Compendium of Mathematics & Physics

Nicolas Dewolf

March 13, 2025

Contents

Contents **1**

I Quantum Theory 7

1 Quantum Mechanics **10**

1.1 Introduction . . . 11
1.1.1 Dirac–von Neumann postulates . . . 11
1.2 Schrödinger picture . . . 12
1.3 Heisenberg–Born–Jordan picture . . . 15
1.3.1 Hydrogen atom . . . 16
1.3.2 Molecular dynamics . . . 16
1.4 Mathematical formalism . . . 17
1.4.1 Weyl systems . . . 17
1.4.2 Dirac–von Neumann postulates: revisited . . . 18
1.4.3 Symmetries . . . 19
1.4.4 Symmetric states . . . 20
1.5 Foundations ♣ . . . 20
1.5.1 Measurement problem . . . 20
1.5.2 Copenhagen interpretation . . . 21
1.5.3 Many-worlds interpretation . . . 21
1.5.4 Relational quantum mechanics . . . 21
1.6 Angular Momentum . . . 24
1.6.1 Angular momentum operator . . . 24
1.6.2 Rotations . . . 25
1.6.3 Spinor representation . . . 26
1.6.4 Coupling of angular momenta . . . 26
1.7 Approximation methods . . . 27
1.7.1 WKB approximation . . . 27
1.8 Curved backgrounds ♣ . . . 28
1.8.1 Extending quantum mechanics . . . 28
1.8.2 WKB approximation . . . 30

2 Quantum Information Theory **32**
2.1 Entanglement . . . 32
2.1.1 Introduction . . . 32
2.1.2 Bell states . . . 33
2.1.3 SRE states . . . 34
2.2 Density operators . . . 34
2.3 Channels . . . 35
2.4 Quantum logic . . . 36
2.4.1 Birkhoff–von Neumann logic . . . 36
2.4.2 Linear logic . . . 37
2.5 Topos theory ♣ . . . 42
2.5.1 Bohr topos . . . 42
2.5.2 Internal logic . . . 44

3 Quantization ♣ **45**
3.1 Introduction . . . 45
3.2 Deformation quantization . . . 46
3.3 Geometric Quantization . . . 47
3.3.1 Prequantization . . . 47
3.3.2 Coadjoint orbits . . . 52
3.4 Toeplitz quantization . . . 52
3.5 Constrained systems . . . 54
3.5.1 Dirac procedure . . . 54
3.5.2 BRST quantization . . . 54
3.5.3 BFV quantization . . . 55

4 Perturbation Theory **57**
4.1 Interaction picture . . . 57
4.2 Rayleigh–Schrödinger theory . . . 58
4.3 Adiabatic approximation . . . 59
4.3.1 Berry phase . . . 59

5 Scattering Theory **61**
5.1 Cross sections . . . 61
5.2 Lippman–Schwinger equations . . . 61

II Relativity & Quantum Field Theory **63**

6 Special Relativity **66**
6.1 Lorentz transformations . . . 66
6.2 Energy and momentum . . . 67

6.3 Action principle 67

7 General Relativity **70**
7.1 Causal structure 70
7.2 Einstein field equations 71
7.3 Black holes 72
7.4 Conserved quantities 73
7.5 Tetradic formulation 73

8 Quantum Field Theory **75**
8.1 Klein–Gordon field 76
8.1.1 Lagrangian and Hamiltonian 76
8.1.2 Raising and lowering operators 76
8.1.3 Scalar propagator 78
8.1.4 Normalization constant 78
8.1.5 Invariant integration measure 78
8.2 Contractions and Wick's theorem 78
8.2.1 Bosonic fields 78
8.2.2 Fermionic fields 80
8.3 Quantum chromodynamics 81
8.4 Feynman rules 81
8.4.1 Scalar theory 81
8.4.2 Graph theory 82
8.4.3 Degree of divergence 83
8.4.4 Amplituhedron ♣ 84
8.5 Renormalization 84
8.5.1 Introduction: Statistical physics 85
8.5.2 Dimensional regularization 85
8.5.3 Wilsonian renormalization 87
8.5.4 Connes–Kreimer renormalization 87
8.6 Entanglement in QFT 90
8.6.1 Lattice theories 91

III Appendices **92**

Appendix A *G*-Structures **94**

Appendix B Notes **97**
B.1 Noether's Theorem and Gauge-Gravity Duality by S. De Haro 97
B.1.1 Pseudotensors 97
B.1.2 AdS-CFT duality 99

B.2 A generalization of Noether's theorem and the information-theoretic approach to the study of symmetric dynamics by R. Spekkens 99

List of Symbols **101**

References **108**

Index **110**

Introduction

Goals

This compendium originated out of the necessity for a compact summary of important theorems and formulas during physics and mathematics classes at university. When the interest in more (and more exotic) subjects grew, this collection lost its compactness and became the chaos it now is. Although there should exist some kind of overall structure, it was not always possible to keep every section self-contained or respect the order of the chapters.

It should definitely not be used as a formal introduction to any subject. It is neither a complete work nor a fact-checked one, so the usefulness and correctness is not guaranteed. However, it can be used as a look-up table for theorems and formulas, and as a guide to the literature. To this end, each chapter begins with a list of useful references. At the same time, only a small number of statements are proven in the text (or appendices). This was done to keep the text as concise as possible (a failed endeavour). However, in some cases the major ideas underlying the proofs are provided.

Structure and conventions

Sections and statements that require more advanced concepts, in particular concepts from later chapters or (higher) category theory, will be labelled by the *clubs* symbol ♣. Some definitions, properties or formulas are given with a proof or an extended explanation whenever I felt like it. These are always contained in a blue frame to make it clear that they are not part of the general compendium. When a section uses notions or results from a different chapter at its core, this will be recalled in a green box at the beginning of the section.

Definitions in the body of the text will be indicated by the use of **bold font**. Notions that have not been defined in this summary but that are relevant or that will be defined further on in the compendium (in which case a reference will be provided) are indicated by *italic text*. Names of authors are also written in *italic*.

Objects from a general category will be denoted by a lower-case letter (depending on the context, upper-case might be used for clarity), functors will be denoted by upper-case letters and the categories themselves will be denoted by symbols in **bold font**. In the later chapters on physics, specific conventions for the different types of vectors will often be adopted. Vectors in Euclidean space will be denoted by a bold font letter with an arrow above, e.g. $\vec{\boldsymbol{a}}$, whereas vectors in Minkowski space (4-vectors) and differential forms will be written without the arrow, e.g. $\boldsymbol{a}$. Matrices and tensors will always be represented by capital letters and, dependent on the context, a specific font will be adopted.

Part I

Quantum Theory

Table of Contents

1 Quantum Mechanics **10**

1.1 Introduction . . . 11

1.1.1 Dirac–von Neumann postulates . . . 11

1.2 Schrödinger picture . . . 12

1.3 Heisenberg–Born–Jordan picture . . . 15

1.3.1 Hydrogen atom . . . 16

1.3.2 Molecular dynamics . . . 16

1.4 Mathematical formalism . . . 17

1.4.1 Weyl systems . . . 17

1.4.2 Dirac–von Neumann postulates: revisited . . . 18

1.4.3 Symmetries . . . 19

1.4.4 Symmetric states . . . 20

1.5 Foundations ♣ . . . 20

1.5.1 Measurement problem . . . 20

1.5.2 Copenhagen interpretation . . . 21

1.5.3 Many-worlds interpretation . . . 21

1.5.4 Relational quantum mechanics . . . 21

1.6 Angular Momentum . . . 24

1.6.1 Angular momentum operator . . . 24

1.6.2 Rotations . . . 25

1.6.3 Spinor representation . . . 26

1.6.4 Coupling of angular momenta . . . 26

1.7 Approximation methods . . . 27

1.7.1 WKB approximation . . . 27

1.8 Curved backgrounds ♣ . . . 28

1.8.1 Extending quantum mechanics . . . 28

1.8.2 WKB approximation . . . 30

2 Quantum Information Theory **32**

2.1 Entanglement . . . 32

2.1.1 Introduction . . . 32

2.1.2 Bell states . . . 33
2.1.3 SRE states . . . 34
2.2 Density operators . . . 34
2.3 Channels . . . 35
2.4 Quantum logic . . . 36
2.4.1 Birkhoff–von Neumann logic . . . 36
2.4.2 Linear logic . . . 37
2.5 Topos theory ♣ . . . 42
2.5.1 Bohr topos . . . 42
2.5.2 Internal logic . . . 44

3 Quantization ♣ **45**
3.1 Introduction . . . 45
3.2 Deformation quantization . . . 46
3.3 Geometric Quantization . . . 47
3.3.1 Prequantization . . . 47
3.3.2 Coadjoint orbits . . . 52
3.4 Toeplitz quantization . . . 52
3.5 Constrained systems . . . 54
3.5.1 Dirac procedure . . . 54
3.5.2 BRST quantization . . . 54
3.5.3 BFV quantization . . . 55

4 Perturbation Theory **57**
4.1 Interaction picture . . . 57
4.2 Rayleigh–Schrödinger theory . . . 58
4.3 Adiabatic approximation . . . 59
4.3.1 Berry phase . . . 59

5 Scattering Theory **61**
5.1 Cross sections . . . 61
5.2 Lippman–Schwinger equations . . . 61

Chapter 1

Quantum Mechanics

The main reference for this chapter is Bransden and Joachain (2000). In the first two sections, the two basic formalisms of quantum mechanics are introduced: wave and matrix mechanics. The main reference for the mathematically rigorous treatment of quantum mechanics, in particular in the infinite-dimensional setting, is Moretti (2016). The main reference for the generalization to curved backgrounds is Schuller (2016). The section on the WKB approximation is based on Bates and Weinstein (1997). Relevant chapters in this compendium are, amongst others, **??**, **??** and **??**.

Chapter Contents

1.1 Introduction **11**
1.1.1 Dirac–von Neumann postulates 11
1.2 Schrödinger picture **12**
1.3 Heisenberg–Born–Jordan picture **15**
1.3.1 Hydrogen atom 16
1.3.2 Molecular dynamics 16
1.4 Mathematical formalism **17**
1.4.1 Weyl systems 17
1.4.2 Dirac–von Neumann postulates: revisited 18
1.4.3 Symmetries 19
1.4.4 Symmetric states 20
1.5 Foundations ♣ **20**
1.5.1 Measurement problem 20
1.5.2 Copenhagen interpretation 21
1.5.3 Many-worlds interpretation 21
1.5.4 Relational quantum mechanics 21

1.6 Angular Momentum 24
1.6.1 Angular momentum operator 24
1.6.2 Rotations 25
1.6.3 Spinor representation 26
1.6.4 Coupling of angular momenta 26
1.7 Approximation methods 27
1.7.1 WKB approximation 27
1.8 Curved backgrounds ♣ 28
1.8.1 Extending quantum mechanics 28
1.8.2 WKB approximation 30

1.1 Introduction

This section will give both an introduction and formal treatment of the objects and notions used in quantum mechanics.

1.1.1 Dirac–von Neumann postulates

Axiom 1.1 (States). The states of a (closed) system are represented by vectors in a (complex) Hilbert space $\mathcal{H}$. In the infinite-dimensional setting, one often further restricts to separable spaces, i.e. the spaces are required to admit a countable Hilbert basis.

Notation 1.1.1 (Dirac notation). State vectors $|\psi\rangle$ are called **ket**'s and their duals $\langle\psi|$ are called **bra**'s. The inner product of a state $|\phi\rangle$ and a state $|\psi\rangle$ is denoted by $\langle\phi\,|\,\psi\rangle$. This notation is often called the **braket notation** (or Dirac notation).

Axiom 1.2 (Observables). Every physical property is represented by a bounded, self-adjoint operator. In the finite-dimensional case, this is equivalent to an operator that admits a complete set of eigenfunctions.

Definition 1.1.2 (Compatible observables). Two observables are said to be compatible if they share a complete set of eigenvectors.

Formula 1.1.3 (Closure relation). For a complete set of eigenvectors, the closure relation (also called the **resolution of the identity**) is given by (see also **??**)

$$\sum_n |\psi_n\rangle\langle\psi_n| + \int_X |x\rangle\langle x|\,dx = \mathbb{1}\,, \tag{1.1}$$

where the sum ranges over the discrete spectrum and the integral over the continuous spectrum. For simplicity, the summation will also be used for the continuous part.

