(E_{ij}(c)) = 1 \tag{16.71}$$

which implies that $E_{ij}(c) \in GL_n(K)$.

Property 16.5.59. We find the following results concerning the multiplication by an elementary matrix:

1. Left multiplication by an elementary matrix $E_{ij}(c)$ comes down to replacing the i-th row of the matrix with the i-th row plus c times the j-th row.
2. Right multiplication by an elementary matrix $E_{ij}(c)$ comes down to replacing the j-th column of the matrix with the j-th column plus c times the i-th column.

Theorem 16.5.60. *Every matrix $A \in GL_n(K)$ can be written in the following way:*

$$A = SD$$

where S is a product of elementary matrices and $D = diag(1, \dots, 1, \det(A))$.

Definition 16.5.61 (Special linear group). The following subset of $GL_n(K)$ is called the special linear group:

$$SL_n(K) = \{A \in GL_n(K) \mid \det(A) = 1\} \tag{16.72}$$

Theorem 16.5.62. *Every $A \in SL_n(K)$ can be written as a product of elementary matrices.*[20]

Definition 16.5.63 (Orthogonal group). The orthogonal and special orthogonal group are defined as follows:

$$\begin{aligned} O_n(K) &= \{A \in GL_n(K) \mid AA^T = A^TA = I_n\} \\ SO_n(K) &= O_n(K) \cap SL_n(K) \end{aligned}$$

Property 16.5.64. For orthogonal matrices, conjugacy 16.5.34 and congruency 16.5.37 are equivalent.

Definition 16.5.65 (Unitary group). The unitary and special unitary group are defined as follows:

$$\begin{aligned} U_n(K, \sigma) &= \{A \in GL_n(K) \mid A\overline{A}^T = \overline{A}^TA = I_n\} \\ SU_n(K, \sigma) &= U_n(K) \cap SL_n(K) \end{aligned}$$

where σ denotes the *involution*[21] $a^\sigma \equiv \overline{a}$.

Remark. If $K = \mathbb{C}$ where the involution is taken to be the complex conjugate, the σ is often ommited in the definition: $U_n(K)$ and $SU_n(K)$.

Definition 16.5.66 (Unitary equivalence). Let A, B be two matrices in $M_n(K)$. If there is a unitary matrix U such that

$$A = U^\dagger BU$$

then the matrices A and B are said to be **unitarily equivalent**.

[20] Follows readily from theorem 16.5.60.

[21] An involution is an operator that is its own inverse: $f(f(x)) = x$.

16.6 Eigenvectors

Definition 16.6.1 (Eigenvector). A vector $v \in V \setminus \{0\}$ is called an **eigenvector** of the linear operator $f : V \to V$ if it satisfies the following equation:

$$f(v) = \lambda v \tag{16.73}$$

Where $\lambda \in K$ is the **eigenvalue** belonging to v.

Definition 16.6.2 (Eigenspace). The subspace of V consisting of the zero vector and the eigenvectors of an operator is called the eigenspace associated with that operator. It is given by:

$$\ker(\lambda \mathbf{1}_V - f) \tag{16.74}$$

Theorem 16.6.3 (Characteristic equation[22]). *Let $f \in End_K(V)$ be a linear operator. A scalar $\lambda \in K$ is an eigenvalue of f if and only if it satisfies the characteristic equation 16.68.*

Theorem 16.6.4. *A linear operator $f \in End_K(V)$ defined over an n-dimensional K-vector space V has at most n different eigenvalues.*[23]

Method 16.6.5 (Finding the eigenvectors of a matrix). To calculate the eigenvectors of a matrix one should perform the following steps:

1. First we find the eigenvalues λ_i of $\mathbf{A}$ by applying theorem 16.6.3.
2. Then we find the eigenvector v_i belonging to the eigenvalue λ_i by using the following equation:
$$(\mathbf{A} - \lambda_i \mathbf{1}_V)\, v_i = 0 \tag{16.75}$$

16.6.1 Diagonalization

Definition 16.6.6 (Diagonalizable operator). An operator $f \in \mathrm{End}_K(V)$ on a finite-dimensional K-vector space V is diagonalizable if there exists a matrix representation $A \in M_n(K)$ of f such that A is a diagonal matrix.

Property 16.6.7. Every diagonalizable operator is semisimple[24]. Conversely, in finite dimensions a semisimple operator, over an algebraically closed field, is diagonalizable.

Theorem 16.6.8. *A linear operator f defined on a finite-dimensional K-vector space V is diagonalizable if and only if the set of eigenvectors of f forms a basis of V.*

Theorem 16.6.9. *A matrix $A \in M_n(K)$ is diagonalizable if and only if there exists a matrix $P \in GL_n(K)$ such that $P^{-1}AP$ is diagonal.*

[22] This theorem also holds for the eigenvalues of a matrix $A \in M_n(K)$.
[23] This theorem also holds for a matrix $A \in M_n(K)$.
[24] See 16.3.32.

Corollary 16.6.10. Using the fact that the trace of a linear operator is invariant under similarity transformations (see property 16.63) we get following useful formula:

$$\boxed{\mathrm{tr}(f) = \sum_i \lambda_i} \tag{16.76}$$

where $\{\lambda_i\}_{0\leq i\leq n}$ are the eigenvalues of f.

Property 16.6.11. Let V be an n-dimensional K-vector space. Let $f \in \mathrm{End}_K(V)$ be a linear operator. We find the following properties of the eigenvectors/eigenvalues of f:

1. The eigenvectors of f belonging to different eigenvalues are linearly independent.
2. If f has exactly n eigenvalues, f is diagonalizable.
3. If f is diagonalizable, V is the direct sum of the eigenspaces of f belonging to the different eigenvalues of f.

16.6.2 Multiplicity

Definition 16.6.12 (Multiplicity). Let V be a K-vector space. Let $f \in \mathrm{End}_K(V)$ be a linear operator with characteristic polynomial[25]:

$$\chi_f(x) = \prod_{i=1}^{n} (x - \lambda_i)^{n_i} \tag{16.77}$$

We can define the following multiplicities:

1. The *algebraic multiplicity* of an eigenvalue λ_i is equal to n_i.
2. The *geometric multiplicity* of an eigenvalue λ_i is equal to the dimension of the eigenspace belonging to that eigenvalue.

Remark 16.6.13. The geometric multiplicity is always at least 1.

Property 16.6.14. The algebraic multiplicity is always greater than or equal to the geometric multiplicity.

Theorem 16.6.15. *Let $f \in End_K(V)$ be a linear operator. f is diagonalizable if and only if for every eigenvalue the algebraic multiplicity is equal to the geometric multiplicity.*

Property 16.6.16. Every Hermitian operator $f \in \mathrm{End}_K(\mathbb{C}^n)$ has the following properties:

1. All the eigenvalues of f are real.

[25] We assume that the characteristic polynomial can be written in this form. This depends on the possibility to completely factorize the polynomial in K (i.e. it has 'enough' roots in K). If not, f cannot even be diagonalized. However, there always exists a field F containing K, called a *splitting field*, where the polynomial has 'enough' roots.

2. Eigenvectors belonging to different eigenvalues are orthogonal.
3. f is diagonalizable and there always exists an orthonormal basis of eigenvectors of f.[26]

Property 16.6.17. Let $A, B \in \mathrm{End}_K(V)$ be two linear operators. If the commutator $[A, B] = 0$, then the two operators have a common eigenbasis.

Theorem 16.6.18 (Sylvester's law of inertia). *Let S be a symmetric matrix. The number of positive and negative eigenvalues is invariant with respect to similarity transformations*[27]*.*

16.7 Euclidean space $\mathbb{R}^n$

A finite-dimensional $\mathbb{R}$-vector space is called a **Euclidean space**.

16.7.1 Angle

Definition 16.7.1 (Angle). Let $(V, \langle .|. \rangle)$ be a real inner-product space. For every $u, v \in V \setminus \{0\}$ we can define the angle between them as[28]:

$$\sphericalangle(u, v) = \mathrm{acos} \frac{\langle u|v \rangle}{||u|| \cdot ||v||} \tag{16.78}$$

where we set the range of acos as $[0, \pi]$.

Notation 16.7.2. When working in a Euclidean space the inner product $\langle v|w \rangle$ is often written as $v \cdot w$ or even vw.

16.7.2 Vector product

Definition 16.7.3 (Orientation). Let $\mathcal{B}, \mathcal{B}'$ be two ordered bases of $\mathbb{R}^n$. Let Q be the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$. If $\det(Q) > 0$ then the bases are said to have the same orientation (or be *consistently oriented*). If $\det(Q) < 0$ then the bases are said to have an opposite orientation.

Corollary 16.7.4 (Positive orientation). The previous definition imposes an equivalence relation on the set of bases of $\mathbb{R}^n$. The set of bases consists out of two equivalence classes. Take one class and call the bases in it *positively* or *directly* oriented. The bases in the other class are then said to be *negatively* or *indirectly* oriented.

[26] This implies that the matrix P diagonalizing the Hermitian operator is unitary, i.e. $P^{-1} = P^\dagger$.

[27] Also with respect to conjugation, which are equivalent to similarity transformations according to property 16.5.64.

[28] This formula follows readily from the Cauchy-Schwarz inequality (see theorem 18.2.6).

Remark 16.7.5. It is convenient to take the standard basis $(e_1, \dots, e_n)$ to be positively oriented.

Formula 16.7.6 (Cross product).

$$\boxed{(v \times w)_i = \varepsilon_{ijk} v_j w_k} \tag{16.79}$$

where ε_{ijk} is the 3-dimensional Levi-Civita symbol.

Remark 16.7.7. It is important to note that the previous construction is only valid in 3 dimenensions.

Chapter 17

Vector calculus

17.1 Nabla-operator

Definition 17.1.1 (Nabla).

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \tag{17.1}$$

Following formulas can be found by using basic properties of (vector) calculus.

Formula 17.1.2 (Gradient).

$$\nabla V = \left(\frac{\partial V_x}{\partial x}, \frac{\partial V_y}{\partial y}, \frac{\partial V_z}{\partial z} \right) \tag{17.2}$$

Formula 17.1.3. Let $\varphi(\vec{\boldsymbol{x}})$ be a scalar field. The total differential $d\varphi$ can be rewritten as

$$d\varphi = \nabla\varphi \cdot d\vec{r} \tag{17.3}$$

Property 17.1.4. The gradient of a scalar function V is perpendicular to the level sets 2.10 of V.

Definition 17.1.5 (Directional derivative). Let $\vec{a}$ be a unit vector. The directional derivative $\nabla_{\vec{\boldsymbol{a}}} V$ is defined as the change of the function V in the direction of $\vec{a}$:

$$\nabla_{\vec{\boldsymbol{a}}} V \equiv (\vec{a} \cdot \nabla) V \tag{17.4}$$

Example 17.1.6. Let $\varphi(\vec{\boldsymbol{x}})$ be a scalar field. Let $\vec{t}$ denote the tangent vector to a curve $\vec{r}(s)$ with s natural parameter. The variation of the scalar field $\varphi(\vec{\boldsymbol{x}})$ along $\vec{r}(s)$ is given by

$$\frac{\partial\varphi}{\partial s} = \frac{d\vec{r}}{ds} \cdot \nabla\varphi \tag{17.5}$$

Definition 17.1.7 (Conservative vector field). A vector field obtained as the gradient of a scalar function.

Property 17.1.8. A vector field is conservative if and only if its line integral is path independent.

Formula 17.1.9 (Gradient of tensor). Let T be a tensor field with coordinates x^i. Let $\vec{\boldsymbol{e}}^i(x^1, x^2, x^3)$ be a curvilinear orthogonal frame[1]. The gradient of T is defined as follows:

$$\nabla T = \frac{\partial T}{\partial x^i} \otimes \vec{\boldsymbol{e}}^i \tag{17.6}$$

Formula 17.1.10 (Divergence).

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{17.7}$$

Definition 17.1.11 (Solenoidal vector field). A vector field $\vec{\boldsymbol{V}}(\vec{\boldsymbol{x}})$ is said to be solenoidal if it satisfies:

$$\nabla \cdot \vec{\boldsymbol{V}} = 0 \tag{17.8}$$

It is also known as a **divergence free vector field**.

Formula 17.1.12 (Rotor / curl).

$$\nabla \times \vec{\boldsymbol{A}} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \tag{17.9}$$

Definition 17.1.13 (Irrotational vector field). A vector field $\vec{\boldsymbol{V}}(\vec{\boldsymbol{x}})$ is said to be irrotational if it satisfies:

$$\nabla \times \vec{\boldsymbol{V}} = 0 \tag{17.10}$$

Remark 17.1.14. All conservative vector fields are irrotational but irrotational vector fields are only conservative if the domain is simply-connected[2]

17.1.1 Laplacian

Definition 17.1.15 (Laplacian).

$$\triangle V \equiv \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{17.11}$$

$$\nabla^2 \vec{\boldsymbol{A}} = \nabla \left(\nabla \cdot \vec{\boldsymbol{A}} \right) - \nabla \times \left(\nabla \times \vec{\boldsymbol{A}} \right) \tag{17.12}$$

Remark 17.1.16. Equation 17.12 is called the **vector laplacian**.

Formula 17.1.17 (Laplacian in different coordinate systems).

[1] See definition 26.2.20.
[2] See definition 5.6.9.

- Cylindrical coordinates (ρ, ϕ, z):

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2} \tag{17.13}$$

- Spherical coordinates (r, ϕ, θ):

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] \tag{17.14}$$

17.1.2 Mixed properties[3]

$$\nabla\times(\nabla V)=0 \tag{17.15}$$

$$\nabla\cdot\left(\nabla\times\vec{\boldsymbol{V}}\right)=0 \tag{17.16}$$

In Cartesian coordinates equation 17.12 can be rewritten as follows:

$$\nabla^2\vec{\boldsymbol{A}}=(\triangle A_x,\triangle A_y,\triangle A_z) \tag{17.17}$$

17.1.3 Helmholtz decomposition

Formula 17.1.18 (Helmholtz decomposition). Let $\vec{\boldsymbol{P}}$ be a vector field that decays rapidly (more than $1/r$) when $r\to\infty$. $\vec{\boldsymbol{P}}$ can be written as follows:

$$\vec{\boldsymbol{P}}=\nabla\times\vec{\boldsymbol{A}}+\nabla V \tag{17.18}$$

17.2 Line integrals

Formula 17.2.1 (Line integral of a continuous scalar field). Let f be a continuous scalar field. Let Γ be a piecewise smooth curve with parametrization $\vec{\boldsymbol{\varphi}}(t), t\in[a,b]$. We define the line integral of f over Γ as follows:

$$\int_\Gamma f(s)ds=\int_a^b f(\vec{\boldsymbol{\varphi}}(t))||\vec{\boldsymbol{\varphi}}'(t)||dt \tag{17.19}$$

Formula 17.2.2 (Line integral of a continuous vector field). Let $\vec{\boldsymbol{F}}$ be a continuous vector field. Let Γ be a piecewise smooth curve with parametrization $\vec{\boldsymbol{\varphi}}(t), t\in[a,b]$. We define the line integral of F over Γ as follows:

$$\int_\Gamma\vec{\boldsymbol{F}}(\vec{\boldsymbol{s}})\cdot d\vec{\boldsymbol{s}}=\int_a^b\vec{\boldsymbol{F}}(\vec{\boldsymbol{\varphi}}(t))\cdot\vec{\boldsymbol{\varphi}}'(t)dt \tag{17.20}$$

[3]See remark 26.4.10 for a differential geometric approach.

17.3 Integral theorems[4]

Theorem 17.3.1 (Fundamental theorem of calculus for line integrals). *Let $\vec{\Gamma} : \mathbb{R} \to \mathbb{R}^3$ be a smooth curve.*

$$\int_{\Gamma(a)}^{\Gamma(b)} \nabla f(\vec{\boldsymbol{r}}) \cdot d\vec{\boldsymbol{r}} = \varphi(\Gamma(b)) - \varphi(\Gamma(a)) \tag{17.21}$$

Theorem 17.3.2 (Kelvin-Stokes' theorem).

$$\oint_{\partial S} \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{l}} = \iint_S \left(\nabla \times \vec{\boldsymbol{A}}\right) dS \tag{17.22}$$

Theorem 17.3.3 (Divergence theorem[5]).

$$\oiint_{\partial V} \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{S}} = \iiint_V \left(\nabla \cdot \vec{\boldsymbol{A}}\right) dV \tag{17.23}$$

Corollary 17.3.4 (Green's identity).

$$\oiint_{\partial V} (\psi \nabla \phi - \phi \nabla \psi) \cdot d\vec{\boldsymbol{S}} = \iiint_V \left(\psi \nabla^2 \phi - \phi \nabla^2 \psi\right) dV \tag{17.24}$$

17.4 Curvilinear coordinates

In this section the differential operators are generalized to curvilinear coordinates. To do this we need the scale factors as formally defined in equation 23.14. Also there is no Einstein summation used, all summations are written explicitly.

Formula 17.4.1 (Unit vectors).

$$\frac{\partial \vec{r}}{\partial q^i} = h_i \hat{e}_i \tag{17.25}$$

Formula 17.4.2 (Gradient).

$$\nabla V = \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial V}{\partial q^i} \hat{e}_i \tag{17.26}$$

Formula 17.4.3 (Divergence).

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial q^1}(A_1 h_2 h_3) + \frac{\partial}{\partial q^2}(A_2 h_3 h_1) + \frac{\partial}{\partial q^3}(A_3 h_1 h_2) \right) \tag{17.27}$$

Formula 17.4.4 (Rotor).

$$(\nabla \times \vec{\boldsymbol{A}})_i = \frac{1}{h_j h_k} \left(\frac{\partial}{\partial q^j}(A_k h_k) - \frac{\partial}{\partial q^k}(A_j h_j) \right) \tag{17.28}$$

where $i \neq j \neq k$.

[4] These theorems follow from the more general Stokes' theorem 27.3.

[5] Also known as *Gauss's theorem* or the *Gauss-Ostrogradsky theorem.*

Chapter 18

Banach spaces and Hilbert spaces

In this chapter the term "linear operator", which is normally reserved for maps of the form $f : V \to V$, is used instead of "linear map". This was done to keep the vocabulary in track with that of the standard literature on Banach spaces and operator spaces.

For a revision of inner product spaces see section 16.4.

18.1 Banach spaces

Definition 18.1.1 (Norm). Let V be a K-vector space. A function $||\vec{v}|| : V \to [0, +\infty[$ is called a norm if it satisfies following conditions:

- **Non-degeneracy:** $||\vec{v}|| = 0 \iff \vec{v} = 0$
- **Homogeneity:** $||a\vec{v}|| = |a|||\vec{v}||$ for all scalars $a \in K$
- **Triangle equality (subadditivity):** $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$

Remark 18.1.2. A norm $|| \cdot ||$ clearly induces a metric[1] by setting $d(x, y) = ||x - y||$.

Definition 18.1.3 (Normed vector space). A K-vector space equipped with a norm $|| \cdot ||$.

Definition 18.1.4 (Banach space). A normed vector space that is complete[2] with respect to the norm $|| \cdot ||$.

Definition 18.1.5 (Reflexive space). A Banach space V for which its dual coincides with the dual of its dual, i.e. $V^* = (V^*)^*$.

Property 18.1.6. Every finite-dimensional Banach spaces is reflexive. This follows from property 16.3.37.

[1] See definition 6.1.1.
[2] See condition 6.12.

Property 18.1.7. Let (x_n) be a Cauchy sequence in a normed space V. Then $(||x_n||)$ is a convergent sequence in $\mathbb{R}$. This implies that every Cauchy sequence in a normed space is bounded.

Property 18.1.8. The topological (continuous) dual of a Banach space is also a Banach space.

18.1.1 Theorems

Property 18.1.9. Let X be a general TVR. Every linear map $\varphi : \mathbb{K}^n \to X$ is continuous.

Property 18.1.10. Let X be a finite-dimensional normed vector space. Every linear bijection $\varphi : \mathbb{K}^n \to X$ is a homeomorphism.

Corollary 18.1.11. Two finite-dimensional normed vector spaces with the same dimension are homeomorphic. It follows that all metrics on a finite-dimensional normed vector space are equivalent.

Theorem 18.1.12 (Open mapping theorem[3]). *Let $f : V \to W$ be a continuous linear operator between two Banach spaces. If f is surjective then it also open.*

18.1.2 Bounded operators

Definition 18.1.13 (Bounded operator). Let $L : V \to W$ be a linear operator between two Banach spaces. The operator is said to be bounded if there exists a scalar M that satisfies the following condition:

$$\boxed{\forall v \in V : ||Lv||_W \leq M||v||_V} \tag{18.1}$$

Notation 18.1.14. The space of bounded linear operators from V to W is denoted by $\mathcal{B}(V, W)$.

Property 18.1.15. If V is a Banach space then $\mathcal{B}(V, V)$ is also a Banach space.

Definition 18.1.16 (Operator norm). The operator norm of L is defined as follows:

$$||L||_{op} = \inf\{M | M \text{ satisfies condition 18.1}\} \tag{18.2}$$

As the name suggests it is a norm on $\mathcal{B}(V, W)$. The topology induced by this norm is called the norm topology.

Equivalent definitions of the operator norm are:

$$||L||_{op} = \sup_{||x||\leq 1} ||L(x)|| = \sup_{||x||=1} ||L(x)|| = \sup_{x\neq 0} \frac{||L(x)||}{||x||} \tag{18.3}$$

[3]Sometimes called the *Banach-Schauder* theorem.

Following property reduces the problem of continuity to that of boundedness:

Property 18.1.17. Let $f \in \mathcal{L}(V, W)$. Following statements are equivalent:

- f is bounded.
- f is continuous at 0.
- f is continuous on V.
- f is uniformly continuous.
- f maps bounded sets to bounded sets.

Property 18.1.18. Let A be a bounded linear operator with eigenvalue λ. We then have:

$$|\lambda| \leq ||A||_{op} \tag{18.4}$$

Property 18.1.19. Let A be a bounded linear operator. Let $A^\dagger$ denote its adjoint[4]. Then $A^\dagger$ is bounded and $||A||_{op} = ||A^\dagger||_{op}$.

18.1.3 Spectrum

Definition 18.1.20 (Resolvent set). Let A be a bounded linear operator on a normed space V. The resolvent set $\rho(A)$ consists of all scalar $\lambda \in \mathbb{C}$ such that $(A - \lambda \mathbb{1})^{-1}$ is a bounded linear operator, called the resolvent of A, on a dense subset of V. These scalars λ are called **regular values** of A.

Definition 18.1.21 (Spectrum). The set of scalars $\mu \notin \rho(A)$ is called the spectrum of A.

Remark 18.1.22. It is obvious from the definition of an eigenvalue that every eigenvalue of A belongs to the spectrum of A. The converse however is not true.

Definition 18.1.23 (Point spectrum). The set of scalars $\mu \in \mathbb{C}$ for which the resolvent of A fails to be injective is called the point spectrum of A. This set coincides with the set of eigenvalues of A.

Definition 18.1.24 (Continuous spectrum). The set of scalars $\mu \in \mathbb{C}$ for which the resolvent of A fails to be surjective but for which the range of the resolvent is dense in V is called the continuous spectrum of A. The scalars for which the range is not dense is called the **residual spectrum** $\sigma_r(A)$.

Definition 18.1.25 (Compression spectrum). The set of scalars $\mu \in \mathbb{C}$ for which the resolvent of A fails to have a dense range in V is called the compression spectrum $\sigma(A)$. It follows that $\sigma_r(A) \subseteq \sigma(A)$.

[4]See definition 16.4.6.

18.1.4 Fredholm operators

Definition 18.1.26 (Compact operator). Let V, W be Banach spaces. A linear operator $A : V \to W$ is compact if the image of any bounded set in V is relatively compact[5].

Property 18.1.27. Every compact operator is bounded and hence continuous.

Corollary 18.1.28. Every linear map between finite-dimensional Banach spaces is bounded.

Definition 18.1.29 (Fredholm operator). Let V, W be Banach spaces. A Fredholm operator $F : V \to W$ is a bounded linear operator $F \in \mathcal{B}(V, W)$ for which the kernel and cokernel are finite-dimensional.

Property 18.1.30. An operator $F : V \to W$ is a Fredholm operator if and only if there exists a bounded linear operator $G \in \mathcal{B}(W, V)$ such that $\mathbb{1}_V - FG$ and $\mathbb{1}_W - GF$ are compact on V and W respectively.

18.2 Hilbert space

Definition 18.2.1 (Hilbert space). A vector space that is both a Banach space and an inner product space (where the norm is induced by the inner product).

Example 18.2.2. Let $f, g \in \mathcal{L}^2([a, b], \mathbb{C})$, the inner product of f and g is defined as:

$$\boxed{\langle f|g\rangle = \int_a^b f^*(x)\overline{g(x)}dx} \tag{18.5}$$

Remark 18.2.3. See section 10.4.2 for a more formal treatment of this subject.

Formula 18.2.4. It is also possible to define an inner product with respect to a weight function $\phi(x)$:

$$\int_a^b f^*(x)g(x)\phi(x)dx \tag{18.6}$$

Using this formula it is possible to define orthogonality with respect to a weight function.

18.2.1 Inner products and norms

Formula 18.2.5. Let V be an inner product space. A norm on V can be induced by the inner product in the following way:

$$||v||^2 = \langle v|v\rangle \tag{18.7}$$

However not every norm induces an inner product. Only norms that satisfy the parallellogram law 18.9 induce an inner product. This inner product can be recovered through the polarization identity 18.10 (see below).

[5] See definition 5.5.11.

Property 18.2.6 (Cauchy-Schwarz inequality).

$$\boxed{|\langle v|w\rangle| \leq ||v||\ ||w||} \tag{18.8}$$

where the equality holds if and only if v and w are linearly dependent.

Corollary 18.2.7. The Cauchy-Schwarz inequality can be used to prove the triangle inequality. Together with the properties of an inner product this implies that an inner product space is also a normed space.

Formula 18.2.8 (Parallellogram law).

$$||v+w||^2 + ||v-w||^2 = 2(||v||^2 + ||w||^2) \tag{18.9}$$

Formula 18.2.9 (Polarization identity).

$$4\langle v|w\rangle = ||v+w||^2 - ||v-w||^2 + i\left(||v+iw||^2 - ||v-iw||^2\right) \tag{18.10}$$

Formula 18.2.10 (Pythagorean theorem). In an inner product space the triangle equality reduces to the well-known Pythagorean theorem for orthogonal vectors v, w:

$$||v+w||^2 = ||v||^2 + ||w||^2 \tag{18.11}$$

This formula can be extended to any set of orthogonal vectors $x_1, ..., x_n$:

$$\boxed{\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} ||x_i||^2} \tag{18.12}$$

18.2.2 Generalized Fourier series

Property 18.2.11 (Bessel's inequality). First of all we have following general equality for orthonormal vectors $x_1, ..., x_n$ and complex scalars $a_1, ..., a_n$:

$$\left\|x - \sum_{i=1}^{n} a_i x_i\right\|^2 = ||x||^2 - \sum_{i=1}^{n} |\langle x, x_i\rangle|^2 + \sum_{i=1}^{n} |\langle x, x_i\rangle - a_i|^2 \tag{18.13}$$

This expression becomes minimal for $a_i = \langle x, x_i\rangle$ (last term becomes 0). This leads to Bessel's inequality:

$$\boxed{\sum_{i=1}^{n} |\langle x, x_i\rangle|^2 \leq ||x||^2} \tag{18.14}$$

Corollary 18.2.12. The sum in 18.14 is bounded for all n, so the series $\sum_{i=1}^{+\infty}$ converges for all x. This implies that the sequences $(\langle x, x_n\rangle)$ belongs to the space l^2 of square-summable sequences.

This result does however not imply that the generalized Fourier series $\sum_{i=1}^{+\infty}\langle x, x_i\rangle x_i$ converges to x. The following theorem gives a necessary and sufficient condition for the convergence.

Theorem 18.2.13. *Let $\mathcal{H}$ be a Hilbert space. Let (x_n) be an orthonormal sequence in $\mathcal{H}$ and let (a_n) be a sequence in $\mathbb{C}$. The expansion $\sum_{i=1}^{+\infty} a_i x_i$ converges in $\mathcal{H}$ if and only if $(a_n) \in l^2$. Furthermore the expansion satisfies following equality:*

$$\left\|\sum_{i=1}^{+\infty} a_i x_i\right\|^2 = \sum_{i=1}^{+\infty} |a_i|^2 \tag{18.15}$$

As we noted the sequence $(\langle x, x_n\rangle)$ belongs to l^2 so the generalized Fourier series converges of $x \in \mathcal{H}$ converges in $\mathcal{H}$.

Remark 18.2.14. Although the convergence of the generalized Fourier series of $x \in \mathcal{H}$ can be established using previous theorem, it does not follow that the expansion converges to x itself. We can merely say that the Fourier expansion is the best approximation of x with respect to the norm on $\mathcal{H}$.

18.2.3 Complete sets

Definition 18.2.15 (Complete set). Let $\{e_i\}_{i\in I}$ be a set (possibly a sequence) of orthonormal vectors in an inner product space V. This set is said to be complete if every vector $x \in V$ can be expressed as follows:

$$x = \sum_{i\in I}\langle x, x_i\rangle x_i \tag{18.16}$$

This implies that a complete set is a basis for the vector space.

Another characterization is the following.

Alternative Definition 18.2.16. A complete set of orthonormal vectors is a set $S \subset V$ such that we cannot add another vector w to it satisfying:

$$\forall v_i \in S : \langle v_i, w\rangle = 0 \qquad \wedge \qquad w \neq 0 \tag{18.17}$$

Property 18.2.17. For complete sequences (x_n) the inequality of Bessel 18.14 becomes an equality. Furthermore, the generalized Fourier series with respect to the complete sequence is unique.

Using previous property we can prove the following theorem due to Parceval.

Theorem 18.2.18 (Parceval). *Let (x_n) be a complete sequence in a Hilbert space $\mathcal{H}$. Every vector $x \in \mathcal{H}$ has a unique Fourier series representation $\sum_{i=1}^{+\infty} a_i x_i$ where the Fourier coefficients (a_i) belong to l^2 and the inequality of Bessel is an equality.*

Conversely if the inequality of Bessel becomes an equality for every $x \in \mathcal{H}$ then the sequence (x_n) is complete.

Property 18.2.19. A sequence (x_n) in a Hilbert space $\mathcal{H}$ is complete if and only if $\langle x, x_i\rangle = 0$ for all x_i implies that $x = 0$.

18.2.4 Orthogonality and projections

The basic notions on orthogonality in inner product space can be found in section 16.4.2.

Property 18.2.20. Let S be a subset (not necessarily a subspace) of a Hilbert space $\mathcal{H}$. The orthogonal complement $S^{\perp}$ is closed in $\mathcal{H}$.

Corollary 18.2.21. The previous property implies that the orthogonal complemement of some arbitrary subset of a Hilbert space is a Hilbert space itself.

Theorem 18.2.22 (Projection theorem). *Let H be a Hilbert space and $K \leq H$ a complete subspace. For every $h \in H$ there exists a unique $h' \in K$ such that $h - h'$ is orthogonal to every $k \in K$, i.e $h - h' \in K^{\perp}$.*

Remark 18.2.23. An equivalent definition for the unique $h' \in K$ is $||h-h'|| = \inf\{||h-k|| : k \in K\}$.

Corollary 18.2.24. It follows that given a complete (or closed) subspace S the Hilbert space $\mathcal{H}$ can be decomposed as $\mathcal{H} = S \oplus S^{\perp}$.

18.2.5 Separable Hilbert spaces

The definition of separable spaces in the sense of point-set topology is given in 5.5.19. An equivalent definition for Hilbert spaces is the following.

Alternative Definition 18.2.25 (Separable Hilbert space). A Hilbert space is separable if it contains a complete sequence of orthonormal vectors.

Corollary 18.2.26. Using the Gram-Schmidt method it follows from previous definition that every finite-dimensional Hilbert space is separable.

The following theorem shows that (up to an isomorphism) there are only 2 distinct types of separable Hilbert spaces.

Theorem 18.2.27. *Let $\mathcal{H}$ be separable. If $\mathcal{H}$ is finite-dimensional with dimension n then it is isometrically isomorphic to $\mathbb{C}^n$. If $\mathcal{H}$ is infinite-dimensional then it is isometrically isomorphic to l^2.*

Property 18.2.28. Every orthogonal subset of a separable Hilbert space is countable.

18.2.6 Compact operators

The following definition is equivalent to definition 18.1.26:

Definition 18.2.29 (Compact operator). Let A be a linear operator on a Hilbert space $\mathcal{H}$. A is said to be compact if for every sequence (x_n) in $\mathcal{H}$ the sequence $(A[x_n])$ has a convergent subsequence.

18.2.7 Linear functionals

Property 18.2.30. Let f be a continuous linear functional. Then $\dim(\ker f)^{\perp}$ is 0 or 1 where the former case only arises when $f \equiv 0$.

Theorem 18.2.31 (Riesz' representation theorem). *Let $\mathcal{H}$ be a Hilbert space. For every continuous linear functional $\rho : \mathcal{H} \to \mathbb{R}$ there exists a unique element $x_0 \in \mathcal{H}$ such that*

$$\rho(h) = \langle h, x_0 \rangle \tag{18.18}$$

for all $h \in \mathcal{H}$. This implies that $\mathcal{H}$ and $\mathcal{H}^*$ are isometrically isomorphic. Furthermore the operator norm of ρ is equal to the norm of x_0.

Remark 18.2.32. This theorem justifies the bra-ket notation used in quantum mechanics where one associates to every ket $|\psi\rangle \in \mathcal{H}$ a bra $\langle\psi| \in \mathcal{H}^*$.

Chapter 19

Operator algebras

19.1 Involutive algebras

Definition 19.1.1 (Involution). Let $*$ be an automorphism of an algebra A. If $*(*\ a) = a$ for all $a \in A$ then $*$ is called an involution of A.

Definition 19.1.2 (Involutive algebra[1]). An involutive algebra is an associative algebra A over a commutative ring R with involution $^-$ together with an operator $^* : A \to A$ such that:

- $(a+b)^* = a^* + b^*$
- $(ab)^* = b^* a^*$
- $(\lambda a)^* = \overline{\lambda} a^*$

where $\lambda \in R$.

19.2 C*-algebras

Definition 19.2.1 (C*-algebra). A C*-algebra is a involutive Banach algebra[2] A such that the **C*-identity**

$$||a^* a|| = ||a||\ ||a^*|| \tag{19.1}$$

is satisfied.

Definition 19.2.2 (Positive). An element of a C*-algebra is called positive if it is self-adjoint and if its spectrum is contained in $[0, +\infty[$. A linear functional on a C*-algebra is called positive if every positive element is mapped to a positive number.

[1] Also called a ***-algebra**.

[2] See definition 18.1.4.

Definition 19.2.3 (State). Let A be a C*-algebra. A state ψ on A is a positive linear functional of unit norm.

Chapter 20

Tensor calculus

20.1 Tensor product

20.1.1 Tensor product

There are two possible ways to introduce the components of a tensor (on finite dimensional spaces). One way is to interpret tensors as multilinears maps another way is to interpret the components as expansion coefficients with respect to the tensor space basis.

Definition 20.1.1. The tensor product of vector spaces V and W is defined as[1] the set of multilinear maps on the Cartesian product $V^* \times W^*$. Let v, w be vectors in respectively V and W. Let g, h be vectors in the corresponding dual spaces. The tensor product of v and w is then defined as:

$$\boxed{(v \otimes w)(g, h) = v(g)w(h)} \tag{20.1}$$

Definition 20.1.2 (Tensor component). One way to define the tensor components is as follows: Let $\mathbf{T}$ be a tensor that takes r vectors and s covectors as input and returns a scalar. The different components are given by $\mathbf{T}(e_i, ..., e_j, e^k, ..., e^l) = T_{i...j}{}^{k...l}$.

Property 20.1.3 (Universal property). A set X together with a bilinear map $\mathcal{T} : V \times W \to X$ is said to have the universal property if for every bilinear map $f : V \times W \to Z$, where Z is some other vector space, there exists a unique linear map $f' : X \to Z$ such that $f = f' \circ \mathcal{T}$.

Corollary 20.1.4. The tensor product is unique up to a linear isomorphism. This results in

$$V \otimes W \cong W \otimes V \tag{20.2}$$

[1] "isomorphic to" would be a better terminology. See the "universal property" 20.1.3. For a complete proof and explanation, see [15].

The isomorphism is given by:

$$v(f) \equiv f(v) \tag{20.3}$$

where $v \in V$ and $f \in V^*$.

Notation 20.1.5 (Tensor power).

$$V^{\otimes n} = \underbrace{V \otimes ... \otimes V}_{n \text{ copies}} \tag{20.4}$$

Remark 20.1.6. More generally, the tensor product of r copies of V and s copies of V^* is the vector space $\mathcal{T}^r_s(V) = V^{\otimes r} \otimes V^{*\otimes s}$. These tensors are said to be of **type** (r, s).

Remark 20.1.7. Generally the space $\mathcal{T}^1_1 V$ is only isomorphic to the space $\text{End}(V^*)$. The isomorphism is given by the map $\hat{T} : V^* \to V^* : \omega \mapsto \mathbf{T}(\cdot, \omega)$ for every $\mathbf{T} \in \mathcal{T}^1_1 V$. Furthermore the spaces $\mathcal{T}^0_1 V$ and V^* are isomorphic.

For finite-dimensional vector spaces the space $\mathcal{T}^1_1 V$ is also isomorphic to $\text{End}(V)$ (see property 16.3.37). The space $\mathcal{T}^1_0 V$ will also be isomorphic to V itself.

Definition 20.1.8. The scalars (elements of the base field K) are by definition the $(0, 0)$ tensors.

Alternative Definition 20.1.9. The tensor space $\mathcal{T}^r_s(V)$ is spanned by the basis

$$\underbrace{e_i \otimes ... \otimes e_j}_{r \text{ basis vector}} \otimes \underbrace{\varepsilon^k \otimes ... \otimes \varepsilon^l}_{s \text{ dual basis vectors}}$$

where the operation $\otimes$ satisfies following properties:

1. Associativity: $u \otimes (v \otimes w) = u \otimes v \otimes w$
2. Multilinearity: $a(v \otimes w) = (av) \otimes w = v \otimes (aw)$ and $v \otimes (u + w) = v \otimes u + v \otimes w$

The expansion coefficients in this basis are written as $T^{i...j}{}_{k...l}$

Property 20.1.10 (Dimension of tensor product). From the previous construction it follows that the dimension of $\mathcal{T}^r_s(V)$ is equal to rs.

We now have to proof that the values of the tensor operating on r basis vectors and s basis covectors are equal to the corresponding expansion coefficients:

Proof. Let $\mathbf{T} = T_{i...j}{}^{k...l} e^i \otimes ... \otimes e^j \otimes e_k \otimes ... \otimes e_l$. Applying 20.1.1 and using the definition of the dual vectors 16.23 we have:

$$\begin{aligned} \mathbf{T}(e_a, ..., e_b, \varepsilon^m, ..., \varepsilon^n) &= T_{i...j}{}^{k...l} e^i(e_a)...e^j(e_b)e_k(e^m)...e_l(e^n) \\ &= T_{i...j}{}^{k...l} \delta^i_a ... \delta^j_b \delta^m_k ... \delta^n_l \\ &= T_{a...b}{}^{m...n} \end{aligned}$$

This is exactly the same result as the one we get by applying the first definition. □

20.2 Transformation rules

Let the basis for V transform as $e'_i = A^j{}_i e_j$ and $e_i = B^j{}_i e'_j$. Because the basis transformations A and B should be well-defined, they are each other's inverse: $B = A^{-1}$.

Definition 20.2.1 (Contravariant). A tensor component that transforms by the following rule is called contravariant:

$$v^i = A^i{}_j v'^j \tag{20.5}$$

Definition 20.2.2 (Covariant). A tensor component that transforms by the following rule is called covariant:

$$p_i = B^j{}_i \; p'_j \tag{20.6}$$

Example 20.2.3 (Mixed tensor). As an example of a mixed tensor we give the transformation formula for the mixed third-order tensor T^k_{ij}:

$$T^k_{ij} = A^k{}_w B^u{}_i B^v{}_j T'^w{}_{uv}$$

Theorem 20.2.4 (Quotient rule). *Assume we have an equation such as $K_i A^{jk} = B_i{}^{jk}$ or $K_i{}^j A_{jl}{}^k = B_{il}{}^k$ with A and B two known tensors[2]. The quotient rule asserts the following: "If the equation of interest holds under all transformations, then K is a tensor of the indicated rank and covariant/contravariant character".*

Remark. This rule is a useful substitute for the "illegal" division of tensors.

20.3 Tensor operations

20.3.1 General operations

Definition 20.3.1 (Contraction). Let A be a tensor of type (n, m). Setting a sub- and superscript equal and summing over this index gives a new tensor of type $(n-1, m-1)$. This operation is called the contraction of A. It is given by the evaluation map

$$V \otimes V^* : e_i \otimes e^j \mapsto e^j(e_i) \tag{20.7}$$

Definition 20.3.2 (Direct product). Let A and B be two random tensors (both rank and co-/contravariancy). The tensor constructed by the componentwise multiplication of A and B is called the direct product of A and B.

Example 20.3.3. Let $A^i{}_k$ and $B^j{}_{lm}$ be two tensors. The direct product is equal to:

$$C^{i\ j}_{\ k\ lm} = A^i{}_k B^j{}_{lm}$$

[2]This rule does not necessarily hold when $B = 0$ as transformations rules are not defined for the null-tensor.

Formula 20.3.4 (Operator product). It is also possible to combine operators working on different vector spaces so to make them work on the tensor product space. To do this we use following definition:

$$\boxed{(\hat{A} \otimes \hat{B})(v \otimes w) = (\hat{A}v) \otimes (\hat{B}w)} \tag{20.8}$$

Remark. Consider an operator $\hat{A}$ working on a space V_1. When working with a combined space $V_1 \otimes V_2$ the corresponding operator is in fact $\hat{A} \otimes \mathbb{1}$ but it is often still denoted by $\hat{A}$ in physics.

Notation 20.3.5. Consider a tensor with two indices T_{ij}. The antisymmetric part can then be written as:

$$T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji}) \tag{20.9}$$

20.3.2 Determinant

Definition 20.3.6 (n-form). An n-form is a totally anti-symmetric element $\omega \in \mathcal{T}^0_n V$. $\dim V$-forms are also called **top forms** or **volume forms**.

Definition 20.3.7 (Determinant). Let φ be an element in $\mathcal{T}^1_1 V \cong \mathrm{End}(V)$. Let ω be a volume form and let $\{e_i\}_{i \leq n}$ be a basis for V. The determinant of φ is then defined as:

$$\det \varphi = \frac{\omega(\varphi(e_1), ..., \varphi(e_n))}{\omega(e_1, ..., e_n)} \tag{20.10}$$

This definition is well-defined, i.e. it is independent of the choice of volume form and basis. Furthermore it coincides with definition 16.5.50.

One should note that the determinant is only well-defined for $(1,1)$-tensors. Although other types of tensors can also be represented as matrices, definition 16.5.50 would not be independent of a choice of basis anymore. An alternative concept can be defined using principal bundles and more precisely frame bundles (see section 26.5).

20.3.3 Differentiation

Property 20.3.8.

$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{A}} \otimes \vec{\boldsymbol{B}}) = (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}})\vec{\boldsymbol{B}} + (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} \tag{20.11}$$

20.3.4 Levi-Civita tensor

Definition 20.3.9 (Levi-Civita tensor). Let e^i be the dual vector to e_i. In n dimensions, we define the Levi-Civita tensor as follows:

$$\boldsymbol{\varepsilon} = \varepsilon_{12...n} e^1 \otimes e^2 \otimes ... \otimes e^n \tag{20.12}$$

where

$$\varepsilon_{i\ldots n} = \begin{cases} 1 & \text{if } (i\ldots n) \text{ is an even permutation of } (12\ldots n) \\ -1 & \text{if } (i\ldots n) \text{ is an odd permutation of } (12\ldots n) \\ 0 & \text{if any of the indices occurs more than once} \end{cases}$$

Remark 20.3.10. The Levi-Civita symbol is not a tensor, but a pseudotensor. This means that the sign changes under reflections (or any transformation with determinant -1).

Formula 20.3.11 (Cross product). By using the Levi-Civita symbol, we can define the i-th component of the cross product[3] of two vectors $\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$ as follows:

$$\boxed{(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}})_i = \varepsilon_{ijk} v_j w_k} \tag{20.13}$$

20.3.5 Complexification

Definition 20.3.12 (Complexification). Let V be a real vector space. The complexification of V is defined as the following tensor product:

$$V^{\mathbb{C}} = V \otimes \mathbb{C} \tag{20.14}$$

On its own this remains a real vector space. However we can turn this space into a complex vector space by generalizing the scalar product as follows:

$$\alpha(v \otimes \beta) = v \otimes (\alpha\beta) \tag{20.15}$$

for all $\alpha, \beta \in \mathbb{C}$.

Property 20.3.13. By noting that every element $\overline{v} \in V^{\mathbb{C}}$ can be written as

$$\overline{v} = (v_1 \otimes 1) + (v_2 \otimes i)$$

we can decompose the complexification as follows:

$$V^{\mathbb{C}} \cong V \oplus iV \tag{20.16}$$

20.4 (Anti)symmetric tensors

20.4.1 Symmetric tensors

Notation 20.4.1. The space of symmetric $(0, n)$ tensors is denoted by $S^n(V^*)$. The space of symmetric $(n, 0)$ tensors is denoted by $S^n(V)$.

[3] Following from remark 20.3.10 we can see that the cross product is in fact not a vector, but a pseudovector.

20.4.2 Antisymmetric tensors

Definition 20.4.2 (Antisymmetric tensor). Tensors that change sign under the interchange of any two indices.

Notation 20.4.3. The space of antisymmetric $(0, n)$ tensors is denoted by $\Lambda^n(V^*)$. The space of antisymmetric $(n, 0)$ tensors is denoted by $\Lambda^n(V)$.

Remark. Elements of $\Lambda^2(V)$ are also known as **bivectors**. Elements of $\Lambda^k(V)$ are generally known as k-**blades**.

Property 20.4.4. Let $n = \dim(V)$. $\Lambda^r(V)$ equals the null-space for all $r \geq n$.

20.4.3 Wedge product

Definition 20.4.5 (Wedge product).

$$\boxed{f \wedge g = f \otimes g - g \otimes f} \tag{20.17}$$

From this definition it immediately follows that the wedge product is antisymmetric.

Formula 20.4.6. Let $\{P_i\}_i$ be the set of all permutations of the sequence $(1, ..., k)$.

$$e_1 \wedge ... \wedge e_k = \sum_i \operatorname{sgn}(P_i) e_{P_i(1)} \otimes ... \otimes e_{P_i(k)} \tag{20.18}$$

Construction 20.4.7. Let $\{e_i\}_{1 \leq i \leq n}$ be a basis for V. It is clear from the definition 20.17 that a basis for $\Lambda^r(V)$ is given by

$$\{e_{i_1} \wedge ... \wedge e_{i_r} \mid \forall k : 1 \leq i_k \leq \dim(V)\}$$

The dimension of this space is given by:

$$\dim \Lambda^k(V) = \binom{n}{k} \tag{20.19}$$

Remark 20.4.8. For $k = 0$, the above construction is not useful, so we just define $\Lambda^0(V) = \mathbb{R}$.

Formula 20.4.9 (Levi-Civita symbol). The Levi-Civita tensor in n dimensions as introduced in 20.12 can now be rewritten more concisely as:

$$\boldsymbol{\varepsilon} = e_1 \wedge ... \wedge e_n \tag{20.20}$$

Formula 20.4.10. In 3 dimensions there exists an important isomorphism $J : \Lambda^2(\mathbb{R}^3) \to \mathbb{R}^3$:

$$J(\lambda)^i = \frac{1}{2} \varepsilon^i{}_{jk} \lambda^{jk} \tag{20.21}$$

where $\lambda \in \Lambda^2(\mathbb{R}^3)$.

Looking at the definition of the cross product 16.79, we can see that $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}$ is actually the same as $J(\vec{\boldsymbol{v}} \wedge \vec{\boldsymbol{w}})$. One can thus use the wedge product to generalize the cross product to higher dimensions.

Example 20.4.11. Let A, B and C be three vectors in V. Now consider following expression:

$$(C \wedge B)(L(A), \cdot)$$

where $L(A)$ is the metric dual of A (see 16.30). Evaluating this formula using the properties of the wedge and tensor products leads to the well known BAC-CAB rule of triple cross products:

$$(C \cdot A)B - (B \cdot A)C$$

Remark 20.4.12. The wedge product can also be defined without prior knowledge of tensor products. The wedge product is then defined by letting $V \wedge W$ be the smallest space such that for all elements $v \in V, w \in W$ the following property holds:

$$v \wedge w = -w \wedge v \tag{20.22}$$

Now, let $\{e_i\}_{i \leq m}$ be an ordered basis for V and let $\{d_j\}_{j \leq n}$ be an ordered basis for W. The wedge product $V \wedge W$ is spanned by the basis $\{e_i \wedge d_j\}_{i \leq m, j \leq n}$.

20.4.4 Exterior algebra

Definition 20.4.13 (Exterior power). In the theory of exterior algebras, the space $\Lambda^k(V)$ is often called the k^{th} exterior power of V.

Definition 20.4.14 (Exterior algebra). We can define a graded vector space[4] $\Lambda^*(V)$ as follows:

$$\Lambda^*(V) = \bigoplus_{k \geq 0} \Lambda^k(V)$$

Then we can turn this graded vector space into a graded algebra by taking the wedge product as the multiplication:

$$\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$$

This algebra is called the exterior algebra or **Grassmann algebra** of V.

Alternative Definition 20.4.15 (†). Let $T(V)$ be the (free) tensor algebra over the vector space V, i.e.

$$T(V) = \bigoplus_{k \geq 0} T^{\otimes k}(V) \tag{20.23}$$

where $T^{\otimes k}(V)$ is the k^{th} tensor power of V. The exterior algebra over V is generally defined as the quotient of $T(V)$ by the two-sided ideal I generated by $\{v \otimes v | v \in V\}$.

Property 20.4.16. The exterior algebra is both an associative algebra and a unital algebra with unit element $1 \in \mathbb{R}$. Furthermore it is also commutative in the graded sense (see 3.21).

[4] See definition 16.8.

20.4.5 Hodge star

It follows from equation 20.19 that the spaces $\Lambda^k(V)$ and $\Lambda^{n-k}(V)$ have the same dimension, so there exists an isomorphism between them. This map is given by the Hodge star $*$. However this map can only be defined independent of the choice of (ordered) basis if we restrict ourselves to vector spaces equipped with a non-degenerate Hermitian form 16.4.2.

Definition 20.4.17 (Volume element). Let V be an n-dimensional vector space with ordered basis $\{e_i\}_{i\leq n}$. The volume element on V is defined as:

$$\mathrm{Vol}(V) := e_1 \wedge ... \wedge e_n \tag{20.24}$$

It is clear that this is an element of $\Lambda^n(V)$.

Definition 20.4.18 (Orientation). Let $\omega \in \Lambda(V)$ be an element of degree n. From the previous definition it follows that this k-blade is a scalar multiple of Vol because $\Lambda^n(V)$ is one-dimensional:

$$\omega = r\mathrm{Vol}(V)$$

The k-blade ω induces an orientation on V in the following way. If the scalar $r > 0$ then the orientation is said to be **positive**. If $r < 0$ then the orientation is **negative**.

Formula 20.4.19 (Inner product). Let V be equipped with an inner product $\langle \cdot, \cdot \rangle$. Then we can define an inner product on $\Lambda^k(V)$ by:

$$\boxed{\langle v_1 \wedge ... \wedge v_k | w_1 \wedge ... \wedge w_k \rangle_k = \det(\langle v_i, w_j \rangle)} \tag{20.25}$$

For an orthogonal basis, this formula factorises into:

$$\langle v_1 \wedge ... \wedge v_k | w_1 \wedge ... \wedge w_k \rangle_k = \langle v_1 | w_1 \rangle \cdots \langle v_k | w_k \rangle \tag{20.26}$$

Definition 20.4.20 (Hodge star). The Hodge star $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$ is defined as the isomorphism such that for all $\omega \in \Lambda^k(V)$ and $\rho \in \Lambda^{n-k}(V)$ we have the following equality:

$$\omega \wedge \rho = \langle *\omega, \rho \rangle_{n-k} \mathrm{Vol}(V) \tag{20.27}$$

where $\langle \cdot, \cdot \rangle$ is the inner product 20.25 on $\Lambda^{n-k}(V)$. Furthermore, this isomorphism is unique.

Proof. Because $\omega \wedge \rho$ is an element of $\Lambda^n(V)$ it is a scalar multiple of $\mathrm{Vol}(V)$. This implies that it can be written as

$$c(\rho)\mathrm{Vol}(V)$$

The map $c : \Lambda^{n-k}(V) \to \mathbb{R} : \rho \mapsto c(\rho)$ is a linear map and thus a continuous map, so we can apply Riesz' representation theorem to identify c with a unique element $*\omega \in \Lambda^{n-k}(V)$ such that

$$c(\rho) = \langle *\omega, \rho \rangle_{n-k}$$

□

Formula 20.4.21. Let $\{e_i\}_{i\leq n}$ be a positively oriented ordered orthonormal basis for V. An explicit formula for the Hodge star is given by the following construction. Let $\{i_1, ..., i_k\}$ and $\{j_1, ..., j_{n-k}\}$ be two complementary index sets with increasing subindices. Let $\omega = e_{i_1} \wedge ... \wedge e_{i_k}$.

$$\boxed{*\omega = \text{sgn}(\tau) \prod_{m=1}^{n-k} \langle e_{j_m} | e_{j_m} \rangle e_{j_1} \wedge ... \wedge e_{j_{n-k}}} \tag{20.28}$$

where τ is the permutation that maps $e_{i_1} \wedge ... \wedge e_{i_k} \wedge e_{j_1} \wedge ... \wedge e_{j_{n-k}}$ to $\text{Vol}(V)$

Corollary 20.4.22. Consider three vectors $u, v, w \in \mathbb{R}^3$.

$$*(v \wedge w) = v \times w \tag{20.29}$$
$$*(v \times w) = v \wedge w \tag{20.30}$$
$$*(u \wedge v \wedge w) = u \cdot (v \times w) \tag{20.31}$$

Remark 20.4.23. Formula 20.21 is an explicit evaluation of the first equation 20.29.

Proof. The sign $\text{sgn}(\tau)$ can be written using the Levi-Civita symbol ε_{ijk} as defined in 20.12. The factor $\frac{1}{2}$ is introduced to correct for the double counting due to the contraction over both the indices j and k.

Property 20.4.24. Consider an inner product space V, then

$$\boxed{** \ \omega = (-1)^{k(n-k)}\omega} \tag{20.32}$$

In $n = 4$ this leads to $**\omega = \omega$ which means that the Hodge star is an involution in 4-dimensional inner product spaces.

Definition 20.4.25 (Self-dual). Let V be a 4-dimensional inner product space. Consider $\omega \in \Lambda^2(V)$. Then ω is said to be self-dual if $*\omega = \omega$. Furthermore every $v \in \Lambda^2(V)$ can be uniquely decomposed as the sum of a self-dual and an anti-self-dual 2-form.

20.4.6 Grassmann numbers

Although this section does not really belong to the chapter about tensors, we have included it here as it is an application of the concept of exterior algebras. The concept of Grassmann numbers (or variables) is used in QFT when performing calculations in the fermionic sector.

Definition 20.4.26 (Grassmann numbers). Let V be a complex vector space spanned by a set of generators θ_i. The Grassmann algebra with Grassmann variables θ_i is the exterior algebra over V. The wedge symbol of Grassmann variables is often ommitted when writing the product: $\theta_i \wedge \theta_j \equiv \theta_i\theta_j$.

Remark 20.4.27. Furthermore, from the anti-commutativity it follows that we can regard the Grassmann variables as being non-zero square-roots of zero.

Property 20.4.28. Consider a one-dimensional Grassmann algebra. When constructing the polynomial ring $\mathbb{C}[\theta]$ generated by θ, we see that, due to the anti-commutativity, $\mathbb{C}[\theta]$ is spanned only by 1 and θ. All higher degree terms vanish because $\theta^2 = 0$. This implies that the most general polynomial over a one-dimenisonal Grassmann algebra can be written as

$$p(\theta) = a + b\theta \tag{20.33}$$

Definition 20.4.29. We can equip the exterior algebra Λ with Grassmann variables θ_i with an involution similar to that on $\mathbb{C}$:

$$(\theta_i\theta_j...\theta_k)^* = \theta_k...\theta_j\theta_i \tag{20.34}$$

Elements $z \in \Lambda$ such that $z^* = z$ are called **(super)real**, elements such that $z^* = -z$ are called **(super)imaginary**. This convention is called the *DeWitt* convention.

Chapter 21

Clifford Algebra

21.1 Clifford algebra

Definition 21.1.1 (Clifford algebra). Let V be unital associative algebra. The Clifford algebra over V with quadratic form $Q : V \to K$ is the free algebra generated by V under the following condition:

$$v \cdot v = Q(v)1 \tag{21.1}$$

where 1 is the unit element in V. This condition implies that the square of a vector is a scalar.

Notation 21.1.2. The Clifford algebra corresponding to V and Q is denoted by $C\ell(V, Q)$.

Construction 21.1.3. The previous definition can be given an explicit construction. First we construct the tensor algebra of V:

$$T(V) = \bigoplus_{k\in\mathbb{N}} V^{\otimes k} \tag{21.2}$$

Then we construct a two-sided ideal I of V generated[1] by $\{v \otimes v - Q(v)1_V \mid v \in V\}$. The Clifford algebra $C\ell(V, Q)$ can then be constructed as the quotient algebra $T(V)/I$.

Remark 21.1.4. Looking at definition 20.4.15 we see that the exterior algebra $\Lambda^*(V)$ coincides with the Clifford algebra $C\ell(V, 0)$. If $Q \neq 0$ then the two algebras are still linearly isomorphic when $\text{char}(V) \neq 2$.

Property 21.1.5 (Dimension). If V has dimension n then $C\ell(V, Q)$ has dimension 2^n.

21.2 Geometric algebra

Definition 21.2.1 (Geometric algebra). Let V be a vector space equipped with a symmetric bilinear form $g : V \times V \to K$. The geometric algebra (GA) over V is defined as

[1] See definition 3.2.9.

the Clifford algebra $C\ell(V, g)$. If $\text{char}(V) \neq 2$ then the bilinear form uniquely determines a quadratic form $Q : v \mapsto g(v, v)$ as required in definition 21.1.1.

Definition 21.2.2 (Inner and exterior product). Analogous to the inner product in linear algebra and the wedge product in exterior algebras one can define an (a)symmetric product on the geometric algebra.

First of all we note that the product ab of two vectors a and b can be written as the sum of a symmetric and an antisymmetric part:

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) \tag{21.3}$$

We can then define the inner product as the symmetric part:

$$a \cdot b := \frac{1}{2}(ab + ba) = \frac{1}{2}\left((a + b)^2 - a^2 - b^2\right) = g(a, b) \tag{21.4}$$

Analogously we define the exterior (outer) product as the antisymmetric part:

$$a \wedge b := \frac{1}{2}(ab - ba) \tag{21.5}$$

These definitions allow us the rewrite formula 21.3 as:

$$\boxed{ab = a \cdot b + a \wedge b} \tag{21.6}$$

Remark. Looking at the last equality in the definition of the inner product 21.4 we see that condition 21.1 is satisfied when $a = b$.

Example 21.2.3 (Exterior algebra). When g is fully degenerate, i.e. $g(v, v) = 0$ for all $v \in V$, the inner product is identically zero for all vectors and the geometric algebra coincides with exterior algebra[2] over V. For general forms g the exterior algebra is a subalgebra of the GA.

Definition 21.2.4 (Multivector). Any element of the GA over V is called a multivector. The simple multivectors of grade k, i.e. elements of the form $v_1 v_2 ... v_k$ with $v_i \in V$ for all i, are called k-blades. This generalizes the remark underneath 20.4.3. Sums of multivectors of different grades are called mixed multivectors[3].

Let $n = \dim(V)$. Multivectors of grade n are also called **pseudoscalars** and multivectors of grade $n - 1$ are also called **pseudovectors**.

Definition 21.2.5 (Grade projection operator). Let a be a general multivector. The grade (projection) operator $\langle \cdot \rangle_k : \mathcal{G} \to \mathcal{G}_k$ is defined as the projection of a on the k-vector part of a.

[2] See definition 20.4.14.

[3] These elements do not readily represent a geometric structure.

Using the grade operators we can extend the inner and exterior product to the complete GA as follows.

Formula 21.2.6. Let A, B be two multivectors of respectively grade m and n. Their inner product is defined as:

$$A \cdot B = \langle AB \rangle_{|m-n|} \tag{21.7}$$

Their exterior product is defined as:

$$A \wedge B = \langle AB \rangle_{m+n} \tag{21.8}$$

21.3 Pin group

21.3.1 Clifford group

Definition 21.3.1 (Main involution). Let V_0, V_1 be respectively the grade 0 and 1 components of the Clifford algebra $C\ell(V, Q)$. Consider the following operator:

$$\hat{v} = \begin{cases} v & v \in V_0 \\ -v & v \in V_1 \end{cases} \tag{21.9}$$

This operation can be generalized to all of $C\ell(V, Q)$ using linearity. The resulting operator is called the main involution or **inversion** on $C\ell(V, Q)$. Furthermore it turns the Clifford algebra into a superalgebra[4].

Formula 21.3.2 (Twisted conjugation). Let $v \in V$ be a vector and let $s \in C\ell(V, Q)$ be an invertible element of the Clifford algebra over V. The twisted conjugation of v by s is given by the map:

$$\chi : C\ell(V, Q) \times V : \chi(s)v = sv\hat{s}^{-1} \tag{21.10}$$

Definition 21.3.3 (Clifford group). The Clifford group $\Gamma(V, Q)$ is defined as follows:

$$\Gamma(V, Q) = \{s \in C\ell(V, Q) : sv\hat{s}^{-1} \in V, v \in V\} \tag{21.11}$$

It is the set of Clifford algebra elements that stabilize V under twisted conjugation. Furthermore this set is closed under multiplication and forms a group. It contains all invertible elements $s \in V$, i.e. all elements $s \in V$ for which $Q(v) \neq 0$.

Property 21.3.4. If V is finite-dimensional, the map

$$\chi : \Gamma(V, Q) \to O(V, Q) : s \mapsto \chi(s) \tag{21.12}$$

defines a representation[5] called the *vectorial representation*. Furthermore, from the first isomorphism theorem 3.1.34 it follows that $O(V, Q)$ is isomorphic to $\Gamma(V, Q)/\ker\chi$ where

[4] See definition 16.2.25.

[5] The surjectiveness of the map χ follows from the *Cartan-Dieudonné theorem*.

$\ker \chi = \mathbb{R}\backslash\{0\}$. This isomorphism[6] also implies that the Clifford group is given by the the set of finite products of invertible elements $v \in V$:

$$\Gamma(V,Q) = \left\{ \prod_i^k s_i : s_i \text{ invertible in } V, n \in \mathbb{N} \right\} \tag{21.13}$$

Corollary 21.3.5. By nothing that pure rotations can be decomposed as an even number of reflections we find that:

$$\Gamma^+(V,Q)/\mathbb{R}_0 = SO(V,Q) \tag{21.14}$$

where Γ^+ is the intersection of the even Clifford algebra and the Clifford group.

21.3.2 Pin and Spin groups

Formula 21.3.6 (Spinor norm). On $\Gamma(V,Q)$ (and in fact on al of $C\ell(V,Q)$ by using the linearity) one can define the spinor norm:

$$\mathcal{N}(x) = \hat{x}x \tag{21.15}$$

where $\hat{}$ is the main involution. The map $|\mathcal{N}|$ then gives a group homomorphism from $\Gamma(V,Q)$ to $\mathbb{R}_0^+$.

Definition 21.3.7 (Pin and spin groups). Using the spinor norm $\mathcal{N}$ we can now define the pin and spins groups as follows:

$$\mathrm{Pin}(V) = \{s \in \Gamma(V,Q) : |\mathcal{N}|(s) = 1\} \tag{21.16}$$

and

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \Gamma^+(V,Q) \tag{21.17}$$

Remark 21.3.8. The Pin group can also be defined as the set of elements in $\Gamma(V,Q)$ that can be written as a product of unit Clifford vectors. The Spin group is then defined as the elements that can be written as the product of an even number of unit Clifford vectors.

Property 21.3.9. The Pin group satisfies following isomorphicity relation:

$$\mathrm{Pin}(V,Q)/\mathbb{Z}_2 \cong O(V,Q) \tag{21.18}$$

and analogously for the Spin group and $SO(V,Q)$. These relations also imply that the Pin and Spin groups form a double covering group[7] for respectively the orthogonal and special orthogonal groups.

Definition 21.3.10 (Spinor). Elements of the Spin group are called spinors.

[6] Together with the *Cartan-Dieudonné theorem*.

[7] A covering group is a topological group that is also a covering space. See definition 5.3.13 for more information about the latter.

Chapter 22

Representation Theory

22.1 Group representations

Definition 22.1.1 (Representation). A representation of a group G, acting on a vector space V, is a homomorphism $\rho : G \to GL(V)$ from G itself to the automorphism group[1] of V. This is a specific case of a group action[2].

Property 22.1.2. Because every linear map maps the zero vector to the zero vector, a group representation can never be free[3].

Definition 22.1.3 (Subrepresentation). A subrepresentation of a representation V is a subspace of V invariant under the action of the group G.

Example 22.1.4 (Permutation representation). Consider a vector space V equipped with a basis $\{e_i\}_{i\in I}$ with $|I| = n$. Let $G = S^n$ be the symmetric group of dimension n. Based on remark 3.1.36 we can consider the action of G on the index set I. This representation is given by

$$\rho(g) : \sum_{i\in I} v_i e_i \mapsto \sum_{i\in I} v_i e_{g\cdot i} \tag{22.1}$$

Example 22.1.5. Consider a representation ρ on V. There exists a natural representation on the dual space V^*. The homomorphism $\rho^* : G \to GL(V^*)$ is given by:

$$\rho^*(g) = \rho^T(g^{-1}) : V^* \to V^* \tag{22.2}$$

where ρ^T is the transpose as defined in 16.3.40. This map satifies the following defining property:

$$\Big\langle \rho^*(g)(v^*), \rho(g)(v) \Big\rangle = \langle v^*, v\rangle \tag{22.3}$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing of V and its dual.

[1] See definition 16.3.11.
[2] See definition 3.1.35.
[3] See definition 3.1.40.

Example 22.1.6. A representation ρ which acts on spaces V, W can also be extended to the tensor product $V \otimes W$ in the following way:

$$g(v \otimes w) = g(v) \otimes g(w) \tag{22.4}$$

22.2 Irreducible representations

Definition 22.2.1 (Irreducibility). A representation is said to be irreducible if there exist no proper non-zero subrepresentation.

Example 22.2.2 (Standard representation). Consider the action of $\mathrm{Sym}(n)$ on a vector space V. The line generated by $v_1 + v_2 + ... + v_n$ is invariant under the permutation action of $\mathrm{Sym}(n)$. It follows that the permutation representation (on finite-dimensional spaces) is never irreducible.

The $(n-1)$-dimensional complementary subspace

$$W = \{a_1v_1 + a_2v_2 + ... + a_nv_n | a_1 + a_2 + ... + a_n = 0\} \tag{22.5}$$

does form an irreducible representation when we restrict ρ to W. It is called the standard representation of S^n.

Theorem 22.2.3 (Schur's lemma). *Let V, W be two irreducible representations of a finite group G. Let $\varphi : V \to W$ be a G-module homomorphism. We then have:*

- *φ is an isomorphism or $\varphi = 0$*
- *If $V = W$ then φ is constant, i.e. φ is a scalar multiple of the identity map $\mathbb{1}_V$.*

Property 22.2.4. If W is a subrepresentation of V then there exists an invariant complementary subspace W' such that $V = W \oplus W'$.

This space can be found as follows: Choose an arbitrary complement U such that $V = W \oplus U$. From this we construct a projection map $\pi_0 : V \to W$. Averaging over G gives

$$\pi(v) = \sum_{g \in G} g \circ \pi_0(g^{-1}v) \tag{22.6}$$

which is a G-linear map $V \to W$. On W it is given by the multiplication of W by $|G|$. Its kernel is then an invariant subspace of V under the action of G and complementary to W.

Property 22.2.5. Let G be a finite group. A representation V can be uniquely decomposed as

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \tag{22.7}$$

where all V_k's are distinct irreducible representations.

Part V

Differential Geometry

Chapter 23

Curves and Surfaces

23.1 Curves

Property 23.1.1 (Regular curve). Let $\vec{\boldsymbol{c}}(t)$ be a curve defined on an interval I. $\vec{\boldsymbol{c}}(t)$ is said to be regular if $\frac{d\vec{\boldsymbol{c}}}{dt} \neq \vec{\mathbf{0}}$ for all $t \in I$.

Definition 23.1.2 (C^r-parameter transformation). A transformation $\varphi :]c, d[\to]a, b[$: $u \mapsto t$ such that $\varphi(u)$ is a C^r-diffeomorphism[1].

Definition 23.1.3 (Geometric property). A geometric property is a property that is invariant under:

1. parameter transformations
2. positive orthonormal changes of basis

Theorem 23.1.4. *Let $\vec{\boldsymbol{c}}(t), \vec{\boldsymbol{d}}(t)$ be two curves with the same image. We then have the following relation:*

$$\vec{\boldsymbol{c}}(t) \text{ regular} \iff \vec{\boldsymbol{d}}(t) \text{ regular} \tag{23.1}$$

23.1.1 Arc length

Definition 23.1.5 (Natural parameter). Let $\vec{\boldsymbol{c}}(t)$ be a curve. The parameter t is said to be a natural parameter if:

$$\left|\left|\frac{d\vec{\boldsymbol{c}}}{dt}\right|\right| \equiv 1 \tag{23.2}$$

Formula 23.1.6 (Arc length). The following function $\phi(t)$ is a bijective map and a natural parameter of $\vec{\boldsymbol{c}}(t)$:

$$\phi(t) = \int_{t_0}^{t} ||\dot{\vec{\boldsymbol{c}}}(t)||dt \tag{23.3}$$

[1] See definition 16.3.12

Remark. The arc length as defined above is often denoted by 's'.

Theorem 23.1.7. *Let $\vec{\boldsymbol{c}}(t)$ be a curve. Let u be an alternative parameter of $\vec{\boldsymbol{c}}(t)$. It is a natural parameter if and only if there exists a constant α such that:*

$$u = \pm s + \alpha$$

where s is the integral as defined in equation 23.3.

Remark. As the last theorem implies, no unique natural parameter or arc length exists.

23.1.2 Frenet-Serret frame

Definition 23.1.8 (Tangent vector). Let $\vec{\boldsymbol{c}}(s)$ be parametrized by arc length. The tangent vector $\vec{\boldsymbol{t}}(s)$ is defined as:

$$\vec{\boldsymbol{t}}(s) = \vec{\boldsymbol{c}}\,'(s) \tag{23.4}$$

Property 23.1.9. From the definition of the natural parametrization 23.2 and the previous definition it follows that the tangent vector is a unit vector:

$$\vec{\boldsymbol{t}}(s) \cdot \vec{\boldsymbol{t}}(s) = \left|\left| \frac{d\vec{\boldsymbol{c}}(s)}{ds} \right|\right|^2 = 1$$

Definition 23.1.10 (Principal normal vector). Let $\vec{\boldsymbol{c}}(s)$ be parametrized by arc length. The principal normal vector is defined as:

$$\vec{\boldsymbol{n}}(s) = \frac{\vec{\boldsymbol{t}}\,'(s)}{||\vec{\boldsymbol{t}}\,'(s)||} \tag{23.5}$$

Property 23.1.11. From property 23.1.9 and the definition of the principal normal vector it follows that the tangent vector and principal normal vector are orthogonal:

$$\vec{\boldsymbol{t}}(s) \cdot \vec{\boldsymbol{t}}(s) = 1 \implies \vec{\boldsymbol{t}}(s) \cdot \vec{\boldsymbol{t}}\,'(s) = 0 \implies \vec{\boldsymbol{t}}(s) \cdot \vec{\boldsymbol{n}}(s) = 0$$

Definition 23.1.12 (Binormal vector). Let $\vec{\boldsymbol{c}}(s)$ be parametrized by arc length. The binormal vector is defined as:

$$\vec{\boldsymbol{b}}(s) = \vec{\boldsymbol{t}}(s) \times \vec{\boldsymbol{n}}(s) \tag{23.6}$$

Definition 23.1.13 (Frenet-Serret frame). As the vectors $\vec{\boldsymbol{t}}(s), \vec{\boldsymbol{n}}(s)$ and $\vec{\boldsymbol{b}}(s)$ are mutually orthonormal and linearly independent, we can use them to construct a positive orthonormal basis. The ordered basis $\left(\vec{\boldsymbol{t}}(s), \vec{\boldsymbol{n}}(s), \vec{\boldsymbol{b}}(s)\right)$ is called the **Frenet-Serret** frame.

Remark. This basis does not have to be the same in every point of $\vec{\boldsymbol{c}}(s)$.

Definition 23.1.14 (Curvature). Let $\vec{\boldsymbol{c}}(s)$ be parametrized by arc length. The curvature of $\vec{\boldsymbol{c}}(s)$ is defined as:

$$\frac{1}{\rho(s)} = ||\vec{\boldsymbol{t}}\,'(s)|| \tag{23.7}$$

Definition 23.1.15 (Torsion). Let $\vec{\boldsymbol{c}}(s)$ be a curve parametrized by arc length. The torsion of $\vec{\boldsymbol{c}}(s)$ is defined as:

$$\tau(s) = \rho(s)^2(\vec{\boldsymbol{t}} \;\; \vec{\boldsymbol{t}}' \;\; \vec{\boldsymbol{t}}'') \tag{23.8}$$

Formula 23.1.16 (Frenet formulas). The derivatives of the tangent, principal normal and binormal vectors can be written as a linear combination of the those vectors themself as:

$$\begin{cases} \vec{\boldsymbol{t}}\,'(s) &= & \frac{1}{\rho(s)}\vec{\boldsymbol{n}}(s) \\ \vec{\boldsymbol{n}}\,'(s) &= -\frac{1}{\rho(s)}\vec{\boldsymbol{t}}(s) & + \quad \tau(s)\vec{\boldsymbol{b}}(s) \\ \vec{\boldsymbol{b}}\,'(s) &= & -\tau(s)\vec{\boldsymbol{n}}(s) \end{cases} \tag{23.9}$$

Theorem 23.1.17 (Fundamental theorem of curves). *Let $k(s), w(s) : U \to \mathbb{R}$ be two $\mathcal{C}^1$ functions with $k(s) \geq 0, \forall s$. There exists an interval $]-\varepsilon, \varepsilon[\subset U$ and a curve $\vec{\boldsymbol{c}}(s) :]-\varepsilon, \varepsilon[\to \mathbb{R}^3$ with natural parameter s such that $\vec{\boldsymbol{c}}(s)$ has $k(s)$ as its curvature and $w(s)$ as its torsion.*

23.2 Surfaces

Notation 23.2.1. Let $\vec{\boldsymbol{\sigma}}$ be a surface[2]. The derivative of $\vec{\boldsymbol{\sigma}}$ with respect to the coordinate q^i is written as follows:

$$\frac{\partial \vec{\boldsymbol{\sigma}}}{\partial q^i} = \vec{\boldsymbol{\sigma}}_i \tag{23.10}$$

23.2.1 Tangent vectors

Definition 23.2.2 (Tangent plane). Let $P(q_0^1, q_0^2)$ be a point on the surface Σ. The tangent space $T_P\Sigma$ to $\vec{\boldsymbol{\sigma}}$ in P is defined as follows:

$$\forall \vec{\boldsymbol{r}} \in T_P\Sigma : \left[\vec{\boldsymbol{r}} - \vec{\boldsymbol{\sigma}}(q_0^1, q_0^2)\right] \cdot \left[\vec{\boldsymbol{\sigma}}_1(q_0^1, q_0^2) \times \vec{\boldsymbol{\sigma}}_2(q_0^1, q_0^2)\right] = 0 \tag{23.11}$$

Definition 23.2.3 (Normal vector). The cross product in equation 23.11 is closely related to the normal vector to Σ in P. The normal vector in the point (q_0^1, q_0^2) is defined as:

$$\vec{\boldsymbol{N}}(q_0^1, q_0^2) = \frac{1}{||\vec{\boldsymbol{\sigma}}_1 \times \vec{\boldsymbol{\sigma}}_2||}\left(\vec{\boldsymbol{\sigma}}_1 \times \vec{\boldsymbol{\sigma}}_2\right) \tag{23.12}$$

23.2.2 First fundamental form

Definition 23.2.4 (Metric coefficients). Let $\vec{\boldsymbol{\sigma}}$ be a surface. The metric coefficients g_{ij} are defined as follows:

$$\boxed{g_{ij} = \vec{\boldsymbol{\sigma}}_i \cdot \vec{\boldsymbol{\sigma}}_j} \tag{23.13}$$

[2] $\vec{\boldsymbol{\sigma}}$ denotes the surface as a vector field. Σ denotes the geometric image of $\vec{\boldsymbol{\sigma}}$.

Definition 23.2.5 (Scale factor). The following factors are often used in vector calculus:

$$g_{ii} = h_i^2 \tag{23.14}$$

Definition 23.2.6 (First fundamental form). Let $\vec{\boldsymbol{\sigma}}$ be a surface. Define a bilinear form $I_P(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) : T_P\Sigma \times T_P\Sigma \to \mathbb{R}$ that restricts the inner product to $T_P\Sigma$:

$$I_P(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \tag{23.15}$$

This bilinear form is called the first fundamental form or **metric**.

Corollary 23.2.7. All $\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \in T_P\Sigma$ are linear combinations of the tangent vectors $\vec{\boldsymbol{\sigma}}_1, \vec{\boldsymbol{\sigma}}_2$. This leads to the following relation between the first fundamental form and the metric coefficients 23.13:

$$I_P(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = v^i \vec{\boldsymbol{\sigma}}_i \cdot w^j \vec{\boldsymbol{\sigma}}_j = g_{ij} v^i w^j$$

Notation 23.2.8. The length can be written as

$$s = \int \sqrt{||\dot{\vec{\boldsymbol{c}}}(t)||} dt = \int \sqrt{ds^2}$$

where the second equality is formally defined. The two equalities together can be combined into the following notation for the metric:

$$\boxed{ds^2 = g_{ij} dq^i dq^j} \tag{23.16}$$

Formula 23.2.9. Let (g_{ij}) be the metric tensor. We define the matrix (g^{ij}) as its inverse:

$$(g^{ij}) = \frac{1}{\det(g_{ij})} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} \tag{23.17}$$

23.2.3 Isometries

Definition 23.2.10 (Isometry). An isometry is a distance-preserving map, i.e. a diffeomorphism $\Phi : \Sigma \to \Sigma'$ that maps arc segments in Σ to arc segments with the same length in Σ'.

Property 23.2.11. A diffeomorphism Φ is an isometry if and only if the metric coefficients of σ and σ' are the same.

Definition 23.2.12 (Conformal map). A diffeomorphism $\Phi : \Sigma \to \Sigma'$ is said to be conformal or isogonal if it maps two intersecting curves in Σ to intersecting curves in Σ' with the same intersection angle.

Property 23.2.13. A diffeomorphism Φ is conformal if and only if the metric coefficients of σ and σ' are proportional.

Definition 23.2.14 (Surface preserving map). A diffeomorphism $\Phi : \Sigma \to \Sigma'$ is sadi to be surface-preserving if it maps a segment of Σ to a segment of Σ' with the same surface.

Property 23.2.15. A diffeomorphism Φ is surface-preserving if and only if the metric coefficients of σ and σ' satisfy:

$$g'_{11}g'_{22} - (g'_{12})^2 = g_{11}g_{22} - g_{12}^2 \tag{23.18}$$

for all points (q^1, q^2).

Corollary 23.2.16. A map that is surface-preserving and conformal is also isometric.

23.2.4 Second fundamental form

Definition 23.2.17 (Second fundamental form). Let $\vec{\boldsymbol{\sigma}}(q^1, q^2)$ be a surface. The second fundamental form is a bilinear form $II_P(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) : T_P\Sigma \times T_P\Sigma \to \mathbb{R}$ defined as follows:

$$\boxed{II_P(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = L_{ij}(q^1, q^2)v^i w^j} \tag{23.19}$$

where $L_{ij} = \vec{\boldsymbol{N}} \cdot \vec{\boldsymbol{\sigma}}_{ij}$.

Definition 23.2.18 (Normal curvature). Let $\vec{\boldsymbol{c}}$ be a curve parametrized as

$$\vec{\boldsymbol{c}}(s) = \vec{\boldsymbol{\sigma}}\left(q^1(s), q^2(s)\right)$$

The normal curvature of $\vec{\boldsymbol{c}}(s)$ at a point $(q^1(s), q^2(s))$ is defined as:

$$\boxed{\frac{1}{\rho_n(s)} = \vec{\boldsymbol{c}}\,''(s) \cdot \vec{\boldsymbol{N}}(s)} \tag{23.20}$$

From the definition of the second fundamental form it follows that the normal curvature can be written as:

$$\frac{1}{\rho_n(s)} = II(\vec{\boldsymbol{t}}, \vec{\boldsymbol{t}}) = \frac{II\left(\dot{\vec{\boldsymbol{c}}}(t), \dot{\vec{\boldsymbol{c}}}(t)\right)}{I\left(\dot{\vec{\boldsymbol{c}}}(t), \dot{\vec{\boldsymbol{c}}}(t)\right)} \tag{23.21}$$

where the last equality holds for any given parameter t.

Theorem 23.2.19 (Meusnier's theorem). *Let $\vec{\boldsymbol{c}}, \vec{\boldsymbol{d}}$ be two curves on a surface $\vec{\boldsymbol{\sigma}}$. The curves have the same normal curvature in a point $(q^1(t_0), q^2(t_0))$ if $\vec{\boldsymbol{c}}(t_0) = \vec{\boldsymbol{d}}(t_0)$ and if $\dot{\vec{\boldsymbol{c}}}(t_0) \parallel \dot{\vec{\boldsymbol{d}}}(t_0)$. Furthermore, the osculating circles of all curves with the same normal curvature at a given point form a sphere.*

Property 23.2.20. The normal curvature of at a given point is equal to the curvature of the normal section, i.e. the intersection of the surface with a normal plane at the point.

Definition 23.2.21 (Geodesic curvature). Let $\vec{\boldsymbol{c}}$ be a curve parametrized as $\vec{\boldsymbol{c}}(s) = \vec{\boldsymbol{\sigma}}\left(q^1(s), q^2(s)\right)$. The geodesic curvature of $\vec{\boldsymbol{c}}(s)$ at a point $(q^1(s), q^2(s))$ is defined as:

$$\boxed{\frac{1}{\rho_g(s)} = \left(\vec{\boldsymbol{N}}(s)\ \vec{\boldsymbol{t}}(s)\ \vec{\boldsymbol{t}}\,'(s)\right)} \tag{23.22}$$

Formula 23.2.22. Let $\vec{\boldsymbol{c}}$ be a curve on a surface $\vec{\boldsymbol{\sigma}}$. From the definitions of the normal and geodesic curvature it follows that:

$$\boxed{\frac{1}{\rho^2} = \frac{1}{\rho_n^2} + \frac{1}{\rho_g^2}} \tag{23.23}$$

23.2.5 Curvature of a surface

Definition 23.2.23 (Weingarten map). Let P be a point of a surface Σ. The Weingarten map $L_P : T_P\Sigma \to T_P\Sigma$ is a linear map defined as:

$$L_P(\vec{\boldsymbol{\sigma}}_1) = -\vec{\boldsymbol{N}}_1 \qquad \text{and} \qquad L_P(\vec{\boldsymbol{\sigma}}_2) = -\vec{\boldsymbol{N}}_2 \tag{23.24}$$

Formula 23.2.24. Let $\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \in T_P\Sigma$. The following equalities hold:

$$L_P(\vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{w}} = L_P(\vec{\boldsymbol{w}}) \cdot \vec{\boldsymbol{v}} = II_P(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \tag{23.25}$$

Formula 23.2.25 (Matrix elements of L_P). Let (g^{ij}) be the inverse of the metric tensor. The matrix elements of L_P are defined as:

$$L^k_j = g^{ki} L_{ij}$$

Formula 23.2.26 (Weingarten formulas).

$$\vec{\boldsymbol{N}}_j = -L^k_j \vec{\boldsymbol{\sigma}}_k \tag{23.26}$$

Theorem 23.2.27. *For every point P on the surface Σ there exists a basis $\{\vec{\boldsymbol{h}}_1, \vec{\boldsymbol{h}}_2\} \subset T_P\Sigma$ of eigenvectors of L_P. Furthermore, the corresponding eigenfunctions are given by $II_P(\vec{\boldsymbol{h}}_i, \vec{\boldsymbol{h}}_i)$ and these eigenvalues are the extreme values of the normal curvature at the point P.*

Definition 23.2.28 (Principal curvatures). The eigenvalues of the Weingarten map are called the principal curvatures of the surface and they are denoted by $\frac{1}{R_1}$ and $\frac{1}{R_2}$. The tangent vectors corresponding to these curvatures are called the **principal directions**.

Remark. If the principal curvatures are not equal, the principal directions are orthogonal. If they are equal, the point P is said to be an **umbilical point** or **umbilic**.

Property 23.2.29. For the principal directions we have $L_P(\vec{\boldsymbol{h}}_1) \cdot \vec{\boldsymbol{h}}_2 = 0$[3]. If P is an umbilic then every tangent vector in P is a principal direction and the equality is satisfied for every two tangent vectors.

Definition 23.2.30 (Gaussian curvature). The Gaussian curvature K of a surface is defined as the determinant of the Weingarten map, i.e.:

$$K = \frac{1}{R_1 R_2} \tag{23.27}$$

Definition 23.2.31 (Mean curvature). The mean curvature H of a surface is defined as the trace of the Weingarten map, i.e.:

$$H = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{23.28}$$

[3]Tangent vectors that satisfy this equation are called **adjoint** tangent vectors.

Property 23.2.32. The principal curvatures are the solutions of the following equation:

$$x^2 - 2Hx + K = 0$$

This is the characteristic equation (16.68) of the Weingarten map.

Definition 23.2.33. Let P be a point on the surface Σ.

- P is said to be **elliptic** if $K > 0$ in P.
- P is said to be **hyperbolic** if $K < 0$ in P.
- P is said to be **parabolic** if $K = 0$ and $\frac{1}{R_1}$ or $\frac{1}{R_2} \neq 0$ in P.
- P is said to be **flat** if $\frac{1}{R_1} = \frac{1}{R_2} = 0$ in P.
- P is said to be **umbilical** if $\frac{1}{R_1} = \frac{1}{R_2}$ in P.

Remark. From previous definition it follows that a flat point is a special type of umbilic.

Theorem 23.2.34. *A surface Σ containing only umbilics is a part of a sphere or a part of a plane.*

Theorem 23.2.35. *In the neighbourhood of a point P of a surface with principal curvatures $1/R_1$ and $1/R_2$ is locally the same as the following quadric:*

$$x_3 = \frac{1}{2}\left(\frac{x_1^2}{R_1} + \frac{x_2^2}{R_2}\right) \tag{23.29}$$

if we ignore terms of order > 2.

Theorem 23.2.36 (Euler's formula). *The normal curvature of a couple $(P, \vec{\boldsymbol{e}})$ where is $\vec{\boldsymbol{e}} = \vec{\boldsymbol{h}}_1 \cos\theta + \vec{\boldsymbol{h}}_2 \sin\theta \in T_P\Sigma$ is given by:*

$$\frac{1}{\rho_n} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2} \tag{23.30}$$

Definition 23.2.37 (Asymptotic curve). An asymptotic curve is a curve which is in every point P tangent to a direction with zero normal curvature.

Formula 23.2.38 (Differential equation for asymptotic curves).

$$L_{11}\left(\dot{q}^1(t)\right)^2 + 2L_{12}\dot{q}^1(t)\dot{q}^2(t) + L_{22}\left(\dot{q}^2(t)\right)^2 = 0 \tag{23.31}$$

Property 23.2.39. A curve on a surface is an asymptotic curve if and only if the tangent plane and the osculation plane coincide in every point P of the surface.

Definition 23.2.40 (Line of curvature). A curve is a line of curvature if the tangent vector in every point P is a principal direction of the surface in P.

Formula 23.2.41 (Rodrigues' formula). A curve is a line of curvature if and only if it satisfies the following formula:

$$\frac{d\vec{\boldsymbol{N}}}{dt}(t) = -\frac{1}{R(t)}\frac{d\vec{\boldsymbol{c}}}{dt}(t) \tag{23.32}$$

If the curve satisfies this formula, then the scalar function $1/R(t)$ coincides with the principal curvature along the curve.

Formula 23.2.42 (Differential equation for curvature lines).

$$\begin{vmatrix} (\dot{q}^2)^2 & -\dot{q}^1\dot{q}^2 & (\dot{q}^1)^2 \\ g_{11} & g_{12} & g_{22} \\ L_{11} & L_{12} & L_{22} \end{vmatrix} = 0 \tag{23.33}$$

Property 23.2.43. From theorem 23.2.27 we know that the principal directions are orthogonal vectors. It follows that on a surface containing no umbilics the curvature lines form an orthogonal web and in every point P we find 2 orthogonal curvature lines.

23.2.6 Christoffel symbols and geodesics

Formula 23.2.44 (Gauss' formulas).

$$\vec{\boldsymbol{\sigma}}_{ij} = L_{ij}\vec{\boldsymbol{N}} + \Gamma^k{}_{ij}\vec{\boldsymbol{\sigma}}_k \tag{23.34}$$

where the **Christoffel symbols** $\Gamma^k{}_{ij}$ are defines as:

$$\boxed{\Gamma^k{}_{ij} = g^{kl}\vec{\boldsymbol{\sigma}}_l \cdot \vec{\boldsymbol{\sigma}}_{ij}} \tag{23.35}$$

Corollary 23.2.45. From the expression of the Christoffel symbols we can derive an alternative expression using only the metric tensor g_{ij}:

$$\Gamma^k{}_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^l} + \frac{\partial g_{jl}}{\partial q^i}\right) \tag{23.36}$$

Definition 23.2.46 (Geodesic). A geodesic is a curve with zero geodesic curvature.

Theorem 23.2.47. *A curve on a surface is an geodesic if and only if the tangent plane and the osculation plane are orthogonal in every point P of the surface.*

Formula 23.2.48 (Differential equation for geodesic). If the curve is parametrized by arc length, then it is a geodesic if the functions $q^1(s)$ and $q^2(s)$ satisfy the following differential equation:

$$\boxed{q''^k + \Gamma^k{}_{ij}q'^i q'^j = 0} \tag{23.37}$$

23.2.7 Theorema Egregium

Formula 23.2.49 (Codazzi-Mainardi equations).

$$\frac{\partial L_{ij}}{\partial q^k} - \frac{\partial L_{ik}}{\partial q^j} = \Gamma^l{}_{ik} L_{lj} - \Gamma^l{}_{ij} L_{lk} \tag{23.38}$$

Definition 23.2.50 (Riemann curvature tensor).

$$\boxed{R^l{}_{ijk} = \frac{\partial \Gamma^l{}_{ik}}{\partial q^j} - \frac{\partial \Gamma^l{}_{ij}}{\partial q^k} + \Gamma^s{}_{ik}\Gamma^l{}_{sj} - \Gamma^s{}_{ij}\Gamma^l{}_{ks}} \tag{23.39}$$

Formula 23.2.51 (Gauss' equations).

$$R^l{}_{ijk} = L_{ik}L^l_j - L_{ij}L^l_k \tag{23.40}$$

Theorem 23.2.52 (Theorema Egregium). *The Gaussian curvature K (formula 23.27) is completely determined by the metric tensor g_{ij} and its derivatives:*

$$\boxed{K = \frac{R^l{}_{121} g_{l2}}{g_{11}g_{22} - g_{12}^2}} \tag{23.41}$$

Remark. This theorem is remarkable due to the fact that the coefficients L_{ij}, which appear in the general formula of the Gaussian curvature, cannot be expressed in terms of the metric tensor.

Property 23.2.53. From the condition of isometries 23.2.11 and the previous theorem it follows that if two surfaces are connected by an isometric map, the corresponding points in Σ and Σ' have the same Gaussian curvature.

Corollary 23.2.54. There exists no isometric projection from the sphere to the plane. This also implies that a perfect (read: isometric) map of the Earth can not be created.

Chapter 24

Manifolds

24.1 Charts

Definition 24.1.1 (Chart). Let M be a set. Let U be an open subset of M and let O be an open subset of $\mathbb{R}^n$. Let $\varphi : U \to O$ be a homeomorphism. The pair (U, φ) is called a chart on M.

Definition 24.1.2 (Transition map). Let (U_1, φ_1) and (U_2, φ_2) be two charts in $\mathcal{A}$. The mapping $\varphi_1^{-1} \circ \varphi_2$ is called a transition map.

If $\varphi_1^{-1} \circ \varphi_2$ is continuous then the charts are said to be C^0-compatible. However the composition of any two continuous functions is also continuous so it follows that every two charts on a topological manifold are C^0-compatible.

Definition 24.1.3 (Atlas). Let M be a set. Let $\{(U_i, \varphi_i)\}_i$ be a set of (pairwise) $\diamond$-compatible charts (where $\diamond$ denotes any compatibility relation) such that $\bigcup_i U_i = M$. This set of charts is called a $\diamond$-atlas on M. From the remark on C^0-compatibility of charts in previous definition it is then obvious that every atlas is a C^0-atlas.

Definition 24.1.4 (Maximal Atlas). Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two atlasses covering the same set M. If $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$ is again an atlas then the atlasses are said to be equivalent or compatible. The largest such union is called a maximal atlas.

Definition 24.1.5 (Manifold). A set M equipped with a maximal C^0-atlas $\mathcal{A}$ is called a topological manifold. An alternative definition (often used in topology) is that of a locally Euclidean Hausdorff space. The topology on M is given by the collection of open sets contained in the charts.

Remark. In the literature second-countability is often added to the definition of a topological manifold. This ensures that the space has (among others) the property of paracompactness.

Definition 24.1.6 (C^k-manifold). If all transition maps are C^k-diffeomorphisms than the manifold is called a C^k-manifold. A C^∞-manifold is also called a smooth manifold.

Theorem 24.1.7 (Whitney). *Every C^k-atlas contains a C^∞-atlas. Furthermore, if two C^k-atlasses contain the same C^∞-atlas then they are identical. It follows that every differentiable manifold is automatically smooth.*

Theorem 24.1.8 (Radó-Moise). *In the dimensions $1, 2$ and 3 there exists for every topological manifold a unique smooth structure.*

Theorem 24.1.9. *For dimensions higher than 4, there exist only finitely many distinct smooth structures.*

Remark. In $\dim M = 4$ there are only partial results. For non-compact manifolds there exist uncountably many distinct smooth structures. For compact manifolds there exists no complete characterization.

Formula 24.1.10 (Smooth[1] function). Let $f : M \to N$ be a function between two smooth manifolds. f is said to be smooth if there exist charts (U, φ) and (V, ψ) for M and N with $f(U) \subseteq V$ such that the function

$$f_{\varphi\psi} = \psi \circ f \circ \varphi^{-1} \tag{24.1}$$

is smooth on $\mathbb{R}^n$.

Remark. The function $f_{\varphi\psi}$ in equation 24.1 is called the **local representation** of f.

Notation 24.1.11. The set of all C^∞ functions on a manifold M defined on a neighbourhood of $m \in M$ is denoted by $C^\infty_m(M)$. This set forms a commutative unital ring when equipped with the usual sum and product (composition) of functions.

24.2 Tangent vectors

Definition 24.2.1 (Tangent vector). Let M be a smooth manifold and $p \in M$. Let $f, g : M \to \mathbb{R} \in C^\infty_p(M)$. A tangent vector on M is a differential operator v_p satisfying the following properties:

1. Linearity: $v_p(af + g) = av_p(f) + v_p(g)$
2. Leibniz property: $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$

Maps with these properties are also called **derivations**[2].

Property 24.2.2. For every constant function $c : p \mapsto c$ we have:

$$v_p(c) = 0 \tag{24.2}$$

[1] In this definition one can replace 'smooth' by 'C^k-differentiable'.

[2] Generally, every operation that satisfies the Leibniz property is called a derivation.

Definition 24.2.3 (Tangent space). Following from the previous definition, we can construct a tangent (vector) space T_pM in each point $p \in M$. The basis vectors are given by:

$$\boxed{\left.\frac{\partial}{\partial q^i}\right|_p : C_p^\infty(M,\mathbb{R}) \to \mathbb{R} : f \mapsto \frac{\partial}{\partial q^i}\left(f \circ \varphi^{-1}\right)(\varphi(p))} \tag{24.3}$$

where (U, φ) is a coordinate chart such that $p \in U$ and $(q^1, ..., q^n)$ are local coordinates.

Remark 24.2.4. Due to the explicit dependence of the tangent vectors on the point $p \in M$, it is clear that for curved manifolds the tangent spaces belonging to different points will not be the same.

Property 24.2.5. From the above tangent space construction it follows that:

$$\boxed{\dim(T_pM) = \dim(M)} \tag{24.4}$$

This also implies that the tangent spaces over two distinct points $p, q \in M$ are isomorphic.

Definition 24.2.6 (Curve). A smooth function $\gamma : \mathbb{R} \to M$ with $\gamma(0) = m$ is called a smooth curve through $m \in M$.

Alternative Definition 24.2.7 (Tangent space). The alternative construction goes as follows. Let (U, φ) be a chart for the point $p \in M$. Two smooth curves γ_1, γ_2 through $p \in M$ are said to be tangent at p if:

$$\frac{d(\varphi \circ \gamma_1)}{dt}(0) = \frac{d(\varphi \circ \gamma_2)}{dt}(0) \tag{24.5}$$

or equivalently, if their local representatives are tangent in 0. This relation imposes an
def equivalence relation[3] on the set of smooth curves through p. One then defines the tangent space at p as the set of equivalence classes of tangent curves through p. Explicitly these equivalence classes are constructed as follows:

We can define the following tangent vector to the curve $c(t)$ through p as:

$$\boxed{v_p(f) = \left.\frac{d(f \circ c)}{dt}\right|_{t=0}} \tag{24.6}$$

Applying the chain rule gives us

$$v_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial q^i}(\varphi(p))\frac{dq^i}{dt}(0) \tag{24.7}$$

where $q^i = (\varphi \circ c)^i$. The first factor depends only on the point p and the second factor is equal for all tangent curves through p. We thus see that tangent curves define the same tangent vector.

The proof that both definitions of the tangent space are in fact equivalent is given in the appendices.

[3]The relation is well-defined (under a change of chart) because the transition maps (and their Jacobian matrices) are invertible and thus non-singular.

24.3 Curvature

Formula 24.3.1 (Riemann Curvature Tensor). Let $V \in TM$. Let D_μ be the covariant derivative.

$$\boxed{[D_\mu, D_\nu]V^\rho = R^\rho{}_{\kappa\mu\nu}V^\kappa} \tag{24.8}$$

Formula 24.3.2 (Ricci tensor).

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \tag{24.9}$$

Formula 24.3.3 (Ricci scalar).

$$R = R^\mu{}_\mu \tag{24.10}$$

This scalar quantity is also called the **scalar curvature**.

Formula 24.3.4 (Einstein tensor).

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R} \tag{24.11}$$

Theorem 24.3.5. *For 4-dimensional manifolds the Einstein tensor $G_{\mu\nu}$ is the only tensor containing at most second derivatives of the metric $g_{\mu\nu}$ and satisfying:*

$$\nabla_\mu G^{\mu\nu} = 0 \tag{24.12}$$

24.4 Submanifolds

Definition 24.4.1 (Submanifold). Let M be a manifold. A subset $N \subset M$ is called a submanifold of M if N, equipped with the subspace topology, is a topological manifold on its own.

Definition 24.4.2 (Immersion). Let $f : M \to N$ be a differentiable function between smooth manifolds. f is called an immersion if its derivative[4] is everywhere injective, or equivalently if its derivative has maximal rank[5] everywhere:

$$\mathrm{rk}_p(f) = \dim(M), \forall p \in M \tag{24.13}$$

Definition 24.4.3 (Submersion). Let $f : M \to N$ be a differentiable map between smooth manifolds. A **regular point**[6] of f is a point $x \in M$ such that $T_x f$ is surjective. f is called a submersion if its derivative is everywhere surjective, or equivalently if

$$\mathrm{rk}_p(f) = \dim(N), \forall p \in M \tag{24.14}$$

[4] This is formally defined in 26.17. For now it is the map represented by the Jacobian matrix.
[5] See definition 26.2.9.
[6] $f(x)$ is then called a **regular value**.

Definition 24.4.4 (Embedding). A differentiable function between smooth manifolds is called a smooth embedding if its both an injective immersion and an embedding in the topological sense 5.3.11. This implies that the submanifold topology coincides with the subspace topology 5.1.

Definition 24.4.5 (Embedded submanifold). Let M be a manifold. A subset N is an embedded[7] submanifold if the inclusion map $f : M \hookrightarrow N$ is a smooth embedding.

Definition 24.4.6 (Slice). Let $m < n$ be two positive integers. The space $\mathbb{R}^m$ can be viewed as a subspace of $\mathbb{R}^n$ by identifying them in the following way:

$$\mathbb{R}^m \cong \mathbb{R}^m \times \{\underbrace{0, ..., 0}_{n-m}\} \overset{\iota}{\hookrightarrow} \mathbb{R}^m \times \mathbb{R}^{n-m} \cong \mathbb{R}^n \tag{24.15}$$

where $\iota : (x_1, ..., x_m) \mapsto (x_1, ..., x_m, \underbrace{0, ..., 0}_{n-m})$ is the canonical inclusion map.

Alternative Definition 24.4.7. A k-dimensional embedded manifold N of M can now be defined equivalently as a subset of M such that there exists a positive integer k and such that for every point $p \in N$ there exists a chart (U, φ) with

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^k \times \{\underbrace{0, ..., 0}_{n-k}\}) \tag{24.16}$$

where $n = \dim(M)$. The set $U \cap N$ is called a slice of (U, φ) in analogy with the previous definition of a (standard) slice.

Theorem 24.4.8 (Submersion theorem[8]). *Consider a smooth map $f : M_1 \to M_2$ between smooth manifolds. Let $y \in M_2$ be a regular value. Then $N = f^{-1}(y)$ is a submanifold of M_1 with codimension* $\dim(M_2)$.