Axiom 1.3 (Born rule). Let $\mathcal{H}$ be the Hilbert space of a physical system and consider an observable $\widehat{O}$. If $|\psi\rangle$ is a state vector and $\widehat{P}_\phi$ is the projection onto an eigenvector $|\phi\rangle$ of $\widehat{O}$, the probability of observing the state $|\phi\rangle$ is given by:

$$\frac{\langle\psi \mid \widehat{P}_\phi \mid \psi\rangle}{\langle\psi \mid \psi\rangle} = \frac{|\langle\psi \mid \phi\rangle|^2}{\langle\psi \mid \psi\rangle}. \tag{1.2}$$

Property 1.1.4 (Projectivization). In light of the Born rule, the dynamics of a system does not depend on the global phase or normalization, i.e. states are represented by rays in a projective Hilbert space $\mathcal{H}\mathbb{P}$ (**??**).

Combining Born's rule with **??**, gives the following definition.

Definition 1.1.5 (Expectation value). The expectation value of an observable $\widehat{A}$ in a (normalized) state $|\psi\rangle$ is defined as follows:

$$\langle\widehat{A}\rangle_\psi := \langle\psi|\widehat{A}|\psi\rangle. \tag{1.3}$$

The subscript ψ is often left implicit. As in ordinary statistics (**??**), the uncertainty or variance is defined as follows:

$$\Delta A := \langle\widehat{A}^2\rangle - \langle\widehat{A}\rangle^2. \tag{1.4}$$

Formula 1.1.6 (Uncertainty relation). Let $\widehat{A}, \hat{B}$ be two observables and let $\Delta A, \Delta B$ be the corresponding uncertainties. The (**Robertson**) uncertainty relation reads as follows:

$$\Delta A \Delta B \geq \frac{1}{4}\left|\left\langle\left[\widehat{A}, \hat{B}\right]\right\rangle\right|^2. \tag{1.5}$$

Axiom 1.4 (Projection[1]). Let $\mathcal{H}$ be the Hilbert space of a physical system and consider an observable $\widehat{O}$ with eigenvalues $\{o_i\}_{i\in I}$. After measuring the observable $\widehat{O}$ in the state $|\psi\rangle$, the outcome will be one of the eigenvalues o_i and system will 'collapse' to, i.e. get projected onto, the eigenstate $\widehat{P}_{o_i}|\psi\rangle \equiv |o_i\rangle$.

Axiom 1.5 (Unitary evolution). The evolution of a closed system is unitary, i.e. there exists a unitary operator $\widehat{U}(t,t') \in \mathrm{Aut}(\mathcal{H})$, for all times $t \leq t'$, such that[2]

$$|\psi(t)\rangle = \widehat{U}(t,t')|\psi(t')\rangle. \tag{1.6}$$

1.2 Schrödinger picture

Since the energy is of paramount importance in physics, the associated eigenvalue equation deserves its own name.

[1] Also called the **measurement postulate**.

[2] Note that some authors use a different convention whereby the two arguments are interchanged.

Formula 1.2.1 (Time-independent Schrödinger equation).

$$\widehat{H}|\psi\rangle = E|\psi\rangle \tag{1.7}$$

The operator $\widehat{H}$ is called the **Hamiltonian** of the system. The wave function ψ is an element of the vector space $L^2(\mathbb{R},\mathbb{C})\otimes\mathcal{H}$ with $\mathcal{H}$ the internal Hilbert space (describing, for example, the spin or charge of a particle). This is an eigenvalue equation for the energy levels of the system.

@@ INTRODUCE POSITION/CONFIGURATION REPRESENTATION @@

The time evolution of a wave function was governed by Axiom 1.5. By passing to generators, the following equation is obtained.

Formula 1.2.2 (Time-dependent Schrödinger equation).

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle\,. \tag{1.8}$$

In case $\widehat{H}$ is time independent, the TISE can be obtained from this equation by separation of variables (see below).

Proof (Derivation of TISE from TDSE). Starting from the one-dimensional TDSE in position space with a time-independent Hamiltonian, one can perform a separation of variables and assert a solution of the form $\psi(x,t) = X(x)T(t)$. Inserting this in the previous equation gives

$$i\hbar X(x)T'(t) = \big(\widehat{H}X(x)\big)T(t)\,.$$

Dividing both sides by $X(x)T(t)$ and rearranging the terms gives

$$i\hbar\frac{T'(t)}{T(t)} = \frac{\widehat{H}X(x)}{X(x)}\,.$$

Because the left side only depends on t and the right side only depends on x, one can conclude that they both have to equal a constant $E \in \mathbb{C}$. This leads to the following system of differential equations:

$$\begin{cases} i\hbar T'(t) = ET(t)\,, \\ \widehat{H}X(x) = EX(x)\,. \end{cases}$$

The first equation immediately gives a solution for T:

$$T(t) = C\exp\left(-\frac{iE}{\hbar}t\right). \tag{1.9}$$

The second equation is exactly the TISE (Formula 1.2.1). □

Example 1.2.3 (Massive particle in a stationary potential).

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right)\psi(x,t) \tag{1.10}$$

In this case, the TISE reads as follows:

$$\psi''(x) = -\frac{2m}{\hbar^2}\big(E - V(x)\big)\psi(x)\,. \tag{1.11}$$

Formula 1.2.4 (General solution). A general solution of the TDSE (for time-independent Hamiltonians) is given by the following formula (cf. **??**):

$$\psi(x,t) = \sum_E c_E \psi_E(x) e^{-\frac{i}{\hbar}Et}\,, \tag{1.12}$$

where the functions $\psi_E(x)$ are the eigenfunctions of the TISE. The coefficients c_E can be found using the orthogonality relations

$$c_E = \left(\int_{\mathbb{R}} \overline{\psi_E}(x)\psi(x,t_0)\,dx\right)e^{\frac{i}{\hbar}Et_0}\,. \tag{1.13}$$

Formula 1.2.5 (Dyson series). Inserting the evolution operator from Axiom 1.5 into the TDSE 1.2.2, even generalising to time-dependent Hamiltonians, the following operator equation is obtained:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}(t,t') = \widehat{H}(t)\widehat{U}(t,t')\,. \tag{1.14}$$

With the initial condition $\widehat{U}(t,t) = 1$ for all $t \in \mathbb{R}$, this can be formally solved as follows (for $t \geq t'$):

$$\widehat{U}(t,t') = \mathbb{1} - \frac{i}{\hbar}\int_{t'}^{t} \widehat{H}(\tau)\widehat{U}(\tau,t')\,d\tau\,. \tag{1.15}$$

This solution can be iterated to obtain a series expansion of the evolution operator:

$$\widehat{U}(t,t') = 1 - \frac{i}{\hbar}\int_{t'}^{t}\widehat{H}(\tau)\,d\tau + \left(-\frac{i}{\hbar}\right)^2\int_{t'}^{t}\int_{t'}^{\tau}\widehat{H}(\tau)\widehat{H}(\tau')\,d\tau'\,d\tau + \cdots\,. \tag{1.16}$$

It is clear that the integrands are time-ordered. By explicitly introducing the **time-ordering operator**

$$\mathcal{T}\big(\widehat{H}(t_1)\widehat{H}(t_2)\big) = \begin{cases}\widehat{H}(t_1)\widehat{H}(t_2) & \text{if } t_1 \geq t_2\,,\\ \widehat{H}(t_2)\widehat{H}(t_1) & \text{if } t_2 > t_1\,,\end{cases} \tag{1.17}$$

the integrals can be rewritten in a more symmetric form:

$$\widehat{U}(t,t') = 1 - \frac{i}{\hbar}\int_{t'}^{t}\widehat{H}(\tau)\,d\tau + \frac{1}{2!}\left(-\frac{i}{\hbar}\right)\int_{t'}^{t}\int_{t'}^{t}\mathcal{T}\big(\widehat{H}(\tau)\widehat{H}(\tau')\big)\,d\tau'\,d\tau + \cdots\,. \tag{1.18}$$

By comparing this expression to the series expansion for exponential functions, the following concise formula is obtained:

$$\widehat{U}(t,t') = \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t'}^{t}\widehat{H}(\tau)\,d\tau\right). \tag{1.19}$$

The expression on the right-hand side is called the **Dyson series**.

1.3 Heisenberg–Born–Jordan picture

In the previous section, the central object was the wave function. It was this object that evolved in time and the operators acting on the Hilbert space of physical states were assumed to be fixed. However, it is also possible to transfer this dependence on time to the operators.

Formula 1.3.1 (Time-dependent observables).

$$\widehat{O}_H(t) := e^{\frac{i}{\hbar}\widehat{H}t}\widehat{O}_S(t)e^{-\frac{i}{\hbar}\widehat{H}t} \tag{1.20}$$

The equivalence between the Schödinger and Heisenberg pictures essentially come from the fact that the time-evolving expectation values of operators are given by the following formula:

$$\langle\widehat{O}(t)\rangle = \langle\psi|e^{\frac{i}{\hbar}\widehat{H}t}\widehat{O}(t)e^{-\frac{i}{\hbar}\widehat{H}t}|\psi\rangle\,. \tag{1.21}$$

The difference between the pictures is simply the choice of whether to include the evolution operator in the states or in the operators.

Using the above transformation, the Schödinger equation (Formula 1.2.2) can also be reexpressed.

Formula 1.3.2 (Time-dependent Schrödinger equation).

$$\frac{\partial\widehat{O}_H}{\partial t}(t) = \frac{i}{\hbar}\left[\widehat{H}_H(t),\widehat{O}_H(t)\right] + \left(\frac{\partial\widehat{O}}{\partial t}(t)\right)_H \tag{1.22}$$

Taking this expression for the Schrödinger equation and taking expectation values (using the linearity of the equation), gives the following (interaction-independent) result.

Theorem 1.3.3 (Ehrenfest). *Let $\widehat{H}$ be the Hamiltonian and consider an observable $\widehat{O}$. The expectation value of this operator evolves as follows:*

$$\frac{\mathrm{d}\langle\widehat{O}\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\langle[\widehat{O},\widehat{H}]\rangle + \left\langle\frac{\partial\widehat{O}}{\partial t}\right\rangle. \tag{1.23}$$

Remark 1.3.4 (Equivalence). It is important to note that the Schrödinger equation could be replaced by Ehrenfest's theorem. They are entirely equivalent.

But, given the abstract state vectors $|\psi\rangle$ from Section 1.1.1, how does one recover the position (configuration) representation $\psi(x)$? This is simply the projection of the state vector $|\psi\rangle$ on the 'basis function' $\delta(x)$, i.e. $\psi(x)$ represents an expansion coefficient in terms of a 'basis' for the physical Hilbert space. In the same way, one can obtain the momentum representation $\psi(p)$ by projecting onto the plane waves e^{ipx}.

Remark 1.3.5. It should be noted that neither the 'basis states' $\delta(x)$, nor the plane waves e^{ipx} are square integrable and, hence, they are not elements of the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$. This issue can be resolved through the concept of *rigged Hilbert spaces*.

@@ COMPLETE @@

1.3.1 Hydrogen atom

Consider the hydrogen atom, i.e. a single proton (the nucleus) orbited by a single electron with only the electrostatic Coulomb force acting between them (gravity can safely be neglected):

$$\widehat{H} := \frac{\hat{p}_p^2}{2m_p} + \frac{\hat{p}_e^2}{2m_e} - \frac{e^2}{4\pi\varepsilon r^2}\,. \tag{1.24}$$

It is not hard to see that this is the quantum mechanical version of the Kepler problem (**??**). The special property of the Kepler problem was that it contained a 'hidden' symmetry that gave rise to the conserved Laplace–Runge–Lenz vector (**??**). As is the case for all conserved charges in quantum mechanics, this symmetry induces a degeneracy of the energy eigenvalues. Degeneracy of the magnetic quantum number $m \in \mathbb{N}$ follows from rotational symmetry, but the energy levels of the hydrogen atom only depend on the principal quantum number $n \in \mathbb{N}$. It is the degeneracy of the total angular quantum number $l \in \mathbb{N}$ that is due to this 'hidden' SO(4)-symmetry. It is often called an 'accidental degeneracy' for this reason.