24.5 Manifolds with boundary

Definition 24.5.1 (Manifold with boundary). Let $\mathbb{H}^n$ denote the upper half space, i.e.:

$$\mathbb{H}^n = \{(x_1, ..., x_n) | x_n \geq 0\} \subset \mathbb{R}^n \tag{24.17}$$

An n-dimensional manifold with boundary is then given by a set M together with a maximal atlas consisting of (regular) charts (U, φ) such that U is diffeomorphic to $\mathbb{R}^n$, these points are called **interior points**, and (boundary) charts (V, ϕ) such that V is diffeomorphic to $\mathbb{H}^n$, these points are called **boundary points**.

[7] An immersed submanifold is defined analogously. The requirement of the inclusion map being a smooth embedding is relaxed to it being an (injective) immersion. However the submanifold topology will no longer coincide with the subspace topology.

[8] Also called the **regular value theorem**.

Remark 24.5.2 (Manifold boundary). The boundary ∂M, consisting of all boundary points of M as defined in the above definition, should not be confused with the topological boundary of M. In general these are different sets. Similarly, the interior $\text{Int}(M) = M \setminus \partial M$, in the sense of manifolds, should not be confused with the topological interior.

Property 24.5.3. Let M be an n-dimensional manifold with boundary. Let (U, φ) be a chart for $p \in \partial M$. Then

$$\varphi(p) \in \partial \mathbb{H}^n = \{(x_1, ..., x_n) | x_n = 0\} \tag{24.18}$$

Chapter 25

Lie groups and Lie algebras

25.1 Lie groups

Definition 25.1.1 (Lie group). A Lie group is a group that is also a differentiable manifold such that both the multiplication and inversion are smooth functions.

Definition 25.1.2 (Lie subgroup). A subset of a Lie group is a Lie subgroup if it is both a subgroup and a closed submanifold.

Theorem 25.1.3 (Closed subgroup theorem[1]). *If H is a closed[2] subgroup of a Lie group G then H is a Lie subgroup of G.*

Property 25.1.4. Let G be a connected Lie group. Every neighbourhood U_e of the identity e generates G, i.e. every element $g \in G$ can be written as a word in U_e.

Definition 25.1.5 (Isogeny). Let G, H be two Lie groups. G and H are said to be isogenous if one is a covering space[3] of the other. The covering map is then called an isogeny between G and H.

25.1.1 Left invariant vector fields

Definition 25.1.6 (Left Invariant Vector Field (LIVF)). Let G be a Lie group. Let X be a vector field on G. X is left invariant if the following equivariance relation holds for all $g \in G$:

$$L_{g,*}X(h) = X(g \cdot h) \tag{25.1}$$

where L_g denotes the left action map associated with g.

Property 25.1.7. The set $\mathcal{L}(G)$ of LIVF's on a Lie group G is a vector space over $\mathbb{R}$.

[1] Sometimes called **Cartan's theorem**.
[2] With respect to the group topology on G.
[3] See definition 5.3.13.

Property 25.1.8. The map $L_{g,*}$ is an isomorphism for every $g \in G$. It follows that a LIVF is uniquely determined by its value at the identity of G. Furthermore, for every $v \in T_e(G)$, there exists a LIVF $X \in \mathcal{L}(G)$ such that $X(e) = v$ and this mapping is an isomorphism from $T_e(G)$ to $\mathcal{L}(G)$.

25.1.2 One-parameter subgroups

Definition 25.1.9 (One-parameter subgroup). A one-parameter (sub)group is a Lie group homomorphism $\Phi : \mathbb{R} \to G$ from the additive group of real numbers to a Lie group G.

Property 25.1.10. Let $\Phi : \mathbb{R} \to G$ be a one-parameter subgroup of G. Let $\Psi : G \to H$ be a continuous group homomorphism. Then $\Psi \circ \Phi : \mathbb{R} \to H$ is a one-parameter subgroup of H.

Property 25.1.11. All LIVF's X are complete[4]. Hence for every LIVF X we can find an integral curve γ^X with initial condition $\gamma^X(0) = e$ for which the maximal flow domain[5] $D(X)$ is $]-\infty, +\infty[$. This implies that the associated flow σ_t determines a one-parameter subgroup of G. Conversely, for every one-parameter subgroup $\phi(t)$ we can construct a LIVF $X = \phi'(0)$. This correspondence is a bijection.

25.1.3 Cocycles

Definition 25.1.12 (Cocycle). Let M be a smooth manifold and G a Lie group. A cocycle on M with values in G is a family of smooth functions $g_{ij} : U_i \cap U_j \to G$ that satisfy the following condition:

$$g_{ij} = g_{ik} \circ g_{kj} \tag{25.2}$$

Property 25.1.13. Let $\{g_{ij}\}_{i,j}$ be a cocycle on M. We have the following properties:

- $g_{ii}(x) = \mathbb{1}_M$
- $g_{ij}(x) = (g_{ji}(x))^{-1}$

for all $x \in M$.

25.2 Lie algebras

There are two ways to define a Lie algebra. The first one is a stand-alone definition using a vector space equipped with a multiplication operation. The second one establishes a direct relation between Lie groups (see 25.1.1) and real Lie algebras.

[4]See definition 26.3.9.
[5]See definition 26.3.7.

25.2.1 Definitions

Definition 25.2.1 (Lie algebra). Let V be a vector space equipped with a binary operation $[\cdot,\cdot] : V \times V \to V$ is a Lie algebra if the Lie bracket $[\cdot,\cdot]$ satisfies the following conditions:

1. Bilinearity: $[ax + y, z] = a[x, z] + [y, z]$
2. Alternativity: $[v, v] = 0$
3. Jacobi identity: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

Definition 25.2.2 (Lie algebra of LIVF's). Consider the vector space $\mathcal{L}(G)$ of LIVF's on a Lie group G. Using property 26.35 we can show that the commutator (Lie bracket) also defines a LIVF on G. It follows that $\mathcal{L}(G)$ is closed under Lie brackets and hence is also a Lie algebra.

Alternative Definition 25.2.3 (Lie algebra of Lie group). Let G be a Lie group. The tangent space $\mathfrak{g} := T_eG$ has the structure of a Lie algebra where the Lie bracket is induced by the commutator of vector fields 26.35 in the following way:

$$[\![A, B]\!] := [l_{g,*}A, l_{g,*}B]|_{g=e} \tag{25.3}$$

where $A, B \in T_eG$ and where $[\cdot,\cdot]$ is the Lie bracket on $\mathcal{L}(G)$. This induces an isomorphism of Lie algebras: $\mathfrak{g} \cong_{\text{Lie}} \mathcal{L}(G)$.

Notation 25.2.4. Lie algebras are generally denoted by fraktur symbols. For example, the Lie algebra associated with the Lie group G is often denoted by $\mathfrak{g}$.

Theorem 25.2.5 (Ado). *Every finite-dimensional Lie algebra can be embedded as a subalgebra of* $\mathfrak{gl}_n$.

Theorem 25.2.6 (Lie's third theorem). *Every finite-dimensional Lie algebra* $\mathfrak{g}$ *is the Lie algebra of a unique simply-connected Lie group* G.

Definition 25.2.7 (Lie algebra homomorphism). A map $\Phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if it satisfies following condition

$$\Phi([X, Y]) = [\Phi(X), \Phi(Y)] \tag{25.4}$$

for all $X, Y \in \mathfrak{g}$.

Property 25.2.8 (Homomorphisms theorem[6]). Let G, H be Lie groups with G simply-connected. If a linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism then there exists a unique Lie group homomorphism $\phi : G \to H$ such that $\Phi = \phi_*$.[7]

[6] See also formula 25.3.7.

[7] The converse is trivial: every Lie group homomorphism induces a Lie algebra homomorphism through its differential.

25.2.2 Exponential map

Formula 25.2.9 (Exponential map). Let $X \in \mathfrak{g}$ be a LIVF on G. We define the exponential map $\exp : \mathfrak{g} \to G$ as:

$$\boxed{\exp(X) := \gamma_X(1)} \tag{25.5}$$

where γ_X is the associated one-parameter subgroup defined in property 25.1.11.

Property 25.2.10. The exponential map is the unique map $\mathfrak{g} \to G$ such that $\exp(0) = e$ and for which the restrictions to the lines through the origin in $\mathfrak{g}$ are one-parameter subgroups of G.

Corollary 25.2.11. Because the identity element $\mathbb{1}_{\mathfrak{g}} = (\exp_*)_e$ is an isomorphism, the inverse function theorem 26.2.10 implies that the image of exp will contain a neighbourhood of the identity $e \in G$. If G is connected then property 25.1.4 implies that exp generates all of G.

Together with the property that $\psi \circ \exp = \exp \circ \psi_*$ for every Lie group homomorphism $\psi : G \to H$ it follows that *if G is connected, a Lie group homomorphism $\psi : G \to H$ is completely determined by its differential ψ_* at the identity $e \in G$.*

Example 25.2.12 (Matrix Lie groups). For matrix Lie groups we define the classic matrix exponential:

$$e^{tX} = \sum_{k=0}^{+\infty} \frac{(tX)^k}{k!} \tag{25.6}$$

This operation defines a curve $\gamma(t)$ which can be used as a one-parameter subgroup on G. It should be noted that this formula converges for every $X \in M_{m,n}$ and is invertible with the inverse given by $\exp(-X)$. Using Ado's theorem 25.2.5 one can then use this matrix exponential to represent the exponential map for any (finite-dimensional) Lie algebra.

Property 25.2.13. Let G be a compact Lie group. The exponential map is surjective. However, because the asocciated Lie algebra $\mathfrak{g}$ is non-compact, the exponential map cannot be homeomorphic and hence cannot be injective.

25.2.3 Structure

Definition 25.2.14 (Structure constants). As Lie algebras are closed under Lie brackets, every Lie bracket can be expanded in term of a basis $\{X_k\}_{k \in I}$ as follows:

$$[X_i, X_j] = \sum_{k \in I} {c_{ij}}^k X_k \tag{25.7}$$

where the factors ${c_{ij}}^k$ are called the structure constants[8] of the Lie algebra.

[8] Note that these constants are basis-dependent.

Property 25.2.15. Two Lie algebras $\mathfrak{g}, \mathfrak{h}$ are isomorphic if one can find bases $\mathcal{B}$ for $\mathfrak{g}$ and $\mathcal{C}$ for $\mathfrak{h}$ such that the associated structure constants are equal for all indices i, j and k.

Formula 25.2.16 (Baker-Campbell-Hausdorff formula). This formula is the solution of the equation

$$Z = \log(\exp(X)\exp(X)) \tag{25.8}$$

for $X, Y \in \mathfrak{g}$. The solution is given by following formula

$$e^X e^Y = \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdot\right) \tag{25.9}$$

One should note that this formula will only converge if X, Y are sufficiently small (for matrix Lie algebras this means that $||X|| + ||Y|| < \frac{\ln(2)}{2}$ under the Hilbert-Schmidt norm 16.46). Due to the closure under commutators (see Lie algebra definition) the exponent in the BCH formula is also an element of the Lie algebra. So the formula gives an expression for Lie group multiplication in terms of Lie algebra elements (whenever the formula converges).

Corollary 25.2.17 (Lie product formula[9]). Let $\mathfrak{g}$ be a Lie algebra. The following formula applies to any $X, Y \in \mathfrak{g}$:

$$e^{X+Y} = \lim_{n \to +\infty} \left(e^{\frac{X}{n}} e^{\frac{Y}{n}}\right)^n \tag{25.10}$$

25.2.4 Examples

Example 25.2.18. The cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ turns $\mathbb{R}^3$ into a Lie algebra.

Example 25.2.19. An interesting example is the Lie algebra associated to the Lie group of invertible complex[10] matrices $GL(n, \mathbb{C})$. This Lie group is a subset of its own Lie algebra $\mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$. It follows that for every $A \in GL(n, \mathbb{C})$ and every $B \in \mathfrak{gl}(n, \mathbb{C})$ the following equality holds:

$$L_{A,*}(B) = L_A(B) \tag{25.11}$$

Corollary 25.2.20. By noting that the endomorphism ring $\mathrm{End}(V)$ of an n-dimensional vector space V is given by the matrix ring $M_n(K)$, we see that $\mathrm{End}(V)$ also forms a Lie algebra when equipped with the commutator of linear maps.

Following two examples of Lie algebras can be checked using condition 16.35:

Example 25.2.21 (Lie algebra of $O(3)$). The set of 3×3 anti-symmetric matrices. It is also important to note that $\mathfrak{o}(3) = \mathfrak{so}(3)$. The structure constants of this Lie algebra are given by 20.12, i.e. $C_{ijk} = \varepsilon_{ijk}$.

[9] Also called the Lie-Trotter formula.

[10] As usual, this result is also valid for real matrices.

Example 25.2.22 (Lie algebra of $SU(2)$). The set of 2×2 traceless anti-Hermitian matrices. This result can be generalized to arbitrary $n \in \mathbb{N}$.

Example 25.2.23 ($SL(2,\mathbb{C})$). To compute the Lie bracket in the Lie algebra $\mathfrak{sl}(2,\mathbb{C}) = T_e(SL(2,\mathbb{C}))$ we need to find the action of $l_{g,*}$ on any vector $Y \in \mathfrak{sl}(2,\mathbb{C})$. This is given by:

$$l_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),*} \left(\frac{\partial}{\partial x^i}\bigg|_e \right) = \begin{pmatrix} a & 0 & b \\ -b & a & 0 \\ c & 0 & \frac{1+bc}{a} \end{pmatrix}_i^m \frac{\partial}{\partial x^m}\bigg|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \tag{25.12}$$

where we used the coordinate chart (U, ϕ) defined by:

$$U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) : a \neq 0 \right\}$$

and

$$\phi : U \to \mathbb{C}^3 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c)$$

One can then use this formula to work out the Lie bracket of the basis vectors $X_i = \frac{\partial}{\partial x^i}\big|_e$ to obtain the structure constants:

$$\begin{cases} [\![X_1, X_2]\!] = 2X_2 \\ [\![X_1, X_3]\!] = -2X_3 \\ [\![X_2, X_3]\!] = X_1 \end{cases} \tag{25.13}$$

25.2.5 Solvable Lie algebras

Definition 25.2.24 (Derived algebra). Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot,\cdot]$. The derived Lie algebra is defined as follows:

$$[\mathfrak{g}, \mathfrak{g}] = \{[x, y] : x, y \in \mathfrak{g}\} \tag{25.14}$$

Definition 25.2.25 (Solvable Lie algebra). Consider the sequence of derived Lie algebras

$$g \geq [\mathfrak{g}, \mathfrak{g}] \geq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \geq \cdots \tag{25.15}$$

If this sequence ends in the zero-space then the Lie algebra $\mathfrak{g}$ is said to be solvable.

Definition 25.2.26 (Radical). Let $\mathfrak{g}$ be a Lie algebra. The radical of $\mathfrak{g}$ is the largest solvable ideal in $\mathfrak{g}$.

25.2.6 Simple Lie algebras

Definition 25.2.27 (Direct sum). The direct sum of two Lie algebras $\mathfrak{g}, \mathfrak{h}$ is defined as the direct sum in the sense of vector spaces (see 16.2.17) together with the condition

$$[x, y] = 0 \tag{25.16}$$

for all $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$.

Definition 25.2.28 (Semidirect product). The semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$ of two Lie algebras $\mathfrak{g}, \mathfrak{h}$ is defined as the direct sum in the sense of vector spaces (see 16.2.17) together with the condition that $\mathfrak{g}$ is an ideal of $\mathfrak{h}$ under the Lie bracket.

Definition 25.2.29 (Simple Lie algebra). A Lie algebra is said to be simple if it is non-Abelian and if it has no non-trivial ideals.

Definition 25.2.30 (Semisimple Lie algebra). A Lie algebra is said to be semisimple if it is the direct sum of simple algebras.

Theorem 25.2.31 (Levi decomposition). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. This algebra can be decomposed as follows:*

$$\mathfrak{g} = \mathfrak{R} \ltimes (\mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n) \tag{25.17}$$

where $\mathfrak{R}$ is the radical of $\mathfrak{g}$ and the algebras $\mathfrak{L}_i$ are simple subalgebras.

Definition 25.2.32. The semisimple subalgebra $\mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n$ in the Levi decomposition of $\mathfrak{g}$ is called the **Levi subalgebra** or **Levi factor** of $\mathfrak{g}$.

25.3 Representation theory

25.3.1 Lie groups

Definition 25.3.1 (Representation of Lie groups). Let G be a Lie group and let V be a vector space. A representation of G on V is a Lie group homomorphism $\rho : G \to GL(V)$.

Definition 25.3.2 (Adjoint representation of Lie groups). Let G be a Lie group. Consider the conjugation map $\Psi_g : h \mapsto ghg^{-1}$. The adjoint representation of G is defined by the differential of the conjugation $T_e\Psi_g$:

$$\mathrm{Ad}_g : T_eG \to T_eG : X \mapsto gXg^{-1} \tag{25.18}$$

It is a representation of G on its own tangent space $T_eG \equiv \mathfrak{g}$.

25.3.2 Lie algebras

Definition 25.3.3 (Representation of Lie algebras). Let $\mathfrak{g}$ be a Lie algebra and let V be a vector space. A representation of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$.

Formula 25.3.4 (Adjoint representation of Lie algebras). Using the fact that the adjoint representation of Lie groups is smooth we can define the adjoint representation of Lie algebras as:

$$\mathrm{ad}_X := T_e(\mathrm{Ad}_g) \tag{25.19}$$

where $g = e^{tX}$. Explicitly, let $\mathfrak{g}$ be a Lie algebra. For every element $X \in \mathfrak{g}$ the adjoint map is given by:

$$\mathrm{ad}_X(Y) = [X, Y] \tag{25.20}$$

This representation is faithful.

Property 25.3.5. Given the antisymmetry of the Lie bracket the Jacobi identity is equivalent to ad: $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ being a Lie algebra homomorphism, i.e. $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$.

Formula 25.3.6. Let $\{e_i\}_{i \leq n}$ be a basis of a Lie algebra $\mathfrak{g}$. The structure coefficients can be calculated using the adjoint map as follows:

$$(\mathrm{ad}_{e_i})^j_k = C_{ik}{}^j \tag{25.21}$$

Formula 25.3.7 (Induced homomorphism). Let $\phi : G \to H$ be a Lie group homomorphism[11] with G connected and simply-connected. This homomorphism induces a a Lie algebra homomorphism[12] $\Phi : \mathfrak{g} \to \mathfrak{h}$ given by:

$$\Phi(X) = \frac{d}{dt}\phi\left(e^{tX}\right)\Big|_{t=0} \tag{25.22}$$

or equivalently:

$$\phi\left(e^{tX}\right) = e^{t\Phi(X)} \tag{25.23}$$

Remark 25.3.8. The homomorphism induced by $\mathrm{Ad} : G \to H$ is precisely $\mathrm{ad} : \mathfrak{g} \to \mathfrak{h}$. Informally we can thus say that the infinitesimal version of the similarity transformation is given by the commutator (in case of $G = \mathrm{GL}_n$).

Corollary 25.3.9 (Commutator). For the general linear group GL_n the Lie bracket is given by the commutator:

$$\boxed{[X, Y] = XY - YX} \tag{25.24}$$

This follows from definition 25.20: $[X, Y] = \frac{d}{dt}\mathrm{Ad}_{\gamma(t)}(Y)\big|_{t=0}$ with $\gamma(0) = e$ and $\gamma'(0) = X$.

25.3.3 Killing form

Definition 25.3.10 (Killing form[13]). Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The Killing form on $\mathfrak{g}$ is defined as the following symmetric bilinear form[14]:

$$\boxed{K(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)} \tag{25.25}$$

The trace can be found by representing the Lie algebra elements as matrices using Ado's theorem 25.2.5. From equation 25.21 we can work out the action of the Killing form on the basis $\{e_i\}_{i \leq n}$:

$$K_{ij} = C_{ik}{}^l C_{jl}{}^k \tag{25.26}$$

where C^k_{ij} are the structure constants of the Lie algebra.

[11] Continuity (inherent to the definition of a Lie group homomorphism) is needed to ensure that $\phi(e^{tX})$ is also a one-parameter subgroup (see 25.1.10).

[12] See also property 25.2.8.

[13] Also called the **Cartan-Killing form**.

Theorem 25.3.11 (Cartan's criterion). *A Lie algebra is semisimple if and only if its Killing form is non-degenerate.*

Theorem 25.3.12. *If a Lie group G is compact then the Killing form of its associated Lie algebra $\mathfrak{g}$ is negative-definite.*

Corollary 25.3.13. Let G be a compact Lie group. If its Lie algebra is semisimple the Killing form K induces a metric

$$g : (X, Y) \mapsto -\operatorname{tr}(\operatorname{ad}_X, \operatorname{ad}_Y) \tag{25.27}$$

which turns the corresponding Lie group G into a Riemannian manifold.

Property 25.3.14. The adjoint map ad_Z is antisymmetric with respect to the Killing form:

$$K(\operatorname{ad}_Z X, Y) = -K(X, \operatorname{ad}_Z Y) \tag{25.28}$$

Property 25.3.15. The Killing-form is Ad-invariant, i.e.

$$K(\operatorname{Ad}_g(X), \operatorname{Ad}_g(Y)) = K(X, Y) \tag{25.29}$$

for all $g \in G$. From this it follows that Ad is a map from G to the isometry group $\operatorname{Isom}(\mathfrak{g})$.

Definition 25.3.16. Let $\mathfrak{g}$ be a Lie algebra and let V be a vector space equipped with a Lie algebra representation $\rho : \mathfrak{g} \to \operatorname{End}(V)$. One can then define a Killing form associated with ρ in the following way:

$$K_\rho(X, Y) = \operatorname{tr}\Big(\rho(X) \circ \rho(Y)\Big) \tag{25.30}$$

Remark 25.3.17. This definition is clearly more general than 25.25 as K is simply given by K_{ad}.

25.3.4 Roots and Dynkin diagrams

Definition 25.3.18 (Cartan subalgebra). Let $\mathfrak{g}$ be a Lie algebra. A subalgebra $\mathfrak{h}$ is called a Cartan subalgebra if there exists a basis $\{h_i\}_{i\in I}$ of $\mathfrak{h}$ that can be extended to a basis $\{h_i\}_{i\in I} \cup \{g_j\}_{j\in J}$ of $\mathfrak{g}$ such that every g_j is an eigenvector of the adjoint map ad_h for all $h \in \mathfrak{h}$.

Property 25.3.19. Every finite-dimensional Lie algebra contains a Cartan subalgebra.

Property 25.3.20. If $\mathfrak{g}$ is semisimple then its Cartan subalgebra is Abelian.

Definition 25.3.21 (Root). Let $\mathfrak{g}$ be a Lie algebra with Cartan subalgebra $\mathfrak{h}$. From the definition of a Cartan subalgebra it follows that for all $h \in \mathfrak{h}$:

$$[h, g_j] = \alpha_j(h) g_j \tag{25.31}$$

where $\{g_j\}_{j\in J}$ is the basis extension of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Because $\alpha_j(h)$ is an eigenvector it is an element of the base field $\mathbb{C}$ and hence we can view α_j as a linear map $\mathfrak{h} \to \mathbb{C}$, or equivalently $\alpha_j \in \mathfrak{h}^*$. These linear maps are called the roots of $\mathfrak{g}$.

[14] i.e. a symmetric $(0, 2)$-tensor in $\mathfrak{g}^* \otimes \mathfrak{g}^*$ (See definition 20.1.9)

Property 25.3.22. From equation 25.28 it follows that if λ is a root of $\mathfrak{g}$ then $-\lambda$ is also a root of $\mathfrak{g}$.

Previous property implies that the root set Φ is not linearly independent. Therefore we introduce following concept:

Definition 25.3.23 (Simple root). The set of simple roots[15] Δ is a linearly independent subset of Φ such that every element $\lambda \in \Phi$ can be written as:

$$\lambda = \pm \sum_{i}^{n} a_i \lambda_i \tag{25.32}$$

where $a_i \in \mathbb{N}$ and $\lambda_i \in \Delta$. This definition requires the expansion coefficients a_i of a certain root λ to be either all positive or all negative.

Property 25.3.24. Let $\mathfrak{h}$ be a Cartan subalgebra. The set of simple roots Δ forms a basis for the dual space $\mathfrak{h}^*$ (over $\mathbb{C}$).

Definition 25.3.25 (Weyl group). For every simple root λ we define the linear map σ_λ as follows[16]:

$$\sigma_\lambda : \text{span}_{\mathbb{R}}(\Delta) \to \text{span}_{\mathbb{R}}(\Delta) : \mu \mapsto \mu - 2\frac{\langle \lambda, \mu \rangle}{\langle \lambda, \lambda \rangle}\lambda \tag{25.33}$$

where the inner product $\langle \cdot, \cdot \rangle$ is given by the dual Killing form.[17] The Weyl group W is then defined as the group generated by all the σ_λ's.

Property 25.3.26. Every root $\phi \in \Phi$ can be written as $\phi = \sigma(\mu)$ for some $\mu \in \Delta$ and $\sigma \in W$. Furthermore, the root set Φ is closed under the action of W.

Definition 25.3.27 (Cartan matrix). Let $\lambda_i, \lambda_j \in \Delta$ be simple roots. It follows from previous property that

$$\sigma_{\lambda_i}(\lambda_j) = \lambda_j - 2\frac{\langle \lambda_i, \lambda_j \rangle}{\langle \lambda_i, \lambda_i \rangle}\lambda_i$$

is a root. From the definition of simple roots it then follows that the quantity

$$C_{ij} = 2\frac{\langle \lambda_i, \lambda_j \rangle}{\langle \lambda_i, \lambda_i \rangle} \tag{25.34}$$

is an integer. The matrix formed by these numbers is called the Cartan matrix.

Property 25.3.28. The Cartan matrix C_{ij} has the following properties:

[15] For every root set Φ one can find a set of simple roots.

[16] This comes down to a reflection through the hyperplane orthogonal to the vector λ.

[17] Consider the *sharp* map 28.2 where one replaces the metric g by the Killing form K. The dual Killing form K^* is then a proper inner product defined as:

$$K^*(\cdot, \cdot) = K(\cdot^\sharp, \cdot^\sharp)$$

- $C_{ii} = 2$
- $C_{ij} \leq 0$ if $i \neq j$
- $C_{ij} = 0 \iff C_{ji} = 0$

This last property however does not imply that the Cartan matrix is symmetric. The fact that it is not symmetric can immediately be seen from its definition.

Definition 25.3.29 (Bond number). For all indices $i \neq j$ the bond number n_{ij} is defined as follows:

$$n_{ij} = C_{ij}C_{ji} \tag{25.35}$$

Using the definition of the coefficients C_{ij} we see that n_{ij} is an integer equal to $4\cos^2 \sphericalangle(\lambda_i, \lambda_j)$. This implies that n_{ij} can only take on the values $0, 1, 2, 3$.[18]

Remark 25.3.30. In the case of $n_{ij} = 2$ or $n_{ij} = 3$ there arise two possibilities. Namely that $C_{ij} > C_{ji}$ or $C_{ij} < C_{ji}$. From the definition of the Cartan integers and the symmetry of the dual Killing form these cases correspond to $\langle \lambda_i, \lambda_i \rangle < \langle \lambda_j, \lambda_j \rangle$ and $\langle \lambda_i, \lambda_i \rangle > \langle \lambda_j, \lambda_j \rangle$.

Construction 25.3.31 (Dynkin diagram). For a semisimple Lie algebra $\mathfrak{g}$ with simple root set Δ one can draw a so-called Dynkin diagram by using the following rules:

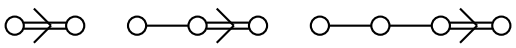
1. For every simple root $\lambda \in \Delta$ draw a circle: ◯
2. If ◯ and ◯ denote the simple roots λ_i and λ_j, draw n_{ij} lines between them.
3. When $n_{ij} = 2$ or $n_{ij} = 3$ add a $<$ or $>$ sign to relate the roots based on their lengths (see previous remark).

Theorem 25.3.32 (Cartan & Killing). *Every finite-dimensional simple $\mathbb{C}$-Lie algebra can be reconstructed from its set of simple roots Δ.*

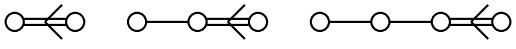
Method 25.3.33. The Dynkin diagrams can be classified as follows (for every type the first three examples are given):

- A_n:

- $\mathrm{B}_n, n \geq 2$:

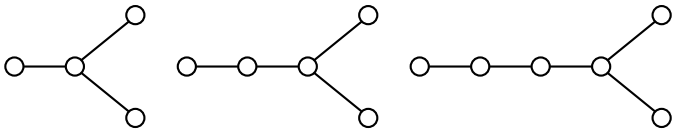


- $\mathrm{C}_n, n \geq 2$:



- $\mathrm{D}_n, n \geq 4$:

[18] The value 4 would only be possible if the angle between λ_i and λ_j is 0 but this can only occur in the case that $i = j$, which was excluded from the definition.



These are the only possible diagrams for simple Lie algebras.[19]

Example 25.3.34 ($SL(2,\mathbb{C})$). By looking at the Lie brackets in 25.13 we see that the one-element set $\{X_1\}$ forms a Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C})$. From 25.13 it is also immediately clear that the simple root set Δ is given by the one-element set $\{\lambda \in \mathfrak{sl}^*(2,\mathbb{C}) : \lambda(X_1) \mapsto 2\}$. Hence the Dynkin diagram for $\mathfrak{sl}(2,\mathbb{C})$ is A_1.

25.4 Universal enveloping algebra

Definition 25.4.1 (Universal enveloping algebra). Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot,\cdot]$. First construct the tensor algebra $T(\mathfrak{g})$. The universal enveloping algebra $U(\mathfrak{g})$ is defined as quotient of $T(\mathfrak{g})$ by the two-sided ideal generated by the elements $g\otimes h - h\otimes g - [g,h]$.

Definition 25.4.2 (Casimir invariant[20]). Let $\mathfrak{g}$ be a Lie algebra. A Casimir invariant J is an element of the center of $U(\mathfrak{g})$.

Formula 25.4.3 (Quadratic Casimir invariant). Consider a Lie algebra representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ on an n-dimensional vector space V. Let $\{X_i\}_{i\leq n}$ be a basis for $\mathfrak{g}$. The (quadratic) Casimir invariant associated with ρ is given by:

$$\Omega_\rho = \sum_{i=0}^{n} \rho(X_i) \circ \rho(\xi_i) \tag{25.36}$$

where the set $\{\xi_i\}_{i\leq n}$ is defined by the relation $K_\rho(X_i,\xi_j) = \delta_{ij}$ using the Killing form 25.30.

Property 25.4.4. When the representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is irreducible Schur's lemma 22.2.3 tells us that:

$$\Omega_\rho = c_\rho \mathbb{1}_V \tag{25.37}$$

By taking the trace of this formula and using formula 25.30 we see that $c_\rho = \frac{\dim \mathfrak{g}}{\dim V}$.

25.5 Poisson algebras and Lie superalgebras

Definition 25.5.1 (Poisson algebra). Let V be a vector space equipped with two bilinear operations $\star$ and $\{\cdot,\cdot\}$ that satisfy the following conditions:

- The couple $(V,\star)$ is an associative algebra.
- The couple $(V,\{\cdot,\cdot\})$ is a Lie algebra.

[19] With exception of E_6, E_7, E_8, F_4 and G_2, the so-called *exceptional Lie algebras*.
[20] Also known as a **Casimir operator** or **Casimir element**.

- the **Poisson bracket** $\{\cdot,\cdot\}$ acts as a derivation[21] with respect to the operation $\star$, i.e.

$$\{x, y \star z\} = \{x, y\} \star z + y \star \{x, z\}$$

[21] See definition 24.2.1.

Chapter 26

Bundle theory

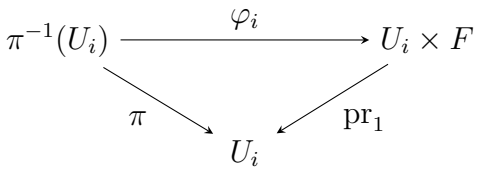
26.1 Fibre bundles

Definition 26.1.1 (Bundle). A bundle is a triple (E, B, π) where E, B are topological spaces and π is a continuous surjective map.

Definition 26.1.2 (Fibered manifold). A fibered manifold is a surjective submersion[1] $\pi : E \to B$ where E is called the **total space**, B the **base space** and π the **projection**. For every point $p \in B$, the set $\pi^{-1}(p)$ is called the **fibre** over p.

The most important example of a fibered manifold is a fibre bundle:

Definition 26.1.3 (Fibre bundle). A fibre bundle is a tuple (E, B, π, F, G) where E, B and F are topological spaces and G is a topological group (called the **structure group**), such that there exists a smooth surjective map $\pi : E \to B$ and an open cover $\{U_i\}_{i \in I}$ of B for which there exists a family of homeomorphisms $\{\varphi_i : \pi^{-1}(U_i) \to U_i \times F\}_{i \in I}$ that make the following diagram commute:



As for topological bundles and fibered manifolds we call E and B the total space and base space respectively. The space F is called the **(typical) fibre**. We also call φ_i a **local trivialization**[2], (U_i, φ_i) a **bundle chart**[3] and the set $\{(U_i, \varphi_i)\}_{i \in I}$ a **trivializing cover**.

[1] See definition 24.4.3.

[2] This name follows from the fact that the bundle is locally homeomorphic to a (trivial) product space: $E \cong U \times F$.

[3] This is due to the similarities with the charts defined for manifolds.

The transition maps $\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ can be identified with the cocycle[4] $g_{ji} : U_i \cap U_j \to G$, associated to the (left) action (which we require to be faithful[5]) of G on every fibre, by the following relation:

$$\varphi_j \circ \varphi_i^{-1}(b, x) = (b, g_{ji}(b) \cdot x) \tag{26.1}$$

Remark 26.1.4. One should pay attention that the bundle charts are not coordinate charts in the original sense 24.1.1 because the image of φ_i is not an open subset of $\mathbb{R}^n$. However they serve the same purpose and we can still use them to locally inspect the total space P.

Notation 26.1.5. A fibre bundle (E, B, π, F, G) is often indicated by the following diagram:

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

or more compactly $F \hookrightarrow E \xrightarrow{\pi} B$. A drawback of these notations is that we do not immediately know what the structure group of the bundle is.

Definition 26.1.6 (Fibre). Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibre bundle over a base space B. The fibre over $b \in B$ is defined as the set $\pi^{-1}(b)$. It is often denoted by F_b.

Definition 26.1.7 (Smooth fibre bundle). A smooth fibre bundle is a fibre bundle (E, B, π, F, G) with the following constraints:

- The base space B and typical fibre F are smooth manifolds.
- The structure goup G is a Lie group.
- The projection map, trivializing maps and transition functions are diffeomorphisms.

Remark 26.1.8. A smooth fibre bundle is also a smooth manifold.

Construction 26.1.9 (Fibre bundle construction theorem). Let M and F be topological spaces and let G be a topological group equipped with a left action on F. Suppose that we are given a cover $\{U_i\}_{i \in I}$ of M and a set of continuous functions $\{g_{ji} : U_i \cap U_j \to G\}$ that satisfy the cocycle condition 25.1.12. A fibre bundle over M can then be constructed as follows:

1. We first construct for every set U_i an associated set $U_i \times F$.
2. We then construct the disjoint union $T \equiv \bigsqcup_{i \in I} U_i \times F$ equipped with the disjoint union topology[6].

[4] See definition 25.1.12.
[5] See definition 3.1.41.
[6] See definition 5.1.4.

3. From this disjoint union we construct a quotient space[7] (equipped with the quotient space topology) by applying following equivalence relation for every i, j:

$$(p, f) \sim (p, g_{ji}(x) \cdot f) \tag{26.2}$$

for all $x \in U_i \cap U_j$ and $f \in F$. The fibre bundle is equal to this quotient space $T/\sim$ together with the projection π that maps the equivalence class of $(x, f) \in T$ to $x \in M$.

4. Local trivializations are given by the maps $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$ that satisfy:

$$\varphi_i^{-1} : (x, f) \mapsto [(x, f)] \tag{26.3}$$

where $[A]$ means the equivalence class of A in $T/\sim$.

Definition 26.1.10 (Compatible[8] bundle charts). A bundle chart (V, ψ) is compatible with a trivializing cover $\{(U_i, \varphi_i)\}_{i\in I}$ if whenever $V \cap U_i \neq \emptyset$ their exists a map $h_i : V \cap U_i \to G$ such that:

$$\psi \circ \varphi_i^{-1}(b, x) = (b, h_i(b)x) \tag{26.4}$$

for all $b \in V \cap U_i$ and $x \in F$. Two trivializing covers are *equivalent* if all bundle charts are cross-compatible. As in the case of manifolds, this gives rise to the notion of a **G-atlas**. A **G-bundle** is then defined as a fibre bundle eqipped with an equivalence class of G-atlases.

Definition 26.1.11 (Bundle map). A bundle map between two fibre bundles $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ is a pair (f_E, f_B) of continuous maps that make diagram 26.1 commute:

$$\begin{array}{ccc} E_1 & \xrightarrow{f_E} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow[f_B]{} & B_2 \end{array}$$

Figure 26.1: Bundle map between fibre bundles.

Definition 26.1.12 (Isomorphic fibre bundles). Two fibre bundles F and G are isomorphic if there exist bundle maps $f : F \to G$ and $g : G \to F$ such that $f \circ g = \mathbb{1}_G$ and $g \circ f = \mathbb{1}_F$.

Definition 26.1.13 (Equivalent fibre bundles). Two fibre bundles $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ (with the same typical fibre and structure group) are equivalent if there exist trivializing covers[9] $\{(U_i, \varphi_i)\}_{i\in I}$ and $\{(U_i, \varphi_i')\}_{i\in I}$ and a family of smooth functions $\{\rho_i : U_i \to G\}_{i\in I}$ such that:

$$g'_{ji}(b) = \rho_j(b) \circ g_{ji}(b) \circ \rho_i^{-1}(b) \tag{26.5}$$

[7] See definition 5.4.

[8] Also called an **admissible chart**.

for every $b \in U_i \cap U_j$. An explicit form of these functions is given by:

$$\rho_i = \varphi_i' \circ \varphi_i^{-1} \tag{26.6}$$

This transformation is called a **gauge transformation** (notably in physics) .

Property 26.1.14. Two fibre bundles over the same base space are equivalent if and only if they are isomorphic. Furthermore, every bundle map between bundles over the same base space induces an equivalence (and thus also an isomorphism).

Definition 26.1.15 (Trivial bundle). A fibre bundle (E, B, π, F) is trivial if $E = B \times F$.

Definition 26.1.16 (Trivialization). A trivialization of a fibre bundle ξ is an equivalence $\xi \to B \times F$. Bundles for which a trivialization can be found are also called *trivial bundles*.

Definition 26.1.17 (Subbundle). A subbundle of a fibre bundle $\pi : E \to B$ is a triple (E', B', π') such that $E' \subset E$, $B' \subset B$ (where $\subset$ now means 'submanifold of') and $\pi' = \pi|_{E'}$.

Definition 26.1.18 (Pullback bundle). Let $\pi : E \to B$ be a fibre bundle. Let $f : B \to B'$ be a continuous map between topological spaces. The pullback bundle f^*E is defined as follows:

$$f^*E = \{(b', e) \in B' \times E : f(b') = \pi(e)\} \tag{26.7}$$

The topology on f^*E is given by the subspace topology.

Definition 26.1.19 (Fibre product). Let (F_1, B, π_1) and (F_2, B, π_2) be two fibre bundles on a base space B. Their fibre product is defined as:

$$F_1 \diamond F_2 = \{f \times g \in F_1 \times F_2 : \pi_1(f) = \pi_2(g)\} \tag{26.8}$$

26.1.1 Sections

Definition 26.1.20 (Section). A **global** section on a fibre bundle $\pi : E \to B$ is a smooth function $s : B \to E$ such that $\pi \circ s = \mathbb{1}_B$. For any open subset $U \subset B$ we define a local section as a smooth function $s_U : U \to E$ such that $\pi \circ s_U(b) = b$ for all $b \in U$.

Notation 26.1.21. The set of all global sections on a bundle E is denoted by $\Gamma(E)$. The set of local sections on $U \subset E$ is similarly denoted by $\Gamma(U)$.

Property 26.1.22. The sections on a fibre bundle E pullback to the pullback bundle f^*E by setting $f^*s = s \circ f$.

[9] Remark that the collection $\{U_i\}_{i \in I}$ is the same for both trivializing covers.

26.1.2 Jet bundles

Definition 26.1.23 (Jet). Consider a fibre bundle (E, B, π) with its sections $\Gamma(E)$. Two sections $\sigma, \xi \in \Gamma(E)$ with local coordinates (σ^i) and (ξ^i) define the same r-jet at a point $p \in B$ if and only if:

$$\left.\frac{\partial^\alpha \sigma^i}{\partial x^\alpha}\right|_p = \left.\frac{\partial^\alpha \xi^i}{\partial x^\alpha}\right|_p \tag{26.9}$$

for all $0 \leq i \leq \dim E$ and every multi-index α such that $0 \leq |\alpha| \leq r$. It is clear that this relation defines an equivalence relation. The r-jet at $p \in B$ with representative σ is denoted by $j^r_p\sigma$. The number r is called the **order** of the jet.

Definition 26.1.24 (Jet manifold). Consider a fibre bundle (E, B, π). The r-jet manifold $J^r(\pi)$ of the projection π is defined as:

$$J^r(\pi) = \{j^r_p\sigma : \sigma \in \Gamma(E), p \in B\} \tag{26.10}$$

The set $J^0(\pi)$ is identified with the total space E.

Definition 26.1.25 (Jet projections). Let (E, B, π) be a fibre bundle with r-jet manifolds $J^r(\pi)$. The **source projection** π_r and **target projection** $\pi_{r,0}$ are defined as the maps

$$\pi_r : J^r(\pi) \to B : j^r_p\sigma \mapsto p \tag{26.11}$$

$$\pi_r : J^r(\pi) \to E : j^r_p\sigma \mapsto \sigma(p) \tag{26.12}$$

These projections satisfy $\pi_r = \pi \circ \pi_{r,0}$. We can also define a **k-jet projection** $\pi_{r,k}$ as the map

$$\pi_{r,k} : J^r(\pi) \to J^k(\pi) : j^r_p\sigma \mapsto j^k_p\sigma \tag{26.13}$$

where $k \leq r$. The k-jet projections satisfy a transitivity property $j_{k,m} = j_{r,m} \circ j_{k,r}$.

Definition 26.1.26 (Jet prolongation). Let σ be a section on a fibre bundle (E, B, π). The r-jet prolongation $j^r\sigma$ corresponding to σ is defined as the following map:

$$j^r\sigma : B \to J^r(\pi) : p \mapsto j^r_p\sigma \tag{26.14}$$

Definition 26.1.27 (Jet bundle). The r-jet bundle corresponding to the projection π is then defined as the triple $(J^r(\pi), B, \pi_r)$. The bundle charts (U_i, φ_i)[10] on E define *induced* bundle charts on $J^r(\pi)$ in the following way:

$$U^r_i = \{j^r_p\sigma : \sigma(p) \in U_i\} \tag{26.15}$$

$$\varphi^r_i = \left(x^k, u^\alpha, \left.\frac{\partial^I u^\alpha}{\partial x^I}\right|_p\right) \tag{26.16}$$

where I is a multi-index such that $0 \leq |I| \leq r$. The partial derivatives $\left.\frac{\partial^I u^\alpha}{\partial x^I}\right|_p$ are called the **derivative coordinates** on $J^r(\pi)$.

[10]Where $\varphi_i = (x^k, u^\alpha)$ with x^k the base space coordinates and u^α the total space coordinates.

26.2 Vector bundles

The tangent space, as introduced in subsection 24.2, can also be introduced in a more topological way:

26.2.1 Tangent bundle

Construction 26.2.1 (Tangent bundle). Let M be an n-dimensional manifold with atlas $\{(U_i, \varphi_i)\}_{i\leq n}$. Construct for every open set O an associated set $TO = O \times \mathbb{R}^n$ and construct for every smooth function f an associated smooth function on TO, called the **differential** or **derivative** of f, by:

$$Tf : O \times \mathbb{R}^n \to f(O) \times \mathbb{R}^n : (x, v) \mapsto (f(x), Df(x)v) \tag{26.17}$$

where $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ is the linear operator represented by the Jacobian matrix of f in x.

Applying this definition to the transition functions ψ_{ji} we obtain a new set of functions $\widetilde{\psi}_{ji} := T\psi_{ji} : U_i \times \mathbb{R}^n \to U_j \times \mathbb{R}^n$ given by:

$$\widetilde{\psi}_{ji}(\varphi_i(x), v) = \big(\varphi_j(x), D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(x))v\big) \tag{26.18}$$

Because the transition functions are diffeomorphisms, the Jacobians are invertible. This implies that the maps $\widetilde{\psi}_{ji}$ are elements of $GL(\mathbb{R}^n)$. The tangent bundle is now obtained by applying the fibre bundle construction theorem 26.1.9 to the triple $(M, \mathbb{R}^n, GL(\mathbb{R}^n))$ together with the base cover $\{U_i\}_{i\leq n}$ and the cocycle $\{\widetilde{\psi}_{ji}\}_{i,j\leq n}$.

Definition 26.2.2 (Natural chart). The charts in the atlas of the constructed bundle are sometimes called **natural charts** because the first n coordinates are equal to the coordinates of the base space.

Alternative Definition 26.2.3. The above construction eventually comes down to the following, more intuitive, definition of the tangent bundle:

$$TM = \bigsqcup_{p\in M} T_pM \tag{26.19}$$

equipped with the disjoint union topology 5.1.4 and the projection map[11]

$$\pi : TM \to M : (p, X) \mapsto p \tag{26.20}$$

where X is a tangent vector in T_pM. An atlas on TM is then given by the charts $(\pi^{-1}(U_i), \theta)$ with

$$\theta : TM \to \mathbb{R}^{2n} : (p, X) \mapsto (\varphi_i(p), X^1, ..., X^n) \tag{26.21}$$

where $X = X^i \left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$ and where (U_i, φ_i) is a chart on M covering the point $p \in M$.

[11] The map π is single-valued because the tangent bundle is defined as the disjoint union of the tangent spaces.

Property 26.2.4. Let M be an n-dimensional manifold. Using the natural charts on TM, which give a local homeomorphism

$$\psi_i : TM \to U_i \times \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$$

we can see that TM is isomorphic to $\mathbb{R}^{2n}$. This implies that:

$$\boxed{\dim TM = 2 \dim M} \tag{26.22}$$

Definition 26.2.5 (Tangent space). Let $x \in M$. The topological definition of the tangent space is given by the fibre

$$T_x M := \tau_M^{-1}(x) \tag{26.23}$$

If we use the natural charts to map $T_x M$ to the set $\varphi_i(x) \times \mathbb{R}^n$, we see that $T_x M$ is isomorphic to $\mathbb{R}^n$ and thus also to M itself. Furthermore, we can equip every fibre with the following vector space structure:

$$\begin{aligned}(x, v_1) + (x, v_2) &:= (x, v_1 + v_2)\\ r(x, v) &:= (x, rv)\end{aligned}$$

Remark 26.2.6. Now it is clear that the rule "*a vector is something that transforms like a vector*" stems from the fact that:

$$\text{a vector } v \in T_x M \text{ is tangent to } \varphi_i(x) \text{ in a chart } (U_i, \varphi_i)$$

if and only if

$$D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(x))v \text{ is tangent to } \varphi_j(x) \text{ in a chart } (U_j, \varphi_j)$$

Definition 26.2.7 (Differential). The map T defined in 26.17 can be generalized to arbitrary smooth manifolds as the map $Tf : TM \to TN$. Furthermore, let $x \in U \subseteq M$ and let $V = f(U)$. By looking at the restriction of Tf to $T_x M$, denoted by $T_x f$, we see that it maps $T_x U$ to $T_{f(x)} V$ linearly.

Property 26.2.8. The map $Tf : TM \to TN$ (see 26.17) has following properties[12]:

- $T(\mathbb{1}_M) = \mathbb{1}_{TM}$
- Let f, g be two smooth functions on smooth manifolds. Then $T(f \circ g) = Tf \circ Tg$.

Definition 26.2.9 (Rank). Let $f : M \to N$ be a differentiable map between smooth manifolds. Using the fact that Tf is a linear map of fibres[13], we define the rank of f at $p \in M$ as the rank (in the sense of 16.3.16) of the differential $Tf : T_p M \to T_{f(p)} N$.

Theorem 26.2.10 (Inverse function theorem). *A C^∞ map $f : M \to N$ between smooth manifolds is locally homeomorphic (resp. locally diffeomorphic) if and only if its differential $Tf : T_p M \to T_p N$ is an isomorphism (resp. diffeomorphism) at p.*

[12] This turns the map T into a functor on the category of smooth manifolds. Hence we can view T as a functorial derivative.

[13] See definition 26.2.7.

26.2.2 Vector bundles

Instead of restricting ourselves by letting the typical fibre be a Euclidean space with the same dimension as the base manifold, we can generalize the construction of the tangent bundle in the following way:

Construction 26.2.11 (Vector bundle). Consider a smooth n-dimensional manifold M with atlas $\{(U_i, \varphi_i)\}_{i\leq n}$, a cocycle $\{g_{ji} : U_i \cap U_j \to G\}_{i,j\leq n}$ with values in a Lie group G and a smooth representation $\rho : G \to GL(V)$, where V is a vector space. A bundle can then be constructed using 26.1.9

Remark 26.2.12. As is also the case for tangent bundles (which are specific cases of vector bundles where the typical fibre has the same dimension as the manifold) the choice of charts on E is not random. To preserve the structure of fibres, the use of the natural charts is imperative.

Example 26.2.13 (Line bundle). A line bundle is a vector bundle with a one-dimensional fibre V. A common example is the $\mathbb{C}$-line bundle over configuration space for which, in quantum mechanics, the sections correspond to the physical "wave functions".

Definition 26.2.14 (Whitney sum). Consider two vector bundles E, E' with fibres W, W' respectively. Then we can construct a new vector bundle $E \oplus E'$ by defining the new typical fibre to be the direct sum $W \oplus W'$, i.e. the fibre above b is given by $W_b \oplus W'_b$. This operation is called the Whitney sum or direct sum of vector bundles.

26.2.3 Associated vector bundles

Definition 26.2.15 (Associated vector bundle). Consider a representation $\rho : GL(\mathbb{R}^n) \to GL(\mathbb{R}^l)$ together with the cocycle $\{t_{ji} := D(\psi_{ji}) \circ \varphi_i\}_{i,j\leq n}$ as defined for the tangent bundle. The composition $\rho \circ t_{ji} : U_i \cap U_j \xrightarrow{t_{ji}} GL(\mathbb{R}^n) \xrightarrow{\rho} GL(\mathbb{R}^l)$ is again a cocycle and can thus be used to define a new vector bundle on M. The vector bundle $E = \rho(TM)$ so obtained is called the associated bundle of the tangent bundle induced by ρ.

Example 26.2.16 (Contravariant vectors). By noting that the k^{th} tensor power $\otimes^k$ induces a representation given by the tensor product of the representations, we can construct the bundle of k^{th} order contravariant vectors $\otimes^k(TM)$ with the cocycle given by $x \mapsto t_{ji}(x) \otimes \cdots \otimes t_{ji}(x)$.

Example 26.2.17 (Cotangent bundle). Another (smooth) representation is given by $A \mapsto (A^T)^{-1} = (A^{-1})^T$ for every linear map A. The vector bundle constructed this way, where the cocycle is given by $(t^T_{ji})^{-1}$, is called the cotangent bundle on M and is denoted by T^*M. Elements of the fibres are called **covariant vectors** or **covectors**.

Notation 26.2.18. A combination of the cocycle t_{ji} and its dual $(t^T_{ji})^{-1}$ can also be used to define the bundle of k^{th} order contravariant and l^{th} order covariant vectors on M. This bundle is denoted by $T^{(k,l)}M$.

Example 26.2.19 (Pseudovectors). If we consider the representation

$$\rho : A \mapsto \operatorname{sgn}\det(A)A \tag{26.24}$$

we can construct a bundle similar to the tangent bundle. The sign of the cocycle functions t_{ji} now has an influence on the fibres. Elements of these fibres are called **pseudovectors**.

26.2.4 Sections

Definition 26.2.20 (Frame). A frame of a vector bundle E is a tuple $(s_1, ..., s_n)$ of smooth sections such that $(s_1(b), ..., s_n(b))$ is a basis of the fibre $\pi^{-1}(b)$ for all $b \in B$.

Property 26.2.21. A vector bundle is trivial if and only if there exists a frame of global sections.

Theorem 26.2.22 (Serre & Swan). *The set of all smooth sections $\Gamma(E)$ over a vector bundle E with base space M is a finitely generated projective $C^\infty(M)$-module.*

26.3 Vector fields

Definition 26.3.1 (Vector field). A smooth section $s \in \Gamma(TM)$ of the tangent bundle is called a vector field. The set of vector fields forms a $C^\infty(M)$-module.

Notation 26.3.2. The set of all vector fields on a manifold M is often denoted by $\mathfrak{X}(M)$.

Theorem 26.3.3 (Hairy ball theorem). *There exists no nowhere vanishing vector field on an even-dimensional sphere S^{2n}.*

Definition 26.3.4 (Pullback). Let X be vector field on M and let $\varphi : M \to N$ be a diffeomorphism between smooth manifolds. The pullback of X along φ is defined as:

$$(\varphi^* X)_p = T\varphi^{-1}(X_{\varphi(p)}) \tag{26.25}$$

Definition 26.3.5 (Pushforward). Let $X \in \mathfrak{X}(M)$ and let $\varphi : M \to N$ be a diffeomorphism between smooth manifolds. Using the differential $T\varphi$ we can define the pushforward of X along φ as:

$$(\varphi_* X)_{\varphi(p)} = T\varphi(X_p) \tag{26.26}$$

which we can rewrite using the pullback as:

$$\varphi_* X = \varphi^{-1*} X \tag{26.27}$$

Or equivalently we can define a vector field on N by:

$$(\varphi_* X)_q(f) = X_{\varphi^{-1}(q)}(f \circ \varphi) \tag{26.28}$$

for all smooth functions $f : N \to \mathbb{R}$ and points $q \in N$.

26.3.1 Integral curves

Definition 26.3.6 (Integral curve). Let $X \in \mathfrak{X}(M)$ and let $\gamma :]a, b[\to M$ be a smooth curve on M. γ is said to be an integral curve of X if:

$$\boxed{\gamma'(t) = X(\gamma(t))} \tag{26.29}$$

for all $t \in]a, b[$ where we defined $\gamma'(t) := T\gamma(t, 1)$ using the functorial derivative 26.17.

This equation can be seen as a system of ordinary differential equations in the second argument. Using Picard's existence theorem[14] together with the initial value condition $\gamma(0) = p$ we can find a unique curve on $]a, b[$ satisfying the defining equation 26.29. Furthermore we can extend the interval $]a, b[$ to a maximal interval such that the solution is still unique. This solution, denoted by γ_p, is called the **integral curve of X through p**.

Definition 26.3.7 (Flow). Let $X \in \mathfrak{X}(M)$. The function σ_t:

$$\sigma_t(p) = \gamma_p(t) \tag{26.30}$$

is called the flow of X at time t. The **flow domain** is defined as the set $D(X) = \{(t, p) \in \mathbb{R} \times M \mid t \in]a_p, b_p[\}$ where $]a_p, b_p[$ is the maximal interval on which $\gamma_p(t)$ is defined.

Property 26.3.8. Suppose that $D(X) = \mathbb{R} \times M$. The flow σ_t has following properties for all $s, t \in \mathbb{R}$:

- $\sigma_0 = \mathbb{1}_M$
- $\sigma_{s+t} = \sigma_s \circ \sigma_t$
- $\sigma_{-t} = (\sigma_t)^{-1}$

These three properties[15] say that σ_t is a bijective group action from M to the additive group of real numbers. This implies that σ_t is indeed a **flow** in the general mathematical sense.

Definition 26.3.9 (Complete vector field). A vector field X is called complete if the flow domain for every flow is all of $\mathbb{R}$.

Property 26.3.10. The flow σ_t of a vector field is of class C^∞. If X is complete it follows from previous definition that the flow is a diffeomorphism from M onto itself.

Property 26.3.11. If the manifold M is compact then every vector field $X \in \mathfrak{X}(M)$ is complete.

[14] Also Picard-Lindelöf theorem.

[15] The third property follows from the other two.

26.3.2 Lie derivative

Formula 26.3.12 (Lie derivative for smooth functions). Let $X \in \mathfrak{X}(M)$ and let $f : M \to \mathbb{R}$ be a smooth function. The Lie derivative of f with respect to X at $p \in M$ is defined as:

$$\boxed{(\mathcal{L}_X f)(p) = \lim_{t\to 0} \frac{f(\gamma_p(t)) - f(p)}{t}} \tag{26.31}$$

which closely resembles the standard derivative in Euclidean space.

Formula 26.3.13 (†). Working out previous formula and rewriting it as an operator equality gives:

$$\boxed{\mathcal{L}_X = \sum_k X_k \frac{\partial}{\partial x^k}} \tag{26.32}$$

It is clear that this is just the vector field X expanded in the basis 24.2.3. We also recover the behaviour of a tangent vector as a derivation. So for smooth functions $f : M \to \mathbb{R}$ we obtain:

$$\mathcal{L}_X f(p) = X_p(f) \tag{26.33}$$

Formula 26.3.14 (Lie derivative for vector fields†). Let $X, Y \in \mathfrak{X}(M)$

$$\boxed{\mathcal{L}_X Y = \frac{d}{dt}(\sigma_t^* X)(\gamma_p(t))\Big|_{t=0}} \tag{26.34}$$

Property 26.3.15. Let $X, Y \in \mathfrak{X}(M)$ be vector fields of class C^k. The Lie derivative has following properties:

- $\mathcal{L}_X Y$ is a vector field.
- **Lie bracket**:

  $$\mathcal{L}_X Y = [X, Y] \tag{26.35}$$

  which is also a derivation on $C^{k-1}(M, \mathbb{R})$ due to the cancellation of second-order derivatives in the local representation. It follows that the Lie derivative on vector fields turns the space $\mathfrak{X}(M)$ into a real Lie algebra.
- The Lie derivative is antisymmetric:

  $$\mathcal{L}_X Y = -\mathcal{L}_Y X \tag{26.36}$$

  This follows from the previous property.

26.3.3 Grassmann bundle

Looking at property 16.2.11 and noting that $\mathrm{GL}_n(\mathbb{R})$ is a Lie group, we can endow the Grassmannian $\mathrm{Gr}(k, \mathbb{R}^n)$ 16.2.10 with a differentiable structure, turning it into a smooth

manifold. With this we can construct a new bundle[16] by applying the usual construction theorem 26.1.9:

Construction 26.3.16 (Grassmann bundle). First define the transition functions:

$$\psi_{ji} : \varphi_i(U_i \cap U_j) \times \mathrm{Gr}(k, \mathbb{R}^n) \to \varphi_j(U_i \cap U_j) \times \mathrm{Gr}(k, \mathbb{R}^n) : (\varphi_i(x), V) \mapsto (\varphi_j(x), t_{ji}(x) \cdot V) \tag{26.37}$$

where $\{t_{ji}\}_{i,j \leq n}$ is the tangent bundle cocycle, but now with an action on the compact manifold $\mathrm{Gr}(k, \mathbb{R}^n)$ instead of the vector space $\mathbb{R}^n$. This set of transition functions is then used to create a new fibre bundle where every fibre is diffeomorphic to $\mathrm{Gr}(k, \mathbb{R}^n)$, namely it is the Grassmannian $\mathrm{Gr}(k, T_pM)$ associated to the tangent space in every point $p \in M$.

Notation 26.3.17. The Grassman k-plane bundle is denoted by $\mathrm{Gr}(k, TM)$.

26.3.4 Frobenius theorem

Definition 26.3.18 (Distribution). A smooth section of the Grassman k-plane bundle is called a distribution of k-planes.

Definition 26.3.19 (Integrable). Let M be a smooth manifold and let $W \in \Gamma(\mathrm{Gr}(k, TM))$ be a distribution of k-planes. A submanifold $N \subseteq M$ is said to integrate W with initial condition $p_0 \in M$ if for every $p \in N$ we find that $W(p) = T_pN$ and $p_0 \in N$. W is said to be integrable if there exists such a submanifold N.

Definition 26.3.20 (Frobenius integrability condition). A distribution of k-planes W over a smooth manifold M is said to satisfy the Frobenius integrability condition in an open set $U \subseteq M$ if for every two vector fields X, Y defined on U, such that $X(p) \in W(p)$ and $Y(p) \in W(p)$ for all $p \in U$, there Lie bracket $[X, Y](p)$ is also an element of $W(p)$ for all $p \in U$.

Theorem 26.3.21 (Frobenius' integrability theorem). *Let W be a distribution of k-planes over a smooth manifold M. Then W is integrable if and only if W satisfies the Frobenius integrability condition.*

26.4 Differential k-forms

Definition 26.4.1 (Differential form). A differential k-form is a map

$$\boxed{\omega : T^{\diamond k}M \to \mathbb{R}} \tag{26.38}$$

such that the restriction of ω to each fibre of the fibre product[17] $T^{\diamond k}M$ is multilinear and antisymmetric.

[16] Due to the fact that the Grassmannian is not a vector space, we construct a general fibre bundle and not a vector bundle.

The space of all differential k-forms on a manifold M is denoted by $\Omega^k(M)$. Just like $\mathfrak{X}(M)$ it forms a $C^\infty(M)$-module. The set $\Omega^0(M)$ is defined as the space of smooth functions $C^\infty : M \to \mathbb{R}$.

Alternative Definition 26.4.2. An alternative definition goes as follows. Consider the representation

$$\rho_k : GL(R^{m*}) \to GL(\Lambda^k(\mathbb{R}^{m*})) : T \mapsto T \wedge ... \wedge T$$

where T is a linear map. This representation induces an associated vector bundle[18] $\rho_k(\tau_M^*)$ of the cotangent bundle on M. A differential k-form is then given by a section of $\rho_k(\tau_M^*)$. $\Omega^k(M)$ can then be defined as follows:

$$\Omega^k(M) = \Gamma(\rho_k(\tau_M^*))$$

Construction 26.4.3. We can construct a Grassmann algebra[19] by equipping the graded vector space

$$\Omega(M) = \bigoplus_{k\geq 0} \Omega^k(M) \tag{26.39}$$

with the wedge product of differential forms (which is induced by the wedge product on $\Lambda^k(\mathbb{R}^{m*})$ through the alternative definition). This graded algebra is associative, graded-commutative and unital with the constant function $1 \in C^\infty(M)$ as identity element.

Definition 26.4.4 (Pullback). Let $f : M \to N$ be a smooth function between smooth manifolds and let ω be a differential k-form on N. The pullback of ω by f is defined as:

$$\boxed{f^*(\omega) = \omega \circ f_* : TM \to \mathbb{R}} \tag{26.40}$$

So f^* can be seen as a map pulling elements from T^*N back to T^*M.

Definition 26.4.5 (Pushforward). Let $f : M \to N$ be a diffeomorphism between smooth manifolds and let ω be a differential k-form on M. The pushforward ω by f is defined as:

$$f_*(\omega) : \omega \circ (f^{-1})_* : TN \to \mathbb{R} \tag{26.41}$$

Remark. Note that the pushforward of differential k-form is only defined for diffeomorphisms, in constrast to pullbacks which only require smooth functions. Furthermore this also explains why differential forms are the most valuable elements in differential geomeotry. Vector fields can't even be pulled back in general by smooth maps.

Formula 26.4.6 (Dual basis). Consider the basis $\{ \frac{\partial}{\partial x_i}\Big|_p \}_{i\leq n}$ from definition 24.2.3 for the tangent space T_pM. From this set we can construct[20] a natural dual basis for the cotangent space T_p^*M using the natural pairing of these spaces:

$$\left\langle \frac{\partial}{\partial x^i}, dx^j \right\rangle = \delta_i^j \tag{26.42}$$

[17] See definition 26.8.

[18] See definition 26.2.15.

[19] As in definition 20.4.14.

[20] It should however be noted that dx^i is not just a notation. These basis vectors are in fact constructed by applying the exterior derivative 26.4.7 to the coordinate maps x^i.

26.4.1 Exterior derivative

Definition 26.4.7 (Exterior derivative). The exterior derivative d_k is a map defined on the graded algebra of differential k-forms:

$$d_k : \Omega^k(M) \to \Omega^{k+1}(M) \tag{26.43}$$

For $k = 0$ it is given by[21]:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \tag{26.44}$$

where we remark that the 'infinitesimals' are in fact unit vectors with norm 1. This formula can be generalized to higher dimensions as follows:

$$\boxed{d(f dx_{i_1} \wedge ... \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}} \tag{26.45}$$

Corollary 26.4.8. It follows immediately from 26.45 that

$$d(dx_i) = 0 \tag{26.46}$$

for all $i \leq n$.

Property 26.4.9. The exterior derivatives have following properties:

- For all $k \geq 0$, for all $\omega \in \Omega^k(M)$: $d_k \circ d_{k+1} = 0$, so $\text{im}(d_k) \subseteq \ker(d_{k+1})$.
- The exterior derivative is an $\mathbb{R}$-linear map.
- Graded Leibniz rule:

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^j \omega_1 \wedge d\omega_2 \tag{26.47}$$

  where $\omega_1 \in \Omega^j(M), \omega_2 \in \Omega^k(M)$.
- Let $f \in C^\infty(M)$: $f^*(d\omega) = d(f^*\omega)$ where f^* denotes the pullback 26.40.

Remark 26.4.10 (†). The gradient, rotor (curl) and divergence from standard vector calculus[22] can be rewritten using exterior derivatives as follows: Let $\vec{\boldsymbol{f}} = (f_1, f_2, f_3)$ with f_i smooth for every i and let f be a smooth function. Denote the canonical isomorphism between $\mathbb{R}^3$ and $\mathbb{R}^{3*}$ by $\sim$.

$$\boxed{\begin{aligned} \sim df &= \nabla f \\ \sim (*d\alpha) &= \nabla \times \vec{\boldsymbol{f}} \\ *d\omega &= \nabla \cdot \vec{\boldsymbol{f}} \end{aligned}} \tag{26.48, 26.49, 26.50}$$

The properties in section 17.1.2 then follow from the identity $d^2 = 0$.

[21] For $f \in \Omega^0(M)$, we call df the **differential** of f.

[22] See section 17.1.

Example 26.4.11. Let $f \in C^\infty(M, \mathbb{R})$. Let γ be a curve on M. From the definition 26.42 of the basis $\{dx_k\}_{k\leq n}$ we obtain following result:

$$\langle df(x), \gamma'(t)\rangle = \sum_k \frac{\partial f}{\partial x_k}(x)\gamma_k'(t) = (f \circ \gamma')(t) \tag{26.51}$$

Example 26.4.12. An explicit formula for the exterior derivative of a k-form Φ is:

$$\begin{aligned} d\Phi(X_1, ...nX_{k+1}) = &\sum_{i=0}^{k+1}(-1)^i X_i(\Phi(X_1, ..., \hat{X}_i, ..., X_{k+1})) \\ &- \sum_{i<j} \Phi([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{k+1}) \end{aligned} \tag{26.52}$$

where $\hat{X}$ means that this argument is omitted.

26.4.2 Lie derivative

Formula 26.4.13 (Lie derivative of differential forms).

$$\boxed{\mathcal{L}_X\omega(p) = \lim_{t\to 0}\frac{\sigma_t^*\omega - \omega}{t}(p)} \tag{26.53}$$

Formula 26.4.14 (Lie derivative of smooth functions). Using the definition of the exterior derivative of smooth functions 26.44 and the definition of the dual (cotangent) basis 26.42 we can rewrite the Lie derivative 26.32 as:

$$Xf(p) = df_p(X(p)) \tag{26.54}$$

Property 26.4.15. The Lie derivative also has following Leibniz-type property with respect to differential forms:

$$\mathcal{L}_X(\omega(Y)) = (\mathcal{L}_X\omega)(Y) + \omega(\mathcal{L}_X Y) \tag{26.55}$$

where X, Y are two vector fields and ω is a 1-form.

Formula 26.4.16 (Lie derivative of tensor fields). By comparing the definitions of the Lie derivatives of vector fields 26.34 and differential forms 26.53 we can see that both definitions are identical upon replacing X by ω. This leads to the definition of a Lie derivative of a general tensor field $\mathcal{T} \in \Gamma(T^{(k,l)}M)$:

$$\boxed{\mathcal{L}_X\mathcal{T}(p) = \frac{d}{dt}\sigma_t^*\mathcal{T}(\gamma_p(t))\Big|_{t=0}} \tag{26.56}$$

26.4.3 Interior product

Definition 26.4.17 (Interior product). Aside from the differential (exterior derivative) we can also define another operation on the algebra of differential forms:

$$\iota_X : (\iota_X\omega)(v_1, ..., v_{k-1}) \mapsto \omega(X, v_1, ..., v_{k-1}) \tag{26.57}$$

This antiderivation (of degree -1) from $\Omega^k(M)$ to $\Omega^{k-1}(M)$ is called the **interior product** or **interior derivative**. This can be seen as a generalization of the contraction map 20.7.

Formula 26.4.18 (Cartan's magic formula[23]). Let X be a vector field and let ω be a differential k-form. The Lie derivative of ω along X is given by the following formula:

$$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega) \tag{26.58}$$

26.4.4 de Rham Cohomology

Definition 26.4.19 (Exact form). Let $\omega \in \Omega^k(M)$. If ω can be written as $\omega = d\chi + 0$ for some $\chi \in \Omega^{k-1}(M)$ and $0 \in \Omega^0(M)$ the zero function then ω is said to be exact. It follows that

$$\mathrm{im}(d_k) = \{\omega \in \Omega^{k+1}(M) : \omega \text{ is exact}\} \tag{26.59}$$

Definition 26.4.20 (Closed form). Let $\omega \in \Omega^k(M)$. If $d\omega = 0$ then ω i said to be closed. It follows that

$$\{\omega \in \Omega^k(M) : \omega \text{ is closed}\} \subseteq \ker(d_k) \tag{26.60}$$

Remark 26.4.21. From the first item of property 26.4.9 it follows that every exact form is closed. The converse however is not true[24].

Definition 26.4.22 (de Rham complex). The structure given by the chain

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} ... \tag{26.61}$$

together with the sequence of exterior derivatives d_k forms a cochain complex. This complex is called the de Rham complex.

The relation between closed and exact forms can be used to define the de Rham cohomology groups.

Definition 26.4.23 (de Rham cohomology). The k^{th} de Rham cohomology group on M is defined as the following quotient space:

$$\boxed{H^k_{\mathrm{dr}}(M) = \frac{\ker(d_{k+1})}{\mathrm{im}(d_k)}} \tag{26.62}$$

[23] Sometimes called **Cartan's (infinitesimal) homotopy formula**.
[24] See result 26.4.25 for more information.

This quotient space is a vector space. Two elements of the same equivalence class in $H^k_{dr}(M)$ are said to be **cohomologous**.

One can construct a graded ring 3.2.15 from these cohomology groups, called the cohomology ring H^*. The product is called the **cup product** $\smile$ and it is a graded-commutative product (see 3.21).

Definition 26.4.24 (Cup product). Let $[\nu] \in H^k_{\mathrm{dr}}$ and $[\omega] \in H^l_{\mathrm{dr}}$, where we used $[\cdot]$ to show that the elements are in fact equivalence relations belonging to differential forms ν and ω. The cup product is defined as follows: $[\nu] \smile [\omega] = [\nu \wedge \omega]$.

Theorem 26.4.25 (Poincaré's lemma[25]). *For every point $p \in M$ there exists a neighbourhood on which the de Rham cohomology is trivial:*

$$\forall p \in M : \exists U \subseteq M : H^k_{dr}(U) = 0 \tag{26.63}$$

This implies that every closed form is locally exact.

26.4.5 Vector-valued differential forms

Definition 26.4.26 (Vector-valued differential form). Let V be a vector space and E a vector bundle with V as typical fibre. A vector-valued differential form can be defined in two ways. Firstly we can define a vector-valued k-form as a map $\omega : \bigotimes^k TM \to V$. A more general definition is based on sections of a corresponding vector bundle:

$$\Omega^k(M, E) = \Gamma(E \otimes \Lambda^k T^*M) \tag{26.64}$$

Formula 26.4.27 (Wedge product). Let $\omega \in \Omega^k(M, E_1)$ and $\nu \in \Omega^p(M, E_2)$. The wedge product of these differential forms is defined as:

$$\omega \wedge \nu(v_1, ..., v_{k+p}) = \frac{1}{(k+p)!} \sum_{\sigma \in S_{k+p}} \mathrm{sgn}(\sigma) \omega(v_{\sigma(1)}, ..., v_{\sigma(k)}) \otimes \nu(v_{\sigma(k+1)}, ..., v_{\sigma(p)}) \tag{26.65}$$

This is a direct generalization of the formula for the wedge product of ordinary differential forms where we replaced the (scalar) product (product in the algebra $\mathbb{R}$) by the tensor product (product in the tensor algebra). It should be noted that the result of this operation is not an element of any of the original bundles E_1 or E_2 but of the product bundle $E_1 \otimes E_2$.

Definition 26.4.28 (Lie-algebra-valued differential form). A vector-valued differential form where the vector space V is equipped with a Lie algebra structure.

Formula 26.4.29 (Wedge product). Let $\omega \in \Omega^k(M, \mathfrak{g})$ and $\nu \in \Omega^p(M, \mathfrak{g})$. The wedge product of these differential forms is defined as:

$$[\omega \wedge \nu](v_1, ..., v_{k+p}) = \frac{1}{(k+p)!} \sum_{\sigma \in S_{k+p}} \mathrm{sgn}(\sigma) [\omega(v_{\sigma(1)}, ..., v_{\sigma(k)}), \nu(v_{\sigma(k+1)}, ..., v_{\sigma(p)})] \tag{26.66}$$

where $[\cdot, \cdot]$ is the Lie bracket in $\mathfrak{g}$.

[25] The original theorem states that on a contractible space (see definition 5.6.5) every closed form is exact.

26.5 Principal bundles

Definition 26.5.1 (Principal bundle). A principal bundle is a fibre bundle (E, B, π, G, F) together with a right action $\rho : P \times G \to P$ that satisfies two properties:

- Free[26]: This implies that the orbits are isomorphic to the structure group.
- Fibrewise transitivity[27]: This action preserves fibres, i.e. $y \cdot g \in F_b$ for all $y \in F_b, g \in G$ which implies that the fibres over B are exactly the orbits of ρ.

Together these properties imply that the typical fibre F and structure group G can be identified.

Remark 26.5.2. A corrolary of this definition is that the bundle $E \xrightarrow{\pi} M$ is isomorphic to the bundle $E \xrightarrow{\rho} E/G$ where E/G denotes the orbit space of E with respect to the G-action and ρ is the projection onto an equivalence class in the orbit space. (This property could have been used as part of the definition instead of the fibrewise transitivity.)

Remark 26.5.3. We remark that although the fibres are homeomorphic to G, they do not carry a group structure due to the lack of a distinct identity element. This turns them into **G-torsors**. However it is possible to locally (i.e. in a neighbourhood of a point $p \in M$), but not globally, endow the fibres with a group structure by choosing an element of every fibre to be identity element.

Property 26.5.4. The dimension of P is given by:

$$\dim P = \dim M + \dim G \tag{26.67}$$

Property 26.5.5. Every local trivialization φ_i is G-equivariant:

$$\varphi_i(z \cdot g) = \varphi_i(z) \cdot g \tag{26.68}$$

Definition 26.5.6 (Principal bundle map). A bundle map $F : P_1 \to P_2$ between principal G-bundles is a pair of smooth maps (f_B, f_P) such that:

1. (f_B, f_P) is a bundle map in the sense of fibre bundles.
2. f_P is G-equivariant[28].

The map f_P is said to **cover** f_B.

26.5.1 Associated bundles

Construction 26.5.7 (Associated principal bundle). For every fibre bundle we can construct an associated principal G-bundle by replacing the fibre F by G itself.

[26]See definition 3.1.40.
[27]See definition 3.1.42.
[28]See definition 3.1.46.

Property 26.5.8. A fibre bundle ξ is trivial if and only if the associated principal bundle is trivial. More generally, two fibre bundles are isomorphic if and only if their associated principal bundles are isomorphic.

Example 26.5.9 (Frame bundle). Let V be an n-dimensional vector space. Denote the set of ordered bases (or **frames**) of V by $F(V)$. It follows from the fact that every basis transformation is given by the action of an element of the general linear group that $F(V)$ is isomorphic to $GL(V) \cong GL(\mathbb{R}^n)$.

Given a vector bundle E we can thus construct a principal bundle associated to the vector bundle E by replacing every fibre $\pi^{-1}(b)$ by $F(\pi^{-1}(b)) \cong \mathrm{GL}(\mathbb{R}^n)$. The right action on this bundle by $g \in GL(\mathbb{R}^n)$ is given by the basis transformation $\widetilde{e}_j = g^i_j e_i$.

Construction 26.5.10 (Associated bundle to a principal bundle). Consider a principal bundle $G \hookrightarrow P \to M$ and let F be a smooth manifold equipped with a left G-action $\triangleright$. One can then construct an associated bundle $P_F \equiv P \times_\triangleright F$ in the following way:

1. Define an equivalence relation $\sim_G$ on the product manifold $P \times F$ given by:
$$(p, f) \sim_G (p', f') \iff \exists g \in G : (p', f') = (p \cdot g, g^{-1} \triangleright f) \tag{26.69}$$
2. The total space of the associated bundle is then given by the following quotient manifold:
$$P_F := (P \times F)/\sim_G \tag{26.70}$$
3. The projection map $\pi_F : P_F \to M$ is defined as
$$\pi_F : [p, f] \mapsto \pi(p) \tag{26.71}$$
where $[p, f]$ is the equivalence class of $(p, f) \in P \times F$ in the quotient manifold P_F.

Example 26.5.11 (Tangent bundle). Starting from the frame bundle $F(M)$ over a manifold M one can reconstruct (up to a bundle isomorphism) the tangent bundle TM in the following way:

Consider the left G-action $\triangleright$ given by:
$$\triangleright\colon G \times \mathbb{R}^n \to \mathbb{R}^n : (g \triangleright f)^i = (g^{-1})^i_j f^j \tag{26.72}$$
The tangent bundle is bundle isomorphic to the associated bundle $LM \times_\triangleright \mathbb{R}^n$ where the bundle map is defined as $[e, v] \mapsto v^i e_i \in TM$.

Construction 26.5.12 (Associated bundle map). Given principal bundle map, denoted by (f_P, f_B), between two principal bundles one can construct an associated bundle map between any two of their associated bundles with the same typical fibre in the following way:

1. The total space map $\widetilde{f}_P : P \times_G F \to P \times_{G'} F$ is given by:
$$\widetilde{f}_P([p, f]) = [f_P(p), f] \tag{26.73}$$
2. the base space map is simply given by f_B itself:
$$\widetilde{f}_B(b) = f_B(b) \tag{26.74}$$

26.5.2 Sections

Where every vector bundle has at least one global section, the **zero section**[29], a general principal bundle does not necessarily have a global section. This is made clear by the following property:

Property 26.5.13. A principal G-bundle P is trivial if and only if there exists a global section on P. Furthermore, there exists a bijection between the set of all global sections $\Gamma(P)$ and the set of trivializations $\text{Triv}(P)$.

26.5.3 Reduction of the structure group

Construction 26.5.14. Consider a fibre bundle $\mathcal{F} = (E, B, \pi, F, G)$. Let H be a subgroup of G. If there exists a fibre bundle with structure group H equivalent to $\mathcal{F}$ then we say that the structure group G can be reduced to H.

Property 26.5.15. An n-dimensional manifold is orientable if and only if the structure group $GL(\mathbb{R}^n)$ of its frame bundle $F(M)$ is reducible to $GL^+(\mathbb{R}^n)$, i.e. the group of invertible matrices with positive determinant.

Definition 26.5.16 (G-structure). Consider a manifold M. A G-structure on M is the reduction of the structure group $GL(\mathbb{R}^n)$ of the frame bundle $F(M)$ to a subgroup $G \subset GL(\mathbb{R}^n)$.

Example 26.5.17. An $O(n)$-structure on M turns the manifold into a Riemannian manifold[30]. Because the cotangent bundle T^*M transforms[31] using the transpose inverse of the transition maps of the tangent bundle TM, which for maps in $O(n)$ is equal to the oringinal maps, these two bundles are equivalent. The isomorphism is given by the *musical isomorphisms*[32].

26.6 Connections

26.6.1 Vertical bundle

Because smooth fibre bundles (which include smooth principal G-bundles) are also smooth manifolds we can define the traditional notions for them, such as the tangent bundle. We use this to construct the notions of horizontal and vertical bundles:

Definition 26.6.1 (Vertical vector). Let $\pi : E \to B$ be a smooth fibre bundle. The subbundle $\ker(T\pi)$ of TE is called the vertical bundle of E. Fibrewise this gives us $V_x = T_x(E_{\pi(x)})$.

[29] This is the map $s : b \to \vec{\mathbf{0}}$ for all $b \in B$.
[30] See definition 28.1.3.
[31] See example 26.2.17.
[32] See definition 28.1.2.

For principal G-bundles we can use an equivalent definition:

Alternative Definition 26.6.2. Consider a smooth principal G-bundle $G \hookrightarrow P \xrightarrow{\pi} M$. We first construct a map ι_p for every element $p \in P$:

$$\iota_p : G \to P : g \mapsto p \cdot g \tag{26.75}$$

We then define a tangent vector $v \in T_pP$ to be vertical if it lies in the image of $T_e\iota_p$, i.e. $\mathrm{Vert}(T_pP) = \mathrm{im}(T_e\iota_p)$. This construction is supported by the exactness of following short sequence:

$$0 \to \mathfrak{g} \xrightarrow{T_e\iota_p} T_pP \xrightarrow{T_p\pi} T_xM \to 0 \tag{26.76}$$

Property 26.6.3 (Dimension). It follows from the second definition that the vertical vectors of a principal G-bundle are nothing but the pushforward of the Lie algebra $\mathfrak{g}$ under the right action of G on P. Furthermore, the exactness of the sequence implies that $T_e\iota_p : \mathfrak{g} \to \mathrm{Vert}(T_pP)$ is an isomorphism of vector spaces. In particular, it implies that

$$\dim \mathrm{Vert}(T_pP) = \dim \mathfrak{g} = \dim G \tag{26.77}$$

Definition 26.6.4 (Fundamental vector field). Consider a principle G-bundle. Let $A \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra corresponding to G. The vertical vector field $A^\# : P \to TP$ given by

$$A^\#(p) = \iota_{p,*}(A) \in \mathrm{Vert}(T_pP) \tag{26.78}$$

is called the fundamental vector field associated to A.

Property 26.6.5. The map $(\cdot)^\# : \mathfrak{g} \to \Gamma(TP)$ is a Lie algebra morphism:

$$[A, B]^\# = [A^\#, B^\#] \tag{26.79}$$

where the Lie bracket on the left is that in $\mathfrak{g}$ and the Lie bracket on the right is that in $\mathfrak{X}(M)$ given by 26.35.

Property 26.6.6. The vertical bundle satisfies the following G-equivariance condition:

$$R_{g,*}(\mathrm{Vert}(T_pP)) = \mathrm{Vert}(T_{pg}P) \tag{26.80}$$

By differentiating the equality

$$R_g \circ \iota_p = \iota_{pg} \circ \mathrm{ad}_{g^{-1}}$$

and using 25.3.2, 26.78 we obtain the following algebraic formulation of the G-equivariance condition:

$$R_{g,*}\left(A^\#(p)\right) = \left(\mathrm{Ad}_{g^{-1}}A\right)^\#(pg) \tag{26.81}$$

26.6.2 Horizontal bundle

Definition 26.6.7 (Connection). Consider a principal bundle P with structure group G. A connection on P is the selection of a subspace $\mathrm{Hor}(T_pP) \leq T_pP$ for every $p \in P$ such that:

- $\mathrm{Vert}(T_pP) \oplus \mathrm{Hor}(T_pP) = T_pP$
- The selection depends smoothly on p.[33]
- The subspace $\mathrm{Hor}(T_pP)$ is G-equivariant:

$$R_{g,*}(\mathrm{Hor}(T_pP)) = \mathrm{Hor}(T_{pg}P) \tag{26.82}$$

The elements of $\mathrm{Hor}(T_pP)$ are said to be **horizontal vectors** with respect to the connection.

Remark 26.6.8. Remark that the G-invariance condition for vertical bundles is an intrinsic property while we have to require it by definition for the horizontal bundle.

Definition 26.6.9 (Horizontal bundle). The horizontal bundle $\mathrm{Hor}(TP)$ is defined as $\bigcup_{p\in P} \mathrm{Hor}(T_pP)$. It is a subbundle of TP. The G-invariance condition then implies that this subbundle is invariant under (the pushforward of) the right action of G.

Property 26.6.10 (Dimension). Properties 26.67, 26.77 and the direct sum decomposition of T_pP imply the following relation:

$$\dim \mathrm{Hor}(T_pP) = \dim M \tag{26.83}$$

Property 26.6.11. We take some time to summarize all dimensional relations between the components of a principal G-bundle over a base manifold M:

$$\boxed{\begin{aligned} \dim P &= \dim M + \dim G && (26.84)\\ \dim M &= \dim \mathrm{Hor}(T_pP) && (26.85)\\ \dim G &= \dim \mathrm{Vert}(T_pP) && (26.86) \end{aligned}}$$

for all $p \in P$.

Definition 26.6.12 (Dual connection). First we define the dual of the horizontal bundle:

$$\mathrm{Hor}(T_p^*P) = \{h^* \in T_p^*P | h^*(v) = 0, v \in \mathrm{Vert}(T_pP)\} \tag{26.87}$$

Equivalently, the horizontal covector bundle is defined as the set of linear functionals that annihilate vertical vectors. Just as with the vertical bundle this structure is independent of any connection on P.

A dual connection can then be defined as the selection of a vertical covector bundle $\mathrm{Vert}(T_p^*P)$ satisfying the conditions of definition 26.6.7 (where Vert and Hor should be interchanged).

[33] See the definiton of a (smooth) distribution 26.3.18.

26.6.3 Ehresmann connection

Definition 26.6.13 (Ehresmann connection). Let (P, M, π, G) be a principal bundle. An Ehresmann connection is a $\mathfrak{g}$-valued 1-form $\omega : TP \to \mathfrak{g}$ that satisfies the following 2 conditions:

1. G-equivariance:
$$\omega \circ R_{g,*} = \mathrm{Ad}_{g^{-1}} \circ \omega \tag{26.88}$$
2. Annihilation of fundamental vector fields:
$$\omega(A^{\#}) = A \tag{26.89}$$

The horizontal subspaces are then defined as $\mathrm{Hor}(T_pP) = \ker \omega|_p$.

Property 26.6.14. Consider two principal G-bundles ξ_1 and ξ_2. Let ω be an Ehresmann connection on ξ_1 and let $F : \xi_1 \to \xi_2$ be a bundle map. The map $F^*\omega$ defines an Ehresmann connection on ξ_2.

Example 26.6.15. Consider a principal G-bundle. An Ehresmann connection on this bundle is given by the following map:
$$\omega = (T_e\iota_p)^{-1} \circ \mathrm{pr}_V \tag{26.90}$$
where pr_V is the projection $TP \to \mathrm{Vert}(TP)$ associated to the decomposition from definition 26.6.7.

Definition 26.6.16 (Horizontal and vertical forms). Let ω be an Ehresmann connection on a principal bundle $G \hookrightarrow P \to M$. Let $\theta \in \Omega^k(P)$ be a k-form. We define following notions:

- θ is said to be horizontal if
$$\theta(v_1, ..., v_k) = 0 \tag{26.91}$$
whenever at least 1 of the v_i lies in $\mathrm{Vert}(T_pP)$.
- θ is said to be vertical if
$$\theta(v_1, ..., v_k) = 0 \tag{26.92}$$
whenever at least 1 of the v_i lies in $\mathrm{Hor}(T_pP)$.

For functions $f \in \Omega^0(P)$ it is vacuously true that they are both vertical and horizontal.

26.6.4 Maurer-Cartan form

Definition 26.6.17 (Maurer-Cartan form). For every $g \in G$ we have that the tangent space T_gG is isomorphic to $T_eG = \mathfrak{g}$. The isomorphism $T_gG \to \mathfrak{g}$ is given by the Maurer-Cartan form:
$$\boxed{\Omega := L_{g^{-1},*}} \tag{26.93}$$

Definition 26.6.18. Consider a manifold $M = \{x\}$. When constructing a principal G-bundle over M we see that the total space $P = \{x\} \times G$ can be identified with the structure group G. From the relations in property 26.6.11 we see that the horizontal spaces are null-spaces (this defines a smooth distribution and thus a connection according to 26.6.7) and that the vertical spaces are equal to the tangent spaces, i.e. $\text{Vert}(T_gG) = T_gG$ (where we used the association $P \cong G$).

The simplest way to define a connection form ω on this bundle would be the trivial projection $TP \to \text{Vert}(TP) = \mathbb{1}_{TP}$. The image of this map would however be T_gG and not $\mathfrak{g}$ as required.
def This can be solved by using the Maurer-Cartan form $\Omega : T_gG \to \mathfrak{g}$, i.e. we define $\omega(v) = \Omega(v)$.

Property 26.6.19. The Maurer-Cartan form is the unique Ehresmann connection on the bundle $G \hookrightarrow G \to \{x\}$.

26.6.5 Horizontal lifts and parallel transport

Property 26.6.20. Consider a principal G-bundle $G \hookrightarrow P \to M$ and a curve $\gamma : [0,1] \to M$. Let $p_0 \in \pi^{-1}(\gamma(0))$. There exists a unique curve $\widetilde{\gamma}_{p_0} : [0,1] \to P$ satisfying the following conditions:

- $\widetilde{\gamma}_{p_0}(0) = p_0$
- $\pi \circ \widetilde{\gamma}_{p_0} = \gamma$
- $\widetilde{\gamma}'_{p_0}(t) \in \text{Hor}(TP)$ for all $t \in [0,1]$

The curve $\widetilde{\gamma}_{p_0}$ is said to be the horizontal lift of γ starting at p_0. When it is clear from the context what the basepoint p_0 is, the subscript is often ommited and we write $\widetilde{\gamma}$ instead of $\widetilde{\gamma}_{p_0}$.

Remark 26.6.21 (Horizontal curve). Curves satisfying the last condition are said to be horizontal.

Definition 26.6.22 (Parallel transport on principal bundles). The parallel transport map with respect to the curve γ is defined as follows:

$$\text{Par}_t^\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(t)) : p_0 \mapsto \widetilde{\gamma}_{p_0}(t) \tag{26.94}$$

This map is G-equivariant and it is a diffeomorphism of fibres.

Formula 26.6.23. Consider a principal bundle $G \hookrightarrow P \to M$. Let $\gamma(t)$ be a curve in M and let ω be an Ehresmann connection on this bundle. The horizontal lift of $\gamma(t)$ can locally be parametrized as $\widetilde{\gamma}(t) = (\gamma(t), g(t))$ where $g(t)$ is some unique curve in G. To determine $\widetilde{\gamma}(t)$ it is thus sufficient to find $g(t)$. The following parametrization uniquely characterizes $g(t)$:

$$g'(t) = -\omega(\gamma(t), e, \gamma'(t), 0)g(t) \tag{26.95}$$

Using the trivial section $s : U \to U \times G : x \mapsto (x, e)$ where U is an open subset of M we can rewrite this formula as follows: First we consider the action of the pullback $s^*\omega$ on the

derivative $\gamma_* : \mathbb{R} \times \mathbb{R} \to TM : (t, 1) \mapsto (\gamma(t), \gamma'(t))$. Using the fact that it is linear in the second argument we can write

$$s^*\omega(\gamma(t), \gamma'(t)) = A(\gamma(t))\gamma'(t)$$

where $A : M \to \mathrm{Hom}(\mathbb{R}^{\dim M}, \mathfrak{g})$ gives a linear map for each point $\gamma(t) \in M$. The action can also be rewritten using the relation $f^*\omega = \omega \circ f_*$ as

$$s^*\omega(\gamma(t), \gamma'(t)) = \omega\Big(s_*(\gamma(t), \gamma'(t))\Big) = \omega(\gamma(t), e, \gamma'(t), 0)$$

Combining these relations with the ODE for $g(t)$ gives

$$\left(\frac{d}{dt} + A(\gamma(t))\gamma'(t)\right) g(t) = 0 \tag{26.96}$$

where $\frac{d}{dt}$ is a matrix given by the scalar multiplication of the derivative $\frac{d}{dt}$ and the unit matrix I.

Definition 26.6.24 (Holonomy group). Consider a principal bundle $G \hookrightarrow P \to M$. Let $\Omega^{ps}_m M \subset \Omega M$ be the loop space with basepoint $m \in M$ of piecewise smooth loops. The holonomy group $\mathrm{Hol}_p(\omega)$ based at $p \in \pi^{-1}(m) \subset P$ with respect to the connection form ω is given by:

$$\mathrm{Hol}_p(\omega) = \{g \in G : p \text{ and } p \cdot g \text{ can be connected by a } \widetilde{\gamma}, \gamma \in \Omega^{ps}_m M\} \tag{26.97}$$

Definition 26.6.25 (Reduced holoomy group). The reduced holonomy group $\mathrm{Hol}^0_p(\omega)$ is defined as the subset of $\mathrm{Hol}_p(\omega)$ using only contractible loops.

26.6.6 Koszul connections and covariant derivatives

Definition 26.6.26 (Parallel transport on vector bundles). Consider a principal bundle $G \hookrightarrow P \to M$ where we explicitly require P to be trivial, i.e. $P = M \times G$. Suppose that the Lie group G acts on a vector space V by a representation $\rho : G \to GL_m$. We can then construct an associated vector bundle $\pi_1 : M \times V \to M$.

Parallel transport on this vector bundle is then defined as follows. Let $\gamma(t)$ be a curve in M such that $\gamma(0) = x_0$ and $x_1 = \gamma(1)$. Furthemore, let the horizontal lift $\widetilde{\gamma}(t)$ have $\widetilde{\gamma}(0) = (x_0, h)$ as initial condition. The parallel transport of the point $(x_0, v_0) \in M \times V$ along γ is given by the following map:

$$\mathrm{Par}^\gamma_t : \pi_1^{-1}(x_0) \to \pi_1^{-1}(\gamma(t)) : (x_0, v_0) \mapsto \big(\gamma(t), \rho\big(g(t)h^{-1}\big)v_0\big) \tag{26.98}$$

It should be noted that this map is independent of the initial element $h \in G$. Furthermore, Par^γ_t is an isomorphism of vector spaces and can thus be used to identify distant fibers (as long as they lie in the same path-component).

Remark 26.6.27. Two remarks have to be made. First of all, although the previous construction explicitly used trivial bundles, it is also valid for general non-trivial vector bundles. Secondly, for every vector bundle one can construct the frame bundle and use the parallel transport map on this bundle to define parallel transport of vectors. Hence the previous construction is valid for any vector bundle.

Definition 26.6.28 (Covariant derivative). Consider a vector bundle with model fibre space V and its associated principal $GL(V)$-bundle with Ehresmann connection ω, both over a base manifold M. Let $\sigma : M \to E$ be a section of the vector bundle and let X be a vector field on M. The covariant derivative of σ with respect to X is defined as:

$$\boxed{\nabla_X \sigma(x_0) = \lim_{t \to +\infty} \frac{(\mathrm{Par}_t^\gamma)^{-1}\sigma(\gamma(t)) - \sigma(x_0)}{t}} \tag{26.99}$$

where $\gamma(t)$ is any curve such that $\gamma(0) = x_0$ and $\gamma'(0) = X(x_0)$.

Property 26.6.29. Let $\pi : E \to M$ be a vector bundle. Let σ, X and f be respectively a section on E, a vector field on M and a C^∞ function on M. The covariant derivative along X satisfies following properties[34]:

- $\nabla_X \sigma$ is a smooth section on E.
- The map $(X, \sigma) \mapsto \nabla_X \sigma$ is bilinear over $\mathbb{R}$.
- $\nabla_{(fX)}\sigma = f \nabla_X \sigma$
- $\nabla_X (f\sigma) = f \nabla_X \sigma + X(f)\sigma$

Remark 26.6.30. The last two properties show the major difference between the Lie derivative and the covariant derivative when σ is a section of the tangent bundle, i.e. a vector field. Lie derivatives depend on the local behaviour of both X and σ. The covariant derivative on the other hand only depends on the value of X at $p \in M$ and on the local behaviour of σ.

Definition 26.6.31 (Koszul connection). The map

$$\Gamma(TM) \times \Gamma(E) \to \Gamma(E) : (X, \sigma) \mapsto \nabla_X \sigma \tag{26.100}$$

is called a Koszul connection if the above properties hold. From the above constructions it also follows that every Ehresmann connection on a principal bundle induces a Koszul connection on all of its associated vector bundles.

Definition 26.6.32 (Exterior covariant derivative). Consider a principal bundle $G \hookrightarrow P \to M$ equipped with an Ehresmann connection ω. Let $\theta \in \Omega^k(P)$ be a differential k-form. The exterior covariant derivative $D\theta$ is defined as follows:

$$D\theta(v_0, ..., v_k) = d\theta(v_0^H, ..., v_k^H) \tag{26.101}$$

[34]The last property, the Leibniz property, turns the covariant derivative into a derivation of the space of sections.

where d is the exterior derivative 26.4.7 and v_i^H is the projection of v_i on the horizontal subspace $\mathrm{Hor}(T_pP)$ associated to the Ehresmann connection ω. From the definition it follows that the covariant derivative $D\theta$ is a horizontal form[35].

Remark 26.6.33. The exterior covariant derivative can also be defined for general W-valued k-forms where W is a vector space. This can be done by defining it component-wise with respect to a given basis on W. Afterwards one can prove that the choice of basis plays no role.

Formula 26.6.34. Using the Koszul connection on the tangent bundle TP we can rewrite the action of the exterior covariant derivative as follows:

$$D\theta(v_0,...,v_k) = \sum_i^k (-1)^i \nabla_{v_i}\theta(v_0,...,\hat{v}_i,...,v_k) + \sum_{i<j}^k (-1)^{i+j}\theta([v_i,v_j],v_0,...,\hat{v}_i,...,\hat{v}_j,...,v_k) \tag{26.102}$$

where $\hat{v}_i$ means that this vector is omitted. As an example we explicitly give the formula for a 1-form Φ:

$$D\Phi(X,Y) = \nabla_X(\Phi(Y)) - \nabla_Y(\Phi(X)) - \Phi([X,Y]) \tag{26.103}$$

which should remind the reader of the analogous formula for the ordinary exterior derivative 26.52.

26.6.7 Curvature of a connection

Definition 26.6.35 (Curvature). Let ω be an Ehresmann connection on a principal bundle $G \hookrightarrow P \to M$. The curvature Ω of ω is defined as the exterior covariant derivative $D\omega$.

Definition 26.6.36 (Flat connection). An Ehresmann connection ω is said to be flat if its curvature Ω vanishes everywhere.

Property 26.6.37 (Bianchi's identity). Let ω be an Ehresmann connection with curvature Ω.

$$\boxed{D\Omega = 0} \tag{26.104}$$

Remark 26.6.38. One should however pay attention not to generalize this result to general differential forms. Only the exterior derivative satisfies the coboundary condition $d^2 \equiv 0$, the exterior covariant derivative does not.

Formula 26.6.39 (Cartan structure equation). Let ω be an Ehresmann connection and let Ω be its curvature form.

$$\boxed{\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]} \tag{26.105}$$

The following property is an immediate consequence of the Frobenius integrability theorem 26.3.21 and the fact that an Ehresmann connection vanishes on the horizontal subbundle.

[35] See definition 26.6.16.

Property 26.6.40. Let ω be an Ehresmann connection. The associated horizontal distribution[36]

$$p \mapsto \mathrm{Hor}(T_pP)$$

is integrable if and only if the connection ω is flat. Furthermore, the vertical distribution is always integrable.

Example 26.6.41. Let ω_G be the Maurer-Cartan form on a Lie group G. It follows from the fact that the only horizontal vector on the bundle $G \hookrightarrow G \to \{x\}$ is the zero vector, that the curvature of ω_G is 0.

[36]See 26.3.18 for the definition of a distribution of vector spaces.

Chapter 27

Integration on manifolds

27.1 Orientation

Definition 27.1.1 (Orientation). Similar to definition 20.4.18 we can define an orientation on a differentiable manifold M. First we modify the definition of the volume element a little bit. A **volume form** on M is a nowhere-vanishing top-dimensional form $\mathrm{Vol} \in \Omega^n(M)$ where $n = \dim(M)$. The definition of an orientation is then equivalent to that in 20.4.18.

An **oriented atlas** is given by all charts of M for which the pullback of the Euclidean volume form is a positive multiple of Vol. This also means that the transition functions have a positive Jacobian determinant[1]. The existence of a volume form turns a differentiable manifold into an **orientable manifold**.

An orientable manifold with volume form ω is said to be **positively oriented** if $\omega(v_1, ..., v_n) > 0$ where $(v_1, ..., v_n)$ is a basis for T_pM.

Example 27.1.2. Let $M = \mathbb{R}^n$. The canonical Euclidean volume form is given by the determinant map

$$\det : (u_1, ..., u_n) \mapsto \det(u_1, ..., u_n) \tag{27.1}$$

where the u_n's are expressed in the canonical basis $(e_1, ..., e_n)$. The name 'volume form' is justified by noting that the determinant map gives the signed volume of the n-dimensional parallelotope spanned by the vectors $\{u_1, ..., u_n\}$.

Property 27.1.3. Let ω_1, ω_2 be two volume forms on M. Then there exists a smooth function f such that

$$\omega_1 = f\omega_2$$

Furthermore, the sign of this function is constant on every connected component of M.

[1] This is in fact an equivalent definition.

27.2 Integration of top-dimensional forms

Formula 27.2.1. Let θ be a top-dimensional form on M with compact support. Let $\{\varphi_i\}_i$ be a partition of unity[2] subordinate to an atlas $\{(U_i, \varphi_i)\}_i$.

$$\int_M \theta = \sum_i \int_{U_i} \varphi_i \theta dx_1 ... dx_n \tag{27.2}$$

27.3 Stokes' theorem

Theorem 27.3.1 (Stokes' theorem). *Let Σ be an orientable smooth manifold. Denote the boundary of Σ by $\partial\Sigma$. Let ω be a differential k-form on Σ. We have the following equality:*

$$\boxed{\int_{\partial\Sigma} \omega = \int_\Sigma d\omega} \tag{27.3}$$

Corollary 27.3.2. The Kelvin-Stokes theorem 17.22, the divergence theorem 17.23 and Green's identity 17.24 are immediate results of this (generalized) Stokes' theorem.