@@ COMPLETE @@

1.3.2 Molecular dynamics

Consider the Hamiltonian of two interacting atoms:

$$\widehat{H} = \frac{\widehat{P}_1^2}{2M_1} + \frac{\widehat{P}_2^2}{2M_2} + \frac{\hat{q}_1\hat{q}_2}{4\pi\varepsilon R^2} + \sum_i \frac{\hat{p}_i^2}{2m} - \frac{e\hat{q}_1}{4\pi\varepsilon r_{i1}^2} - \frac{e\hat{q}_2}{4\pi\varepsilon r_{i2}^2} + \sum_{i\neq j} \frac{e^2}{4\pi\varepsilon r_{ij}^2}\,, \tag{1.25}$$

where the indices i, j indicate the electrons and uppercase symbols denote operators associated to the nuclei.

Except for the most simple situations, solving the Schrödinger equation for this Hamiltonian becomes intractable (both analytically and numerically). However, in general,

one can approximate the situation. The masses of nuclei are much larger than those of the electrons and this influences their motion, they move much slower than the electrons. In essence, the nuclei and electrons live on different time scales and this allows to decouple their dynamics:

$$\widehat{H}_{\text{nucl}} = \frac{\widehat{P}_1^2}{2M_1} + \frac{\widehat{P}_2^2}{2M_2} + \frac{Q_1 Q_2}{4\pi\varepsilon R^2} + V_{\text{eff}}(R_1, R_2). \tag{1.26}$$

The electrons generate an effective potential for the nuclei and the Schrödinger equation decouples as follows:

$$\begin{aligned} \widehat{H}_{\text{nucl}}(R)\psi(R) &= E\psi(R)\,, \\ \widehat{H}_{\text{el}}(r,R)\phi(r,R) &= E_{\text{el}}\phi(r,R)\,. \end{aligned} \tag{1.27}$$

This is the so-called **Born–Oppenheimer approximation**. From a more modern physical perspective, this approximation can also be seen to be a specific instance of renormalization theory, where the short time-scale (or, equivalently, the high energy-scale) degrees of freedom are integrated out of the theory.

1.4 Mathematical formalism

1.4.1 Weyl systems

Definition 1.4.1 (Canonical commutation relations). Two observables $\widehat{A}, \widehat{B}$ are said to obey a canonical commutation relation (CCR) if they satisfy (up to a constant factor $\hbar$)

$$[\widehat{A}, \widehat{B}] = i\,. \tag{1.28}$$

The prime examples are the position and momentum operators $\hat{x}, \hat{p}$. Through functional calculus, one can also define the exponential operators $e^{is\widehat{A}}$ and $e^{it\widehat{B}}$. The above relation then induces the so-called **Weyl form** of the CCR:

$$e^{is\widehat{A}} e^{it\widehat{B}} = e^{ist} e^{it\widehat{B}} e^{is\widehat{A}}\,. \tag{1.29}$$

Theorem 1.4.2 (Stone–von Neumann). *All pairs of irreducible, unitary one-parameter subgroups satisfying the Weyl form of the CCRs are unitarily equivalent.*
Corollary 1.4.3. The Schrödinger and Heisenberg pictures are unitarily equivalent.

In fact, one can generalize the Weyl form of the CCRs.
Definition 1.4.4 (Weyl system). Let (L, ω) be a symplectic vector space and let K be a complex vector space. Consider a map W from L to the space of unitary operators on K. The pair (K, W) is called a Weyl system over (L, ω) if it satisfies

$$W(z)W(z') = e^{i/2\omega(z,z')} W(z+z') \tag{1.30}$$

for all $z, z' \in L$, i.e. W is a projective representation of the Abelian group L and ω is, up to rescaling, the group cocycle inducing it (**??**). The relation itself is called a **Weyl relation**.

Definition 1.4.5 (Heisenberg system). Let W be a Weyl system. The selfadjoint generators $\phi(z)$, which exist by Stone's theorem **??**, of the maps $t \mapsto W(tz)$ are said to form a Heisenberg system. These operators satisfy the following properties:

1. **Positive homogeneity**: $\lambda\phi(z) = \phi(\lambda z)$ for all $\lambda > 0$,
2. **Commutator**: $[\phi(z), \phi(z')] = -i\omega(z, z')$, and
3. **Weak additivity**: $\phi(z + z')$ is the closure (**??**) of $\phi(z) + \phi(z')$.

Remark 1.4.6. It should be noted that the Weyl relations are more fundamental than their infinitesimal counterparts. Only the Weyl relations are well defined on more general spaces and when passing to a relativistic setting.

Recall **??**, where the framework of measure theory and distributions was generalized to the noncommutative context.

Property 1.4.7 (Schrödinger representation). Consider a distribution d on a (real) TVS V. There exists a unique unitary representation U of the additive group V^* on $L^2(V, d)$ such that

$$U(\lambda)f = e^{id(\lambda)}f \tag{1.31}$$

for all bounded tame functions f and such that 1 is cyclic for U in $L^2(V, d)$. Moreover, this representation is continuous with respect to the finest locally convex topology on V (the one generated by all seminorms on V)[3].

@@ EXPLAIN RELEVANCE e.g. Baez, Segal, and Zhou (2014) @@

1.4.2 Dirac–von Neumann postulates: revisited

Section 1.1.1 presented the axioms of quantum mechanics in terms of Hilbert spaces and the operators thereon. However, the incredible insight of *von Neumann* was that one can do away with the Hilbert space. By **??**, the observables of a quantum-mechanical system form a C^*-algebra. Consequently, the idea is to rephrase the axioms in purely C^*-algebraic terms (**??**). By **??**, these two approaches are equivalent.

Axiom 1.6 (Observables). A physical system is characterized by a C^*-algebra, with the observables corresponding to the self-adjoint elements.

Axiom 1.7 (States). A state of a quantum-mechanical system is given by a state of the associated C^*-algebra (**??**).

Axiom 1.8 (Born rule). The expectation value of an observable a in a state ω is given by the evaluation $\omega(a)$.

[3] This topology is also known as the **algebraic topology**

Remark 1.4.8. Section 2.2 will link this axiom to traces and operator theory through **??** and **??**.

Axiom 1.9 (Projection).

Axiom 1.10 (Unitary evolution).

@@ CORRECT ALL AXIOMS @@

1.4.3 Symmetries

Property 1.4.9 (States). By the postulates of quantum mechanics, states are represented by rays in the projective Hilbert space $\mathcal{H}\mathbb{P}$. The probabilities, given by the Born rule (Axiom 1.3), can be expressed in terms of the *Fubini–Study metric* on $\mathcal{H}\mathbb{P}$ as follows:

$$\mathcal{P}(\psi,\phi) := \cos^2\big(d_{\mathrm{FS}}(\psi,\phi)\big) = \frac{|\langle\psi\,|\,\phi\rangle|^2}{\langle\psi\,|\,\psi\rangle\langle\phi\,|\,\phi\rangle}\,, \tag{1.32}$$

where $|\psi\rangle, |\phi\rangle$ are representatives of the states ψ, ϕ in $\mathcal{H}\mathbb{P}$.

Definition 1.4.10 (Symmetry). A quantum symmetry (or **quantum automorphism**) is an isometric automorphism of $\mathcal{H}\mathbb{P}$. The group of these symmetries is denoted by $\mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P})$.

The following theorem due to *Wigner* gives a (linear) characterization of quantum symmetries.[4]

Theorem 1.4.11 (Wigner). *Every quantum automorphism of $\mathcal{H}\mathbb{P}$ is induced by a unitary or anti-unitary operator on $\mathcal{H}$.*

This is equivalent to saying that the group morphism

$$\pi : \mathsf{Aut}(\mathcal{H},\mathcal{P}) := \mathsf{U}(\mathcal{H}) \times \mathsf{AU}(\mathcal{H}) \to \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \tag{1.33}$$

is surjective. Together with the kernel $\mathsf{U}(1)$, given by phase shifts, this forms a short exact sequence:

$$1 \longrightarrow \mathsf{U}(1) \longrightarrow \mathsf{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \longrightarrow 1\,. \tag{1.34}$$

In the case of symmetry breaking (e.g. lattice systems), the full symmetry group is reduced to a subgroup $G \subset \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P})$. The group of operators acting on $\mathcal{H}$ is then given by the pullback $\widetilde{G}$ of the diagram

$$\mathsf{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \longleftarrow G\,. \tag{1.35}$$

It should also be noted that the kernel of the homomorphism $\widetilde{G} \to G$ is again $\mathsf{U}(1)$. This leads to the property that $\widetilde{G}$ is a $\mathbb{Z}_2$-twisted (hence noncentral) $\mathsf{U}(1)$-extension of G, where the twist is induced by the homomorphism $\phi : \mathsf{Aut}(\mathcal{H},\mathcal{P}) \to \mathbb{Z}_2$ that says whether an operator is implemented unitarily or anti-unitarily.

@@ COMPLETE @@

[4] It is a particular case of a more general theorem in projective geometry.

1.4.4 Symmetric states

Axiom 1.11 (Symmetrization postulate). Let $\mathcal{H}$ be the single-particle Hilbert space. A system of $n \in \mathbb{N}$ identical particles is described by a state $|\Psi\rangle$ belonging to either $S^n\mathcal{H}$ or $\Lambda^n\mathcal{H}$. These **bosonic** and **fermionic** states are, respectively, of the form

$$|\Psi_B\rangle = \sum_{\sigma\in S_n} |\psi_{\sigma(1)}\rangle \cdots |\psi_{\sigma(n)}\rangle \tag{1.36}$$

and

$$|\Psi_F\rangle = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)|\psi_{\sigma(1)}\rangle \cdots |\psi_{\sigma(n)}\rangle\,, \tag{1.37}$$

where the $|\psi_i\rangle$ are single-particle states and S_n is the permutation group on n elements.

Remark 1.4.12. In ordinary quantum mechanics, this is a postulate, but in quantum field theory, this is a consequence of the *spin-statistics theorem*. @@ ADD THIS THEOREM TO [QFT] @@

Definition 1.4.13 (Slater determinant). Let $\{\phi_i(\vec{q})\}_{i\leq n}$ be a set of wave functions, called **spin orbitals**, describing a system of n identical fermions. The totally antisymmetric wave function of the system is given by

$$\psi(\vec{q}_1,\dots,\vec{q}_n) = \frac{1}{\sqrt{n!}}\det\begin{pmatrix}\phi_1(\vec{q}_1) & \cdots & \phi_n(\vec{q}_1)\\ \vdots & \ddots & \vdots\\ \phi_1(\vec{q}_n) & \cdots & \phi_n(\vec{q}_n)\end{pmatrix}. \tag{1.38}$$

A similar function can be defined for bosonic systems using the concept of *permanents*.

1.5 Foundations ♣

1.5.1 Measurement problem

If one looks at the Schrödinger equation (Formula 1.2.2) or Ehrenfest's theorem (Dummy 1.3.3), it is easy to see that time evolution is entirely linear and deterministic. Superpositions are preserved under Hamiltonian flow (a crucial ingredient of quantum mechanics) and, given an initial state, time evolution will always lead to the same final state. However, the Born rule (Axiom 1.3), which governs 'measurements' is very nonlinear and nondeterministic. It is probabilistic and, once a 'measurement' has been performed, the state has 'collapsed' onto an eigenstate of the observable under consideration.

The issue of what constitutes a 'measurement' — Is it a conscious human doing an experiment? Is it a mouse interfering with an experiment? Is it two particles interacting? ...[5] — and why exactly the Born rule holds and what it entails, i.e. how probabilities arise, is known as the measurement problem. On a historical note, it should

[5] This (perhaps artificial) boundary between classical and quantum is sometimes called the **Heisenberg cut**.

be noted that, after an initial surge of interest shortly after the 5th Solvay Conference (1927), where quantum mechanics was formally established, the study of the foundations of quantum mechanics (the measurement problem specifically) became an infamous topic due to the pragmatic mentality of nuclear physics during the 20th century.

@@ ADD (dynamical collapse, epistemic) @@

1.5.2 Copenhagen interpretation

The Copenhagen interpretation[6] takes the foundations of quantum mechanics as presented above very literally.

@@ COMPLETE (e.g. collapse) @@

1.5.3 Many-worlds interpretation

This interpretation, originating with *Everett*, posits a different idea, which does away with the need of the explicit Born rule axiom. In this interpretation, there is a kind of 'universal wave function', which governs both the observer and the experiment. A 'measurement' is then simply an entanglement-inducing interaction between these two subsystems.