27.4 de Rham Cohomology

Now we can also give a little side note about why the de Rham cohomology groups 26.62 really form a cohomology theory. For this we need some concepts from homology which can be found in section 5.7. Let M be a compact differentiable manifold and let $\{\lambda_i : \Delta^k \to M\}$ be the set of singular k-simplexes on M.

Now suppose that we want to integrate over a singular k-chain C on M, i.e. $C = \sum_{i=0}^k a_i \lambda_i$. Formula 27.2 says that we can pair the k-form ω and the chain C such that they act as duals to each other (hence p-forms are also called p-**cochains**), producing a real number[3]:

$$\langle \cdot, C \rangle : \Omega^n(M) \to \mathbb{R} : \omega \mapsto \int_C \omega = \sum_{i=0}^k a_i \int_{\Delta_k} \lambda_i^* \omega \tag{27.4}$$

where λ_i^* pulls back ω to Δ^k which is a subset of $\mathbb{R}^k$ as required. Now Stokes' theorem 27.3 tells us that

$$\int_C d\omega = \int_{\partial C} \omega \tag{27.5}$$

[2] See definition 5.5.16.

[3] This requires the chain group to have real coefficients instead of integer coefficients as is mostly used in homology.

Using the paring $\langle \cdot, \cdot \rangle$ this becomes

$$\langle d\omega, C \rangle = \langle \omega, \partial C \rangle \tag{27.6}$$

The operators d and ∂ can thus be interpreted as formal adjoints. After checking (again using Stokes' theorem) that all chains C and cochains ω belonging to the same equivalence classes $[C] \in H_k(M, \mathbb{R})$ and $[\omega] \in H^k(M, \mathbb{R})$ give rise to the same number[4] $\langle \omega, C \rangle$ we see that the singular homology groups and the de Rham cohomology groups on M are well defined dual groups. The name cohomology is thus wel chosen for 26.62.

[4]Suppose that $A, B \in [C]$ and $\phi, \chi \in [\omega]$ then $\langle \phi, A \rangle = \langle \chi, B \rangle$.

Chapter 28

Riemannian Geometry

28.1 Riemannian manifolds

28.1.1 Metric

Definition 28.1.1 (Bundle metric). Consider the bundle of second order covariant vectors. Following from 20.1.1 every section g of this bundle gives a bilinear map

$$g_x : T_xM \times T_xM \to \mathbb{R}$$

for all $x \in M$. If this map is symmetric and non-degenerate and if it depends smoothly on p it is called a **(Lorentzian)** metric.[1]

The maps $\{g_x\}_{x\in M}$ can be 'glued' together to form a global metric g, defined on the fibre product[2] $TM \diamond TM$. Defining this map on $TM \times TM$ is not possible as tangent vectors belonging to different points in M cannot be 'compared'. The collection $\{\langle\cdot|\cdot\rangle_x | x \in M\}$ is called a **bundle metric**.

A Riemannian metric also induces a duality between TM and T^*M. This is given by the *flat* and *sharp* isomorphisms:

Definition 28.1.2 (Musical isomorphisms). Let $g : TM \times TM \to \mathbb{R}$ be the Riemannian metric on M. The **flat** isomorphism is defined as:

$$\flat : v \mapsto g(v, \cdot) \tag{28.1}$$

The **sharp** isomorphism is defined as the inverse map:

$$\sharp : g(v, \cdot) \mapsto v \tag{28.2}$$

These 'musical' isomorphisms can be used to lower and raise tensor indices.

[1] See also the section about Hermitian forms and metric forms 16.4.
[2] See definition 26.8.

28.1.2 Riemannian manifold

Definition 28.1.3 (Pseudo-Riemannian manifold). Let M be a smooth manifold. This manifold is called pseudo-Riemannian if it is equipped with a pseudo-Riemannian metric. A **Riemannian manifold** is similarly defined.

Definition 28.1.4 (Riemannian isometry). Let (M, g_M) and (N, g_N) be two Riemannian manifolds. An isometry 23.2.10 $f : M \to N$ is said to be Riemannian if $F^* g_N = g_M$.

Property 28.1.5. Let M be a pseudo-Riemannian manifold. For every $p \in M$ there exists a splitting $T_pM = P \oplus N$ where P is a subspace on which the pseudometric is positive-definite and N is a subspace on which the pseudometriv is negative-definite. This splitting is however not unique, only the dimensions of the two subspaces are well-defined.

Due to the continuity of the pseudometric, the dimensions of this splitting wil be the same for points in the same neighbourhood. For connected manifolds this amounts to a global invariant:

Definition 28.1.6 (Index). Let M be a connected pseudo-Riemannian manifold. The dimension of the *negative* subspace N in the above splitting $T_pP = P \oplus N$ is called the index of the pseudo-Riemannian manifold.

Theorem 28.1.7 (Whitney's embedding theorem). *Every smooth paracompact*[3] *manifold M can be embedded in $\mathbb{R}^{2 \dim M}$.*

Theorem 28.1.8 (Whitney's immersion theorem). *Every smooth paracompact manifold M can be immersed in $\mathbb{R}^{2 \dim M - 1}$.*

Theorem 28.1.9 (Immersion conjecture). *Every smooth paracompact manifold M can be immersed in $\mathbb{R}^{2 \dim M - a(\dim M)}$ where $a(n)$ is the number of 1's in the binary expansion of n.*

28.2 Sphere bundle

Definition 28.2.1 (Unit sphere bundle). Let V be a normed vector space. Consider a vector bundle $V \hookrightarrow E \to B$. From this bundle we can derive a new bundle where we replace the typical fibre V by the unit sphere $\{v \in V : ||v|| = 1\}$. It should be noted that this new bundle is not a vector bundle as the unit sphere is not a vector space.

Remark 28.2.2 (Unit disk bundle). A similar construction can be made by replacing the unit sphere by the unit disk $\{v \in V : ||v|| \leq 1\}$.

[3] See definition 5.5.14.

28.3 Hilbert bundles

Definition 28.3.1 (Hilbert bundle). A vector bundle for which the typical fibre is a Hilbert space is called a Hilbert bundle.

Definition 28.3.2 (Compatible Hilbert bundle). Consider the isomorphisms

$$l_x : F_x \to \mathcal{H} : h \mapsto \varphi_i(x, h) \in \pi(x) \tag{28.3}$$

where $\mathcal{H}$ is the typical fibre and where $\{(U_i, \varphi_i)\}_{i \in I}$ is a trivializing cover. These maps l_x are called **point-trivializing maps**.

Using these maps we can extend the metric structure of the typical fibre $\mathcal{H}$ to the fibres F_x for all x by:

$$\langle v|w\rangle_x = \langle l_x(v)|l_x(w)\rangle_{\mathcal{H}} \tag{28.4}$$

The Hilbert bundle is said to be compatible (with the metric structure on $\mathcal{H}$) if the above extension is valid for all $v, w \in F_x$.

Remark. For compatible Hilbert bundles, the transition maps $l_{x\to y} = l_y^{-1} \circ l_x : \pi^{-1}(x) \to \pi^{-1}(y)$ are easily seen to be isometries.

Chapter 29

Complex bundles

29.1 Almost complex structure

Definition 29.1.1 (Almost complex structure). Let M be a real manifold. An almost complex strcuture on M is a smooth $(1,1)$-tensor field $J : TM \to TM$ such that $J|_p : T_pM \to T_pM$ satisifes $J|_p^2 = -1$ for all $p \in M$. An **almost complex manifold** is a real manifold equipped with an almost complex structure.

Property 29.1.2. An almost complex manifold is even-dimensional and orientable.

Property 29.1.3. A manifold M admits an almost complex structure if and only if the structure group of the tangent bundle TM can be reduced from $GL(\mathbb{R}^{2n})$ to $GL(\mathbb{C}^n)$.

Chapter 30

Symplectic Topology

30.1 Symplectic manifolds

Definition 30.1.1 (Symplectic form). Let $\omega \in \Omega^2(M)$ be a differential 2-form. ω is said to be a symplectic form if it satisfies following properties:

- Closed: $d\omega = 0$
- Non-degeneracy: if $\omega(u, v) = 0, \forall u \in TM$ then $v = 0$

Definition 30.1.2 (Symplectic manifold). A manifold M equipped with a symplectic 2-form ω is called a symplectic manifold. This structure is often denoted as a pair (M, ω).

Property 30.1.3. From the antisymmetry (valid for all differential k-forms) and the non-degeneracy of the symplectic form, it follows that M is even dimensional.

Theorem 30.1.4 (Darboux). *Let (M, ω) be a symplectic manifold. For every neighbourhood Ω in T^*M there exists a fibered chart (x^i, y^i) such that*

$$\omega|_\Omega = \sum_i dx^i \wedge dy^i \tag{30.1}$$

The charts in Darboux's theorem are called **Darboux charts** and they form a cover for M.

30.2 Lagrangian submanifolds

Definition 30.2.1 (Symplectic complement). Let (M, ω) be a symplectic manifold and let $S \subset M$ be an embedded submanifold $\iota : S \hookrightarrow M$. The symplectic orthogonal complement $T_p^\perp S$ at the point $p \in S$ is defined as:

$$T_p^\perp S = \{v \in T_pM : \omega(v, \iota_* w) = 0, \forall w \in T_pS\} \tag{30.2}$$

Definition 30.2.2 (Isotropic submanifold). Let (M, ω) be a symplectic manifold. An embedded submanifold $\iota : S \hookrightarrow M$ is called isotropic if $T_pS \subset T_p^{\perp}S$.

Definition 30.2.3 (Isotropic submanifold). Let (M, ω) be a symplectic manifold. An embedded submanifold $\iota : S \hookrightarrow M$ is called co-isotropic if $T_p^{\perp}S \subset T_pS$.

Definition 30.2.4 (Larangian submanifold). Let (M, ω) be a symplectic manifold. An embedded submanifold $\iota : S \hookrightarrow M$ is called Lagrangian if $T_pS = T_p^{\perp}S$. Therefore they are sometimes called maximal isotropic submanifolds.

Chapter 31

K-theory

31.1 Basic definitions

Definition 31.1.1 (K-theory). Let X be a compact Hausdorff space and let $\mathrm{Vect}(X)$ be the set of isomorphism classes of finite-dimensional vector bundles over X. Because this set is well-behaved with respect to Whitney sums, the structure $(\mathrm{Vect}(X), \oplus)$ forms an Abelian monoid. The Grothendieck completion[1] of $(\mathrm{Vect}(X), \oplus)$ is called the K-theory of X.

Notation 31.1.2. The K-theory of a space X is denoted by $K^0(X)$.

[1] See definition 3.1.5.

Part VI

Probability Theory & Statistics

Chapter 32

Probability

32.1 Probability

Definition 32.1.1 (Axioms of probability).

- $P(E) \geq 0$
- $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$ if E_1 and E_2 are exclusive.
- $\sum_S P(E_i) = 1$ where the summation runs over all exclusive events.

Remark 32.1.2. The second axiom can be defined more generally by saying that the probability P should be σ-additive. Together with the first axiom and the consequence that $P(\emptyset) = 0$ means that the probability is a measure 10.1.1.

Definition 32.1.3 (Sample space). Let X be a random variable. The set of all possible outcomes of X is called the sample space. The sample space is often denoted by Ω.

Definition 32.1.4 (Probability space). Let (Ω, Σ, P) be a measure space[1]. This measure space is called a probability space if $P(X) = 1$. Furthermore, the measure P is called a probability measure or simply probability.

Definition 32.1.5 (Event). Let (Ω, Σ, P) be a probability space. An element S of the σ-algebra Σ is called an event.

Remark. From the definition of an event it is clear that a single possible outcome of a measurement can be a part of multiple events. So although only one outcome can occur at the same time, multiple event can occur simultaneously.

Remark. When working with measure-theoretic probability spaces it is more convenient to use the σ-algebra (see 2.4.2) of events instead of the power set (see 2.1.1) of all events. Intuitively this seems to mean that some possible outcomes are not treated as events. However

[1] See definition 10.1.2.

we can make sure that the σ-algebra still contains all 'useful' events by using a 'nice' definition of the used probability space. Further information concerning probability spaces can be found in chapter 10.

Formula 32.1.6. Let A, B be two events.

$$P(A \cup B) = P(A) + P(B) + P(A \cap B) \tag{32.1}$$

Definition 32.1.7 (Disjoint events). Two events A and B are said to be disjoint if they cannot happen at the same time:

$$P(A \cap B) = 0 \tag{32.2}$$

Corollary 32.1.8. If A and B are disjoint, the probability that both A and B are true is just the sum of their individual propabilities.

Formula 32.1.9 (Complement). Let A be an event. The probability of A being false is denoted as $P\left(\overline{A}\right)$ and is given by:

$$P\left(\overline{A}\right) = 1 - P(A) \tag{32.3}$$

Corollary 32.1.10. From the previous equation and de Morgan's laws (equations 2.6 and 2.7) we derive the following formula[2]:

$$P\left(\overline{A} \cap \overline{B}\right) = 1 - P(A \cup B) \tag{32.4}$$

32.2 Conditional probability

Definition 32.2.1 (Conditional probability). Let A, B be two events. The probability of A given that B is true is denoted as $P(A|B)$.

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \tag{32.5}$$

Corollary 32.2.2. By interchanging A and B in previous equation and by remarking that this has no effect on the quantity $P(A \cap B)$ the following rsult can be deducued:

$$P(A|B)P(B) = P(B|A)P(A) \tag{32.6}$$

Theorem 32.2.3 (Bayes' theorem). *Let A, B be two events. From the conditional probability 32.5 it is possible to derive following important theorem:*

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}} \tag{32.7}$$

[2]Switching the union and intersection has no impact on the validity of the formula.

Definition 32.2.4 (Independent events). Let A, B be two events. A and B are said to be independent if they satisfy the following relation:

$$P(A \cap B) = P(A)P(B) \tag{32.8}$$

Corollary 32.2.5. If A and B are two independent events, then equation 32.7 simplifies to:

$$P(A|B) = P(A) \tag{32.9}$$

Property 32.2.6. The events $A_1, ..., A_n$ are independent if for all $k \leq n$ for each choice of k events the probability of their intersection is equal to the product of their indivudal propabilities.

Property 32.2.7. The σ-algebras $\mathcal{F}_1, ..., \mathcal{F}_n$ defined on probability space $(\Omega, \mathcal{F}, P)$ are independent if for all choices of distinct indices $i_1, ..., i_k$ from $\{1, ..., n\}$ and for all choices of sets $F_{i_n} \in \mathcal{F}_{i_n}$ the following equation holds:

$$P(F_{i_1} \cap ... \cap F_{i_k}) = P(F_{i_1})...P(F_{i_k}) \tag{32.10}$$

Formula 32.2.8. Let $(B_i)_{i \in \mathbb{N}}$ be a sequence of pairwise disjoint events. If $\bigcup_{i=1}^{+\infty} B_i = \Omega$ then the total probability of a given event A can be calculated as follows:

$$P(A) = \sum_{i=1}^{+\infty} P(A|B_i)P(B_i) \tag{32.11}$$

32.3 Random variables

Definition 32.3.1 (Random variable). Let (Ω, Σ, P) be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if $\forall a \in \mathbb{R} : X^{-1}\big([a, +\infty[\big) \in \Sigma$.[3]

Definition 32.3.2 (σ-algebra generated by a random variable). Let X be a random variable defined on a probability space (Ω, Σ, P). The following family of sets is a σ-algebra:

$$X^{-1}(\mathcal{B}) = \{S \in \Sigma : S = X^{-1}(B \in \mathcal{B})\} \tag{32.12}$$

This measure is called the probability distribution of X.

Notation 32.3.3. The σ-algebra generated by the random variable X is often denoted by $\mathcal{F}_X$, analogous to notation 2.4.7.

Theorem 32.3.4. *Let X, Y be two random variables. X and Y are independent if the σ-algebras generated by them are independent*[4].

[3] $X^{-1}\big([a, +\infty[\big) = \{\omega \in \Omega : X(\omega) \geq a\}$.

[4] See equation 32.10.

32.4 Probability distribution

Definition 32.4.1 (Probability distribution). Let X be a random variable defined on a probability space (Ω, Σ, P). The following function is a measure on the σ-algebra of Borel sets:

$$\boxed{P_X(B) = P(X^{-1}(B))} \tag{32.13}$$

Formula 32.4.2 (Change of variable). Let X be a random variable defined on a probability space (Ω, Σ, P).

$$\int_\Omega g(X(\omega))dP(\omega) = \int_{\mathbb{R}} g(x)dP_X(x) \tag{32.14}$$

Definition 32.4.3 (Density). Let f be a non-negative integrable function and recall theorem 10.2.20. The function f is called the **density** of P with respect to the Lebesgue measure m.

For P to be a probability, f should satisfy the following condition:

$$\int f dm = 1 \tag{32.15}$$

Definition 32.4.4 (Cumulative distribution function). Let f be a density. The c.d.f. corresponding to f is given by:

$$F(y) = \int_{-\infty}^{y} f(x)dx \tag{32.16}$$

Theorem 32.4.5 (Skorokhod's representation theorem). *Let $F : \mathbb{R} \to [0,1]$ be a function that satisfies following 3 properties:*

- *$F(x)$ is non-decreasing.*
- *$\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$*
- *$F(x)$ is right-continuous: $y \geq y_0, y \to y_0 \implies F(y) \to F(y_0)$.*

There exists a random variable $X : [0,1] \to \mathbb{R}$ defined on the probability space $([0,1], \mathcal{B}, m_{[0,1]})$ such that $F = F_X$.

Formula 32.4.6. Let the absolutely continuous probability P_X be defined on the product space $\mathbb{R}^n$. Let f_X be the density associated with P_X. Let $g : \mathbb{R}^n \to \mathbb{R}$ be integrable with respect to P_X.

$$\int_{\mathbb{R}^\ltimes} g(x)dP_X(x) = \int_{\mathbb{R}^n} f_X(x)g(x)dx \tag{32.17}$$

Corollary 32.4.7. Previous formula together with formula 32.14 gives rise to:

$$\int_\Omega g(X)dP = \int_{\mathbb{R}^\ltimes} f_X(x)g(x)dx \tag{32.18}$$

32.5 Moments

32.5.1 Expectation value

Definition 32.5.1 (Expectation value). Let X be random variable defined on a probability space (Ω, Σ, P).

$$\boxed{E(X) = \int_\Omega X dP} \tag{32.19}$$

Notation 32.5.2. Other often used notations are $\langle X \rangle$ and μ.

Definition 32.5.3 (Moment of order r). The moment of order r is defined as the expectation value of the r^{th} power of X and by equation 32.18 this becomes:

$$E(X^r) = \int x^r f_X(x) dx \tag{32.20}$$

Definition 32.5.4 (Central moment of order r).

$$E((X-\mu)^r) = \int (x-\mu)^r f_X(x) dx \tag{32.21}$$

Definition 32.5.5 (Variance). The central moment of order 2 is called the variance: $V(X) = E((X-\mu)^2)$.

Property 32.5.6. If $E(X^n)$ are finite for $n > 0$ then for all $k \leq n$, $E(X^k)$ are also finite. If $E(X^n)$ is infinite then for all $k \geq n$, $E(X^k)$ are also infinite.

Property 32.5.7. Moments of order n are determined by central moments of order $k \leq n$ and central moments of order n are determined by moments of order $k \leq n$.

Definition 32.5.8 (Moment generating function).

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tX} P(X) dX \tag{32.22}$$

Theorem 32.5.9. *If the above function exists we can derive the following useful result*[5] *by using the series expansion 7.15:*

$$E[X^n] = \left. \frac{d^n M_X(t)}{dt^n} \right|_{t=0} \tag{32.23}$$

Definition 32.5.10 (Characteristic function). Let X be a random variable. The characteristic function of X is defined as follows:

$$\varphi_X(t) = E(e^{itX}) \tag{32.24}$$

[5]This property is the reason why 32.22 is called the moment generating function.

Property 32.5.11. The characteristic function has the following properties:

- $\varphi_X(0) = 1$
- $|\varphi_X(t)| \leq 1$
- $\varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$

Formula 32.5.12. If $\varphi_X(t)$ is k times continuously differentiable then X has finite k^{th} moment and

$$E(X^k) = \frac{1}{i^k}\frac{d^k}{dt^k}\varphi_X(0) \tag{32.25}$$

Conversely, if X has finite k^{th} moment then $\varphi_X(t)$ is k times continuously differentiable and the above formula holds.

Formula 32.5.13 (Inversion formula). Let X be a random varibale. If the c.d.f. of X is continuous at $a, b \in \mathbb{R}$ then

$$F_X(b) - F_X(a) = \lim_{c\to+\infty} \frac{1}{2\pi}\int_{-c}^{c} \frac{e^{-ita} - e^{-itb}}{it}\varphi_X(t)dt \tag{32.26}$$

Formula 32.5.14. If $\varphi_X(t)$ is integrable then the c.d.f. is given by:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\varphi_X(t)dt \tag{32.27}$$

Remark 32.5.15. From previous formula it is clear that the density function and the characteristic function are Fourier transformed quantities.

32.5.2 Correlation

Theorem 32.5.16. *Let X, Y be two random variables. They are independent if and only if $E(f(X)g(Y)) = E(f(X))E(g(Y))$ holds for all Borel measurable[6] bounded functions f, g.*

The value $E(XY)$ is equal to the inner product $\langle X|Y\rangle$ as defined in 10.36. It follows that independence of random variables implies orthogonality. To generalize this concept, we introduce following notions.

Definition 32.5.17 (Centred random variable). Let X be a random variable with finite expectation value $E(X)$. The centred random variable X_c is defined as $X_c = X - E(X)$.

Definition 32.5.18 (Covariance). Let X, Y be two random variables. The covariance of X and Y is defined as follows:

$$\mathrm{cov}(X, Y) = \langle X_c|Y_c\rangle = E\big((X - E(X))(Y - E(Y))\big) \tag{32.28}$$

Some basic math gives:

$$\mathrm{cov}(X, Y) = E(XY) - E(X)E(Y) \tag{32.29}$$

[6] See definition 10.1.33.

Definition 32.5.19 (Correlation). Let X, Y be two random variables. The correlation is defined as the cosine of the angle between X_c and Y_c:

$$\rho_{XY} = \frac{\text{cov}(X,Y)}{||X||_2||Y||_2} \tag{32.30}$$

Corollary 32.5.20. From theorem 32.5.16 it follows that independent random variables are also uncorrelated.

Corollary 32.5.21. Uncorrelated X and Y satisfy the following equality: $E(XY) = E(X)E(Y)$.

Property 32.5.22. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables. Their variances satisfy the following equation:

$$V\left(\sum_{i=1}^{+\infty} X_i\right) = \sum_{i=1}^{+\infty} V(X_i) \tag{32.31}$$

32.5.3 Conditional expectation

Let (Ω, Σ, P) be a probability space. Let the random variable $X \in L^2(\Omega, \Sigma, P)$[7]. Consider the sub-$\sigma$-algebra $\mathcal{G} \subset \Sigma$, The spaces $L^2(\Sigma)$ and $L^2(\mathcal{G})$ are complete (see property 10.4.3). The projection theorem 18.2.22 can thus be applied, i.e. there exists for every X a random variable $Y \in L^2(\mathcal{G})$ such that $X - Y$ is orthogonal to $L^2(\mathcal{G})$. This has the following result:

$$\forall Z \in L^2(\mathcal{G}) : \langle X - Y | Z \rangle = \int_\Omega (X - Y) Z dP = 0 \tag{32.32}$$

And since $\mathbb{1}_G \in L^2(\mathcal{G})$ for every $G \in \mathcal{G}$ we find by applying 10.25:

$$\int_G X dP = \int_G Y dP \tag{32.33}$$

This leads us to introducing the following notion of conditional expectations:

Definition 32.5.23 (Conditional expectation). Let (Ω, Σ, P) be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of Σ. For every Σ-measurable random variable $X \in L^2(\Sigma)$ there exists a unique (up to a null set) random variable $Y \in L^2(\mathcal{G})$ that satisfies equation 32.33 for every $G \in \mathcal{G}$. This Y is called the conditional expectation of X given $\mathcal{G}$ and it is denoted by $Y = E(X|\mathcal{G})$:

$$\boxed{\int_G E(X|\mathcal{G}) dP = \int_G X dP} \tag{32.34}$$

Remark 32.5.24. Allthough our derivation was based on random variables from the L^2 class, it is also possible to construct (unique) conditional expectations for random variables from the L^1 class by using method 10.2.23.

[7] This vector space has the same interpretation as the one in section 10.4.2. The difference is that all sets are elements of Σ instead of $\mathcal{M}$, that the functions are Σ-measurable instead of $\mathcal{M}$-measurable and that the integral is calculated with respect to the measure P instead of the Lebesgue measure m.

32.6 Joint distributions

Definition 32.6.1 (Joint distribution). Let X, Y be two random variables defined on the same probability space (Ω, Σ, P). Consider the vector $(X, Y) : \Omega \to \mathbb{R}^2$. The distribution of (X, Y) is defined on the Borel sets of the plane $\mathbb{R}^2$ and it is given by the following measure:

$$P_{(X,Y)}(B) = P((X, Y) \in B) \tag{32.35}$$

Definition 32.6.2 (Joint density). If the probability measure from previous definition can be written as

$$P_{(X,Y)}(B) = \int_B f_{(X,Y)}(x, y) dm_2(x, y) \tag{32.36}$$

for some integrable $f_{(X,Y)}$ it is said that X and Y have a joint density.

Definition 32.6.3 (Marginal distribution). The distributions of one-dimensional random variables is determined by the joint distribution:

$$P_X(A) = P_{(X,Y)}(A \times \mathbb{R}) \tag{32.37}$$

$$P_Y(A) = P_{(X,Y)}(\mathbb{R} \times A) \tag{32.38}$$

where $A \subset \mathcal{B}$.

Corollary 32.6.4. If the joint density exists then the marginal distributions are absolutely continuous and given by

$$f_X(x) = \int_{\mathbb{R}} f_{(X,Y)}(x, y) dy \tag{32.39}$$

$$f_Y(y) = \int_{\mathbb{R}} f_{(X,Y)}(x, y) dx \tag{32.40}$$

The converse however is not always true. The one-dimensional densities can be absolutely continuous without the existence of the joint density.

32.6.1 Independence

Theorem 32.6.5. *Let X, Y be two random variables with joint distribution $P_{(X,Y)}$. X and Y are independent if and only if the joint distribution coincides with the product measure, i.e.:*

$$P_{(X,Y)} = P_X \times P_Y$$

Remark 32.6.6. If X and Y are absolutely continuous then the previous theorem also applies with the densities instead of the distributions.

32.6.2 Conditional probability

Formula 32.6.7 (Conditional density). Let X, Y be two random variables with joint density $f_{(X,Y)}$. The conditional density of Y given $X \in A$ is:

$$h(y|X \in A) = \frac{\int_A f_{(X,Y)}(x,y)dx}{\int_A f_X(x)dx} \tag{32.41}$$

For $X = \{a\}$ this equation fails as the denominator would become 0. However it is possible to avoid this problem by formally putting

$$h(y|A = a) = \frac{f_{(X,Y)}(a,y)}{f_X(a)} \tag{32.42}$$

with $f_X(a) \neq 0$ which is non-restrictive because the probability of having a measurement $(X,Y) \in \{(x,y) : f_X(x) = 0\}$ is 0. We can thus define the conditional probability of Y given $X = a$:

$$P(Y \in B|X = a) = \int_B h(y|X = a)dy \tag{32.43}$$

Formula 32.6.8 (Conditional expectation).

$$E(Y|X)(\omega) = \int_{\mathbb{R}} yh(y|X(\omega))dy \tag{32.44}$$

Furthermore, let $\mathcal{F}_X$ denote the σ-algebra generated by the random variable X. Using Fubini's theorem we can prove that for all sets $A \in \mathcal{F}_X$ the following equality, which should be compared with equation 32.34, holds:

$$\int_A E(Y|X)dP = \int_A YdP \tag{32.45}$$

Remark 32.6.9. Following from previous two equations we can say that the conditional expectation $E(Y|X)$ is the best representation of the random variable Y as a function of X (i.e. measurable with respect to $\mathcal{F}_X$).

Property 32.6.10. As mentioned above, applying Fubini's theorem gives:

$$\boxed{E(E(Y|X)) = E(Y)} \tag{32.46}$$

Chapter 33

Statistics

In this chapter, most definitions and formulas will be based on either a standard calculus approach or a data-driven approach. For a measure-theory based approach see chapter 32.

33.1 Data samples

33.1.1 Moment

Formula 33.1.1 (r^{th} sample moment).

$$\overline{x^r} = \frac{1}{N}\sum_{i=1}^{N} x_i^r \tag{33.1}$$

Formula 33.1.2 (r^{th} central sample moment).

$$m_r = \frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^r \tag{33.2}$$

33.1.2 Mean

Definition 33.1.3 (Arithmetic mean). The arithmetic mean is used to average out differences between measurements. It is equal to the 1^{st} sample moment:

$$\boxed{\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i} \tag{33.3}$$

Formula 33.1.4 (Weighted mean). Let $f : \mathbb{R} \to \mathbb{R}^+$ be a weight function. The weighted mean is given by:

$$\overline{x} = \frac{\sum_i f(x_i)x_i}{\sum_i f(x_i)} \tag{33.4}$$

Corollary 33.1.5. If the data has been grouped in bins, the weight function is given by the number of elements in each bin. Knowing this the (binned) mean becomes:

$$\overline{x} = \frac{1}{N}\sum_{i=1} n_i x_i \tag{33.5}$$

Remark 33.1.6. In the above defintions the measurements x_i can be replaced by function values $f(x_i)$ to calculate the mean of the function $f(x)$.

Remark 33.1.7. It is also important to notice that $\overline{f}(x) \neq f(\overline{x})$. The equality only holds for linear functions.

Definition 33.1.8 (Geometric mean). Let $\{x_i\}$ be a positive data set[1]. The geometric mean is used to average out *normalized* measurements, i.e. ratios with respect to a reference value.

$$g = \left(\prod_{i=1}^{N} x_i\right)^{1/N} \tag{33.6}$$

The following relation exists between the arithmetic and geometic mean:

$$\ln g = \overline{\ln x} \tag{33.7}$$

Definition 33.1.9 (Harmonic mean).

$$h = \left(\frac{1}{N}\sum_{i=1}^{N} x_i^{-1}\right)^{-1} \tag{33.8}$$

The following relation exists between the arithmetic and harmonic mean:

$$\frac{1}{h} = \overline{x^{-1}} \tag{33.9}$$

Property 33.1.10. Let $\{x_i\}$ be a positive data set.

$$h \leq g \leq \overline{x} \tag{33.10}$$

where the equalities only hold when all x_i are equal.

Definition 33.1.11 (Mode). The most occurring value in a dataset.

Definition 33.1.12 (Median). The median of dataset is the value x_i such that half of the values is greater than x_i and the other half is smaller than x_i.

[1] A negative data set is also allowed. The real condition is that all values should have the same sign.

33.1.3 Dispersion

Definition 33.1.13 (Range). The simplest indicator for statistical dispersion. It is however very sensitive for extreme values.

$$R = x_{max} - x_{min} \tag{33.11}$$

Definition 33.1.14 (Mean absolute difference).

$$\mathrm{MD} = \frac{1}{N}\sum_{i=1}^{N}|x_i - \overline{x}| \tag{33.12}$$

Definition 33.1.15 (Sample variance).

$$V(X) = \frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2 \tag{33.13}$$

Formula 33.1.16. The variance can also be written in the following way:

$$\boxed{V(X) = \overline{x^2} - \overline{x}^2} \tag{33.14}$$

Remark 33.1.17. A better estimator for the variance of a sample is the following formula:

$$\hat{s} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2 \tag{33.15}$$

Equation 33.13 gives a good estimation when the sample mean $\overline{x}$ is replaced by the "true" mean μ. Otherwise one should use the estimator 33.15.

Definition 33.1.18 (Standard deviation).

$$\sigma(X) = \sqrt{V(x)} \tag{33.16}$$

Definition 33.1.19 (Skewness). The skewness γ describes the asymmetry of a distribution. It is defined in relation to the third central moment m_3;

$$m_3 = \gamma\sigma^3 \tag{33.17}$$

where σ is the standard deviation. A positive skewness indicates a tail to the right or alternatively a median smaller than $\overline{x}$. A negative skewness indicates a median larger than $\overline{x}$.

Definition 33.1.20 (Pearson's mode skewness).

$$\gamma_P = \frac{\overline{x} - \mathrm{mode}}{\sigma} \tag{33.18}$$

Definition 33.1.21 (Kurtosis). The kurtosis c is an indicator for the "tailedness". It is defined in relation to the fourth central moment m_4:

$$m_4 = c\sigma^4 \tag{33.19}$$

Definition 33.1.22 (Excess kurtosis). The excess kurtosis is defined as $c - 3$. This fixes the excess kurtosis of all univariate normal distributions at 0. A positive excess is an indicator for long "fat" tails, a negative excess indicates short "thin" tails.

Definition 33.1.23 (Percentile). The p^{th} percentile c_p is defined as the value that is larger than $p\%$ of the measurements. The median is the 50^{th} percentile.

Definition 33.1.24 (Interquartile range). The interquartile range is the difference between the upper and lower quartile (75^{th} and 25^{th} percentile respectively).

Definition 33.1.25 (FWHM). The **Full Width at Half Maximum** is the difference between the two values of the independent variable where the dependent variable is half of its maximum.

Property 33.1.26. For Gaussian distributions the following relation exists between the FWHM and the standard deviation σ:

$$\text{FWHM} = 2.35\sigma \tag{33.20}$$

33.1.4 Multivariate datasets

When working with bivariate (or even multivariate) distributions it is useful to describe the relationship between the different random variables. The following two definitions are often used.

Definition 33.1.27 (Covariance). Let X, Y be two random variables. The covariance of X and Y is defined as follows:

$$\text{cov}(X, Y) = \frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})(y_i - \overline{y}) = \overline{xy} - \overline{x}\ \overline{y} \tag{33.21}$$

The covariance of X and Y is often denoted by σ_{XY}.

Formula 33.1.28. The covariance and standard deviation are related by the following equality:

$$\sigma_X^2 = \sigma_{XX} \tag{33.22}$$

Definition 33.1.29 (Correlation coefficent).

$$\rho_{XY} = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} \tag{33.23}$$

The correlation coefficient is bounded to the interval $[-1, 1]$. It should be noted that its magnitude is only an indicator for the linear dependence.

Remark 33.1.30. For multivariate distributions the above definitions can be generalized using matrices:

$$V_{ij} = \text{cov}(x_{(i)}, x_{(j)}) \tag{33.24}$$

$$\rho_{ij} = \rho_{(i)(j)} \tag{33.25}$$

where $\text{cov}(x_{(i)}, x_{(j)})$ and $\rho_{(i)(j)}$ are defined using equations 33.21 and 33.23. The following general equality exists:

$$V_{ij} = \rho_{ij}\sigma_i\sigma_j \tag{33.26}$$

33.2 Law of large numbers

Theorem 33.2.1 (Law of large numbers). *If the size N of a sample tends towards infinity, then the observed frequencies tend towards the theoretical propabilities.*

Corollary 33.2.2 (Frequentist probability[2]).

$$P(X) = \lim_{n\to\infty} \frac{f_n(X)}{n} \tag{33.27}$$

33.3 probability densities

Remark. In the following sections and subsections, all distributions will be taken to be continuous. The formulas can be modified for use with discrete distributions by replacing the integral with a summation.

Definition 33.3.1 (probability density functions p.d.f). Let X be a random variable and $P(X)$ the associated probability distribution. The p.d.f. $f(X)$ is defined as follows:

$$\boxed{P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(X)dX} \tag{33.28}$$

An alternative definition[3] is the following:

$$f(X) = \lim_{\delta x\to 0} \frac{P(x \leq X \leq x + \delta x)}{\delta x} \tag{33.29}$$

Definition 33.3.2 (Cumulative distribution function c.d.f.). Let X be a random variable and $f(X)$ the associated p.d.f. The cumulative distribution function $F(X)$ is defined as follows:

$$F(x) = \int_{-\infty}^{x} f(X)dX \tag{33.30}$$

[2]Also called the **empirical probability**.

[3]A more formal definition uses *measure theory* and the *Radon-Nikodym derivative*.

Theorem 33.3.3. *Let X be a random variable. Let $P(X)$ and $F(X)$ be the associated p.d.f. and c.d.f. Using standard calculus the following equality can be proven:*

$$P(x_1 \leq X \leq x_2) = F(x_2) - F(x_1) \tag{33.31}$$

Theorem 33.3.4. *$F(X)$ is continuous if and only if $P_X(\{y\}) = 0$ for every $y \in \mathbb{R}$.*

Remark 33.3.5 (Normalization).

$$F(\infty) = 1 \tag{33.32}$$

Formula 33.3.6. The p^{th} percentile c_p can be computed as follows[4]:

$$c_p = F^{-1}(p) \tag{33.33}$$

Definition 33.3.7 (Parametric family). A parametric family of probability densities $f(X;\vec{\theta})$ is a set of densities described by one or more parameters $\vec{\theta}$.

33.3.1 Function of a random variable

Formula 33.3.8. Let X be random variable and $f(X)$ the associated p.d.f. Let $a(X)$ be a function of X. The random variable $A = a(X)$ has an associated p.d.f. $g(A)$. If the function $a(x)$ can be inverted, then $g(A)$ can be computed as follows:

$$\boxed{g(a) = f(x(a)) \left| \frac{dx}{da} \right|} \tag{33.34}$$

33.3.2 Multivariate distributions

Remark. In this section all defintions and thereoms will be given for bivariate distributions, but can be easily generalized to more random variables.

Definition 33.3.9 (Joint density). Let X, Y be two random variables. The joint p.d.f. $f_{XY}(x, y)$ is defined as follows:

$$f_{XY}(x, y)dxdy = \begin{cases} f_x(x \in [x, x + dx]) \\ f_y(y \in [y, y + dy]) \end{cases} \tag{33.35}$$

Remark 33.3.10. As f_{XY} is a probability density, the normalization condition 33.32 should be fulfilled.

Definition 33.3.11 (Conditional density). The conditional p.d.f. of X when Y has the value y is given by the following formula:

$$g(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} \tag{33.36}$$

where we should pay attention to the remark made when we defined 32.42.

[4]This is clear from the definition of a percentile, as this implies that $F(c_p) = p$.

Corollary 33.3.12. If X and Y are independent, then by remark 32.6.6 the marginal p.d.f is equal to the conditional p.d.f.

Theorem 33.3.13 (Bayes' theorem). *The conditional p.d.f. can be computed without prior knowledge of the joint p.d.f:*

$$\boxed{g(x|y) = \frac{h(y|x)f_X(x)}{f_Y(y)}} \tag{33.37}$$

Remark. This theorem is the statistical (random variable) analogon of theorem 32.7.

Formula 33.3.14. Let $Z = XY$ with X, Y two independent random variables. The distribution $f(z)$ is given by

$$f(z) = \int_{-\infty}^{+\infty} g(x)h(z/x)\frac{dx}{|x|} = \int_{-\infty}^{+\infty} g(z/y)h(y)\frac{dy}{|y|} \tag{33.38}$$

Corollary 33.3.15. Taking the Mellin transform 11.17 of both the positive and negative part of the above integrand (to be able to handle the absolute value) gives following relation

$$\mathcal{M}\{f\} = \mathcal{M}\{g\}\mathcal{M}\{h\} \tag{33.39}$$

Formula 33.3.16. Let $Z = X + Y$ with X, Y two independent random variables. The distribution $f(z)$ is given by the convolution of $g(x)$ and $h(y)$:

$$f(z) = \int_{-\infty}^{+\infty} g(x)h(z-x)dx = \int_{-\infty}^{+\infty} g(z-y)h(y)dy \tag{33.40}$$

33.3.3 Important distributions

Formula 33.3.17 (Uniform distribution).

$$\boxed{P(x;a,b) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}} \tag{33.41}$$

$$E(x) = \frac{a+b}{2} \tag{33.42}$$

$$V(x) = \frac{(b-a)^2}{12} \tag{33.43}$$

Formula 33.3.18 (Normal distribution). Also called the Gaussian distribution.

$$\boxed{\mathcal{G}(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}} \tag{33.44}$$

Formula 33.3.19 (Standard normal distribution).

$$\boxed{\mathcal{N}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}} \tag{33.45}$$

Remark 33.3.20. Every Gaussian distribution can be rewritten as a standard normal distribution by setting $Z = \frac{X - \mu}{\sigma}$.

Remark 33.3.21. The c.d.f. of the standard normal distribution is given by the error function: $F(z) = \text{Erf}(z)$.

Formula 33.3.22 (Exponential distribution).

$$\boxed{P(x;\tau) = \frac{1}{\tau} e^{-\frac{x}{\tau}}} \tag{33.46}$$

$$E(x) = \tau \tag{33.47}$$

$$V(x) = \tau^2 \tag{33.48}$$

Theorem 33.3.23. *The exponential distribution is memoryless:*

$$P(X > x_1 + x_2 | X > x_2) = P(X > x_1) \tag{33.49}$$

Formula 33.3.24 (Bernoulli distribution). A radnom variable that can only take 2 possible values is described by a Bernoulli distribution. When the possible values are 0 and 1, with respective chances p and $1 - p$, the distribution is given by:

$$\boxed{P(x;p) = p^x (1-p)^{1-x}} \tag{33.50}$$

$$E(x) = p \tag{33.51}$$

$$V(x) = p(1-p) \tag{33.52}$$

Formula 33.3.25 (Binomial distribution). A process with n identical independent trials, all Bernoulli processes $P(x;p)$, is described by a binomial distribution:

$$\boxed{P(r;p,n) = p^r (1-p)^{n-r} \frac{n!}{r!(n-r)!}} \tag{33.53}$$

$$E(r) = np \tag{33.54}$$

$$V(r) = np(1-p) \tag{33.55}$$

Formula 33.3.26 (Poisson distribution). A process with known possible outcomes but an unknown number of events is described by a Poisson distribution $P(r;\lambda)$ with λ the average expected number of events.

$$\boxed{P(r;\lambda) = \frac{e^{-\lambda}\lambda^r}{r!}} \tag{33.56}$$

$$E(r) = \lambda \tag{33.57}$$

$$V(r) = \lambda \tag{33.58}$$

Theorem 33.3.27. *If two Poisson processes $P(r;\lambda_a)$ and $P(r;\lambda_b)$ occur simultaneously and if there is no distinction between the two, then the probability of r events is also described by a Poisson distribution with average $\lambda_a + \lambda_b$.*

Corollary 33.3.28. The number of events coming from A is given by a binomial distribution $P(r_a;\Lambda_a,r)$ where $\Lambda_a = \dfrac{\lambda_a}{\lambda_a+\lambda_b}$.

Remark 33.3.29. For large values of λ ($\lambda \to \infty$), the Poisson distribution $P(r;\lambda)$ can be approximated by a Gaussian distribution $\mathcal{G}(x;\lambda,\sqrt{\lambda})$.

Formula 33.3.30 (χ^2 distribution). The sum of k squared independent (standard) normally distributed random variables Y_i defines the random variable:

$$\chi^2_k = \sum_{i=1}^{k} Y_i^2 \tag{33.59}$$

where k is said to be the number of **degrees of freedom**.

$$\boxed{P(\chi^2;n) = \frac{\chi^{n-2}e^{-\frac{\chi^2}{2}}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}} \tag{33.60}$$

Remark 33.3.31. Due to the CLT the χ^2 distribution approximates a Guassian distribution for large k: $P(\chi^2;k) \xrightarrow{k>30} \mathcal{G}(\sqrt{2\chi^2};\sqrt{2k-1},1)$

Formula 33.3.32 (Student-t distribution). The student-t distribution describes a sample with estimated standard deviation $\hat{\sigma}$.

$$\boxed{P(t;n) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}} \tag{33.61}$$

where

$$t = \frac{(x-\mu)/\sigma}{\hat{\sigma}/\sigma} = \frac{z}{\sqrt{\chi^2/n}} \tag{33.62}$$

Remark. The significance of a difference between the sample mean $\overline{x}$ and the true mean μ is smaller due to the (extra) uncertainty of the estimated standard deviation.

Formula 33.3.33 (Cauchy distribution[5]). The general form $f(x; x_0, \gamma)$ is given by:

$$f(x; x_0, \gamma) = \frac{1}{\pi} \frac{\gamma}{(x - x_0)^2 + \gamma^2} \tag{33.63}$$

The characteristic function 32.24 is given by:

$$E\left(e^{itx}\right) = e^{ix_0 t - \gamma|t|} \tag{33.64}$$

Property 33.3.34. Both the mean and variance of the Cauchy distribution are undefined.

33.4 Central limit theorem (CLT)

Theorem 33.4.1 (Central limit theorem). *A sum of n independent random variables X_i has the following properties:*

1. $\mu = \sum_i \mu_i$
2. $V(X) = \sum_i V_i$
3. *The sum will be approximately (!!) normally distributed.*

Remark 33.4.2. If the random variables are not independent, property 2 will not be fulfilled.

Remark 33.4.3. The sum of Gaussians will be Gaussian to.

33.4.1 Distribution of sample mean

The difference between a sample mean $\overline{x}$ and the true mean μ is described by a distribution with following mean and variance:

Property 33.4.4.

$$< \overline{x} >= \mu \tag{33.65}$$

Property 33.4.5.

$$V(\overline{x}) = \frac{\sigma^2}{N} \tag{33.66}$$

[5]Also known (especially in particle physics) as the **Breit-Wigner** distribution.

33.5 Errors

33.5.1 Different measurement types

When performing a sequence of measurements x_i with different variances σ_i^2, it is impossible to use the arithmetic mean 33.3 in a meaningful way because the measurements are not of the same type. Therefore it is also impossible to apply the CLT 33.4.1.

Definition 33.5.1 (Weighted mean). The appropriate alternative is the *weighted mean*:

$$\boxed{\overline{x} = \frac{\sum_i \frac{x_i}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2}}} \tag{33.67}$$

The resolution of the weighted mean is given by:

$$V(\overline{x}) = \frac{1}{\sum_i \sigma_i^{-2}} \tag{33.68}$$

33.5.2 Propagation of errors

Formula 33.5.2. Let X be random variable with variance $V(x)$. The variance of a linear function $f(X) = aX + b$ is given by:

$$V(f) = a^2 V(x) \tag{33.69}$$

Formula 33.5.3. Let X be random variable with **small** (!!) variance $V(x)$. The variance of a general function $f(X)$ is given by:

$$V(f) \approx \left(\frac{df}{dx}\right)^2 V(x) \tag{33.70}$$

Corollary 33.5.4. The correlation coefficient ρ (33.23) of a random variable X and a **linear** function of X is independent of σ_x and is always equal to ± 1.

Formula 33.5.5 (Law of error propagation). Let $\vec{X}$ be a set of random variables with **small** variances. The variance of a general function $f(\vec{X})$ is given by:

$$\boxed{V(f) = \sum_p \left(\frac{\partial f}{\partial X_{(p)}}\right)^2 V(X_{(p)}) + \sum_p \sum_{q \neq p} \left(\frac{\partial f}{\partial X_{(p)}}\right)\left(\frac{\partial f}{\partial X_{(q)}}\right) \operatorname{cov}(X_{(p)}, X_{(q)})} \tag{33.71}$$

Definition 33.5.6 (Fractional error). Let X, Y be two **independent** random variables. The standard deviation of $f(x, y) = xy$ is given by the fractional error:

$$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2 \tag{33.72}$$

Remark 33.5.7. The fractional error of quantity is equal to the fractional error of the reciprocal of that quantity.

Property 33.5.8. Let X be a random variable. The error of the logarithm of X is equal to the fractional error of X.

Definition 33.5.9 (Covariance of functions).

$$\text{cov}(f_1, f_2) = \sum_p \sum_q \left(\frac{\partial f_1}{\partial X_{(p)}}\right)\left(\frac{\partial f_2}{\partial X_{(q)}}\right) \text{cov}(X_{(p)}, X_{(q)}) \tag{33.73}$$

Corollary 33.5.10. Let $\vec{f} = \{f_1, ..., f_k\}$. The covariance matrix V_f of the k functions is given by:

$$V_f = G\ V_X\ G^T \tag{33.74}$$

where G is the Jacobian matrix of $\vec{f}$.

33.5.3 Systematic errors

Systematic errors are errors that always have the same influence (they shift all values in the same way), that are not independent of eachother and that cannot be directly inferred from the measurements.

33.6 Estimators

Definition 33.6.1 (Estimator). An estimator is a procedure that, given a sample, produces a numerical value for a property of the parent population.

33.6.1 General properties

Property 33.6.2 (Consistency).

$$\boxed{\lim_{N\to\infty} \hat{a} = a} \tag{33.75}$$

Property 33.6.3 (Unbiased estimator).

$$\boxed{< \hat{a} >= a} \tag{33.76}$$

Definition 33.6.4 (Bias).

$$B(\hat{a}) = | < \hat{a} > -a| \tag{33.77}$$

Property 33.6.5 (Efficiency). An estimator $\hat{a}$ is said to be efficient if its variance $V(\hat{a})$ is equal to the minimum variance bound 33.86.

Definition 33.6.6 (Mean squared error).

$$\Upsilon(\hat{a}) = B(\hat{a})^2 + V(\hat{a}) \tag{33.78}$$

Remark 33.6.7. If an estimator is unbiased, the MSE is equal to the variance of the estimator.

33.6.2 Fundamental estimators

Property 33.6.8 (Mean estimator). The sample mean $\overline{x}$ is a consistent and unbiased estimator for the true mean μ due to the CLT. The variance $V(\overline{x})$ of the estimator is given by equation 33.66.

Property 33.6.9 (Variance estimator for known mean). If the true mean μ is known then a consistent and unbiased estimator for the variance is given by:

$$\widehat{V(x)} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2 \tag{33.79}$$

Property 33.6.10 (Variance estimator for unknown mean). If the true mean μ is unknown and the sample mean has been used to estimate μ, then a consistent and unbiased estimator is given by[6]:

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2 \tag{33.80}$$

33.6.3 Estimation error

Formula 33.6.11 (Variance of estimator of variance).

$$V(\widehat{V(x)}) = \frac{(N-1)^2}{N^3} < (x- < x >)^4 > -\frac{(N-1)(N-3)}{N^3} < (x- < x >)^2 >^2 \tag{33.81}$$

Formula 33.6.12 (Variance of estimator of standard deviation).

$$V(\hat{\sigma}) = \frac{1}{4\sigma^2}V(\widehat{V(x)}) \tag{33.82}$$

[6]The modified factor $\frac{1}{N-1}$ is called the Bessel correction. It corrects the bias of the estimator given by the sample variance 33.13. The consistency however is guaranteed by the CLT.

Remark 33.6.13. The previous result is a little odd, as one has to know the true standard deviation to compute the variance of the estimator. This problem can be solved in two ways. Or a value (hopefully close to the real one) inferred from the sample is used as an estimator or a theoretical one is used in the design phase of an experiment to see what the possible outcomes are.

33.6.4 Likelihood function

Definition 33.6.14 (Likelihood). The likelihood $\mathcal{L}(a;\vec{x})$ is the probability to find a set of measurements $\vec{x} = \{x_1, ..., x_N\}$ given a distribution $P(X;a)$:

$$\mathcal{L}(a;\vec{x}) = \prod_{i=1}^{N} P(x_i;a) \tag{33.83}$$

Definition 33.6.15 (Log-likelihood).

$$\log \mathcal{L}(a;\vec{x}) = \sum_i \ln P(x_i;a) \tag{33.84}$$

Property 33.6.16. The expectation value of an estimator $\hat{}$ is given by:

$$< \hat{a} >= \int \hat{a}\mathcal{L}(\hat{a};X)dX \tag{33.85}$$

Theorem 33.6.17 (Minimum variance bound). *The variance of an* ***unbiased*** *estimator has a lower bound: the minimum variance bound*[7] *(MVB).*

$$\boxed{V(\hat{a}) \geq \frac{1}{\left\langle \left(\frac{d\ln\mathcal{L}}{da} \right)^2 \right\rangle}} \tag{33.86}$$

For a biased estimator with bias b the MVB takes on the following form:

$$V(\hat{a}) \geq \frac{\left(1 + \frac{db}{da}\right)^2}{\left\langle \left(\frac{d\ln\mathcal{L}}{da} \right)^2 \right\rangle} \tag{33.87}$$

Remark 33.6.18.

$$\left\langle \left(\frac{d\ln\mathcal{L}}{da} \right)^2 \right\rangle = -\left\langle \frac{d^2\ln\mathcal{L}}{da^2} \right\rangle \tag{33.88}$$

Definition 33.6.19 (Fisher information).

$$I_X(a) = \left\langle \left(\frac{d\ln\mathcal{L}}{da} \right)^2 \right\rangle = N\int \left(\frac{d\ln P}{da} \right)^2 PdX \tag{33.89}$$

[7] It is also known as the Cramer-Rao bound.

33.6.5 Maximum likelihood estimator

Following from definition 33.6.14 it follows that the best estimator $\hat{a}$ is the value for which the likelihood function is maximal. It is the value that makes the measurements the most propable, but it is therefore not the most propable estimator.

Method 33.6.20 (Maximum likelihood estimator). The maximum likelihood estimator $\hat{a}$ is obtained by solving following equation:

$$\left.\frac{d \ln \mathcal{L}}{da}\right|_{a=\hat{a}} = 0 \tag{33.90}$$

Remark 33.6.21. MLH estimators are mostly consistent but often biased.

Property 33.6.22. MLH estimators are invariant under parameter transformations.

Corollary 33.6.23. The invariance implies that the two estimators $\hat{a}$ and $\widehat{f(a)}$ cannot both be unbiased at the same time.

Property 33.6.24. Asymptotically ($N \to \infty$) every **consistent** estimator becomes unbiased and efficient.

33.6.6 Least squares

Method 33.6.25 (Least squares).

1. Fitting a function $y = f(x; a)$ to a set of 2 variables (x, y) where the x values are exact and the y valiues have an uncertainty σ_i to estimate the value a.
2. For every event (x_i, y_i) define the residual $d_i = y_i - f(x_i; a)$.
3. Determine (analytically) the χ^2 value: $\chi^2 = \sum_i \left[\frac{d_i}{\sigma_i}\right]^2$
4. Find the most propably value of $\hat{a}$ by solving the equation $\frac{d\chi^2}{da} = 0$.

Property 33.6.26. The optimalized (minimized) χ^2 is distributed according to a χ^2 dsitribution 33.60 $P(\chi^2; n)$. The number of degrees of freedom n is equal to the number of events N minus the number of fitted parameters k.

Formula 33.6.27 (Linear fit). When all uncertainties σ_i are equal, the slope $\hat{m}$ and intercept $\hat{c}$ are given by following formulas:

$$\hat{m} = \frac{\overline{xy} - \overline{x}\ \overline{y}}{\overline{x^2} - \overline{x}^2} = \frac{\operatorname{cov}(x, y)}{V(x)} \tag{33.91}$$

$$\hat{c} = \overline{y} - \hat{m}\overline{x} = \frac{\overline{x^2} - \overline{x}\ \overline{y}}{\overline{x^2} - \overline{x}^2} \tag{33.92}$$

Remark 33.6.28. The equation $\overline{y} = \hat{c} + \hat{m}\overline{x}$ means that the linear fit passes through the center of mass $(\overline{x}, \overline{y})$.

Formula 33.6.29 (Errors of linear fit).

$$V(\hat{m}) = \frac{1}{N(\overline{x^2} - \overline{x}^2)}\sigma^2 \tag{33.93}$$

$$V(\hat{c}) = \frac{\overline{x^2}}{N(\overline{x^2} - \overline{x}^2)}\sigma^2 \tag{33.94}$$

$$\mathrm{cov}(\hat{m}, \hat{c}) = \frac{-\overline{x}}{N(\overline{x^2} - \overline{x}^2)}\sigma^2 \tag{33.95}$$

Remark 33.6.30. When there are different uncertainties σ_i, the arithmetic means have to be replaced with weighted means, but the expressions remain the same. The quantity σ^2 has to be replaced by its weighted variant:

$$\overline{\sigma^2} = \frac{\sum \sigma_i^2/\sigma_i^2}{\sum \sigma_i^{-2}} = \frac{N}{\sum \sigma_i^{-2}}$$

33.6.7 Binned least squares

The least squares method is very useful to fit data which has been grouped in bins (histograms).

Method 33.6.31 (Binned least squares).

1. N events with distributions $P(X; a)$ divided in N_B intervals. Interval j is centered on the value x_j, has a width W_j and contains n_j events.
2. The ideally expected number of events in the j^{th} interval: $f_j = NW_jP(x_j; a)$
3. The real number of events has a Poisson distribution: $\overline{n}_j = \sigma_j^2 = f_j$
4. Define the binned χ^2 as: $\chi^2 = \sum_i^{N_B} \frac{(n_i - f_i)^2}{f_i^2}$

33.7 Confidence

The real value of a parameter ε can never be known exactly. But it is possible to construct an interval I in which the real value should lie with a certain confidence C.

Example 33.7.1. Let X be a random variable with distribution $\mathcal{G}(x;\mu,\sigma)$. The measurement x lies in the interval $[\mu-2\sigma;\mu+2\sigma]$ with 95% **chance**. The real value μ lies in the interval $[x-2\sigma;x+2\sigma]$ with 95% **confidence**.

Remark. In the previous example there are some Bayesian assumptions: all possible values (left or right side of peak) are given the same possiblity due to the Gaussian distribution, but if one removes the symmetry it is mandatory to use a more careful approach. The symmetry between uncertainties σ and confidence levels is only valid for Gaussian distributions.

33.7.1 Interval types

Definition 33.7.2 (Two-sided confidence interval).

$$\boxed{P(x_- \leq X \leq x_+) = \int_{x_-}^{x_+} P(x)dx = C} \tag{33.96}$$

There are three possible (often used) two-sided intervals:

- Symmetric interval: $x_+ - \mu = \mu - x_-$
- Shortest interval: $|x_+ - x_-|$ is minimal
- Central interval: $\int_{-\infty}^{x_-} P(x)dx = \int_{x_+}^{\infty} P(x)dx = \frac{1-C}{2}$

Remark 33.7.3. For Gaussian distributions these three definitions are equivalent.

Remark. The central interval is the (best and) most widely used confidence interval.

Definition 33.7.4 (One-sided confidence interval).

$$\boxed{P(x \geq x_-) = \int_{x_-}^{+\infty} P(x)dx = C} \tag{33.97}$$

$$\boxed{P(x \leq x_+) = \int_{-\infty}^{x_+} P(x)dx = C} \tag{33.98}$$

Remark 33.7.5. For a discrete distribution it is often impossible to find integers $x_\pm$ such that the real value lies with exact confidence C in the interval $[x_-;x_+]$.

Definition 33.7.6 (Discrete central confidence interval).

$$x_- = \max_\theta \left[\sum_{x=0}^{\theta-1} P(x;X)\right] \leq \frac{1-C}{2} \tag{33.99}$$

$$x_+ = \min_\theta \left[\sum_{x=\theta+1}^{+\infty} P(x;X)\right] \leq \frac{1-C}{2} \tag{33.100}$$

33.7.2 General construction

For every value of the true parameter X it is possible to construct a confidence interval. This leads to the construction of the two functions $x_-(X)$ and $x_+(X)$. The 2D diagram obtained by plotting $x_-(X)$ and $x_+(X)$ with the x-axis horizontally and X-axis vertically is called the confidence region.

Theorem 33.7.7. *Let x_0 be the measured value of a parameter X. From the confidence region, it is possible to infere a confidence interval $[X_-(x); X_+(x)]$. The upper limit X_+ is not the limit such that there is only a $\frac{1-C}{2}$ chance of having a true parameter $X \geq X_+$, but it is the limit such that if the true parameter $X \geq X_+$ then there is a chance of $\frac{1-C}{2}$ to have a measurement x_0 or smaller.*

33.7.3 Extra conditions

Method 33.7.8 (Bayesian statistics).

$$p(\text{theory}|\text{result}) = p(\text{result}|\text{theory})\frac{p(\text{theory})}{p(\text{result})} \tag{33.101}$$

or more mathematically:

$$\boxed{p(X|x) = p(x|X)\frac{p(X)}{p(x)}} \tag{33.102}$$

- The denominator $p(\text{result})$ does not play a real role, it is a normalization constant.
- The probability $p(x|X)$ to have measurement x when the true parameter is X is a Gaussian distribution $\mathcal{G}(x; X, \sigma)$

Remark 33.7.9. If nothing is known about the theory, $p(X)$ is (exaggerated assumption) a uniform probability 33.41.

33.7.4 Interval for a sample mean

Formula 33.7.10 (Interval with known variance). If the sample size is large enough, the real distribution is unimportant, because the CLT ensures a Gaussian distribution of the sample mean $\overline{X}$. The α-level confidence interval such that $P(-z_{\alpha/2} < Z < z_{\alpha/2})$ with $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{N}}$ is given by:

$$\left[\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{N}}; \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right] \tag{33.103}$$

Remark 33.7.11. If the sample size is not sufficiently large, the measured quantity must follow a normal distribution.

Formula 33.7.12 (Interval with unknown variance). To account for the uncertainty of the estimated standard deviation $\hat{\sigma}$, the student-t distribution 33.61 is used instead of a Gaussian distribution to describe the sample mean $\overline{X}$. The α-level confidence interval is given by:

$$\left[\overline{X} - t_{\alpha/2;(n-1)}\frac{s}{\sqrt{N}}; \overline{X} + t_{\alpha/2;(n-1)}\frac{s}{\sqrt{N}}\right] \tag{33.104}$$

where s is the estimated standard deviation 33.80.

Formula 33.7.13 (Wilson score interval). For a sufficiently large sample, a sample proportion $\hat{P}$ is approximately Gaussian distributed with expectation value π and variance $\frac{\pi(\pi-1)}{N}$. The α-level confidence interval is given by:

$$\left[\frac{(2N\hat{P} + z^2_{\alpha/2}) - z_{\alpha/2}\sqrt{z^2_{\alpha/2} + 4N\hat{P}(1-\hat{P})}}{2(N + z^2_{\alpha/2})}; \frac{(2N\hat{P} + z^2_{\alpha/2}) + z_{\alpha/2}\sqrt{z^2_{\alpha/2} + 4N\hat{P}(1-\hat{P})}}{2(N + z^2_{\alpha/2})}\right] \tag{33.105}$$

Remark. The expectation value and variance are these of a binomial distribution 33.53 with $r = X/N$.

33.7.5 Confidence region

33.8 Hypotheses and testing

33.8.1 Hypothesis

Definition 33.8.1 (Simple hypothesis). A hypothesis is called simple if the distribution is fully specified.

Definition 33.8.2 (Composite hypothesis). A hypothesis is called composite if the distribution is given relative to some parameter values.

Definition 33.8.3 (Null hypothesis H_0).

Definition 33.8.4 (Alternative hypothesis H_1).

33.8.2 Testing

Definition 33.8.5 (Type I error). Rejecting a true null hypothesis.

Definition 33.8.6 (Type II error). Accepting/retaining a false null hypothesis.

Definition 33.8.7 (Significance). The probability of making a type I error:

$$\boxed{\alpha = \int P_{H_0}(x)dx} \tag{33.106}$$

Property 33.8.8. Let $a_1 > a_2$. An a_2-level test is also significant at the a_1-level.

Remark 33.8.9. For discrete distributions it is not always possible to achieve an exact level of significance.

Remark. Type I errors occur occasionally. They cannot be prevented, they should however be controlled.

Definition 33.8.10 (Power). The probability of not making a type II error:

$$\boxed{\beta = \int P_{H_1}(x)dx \qquad \rightarrow \qquad \text{power: } 1-\beta} \tag{33.107}$$

Theorem 33.8.11. *A good test is a test with a small significance and a large power. The propabilities $P_{H_0}(x)$ and $P_{H_1}(x)$ should be as different as possible.*

Theorem 33.8.12 (Neyman-Pearson test). *The following test is the most powerful test at significance level α for a threshold η:*

The null hypothesis H_0 is rejected in favour of the alternative hypothesis H_1 if the likelihood ratio Λ satisfies the following condition:

$$\Lambda(x) = \frac{L(x|H_0)}{L(x|H_1)} \leq \eta \tag{33.108}$$

where $P(\Lambda(x) \leq \eta|H_0) = \alpha$

Remark. In some references the reciprocal of $\Lambda(x)$ is used as the definition of the likelihood ratio.

33.8.3 Confindence intervals and decisions

33.9 Goodness of fit

Let $f(x|\vec{\theta})$ be the fitted function with N measurements.

33.9.1 χ^2-test

Formula 33.9.1.

$$\chi^2 = \sum_{i=1}^{N} \frac{\left[y_i - f\left(x_i\right)\right]^2}{\sigma_i^2} \tag{33.109}$$

Property 33.9.2. If there are $N - n$ fitted parameters we have:

$$\int_{\chi^2}^{\infty} f(\chi^2|n)d\chi^2 \approx 1 \implies \begin{cases} \circ \text{ good fit} \\ \circ \text{ errors were overestimated} \\ \circ \text{ selected measurements} \\ \circ \text{ lucky shot} \end{cases} \tag{33.110}$$

Property 33.9.3 (Reduced chi-squared $\chi^2_{\mathbf{red}}$). Define the reduced chi-squared value as follows: $\chi^2_{\text{red}} = \chi^2/n$ where n is the number of degrees of freedom.

- $\chi^2_{\text{red}} >> 1$: Poor modelling.
- $\chi^2_{\text{red}} > 1$: Bad modelling or underestimation of the uncertainties.
- $\chi^2_{\text{red}} = 1$: Good fit.
- $\chi^2_{\text{red}} < 1$: Impropable, overestimation of the uncertainties.

33.9.2 Runs test

A good χ^2-test does not mean that the fit is good. As mentioned in property 33.110 it is possible that the errors were overestimated. Another condition for a good fit is that the data points vary around the fit, i.e. there are no long sequences of points that lie above/underneath the fit. (It is a result of the 'randomness' of a data sample') This condition is tested with a runs test 33.111/33.112.

Remark 33.9.4. The χ^2-test and runs test are complementary. The χ^2-test only takes the absolute value of the differences between the fit and data points into account, the runs test only takes the signs of the differences into account.

Formula 33.9.5 (Runs distribution).

$$\boxed{P(r_{even}) = 2\frac{C^{N_B-1}_{\frac{r}{2}-1}C^{N_O-1}_{\frac{r}{2}-1}}{C^N_{N_B}}} \tag{33.111}$$

$$\boxed{P(r_{odd}) = \frac{C^{N_B-1}_{\frac{r-3}{2}}C^{N_O-1}_{\frac{r-1}{2}} + C^{N_O-1}_{\frac{r-3}{2}}C^{N_B-1}_{\frac{r-1}{2}}}{C^N_{N_B}}} \tag{33.112}$$

$$E(r) = 1 + 2\frac{N_B N_O}{N} \tag{33.113}$$

$$V(r) = 2\frac{N_B N_O}{N}\frac{2N_B N_O - 1}{N(N-1)} \tag{33.114}$$

Remark 33.9.6. For $r > 10-15$ the runs distribution approximates a Gaussian distribution.

33.9.3 Kolmogorov test

Definition 33.9.7 (Empirical distribution function).

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{]-\infty,x]}(x_i) \tag{33.115}$$

where $\mathbb{1}_A(x)$ is the indicator function 10.19.

Definition 33.9.8 (Kolmogorov-Smirnov statistic). Let $F(x)$ be a given cumulative distribution function. The n^{th} Kolmogorov-Smirnov statistic is defined as:

$$D_n = \sup_x |F_n(x) - F(x)| \tag{33.116}$$

Definition 33.9.9 (Kolmogorov distribution).

$$P(K \leq x) = 1 - 2\sum_{i=1}^{+\infty}(-1)^{i-1}e^{-2i^2x^2} = \frac{\sqrt{2\pi}}{x}\sum_{i=1}^{+\infty}e^{-(2i-1)^2\pi^2/(8x^2)} \tag{33.117}$$

Property 33.9.10 (Kolmogorov-Smirnov test). Let the null hypothesis H_0 state that a given data sample is described by a continuous distribution $P(x)$ with cumulative distribution function $F(x)$. The null hypothesis is rejected at significance level α if:

$$D_n\sqrt{n} > K_\alpha \tag{33.118}$$

where K_α is defined by using the Kolmogorov distribution: $P(K \leq K_\alpha) = 1 - \alpha$

Chapter 34

Stochastic calculus

Part VII

Classical Mechanics

Chapter 35

Equations of motion

35.1 General quantities

35.1.1 Linear quantities

Formula 35.1.1 (Force).

$$\boxed{\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{dt}} \tag{35.1}$$

Remark. In classical mechanics, this formula is given by Newton's second law.

Formula 35.1.2 (Work).

$$W = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} \tag{35.2}$$

Definition 35.1.3 (Conservative force). If the work done by a force is independent of the path taken, the force is said to be **conservative**.

$$\oint_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = 0 \tag{35.3}$$

Stokes' theorem 17.22 together with relation 17.15 lets us rewrite the conservative force as the gradient of a scalar field:

$$\boxed{\vec{\boldsymbol{F}} = -\nabla V} \tag{35.4}$$

Formula 35.1.4 (Kinetic energy).

$$E_{kin} = \frac{p^2}{2m} \tag{35.5}$$

35.1.2 Angular quantities

Formula 35.1.5 (Angular velocity).

$$\omega = \frac{v}{r} \tag{35.6}$$

Formula 35.1.6 (Angular frequency).

$$\nu = \frac{\omega}{2\pi} \tag{35.7}$$

Formula 35.1.7 (Moment of inertia). For a symmetric object the moment of inertia is given by:

$$I = \int_V r^2 \rho(r) dV \tag{35.8}$$

For a general body we can define the moment of inertia tensor:

$$\boxed{\mathcal{I} = \int_V \rho(\vec{\boldsymbol{r}}) \left(r^2 \mathbb{1} - \vec{\boldsymbol{r}} \otimes \vec{\boldsymbol{r}} \right) dV} \tag{35.9}$$

Definition 35.1.8 (Principal axes of inertia). Let $[I]$ be the matrix of inertia[1]. This is a real symmetric matrix, which means that it admits an eigendecomposition[2] of the form:

$$[I] = [Q][\Lambda][Q]^T \tag{35.10}$$

The columns of $[Q]$ are called the principal axes of inertia. The eigenvalues are called the **principal moments of inertia**.

Example 35.1.9 (Objects with azimuthal symmetry†). Let r denote the radius of the object.

- Solid disk: $I = \frac{1}{2}mr^2$
- Cylindrical shell: $I = mr^2$
- Hollow sphere: $I = \frac{2}{3}mr^2$
- Solid sphere: $I = \frac{2}{5}mr^2$

Theorem 35.1.10 (Parallel axis theorem). *Consider a rotation about an axis ω through a point A. Let ω_{CM} be a parallel axis through the center of mass. The moment of inertia about ω is related to the moment of inertia about ω_{CM} in the following way:*

$$\boxed{I_A = I_{CM} + M||\vec{\boldsymbol{r}}_A - \vec{\boldsymbol{r}}_{CM}||^2} \tag{35.11}$$

where M is the mass of the rotating body.

[1] The matrix associated with the inertia tensor 35.9.

[2] See 16.6.16.

Formula 35.1.11 (Angular momentum).

$$\boxed{\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}} \tag{35.12}$$

Given the angular velocity vector we can compute the angular momentum as follows:

$$\vec{\boldsymbol{L}} = \mathcal{I}(\vec{\boldsymbol{\omega}}) \tag{35.13}$$

where $\mathcal{I}$ is the moment of inertia tensor. If $\vec{\boldsymbol{\omega}}$ is parallel to a principal axis, then the formula reduces to:

$$\vec{\boldsymbol{L}} = I\vec{\boldsymbol{\omega}} \tag{35.14}$$

Formula 35.1.12 (Torque).

$$\vec{\boldsymbol{\tau}} = \frac{d\vec{\boldsymbol{L}}}{dt} \tag{35.15}$$

For constant bodies, this formula can be rewritten as follows:

$$\vec{\boldsymbol{\tau}} = I\vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} \tag{35.16}$$

Remark 35.1.13. From the previous definitions it follows that both the angular momentum and torque vectors are in fact pseudo-vectors and thus change sign under coordinate transforms with $\det = -1$.

Formula 35.1.14 (Rotational energy).

$$E_{\text{rot}} = \frac{1}{2} I\omega^2 \tag{35.17}$$

35.2 Central force

Definition 35.2.1 (Central force). A central force is a force that only depends on the relative position of two objects:

$$\vec{\boldsymbol{F}}_c \equiv F\left(||\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1|\,|\right)\hat{\boldsymbol{e}}_{\boldsymbol{r}} \tag{35.18}$$

35.3 Kepler problem

Formula 35.3.1 (Potential for a point mass).

$$\boxed{V = -G\frac{M}{r}} \tag{35.19}$$

where $G = 6.67 \times 10^{-11} \frac{Nm^2}{\text{kg}^2}$ is the **gravitational constant**.

35.4 Harmonic oscillator

Formula 35.4.1 (Harmonic potential).

$$\boxed{V = \frac{1}{2}kx^2} \tag{35.20}$$

or

$$V = \frac{1}{2}m\omega^2 x^2 \tag{35.21}$$

where we have set $\omega = \sqrt{\frac{k}{m}}$.

Formula 35.4.2 (Solution).

$$x(t) = A\sin\omega t + B\cos\omega t \tag{35.22}$$

$$= Ce^{i\omega t} + De^{-i\omega t} \tag{35.23}$$

Chapter 36

Lagrangian and Hamiltonian formalism

Definition 36.0.1 (Generalized coordinates). The generalized coordinates q_k are independent coordinates that completely describe the current configuration of a system relative to a reference configuration.

When a system has N degrees of freedom and n_c constraints, there are $(N - n_c)$ generalized coordinates. Furthermore, every set of generalized coordinates, describing the same system, should contain exactly $(N - n_c)$ coordinates.

Definition 36.0.2 (Generalized velocities). The generalized velocities $\dot{q}_k$ are the derivatives of the generalized coordinates with respect to time.

Notation 36.0.3.

$$L\left(\vec{\boldsymbol{q}}(t), \dot{\vec{\boldsymbol{q}}}(t), t\right) \equiv L\left(q_1(t), ..., q_n(t), \dot{q}_1(t), ..., \dot{q}_n(t), t\right) \tag{36.1}$$

Definition 36.0.4 (Action).

$$\boxed{S = \int_{t_1}^{t_2} L\left(\vec{\boldsymbol{q}}(t), \dot{\vec{\boldsymbol{q}}}(t), t\right) dt} \tag{36.2}$$

36.1 Euler-Lagrange equations[†]

Formula 36.1.1 (Euler-Lagrange equation of the first kind).

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k} = Q_k} \tag{36.3}$$

where T is the total kinetic energy.

Formula 36.1.2 (Euler-Lagrange equation of the second kind).

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\right) - \frac{\partial L}{\partial q^k} = 0} \tag{36.4}$$

36.2 Conservation laws and symmetry properties

Definition 36.2.1 (Conjugate momentum). Also called the **canonically conjugate momentum**.

$$p_k = \frac{\partial L}{\partial \dot{q}^k} \tag{36.5}$$

Definition 36.2.2 (Cyclic coordinate). If the lagrangian L does not explicitly depend on a coordinate q_k, the coordinate is called a cyclic coordinate.

Property 36.2.3. The conjugate momentum of a cyclic coordinate is a conserved quantity.

$$\dot{p}_k \overset{36.5}{=} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\right) \overset{36.4}{=} \frac{\partial L}{\partial q^k} \overset{\text{cyclic coord.}}{=} 0 \tag{36.6}$$

36.3 Noether's theorem

Theorem 36.3.1 (Noether's theorem†). *Consider a field transformation*

$$\phi(x) \to \phi(x) + \alpha\delta\phi(x) \tag{36.7}$$

where α is an infinitesimal quantity and $\delta\phi$ is a small deformation. In case of a symmetry we obtain the following conservation law:

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - \mathcal{J}^\mu\right) = 0 \tag{36.8}$$

The factor between parentheses can be interpreted as a conserved current $j^\mu(x)$. Noether's theorem states that every symmetry of the form 36.7 leads to such a current.

The conservation can also be expressed in terms of a charge[1]*:*

$$\frac{dQ}{dt} = \frac{d}{dt}\int j^0 d^3x = 0 \tag{36.9}$$

Definition 36.3.2 (Stress-energy tensor). Consider a field transformation

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu\partial_\mu\phi(x)$$

[1]The conserved current and its associated charge are called the **Noether current** and **Noether charge**.

Because the Lagrangian is a scalar it transforms similarly:

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L}) \tag{36.10}$$

This leads to the existence of 4 conserved currents. These can be used to define the stress-energy tensor:

$$\boxed{T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L}\delta^\mu{}_\nu} \tag{36.11}$$

36.4 Hamilton's equations

Definition 36.4.1 (Canonical coordinates). Consider the generalized coordinates $(q, \dot{p}, t)$ from the Lagrangian formalism. Using these we can define a new set of coordinates, called canonical coordinates, by exchanging the time-derivatives $\dot{q}^i$ in favour of the conjugate momenta p_i (see definition 36.5) and leaving the coordinates q^i and t invariant.

Definition 36.4.2 (Hamiltonian function). The (classical) Hamiltonian function is defined as follows:

$$H(q, p, t) = \sum_i p_i \dot{q}^i - L(q, p, t) \tag{36.12}$$

Formula 36.4.3 (Hamilton's equations[2]).

$$\boxed{\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} && (36.13)\\ -\dot{p}_i &= \frac{\partial H}{\partial q^i} && (36.14)\\ -\frac{\partial L}{\partial t} &= \frac{\partial H}{\partial t} && (36.15) \end{aligned}}$$

The formula to obtain the Hamiltonian from the Lagrangian is an application of the following more general Legendre transformation:

Definition 36.4.4 (Legendre transformation). Consider an equation of the following form:

$$df = udx + vdy \tag{36.16}$$

where $u = \frac{\partial f}{\partial x}$ and $v = \frac{\partial f}{\partial y}$.

Suppose we want to perform a coordinate transformation $(x, y) \to (u, y)$ while preserving the general form of 36.16 for differential quantities. To do this we consider the function

$$g = f - ux \tag{36.17}$$

[2]Also known as the *canonical equations of Hamilton*.

Differentiating gives

$$\begin{aligned} dg &= df - udx - xdu \\ &= (udx + vdy) - udx - xdu \\ &= vdy - xdu \end{aligned}$$

which has the form of 36.16 as desired. The quantities v and x are now given by

$$x = -\frac{\partial g}{\partial u} \qquad \text{and} \qquad v = \frac{\partial g}{\partial y} \tag{36.18}$$

The transition $f \to g$ defined by equations 36.16 and 36.17 is called a Legendre transformation.

Remark 36.4.5. Although the previous derivation used only 2 coordinates, the definition of Legendre transformations can easily be generalized to more coordinates.

36.5 Hamilton-Jacobi equation

36.5.1 Canonical transformations

Definition 36.5.1 (Canonical transformations). A canonical transformation is a transformation that leaves the Hamiltonian equations of motion unchanged. Mathematically this means that the transformations leave the action invariant up to a constant, or equivalently, they leave the Lagrangian invariant up to a complete time-derivative:

$$\sum_i \dot{q}^i p_i - H(q,p,t) = \sum_i \dot{Q}^i P_i - K(Q,P,t) - \frac{dG}{dt}(Q,P,t) \tag{36.19}$$

The function G is called the generating function of the canonical transformation. The choice of G uniquely determines the transformation.

Formula 36.5.2 (Hamilton-Jacobi equation). Sufficient conditions for the generating function S are given by:

$$P_i = \frac{\partial S}{\partial Q^i}$$
$$Q^i = \frac{\partial S}{\partial P_i}$$

and

$$K = H + \frac{\partial S}{\partial t}$$

Choosing the new Hamiltonian function K to be 0 gives the Hamilton-Jacobi equation:

$$\boxed{H\left(q, \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0} \tag{36.20}$$

The function S is called **Hamilton's principal function**.

Property 36.5.3. The new coordinates P_i and Q^i are all constants of motion. This follows immediately from the choice $K = 0$.

Definition 36.5.4 (Hamilton's characteristic function). If the system is time-independent it follows from the HJE that the principal function is of the form

$$S(q, p, t) = W(q, p) - Et \tag{36.21}$$

where E is a constant. The time-independent function W is called Hamilton's characteristic function.

Substituting this result in the HJE results in

$$H\left(q, \frac{\partial S}{\partial q}\right) = E \tag{36.22}$$

In time-independent systems the Hamiltonian function is thus a constant of motion and we call it the **energy** of the system.

36.5.2 Stäckel potential

Remark 36.5.5. If the principal function can be separated into n equations, the HJE splits up into n equations of the form

$$h_i\left(q^i, \frac{dS}{dq^i}, \alpha_i\right) = 0 \tag{36.23}$$

The partial differential equation for S can thus be rewritten as a system of n ordinary differential equations.

Theorem 36.5.6 (Stäckel condition). *Using an orthogonal coordinate system, the Hamilton-Jacobi equation is separable if and only if the potential is of the following form:*

$$V(q) = \sum_{i=1}^{n} \frac{1}{G_i^2(q)} W_i(q^i) \tag{36.24}$$

whenever the Hamiltonian function can be written as

$$H(q, p) = \frac{1}{2} \sum_i \frac{p_i^2}{G_i^2(q)} + V(q) \tag{36.25}$$

These potentials are called ***Stäckel potentials****.*

Chapter 37

Phase space

37.1 Phase space

Definition 37.1.1 (Phase space). The set of all possible n-tuples[1] (q^i, p_i) of generalized coordinates and associated momenta is called the phase space of the system.

Definition 37.1.2 (Rotation). A rotation is the change of a coordinate for which every possible value is allowed.

Definition 37.1.3 (Libration). A libration is the change of coordinate for which only a subset of the total range is allowed. It is the generalization of an oscillation.

37.2 Material derivative

Definition 37.2.1 (Lagrangian derivative[2]). Let $a(\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}}, t)$ be a property of a system defined at every point of the system. The Lagrangian derivative along a path $(\vec{\boldsymbol{r}}(t), \vec{\boldsymbol{v}}(t))$ in phase space is given by:

$$\begin{aligned}\frac{Da}{Dt} &= \lim_{\Delta t \to 0} \frac{a(\vec{\boldsymbol{r}} + \Delta\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}} + \Delta\vec{\boldsymbol{v}}, t + \Delta t) - a(\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}}, t)}{\Delta t} \\ &= \frac{\partial a}{\partial t} + \frac{d\vec{\boldsymbol{r}}}{dt} \cdot \frac{\partial a}{\partial \vec{\boldsymbol{r}}} + \frac{d\vec{\boldsymbol{v}}}{dt} \cdot \frac{\partial a}{\partial \vec{\boldsymbol{v}}} \\ &= \frac{\partial a}{\partial t} + \vec{\boldsymbol{v}} \cdot \nabla a + \frac{d\vec{\boldsymbol{v}}}{dt} \cdot \frac{\partial a}{\partial \vec{\boldsymbol{v}}} \end{aligned} \tag{37.1}$$

The second term $\vec{\boldsymbol{v}} \cdot \nabla a$ in this equation is called the **advective** term.

Remark 37.2.2. In the case that $a(\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}}, t)$ is a tensor field the gradient ∇ has to be replaced by the covariant derivative. The advective term is then called the **convective** term.

[1] Not only those as given by the equations of motion.

[2] Also known as the **material derivative**, especially when applied to fluidum mechanics.

Corollary 37.2.3. If we take $a(\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}}, t) = \vec{\boldsymbol{r}}$ we obtain:

$$\frac{D\vec{\boldsymbol{r}}}{Dt} = \vec{\boldsymbol{v}} \tag{37.2}$$

37.3 Liouville's theorem

Formula 37.3.1 (Liouville's lemma). Consider a phase space volume element dV_0 moving along a path $(\vec{\boldsymbol{r}}(t), \vec{\boldsymbol{v}}(t)) \equiv (\vec{\boldsymbol{x}}(t))$. The Jacobian $J(\vec{\boldsymbol{x}}, t)$ associated with this motion is given by:

$$J(\vec{\boldsymbol{x}}, t) = \frac{dV}{dV_0} = \det\left(\frac{\partial \vec{\boldsymbol{x}}}{\partial \vec{\boldsymbol{x}}_0}\right) = \sum_{ijklmn} \varepsilon_{ijklmn} \frac{\partial x^1}{\partial x_0^i} \frac{\partial x^2}{\partial x_0^j} \frac{\partial x^3}{\partial x_0^k} \frac{\partial x^4}{\partial x_0^l} \frac{\partial x^5}{\partial x_0^m} \frac{\partial x^6}{\partial x_0^n} \tag{37.3}$$

The Lagrangian derivative of this Jacobian then becomes:

$$\frac{DJ}{Dt} = (\nabla \cdot \vec{\boldsymbol{x}})J \tag{37.4}$$

Furthermore using the Hamiltonian equations 36.13 it is easy to prove that

$$\nabla \cdot \vec{\boldsymbol{x}} = 0 \tag{37.5}$$

Theorem 37.3.2 (Liouville's theorem). *Let $V(t)$ be a phase space volume containing a fixed set of particles. Application of Liouville's lemma gives:*

$$\frac{DV}{Dt} = \frac{D}{Dt}\int_{\Omega(t)} d^6x = \frac{D}{Dt}\int_{\Omega_0} J(\vec{\boldsymbol{x}}, t) d^6x_0 = 0 \tag{37.6}$$

It follows that the phase space volume of a Hamiltonian system[3] is invariant with respect to time-evolution.

Formula 37.3.3 (Boltzmann's transport equation). Let $F(\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}}, t)$ be the mass distribution function:

$$M_{tot} = \int_{\Omega(t)} F(\vec{\boldsymbol{x}}, t) d^6x \tag{37.7}$$

From the conservation of mass we can derive the following formula:

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \frac{d\vec{\boldsymbol{r}}}{dt} \cdot \frac{\partial F}{\partial \vec{\boldsymbol{r}}} - \nabla V \cdot \frac{\partial F}{\partial \vec{\boldsymbol{v}}} = 0 \tag{37.8}$$

This formula is a partial differential equation in 7 variables which can be solved to obtain $F(\vec{\boldsymbol{x}}, t)$.

Theorem 37.3.4 (Poincaré recurrence theorem). *Consider a Hamiltonian system with a finite phase space $\mathcal{V}$ (for example when the system is trapped in a potential well). By Liouville's theorem, the phase flow generated by the equations of motion is a volume preserving map $g : \mathcal{V} \to \mathcal{V}$. Let $\mathcal{V}_0$ be the phase space volume of the system. For every point $x_0 \in \mathcal{V}_0$ and for every neighbourhood U of x_0 there exists a point $y \in U$ such that $g^n y \in U$ for every $n \in \mathbb{N}$.*

[3] A system that satisfies Hamilton's equations of motion.

37.4 Continuity equation

Formula 37.4.1 (Reynolds transport theorem[4]). Consider a quantity

$$F = \int_{V(t)} f(\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}}, t) dV$$

Using equation 37.4 and the divergence theorem 17.23 we can obtain:

$$\boxed{\frac{DF}{Dt} = \int_V \frac{\partial f}{\partial t} dV + \oint_S f\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{S}}} \tag{37.9}$$

Formula 37.4.2 (Continuity equations). For a conserved quantity the equation above becomes:

$$\frac{Df}{Dt} + (\nabla \cdot \vec{\boldsymbol{v}}) f = 0 \tag{37.10}$$

$$\frac{\partial f}{\partial t} + \nabla \cdot (f\vec{\boldsymbol{v}}) = 0 \tag{37.11}$$

If we set $f = \rho$ (mass density) then the first equation is called the **Lagrangian continuity equation** and the second equation is called the **Eulerian continuity equation**. Both equations can be found by pulling the Lagrangian derivative inside the integral on the left-hand side of 37.9.

Corollary 37.4.3. Combining the Reynolds transport theorem with the Lagrangian continuity equation gives the following identity for an arbitrary function f:

$$\frac{D}{Dt} \int_V \rho f dV = \int_V \rho \frac{Df}{Dt} dV \tag{37.12}$$

[4]This is a 3D extension of the *Leibniz integral rule*.

Chapter 38

Fluid mechanics

38.1 Cauchy stress tensor

Theorem 38.1.1 (Cauchy's stress theorem[1]). *Knowing the stress vectors acting on the coordinate planes through a point A is sufficient to calculate the stress vector acting on an arbitrary plane passing through A.*

The *Cauchy stress theorem* is equivalent to the existence of the following tensor:

Definition 38.1.2 (Cauchy stress tensor). The Cauchy stress tensor is a $(0,2)$-tensor $\mathbf{T}$ that gives the relation between a stress vector associated to a plane and the normal vector $\vec{\boldsymbol{n}}$ to that plane:

$$\vec{\boldsymbol{t}}_{(\vec{\boldsymbol{n}})} = \mathbf{T}(\vec{\boldsymbol{n}}) \tag{38.1}$$

Example 38.1.3. For identical particles, the stress tensor is given by:

$$\mathbf{T} = -\rho\langle \vec{\boldsymbol{w}} \otimes \vec{\boldsymbol{w}} \rangle \tag{38.2}$$

where $\vec{\boldsymbol{w}}$ is the random component of the velocity vector and $\langle\cdot\rangle$ denotes the expectation value (see 32.19).

Theorem 38.1.4 (Cauchy's lemma). *The stress vectors acting on opposite planes are equal in magnitude but opposite in direction:*

$$\vec{\boldsymbol{t}}_{(-\vec{\boldsymbol{n}})} = -\vec{\boldsymbol{t}}_{(\vec{\boldsymbol{n}})} \tag{38.3}$$

Formula 38.1.5 (Cauchy momentum equation). From Newton's second law 35.1 it follows that:

$$\frac{D\vec{\boldsymbol{P}}}{Dt} = \int_V \vec{\boldsymbol{f}}(\vec{\boldsymbol{x}}, t)dV + \oint_S \vec{\boldsymbol{t}}(\vec{\boldsymbol{x}}, t)dS \tag{38.4}$$

[1] Also known as **Cauchy's fundamental theorem**.

where $\vec{\boldsymbol{P}}$ is the momentum density, $\vec{\boldsymbol{f}}$ are body forces and $\vec{\boldsymbol{t}}$ are surface forces (such as shear stress). Using Cauchy's stress theorem and the divergence theorem 17.23 we get

$$\frac{D\vec{\boldsymbol{P}}}{Dt} = \int_V \left[\vec{\boldsymbol{f}}(\vec{\boldsymbol{x}}, t) + \nabla \cdot \mathbf{T}(\vec{\boldsymbol{x}}, t)\right] dV \tag{38.5}$$

The left-hand side can be rewritten using 37.12 as

$$\int_V \rho \frac{D\vec{\boldsymbol{v}}}{Dt} dV = \int_V \left[\vec{\boldsymbol{f}}(\vec{\boldsymbol{x}}, t) + \nabla \cdot \mathbf{T}(\vec{\boldsymbol{x}}, t)\right] dV \tag{38.6}$$

Chapter 39

Optics

39.1 General

39.1.1 Conservation of energy

From the law of conservation of energy we can derive the following formula:

$$\boxed{T + R + A = 1} \tag{39.1}$$

where

T : Transmission coefficient
R : Reflection coefficient
A : Absorption coefficient

39.1.2 Photon

Formula 39.1.1 (Energy).

$$E = h\nu = \hbar\omega = \frac{hc}{\lambda} \tag{39.2}$$

Formula 39.1.2 (Momentum).

$$p = \frac{h}{\lambda} = \hbar k \tag{39.3}$$

where formula 39.4 was used in the last step.

Remark. These formulas can also be (approximately) used for particles for which the rest mass (energy) is negligible.

39.2 Plane wave

Formula 39.2.1 (Wave number).

$$k = \frac{2\pi}{\lambda} \tag{39.4}$$

Formula 39.2.2 (Plane wave). Following equations represent a plane wave moving in the x-direction and polarized in the xy-plane:

$$\vec{\boldsymbol{E}}(x,t) = \mathrm{Re}\left\{A\,\exp\left[i\left(kx - \omega t + \phi\right)\right]\right\}\vec{\boldsymbol{e}}_y \tag{39.5}$$

$$\vec{\boldsymbol{E}}(x,t) = \mathrm{Re}\left\{A\,\exp\left[2\pi i\left(\frac{x}{\lambda} - \frac{t}{T} + \frac{\phi}{2\pi}\right)\right]\right\}\vec{\boldsymbol{e}}_y \tag{39.6}$$

39.3 Refraction

Formula 39.3.1 (Refraction).

$$v_2 = \frac{v_1}{n} \tag{39.7}$$

Formula 39.3.2 (Diëlectric function). In the case of non-magnetic materials ($\mu_r \approx 1$), we can write the diëlectric function as following:

$$\epsilon = \epsilon_r + i\epsilon_i = \widetilde{n}^2 = (n + ik)^2 \tag{39.8}$$

Where $\widetilde{n}$ is the **complex refractive index** and k is the **extinction coefficient**.

39.4 Absorption

Theorem 39.4.1 (Law of Lambert-Beer[†]).

$$\frac{I(x)}{I(0)} = exp\left(-\frac{4\pi\nu k}{c}x\right) \tag{39.9}$$

Definition 39.4.2 (Absorption coefficient). The constant factor in the Lambert-Beer law is called the absorption coefficient.

$$\alpha = \frac{4\pi\nu k}{c} \tag{39.10}$$

39.5 Diffraction

Chapter 40

Astronomy

40.1 Ellipsoidal coordinates

We start from folowing parametrized equation:

$$f(\tau) = \frac{x^2}{\tau+\alpha} + \frac{y^2}{\tau+\beta} + \frac{z^2}{\tau+\gamma} - 1 \tag{40.1}$$

where $\alpha < \beta < \gamma < 0$. By multiplying away the denominators and setting $f(\tau) = 0$ we obtain a polynomial equation of degree 3 in τ. This polynomial can be formally factorised as

$$-(\tau-\lambda)(\tau-\mu)(\tau-\nu) \tag{40.2}$$

such that the solutions (λ, μ, ν) obey following rules:

$$\begin{cases} \nu & \in &]-\gamma, -\beta[\\ \mu & \in &]-\beta, -\alpha[\\ \lambda & \in &]-\alpha, +\infty[\end{cases}$$

From previous two equations we can find a solution for x^2 by multiplying by $(\tau+\alpha)$ and letting $\tau \to -\alpha$. Solutions for y^2 and z^2 can be found in a similar way:

$$\begin{cases} x^2 &= \dfrac{(\lambda+\alpha)(\mu+\alpha)(\nu+\alpha)}{(\beta-\alpha)(\gamma-\alpha)} \\ y^2 &= \dfrac{(\lambda+\beta)(\mu+\beta)(\nu+\beta)}{(\beta-\alpha)(\beta-\gamma)} \\ z^2 &= \dfrac{(\lambda+\gamma)(\mu+\gamma)(\nu+\gamma)}{(\alpha-\gamma)(\beta-\gamma)} \end{cases} \tag{40.3}$$

For these solutions multiple cases can be considered. We can define different surfaces by fixing τ at different values.

40.1.1 Ellipsoid: $\tau = \lambda$

First we look at the surfaces defined by fixing $\tau = \lambda$ in equation 40.1. By noting that all denominators are positive in this case, we see that the obtained surface is an ellipsoid with the x-axis as the shortest axis. By letting $\lambda \to +\infty$ we obtain the equation of a sphere with radius $\sqrt{\lambda}$. If $\lambda \to -\alpha$ we get an ellipse in the yz-plane. This ellipse is called the **focal ellipse**.

40.1.2 One-sheet hyperboloid: $\tau = \mu$

By fixing $\tau = \mu$ in 40.1 we obtain the equation of one-sheet hyperboloid (also called a **hyperbolic hyperboloid**) around the x-axis. By letting $\mu \to -\alpha$ the hyperboloid collapses in the yz-plane and we obtain the surface outside the focal ellipse. If $\mu \to -\beta$ the hyperboloid becomes degenerate and we get the surface inside the **focal hyperbola** defined by

$$\frac{x^2}{\alpha - \beta} + \frac{z^2}{\gamma - \beta} = 1 \tag{40.4}$$

This hyperbola intersects the z-plane in the foci of the focal ellipse.

40.1.3 Two-sheet hyperboloid: $\tau = \nu$

By fixing $\tau = \nu$ in 40.1 we obtain the equation of two-sheet hyperboloid (also called an **elliptic hyperboloid**) around the z-axis. By letting $\nu \to -\beta$ the hyperboloid becomes degenerate and we obtain the surface outside the focal hyperbola 40.4. If $\nu \to -\gamma$ the two sheets coincide in the xy-plane.

40.1.4 Hamiltonian function

When wrting out the kinetic energy in ellipsoidal coordinates by applying the chain rule for differentiation to the Cartesian kinetic energy while noting that mixed terms of the form $\frac{\partial x^a}{\partial \lambda^i}\frac{\partial x^a}{\partial \lambda^j}$ cancel out when writing them out using 40.3 it is clear that the Hamiltonian function can be spearated as follows:

$$H = \frac{1}{2}\left(\frac{p_\lambda^2}{Q_\lambda^2} + \frac{p_\mu^2}{Q_\mu^2} + \frac{p_\nu^2}{Q_\nu^2}\right) + V \tag{40.5}$$

where $Q_j^2 = \sum_i \left(\frac{\partial x^i}{\partial \lambda^j}\right)^2$ are the metric coefficients in ellipsoidal coordinates.

These coefficients can be calculated by noting that $\frac{\partial x^i}{\partial \lambda} = \frac{1}{x^i}\frac{\partial (x^i)^2}{\partial \lambda}$ and putting $\frac{1}{(\lambda+\alpha)(\lambda+\beta)(\lambda+\gamma)}$ in the front. Furthermore the coefficient belonging to λ^2, μ^2, ν^2, mixed terms and others can

be calculated easily. By doing so we obtain following result

$$Q_\lambda^2 = \frac{1}{4}\frac{(\lambda-\mu)(\lambda-\nu)}{(\lambda+\alpha)(\lambda+\beta)(\lambda+\gamma)} \tag{40.6}$$

which is also valid for μ and ν by applying cyclic permutation to the coordinates.

Following from the Stäckel conditions 36.24 the potential must be of the form

$$V = \sum_i \frac{W_i(\lambda^i)}{Q_i^2} \tag{40.7}$$

if we want to obtain a seperable Hamilton-Jacobi equation. Due to the disjunct nature of λ, μ and ν we can consider W_λ, W_μ and W_ν as three parts of a single function $G(\tau)$ given by:

$$G(\tau) = -4(\tau+\beta)W_\tau(\tau) \tag{40.8}$$

The 3D potential is thus completely determined by a 1D function $G(\tau)$.

40.1.5 Hamilton-Jacobi equation

If we consider a time-independent system we can use 36.22 as our starting point. If we multiply this equation by $(\lambda-\mu)(\lambda-\nu)(\mu-\nu)$ we obtain

$$(\mu-\nu)\left[2(\lambda+\alpha)(\lambda+\beta)(\lambda+\gamma)\left(\frac{dS^\lambda(\lambda)^2}{d\lambda}\right)\right.$$
$$\left. -(\lambda+\alpha)(\lambda+\gamma)G(\lambda) - \lambda^2 E\right] + \text{cyclic permutations} = 0 \tag{40.9}$$

where we rewrote the multiplication factor in the form $a\lambda^2 + b\mu^2 + c\nu^2$ before multiplying the RHS of 36.22. This equation can be elegantly rewritten as

$$(\mu-\nu)U(\lambda) + (\lambda-\mu)U(\nu) + (\nu-\lambda)U(\mu) = 0 \tag{40.10}$$

Differentiating twice with respect to any λ^i gives $U''(\tau) = 0$ or equivalently

$$U(\tau) = I_3 - I_2\tau \tag{40.11}$$

where I_2 and I_3 are two new first integrals of motion.

From the Hamiltonian-Jacobi equations of motion one can calculate the conjugate momenta $p_\tau = \frac{dS^\tau}{d\tau}$. After a lengthy calculation we obtain

$$p_\tau^2 = \frac{1}{2(\tau+\beta)}\left[E - V_{\text{eff}}(\tau)\right] \tag{40.12}$$

where the effective potential is given by

$$\boxed{V_{\text{eff}} = \frac{J}{\tau+\alpha} + \frac{K}{\tau+\gamma} - G(\tau)} \tag{40.13}$$

where J and K are two conserved quantities given by

$$J = \frac{\alpha^2 E + \alpha I_2 + I_3}{\alpha - \gamma} \quad \text{and} \quad K = \frac{\gamma^2 E + \gamma I_2 + I_3}{\gamma - \alpha}$$

To be physically acceptable, p_τ^2 should be positive. This leads to following conditions on the energy:

$$\begin{cases} E \geq V_{\text{eff}}(\lambda) \\ E \geq V_{\text{eff}}(\mu) \\ E \leq V_{\text{eff}}(\nu) \end{cases} \tag{40.14}$$

The generating $G(\tau)$ function should also satisfy some conditions. First we note that we can rewrite our Stäckel potential $V(\lambda, \mu, \nu)$ as

$$V = -\frac{1}{\lambda - \nu}\left(\frac{F(\lambda) - F(\mu)}{\lambda - \mu} - \frac{F(\mu) - F(\nu)}{\mu - \nu}\right) \leq 0 \tag{40.15}$$

where $F(\tau) = (\tau + \alpha)(\tau + \gamma)G(\tau)$.

For $\lambda \to +\infty$ (or $r^2 \to +\infty$) we get $V \approx -\frac{F(\lambda)}{\lambda^2} \approx -G(\lambda)$. Because $V \sim \lambda^{-1}$ it is clear that $G(\tau)$ cannot decay faster than $\lambda^{-1/2}$ at infity. Furthermore we can interpret 40.15 as an approximation of $-F''(\tau)$. So it follows that $F(\tau)$ should be convex. For $\tau \to -\gamma$ we get

$$\begin{cases} \alpha + \tau < 0 \\ \tau + \gamma \to 0 \end{cases}$$

So if $G(\tau)$ decays faster than $\dfrac{1}{\tau + \gamma}$ then $F(\tau) \to -\infty$ which is not possible for a convex function.

To fullfil these conditions we assume that the generating function can be written as

$$\boxed{G(\tau) = \frac{GM}{\sqrt{\gamma_0 + \tau}}} \tag{40.16}$$

where G is the gravitational constant and M is the galactic mass.