The main implication of such an interpretation is, however, that the universal wave function branches every time such an interaction occurs. More precisely, assume that 'we', the observers, perform a measurement on some system (for simplicity, assume that the measurement has a binary outcome). The measurement process is then described as follows:

$$|\text{in}\rangle_{\text{obs}}|\text{in}\rangle_{\text{exp}} \longrightarrow \lambda_0|0\rangle_{\text{obs}}|0\rangle_{\text{exp}} + \lambda_1|1\rangle_{\text{obs}}|1\rangle_{\text{exp}}\,. \tag{1.39}$$

Taking this superposition as a physical reality, this means that if we had measured the state 0, a copy of us living on the other branch will have measured 1 (and the other way around).

@@ COMPLETE (e.g. origin of probabilities) @@

1.5.4 Relational quantum mechanics

An important notion in classical physics is that of a *reference frame*, i.e. a choice of axes and scales. Usually, this corresponds to choosing an observer, relative to which one expresses the motion of all other objects. In relativity, the relative treatment of physics was the grand breakthrough by Einstein. However, although this notion had been left

[6] This name stems from the fact that its initial proponents were from the group of physicists centered around *Bohr*.

aside for a long time in the treatment of quantum mechanics and a specific choice of reference frame was silently assumed, this assumption was not as innocuous as it appears. Superposition and complementarity make a definite choice of absolute reference frame impossible.

To understand the relevance of a relational approach to quantum mechanics, consider the following thought experiment.

Definition 1.5.1 (Wigner's friend). Consider two observers, Wigner and his friend, performing an experiment as shown diagrammatically in Fig. 1.1. One envisions Wigner standing outside the laboratory, having no way to observe what happens inside the lab, and his friend who performs an experiment inside the lab. The paradox arises from the two ways one can describe the sequence of the friend performing a measurement and Wigner checking up on the results in the classical (Copenhagen) interpretation.

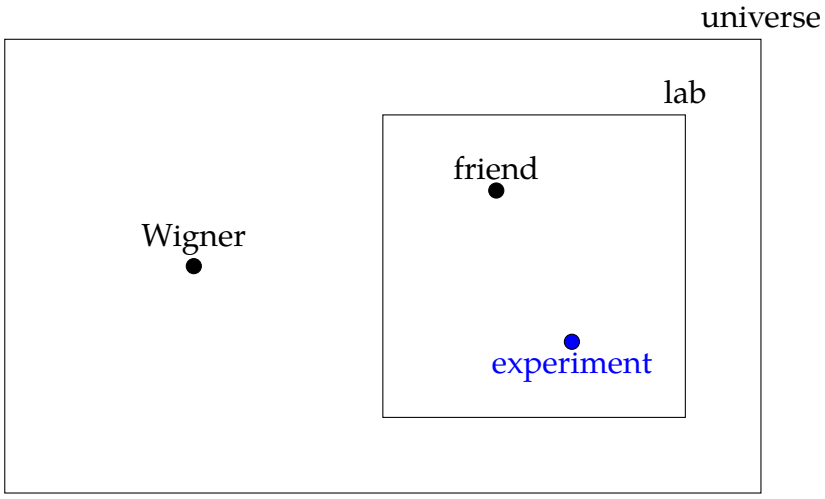


Figure 1.1: Wigner's friend thought experiment.

From the point of view of the friend, at the moment of measurement, the projection/collapse axiom states that the wave function describing friend + experiment 'collapses' to:

$$|\psi\rangle = |\uparrow\rangle_{\text{friend}}|\uparrow\rangle_{\text{exp}}\,. \tag{1.40}$$

However, from the point of view of Wigner, who has not observed the measurement, the state is described by

$$|\psi\rangle' = \alpha|\uparrow\rangle_{\text{friend}}|\uparrow\rangle_{\text{exp}} + \beta|\downarrow\rangle_{\text{friend}}|\downarrow\rangle_{\text{exp}}\,. \tag{1.41}$$

Whereas in the many-world approach one would simply take the branching approach, which is fully unitary and resolves this issue by avoiding collapse, the relational approach takes collapse at face value, but states that observations are relative, i.e. always with respect to some fixed observer (be it a person, a classical object or another quantum-mechanical system). From this point of view, textbook Copenhagen QM is

simply quantum mechanics with respect to some god-given observer, and collapse and unitary evolution do not have to be reconciled.

The more general idea is that information and, hence, the values of observables are a relative notion, i.e. variables only attain their values when considered with respect to a certain observer. As such, RQM is an epistemic interpretation of quantum mechanics in that the wave function only captures 'our' information about the system (or universe) and not the 'true' physical state. When applied to the notion of instantaneity (or velocity), this line of thinking will give rise to (special) relativity as in Chapter 6 (and Chapter 7).

@@ COMPLETE @@

For example, consider three observers: Alice, Bob and Charlie. Assume that each observer has a spin-$\frac{1}{2}$ particle and that, relative to Alice, the joint state is given by

$$|\psi\rangle^A_{ABC} = |\uparrow\rangle^A_A\Big(|\uparrow\rangle^A_B + |\downarrow\rangle^A_B\Big)|\downarrow\rangle^A_C\,. \tag{1.42}$$

Note that this state is separable. Now, what would the state be relative to Bob? If one supposes that changes of reference frame are *coherent* (to be formalized below), the joint state will be

$$|\psi\rangle^B_{ABC} = |\uparrow\rangle^B_B\Big(|\uparrow\rangle^B_A|\downarrow\rangle^B_C + |\downarrow\rangle^B_A|\uparrow\rangle^B_C\Big)\,. \tag{1.43}$$

A mere change of reference frame, an operation that would classically leave the physics invariant, has transformed a product state into an entangled state.

Axiom 1.12 (Relational physics). Given $n \in \mathbb{N}$ systems[7], any state is described relative to one of these systems. Given a choice of 'observing system', let it be system i, the state of system i is given by a fiducial state $|0\rangle^i_i$.

Axiom 1.13 (Coherent change). Consider a change of reference frame $0 \longrightarrow i$ such that

$$\begin{cases} |\psi\rangle^0 \longrightarrow |\psi\rangle^i \\ |\phi\rangle^0 \longrightarrow |\phi\rangle^i\,. \end{cases} \tag{1.44}$$

Then

$$\alpha|\psi\rangle^0 + \beta|\phi\rangle^0 \longrightarrow \alpha|\psi\rangle^i + \beta|\phi\rangle^i \tag{1.45}$$

for all $\alpha, \beta \in \mathbb{C}$.

Abstractly, a (classical) reference frame is defined as follows in the spirit of **??** and **??**.

[7] An abstraction of the notion of observer.

Definition 1.5.2 (Reference frame). Let X be an object of interest. Whereas a coordinate chart on X, modeled on an object Y, is given by a morphism $Y \to X$, a **coordinate system** on X is given by an isomorphism $Y \cong X$, i.e. a global coordinate chart. A reference frame is coordinate system for which Y corresponds the a physical system.

Let the system of interest X admit a group action that is both free and transitive, turning it into a G-torsor (**??**). At the level of sets, one has $X \cong G$ and a choice of origin, i.e. a specific choice of isomorphism, corresponds to a choice of reference frame (the identity element corresponding to the fiducial state above). A change of reference frames $s^0 \longrightarrow s^i$, frome system 0 to system i, is given by the right regular action of the relative coordinate of i on all relative coordinates:

$$\phi^{0\to i}(e, g_1^0, \dots, g_n^0) \mapsto (g_0^i, g_1^0 g_0^i, \dots, e, \dots, g_n^0 g_0^i)\,, \tag{1.46}$$

where the relation $g_i^0 = (g_0^i)^{-1}$ was used. It should be noted that this boils down to a *passive transformation*. When passing to the quantization of these systems, one should assume that G is locally compact and comes equipped with the canonical Haar measure (**??**). In this case, a quantization is given by the space of square-integrable functions $L^2(G)$, where basis states are labeled by group elements.

@@ VERIFY THIS STATEMENT @@

The change-of-reference-frame operator is given as follows:

$$\widehat{U}^{0\to i} := \mathsf{SWAP}_{0,i} \circ \int_G \mathbb{1}_{L^2(G)} \otimes \widehat{U}_R(g_i^0)^{\otimes i-2} \otimes |g_0^i\rangle\langle g_i^0| \otimes \widehat{U}_R(g_i^0)^{\otimes n-i-2}\, dg_i^0\,, \tag{1.47}$$

where

$$\widehat{U}_R(g) : |x\rangle \mapsto |xg^{-1}\rangle \tag{1.48}$$

is the unitary implementation of the right regular action and dg denotes integration with respect to the Haar measure on G. It can be shown that $\widehat{U}^{0\to i}$ is unitary, its inverse being given by $\widehat{U}^{i\to 0}$ and composition is transitive. It can be shown that this procedure can be extended to any one-particle Hilbert space $\mathcal{H}$ as long as the inclusion $G \to \mathcal{H}$ is injective and maps G to an orthonormal basis of (a subset of) $\mathcal{H}$.

1.6 Angular Momentum

1.6.1 Angular momentum operator

Property 1.6.1 (Lie algebra). The angular momentum operators generate a Lie algebra (**??**). The Lie bracket is defined by the following commutation relation:

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk}\hat{J}_k\,. \tag{1.49}$$

Since rotations correspond to actions of the orthogonal group SO(3), it should not come as a surprise that the above relation is exactly the defining relation of the Lie algebra $\mathfrak{so}(3)$ from **??**.

Property 1.6.2. The mutual eigenbasis of $\hat{J}^2$ and $\hat{J}_z$ is defined by the following two eigenvalue equations:

$$\hat{J}^2|j,m\rangle = j(j+1)\hbar^2\,|j,m\rangle\,, \tag{1.50}$$

$$\hat{J}_z|j,m\rangle = m\hbar\,|j,m\rangle\,. \tag{1.51}$$

Definition 1.6.3 (Ladder operators[8]). The raising and lowering operators $\hat{J}_+$ and $\hat{J}_-$ are defined as follows:

$$\hat{J}_+ := \hat{J}_x + i\hat{J}_y \qquad \text{and} \qquad \hat{J}_- := \hat{J}_x - i\hat{J}_y\,. \tag{1.52}$$

These operators only change the quantum number $m_z \in \mathbb{N}$, not the total angular momentum.

Corollary 1.6.4. From the commutation relations of the angular momentum operators, one can derive the commutation relations of the ladder operators:

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z\,. \tag{1.53}$$

Formula 1.6.5. The total angular momentum operator $\hat{J}^2$ can now be expressed in terms of $\hat{J}_z$ and the ladder operators using the commutation relation (1.49):

$$\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z\,. \tag{1.54}$$

Remark 1.6.6 (Casimir operator). From the definition of $\hat{J}^2$, it follows that this operator is a Casimir invariant (**??**) of $\mathfrak{so}(3)$.

1.6.2 Rotations

Formula 1.6.7. An infinitesimal rotation $\widehat{R}(\delta\vec{\boldsymbol{\varphi}})$ is given by the following formula:

$$\widehat{R}(\delta\vec{\boldsymbol{\varphi}}) = \mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\delta\vec{\boldsymbol{\varphi}}\,. \tag{1.55}$$

A finite rotation can be generated by applying this infinitesimal rotation repeatedly:

$$\widehat{R}(\vec{\boldsymbol{\varphi}}) = \left(\mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\frac{\vec{\boldsymbol{\varphi}}}{n}\right)^n = \exp\left(-\frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{\varphi}}\right). \tag{1.56}$$

Formula 1.6.8 (Matrix elements). Applying a rotation over an angle φ about the z-axis to a state $|j,m\rangle$ gives

$$\widehat{R}(\varphi\vec{\boldsymbol{e}}_z)|j,m\rangle = \exp\left(-\frac{i}{\hbar}\hat{J}_z\varphi\right)|j,m\rangle = \exp\left(-\frac{i}{\hbar}m\varphi\right)|j,m\rangle\,. \tag{1.57}$$

[8] Also called the **creation** and **annihilation** operators (especially in quantum field theory).

Multiplying these states with a bra $\langle j', m'|$ and using the orthonormality of the eigenstates, gives the matrix elements of the rotation operator:

$$\widehat{R}_{ij}(\varphi\vec{e}_z) = \exp\left(-\frac{i}{\hbar}m\varphi\right)\delta_{jj'}\delta_{mm'}\,. \tag{1.58}$$

From the expression of the angular momentum operators and the rotation operator, it is clear that a general rotation has no effect on the total angular momentum number $j \in \mathbb{N}$. This means that the rotation matrix will be block diagonal with respect to j. This amounts to the following reduction of the representation of the rotation group:

$$\langle j, m'|\widehat{R}(\varphi\vec{n})|j, m\rangle = \mathcal{D}^{(j)}_{m,m'}(\widehat{R})\,, \tag{1.59}$$

where the functions $\mathcal{D}^{(j)}_{m,m'}(\widehat{R})$ are called the **Wigner D-functions**. For every value of j, there are $(2j+1)$ values for m. This implies that the matrix $\mathcal{D}^{(j)}(\widehat{R})$ is a $(2j+1)\times(2j+1)$-matrix.

1.6.3 Spinor representation

Definition 1.6.9 (Pauli matrices).

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.60}$$

From this definition, it is clear that the Pauli matrices are Hermitian and unitary. Together with the 2×2 identity matrix, they form a basis for the space of 2×2 Hermitian matrices. For this reason, the identity matrix is often denoted by σ_0 (especially in the context of relativistic QM).

Formula 1.6.10. In the spinor representation ($J = \frac{1}{2}$), the Wigner-D matrix reads as follows:

$$\mathcal{D}^{(1/2)}(\varphi\vec{e}_z) = \begin{pmatrix} e^{-i/2\varphi} & 0 \\ 0 & e^{i/2\varphi} \end{pmatrix}. \tag{1.61}$$

1.6.4 Coupling of angular momenta

Due to the tensor product structure of a coupled Hilbert space, the angular momentum operator $\hat{J}_i$ should now be interpreted as $\mathbb{1}\otimes\cdots\otimes\hat{J}_i\otimes\cdots\otimes\mathbb{1}$ (cf. **??**). Because the angular momentum operators $\hat{J}_{k\neq i}$ do not act on the space $\mathcal{H}_i$, one can pull these operators through the tensor product:

$$\hat{J}_i|j_1\rangle\otimes\cdots\otimes|j_n\rangle = |j_1\rangle\otimes\cdots\otimes\hat{J}_i|j_i\rangle\otimes\cdots\otimes|j_n\rangle\,. \tag{1.62}$$

The basis used above is called the **uncoupled basis**.

For simplicity, the total Hilbert space is, from here on, assumed to be that of a two-particle system. Let $\hat{J}$ denote the total angular momentum:

$$\hat{J} = \hat{J}_1 + \hat{J}_2 \,. \tag{1.63}$$

With this operator, one can define a **coupled** state $|J, M\rangle$, where M is the total magnetic quantum number which ranges from $-J$ to J.

Formula 1.6.11 (Clebsch–Gordan coefficients). Because both bases (coupled and uncoupled) span the total Hilbert space $\mathcal{H}$, there exists an invertible transformation between them. The transformation coefficients can be found by using the resolution of the identity:

$$|J, M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2 \,|\, J, M\rangle \,. \tag{1.64}$$

These coefficients are called the Clebsch–Gordan coefficients.

Property 1.6.12. By acting with the operator $\hat{J}_z$ on both sides of Formula 1.6.11, it is possible to prove that the Clebsch–Gordan coefficients are nonzero if and only if $M = m_1 + m_2$.

1.7 Approximation methods

1.7.1 WKB approximation

The Wentzel–Kramers–Brillouin (WKB) approximation[9] starts from the ansatz

$$\psi(\vec{q}) := \exp\big(iS(\vec{q})/\hbar\big) \,, \tag{1.65}$$

with $S : \mathbb{R}^n \to \mathbb{R}$ a phase function that is to be determined. Inserting this in the TDSE (in configuration representation) gives:

$$\left[\frac{\|\vec{\nabla} S(\vec{q})\|^2}{2m} + \big(V(\vec{q}) - E\big) - \frac{i\hbar \Delta S(\vec{q})}{2m}\right] \exp\big(iS(\vec{q})/\hbar\big) = 0 \,. \tag{1.66}$$

To first order, i.e. for slowly varying potentials, the last term can be ignored. In this case, the phase function satisfies the Hamilton–Jacobi equation (**??**):

$$H\big(\vec{q}, S'(\vec{q})\big) = \frac{\|\vec{\nabla} S(\vec{q})\|^2}{2m} + \big(V(x) - E\big) = 0 \,. \tag{1.67}$$

In physics, the Hamilton–Jacobi equation without time derivative is often called the **eikonal equation**[10]. This leads to the following result.

[9] This approach to solving second-order ODEs was essentially introduced a century earlier by *Green* and *Liouville*.

[10] This name stems from optics.

Property 1.7.1. A function $S : \mathbb{R}^n \to \mathbb{R}$ is a phase function for a first-order solution to the Schrödinger equation if its differential lies in a level set of the classical Hamiltonian $H : T^*\mathbb{R}^n \to \mathbb{R}$. These solutions are said to be **admissible**.

To obtain higher-order approximations, the solution has to be generalized beyond a pure phase function:

$$\psi(\vec{q}) = a(\vec{q}) \exp\left(iS(\vec{q})/\hbar\right). \tag{1.68}$$

Assuming S is admissible, the factor $a : \mathbb{R}^n \to \mathbb{R}$ satisfies the **homogeneous transport equation**:

$$a\Delta S + 2\vec{\nabla}a \cdot \vec{\nabla}S = 0. \tag{1.69}$$

If a satisfies this equation, ψ is called a **semiclassical state**. Note that this equation is equivalent to $a^2\vec{\nabla}S$ being divergence free or, equivalently:

$$\mathcal{L}_{\pi_* X^H}(a^2 \operatorname{Vol}) = 0. \tag{1.70}$$

Since Lie derivatives pull back under diffeomorphisms (**??**) and the image $\operatorname{im}(\mathrm{d}S)$ gives a trivial subbundle of $T^*\mathbb{R}^n$, this is also equivalent to

$$\mathcal{L}_{X^H}(a^2\pi^* \operatorname{Vol}) = 0. \tag{1.71}$$

This quadratic behaviour in a leads to the idea that the correct object for representing quantum states is a half-density (see also Method 3.3.7). This leads to the following statement:

> A second-order solution to the Schrödinger equation is given by a pair (S, a), where S is an admissible phase function and $a \in \Omega^{1/2}(\operatorname{im}(\mathrm{d}S))$ is a half-form that is invariant under the (classical) Hamiltonian flow.

The generalization to curved spaces, i.e. replacing $\mathbb{R}^{2n}$ by a symplectic manifold M, will be covered in Section 1.8.2.

1.8 Curved backgrounds ♣

Using the tools of distribution theory and differential geometry (**?? ??**, **??** and onwards), one can introduce quantum mechanics on curved backgrounds (in the sense of 'space', not 'spacetime').

1.8.1 Extending quantum mechanics

Remark 1.8.1 (Rigged Hilbert spaces). A first important remark to be made is that the classical definition of the wave function as an element of $L^2(\mathbb{R}^d, \mathbb{C})$ is not sufficient,

even in flat Cartesian space. A complete description requires the introduction of so-called *Gel'fand triples* or *rigged Hilbert spaces*, where the space of square-integrable functions is replaced by the Schwartz space (**??**) of rapidly decreasing functions. The linear functionals on this space are then given by the tempered distributions.

When working on curved spaces or even in non-Cartesian coordinates on flat space, one can encounter problems with the definition of the self-adjoint operators $\hat{q}^i$ and $\hat{p}_i$. The naive definition $\hat{q}^i = q^i, \hat{p}_i = -i\partial_i$ gives rise to extra terms that break the canonical commutation relations and the selfadjointness of the operators (e.g. the angular position operator $\hat{\varphi}$ on the circle together with its conjugate $\hat{L}$) when calculating inner products.

An elegant solution to this problem is obtained by giving up the definition of the wave function as a well-defined function $\psi : \mathbb{R}^d \to \mathbb{C}$. Assume that the physical space has the structure of a Riemannian manifold (M, g) and that the 'naive' wave functions take values in a vector space V. Then, construct a vector bundle E with typical fibre V over M. By **??**, an invariant description of the 'true' wave function is a map $\Psi : F(E) \to V$ or, locally, the pullback $\psi := \varphi^*\Psi$ for some local section $\varphi : U \subseteq M \to F(E)$. The Levi-Civita connection on M also induces a covariant derivative ∇ on E that can be used to define differential operators.

Now, a general inner product can be introduced:

$$\langle \psi, \phi \rangle := \int_M \overline{\psi(x)}\phi(x)\,\mathrm{Vol}_M \ . \tag{1.72}$$

Because the factor $\sqrt{\det(g)}$ transforms in the inverse manner of the measure dx, the integrand is invariant under coordinate transforms (something that is generally required of physical laws). Using this new inner product, one can for example check the selfad-

jointness of the momentum operator $\widehat{P}_i := -i\nabla_i$:

$$\begin{aligned}
\langle \psi, \widehat{P}_i \phi \rangle &= \int_M \overline{\psi(x)}(-i\nabla_i)\phi(x)\sqrt{\det(g)}\,dx \\
&\overset{??}{=} \int_M \overline{\psi(x)}(-i\partial_i - i\omega_i)\phi(x)\sqrt{\det(g)}\,dx \\
&= \int_M \overline{(-i\partial_i\psi)(x)}\phi(x)\sqrt{\det(g)}\,dx + i\int_M \overline{\psi(x)}\phi(x)\Big(\partial_i\sqrt{\det(g)}\Big)\,dx \\
&\qquad - i\int_M \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx \\
&= \langle \widehat{P}_i\psi, \phi\rangle - i\int_M \overline{\psi(x)}\overline{\omega_i}\phi(x)\sqrt{\det(g)}\,dx \\
&\qquad + i\int_M \overline{\psi(x)}\phi(x)\Big(\partial_i\sqrt{\det(g)}\Big)\,dx \\
&\qquad - i\int_M \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx\,.
\end{aligned}$$

Selfadjointness then requires that

$$\sqrt{\det(g)}(\omega_i + \overline{\omega_i}) = \partial_i\sqrt{\det(g)} \tag{1.73}$$

or

$$2\mathrm{Re}(\omega_i) = \partial_i \ln\Big(\sqrt{\det(g)}\Big)\,. \tag{1.74}$$

@@ COMPLETE (rewrite in global terms) @@

1.8.2 WKB approximation

Property 1.7.1 is generalized quite trivially after replacing $\mathbb{R}^n$ by a configuration manifold Q. A further step is provided by also generalizing **??**.

Property 1.8.2. A Lagrangian submanifold $\iota : L \hookrightarrow T^*Q$ will be called an admissible phase function for a first-order solution to the Schrödinger equation if it satisfies the classical Hamilton–Jacobi equation, i.e. lies in a level set of the classical Hamiltonian $H : T^*Q \to \mathbb{R}$, for a regular value.

To obtain a second-order solution, one also needs prefactor for the semiclassical states. The homogeneous transport equation (1.69) is generalized as follows:

$$a\Delta S + 2\mathcal{L}_{\nabla S}a = 0\,, \tag{1.75}$$

where Δ is the Laplace–Beltrami operator on Q. As before, a general second-order solution, assuming S is admissible, is given by a half-form $a \in \Omega^{1/2}(L)$ satisfying

$$\mathcal{L}_Y a = 0\,, \tag{1.76}$$

where Y is the (nonsingular) vector field on L induced by X^H. This then gives a second-order solution on Q by pulling back along the inverse $(\pi \circ \iota)^{-1}$, which is a diffeomorphism since L is projectable. Moreover, if L is exact (**??**), then S is induced by a primitive of the induced Liouville form $\iota^*\alpha$. If both the exactness and projectability conditions are dropped, the notion of a **geometric solution** are obtained.

To pass to this more general situation, some more structure is needed. If L is not exact, the Liouville form does not admit a global primitive. However, L does admit a (good) cover $\{U_k\}_{k\in I}$ such that on every patch, a second-order solution can be found, and then the problem becomes how to glue these together. The gluing condition is the following integrality condition:

$$\phi_k(x) - \phi_l(x) \in 2\pi\hbar\mathbb{Z}\,, \tag{1.77}$$

where ϕ_k is the phase function on U_k, for all $x \in U_k \cap U_l$. Note that this condition can only be satisfied for all $\hbar \in \mathbb{R}^+$ if $[\alpha] = 0$. However, this is exactly the condition that should be relaxed. Luckily, $\hbar$ should be a fixed value.