Theorem 40.1.1 (Kuzmin's theorem). *The spatial mass density function generated by a Stäckel potential is completely determined by a function of the form $\rho(z)$.*

Corollary 40.1.2. For triaxial mass models in ellipsoidal coordinates the axial ratios are inversely proportional to the axial ratios of the coordinate system.

Part VIII

Electromagnetism

Chapter 41

Electricity

41.1 Resistance R

41.1.1 Conductivity

Definition 41.1.1 (Drift velocity). The average speed of the independent charge carriers is the drift velocity $\vec{\boldsymbol{v_d}}$. It is important to remark that v_d is not equal to the propagation speed of the electricity[1].

Formula 41.1.2 (Conductivity).

$$\sigma = nq\mu \tag{41.1}$$

Formula 41.1.3 (Resistivity).

$$\rho = \frac{1}{\sigma} \tag{41.2}$$

Formula 41.1.4 (Mobility).

$$\mu = \frac{v_d}{E} \tag{41.3}$$

41.1.2 Current density

Formula 41.1.5. Let A be the cross section of a conductor. Let $\vec{\boldsymbol{J}}$ be the current density though A. The current trough A is then given by:

$$I = \iint_A \vec{\boldsymbol{J}} \cdot \hat{n} dS \tag{41.4}$$

Formula 41.1.6 (Free current). The current density generated by free charges is given by:

$$\vec{\boldsymbol{J}} = nq\vec{\boldsymbol{v}}_d \tag{41.5}$$

[1] It is several orders of magnitude smaller.

41.1.3 Pouillet's law

$$R = \rho \frac{l}{A} \tag{41.6}$$

where:

ρ : resistivity of the material
l : length of the resistor
A : cross-sectional area

41.2 Ohm's law

Formula 41.2.1 (Ohm's law).

$$\boxed{\vec{\boldsymbol{J}} = \overset{\leftrightarrow}{\sigma} \cdot \vec{\boldsymbol{E}}} \tag{41.7}$$

where $\overset{\leftrightarrow}{\sigma}$ is the conductivity tensor.

Formula 41.2.2 (Ohm's law in wires). The following formula can be found by combining equations 41.1, 41.2,41.4 and 41.7 and by assuming that the conductivity tensor can be simplified to a scalar (follows from the isotropic behaviour of normal resistors):

$$U = RI \tag{41.8}$$

41.3 Capacitance C

Definition 41.3.1 (Capacitance). The capacitance is a (geometrical) value that reflects the amount of charge a certain body can store.

$$C = \frac{q}{V} \tag{41.9}$$

41.4 Electric dipole

Formula 41.4.1 (Electric dipole).

$$\vec{\boldsymbol{p}} = q\vec{\boldsymbol{a}} \tag{41.10}$$

Where:

q : charge of the positive particle
$\vec{\boldsymbol{a}}$: vector pointing from the negative to the positive particle

Formula 41.4.2 (Energy). If an electric dipole is placed in an electric field, its potential energy is:

$$U = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}} \tag{41.11}$$

Formula 41.4.3 (Torque). If an electric dipole is placed in an electric field, a torque is generated:

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}} \tag{41.12}$$

Chapter 42

Magnetism

42.1 Magnetic field

42.1.1 Magnetizing field $\vec{H}$

The magnetizing field $\vec{H}$ is the field resulting from all exterior sources.

42.1.2 Magnetization $\vec{M}$

$$\vec{\boldsymbol{M}} = \chi \vec{\boldsymbol{H}} \tag{42.1}$$

where χ is the magnetic susceptibility.

42.1.3 Magnetic induction $\vec{B}$

The magnetic induction $\vec{B}$ is the field resulting from exterior sources and interior magnetization. (It is the 'real', detectable field.) In vacuum we have the following relation between the magnetic induction B, the magnetizing field H and the magnetization M:

$$\vec{\boldsymbol{B}} = \mu_0 \left(\vec{\boldsymbol{H}} + \vec{\boldsymbol{M}} \right) \tag{42.2}$$

By combining this formula with formula 42.1 we get[1]:

$$\vec{\boldsymbol{B}} = \mu_0 \left(1 + \chi \right) \vec{\boldsymbol{H}} \tag{42.3}$$

[1] This equation is only valid in linear media.

Definition 42.1.1 (Magnetic permeability). The proportionality constant in formula 42.3 is called the magnetic permeability:

$$\mu = \mu_0(1 + \chi) \tag{42.4}$$

where μ_0 is the magnetic permeability of the vacuum. The factor $1 + \chi$ is called the relative permeability and it is often denoted by μ_r.

42.1.4 Tensorial formulation

In anistropic materials we have to use the tensorial formulation.

$$B_i = \sum_j \mu_{ij} H_j \tag{42.5}$$

$$M_i = \sum_j \chi_{ij} H_j \tag{42.6}$$

Both μ and χ are tensors of rank 2.

42.2 Magnetic multipoles

42.2.1 Dipole

$$\vec{\boldsymbol{m}} = IS\vec{\boldsymbol{u}}_n \tag{42.7}$$

42.3 Electric charges in a magnetic field

42.3.1 Cyclotron

Formula 42.3.1 (Gyroradius).

$$r = \frac{mv_\perp}{|q|B} \tag{42.8}$$

Formula 42.3.2 (Gyrofrequency[2]).

$$\omega = \frac{|q|B}{m} \tag{42.9}$$

[2]Also called the Larmor frequency.

Chapter 43

Maxwell equations

43.1 Lorentz force

Formula 43.1.1 (Lorentz force).

$$\vec{\boldsymbol{F}} = q\left(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\right) \tag{43.1}$$

Formula 43.1.2 (Lorentz force density).

$$\vec{\boldsymbol{f}} = \rho\vec{\boldsymbol{E}} + \vec{\boldsymbol{J}} \times \vec{\boldsymbol{B}} \tag{43.2}$$

43.2 Differential Maxwell equations

Formula 43.2.1 (Gauss' law for electricity).

$$\nabla \cdot \vec{\boldsymbol{E}} = \frac{\rho}{\varepsilon} \tag{43.3}$$

Formula 43.2.2 (Gauss' law for magnetism).

$$\nabla \cdot \vec{\boldsymbol{B}} = 0 \tag{43.4}$$

Formula 43.2.3 (Faraday's law).

$$\nabla \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \tag{43.5}$$

Formula 43.2.4 (Maxwell's law[1]).

$$\nabla \times \vec{\boldsymbol{H}} = \frac{\partial \vec{\boldsymbol{D}}}{\partial t} + \vec{\boldsymbol{J}} \tag{43.6}$$

[1] Also called the law of Maxwell-Ampère.

43.3 Potentials

43.3.1 Decomposition in potentials

Remembering the Helmholtz decomposition (equation 17.18) we can derive the following general form for $\vec{\boldsymbol{B}}$ starting from Gauss' law 43.4:

$$\boxed{\vec{\boldsymbol{B}} = \nabla \times \vec{\boldsymbol{A}}} \tag{43.7}$$

where $\vec{\boldsymbol{A}}$ is the magnetic potential.

Combining equation 43.7 with Faraday's law 43.5 and rewriting it a bit, gives the following general form for $\vec{\boldsymbol{E}}$:

$$\boxed{\vec{\boldsymbol{E}} = -\nabla V - \frac{\partial \vec{\boldsymbol{A}}}{\partial t}} \tag{43.8}$$

where V is the electrostatic potential.

43.3.2 Conditions

Substituting the expressions 43.7 and 43.8 into Gauss' law 43.3 and Maxwell's law 43.6 gives the following two (coupled) conditions for the electromagnetic potentials:

$$\triangle \vec{\boldsymbol{A}} - \varepsilon\mu \frac{\partial^2 \vec{\boldsymbol{A}}}{\partial t^2} = \nabla \left(\nabla \cdot \vec{\boldsymbol{A}} + \varepsilon\mu \frac{\partial V}{\partial t} \right) - \mu \vec{\boldsymbol{J}} \tag{43.9}$$

$$\triangle V - \varepsilon\mu \frac{\partial^2 V}{\partial t^2} = -\frac{\partial}{\partial t} \left(\nabla \cdot \vec{\boldsymbol{A}} + \varepsilon\mu \frac{\partial V}{\partial t} \right) - \frac{\rho}{\varepsilon} \tag{43.10}$$

43.3.3 Gauge transformations

Looking at equation 43.7, it is clear that a transformation $\vec{\boldsymbol{A}} \to \vec{\boldsymbol{A}} + \nabla\psi$ has no effect on $\vec{\boldsymbol{B}}$ due to property 17.15. To compensate this in equation 43.8, we also have to perform the transformation $V \to V - \frac{\partial \psi}{\partial t}$.

The (scalar) function $\psi(\vec{\boldsymbol{r}}, t)$ is called a **gauge function**. The transformations are called **gauge transformations**.

Definition 43.3.1 (Gauge fixing conditions). Conditions that fix a certain gauge (or class of gauge transformations) are called gauge fixing conditions. These select one of many physically equivalent configurations.

43.3.4 Lorenz gauge

A first example of a gauge fixing condition is the Lorenz gauge[2] :

$$\boxed{\nabla \cdot \vec{\boldsymbol{A}} + \varepsilon\mu \frac{\partial V}{\partial t} = 0} \tag{43.11}$$

When using this gauge fixing condition, equations 43.9 and 43.10 become uncoupled and can be rewritten as:

$$\Box \vec{\boldsymbol{A}} = -\mu \vec{\boldsymbol{J}} \tag{43.12}$$

$$\Box V = -\frac{\rho}{\varepsilon} \tag{43.13}$$

To see which gauge functions ψ are valid in this case we perform a transformation as explained above:

$$\vec{\boldsymbol{A}}' = \vec{\boldsymbol{A}} + \nabla\psi \qquad \text{and} \qquad V' = V - \frac{\partial \psi}{\partial t}$$

Substituting these transformations in equation 43.11 and using the fact that both sets of potentials $(\vec{\boldsymbol{A}}, V)$ and $(\vec{\boldsymbol{A}}', V)$ satisfiy the Lorenz gauge 43.11 gives the following condition for the gauge function ψ:

$$\Box\psi = 0 \tag{43.14}$$

Example 43.3.2 (Alternative gauges). Apart from the Lorenz gauge 43.11, there is also the Coulomb gauge:

$$\nabla \cdot \vec{\boldsymbol{A}} = 0 \tag{43.15}$$

43.4 Energy and momentum

Definition 43.4.1 (Poynting vector).

$$\boxed{\vec{\boldsymbol{S}} = \vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}} \tag{43.16}$$

Definition 43.4.2 (Energy density).

$$\boxed{W = \frac{1}{2}\left(\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{D}} + \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{H}}\right)} \tag{43.17}$$

[2]Named after Ludvig Lorenz. Not to be confused with Hendrik Lorentz.

Part IX

Relativity Theory

Chapter 44

Special relativity

44.1 Lorentz transformations

Formula 44.1.1.

$$\beta = \frac{v}{c} \tag{44.1}$$

Formula 44.1.2 (Lorentz factor).

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{44.2}$$

Formula 44.1.3 (Lorentz transformations). Let $\mathbf{V}$ be a 4-vector. A Lorentz boost along the x^1-axis is given by the following transformation:

$$\boxed{\begin{array}{lcl} V'^0 &=& \gamma\left(V^0 - \beta V^1\right) \\ V'^1 &=& \gamma\left(V^1 - \beta V^0\right) \\ V'^2 &=& V^2 \\ V'^3 &=& V^3 \end{array}} \tag{44.3}$$

Remark 44.1.4. Putting $c = +\infty$ in the previous transformation formulas gives the Galilean transformations from classical mechanics.

44.2 Energy and momentum

Formula 44.2.1 (4-velocity).

$$U^\mu = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right) \tag{44.4}$$

or by applying the formulas for proper time and time dilatation we obtain:

$$U^\mu = (\gamma c, \gamma \vec{\boldsymbol{u}}) \tag{44.5}$$

Formula 44.2.2 (4-momentum).

$$p^\mu = m_0 U^\mu \tag{44.6}$$

or by setting $E = cp^0$:

$$p^\mu = \left(\frac{E}{c}, \gamma m_0 \vec{\boldsymbol{u}}\right) \tag{44.7}$$

Definition 44.2.3 (Relativistic mass). The factor γm_0 in the momentum 4-factor is called the relativistic mass. By introducing this quantity (and denoting it by m), the classic formula $\vec{\boldsymbol{p}} = m\vec{\boldsymbol{u}}$ for the 3-momentum can be generalized to 4-momenta p^μ.

Formula 44.2.4 (Relativistic energy relation).

$$\boxed{E^2 = p^2c^2 + m^2c^4} \tag{44.8}$$

This formula is often called the **Einstein relation**.

Chapter 45

General Relativity

45.1 Einstein field equations

Formula 45.1.1 (Einstein field equations). The Einstein field equations without the cosmological constant Λ read:

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}} \tag{45.1}$$

where $G_{\mu\nu}$ is the Einstein tensor 24.11 and $T_{\mu\nu}$ is the stress-energy tensor 36.11.

45.2 Schwarzschild metric

Formula 45.2.1 (Schwarzschild metric).

$$ds^2 = \left(1 - \frac{R_s}{r}\right) c^2 dt^2 - \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{45.2}$$

where R_s is the Schwarzschild radius given by

$$R_s = \frac{2GM}{c^2} \tag{45.3}$$

Theorem 45.2.2 (Birkhoff). *The Schwarzschild metric is the unique solution of the vacuum field equation under the additional constraints of asymptotic flatness and staticity.*

Part X

Quantum Mechanics

Chapter 46

Schrödinger equation

46.1 One dimension

46.1.1 Time independent Schrödinger equation (TISE)

Formula 46.1.1 (TISE).

$$\boxed{\hat{H}\psi(x) = E\psi(x)} \tag{46.1}$$

where $\hat{H}$ is the Hamiltonian of the system.

Property 46.1.2 (Orthonormality). Let $\{\psi_i\}$ be a set of eigenfunctions of the TISE. These functions obey the following orthogonality relations:

$$\int \psi_i^*(x)\psi_j(x)dx = \delta_{ij} \tag{46.2}$$

46.1.2 Time dependent Schrödinger equation (TDSE)

Formula 46.1.3 (TDSE).

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi} \tag{46.3}$$

where $\hat{H}$ is the Hamiltonian of the system.

Formula 46.1.4 (Massive particle in a time-independent potential).

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(\frac{\hat{p}\cdot\hat{p}}{2m} + \hat{V}(x)\right)\psi(x,t)} \tag{46.4}$$

Formula 46.1.5 (General solution).

$$\boxed{\psi(x,t) = \sum_E c_E \psi_E(x) e^{-\frac{i}{\hbar}Et}} \tag{46.5}$$

where the functions $\psi_E(x)$ are the eigenfunctions of the TISE 46.1. The coefficients c_E can be found using the orthogonality relations:

$$c_E = \left(\int \psi_E^*(x')\psi(x',t_0)dx' \right) e^{\frac{i}{\hbar}Et_0} \tag{46.6}$$

Chapter 47

Mathematical formalism

47.1 Postulates

47.1.1 Postulate 6: eigenfunction expansion

Definition 47.1.1 (Observable). An operator $\hat{A}$ which possesses a complete set of eigenfunctions is called an observable.

Formula 47.1.2. Let $|\Psi\rangle$ be an arbitrary wavefunction representing the system. Let the set $\{|\psi_n\rangle\}$ be a complete set of eigenfunctions of an observable of the system. The wavefunction $|\Psi\rangle$ can then be expanded as a linear combination of those eigenfunctions:

$$\boxed{|\Psi\rangle = \sum_n c_n|\psi_n\rangle + \int c_a|\psi_a\rangle da} \tag{47.1}$$

where the summation ranges over the discrete spectrum and the integral over the continuous spectrum.

Formula 47.1.3 (Closure relation). For a complete set of discrete eigenfunctions the closure relation[1] reads:

$$\sum_n |\psi_n\rangle\langle\psi_n| = \mathbb{1} \tag{47.2}$$

For a complete set of continuous eigenfunctions we have the following counterpart:

$$\int |i\rangle\langle i| di = \mathbb{1} \tag{47.3}$$

For a mixed set of eigenfunctions a similar relation is obtained by summing over the discrete eigenfunctions and integrating over the continuous eigenfunctions.

Remark. To simplify the notation we will almost always use the notation of equation 47.2 but implicitly integrate over the continuous spectrum.

[1] This relation is also called the **resolution of the identity**.

47.2 Uncertainty relations

Definition 47.2.1 (Commutator). Let $\hat{A}, \hat{B}$ be two operators. We define the commutator of $\hat{A}$ and $\hat{B}$ as follows:

$$\boxed{\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}} \tag{47.4}$$

Formula 47.2.2.

$$\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B} \tag{47.5}$$

Definition 47.2.3 (Anticommutator). Let $\hat{A}, \hat{B}$ be two operators. We define the anticommutator of $\hat{A}$ and $\hat{B}$ as follows:

$$\boxed{\left\{\hat{A}, \hat{B}\right\}_{+} = \hat{A}\hat{B} + \hat{B}\hat{A}} \tag{47.6}$$

Definition 47.2.4 (Compatible observables). Let $\hat{A}, \hat{B}$ be two observables. If there exists a complete set of functions $|\psi_n\rangle$ that are eigenfunctions of both $\hat{A}$ and $\hat{B}$ then the two operators are said to be compatible.

Formula 47.2.5 (Heisenberg uncertainty relation). Let $\hat{A}, \hat{B}$ be two observables. Let $\Delta A, \Delta B$ be the corresponding uncertainties.

$$\boxed{\Delta A \Delta B = \frac{1}{4}\left|\left\langle\left[\hat{A}, \hat{B}\right]\right\rangle\right|^2} \tag{47.7}$$

47.3 Matrix representation

Formula 47.3.1. The following formula gives the (m, n)-th element of the matrix representation of $\hat{A}$ with respect to the orthonormal basis $\{\psi_n\}$:

$$\boxed{A_{mn} = \langle\psi_m|\hat{A}|\psi_n\rangle} \tag{47.8}$$

Remark 47.3.2. The basis $\{\psi_n\}$ need not consist out of eigenfunctions of $\hat{A}$.

47.4 Slater determinants

Theorem 47.4.1 (Symmetrization postulate). *Let $\mathcal{H}$ be the Hilbert space belonging to a single particle. A system of n identical particles is described by a wave function Ψ belonging to either $S^n(\mathcal{H})$ or $\Lambda^n(\mathcal{H})$.*

Remark 47.4.2. In ordinary quantum mechanics this is a postulate, but in quantum field theory this is a consequence of the spin-statistics theorem of Pauli.

Formula 47.4.3. Let $\{\sigma\}$ be the set of all permutations of the sequence $(1, ..., n)$. Let $|\psi\rangle$ be the single-particle wave function. Fermionic systems are described by a wave function of the form

$$|\Psi_F\rangle = \sum_{\sigma} |\psi_{\sigma(1)}\rangle \cdots |\psi_{\sigma(n)}\rangle \tag{47.9}$$

Bosonic systems are described by a wave function of the form

$$|\Psi_B\rangle = \sum_{\sigma} \text{sgn}(\sigma)|\psi_{\sigma(1)}\rangle \cdots |\psi_{\sigma(n)}\rangle \tag{47.10}$$

Definition 47.4.4 (Slater determinant). Let $\{\phi_i(\vec{\boldsymbol{q}})\}_{i\leq N}$ be a set of wave functions (spin orbitals) describing a system of N identical particles. The (totally antisymmetric) wave function of the system is given by:

$$\boxed{\psi(\vec{\boldsymbol{q}}_1, ..., \vec{\boldsymbol{q}}_N) = \frac{1}{\sqrt{N!}} \det \begin{pmatrix} \phi_1(\vec{\boldsymbol{q}}_1) & \cdots & \phi_N(\vec{\boldsymbol{q}}_1) \\ \vdots & & \vdots \\ \phi_1(\vec{\boldsymbol{q}}_N) & \cdots & \phi_N(\vec{\boldsymbol{q}}_N) \end{pmatrix}} \tag{47.11}$$

47.5 Interaction picture

Let $\hat{H} = \hat{H}_0 + \hat{V}(t)$ be the total Hamiltonian of a system where $\hat{V}(t)$ is the interaction Hamiltonian. Let $|\psi(t)\rangle$ and $\hat{O}$ be the state vector and operator in the Schrödinger picture.

Formula 47.5.1. In the interaction picture we define the state vector as follows:

$$|\psi(t)\rangle_I = e^{\frac{i}{\hbar}\hat{H}_0 t}|\psi(t)\rangle \tag{47.12}$$

From this it follows that the operators in the interaction picture are given by:

$$\hat{O}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{O}e^{-\frac{i}{\hbar}\hat{H}_0 t} \tag{47.13}$$

Formula 47.5.2 (Schrödinger equation). Using the previous definition the Schrödinger equation can be rewritten as follows:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle_I = \hat{V}_I(t)|\psi(t)\rangle_I \tag{47.14}$$

The time-evolution of operators is given by:

$$\frac{d}{dt}\hat{O}_I(t) = \frac{i}{\hbar}\left[\hat{H}_0, \hat{O}_I(t)\right] \tag{47.15}$$

Chapter 48

Angular Momentum

In this chapter we consider the general angular momentum operator $\hat{J} = \left(\hat{J}_x, \hat{J}_y, \hat{J}_z\right)$. This operator works on the Hilbert space spanned by the eigenbasis $\{|j, m\rangle\}$.

48.1 General operator

Property 48.1.1. The mutual eigenbasis of $\hat{J}^2$ and $\hat{J}_z$ is defined by the following two eigenvalue equations:

$$\hat{J}^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle \tag{48.1}$$

$$\hat{J}_z|j, m\rangle = m\hbar|j, m\rangle \tag{48.2}$$

Property 48.1.2. The angular momentum operators generate a Lie algebra 25.2.1. The Lie bracket is defined by following commutation relation:

$$\boxed{\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk}\hat{J}_k} \tag{48.3}$$

Definition 48.1.3 (Ladder operators[1]). The raising and lowering operators[2] $\hat{J}_+$ and $\hat{J}_-$ are defined as:

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \qquad \text{and} \qquad \hat{J}_- = \hat{J}_x - i\hat{J}_y \tag{48.4}$$

Corollary 48.1.4. From the commutation relations of the angular momentum operators we can derive the commutation relations of the ladder operators:

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z \tag{48.5}$$

[1] Also called the **creation** and **annihilation** operators (especially in quantum field theory).

[2] These operators will only affect the z-projection, not the total angular momentum.

Formula 48.1.5. The total angular momentum operator $\hat{J}^2$ can now be expressed in terms of $\hat{J}_z$ and the ladder operators using commutation relation 48.3:

$$\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z \tag{48.6}$$

Remark 48.1.6 (Casimir operator). From the definition of $\hat{J}^2$ it follows that this operator is a Casimir invariant[3] in the algebra generated by the operators $\hat{J}_i$.

48.2 Rotations

48.2.1 Infinitesimal rotation

Formula 48.2.1. An infinitesimal rotation $\hat{R}(\delta\vec{\boldsymbol{\varphi}})$ is given by the following formula:

$$\boxed{\hat{R}(\delta\vec{\boldsymbol{\varphi}}) = \mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\delta\vec{\boldsymbol{\varphi}}} \tag{48.7}$$

A finite rotation can then be produced by applying this infinitesimal rotation repeatedly, which gives:

$$\hat{R}(\vec{\boldsymbol{\varphi}}) = \left(\mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\frac{\vec{\boldsymbol{\varphi}}}{n}\right)^n = \exp\left(-\frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{\varphi}}\right) \tag{48.8}$$

Formula 48.2.2 (Matrix elements). Applying a rotation over an angle φ around the z-axis to a state $|j,m\rangle$ gives:

$$\hat{R}(\varphi\vec{\boldsymbol{e}}_z)|j,m\rangle = \exp\left(-\frac{i}{\hbar}\hat{J}_z\varphi\right)|j,m\rangle = \exp\left(-\frac{i}{\hbar}m\varphi\right)|j,m\rangle \tag{48.9}$$

Multiplying these states with a bra $\langle j',m'|$ and using the orthonormality of the eigenstates gives the matrix elements of the rotation operator:

$$\boxed{\hat{R}_{ij}(\varphi\vec{\boldsymbol{e}}_z) = \exp\left(-\frac{i}{\hbar}m\varphi\right)\delta_{jj'}\delta_{mm'}} \tag{48.10}$$

From the expression of the angular momentum operators and the rotation operator it is clear that a general rotation has no effect on the total angular momentum number j. This means that the rotation matrix will be a block diagonal matrix with respect to j. This amounts to the following reduction of the representation of the rotation group:

$$\boxed{\langle j,m'|\hat{R}(\varphi\vec{\boldsymbol{n}})|j,m\rangle = \mathcal{D}^{(j)}_{m,m'}(\hat{R})} \tag{48.11}$$

where the values $\mathcal{D}^{(j)}_{m,m'}(\hat{R})$ are the **Wigner D-functions**.

Remark (Wigner D-functions). For every value of j there are $(2j+1)$ values for m. The matrix $\mathcal{D}^{(j)}(\hat{R})$ is thus a $(2j+1)\times(2j+1)$-matrix

[3] See definition 25.4.2.

48.2.2 Spinor representation

Definition 48.2.3 (Pauli matrices).

$$\boxed{\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \tag{48.12}$$

From this definition it is clear that the Pauli matrices are Hermitian and unitary. Together with the 2×2 identity matrix they form a basis for the space of 2×2 Hermitian matrices.

Formula 48.2.4. In the spinor representation ($\mathrm{J} = \frac{1}{2}$) the Wigner-D matrix reads:

$$\mathcal{D}^{(1/2)}(\varphi \vec{\boldsymbol{e}}_z) = \begin{pmatrix} e^{-i/2\varphi} & 0 \\ 0 & e^{i/2\varphi} \end{pmatrix} \tag{48.13}$$

48.3 Coupling of angular momenta

48.3.1 Total Hilbert space

Let $\mathcal{H}_i$ denote the Hilbert space of states belonging to the i^{th} particle. The Hilbert space of the total system is given by the following tensor product:

$$\mathcal{H} = \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_n$$

Due to the tensor product definition above, the angular momentum operator $\hat{J}_i$ should now be interpreted as $\mathbb{1} \otimes ... \otimes \hat{J}_i \otimes ... \otimes \mathbb{1}$. This implies that the angular momentum operators $\hat{J}_{l \neq i}$ do not act on the space $\mathcal{H}_i$, so one can pull these operators through the tensor product:

$$\hat{J}_i |j_1\rangle \otimes ... \otimes |j_n\rangle = |j_1\rangle \otimes ... \otimes \hat{J}_i |j_i\rangle \otimes ... \otimes |j_n\rangle$$

The basis used above is called the **uncoupled basis**.

48.3.2 Clebsch-Gordan series

Let $\vec{\boldsymbol{J}}$ denote the total angular momentum defined as:

$$\vec{\boldsymbol{J}} = \hat{J}_1 + \hat{J}_2 \tag{48.14}$$

With this operator we can define a **coupled** state $|\mathbf{J}, \mathbf{M}\rangle$ where $\mathbf{M}$ is the total magnetic quantum number which ranges from $-\mathbf{J}$ to $\mathbf{J}$.

Formula 48.3.1 (Clebsch-Gordan coefficients). Because both bases (coupled and uncoupled) span the total Hilbert space $\mathcal{H}$ there exists a transformation between them. The transformation coefficients can be found by using the resolution of the identity:

$$\boxed{|\mathbf{J}, \mathbf{M}\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2, m_1, m_2\rangle \langle j_1, j_2, m_1, m_2 | \mathbf{J}, \mathbf{M}\rangle} \tag{48.15}$$

These coefficients are called the Clebsch-Gordan coefficients.

Property 48.3.2. By acting with the operator $\hat{J}_z$ on both sides of equation 48.15 it is possible to proof that the CG coefficient are non-zero if and only if $\mathbf{M} = m_1 + m_2$.

Chapter 49

Dirac equation

49.1 Dirac matrices

Definition 49.1.1 (Dirac matrices). The time-like Dirac matrix γ^0 is defined as:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{49.1}$$

where $\mathbb{1}$ is the 2-dimensional identity matrix. The space-like Dirac matrices γ^k, $k = 1, 2, 3$ are defined using the Pauli matrices[1] σ^k:

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \tag{49.2}$$

This form of the Dirac matrices fixes a basis called the **Dirac basis**. The **Weyl** or **chiral** basis is fixed by replacing the time-like matrix γ^0 by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{49.3}$$

Property 49.1.2. The Dirac matrices satisfy following equality:

$$\{\gamma^\mu, \gamma^\nu\}_+ = 2\eta^{\mu\nu}\mathbb{1} \tag{49.4}$$

This has the form of equation 21.4. The Dirac matrices can thus be used as the generating set of a Clifford algebra[2].

Notation 49.1.3 (Feynman slash notation). Let $a = a_\mu x^\mu \in V$ be a general 4-vector. The Feynman slash $\not{a}$ is defined as follows:

$$\not{a} = \gamma^\mu a_\mu \tag{49.5}$$

A more formal treatment of the Feynman slash notation shows that it gives us a canonical map:

$$/ : V \to C\ell_V : a_\mu x^\mu \mapsto a_\mu \gamma^\mu \tag{49.6}$$

[1] See definition 48.12.

[2] See defiinition 21.1.1.

49.2 Spinors

49.2.1 Dirac equation

Formula 49.2.1 (Dirac equation). In covariant form the Dirac equation reads:

$$\boxed{(i\hbar\partial\!\!\!/ - mc)\psi = 0} \tag{49.7}$$

Definition 49.2.2 (Dirac adjoint).

$$\overline{\psi} = \psi^\dagger \gamma^0 \tag{49.8}$$

Formula 49.2.3 (Parity).

$$\hat{P}(\psi) = \gamma^0 \psi \tag{49.9}$$

49.2.2 Chiral spinors

We can also define a fifth matrix:

Definition 49.2.4 (Chiral matrix).

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{49.10}$$

Definition 49.2.5 (Chiral projection). The chiral projections of a spinor ψ are defined as follows:

$$\psi_L = \frac{1-\gamma^5}{2}\psi \tag{49.11}$$

and

$$\psi_R = \frac{1+\gamma^5}{2}\psi \tag{49.12}$$

Every spinor can then be written as:

$$\psi = \psi_L + \psi_R \tag{49.13}$$

Chapter 50

Perturbation theory

50.1 Rayleigh-Schrödinger perturbation theory

The basic of assumptions of the Rayleigh-Schrödinger perturbation theory are that the perturbation Hamiltonian is time-independent and that the eigenfunctions of the unperturbed Hamiltonian $\hat{H}_0$ also form a complete set for the perturbed Hamiltonian.

Formula 50.1.1. The perturbed eigenfunctions and eigenvalues can be expanded in the following way, where we assume that λ is a small perturbation parameter:

$$|\psi_n\rangle = \sum_{i=0}^{+\infty} \lambda^i |\psi_n^{(i)}\rangle \tag{50.1}$$

$$E_n = \sum_{i=0}^{+\infty} \lambda^i E_n^{(i)} \tag{50.2}$$

where i denotes the order of the perturbation.

50.2 Time-dependent perturbation theory

In this section we consider perturbed Hamiltoninians of the following form:

$$\hat{H}(t) = \hat{H}_0 + \lambda \hat{V}(t) \tag{50.3}$$

50.2.1 Dyson series

Formula 50.2.1 (Tomonaga-Schwinger equation). The evolution operator $\hat{U}(t)$ satisfies the following Schrödinger-type equation in the interaction image[1]:

$$i\hbar\frac{d}{dt}\hat{U}_I|\psi(0)\rangle_I = \hat{V}_I(t)\hat{U}_I|\psi(0)\rangle_I \tag{50.4}$$

Formula 50.2.2 (Dyson series). Together with the initial value condition $\hat{U}_I(0) = \mathbb{1}$ the Tomonaga-Schwinger equation becomes an initial value problem. A particular solution is given by:

$$\hat{U}_I(t) = \mathbb{1} - \frac{i}{\hbar}\int_0^t \hat{V}_I(t')\hat{U}_I(t')dt' \tag{50.5}$$

This solution can be iterated to obtain a series expansion of the evolution operator:

$$\hat{U}(t) = 1 - \frac{i}{\hbar}\int_0^t \hat{V}(t_1)dt_1 + \left(-\frac{i}{\hbar}\right)^2\int_0^t dt_1\int_0^{t_1} dt_2\hat{V}(t_1)\hat{V}(t_2) + ... \tag{50.6}$$

It is clear that the integrands obey a time-ordering. By introducing the **time-ordering operator** $\mathcal{T}$:

$$\mathcal{T}\left(\hat{V}(t_1)\hat{V}(t_2)\right) = \begin{cases} \hat{V}(t_1)\hat{V}(t_2) & , \quad t_1 \geq t_2 \\ \hat{V}(t_2)\hat{V}(t_1) & , \quad t_2 > t_1 \end{cases} \tag{50.7}$$

the integrals can be rewritten in a more symmetric form:

$$\hat{U}(t) = 1 - \frac{i}{\hbar}\int_0^t \hat{V}(t_1)dt_1 + \frac{1}{2!}\left(-\frac{i}{\hbar}\right)\int_0^t dt_1\int_0^t dt_2\mathcal{T}\left(\hat{V}(t_1)\hat{V}(t_2)\right) + ... \tag{50.8}$$

or by comparing with the series expansion for exponential functions:

$$\boxed{\hat{U}(t) = \mathcal{T}\left(e^{-\frac{i}{\hbar}\int_0^t \hat{V}(t')dt'}\right)} \tag{50.9}$$

This expansion is called the **Dyson series**.

50.3 Variational method

Definition 50.3.1 (Energy functional).

$$E(\psi) = \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} \tag{50.10}$$

Property 50.3.2. The energy functional 50.10 satisfies following inequality:

$$E(\psi) \geq E_0 \tag{50.11}$$

where E_0 is the ground state energy.

[1] See section 47.5.

Method 50.3.3. Assume that the trial function $|\psi\rangle$ depends on a set of parameters $\{c_i\}_{i\in I}$. The 'optimal' wave function is found by solving the following system of equations:

$$\frac{\partial \psi}{\partial c_i} = 0 \qquad \forall i \in I \tag{50.12}$$

50.4 Adiabatic approximation

50.4.1 Berry phase

Consider a system for which the adiabatic approximation is valid. We then have a wavefunction of the form

$$\psi(t) = C_a(t)\psi_a(t)\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} E_a(t')dt'\right] \tag{50.13}$$

It follows from the orthonormality of the eigenstates $\psi_k(t)$ that the coefficient $C_a(t)$ is just a phase factor, so we can write it as

$$C_a(t) = e^{i\gamma_a(t)} \tag{50.14}$$

Substituting this ansatz in the wavefunction and the Schödinger equation gives a differential equation for the phase factor $\gamma_a(t)$. It can be readily integrated to obtain:

$$\gamma_a(t) = i\int_{t_0}^{t} \left\langle \psi_a(t') \middle| \frac{\partial \psi_a(t')}{\partial t'} \right\rangle dt' \tag{50.15}$$

Due to time evolution the wavefunction accumulates a phase through the coefficient $C_a(t)$ over the period $t_0 - t_f$. This phase is called the **Berry phase**.

Lets try to apply a phase transformation to remove the Berry phase:

$$\psi'_a(t) = \psi_a(t)e^{i\eta(t)} \tag{50.16}$$

Entering this in equation 50.15 gives

$$\bar{\gamma}'_a(t) = \bar{\gamma}_a(t) - \eta(t_f) + \eta(t_0) \tag{50.17}$$

where the overhead bar denotes the integration between t_0 and t_f in equation 50.15. If the system is cyclic then $\psi_a(t_0) = \psi_a(t_f)$. Combining this with equation 50.16 gives us:

$$\eta(t_f) - \eta(t_0) = 2k\pi \qquad k \in \mathbb{N} \tag{50.18}$$

which implies that the Berry phase cannot be eliminated through a basis transformation and is thus an observable property of the system.

Definition 50.4.1 (Berry connection). The quantity

$$\mathbf{A}(\vec{\boldsymbol{c}}) = i\langle\psi_a(\vec{\boldsymbol{c}})|\nabla_{\vec{\boldsymbol{c}}}\psi_a(\vec{\boldsymbol{c}})\rangle \tag{50.19}$$

where $\nabla_{\vec{\boldsymbol{c}}}$ denotes the gradient in phase space, is called the Berry connection (or Berry gauge potential). Applying Stokes' theorem to 50.15 gives us:

$$\bar{\gamma}_a = \int \boldsymbol{\mathcal{B}} \cdot d\vec{\boldsymbol{S}} \tag{50.20}$$

where $\boldsymbol{\mathcal{B}} = \nabla_{\vec{\boldsymbol{c}}} \times \mathbf{A}(\vec{\boldsymbol{c}})$ is called the **Berry curvature**. Although the Berry connection is gauge dependent, the Berry curvature is gauge invariant!

Chapter 51

Scattering theory

51.1 Cross section

Formula 51.1.1 (Differential cross section).

$$\frac{d\sigma}{d\Omega} = \frac{N(\theta, \varphi)}{F} \tag{51.1}$$

where F is the incoming particle flux and N the detected flow rate[1].

51.1.1 Fermi's golden rule

Formula 51.1.2 (Fermi's golden rule). The transition probability from state i to state f is given by:

$$\boxed{\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\langle f|\hat{H}|i\rangle|^2 \frac{dn}{dE_f}} \tag{51.2}$$

51.2 Lippman-Schwinger equations

In this section we consider Hamiltonians of the following form: $\hat{H} = \hat{H}_0 + \hat{V}$ where $\hat{H}_0$ is the free Hamiltonian and $\hat{V}$ the scattering potential. We will also assume that both the total Hamiltonian and the free Hamiltonian have the same eigenvalues.

Formula 51.2.1 (Lippman-Schwinger equation).

$$|\psi^{(\pm)}\rangle = |\varphi\rangle + \frac{1}{E - \hat{H}_0 \pm i\varepsilon}\hat{V}|\psi^{(\pm)}\rangle \tag{51.3}$$

where $|\varphi\rangle$ is an eigenstate of the free Hamiltonian with the same energy as $|\psi\rangle$.

[1] Because N is not defined as a flux but as a rate, the differential cross section has the dimension of area.

Remark 51.2.2. The term $\pm i\varepsilon$ is added to the denominator because otherwise it would be singular. The term has no physical meaning.

Formula 51.2.3 (Born series). If we rewrite the Lippman-Schwinger equation as:

$$|\psi\rangle = |\varphi\rangle + \hat{G}_0\hat{V}|\psi\rangle \tag{51.4}$$

where $\hat{G}_0$ is the Green's operator, then we can derive the following series expansion by iterating the equation:

$$|\psi\rangle = |\varphi\rangle + \hat{G}_0\hat{V}|\varphi\rangle + \left(\hat{G}_0\hat{V}\right)^2|\varphi\rangle + ... \tag{51.5}$$

Formula 51.2.4 (Born approximation). If we cut off the Born series at the first order term in $\hat{V}$ then we obtain the Born approximation:

$$|\psi\rangle = |\varphi\rangle + \hat{G}_0\hat{V}|\varphi\rangle \tag{51.6}$$

Chapter 52

Entanglement & Quantum computing

52.1 Bipartite systems

52.1.1 Marginal density operators

Definition 52.1.1 (Marginal density operator). Let $|\Psi\rangle_{AB}$ be the state of a bipartite system. The marginal density operator $\hat{\rho}_A$ of system A is defined as follows:

$$\hat{\rho}_A = \mathrm{Tr}_B |\Psi\rangle_{ABAB}\langle\Psi| \tag{52.1}$$

Definition 52.1.2 (Purification). Let $\hat{\rho}_A$ be the desnity operator of a system A. A purification of $\hat{\rho}_A$ is a pure state $|\Psi\rangle_{AB}$ of a composite system AB such that:

$$\hat{\rho}_A = \mathrm{Tr}_B |\Psi\rangle_{ABAB}\langle\Psi| \tag{52.2}$$

Property 52.1.3. Any two purifications of the same density operator $\hat{\rho}_A$ are related by a transformation $\mathbb{1}_A \otimes \hat{V}$, where $\hat{V}$ is a unitary operator on $\mathcal{H}_B$.

Chapter 53

Quantum Field Theory

In this chapter we adopt the standard Minkowskian signature $(+,-,-,-)$ unless otherwise stated. This follows the introductory literature and courses such as [4]. Furthermore we also work in natural units unless stated otherwise, i.e. $\hbar = c = 1$.

53.1 Klein-Gordon Field

53.1.1 Lagrangian and Hamiltonian

The simplest Lagrangian (density) is given by:

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2} \tag{53.1}$$

Using the principle of least action we obtain the following Euler-Lagrange equations[1]:

$$\left(\partial^\mu \partial_\mu + m^2\right)\phi = 0 \tag{53.2}$$

or by introducing the **d'Alembertian** $\Box = \partial^\mu \partial_\mu$:

$$\boxed{(\Box + m^2)\phi = 0} \tag{53.3}$$

This equation is called the Klein-Gordon equation. In the limit $m \to 0$ it reduces to the well-known wave equation.

From the Lagrangian 53.1 we can also derive a Hamiltonian function using relation 36.12:

$$\boxed{H = \int d^3x \frac{1}{2}\left[\pi^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)\right]} \tag{53.4}$$

[1] See formula 36.4.

53.1.2 Raising and lowering operators

Fourier expanding the scalar field $\phi(\vec{\boldsymbol{x}}, t)$ in momentum space and inserting it into the Klein-Gordon equation gives:

$$\left(\partial_t^2 + p^2 + m^2\right)\phi(\vec{\boldsymbol{p}}, t) = 0 \tag{53.5}$$

This is the equation for a simple harmonic oscillator with frequency $\omega_{\vec{\boldsymbol{p}}} = \sqrt{p^2 + m^2}$.

Analogous to ordinary quantum mechanics we define the raising and lowering operators $a^\dagger_{\vec{\boldsymbol{p}}}$ and $a_{\vec{\boldsymbol{p}}}$ such that:

$$\phi(\vec{\boldsymbol{x}}) = \iiint \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{\boldsymbol{p}}}}}\left(a_{\vec{\boldsymbol{p}}}e^{i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}} + a^\dagger_{\vec{\boldsymbol{p}}}e^{-i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}}\right) \tag{53.6}$$

$$\pi(\vec{\boldsymbol{x}}) = \iiint \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{\boldsymbol{p}}}}{2}}\left(a_{\vec{\boldsymbol{p}}}e^{i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}} - a^\dagger_{\vec{\boldsymbol{p}}}e^{-i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}}\right) \tag{53.7}$$

An equivalent definition is obtained by performing the transformation $\vec{\boldsymbol{p}} \to -\vec{\boldsymbol{p}}$ in the second term of $\phi(\vec{\boldsymbol{x}})$ and $\pi(\vec{\boldsymbol{x}})$:

$$\boxed{\begin{aligned}\phi(\vec{\boldsymbol{x}}) &= \iiint \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{\boldsymbol{p}}}}}\left(a_{\vec{\boldsymbol{p}}} + a^\dagger_{-\vec{\boldsymbol{p}}}\right)e^{i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}} &(53.8)\\ \pi(\vec{\boldsymbol{x}}) &= \iiint \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{\boldsymbol{p}}}}{2}}\left(a_{\vec{\boldsymbol{p}}} - a^\dagger_{-\vec{\boldsymbol{p}}}\right)e^{i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}} &(53.9)\end{aligned}}$$

When we impose the commutation relation

$$[a_{\vec{\boldsymbol{p}}}, a^\dagger_{\vec{\boldsymbol{q}}}] = (2\pi)^3\delta(\vec{\boldsymbol{p}} - \vec{\boldsymbol{q}}) \tag{53.10}$$

we obtain the following commutation relation for the scalar field and its conjugate momentum:

$$[\phi(\vec{\boldsymbol{x}}), \pi(\vec{\boldsymbol{y}})] = i\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}) \tag{53.11}$$

Combining the previous formulas gives us the following important commutation relations:

$$\boxed{\begin{aligned}[H, a^\dagger_{\vec{\boldsymbol{p}}}] &= \omega_p a^\dagger_{\vec{\boldsymbol{p}}} &(53.12)\\ [H, a_{\vec{\boldsymbol{p}}}] &= -\omega_p a_{\vec{\boldsymbol{p}}} &(53.13)\end{aligned}}$$

The Hamiltonian can also be explicitly calculated:

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{\boldsymbol{p}}}\left(a^\dagger_{\vec{\boldsymbol{p}}}a_{\vec{\boldsymbol{p}}} + \frac{1}{2}[a_{\vec{\boldsymbol{p}}}, a^\dagger_{\vec{\boldsymbol{p}}}]\right) \tag{53.14}$$

It is however immediately clear from 53.10 that the second term in this integral diverges. This is a consequence of both the fact that space is infinite, i.e. the d^3x integral diverges, and

the "large p" limit in the the d^3p integral. The first divergence can be resolved by applying some kind of boundary and considering the energy density instead of the energy itself. The second divergence follows from the fact that by including very large values for p in the integral we enter a parameter range where our theory is likely to break down. So we should introduce a "high p" cut-off. A more practical solution is to note that only energy differences are physical and so we can drop the second term altogether as it is merely a "constant".

53.1.3 Complex scalar fields

Formula 53.1.1 (Pauli-Jordan function).

$$[\phi(x), \phi(y)] = i \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)}_{\Delta(x-y)} \tag{53.15}$$

In the case that $x^0 = y^0$ (ETCR) or $(x-y)^2 < 0$ the Pauli-Jordan function is identically 0.[2]

53.2 Lorentz invariant integrals

When applying a Lorentz boost Λ the delta function $\delta(\vec{\boldsymbol{p}} - \vec{\boldsymbol{q}})$ transforms[3] as $\delta(\Lambda\vec{\boldsymbol{p}} - \Lambda\vec{\boldsymbol{q}})\frac{\Lambda E}{E}$. This is clearly not a Lorentz invariant quantity and cannot be used for normalisation. It is however also clear that the quantity $2E\delta(\vec{\boldsymbol{p}} - \vec{\boldsymbol{q}})$ is Lorentz invariant. (The constant 2 is merely introduced for future convenience.) The correct normalisation for the momentum representation thus becomes:

$$\langle p|q\rangle = 2E_p(2\pi)^3\delta(\vec{\boldsymbol{p}} - \vec{\boldsymbol{q}}) \tag{53.16}$$

where the factors 2 and $(2\pi)^3$ are again a matter of convention.

The factor $2E_p$ does not only occur in the normalisation conditions. To define a Lorentz invariant measure for evaluating integrals in spacetime we define define the following integral:

$$\int \frac{d^3p}{2E_p} = \int d^4p\ \delta(p^2 - m^2)\Big|_{p^0>0} \tag{53.17}$$

This means that the integral of any Lorentz invariant function $f(p)$ using the measure $\frac{d^3p}{2E_p}$ will be Lorentz invariant.

Computing the quantity $\langle 0|\phi(\vec{\boldsymbol{x}})|\vec{\boldsymbol{p}}\rangle$ gives $e^{i\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{p}}}$. This coincides with the position representation from quantum mechanics of the state $|\vec{\boldsymbol{p}}\rangle$ and so we will also interpret it in QFT as the position representation of the single particle state $|\vec{\boldsymbol{p}}\rangle$.

[2] See also the axiom of microcausality 53.4.1

[3] This follows from property 12.9.

53.3 Wick's theorem

53.3.1 Bosonic fields

Definition 53.3.1 (Contraction for neutral bosonic fields).

$$\overbrace{\phi(x)\phi(y)} = \begin{cases} [\phi(x)^{(+)}, \phi(y)^{(-)}] & x^0 > y^0 \\ [\phi(y)^{(+)}, \phi(x)^{(-)}] & y^0 > x^0 \end{cases} \tag{53.18}$$

Formula 53.3.2 (Feynman propagator).

$$\overbrace{\phi(x)\phi(y)} = \underbrace{i \lim_{\varepsilon \to 0^+} i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon}}_{\Delta_F(x-y)} \tag{53.19}$$

Definition 53.3.3 (Contraction for charged bosonic fields).

$$\overbrace{\phi(x)\overline{\phi}(y)} = \begin{cases} [\phi(x)^{(+)}, \overline{\phi}(y)^{(-)}] & x^0 > y^0 \\ [\phi(y)^{(+)}, \overline{\phi}(x)^{(-)}] & y^0 > x^0 \end{cases} \tag{53.20}$$

Remark 53.3.4. In the case of charged bosons, only contractions of the form $\overbrace{\phi(x)\overline{\phi}(y)}$ will remain because $[a, b^+] = 0$ for charged bosons.

53.3.2 Fermionic fields

Definition 53.3.5 (Contraction).

$$\overbrace{\psi(x)\overline{\psi}(y)} = \begin{cases} \{\psi(x)^{(+)}, \overline{\psi}(y)^{(-)}\}_+ & x^0 > y^0 \\ -\{\psi(y)^{(+)}, \overline{\psi}(x)^{(-)}\}_+ & y^0 > x^0 \end{cases} \tag{53.21}$$

Remark 53.3.6. Only contractions of the form $\overbrace{\psi(x)\overline{\psi}(y)}$ will remain because $\{a, b^+\}_+ = 0$.

Formula 53.3.7 (Feynman propagator).

$$\overbrace{\psi(x)\overline{\psi}(y)} = \underbrace{i \lim_{\varepsilon \to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}}_{S_F(x-y)} \tag{53.22}$$

53.4 Axiomatic approach

Theorem 53.4.1 (Axiom of microcausality). *Let $\hat{O}$ be an observable and let x, y be two spacetime points. If $x - y$ is a space-like vector then $[\hat{O}(x), \hat{O}(y)] = 0$.*

53.4.1 Wightman axioms

53.5 Quantum Chromodynamics

Property 53.5.1 (OZI rule). Decay processes for which the corresponding Feynman diagrams become disconnected (initial states and final states are disconnected) when removing internal gluon lines are suppressed with respect to other processes.

Part XI

Thermal Physics & Statistical Mechanics

Chapter 54

Thermodynamics

54.1 General definitions

Definition 54.1.1 (System). The part of space that we are examining.

Definition 54.1.2 (Surroundings). Everything outside the system.

Definition 54.1.3 (Immediate surrounding). The part of the surroundings that 'lies' immediately next to the system.

Definition 54.1.4 (Environment). Everything outside the immediate surroundings.

Definition 54.1.5 (Thermodynamic coordinates). Macroscopical quantities that describe the system.

Definition 54.1.6 (Intensive coordinate). Coordinate that does not depend on the total amount of material (or system size).

Definition 54.1.7 (Extensive coordinate). Coordinate that does depend on the amount of material.

Definition 54.1.8 (Thermodynamic equilibrium). A system in thermodynamic equilibrium is simultaneously in thermal, mechanical and chemical equilibrium. The system is also described by a certain set of constant coordinates.

Theorem 54.1.9. *During thermodynamic equilibrium, all intensive coordinates are uniform throughout the system.*

Definition 54.1.10 (Isolated system). An isolated system can't interact with its surroundings (due to the presence of impenetrable walls).

Definition 54.1.11 (Diathermic wall). A diathermic wall is a wall that allows heat transfer.

Definition 54.1.12 (Adiabatic wall). An adiabatic wall is a wall that does not allow heat transfer.

Definition 54.1.13 (Open system). An open system is a system that allows matter exchange.

Definition 54.1.14 (Closed system). A closed system is a system that does not allow matter exchange.

Definition 54.1.15 (Quasistatic process). A quasistatic process is a sequence of equilibrium states separated by infinitesimal changes.

Definition 54.1.16 (Path). The sequence of equilibrium states in a process is called the path.

54.2 Postulates

Theorem 54.2.1 (Zeroth law). *If two object are in thermal equilibrium with a third object then they are also in thermal equilibrium with eachother.*

Theorem 54.2.2 (First law).

$$U_f - U_i = W + Q \tag{54.1}$$

$$dU = \delta W + \delta Q \tag{54.2}$$

Remark. The δ in the heat and work differentials implies that these are 'inexact' differentials. This means that they cannot be expressed as functions of the thermodynamic coordinates. More formally a differential form dx is called inexact if the integral $\int dx$ is path dependent.

Theorem 54.2.3 (Kelvin-Planck formulation of the second law). *No machine can absorb an amount of heat and completely transform it into work.*

Theorem 54.2.4 (Clausius formulation of the second law). *Heat cannot be passed from a cooler object to a warmer object without performing work.*

Formula 54.2.5 (Clausius' inequality). In differential form, the inequality reads:

$$\frac{\delta Q}{T} \geq 0 \tag{54.3}$$

Theorem 54.2.6 (Third law). *No process can reach absolute zero in a finite sequence of operations.*

54.3 Gases

54.3.1 Ideal gases

Theorem 54.3.1 (Ideal gas law).

$$\boxed{PV = nRT} \tag{54.4}$$

Chapter 55

Statistical mechanics

55.1 Axioms

Theorem 55.1.1 (Ergodic principle). *All microstates corresponding to the same macroscopic state are equally propable.*

Theorem 55.1.2 (Boltzmann formula). *The central axiom of statistical mechanics gives following formula for the entropy:*

$$\boxed{S = k \ln \Omega(E, V, N, \alpha)} \tag{55.1}$$

where Ω denotes the number of microstates corresponding to the system at energy E, volume V, and so on.

55.2 Temperature

The temperature of a system in contact with a heat bath is defined as:

$$\boxed{T = \left(\frac{\partial E}{\partial S}\right)_V} \tag{55.2}$$

55.3 Maxwell-Boltzmann statistics

Consider a system of N indistinguishable non-interacting particles. Let ε_i be the energy associated with the i-th energy level with degeneracy g_i. The probability p_i of finding a particle in the i-th energy level is given by:

$$\boxed{p_i = \frac{g_i e^{-\beta \varepsilon_i}}{Z}} \tag{55.3}$$

where Z is the single particle **partition function** defined as:

$$\boxed{Z = \sum_i g_i e^{-\beta \varepsilon_i}} \tag{55.4}$$

55.4 Grand canonical system

Formula 55.4.1 (Grand canonical partition function).

$$\mathcal{Z}_i = \sum_{\varepsilon_i} e^{\beta n_i (\mu - \varepsilon_i)} \tag{55.5}$$

Corollary 55.4.2. In the case that $n_i \in \{0, 1\}$ this formula reduces to $\mathcal{Z}_i = e^{\beta\mu} Z_i$.

Definition 55.4.3 (Fugacity).

$$z = e^{\mu N} \tag{55.6}$$

55.5 Energy

Theorem 55.5.1 (Virial theorem).

$$\boxed{\langle T \rangle = -\frac{1}{2} \sum_k \langle \vec{\boldsymbol{r}}_k \cdot \vec{\boldsymbol{F}}_k \rangle} \tag{55.7}$$

Corollary 55.5.2. For potentials of the form $V = ar^{-n}$ this becomes:

$$2\langle T \rangle = -n \langle V \rangle \tag{55.8}$$

Theorem 55.5.3 (Equipartition theorem). *Let x be any generalized coordinate (both position or momentum).*

$$\boxed{\left\langle x^k \frac{\partial H}{\partial x^l} \right\rangle = \delta_{kl} k_b T} \tag{55.9}$$

Corollary 55.5.4. For quadratic Hamiltonians this can be rewritten using Euler's theorem for homogeneous functions 7.10 as:

$$\langle T \rangle = \frac{1}{2} k_b T \tag{55.10}$$

Chapter 56

Photon gas

56.1 Black-body radiation

Formula 56.1.1 (Planck's law).

$$\boxed{B_\nu(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kt}} - 1}} \tag{56.1}$$

Formula 56.1.2 (Wien's displacement law).

$$\boxed{\lambda_{max} T = b} \tag{56.2}$$

where $b = 2.897\,772\,9(17) \times 10^{-3}$ Km is **Wien's displacement constant**.

Part XII

Solid State Physics

Chapter 57

Material physics

57.1 Crystals

Theorem 57.1.1 (Steno's law). *The angles between crystal faces of the same type are constant and do not depend on the total shape of the crystal.*

Definition 57.1.2 (Zone). The collection of faces parallel to a given axis, is called a zone. The axis itself is called the zone axis.

57.1.1 Analytic representation

Definition 57.1.3 (Miller indices). Let a, b, c be the lengths of the (not necessarily orthogonal) basis vectors of the crystal lattice. The lattice plane intersecting the axes at $\left(\frac{a}{h}, \frac{b}{k}, \frac{c}{k}\right)$ is denoted by the Miller indices $(h\ k\ l)$.

Notation 57.1.4. Negative numbers are written as $\overline{a}$ instead of $-a$.

Formula 57.1.5 (Coordinates of axes). Let a, b, c denote the lengths of the basis vectors. The axis formed by the intersection of the planes $(h_1\ k_1\ l_1)$ and $(h_2\ k_2\ l_2)$, pointing in the direction of the point (au, bv, cw) is denoted by $[u\ v\ w]$. Where

$$u = \begin{vmatrix} k_1 & l_1 \\ k_2 & l_2 \end{vmatrix} \qquad v = \begin{vmatrix} l_1 & h_1 \\ l_2 & h_2 \end{vmatrix} \qquad w = \begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix} \tag{57.1}$$

Theorem 57.1.6 (Hauy's law of rational indices). *The Miller indices of every natural face of a crystal will always have rational proportions.*

57.2 Symmetries

Definition 57.2.1 (Equivalent planes/axes). When applying certain symmetries to a plane or axis, it often occurs that we obtain a set of equivalent planes/axes. These equivalence classes are denoted respectively by $\{h\ k\ l\}$ and $< h\ k\ l >$.

Property 57.2.2 (Rotational symmetry). Only $1, 2, 3, 4$ and 6-fold rotational symmetries can occur.

57.3 Crystal lattice

Formula 57.3.1. For an orthogonal crystal lattice, the distance between planes of the family $(h\ k\ l)$ is given by:

$$d_{hkl} = \frac{1}{\sqrt{\left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 + \left(\frac{l}{c}\right)^2}} \tag{57.2}$$

57.3.1 Bravais lattice

Definition 57.3.2 (Bravais lattice). A crystal lattice generated by a certain point group symmetry is called a Bravais lattice. There are 14 different Bravais lattices in 3 dimensions. These are the only possible ways to place (infinitely) many points in 3D space by applying symmetry operations to a point group.

Definition 57.3.3 (Wigner-Seitz cell). The part of space consisting of all points closer to a given lattice point than to any other.

57.3.2 Reciprocal lattice

Formula 57.3.4 (Reciprocal basis vectors). The reciprocal lattice corresponding to a given Bravais lattice with primitive basis $\{\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}\}$ is defined by the following reciprocal basis vectors

$$\vec{\boldsymbol{a}}^* = 2\pi \frac{\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}}{\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}})} \tag{57.3}$$

The vectors $\vec{\boldsymbol{b}}^*$ and $\vec{\boldsymbol{c}}^*$ are obtained by permutation of (a, b, c). These vectors satisfy the relations

$$\begin{aligned} \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}}^* &= 2\pi \\ \vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{b}}^* &= 2\pi \\ \vec{\boldsymbol{c}} \cdot \vec{\boldsymbol{c}}^* &= 2\pi \end{aligned} \tag{57.4}$$

$$\tag{57.5}$$

Notation 57.3.5 (Reciprocal lattice vector). The reciprocal lattice vector $\vec{\boldsymbol{r}}^*_{hkl}$ is defined as follows:

$$\vec{\boldsymbol{r}}^*_{hkl} = h\vec{\boldsymbol{a}}^* + k\vec{\boldsymbol{b}}^* + l\vec{\boldsymbol{c}}^* \tag{57.6}$$

Property 57.3.6. The reciprocal lattice vector $\vec{\boldsymbol{r}}^*_{hkl}$ has the following properties:

- $\vec{\boldsymbol{r}}^{\,*}_{hkl}$ is perpendicular to the family of planes $(h\ k\ l)$ of the direct lattice.
- $||\vec{\boldsymbol{r}}^{\,*}_{hkl}|| = \dfrac{2\pi n}{d_{hkl}}$

57.4 Diffraction

57.4.1 Constructive interference

Formula 57.4.1 (Laue conditions). Suppose that an incident beam makes angles α_0, β_0 and γ_0 with the lattice axes. The diffracted beam making angles α, β and γ with the axes will be observed if following conditions are satisfied:

$$\begin{aligned} a(\cos\alpha - \cos\alpha_0) &= h\lambda \\ b(\cos\beta - \cos\beta_0) &= k\lambda \\ c(\cos\gamma - \cos\gamma_0) &= l\lambda \end{aligned}$$

If these conditions have been met then we observe a diffracted beam of order hkl.

Remark 57.4.2. Further conditions can be imposed on the angles, such as the pythagorean formula for orthogonal axes. This has the consequence that the only two possible ways to obtain a diffraction pattern are:

- a fixed crystal and a polychromatic beam
- a rotating crystal and a monochromatic beam

Formula 57.4.3 (Vectorial Laue conditions). Let $\vec{k}_0, \vec{k}$ denote the wave vector of respectively the incident and diffracted beams. The Laue conditions can be reformulated in the following way:

$$\boxed{\vec{k} - \vec{k}_0 = \vec{r}^{\,*}_{hkl}} \tag{57.7}$$

Formula 57.4.4 (Bragg's law). Another equivalent formulation of the Laue conditions is given by following formula:

$$\boxed{2d_{hkl}\sin\theta = n\lambda} \tag{57.8}$$

where

λ : wavelength of the incoming beam

θ : the **Bragg angle**

d_{hkl} : distance between neighbouring planes

Remark 57.4.5. The angle between the incident and diffracted beams is given by 2θ.

Construction 57.4.6 (Ewald sphere). A simple construction to determine if Bragg difraction will occur is the Ewald sphere: Put the origin of the reciprocal lattice at the tip of the incident wave vector $\vec{\boldsymbol{k}}_i$. Now construct a sphere with radius $\frac{2\pi}{\lambda}$ centered on the start of $\vec{\boldsymbol{k}}_i$. All points on the sphere that coincide with a reciprocal lattice point satisfy the vectorial Laue condition 57.7. Therefore Bragg diffraction will occur in the direction of all the intersections of the Ewald sphere and the reciprocal lattice.

57.4.2 Intensity of diffracted beams

Definition 57.4.7 (Systematic extinctions). Every particle in the motive emits its own waves. These waves will interfere and some will cancel out which leads to the absence of certain diffraction spots. These absences are called systematic extinctions.

Definition 57.4.8 (Atomic scattering factor). The waves produced by the individual electrons of an atom, which can have a different phase, can be combined into a resulting wave. The amplitude of this wave is called the atomic scattering factor.

Definition 57.4.9 (Structure factor). The waves coming from the individual atoms in the motive can also be combined, again taking into account the different phases, into a resulting wave. The amplitude of this wave is called the structure factor and it is given by:

$$F(hkl) = \sum_j f_j \exp\left[2\pi i(hx_j + ky_j + lz_j)\right] \tag{57.9}$$

where f_j is the atomic scattering factor of the j^{th} atom in the motive.

Example 57.4.10. A useful example of systematic extinctions is the structure factor of an FCC or BCC lattice for the following specific situations:

If $h + k + l$ is odd, then $F(hkl) = 0$ for a BCC lattice. If h, k and l are not all even or all odd then $F(hkl) = 0$ for an FCC lattice.

Definition 57.4.11 (Laue indices). Higher order diffractions can be rewritten as a first order diffraction in the following way:

$$2d_{nhnknl} \sin\theta = \lambda \qquad \text{with} \qquad d_{nhnknl} = \frac{d_{hkl}}{n} \tag{57.10}$$

Following from the interpretation of the Bragg law as diffraction being a reflection at the lattice plane $(h\ k\ l)$ we can introduce the (fictitious) plane with indices $(nh\ nk\ nl)$. These indices are called Laue indices.

Remark. In contrast to Miller indices which cannot possess common factors, the Laue indices obviously can.

57.5 Alloys

Theorem 57.5.1 (Hume-Rothery conditions). An element can be dissolved in a metal (forming a solid solution) if the following conditions are met:

- The difference between the atomic radii is $\leq 15\%$.
- The crystal structures are the same.
- The elements have a similar electronegativity.
- The valency is the same.

57.6 Lattice defects

Definition 57.6.1 (Vacancy). A lattice point where an atom is missing. Also called a **Schottky defect**.

Formula 57.6.2 (Concentration of Schottky defects†). Let N denote the number of lattice points and n the number of vacancies. The following relation gives the temperature dependence of Schottky defects:

$$\frac{n}{n+N} = e^{-E_v/kT} \tag{57.11}$$

where T is the temperature and E_v the energy needed to create a vacancy.

Remark. A similar relation holds for interstitials.

Definition 57.6.3 (Interstitial). An atom placed at a position which is not a lattice point.

Definition 57.6.4 (Frenkel pair). An atom displaced from a lattice point to an interstitial location (hereby creating a vacancy-interstitial pair) is called a Frenkel defect.

Formula 57.6.5 (Concentration of Frenkel pairs). Let n_i denote the number of atoms displaced from the bulk of the lattice to any N_i possible interstitial positions and thus creating n_i vacancies. The following relation holds:

$$\frac{n_i}{\sqrt{NN_i}} = e^{-E_{fr}/2kT} \tag{57.12}$$

where E_{fr} denotes the energy needed to create a Frenkel pair.

Remark 57.6.6. In compounds the number of vacancies can be much higher than in monoatomic lattices.

Remark 57.6.7. The existence of these defects creates the possibility of diffusion.

57.7 Electrical properties

57.7.1 Charge carriers

Formula 57.7.1 (Conductivity). Definition 41.1 can be modified to account for both positive and negative charge carriers:

$$\sigma = n_n q_n \mu_n + n_p q_p \mu_p \tag{57.13}$$

Remark. The difference between the concentration of positive and negative charge carriers can differ by orders of magnitude across different materials. It can differ by up to 20 orders of magnitude.

57.7.2 Band structure

Definition 57.7.2 (Valence band). The energy band corresponding to the outermost (partially) filled atomic orbital.

Definition 57.7.3 (Conduction band). The first unfilled energy band.

Definition 57.7.4 (Band gap). The energy difference between the valence and conduction bands (if they do not overlap). It is the energy zone[1] where no electron states can exist.

Definition 57.7.5 (Fermi level). The energy level having a 50% chance of being occupied at thermodynamic equilibrium.

Formula 57.7.6 (Fermi function). The following distribution gives the probability of a state with energy E_i being occupied by an electron:

$$\boxed{f(E_i) = \frac{1}{e^{(E_i - E_f)/kT} + 1}} \tag{57.14}$$

where E_f is the Fermi level as defined above.

57.7.3 Intrinsic semiconductors

Formula 57.7.7. Let n denote the charge carrier density as before. We find the following temperature dependence:

$$n \propto e^{-E_g/2kt} \tag{57.15}$$

where E_g is the band gap. This formula can be directly derived from the Fermi function by noting that for intrinsic semiconductors the Fermi level sits in the middle of the band gap, i.e. $E_c - E_f = E_g/2$, and that for most semiconductors $E_g \gg kT$.

57.7.4 Extrinsic semiconductors

Definition 57.7.8 (Doping). Intentionally introducing impurities to modify the (electrical) properties.

Definition 57.7.9 (Acceptor). Group III element added to create an excess of holes in the valence band. The resulting semiconductor is said to be a **p-type semiconductor**.

Definition 57.7.10 (Donor). Group IV element added to create an excess of electrons in the valence band. The resulting semiconductor is said to be an **n-type semiconductor**.

[1] For a basic derivation see [16].

57.7.5 Ferroelectricity

Some materials can exhibit certain phase transitions between a paraelectric and ferroelectric state.

Paraelectric materials have the property that the polarisation $\vec{P}$ and the electric field $\vec{E}$ are proportional. Ferroelectric materials have the property that they exhibit permanent polarization, even in the absence of an electric field. This permanent behaviour is the result of a symmetry breaking, i.e. the ions in the lattice have been shifted out of their 'central' positions and induce a permanent dipole moment.

The temperature at which this phase transition occurs is called the **ferroelectric Curie temperature**. Above this temperature the material will behave as a paraelectric material.

Remark 57.7.11. Ferroelectricity can only occur in crystals with unit cells that do not have a center of symmetry. This would rule out the possiblity of having the asymmetry needed for the dipole moment.

Definition 57.7.12 (Saturation polarization). The maximum polarization obtained by a ferroelectric material. It it obtained when the domain formation also reaches a maximum.

Definition 57.7.13 (Remanent polarization). The residual polarization of the material when the external electric field is turned off.

Definition 57.7.14 (Coercive field). The electric field needed to cancel out the remanent polarization.

Definition 57.7.15 (Piezoelectricity). Materials that obtain a polarization when exposed to mechanical stress are called piezoelectric materials.

Remark 57.7.16. All ferroelectric materials are piezoelectric, but the converse is not true. All crystals without a center of symmetry are piezoelectric. This property is however only a necessary (and not a sufficient) condition for ferroelectricity, as mentioned above.

Example 57.7.17 (Transducer). A device that converts electrical to mechanical energy (and vice versa).

57.8 Magnetic properties

Definition 57.8.1 (Diamagnetism). In diamagnetic materials, the magnetization is oriented opposite to the applied field, so $B < H$. The susceptibility is small, negative and independent of the temperature.

Remark 57.8.2. All materials exhibit a diamagnetic character.

Definition 57.8.3 (Paramagnetism). The susceptibility is small, positive and inversely proportional to the temperature.

Definition 57.8.4 (Ferromagnetism). Spontaneous magnetization can occur. The susceptibility is large and dependent on the applied field and temperature. Above a certain temperature, the **ferromagnetic Curie temperature**, the materials will behave as if they were only paramagnetic.

57.8.1 Paramagnetism

Formula 57.8.5 (Curie's law). If the interactions between the particles can be neglected, we obtain the following law:

$$\chi = \frac{C}{T} \tag{57.16}$$

Materials that satisfy this law are called **ideal paramagnetics**.

Formula 57.8.6 (Curie-Weiss law). If the interactions between particles cannot be neglected, we obtain the following law:

$$\chi = \frac{C}{T - \theta} \tag{57.17}$$

where $\theta = CN_W$ with N_W the **Weiss-constant**. This deviation of the Curie law is due to the intermolecular interactions that induce an internal magnetic field $H_m = N_W M$.

Formula 57.8.7 (Brillouin function B_J).

$$\boxed{B_J(y) = \frac{2J+1}{2J}\coth\left(\frac{2J+1}{2J}y\right) - \frac{1}{2J}\coth\left(\frac{y}{2J}\right)} \tag{57.18}$$

where $y = \dfrac{g\mu_B JB}{kT}$

Remark 57.8.8. Because $\coth(y \to \infty) \approx 1$ we have:

$$\text{if } T \to 0 \quad \text{then} \quad M = Ng\mu_B JB_J(y \to \infty) = Ng\mu_B J \tag{57.19}$$

This value is called the **absolute saturation magnetization**.

57.8.2 Ferromagnetism

Ferromagnetics are materials that have strong internal interactions which lead to large scale (with respect to the lattice constant) parallel ordering of the atomic magnetic (dipole) moments. This leads to the spontaneous magnetization of the material and consequently a nonzero total dipole moment.

Remark. In reality, ferromagnetic materials do not always spontaneously possess a magnetic moment in the absence of an external field. When stimulated by a small external field, they will however display a magnetic moment, much larger than paramagnetic materials would.

Definition 57.8.9 (Domain). The previous remark is explained by the existence of Weiss domains. These are spontaneously magnetized regions in a magnetic material. The total dipole moment is the sum of the moments of the individual domains. If not all the domains have a parallel orientation then the total dipole moment can be 0, a small external field is however sufficient to change the domain orientation and produce a large total magnetization.

Definition 57.8.10 (Bloch walls). A wall between two magnetic domains.

Definition 57.8.11 (Ferromagnetic Curie temperature). Above this this temperature the material loses its ferromagnetic properties and it becomes a paramagnetic material following the Curie-Weiss law.

Remark 57.8.12. For ferromagnetic (and ferrimagnetic) materials it is impossible to define a magnetic susceptibility as the magnetization is nonzero even in the absence of a magnetic field.[2] Above the critical temperature (Curie/Néel) it is however possible to define a susceptiblity as the materials become paramagnetic in this region.

57.8.3 Antiferromagnetism

When the domains in a magnetic material have an antiparallel ordering[3], the total dipole moment will be small. If the temperature rises, the thermal agitation however will disturb the orientation of the domains and the magnetic susceptibility will rise.

Definition 57.8.13 (Néel temperature). At the Néel temperature, the susceptibility will reach a maximum. Above this temperature ($T > T_N$) the material will become paramagnetic, satisfying the following formula:

$$\chi = \frac{C}{T + \theta} \tag{57.20}$$

This resembles a generalization of the Curie-Weiss law with a negative and therefore virtual critical temperature.

57.8.4 Ferrimagnetism

Materials that are not completely ferromagnetic nor antiferromagnetic, due to an unbalance between the sublattices, will have a nonzero dipole moment even in the absence of an external field. The magnitude of this moment will however be smaller than that of a ferromagnetic material. These materials are called ferrimagnetic materials.

Formula 57.8.14 (Néel hyperbola). Above the Néel temperature it is possible to define a susceptibility given by:

$$\frac{1}{\chi} = \frac{T}{C} - \frac{1}{\chi_0} - \frac{\sigma}{T - \theta'} \tag{57.21}$$

[2] This can be seen from equation 42.1: $M = \chi H$. The susceptibility should be infinite.

[3] This will occur if it is energetically more favourable.

57.9 Mathematical description

Theorem 57.9.1 (Neumann's principle). *The symmetry elements of the physical properties of a crystal should at least contain those of the point group of the crystal.*

Part XIII

Appendices

Appendix A

Derivations: Mathematics

A.1 Group theory

A.1.1 Explanation for property 3.1.44

Pick an element $x \in X$. The stabilizer of x with respect to G is the set

$$S_x = \{g \in G | g \cdot x = x\}$$

Due to the transitivity of the group action we have that

$$\forall x, y \in X : \exists h \in G : h \cdot x = y$$

So for every $z \in X$ we can choose a group element g_z such that $g_z \cdot x = z$. For all elements in the coset $g_z S_x = \{g_z s \in G | s \in S_x\}$ the following equality is satisfied:

$$(g_z s) \cdot x = g_z \cdot (s \cdot x) = g_z \cdot x = z$$

This implies that the map $\Phi : G/S_x \to X$ is surjective.

Now we need to prove that Φ is also injective. We give a proof by contradiction. Choose two distinct cosets gS_x and hS_x. Then there exist two elements $G, H \in X$ such that $g \cdot x = G$ and $h \cdot x = H$. Now assume that $G = H$. This means that

$$\begin{aligned} & g \cdot x = h \cdot x \\ \iff & (h^{-1}g) \cdot x = x \\ \iff & h^{-1}g \in S_x \\ \iff & hS_x \ni h(h^{-1}g) = g \end{aligned}$$

This would imply that $gS_x = hS_x$ which is in contradiction to our assumption. It follows that $G \neq H$ such that Φ is injective. □

A.2 Calculus

A.2.1 Proof of method 8.2.2

The function $F(x)$ is defined as follows:

$$F(x) = \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n \tag{A.1}$$

We now perform a Borel transform:

$$\begin{aligned} \int_0^{+\infty} F(xt)e^{-t}dt &= \sum_{n=0}^{N} \int_0^{+\infty} \frac{a_n}{n!} x^n t^n e^{-t} dt \\ &= \sum_{n=0}^{N} \frac{a_n}{n!} x^n \int_0^{+\infty} t^n e^{-t} dt \\ &= \sum_{n=0}^{N} \frac{a_n}{n!} x^n \Gamma(n+1) \\ &= \sum_{n=0}^{N} a_n x^n \end{aligned} \tag{A.2}$$

where we used the definition of the Gamma function 7.17 on line 3 and the relation between the factorial function and the Gamma function 7.18 on line 4. □

A.3 Linear algebra

A.3.1 Proof for the equality of definitions 20.4.14 and 20.4.15

$$(u+v) \otimes (u+v) - u \otimes u - v \otimes v = u \otimes v + v \otimes u \tag{A.3}$$

The LHS is an element of the ideal I generated by $\{v \otimes v | v \in V\}$. Using the ideal generated by elements such as in the RHS gives the usual definition of the exterior algebra based on the wedge product as defined in 20.17 because it imposes the relation $u \wedge v = -v \wedge u$.

We do however have to pay attention to one little detail. As mentioned in 20.4.15 the general definition uses the ideal I to construct the quotient space. The other construction is only equivalent when working over a field with a characteristic different from 2. This follows from the fact that we have to divide by 2 when trying to obtain the ideal I from the RHS by setting $u = v$.

A.4 Manifolds and bundles

A.4.1 Proof of equivalence of tangent space constructions. (Definitions 24.2.3 and 24.2.7)

Let (U, φ) be a chart around the point $p \in M$. Using the first definition of a tangent vector (24.2.3), i.e.:

$$\left.\frac{\partial}{\partial q^i}\right|_p : \mathcal{F}_p(M, \mathbb{R}) \to \mathbb{R} : f \mapsto \frac{\partial}{\partial q^i}\left(f \circ \varphi^{-1}\right)(\varphi(p))$$

we can rewrite equation 24.7:

$$v_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial q^i}(\varphi(p))\frac{dq^i}{dt}(0)$$

as follows:

$$v_p(f) = \left.\frac{\partial f}{\partial q^i}\right|_p \frac{dq^i}{dt}(0)$$

Because the partial derivatives as defined in 24.2.3 form a basis for the tangent space (by construction), we see that this equation is in fact an expansion of the tangent vector v_p in terms of that basis. It follows that vectors tangent to curves[1] are also tangent vectors according to the first definition.

To prove the other direction we have to show that the partial derivative operators can be constructed as vectors tangent to curves.

A tangent vector can be expanded, according to the first construction, in the following way:

$$v_p = v^i \left.\frac{\partial}{\partial q^i}\right|_p$$

where we also define $v = (v^1, ..., v^n)$. We can then construct the curve $\gamma : t \mapsto \varphi^{-1}(q_0 + vt)$. It is obvious that the tangent vector v_p is tangent to the curve γ. From this it follows that we have an isomorphism between the tangent vectors from to the first definition and the equivalence classes of vectors tangent to curves from the second definition. These definitions are thus equivalent. □

Although the previous equivalence implies that the tangent space construction using germs of curves gives us a vector space we could also check the vector space axioms directly. First we prove that the sum of vectors tangent to the curves γ and δ is again a vector tangent to some curve $\chi : \mathbb{R} \to M$. For this let us define the curve

$$\chi(t) \equiv \varphi^{-1} \circ \Big(\varphi \circ \gamma(t) + \varphi \circ \delta(t) - \varphi(p)\Big)$$

[1] More precisely: representatives of equivalence classes of vectors tangent to curves.

where φ is again the coordinate map in some chart (U, φ) around $p \in M$. Using equation 24.7 we find:

$$\begin{aligned} v_{p,\chi}(f) =& \frac{\partial(f \circ \varphi^{-1})}{\partial q^i}(\varphi(p)) \frac{d(\varphi^i \circ \chi)}{dt}(0) \\ =& \frac{\partial(f \circ \varphi^{-1})}{\partial q^i}(\varphi(p)) \frac{d}{dt}\left(\varphi^i \circ \gamma + \varphi^i \circ \delta - \varphi^i(p)\right) \\ =& \frac{\partial(f \circ \varphi^{-1})}{\partial q^i}(\varphi(p)) \left(\frac{d(\varphi^i \circ \gamma)}{dt} + \frac{d(\varphi^i \circ \delta)}{dt}\right) \\ =& v_{p,\gamma}(f) + v_{p,\delta}(f) \end{aligned}$$

The constant term $-\varphi(p)$ in the definition of $\chi(t)$ is necessary to make sure that $\chi(0) = \gamma(0) = \delta(0) = p$. The scalar multiplication by a number $\lambda \in K$ can be proven by defining the curve $\chi(t) = \varphi^{-1} \circ \Big[\lambda\Big(\varphi \circ \gamma(t)\Big)\Big]$.

A.4.2 Explanation for example 26.3.13

In this derivation we use the Landau little-o notation $o(t)$, i.e.:

$$\lim_{t \to 0} \frac{o(t)}{t} = 0 \tag{A.4}$$

Now assume that X is a smooth vector field and f is a smooth function. Because the Lie derivative is a local operation we can work in a local chart such that γ is (again locally) equivalent to a curve[2] $\beta_p : U \to \mathbb{R}^n$ and such that we can expand $\beta_p(t)$ around $p \in U$:

$$\begin{aligned} \mathcal{L}_X f(p) &= \lim_{t \to 0} \left[\frac{f(\beta_p(0) + t\beta_p'(0) + o(t)) - f(p)}{t}\right] \\ &= \lim_{t \to 0} \left[\frac{f(p + tX(p) + o(t)) - f(p)}{t}\right] \\ &= \lim_{t \to 0} \left[\frac{f(p) + tDf(p) \cdot X(p) + o(t) - f(p)}{t}\right] \\ &= \sum_k \frac{\partial f}{\partial x^k}(p) X_k(p) + \lim_{t \to 0} \frac{o(t)}{t} \\ &= \sum_k \frac{\partial f}{\partial x^k}(p) X_k(p) \end{aligned} \tag{A.5}$$

where we used the defining condition 26.29 for integral curves on line 2. If we now rewrite this equation as an operator equality, we obtain:

$$\boxed{\mathcal{L}_X = \sum_k X_k \frac{\partial}{\partial x^k}} \tag{A.6}$$

[2]The vector field $X(p) = (p, Y(p))$ where Y is a smooth vector field on $\mathbb{R}^n$ can also be identified with Y itself. This is implicitly done in the derivation by using the notation X for both vector fields.

A.4.3 Explanation for formula 26.3.14

For vector fields we cannot just take the difference at two different points because the tangent spaces generally do not coincide. We can solve this by using the flow 26.3.7:

$$\mathcal{L}_X Y = \lim_{t\to 0} \frac{(T\sigma_t)^{-1}[X(\gamma_p(t))] - X(p)}{t} \tag{A.7}$$

where the $T\sigma_t$ is the differential 26.2.7 of the flow which satisfies $(T\sigma)^{-1} = T\sigma_{-t}$. To see that this definition makes sense we have to show that $(T\sigma_t)^{-1}[X(\gamma_p(t))] \in T_pM$. This goes as follows:

$$\begin{aligned}
(T\sigma_t)^{-1}[X(\gamma_p(t))](f) &= T\sigma_{-t}[X(\gamma_p(t))](f) \\
&= X(\sigma_{-t}\circ\gamma_p(t))(f\circ\sigma_{-t}) \\
&= X(\sigma_{-t}\circ\sigma_t(p))(f\circ\sigma_{-t}) \\
&= X(p)(f\circ\sigma_{-t}) \\
&\in T_pM
\end{aligned}$$

for all $f \in C^k(M,\mathbb{R})$. On line 3 we used the definition of the flow 26.3.7.

We can also rewrite the second term in the numerator of A.7 using the flow:

$$X(p) = X(\sigma_0(p)) = T\sigma_0(X)$$

Using the definition of the pushforward of vector fields 26.26 the Lie derivative can be rewritten as:

$$\begin{aligned}
\mathcal{L}_X Y &= \lim_{t\to 0} \frac{\sigma_{-t*}X(\gamma_p(t)) - \sigma_{0*}X(\gamma_p(0))}{t} \\
&= \frac{d}{dt}(\sigma_{-t*}X)(\gamma_p(t))\Big|_{t=0}
\end{aligned}$$

Or finally by using the relation between pushforward and pullback 26.27 this becomes:

$$\boxed{\mathcal{L}_X Y = \frac{d}{dt}(\sigma_t^* X)(\gamma_p(t))\Big|_{t=0}} \tag{A.8}$$

A.4.4 Connection between vector calculus and differential geometry (Remark 26.4.10)

Looking at formula 26.44 for the exterior derivative of a smooth function and remembering the definition of the gradient 17.2 we see that these two definitions appear very similar. The major difference lies in the fact that ∇f is a vector in $\mathbb{R}^3$ and df is a covector in $\mathbb{R}^{*3}$. However there exists an isomorphism between these spaces and so we can identify ∇f and df.

Similar relations hold for the rotor 17.9 and divergence 17.7, however here we have to use a different construction as we will be working with the spaces Λ^1 and Λ^2. However we can use the Hodge star 20.27 to obtain the correct dimensions.

Consider a vector $\vec{\boldsymbol{f}} = (f_1, f_2, f_3)$ where f_i is smooth. Using these functions f_i we can construct a 1-form $\alpha = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ and a 2-form $\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$. After applying the exterior derivative (in the corresponding spaces) we obtain:

$$d\alpha = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

$$d\omega = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$$

Using result 20.4.22 and the isomorphism $\sim : \mathbb{R}^{3*} \to \mathbb{R}^3$ we can rewrite this as:

$$\boxed{\begin{aligned} \sim df &= \nabla f \\ \sim (*d\alpha) &= \nabla \times \vec{\boldsymbol{f}} \\ *d\omega &= \nabla \cdot \vec{\boldsymbol{f}} \end{aligned}} \qquad \begin{aligned} &(A.9) \\ &(A.10) \\ &(A.11) \end{aligned}$$

Appendix B

Derivations: Lagrangian formalism

B.1 d'Alembert's principle

In the following derivation we assume a constant mass.

$$
\begin{aligned}
&\sum_k \left(\vec{\boldsymbol{F}}_k - \dot{\vec{\boldsymbol{p}}}_k\right) \dot{\vec{\boldsymbol{r}}}_k = 0 \\
\iff &\sum_k \left(\vec{\boldsymbol{F}}_k - \dot{\vec{\boldsymbol{p}}}_k\right) \cdot \left(\sum_l \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l} \dot{q}_l\right) = 0 \\
\iff &\sum_l \left(\sum_k \vec{\boldsymbol{F}}_k \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l} - \sum_k m \ddot{\vec{\boldsymbol{r}}} \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l}\right) \dot{q}_l = 0 \\
\iff &\sum_l \left(Q_l - \sum_k m \ddot{\vec{\boldsymbol{r}}} \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l}\right) \dot{q}_l = 0
\end{aligned}
\tag{B.1}
$$

Now we look at the following derivative:

$$
\begin{aligned}
&\frac{d}{dt}\left(\dot{\vec{\boldsymbol{r}}} \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l}\right) = \ddot{\vec{\boldsymbol{r}}} \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l} + \dot{\vec{\boldsymbol{r}}} \cdot \frac{d}{dt}\left(\frac{\partial \vec{\boldsymbol{r}}}{\partial q_l}\right) \\
\iff &\ddot{\vec{\boldsymbol{r}}} \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l} = \frac{d}{dt}\left(\dot{\vec{\boldsymbol{r}}} \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l}\right) - \dot{\vec{\boldsymbol{r}}} \cdot \frac{d}{dt}\left(\frac{\partial \vec{\boldsymbol{r}}}{\partial q_l}\right) \\
\iff &\ddot{\vec{\boldsymbol{r}}} \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l} = \frac{d}{dt}\left(\dot{\vec{\boldsymbol{r}}} \cdot \underbrace{\frac{\partial \vec{\boldsymbol{r}}}{\partial q_l}}_{A}\right) - \dot{\vec{\boldsymbol{r}}} \cdot \left(\frac{\partial \dot{\vec{\boldsymbol{r}}}}{\partial q_l}\right)
\end{aligned}
\tag{B.2}
$$

To evaluate A we can take a look at another derivative:

$$\begin{aligned}\frac{\partial \dot{\vec{\boldsymbol{r}}}}{\partial \dot{q}_l} &= \frac{\partial}{\partial \dot{q}_l}\left(\sum_k \frac{\partial r}{\partial q_k}\dot{q}_k\right) \\ &= \sum_k \frac{\partial r}{\partial q_k}\delta_{kl} \\ &= \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l} \\ &= A\end{aligned}$$

Substituting this in formula B.2 gives:

$$\begin{aligned}\ddot{\vec{\boldsymbol{r}}} \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial q_l} &= \frac{d}{dt}\left(\dot{\vec{\boldsymbol{r}}} \cdot \frac{\partial \dot{\vec{\boldsymbol{r}}}}{\partial \dot{q}_l}\right) - \dot{\vec{\boldsymbol{r}}} \cdot \left(\frac{\partial \dot{\vec{\boldsymbol{r}}}}{\partial q_l}\right) \\ &= \frac{d}{dt}\left(\frac{1}{2}\frac{\partial \dot{\vec{\boldsymbol{r}}}^2}{\partial \dot{q}_l}\right) - \frac{1}{2}\frac{\partial \dot{\vec{\boldsymbol{r}}}^2}{\partial q_l}\end{aligned} \tag{B.3}$$

If we multiply this by the mass m and sum over all particles we get :

$$\begin{aligned}\sum_k m_k \ddot{\vec{\boldsymbol{r}}}_k \cdot \frac{\partial \vec{\boldsymbol{r}}_k}{\partial q_l} &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_l}\left(\sum_k \frac{1}{2} m \dot{\vec{\boldsymbol{r}}}_k^2\right) - \frac{\partial}{\partial q_l}\left(\sum_k \frac{1}{2} m \dot{\vec{\boldsymbol{r}}}_k^2\right) \\ &= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l} - \frac{\partial T}{\partial q_l}\end{aligned} \tag{B.4}$$

Where we have denoted the total kinetic energy in the last line as T.
Plugging this result into formula B.1 gives us:

$$\sum_l \left(Q_l - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l} - \frac{\partial T}{\partial q_l}\right)\dot{q}_l = 0 \tag{B.5}$$

As all the q_l are independent the following relation should hold for all l:

$$\begin{aligned}&Q_l - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_l}\right) - \frac{\partial T}{\partial q_l} = 0 \\ \iff &\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_l}\right) - \frac{\partial T}{\partial q_l} = Q_l}\end{aligned} \tag{B.6}$$

This last equation is known as a **Lagrange equation of the first kind**.
If we have a system with only conservative forces acting on it, we can write the force on the i-th particle as:

$$F_i = -\nabla_i V \tag{B.7}$$

With this in mind, lets take a look at the derivative of the potential V with respect to the l-th generalized coordinate:

$$\begin{aligned}\frac{\partial V}{\partial q_l} &= \sum_i (\nabla_i V) \cdot \frac{\partial \vec{\boldsymbol{r}}_i}{\partial q_l} \\ &= -Q_l\end{aligned} \tag{B.8}$$

The differentiation of V with respect to any generalized velocity $\dot{q}_l$ is trivially zero. This combined with the last formula B.8 and with formula B.6 gives:

$$\begin{aligned} & \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_l}\right) - \frac{\partial T}{\partial q_l} = Q_l \\ \iff & \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_l}\right) - \frac{\partial T}{\partial q_l} = -\frac{\partial V}{\partial q_l} + \frac{\partial V}{\partial \dot{q}_l} \\ \iff & \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_l} - \frac{\partial V}{\partial \dot{q}_l}\right) - \frac{\partial}{\partial q_l}\left(T - V\right) = 0 \end{aligned} \tag{B.9}$$

If we introduce a new variable L, called the **Lagrangian**, we get the **Lagrangian equation of the second kind**:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_l}\right) - \frac{\partial L}{\partial q_l} = 0} \tag{B.10}$$

B.2 Hamilton's principle

In this part we start from the principle of least action. First we the define the **action** as following:

$$\boxed{I = \int_{t_1}^{t_2} L\left(y(t), \dot{y}(t), t\right) dt} \tag{B.11}$$

Then we require that this action is minimal for the physically acceptable path. To do this we define a family of paths:

$$y(t, \alpha) = y(t) + \alpha\eta(t) \tag{B.12}$$

Where $\eta(t)$ is an arbitrary function with the following boundary conditions:

$$\begin{cases} \eta(t_1) = 0 \\ \eta(t_2) = 0 \end{cases} \tag{B.13}$$

If we define the action integral over this family of paths, the integral B.11 becomes a function of α:

$$I(\alpha) = \int_{t_1}^{t_2} L\left(y(t, \alpha), \dot{y}(t, \alpha), t\right) dt \tag{B.14}$$

Requiring that the action integral is stationary for $y(t)$ (thus $\alpha = 0$) is equivalent to:

$$\left(\frac{dI}{d\alpha}\right)_{\alpha=0} = 0 \tag{B.15}$$

This condition combined with formula B.14 gives us:

$$\frac{dI}{d\alpha} = \int_{t_1}^{t_2} \frac{d}{d\alpha} L\left(y(t, \alpha), \dot{y}(t, \alpha), t\right) dt \tag{B.16}$$

As we evaluate this derivative in $\alpha = 0$ we can replace $y(t, \alpha)$ by $y(t)$ due to definition B.12.

$$\begin{aligned}\frac{dI}{d\alpha} &= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right] dt \\ &= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial y}\eta(t) + \frac{\partial L}{\partial \dot{y}}\dot{\eta}(t)\right] dt \end{aligned} \tag{B.17}$$

If we substitute $\frac{\partial L}{\partial \dot{y}} := h(t)$ and apply integration by parts to the second term in this integral, we get:

$$\begin{aligned}\frac{dI}{d\alpha} &= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial y}\eta(x) + h(t)\dot{\eta}(t)\right] dt \\ &= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial y}\eta(t) + h(t)\frac{d\eta}{dt}\right] dt \\ &= \int_{t_1}^{t_2} \frac{\partial L}{\partial y}\eta(t)dt + \eta(t_2)h(t_2) - \eta(t_1)h(t_1) - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)\eta(t)dt \end{aligned} \tag{B.18}$$

Due to the initial conditions B.15 for the function $\eta(t)$, the two terms in the middle vanish and we obtain:

$$\frac{dI}{d\alpha} = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)\right]\eta(t)dt \tag{B.19}$$

Furthermore, as the function $\eta(t)$ was arbitrary, the only possible way that this derivative can become zero is when the integrand is identically zero:

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0} \tag{B.20}$$

If we compare this result with formula B.10 we see that we can also obtain the **Lagrangian equations of the second kind** by starting from the principle of least action. (Where the variable y represents the generalized coordinates q_l and the variable $\dot{y}$ represents the generalized velocities $\dot{q}_l$)

Remark B.2.1. Differential equations of the form

$$\boxed{\frac{\partial f}{\partial y}(y, \dot{y}, x) = \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}(y, \dot{y}, x)\right)} \tag{B.21}$$

are known as **Euler-Lagrange equations**.

B.3 Explanation for Noether's theorem 36.3.1

The general transformation rule for the Lagrangian is:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\delta\mathcal{L}(x) \tag{B.22}$$

To have a symmetry, i.e. keep the action invariant, the deformation factor has to be a 4-divergence:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu \mathcal{J}^\mu(x) \tag{B.23}$$

To obtain formula 36.8 we vary the Lagrangian explicitly:

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) \\
&= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi \\
&= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) + \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right]\delta\phi
\end{aligned}$$

The second term vanishes due to the Euler-Lagrange equation B.20. Combining these formulas gives us:

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) - \partial_\mu \mathcal{J}^\mu(x) = 0 \tag{B.24}$$

From this equation we can conclude that the current

$$\boxed{j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - \mathcal{J}^\mu(x)} \tag{B.25}$$

is conserved.

Appendix C

Derivations: Optics and material physics

C.1 Optics

C.1.1 Law of Lambert-Beer 39.9

From formula 39.8 we now that the complex refractive index can be written as

$$\widetilde{n} = n + ik$$

Where k is called the **extinction coefficient**.
From classical optics we also know that in a material the speed of light obeys the following relation:

$$c = \widetilde{n}v$$

Where we have used the complex refractive index. It readily follows that the wavenumber (sadly also given the letter k) can be written as:

$$k = \frac{\omega}{v} = \widetilde{n}\frac{\omega}{c}$$

From classical electromagnetism we know that a plane wave can be written as:

$$E(x,t) = Re\left\{A\,\exp\left[i(kx - \omega t + \phi)\right]\right\}$$

So everything put together we get:

$$E(x,t) = Re\left\{A\,\exp\left[i\left((n+ik)\frac{\omega}{c}x - \omega t + \phi\right)\right]\right\}$$

or also:

$$E(x,t) = Re\left\{A\,\exp\left[in\frac{\omega}{c}x\right]\cdot\exp\left[-k\frac{\omega}{c}x\right]\cdot\exp\left[-i\omega t\right]\cdot\exp\left[i\phi\right]\right\}$$

We also know that the intensity is given by the following relation:

$$I(x) = |E(x)|^2 = E^*(x) \cdot E(x)$$

So only the second factor will remain. Dividing this by its value for $x = 0$ we get:

$$\frac{I(x)}{I(0)} = \frac{E(x) \cdot E^*(x)}{E(0) \cdot E^*(0)} = \exp\left[-\frac{2k\omega}{c}x\right] = \exp[-\alpha x]$$

Where α is the absorption coefficient as defined in formula 39.10. □

Appendix D

Derivations: Classical and Statistical Mechanics

D.1 Moments of inertia

In this section we will always use formula 35.8 to calculate the moment of inertia.

D.1.1 Disk

The volume of a (solid) disk is given by:

$$V_{disk} = \pi R^2 d \tag{D.1}$$

where R is the radius and d is the thickness. The mass density is then given by:

$$\rho = \frac{M}{\pi R^2 d} \tag{D.2}$$

Using cylindrical coordinates the moment of inertia then becomes:

$$I = \frac{M}{\pi R^2 d} \int_0^{2\pi} d\varphi \int_0^d dz \int_0^R r^3 dr \tag{D.3}$$

$$= \frac{M}{\pi R^2 d} 2\pi d \frac{R^4}{4} \tag{D.4}$$

$$= \frac{1}{2} M R^2 \tag{D.5}$$

D.1.2 Solid sphere

The volume of a solid sphere is given by:

$$V_{sphere} = \frac{4}{3}\pi R^3 \tag{D.6}$$

where R is the radius. The mass density is then given by:

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \tag{D.7}$$

We will use spherical coordinates to derive the moment of inertia, but we have to be carefull. The r in formula 35.8 is the distance between a point in the body and the axis of rotation. So it is not the same as the r in spherical coordinates which is the distance between a point and the origin. However the relation between these two quantities is easily found using basic geometry to be:

$$r = r' \sin\theta \tag{D.8}$$

where r' is the spherical coordinate. Now we can calculate the moment of inertia as follows:

$$I = \frac{M}{\frac{4}{3}\pi R^3} \int_0^{2\pi} d\varphi \int_0^R r'^4 dr' \int_0^{\pi} \sin^3\theta d\theta \tag{D.9}$$

$$= \frac{M}{\frac{4}{3}\pi R^3} 2\pi \frac{R^5}{5} \frac{4}{3} \tag{D.10}$$

$$= \frac{2}{5} M R^2 \tag{D.11}$$

D.2 Schottky defects

Let E_v be the energy needed to remove a particle from its lattice point and move it to the surface. Furthermore we will neglect any surface effects and assume that the the energy E_v is indepedent of the distance to the surface.

The total energy of all vacancies is then given by $E = nE_v$. The number of possible microstates is

$$\Omega = \frac{(N+n)!}{n!N!} \tag{D.12}$$

where we used the fact that the removal of n particles creates n more lattice points at the surface. Using Boltzmann's entropy formula 55.1 and Stirling's formula we obtain

$$S(N,n) = k \ln\Omega = k\big[(N+n)\ln(N+n) - n\ln n - N\ln N\big] \tag{D.13}$$

Using 55.2 we can find the temperature:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{dS}{dn}\frac{dn}{dE} = \frac{k}{E_v}\ln\frac{N+n}{n} \tag{D.14}$$

which can be rewritten as

$$\boxed{\frac{n}{N+n} = \exp\left(-\frac{E_v}{kT}\right)} \tag{D.15}$$

The density of Frenkel defects can be derived analogously.

Appendix E

Units and symbols

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Table E.1: Units

List of Symbols

The following symbols are used throughout the summary:

Operations

$\approx$	is approximately equal to
$\cong$	is isomorphic to
$\hookrightarrow$	is included in
$\mapsto$	mapsto
$\mathbb{1}_X$	The identity map on the set X.
$\sphericalangle(v, w)$	The angle between the vectors v and w.
Im	Imaginary part of a complex number.
Re	Real part of a complex number.
Ad_g	Adjoint representation of a Lie group G.
ad_X	Adjoint representation of a Lie algebra $\mathfrak{g}$.
arg	Argument of a complex number.
Par_t^γ	Parallel transport map with respect to the curve γ.
Res	Residue of a complex function.
e	The identity map on a group.
$X + Y$	Sum of the vector spaces X and Y.
$X \oplus Y$	Direct sum of the vector spaces X and Y.
$X \otimes Y$	Tensor product of the vector spaces X and Y.
$X \times Y$	Cartesian product of the sets X and Y.

Sets

$[a, b]$	Closed interval
$\emptyset$	Empty set
$\mathrm{Hom}(V, W)$	The set of (homo)morphisms from a set V to a set W.
$\Lambda^n(V)$	Space of antisymmetric rank n tensors over a vector space V.
$\mathcal{H}$	Hilbert space
$\mathcal{B}(V, W)$	Space of bounded continuous maps from the space X to the space Y.

$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on the manifold M.
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on the manifold M.
$\pi_n(X, x_0)$	The n^{th} homotopy space on X based at $x_0 \in X$.
$\rho(A)$	The resolvent set of a bounded linear operator A.
$\sigma(A)$	The compression spectrum of a bounded linear operator A.
$\sigma_r(A)$	The residual spectrum of a bounded linear operator A.
$\mathrm{Aut}(V)$	The set of automorphisms (invertible endomorphisms) on a set V.
$\mathrm{End}(V)$	The rinf of endomorphisms on a set V.
$\mathrm{Hol}_p(\omega)$	Holonomy group at p with respect to the connection ω.
$\mathrm{Pin}(V)$	The pin group of the Clifford algebra $C\ell(V, Q)$.
$\mathrm{Open}(X)$	The category of open subsets of a topological space X.
$]a, b[$	Open interval
$C_p^\infty(M)$	Ring of all smooth functions $f : M \to \mathbb{R}$ defined on a neighbourhood of $p \in M$.
D^n	Standard n-disk
$GL(V)$	General linear group: group of all automorphisms on a vector space V.
$GL_n(K)$	General linear group: group of all invertible n-dimensional matrices over the field K.
S^n	Standard n-sphere
$S^n(V)$	Space of symmetric rank n tensors over a vector space V.
$SL(2, \mathbb{C})$	Special linear group of dimension 2 over the field of complex numbers. Is isomorphic to $\mathrm{Spin}(1, 3)$ and hence forms a universal double cover of the Lorentz group $SO(1, 3)$. Important in general relativity as the transformation group of spinors.
T^n	Standard n-torus. Cartesian product of n times S^1.
$U(\mathfrak{g})$	The universal enveloping algebra of a Lie algebra $\mathfrak{g}$.
ΩX	The loop space on X.

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