Definition 1.8.3 (Quantizable Lagrangian). A projectable Lagrangian submanifold $L \subset T^*M$ is said to be quantizable if there exists an $\hbar \in \mathbb{R}^+$ such that the restriction of the Liouville class to L is $\hbar$-integral, i.e. the integrality condition (1.77) is satisfied. All values $\hbar$ for which the integrality condition is satisfied, are said to be **admissible**.

Remark 1.8.4. Note that the admissible values for $\hbar$ will form a decreasing sequence of the form

$$\hbar_0, \frac{\hbar_0}{2}, \dots\,, \tag{1.78}$$

where $\hbar_0$ is the greatest admissible value.

For the weakening of the projectability condition, see Bates and Weinstein (1997). However, even without weakening that condition, there is still a remaining issue to the quantization of classical solutions. This will involve Maslov indices (**??**) and Morse theory (**??**).

Chapter 2

Quantum Information Theory

The section on (quantum) reference frames is based on De La Hamette and Galley (2020).

Chapter Contents

2.1 Entanglement **32**
2.1.1 Introduction 32
2.1.2 Bell states 33
2.1.3 SRE states 34
2.2 Density operators **34**
2.3 Channels **35**
2.4 Quantum logic **36**
2.4.1 Birkhoff–von Neumann logic 36
2.4.2 Linear logic 37
2.5 Topos theory ♣ **42**
2.5.1 Bohr topos 42
2.5.2 Internal logic 44

2.1 Entanglement

2.1.1 Introduction

Construction 2.1.1 (Schmidt decomposition). Consider a bipartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. For any such state, there exist orthonormal sets $\{|e_i\rangle, |f_j\rangle\}_{i,j\leq\kappa}$ such that

$$|\psi\rangle = \sum_{i=1}^{\kappa} \lambda_i |e_i\rangle \otimes |f_i\rangle\,, \tag{2.1}$$

where the coefficients λ_i are nonnegative real numbers. All objects in this expression can be obtained from a singular value decomposition of the coefficient matrix C of $|\psi\rangle$ in some bases of $\mathcal{H}_1$ and $\mathcal{H}_2$. The number $\kappa \in \mathbb{N}$ is called the **Schmidt rank** of $|\psi\rangle$.

Definition 2.1.2 (Entangled states). Consider a state $|\psi\rangle$ and consider its Schmidt decomposition. If the Schmidt rank is 1, i.e. the state can be written as $|\psi\rangle = |v\rangle \otimes |w\rangle$, the state is said to be **separable**. Otherwise, the state is said to be entangled.

The following theorem follow from the linearity of quantum mechanics.

Theorem 2.1.3 (No-cloning). *There is no unitary operator $\widehat{U}$ on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that*

$$\widehat{U}|\psi\rangle_1|\phi\rangle_2 = e^{i\alpha(\psi,\phi)}|\psi\rangle_1|\psi\rangle_2 \tag{2.2}$$

for all (normalized) $|\psi\rangle_1 \in \mathcal{H}_1$ and $|\phi\rangle_1 \in \mathcal{H}_2$.

Theorem 2.1.4 (No-deleting). *Consider a tripartite system $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ such that $\mathcal{H}_1 \cong \mathcal{H}_2$. If $\widehat{U}$ is a unitary operator on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ such that*

$$\widehat{U}|\psi\rangle_1|\psi\rangle_2|\phi\rangle_3 = |\psi\rangle_1|0\rangle_2\left|\phi_\psi\right\rangle_3 \tag{2.3}$$

for all $|\psi\rangle_1 \in \mathcal{H}_1$, where the final ancilla state $\left|\phi_\psi\right\rangle_3$ might depend on the initial state $|\psi\rangle_1$, then $\widehat{U}$ is simply a swap, i.e. $|\psi\rangle_1 \mapsto \left|\phi_\psi\right\rangle_3$ is an isometric embedding.

2.1.2 Bell states

Definition 2.1.5 (Bell state). A (binary) Bell state (also called a **cat state** or **Einstein–Podolsky–Rosen pair**) is defined as the following entangled state:

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)\,. \tag{2.4}$$

In fact, this state can be extended to a full maximally entangled basis for the 2-qubit Hilbert space:

$$\begin{aligned} |\Phi^-\rangle &:= \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big)\,, \\ |\Psi^+\rangle &:= \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)\,, \\ |\Psi^-\rangle &:= \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)\,. \end{aligned} \tag{2.5}$$

Method 2.1.6 (Dense coding[1]). Consider the Bell state $|\Phi^+\rangle$. By acting with one of the (unitary) spin-flip operators $\widehat{X}, \widehat{Y}, \widehat{Z}$, one can obtain any of the other three Bell states:

$$\begin{aligned} \widehat{X}|\Phi^+\rangle &= |\Phi^-\rangle\,, \\ \widehat{Y}|\Phi^+\rangle &= |\Psi^+\rangle\,, \\ \widehat{Z}|\Phi^+\rangle &= |\Psi^-\rangle\,. \end{aligned} \tag{2.6}$$

[1] Sometimes called **superdense coding**.

In a typical Alice-and-Bob-style experiment, one can ask whether this observation allows to achieve a better-than-classical communication channel. If Alice performs a spin flip on her qubit, although the resulting state has instantly 'changed' (cf. *spooky action at a distance*), Bob still cannot uniquely determine what this state is (since the resulting state is still maximally entangled). However, if Alice sends her qubit to Bob, the latter can perform a measurement on the composite system to find out what the state is and in this way determine which operation Alice performed ($\mathbb{1}, \widehat{X}, \widehat{Y}, \widehat{Z}$). Alice has thus effectively sent 2 classical bits of information through 1 qubit. Note that due to the fact that Alice still has to send her qubit through classical means, no faster-than-light communication is achieved.

Definition 2.1.7 (GHZ state). The Greenberger–Horne–Zeilinger state is defined as the multiparticle qudit ($d, N > 2$) version of the Bell state and is, therefore, also referenced to as a cat state:

$$|\mathsf{GHZ}\rangle = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} |i\rangle^{\otimes N} . \tag{2.7}$$

2.1.3 SRE states

@@ ADD @@

2.2 Density operators

Definition 2.2.1 (Density operator). Consider a (finite-dimensional) Hilbert space $\mathcal{H}$. A density operator on $\mathcal{H}$ is a linear operator $\rho \in \mathsf{End}(\mathcal{H})$ satisfying the following properties:

1. **Positivity**: $\langle v \,|\, \rho v \rangle \geq 0$ for all $v \in \mathcal{H}$,
2. **Hermiticity**: $\rho^\dagger = \rho$, and
3. **Unit trace**: $\mathsf{tr}(\rho) = 1$.

More concisely, density operators are the representing objects of normal states (**??**) on $\mathcal{B}(\mathcal{H})$.

Example 2.2.2 (Classical probability). A diagonal density matrix corresponds to a discrete probability distribution.

Formula 2.2.3 (Expectation value). Extending Definition 1.1.5, the expectation of an observable $\widehat{O}$ with respect to a density operator $\hat{\rho}$ is given by

$$\langle \widehat{O} \rangle_{\hat{\rho}} = \mathsf{tr}\big(\hat{\rho}\widehat{O}\big) . \tag{2.8}$$

Definition 2.2.4 (Pure state). A state is said to be pure if it is described by an outer product of a state vector or, equivalently, by an idempotent density matrix:

$$\rho = |\psi\rangle\langle\psi| . \tag{2.9}$$

A density matrix that is not of this form gives rise to a **mixed state**.

Definition 2.2.5 (Reduced density operator). Let $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be the state of a bipartite system. The reduced density operator ρ_A of A is defined as follows:

$$\rho_A := \mathrm{tr}_B |\Psi\rangle\langle\Psi| . \tag{2.10}$$

Definition 2.2.6 (Purification). Let ρ_A be the density operator of a system A. A purification of ρ_A is a pure state $|\Psi\rangle$ of some composite system $A \otimes B$ such that

$$\rho_A = \mathrm{tr}_B |\Psi\rangle\langle\Psi| . \tag{2.11}$$

Property 2.2.7. Any two purifications of the same density operator ρ_A are related by a transformation $\mathbb{1}_A \otimes \widehat{V}$ with $\widehat{V}$ an isometry.

2.3 Channels

The following definition generalizes the content of **??** to the setting of partial information. When generalizing the projections in a PVM (spectral measure), one obtains a POVM.

Definition 2.3.1 (Positive operator-valued measure). First, let $\mathcal{H}$ be a finite-dimensional Hilbert space. A POVM on $\mathcal{H}$ consists of a finite set of positive (semi)definite operators $\{P_i\}_{i \leq n}$ such that

$$\sum_{i=1}^{n} P_i = \mathbb{1}_{\mathcal{H}} . \tag{2.12}$$

The probability to obtain state i, given a general state $\hat{\rho}$, is given by $\mathrm{tr}(\hat{\rho} P_i)$. Note that the operators are not necessarily orthogonal projectors, so n can be greater than $\dim(\mathcal{H})$.

Now, consider a measurable space (X, Σ) and a (possibly infinite-dimensional) Hilbert space $\mathcal{H}$. A POVM on X consists of a function $P : \Sigma \to \mathcal{B}(\mathcal{H})$ satisfying the following conditions:

1. P_E is positive and self-adjoint for all $E \in \Sigma$,
2. $P_X = \mathbb{1}_{\mathcal{H}}$, and
3. for all disjoint $(E_n)_{n \in \mathbb{N}} \subset \Sigma$:

$$\sum_{n \in \mathbb{N}} P_{E_n} = P_{\cup_{n \in \mathbb{N}} E_n} . \tag{2.13}$$

The following theorem can be derived from Stinespring's theorem **??**.

Theorem 2.3.2 (Naimark dilation theorem). *Every POVM P on $\mathcal{H}$ can be realized as a PVM Π on a, possibly larger, Hilbert space $\mathcal{K}$, i.e. there exists a bounded operator $V : \mathcal{K} \to \mathcal{H}$ such that*

$$P(\cdot) = V\Pi(\cdot)V^\dagger . \tag{2.14}$$

In the finite-dimensional setting, V can be chosen to be an isometry.

Recall the content of **??**.

Definition 2.3.3 (Completely positive trace-preserving). Consider a map $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ between bounded operators on two (finite-dimensional) Hilbert spaces. This map preserves density matrices if it positive (**??**) and if it is trace-preserving (**??**). Furthermore, to ensure that an operation applied to a subsystem does not interfere with the positivity of the complete system, they are also required to be completely positive (**??**).

Completely positive, trace-preserving (CPTP) maps are often called **quantum channels** or **superoperators**.

Theorem 2.3.4 (Choi–Jamiołkowski). *The following map between quantum channels $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ and density operators $\rho \in \mathsf{End}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is an isomorphism:*

$$\Phi \mapsto (\mathbb{1}_{\mathcal{H}_1} \otimes \Phi)|\mathsf{GHZ}\rangle\langle\mathsf{GHZ}| , \tag{2.15}$$

where the GHZ state was introduced in Definition 2.1.7.

2.4 Quantum logic

2.4.1 Birkhoff–von Neumann logic

Consider classical propositional logic. This is governed by the Boolean property, i.e. the set of all propositions admits the structure of a complete Boolean algebra (**??**). Now, the question becomes how to treat propositions in quantum mechanics (as needed in, for example, quantum computing). In the sense of *von Neumann*, the propositions should be characterized by $\{0, 1\}$-valued observables or, equivalently, by projection operators. As such, the natural lattice to consider logic is that of closed subspaces of the state space $\mathcal{H}$ (which is a Hilbert space). Birkhoff–von Neumann logic is the study of such lattices.

Note that, in contrast to classical logic, the lattices of closed subspaces are not Boolean. The lattices are merely complete, orthomodular lattices (**??**).[2] Now, although these

[2]It should be noted that complete, orthomodular lattices are, in general, very different from those originating from Hilbert spaces (cf. *Piron's theorem*).

lattices are themselves very interesting, there relevance for quantum logic are heavily discussed for several reasons (e.g. lack of distributivity, lack of a clear implication operator, lack of an extension to predicate logic). In the next section it will be explained how this issue can be avoided by embedding Birkhoff–von Neumann logic into the more general framework of linear logic.

2.4.2 Linear logic

A basic component of standard logic are the *structural inference rules*. These are inference rules that do not involve any logical operations. The following two inference rules for first-order logic control context extension (for an introduction to sequent calculus, see **??**):

- **Contraction**:

$$\frac{\Gamma, p_1 : P, p_2 : P \vdash t_{p_1,p_2} : T}{\Gamma, p : P \vdash t_{p,p} : T\,.} \tag{2.16}$$

  This rule states that, in a valid judgement, premises might be used more than once.

- **Weakening**:

$$\frac{\Gamma \vdash P : \mathsf{Type} \qquad \Gamma \vdash t : T}{\Gamma, P \vdash t : T\,.} \tag{2.17}$$

  This rule states that, any premise can be added to (the premises of) a valid judgement.

In terms of categorical semantics, these two rules correspond (in the independent setting) to the diagonal and projection morphisms in Cartesian categories (**??**).

Now, when considering quantum mechanics, two important results are the no-cloning and no-deleting theorem (Dummy 2.1.3 and Dummy 2.1.4). These correspond to the fact that the categories FinVect and Hilb are monoidal, but not Cartesian monoidal, i.e. the tensor product does not admit diagonal and projection morphisms. The natural type of logic in this setting is than a *substructural* one where the contraction and weakening rules are not valid.

In linear logic, the following propositions exist:

1. **Variables**: Every propositional variable is a proposition.
2. **Negation**: If P is a proposition, so is $P^{\perp}$.
3. **Connectives**: If P, Q are propositions, then

- **Additive conjunction**: $P \& Q$ is a proposition. (Read: P with Q.)
- **Additive disjunction**: $P \oplus Q$ is a proposition. (Read: P plus Q.)
- **Multiplicative conjunction**: $P \otimes Q$ is a proposition. (Read: P times Q.)
- **Multiplicative disjunction**: $P \parr Q$ is a proposition. (Read: P par Q.)

4. **Constants**:
   - **Additive truth**: $\top$,
   - **Additive falsity**: 0,
   - **Multiplicative truth**: 1, and
   - **Multiplicative falsity**: $\bot$.
5. **Exponential connectives**: If P is a proposition, then
   - **Exponential conjunction**: $!P$ is a proposition. (Read: of course P.)
   - **Exponential disjunction**: $?P$ is a proposition. (Read: why not P.)

Given a context, the following inference rules are valid:[3]

1. **Identity***: If P is a propositional variable, then $P \vdash P$.
2. **Exchange***: Sequents remain valid under permutations.
3. **Restricted weakening***: If P is a proposition, then

$$\frac{\Gamma \vdash \Theta}{\Gamma, !P \vdash \Theta} \tag{2.18}$$

   and, dually,

$$\frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, ?P}. \tag{2.19}$$

4. **Restricted contraction***: If P is a proposition, then

$$\frac{!P, !P \vdash \Theta}{!P \vdash \Theta} \tag{2.20}$$

   and, dually,

$$\frac{\Gamma \vdash ?P, ?P}{\Gamma \vdash ?P}. \tag{2.21}$$

[3]The form of these rules heavily depends on the exchange rule (the second item). Care must be taken if this rule is weakened.

5. **Negation**: If P is a proposition, then

$$\frac{\Gamma \vdash \Theta, P}{\Gamma, P^{\perp} \vdash \Theta} \tag{2.22}$$

and, conversely,

$$\frac{\Gamma, P \vdash \Theta}{\Gamma \vdash \Theta, P^{\perp}}. \tag{2.23}$$

Note that these rules allow to write any sequent in right form, i.e. $\vdash \Gamma^{\perp}, P$.

6. **Additive conjunction**: If P, Q are propositions, then

$$\frac{P \vdash \Theta}{P \& Q \vdash \Theta} \qquad \frac{Q \vdash \Theta}{P \& Q \vdash \Theta} \tag{2.24}$$

and, conversely,

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \& Q}. \tag{2.25}$$

7. **Additive disjunction**: If P, Q are propositions, then

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \oplus Q} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \oplus Q} \tag{2.26}$$

and, conversely,

$$\frac{P \vdash \Theta \qquad Q \vdash \Theta}{P \oplus Q \vdash \Theta}. \tag{2.27}$$

8. **Multiplicative conjunction**: If P, Q are propositions, then

$$\frac{P, Q \vdash \Theta}{P \otimes Q \vdash \Theta} \tag{2.28}$$

and, conversely,

$$\frac{\Gamma \vdash P \qquad \Lambda \vdash Q}{\Gamma, \Lambda \vdash P \otimes Q}. \tag{2.29}$$

9. **Multiplicative disjunction**: If P, Q are propositions, then

$$\frac{\Gamma \vdash P, Q}{\Gamma \vdash P \parr Q} \tag{2.30}$$

and, conversely,

$$\frac{P \vdash \Delta \qquad Q \vdash \Theta}{P \parr Q \vdash \Delta, \Theta}. \tag{2.31}$$

10. **Truth and falsity**:

$$\begin{array}{ll} \Gamma \vdash \top & 0 \vdash \Theta \\ \dfrac{\Gamma \vdash \Theta}{\Gamma, 1 \vdash \Theta} & \vdash 1 \\ \dfrac{\vdash \Theta}{\vdash \Theta, \bot} & \bot \vdash \ . \end{array} \tag{2.32}$$

11. **Exponential conjunction**: If P is a proposition, then

$$\frac{P \vdash \Theta}{!P \vdash \Theta} \tag{2.33}$$

and, conversely, whenever Γ consists solely of !-propositions and Θ consists solely of ?-propositions,

$$\frac{\Gamma \vdash P}{\Gamma \vdash !P}. \tag{2.34}$$

12. **Exponential disjunction**: If P is a proposition, then

$$\frac{\Gamma \vdash P}{\Gamma \vdash ?P} \tag{2.35}$$

and, conversely, whenever Γ consists solely of !-propositions and Θ consists solely of ?-propositions,

$$\frac{P \vdash \Theta}{?P \vdash \Theta}. \tag{2.36}$$

The inference rules with an asterisk are the structural rules. Note that a *cut-elimination theorem* holds and, hence, the identity and cut rules for general propositions can be derived from the rules above.

Linear implication is characterized as follows:

$$P \vdash Q \iff \vdash P^{\perp} \parr Q \iff \vdash P \multimap Q\,. \tag{2.37}$$

Remark 2.4.1 (Resource theory). Before passing to the properties that follow from the basic rules and the categorical semantics of linear logic (eventually passing to linear type theory), it is useful to rephrase the connectives and their inference rules in terms of 'resources'.

In this interpretation, an implication $A \implies B$ would mean that the resources A can be used to obtain the resources B. However, in ordinary logic, if A and $A \implies B$ hold,

one can derive that B holds, but A also still holds. This is something that does not work with resources. If you can use resources A to construct B, the resources A are (usually[4]) used up.[5] One, hence, needs a more subtle and nuanced framework to capture these notions: linear logic. The implication that will be used, where resources are spent, is denoted by $A \multimap B$ for clarity. The two conjunctives, $\otimes$ and &, mean that two resources are available concurrently and separately, respectively. So, if $A \multimap B$ holds, then, since the contraction rule is not valid, one does not have $A \multimap B \otimes B$. However, one does have $A \otimes A \multimap B \otimes B$.

The interpretation of the connectives is as follows:

- $A \otimes B$: A and B are both available for use at the same time, e.g. one has a warehouse with A and a warehouse with B at the same time.
- $A \& B$: Either A or B are available for use, but not both.
- $A \parr B$:
- $A \oplus B$:

The connectives in linear logic satisfy (or generalize) many of the properties of ordinary logic.

Property 2.4.2 (Distributivity).

$$\begin{aligned} P \otimes (Q \oplus R) &= (P \otimes Q) \oplus (P \otimes R) \\ P \parr (Q \& R) &= (P \parr Q) \& (P \parr R) \\ P \otimes 0 &= 0 \\ P \parr \top &= \top \end{aligned} \tag{2.38}$$

The exponential connectives can be used to turn additive connectives into multiplicative ones (and the other way around) as with the ordinary exponential function in calculus.

Property 2.4.3.

$$\begin{aligned} !(P \& Q) &= !P \otimes !Q \\ ?(P \oplus Q) &= ?P \parr ?Q \\ !\top &= 1 \\ ?0 &= \bot \end{aligned} \tag{2.39}$$

Moreover, due to the apparent similarity with the operators in (S4) modal logic (**??**), the exponential connectives are sometimes also called **modalities**.

[4] This is not the case with catalysts.

[5] One could also give this a causal flavour (Girard, 1995).

Linear negation can also be defined alternatively.

Property 2.4.4 (Negation). Linear negation admits the following recursive definition:

- $P^{\perp\perp} = P$,
- $(P \& Q)^{\perp} = P^{\perp} \oplus Q^{\perp}$,
- $(P \otimes Q)^{\perp} = P^{\perp} \parr Q^{\perp}$,
- $\top^{\perp} = 0$,
- $1^{\perp} = \bot$, and
- $(!P)^{\perp} = ?P^{\perp}$.

Property 2.4.5 (Categorical semantics). Whereas standard Boolean logic is the internal logic of Cartesian closed categories — where conjunction, disjunction and implication correspond, respectively to products, coproducts and internal homs — linear logic is the internal logic of (a subclass of) $*$-autonomous categories (**??**).

The multiplicative conjunction $\otimes$ corresponds to the tensor product, hence the notation. Similar to ordinary logic, the linear implication $\multimap$ corresponds to taking internal homs. The important part, now, is that negation comes as a separate entity, in this case given by taking duals: $x^{\perp} \equiv x^{*}$. The multiplicative disjunction $\parr$ is then constructed through Property 2.4.4 (which corresponds to de Morgan duality as in **??**).

For the additive connectives, one needs the existence of finite products. The $*$-autonomy then also implies the existence of finite coproducts (again through de Morgan duality).

For the exponential connectives, some more structure is needed. As with modal logic, the structure is given by the existence of a suitable (co)monad.

2.5 Topos theory ♣

2.5.1 Bohr topos

Definition 2.5.1 (Bohr topos). Consider a C^{*}-algebra A and denote by $\mathsf{ComSub}(A)$ the poset (**??**) of commutative C^{*}-subalgebras. This set can be equipped with the **Alexandrov topology**[6], i.e. the topology for which the open sets are the upward closed subsets. The topological space $(\mathsf{ComSub}(A), \tau_{\text{Alex}})$ is called the Bohr site of A.

The sheaf topos over the Bohr site is called the Bohr topos $\mathsf{Bohr}(A)$. It can be turned into a ringed topos, where the ring object (which is even an internal commutative C^{*}-

[6] There exist an equivalences $\mathsf{Pre} \cong \mathsf{AlexTop}$ and $\mathsf{Pos} \cong \mathsf{AlexTop}_{T_0}$.

algebra) is given by the tautological functor

$$\underline{A} : \mathsf{ComSub}(A) \to \mathsf{Set} : C \mapsto C\,. \tag{2.40}$$

Property 2.5.2. A morphism in $\mathsf{C^*Alg}$ is commutativity reflecting if and only if the induced morphism on posets admits a right adjoint. Moreover, there exists a bijection between the following two classes of morphisms:

- Geometric morphisms $f : \mathsf{Bohr}(B) \to \mathsf{Bohr}(A)$ admitting a right adjoint together with epimorphisms of internal algebras $\underline{A} \to f^*\underline{B}$.
- Commutativity-reflecting functions $f : A \to B$ that restrict to algebra morphisms on all commutative subalgebras.

Definition 2.5.3 (Spectral presheaf). The presheaf Σ on a Bohr site assigning to every commutative subalgebra its Gel'fand spectrum.

The idea behind the Bohr topos is that, given a general C^*-algebra A, the Bohr topos $\mathsf{Bohr}(A)$ is interpreted as its quantum phase space. This is similar to **??**, where smooth spaces are also reinterpreted in terms of sheaf topoi.

Theorem 2.5.4 (Kochen–Specker). *For $A = \mathcal{B}(\mathcal{H})$, the spectral presheaf has no global elements if $\dim(\mathcal{H}) > 2$.*

Property 2.5.5 (Gleason's theorem). There exists a natural bijection between the quantum states of a C^*-algebra A and the classical states of $\underline{A}$ internal to $\mathsf{Bohr}(A)$.

Definition 2.5.6 (Bohrification). Consider a C^*-algebra A together with its Bohr topos $\mathsf{Bohr}(A)$. To its internal C^*-algebra $\underline{A}$, one can assign an internal locale $\underline{\Sigma}_A$ by (internal) Gel'fand duality (**??**). Under the equivalence **??**, one then obtains a locale Σ_A. The functor

$$\Sigma : \mathsf{C^*Alg} \to \mathsf{Loc} : A \mapsto \Sigma_A \tag{2.41}$$

is called Bohrification. This locale can be constructed as the disjoint union

$$\Sigma_A = \bigsqcup_{C \in \mathsf{ComSub}(A)} \Phi_C\,, \tag{2.42}$$

the étale locale corresponding to the spectral presheaf (i.e. the spectral presheaf is the internal Gel'fand spectrum of $\underline{A}$). Its open sets are given by those subsets whose restrictions to commutative subalgebras are open in such a way that these restrictions are compatible with subalgebra inclusions.

Example 2.5.7 (Gel'fand spectrum). If A is a commutative C^*-algebra, its Bohrification is not isomorphic to its ordinary Gel'fand spectrum Φ_A. However, after replacing the topology on $\mathsf{Bohr}(A)$ by the double negation topology (**??**) and repeating the above construction, one obtains

$$\Phi_A \cong \Sigma_A^{\neg\neg}\,. \tag{2.43}$$

This locale can also be obtained in another way. Double negation $\neg\neg$ defines an (internal) **nucleus** on the (internal) locale Σ_A, i.e. a left-exact monad. $\Sigma_A^{\neg\neg}$ is then given by the fixed points of $\neg\neg : \Sigma_A \to \Sigma_A$.

By Property 2.5.2 above, the following relation is obtained.
Property 2.5.8 (Observables). Morphisms $\mathsf{Bohr}(A) \to \mathsf{Bohr}(C(\mathbb{R})_0)$ admitting a right adjoint together with an epimorphism $\underline{C_0(\mathbb{R})} \to f^*\underline{A}$ correspond to observables on A.

The topological bundle $\Sigma_A \to \mathsf{Alex}(\mathsf{ComSub}(A))$ also admits a topos-theoretic incarnation. There exists a (canonical) morphism of ringed topoi

$$\pi : \mathsf{Bohr}(A) \to \left(\mathsf{Sh}\left(\mathsf{Alex}(\mathsf{ComSub}(A))\right), \underline{\mathbb{R}}\right), \tag{2.44}$$

whose underlying geometric morphism is simply the identity.
Property 2.5.9 (States). A positive and normalized section of the morphism $\pi : \mathsf{Bohr}(A) \to \left(\mathsf{Sh}\left(\mathsf{Alex}(\mathsf{ComSub}(A))\right), \underline{\mathbb{R}}\right)$ in the category of $\underline{\mathbb{R}}$-*module topoi*.

2.5.2 Internal logic

Whereas Section 2.4 covers quantum logic from the external point of view, the spectral presheaf (Definition 2.5.3) allows to treat it internally. By functional calculus (**??**), every proposition $A \in \Delta$ about a self-adjoint operator on a Hilbert space $\mathcal{H}$, i.e. every measurable subset of the spectrum $\sigma(A)$, corresponds to a projection $P_{A,\Delta}$. Now, the goal will be to assign to every such projection a 'classical interpretation' in the classical **context** given by a commutative subalgebra of A. This is achieved by the subobject of the spectral presheaf Σ constructed as follows:

$$\delta_P(V) := \{\omega \in \Sigma_v \mid \omega(P|_V) = 1\},$$

where $P|_V$ is the smallest projection in V such that $\mathsf{im}(P) \subseteq \mathsf{im}(P|_V)$. The morphism

$$\delta : \mathcal{P}(\mathcal{H}) \to \mathsf{Sub}(\Sigma)$$

is called the **daseinization** map. Note that this map extends to all self-adjoint operators through functional calculus.

A 'pure state' ψ in classical mechanics corresponds to a map from subsets of phase space to the subobject classifier (saying whether the point lies in the given subset or not):

$$T^\psi : \mathsf{Sub}(P) \to \{0, 1\}.$$

By analogy, a pure state $|\psi\rangle$ in quantum mechanics will correspond to a morphism from subobjects of the spectral presheaf:

$$T^{|\psi\rangle} : \mathsf{Sub}(\Sigma) \to \mathsf{Hom}_{\mathsf{Psh}(\mathsf{ComSub}(A))}(1, \Omega),$$

where Ω is the subobject classifier from **??**.

Chapter 3

Quantization ♣

The content of **?? ??**, **??** and **??** are prerequisites for the section on geometric quantization. References for the section on geometric quantization are Bates and Weinstein (1997); Brylinski (1993); Camosso (2021); Tuynman (2016). The section on Toeplitz quantization is mainly based on Hawkins (2000). A general reference comparing deformation and geometric quantization is Hawkins (1998).

Chapter Contents

3.1 Introduction **45**
3.2 Deformation quantization **46**
3.3 Geometric Quantization **47**
3.3.1 Prequantization 47
3.3.2 Coadjoint orbits 52
3.4 Toeplitz quantization **52**
3.5 Constrained systems **54**
3.5.1 Dirac procedure 54
3.5.2 BRST quantization 54
3.5.3 BFV quantization 55

3.1 Introduction

Given the content of Section 1.4, the general quantization procedure for a dynamical system (M, ω, H) can be axiomatized as follows.

Method 3.1.1 (Abstract quantization). A quantization of a symplectic manifold (M, ω) is a pair $(\mathcal{H}, \mathcal{O})$ where:

- $\mathcal{H}$ is a (complex) separable Hilbert space,
- $\mathcal{O}$ takes real functions $C^\infty(M)$ to self-adjoint operators,
- $\mathcal{O}$ is $\mathbb{C}$-linear,
- $\mathcal{O}(1) = \mathbb{1}_{\mathcal{H}}$, and
- **Dirac correspondence**: $[\mathcal{O}(f), \mathcal{O}(g)] = i\hbar\mathcal{O}(\{f, g\}) + O(\hbar^2)$.

Remark 3.1.2 (C^*-bundles). Hawkins (2000) defines a general quantization as a *continuous bundle* $\pi : A \to [0, +\infty[$ *of* C^**-algebras* such that $\mathrm{ev}_0(\Gamma(A)) = C^\infty(M)$. See Section 3.4 for more information.

Because a Hilbert space forms an irreducible representation of any complete set of observables, it makes sense to add an additional irreducibility axiom.

Axiom 3.1 (Irreducibility postulate). Let (M, ω) be a $2n$-dimensional symplectic manifold. If the observables $\{f_i\}_{i\leq n}$ form a complete set, i.e. any function that Poisson commutes with all f_i is necessarily constant, the quantum state space $\mathcal{H}$ is required to be irreducible with respect to the action of $\{\mathcal{O}(f_i)\}_{i\leq n}$. Equivalently, if a group G acts transitively on M, the state space is required to be an irreducible representation of a U(1)-central extension of G.

The simplest example is again $T^*\mathbb{R}^n$. Here, the usual choice of observables are the coordinate and momentum functions $\{q^i, p_i\}_{i\leq n}$ and the group G is given by the *Heisenberg group*, a U(1)-central extension of the translation group $\mathbb{R}^{2n}$. Irreducible representations are characterized by the Stone–von Neumann theorem 1.4.2.

Although the above procedure sounds reasonable, it is known to be inconsistent. *Groenewold* and *Van Hove* showed that there does not exist a map $\mathcal{O}$, satisyfing the axioms 3.1.1, that takes the entire (Lie) algebra of classical observables to the (Lie) algebra of corresponding quantum observables. As such, a different approach should be followed.

3.2 Deformation quantization

Definition 3.2.1 (Formal deformation quantization). Let A be a Poisson algebra. A **star product** on A is an associative, $\mathbb{R}$-bilinear product on $A[[\hbar]]$ such that for all $a, b \in A$:

$$a * b = \sum_{i=0}^{+\infty} B_i(a, b)\hbar^i \tag{3.1}$$

with $B_0(a, b) = ab$. A formal deformation quantization of A is a star product such that

$$\{a, b\} = B_1(a, b) - B_1(b, a)\,. \tag{3.2}$$

Definition 3.2.2 (Moyal deformation quantization). Let V be a Poisson vector space[1], i.e. a vector space equipped with a Poisson bivector $\pi \in V \wedge V$. The ring $C^\infty(V)[[\hbar]]$ can be equipped with the following product:

$$f * g := \mu \circ \exp(\hbar\pi)(f \otimes g)\,, \tag{3.3}$$

where π is viewed as an endomorphism on $C^\infty(V) \otimes C^\infty(V)$ and μ is the ordinary product map.

Definition 3.2.3 (Fedosov deformation quantization). Consider a symplectic manifold (M, ω). Every tangent space T_pM carries the structure of a Poisson vector space and, hence, admits a Moyal quantization $(T_pM, *_\omega)$. These spaces can be turned into an algebra bundle $\mathcal{A}$ over M. It can, moreover, be shown that any such bundle admits at least one flat connection that respects the algebra structure and such that there exists an isomorphism as follows:

$$\ker(\nabla^{\text{Fed}}) \cong C^\infty(M)[[\hbar]]\,. \tag{3.4}$$

This connection is called a **Fedosov connection**. The restriction of the star product on $\Gamma(\mathcal{A})$ to this subbundle gives a formal deformation quantization of $C^\infty(M)$.

Remark 3.2.4 (Kontsevich quantization). *Kontsevich* had previously proven that every finite-dimensional Poisson manifold admits a deformation quantization of its algebra of smooth functions. However, a slight variation of Fedosov's approach can be used even for infinite-dimensional algebras such as those occuring in field theory (where Kontsevich's approach is invalid).

3.3 Geometric Quantization

In this section, the natural constant $\hbar$ has been set to 1.

3.3.1 Prequantization

Definition 3.3.1 (Prequantum line bundle). Consider a symplectic manifold (M, ω). A prequantum line bundle on M is a Hermitian line bundle equipped with a connection ∇ such that $\omega = F_\nabla$, where F_∇ denotes the curvature of ∇.

Property 3.3.2. Complex line bundles are classified by the (first) Chern class $c_1 \in H^2(M; \mathbb{Z})$, which is proportional to the curvature form through Chern–Weil theory. Therefore, a prequantum line bundle exists if and only if the symplectic form is integral (up to a factor of 2π). For simply connected manifolds, this is equivalent to

$$\forall S \in H_2(M) : \int_S \omega \in 2\pi\mathbb{Z}\,. \tag{3.5}$$

This condition resembles the 'old' *Bohr–Sommerfeld condition*[2] and is, in general, known

[1] Also called a **linear Poisson manifold**.

[2] In fact, it is closer to Einstein's quantization rule (Stone, 2005).

as the **Weil integrality condition**.

There are two contributions to the moduli space of prequantum line bundles or, essentially, U(1)-principal bundles with connection. The latter are known to be classified by differential U(1)-cohomology as shown in **??**. Since the curvature is fixed by the sympletic form, prequantum line bundles are classified by lifts from curvature forms to differential cohomology along the curvature projection. Differential cohomology consists of two parts, roughly corresponding to the following two aspects: there can exist topologically inequivalent bundles and there can exist inequivalent connections on the same bundle differing by a flat connection, where two flat connections are in turn inequivalent if they differ by a closed one-form that is neither integral nor exact. The former are classified by integral curvature forms $H^2_{\mathrm{dR}}(M)$, while the latter are classified by the Čech cohomology group $H^1(M;\mathsf{U}(1))$ or, equivalently by isomorphism (**??**), $H^2(M;\mathbb{Z})$.

Corollary 3.3.3 (Dirac quantization condition). One can derive the Dirac quantization condition from Weil integrality. If one couples the system to a gauge potential, the minimal coupling procedure gives $\omega \longrightarrow \omega + eF$. Weil integrality then implies that e is an integer.

Definition 3.3.4 (Prequantum Hilbert space). Consider a symplectic manifold (M,ω) together with a prequantum line bundle L. The prequantum Hilbert space $\mathcal{H}_L$ is defined as (the L^2-completion of) the space of square-integrable sections of L with respect to the metric on L and the Liouville volume form on M.

To every smooth function $f \in C^\infty(M,\mathbb{C})$, one can associate a (**Segal–Kostant–Souriau**) prequantum operator $\hat{f} : \Gamma(L) \to \Gamma(L)$ by the following formula:

$$\hat{f} : \psi \mapsto f \cdot \psi - i\nabla_{X_f}\psi\,, \tag{3.6}$$

where X_f is the Hamiltonian vector field associated to f and, locally,

$$\nabla = \mathsf{d} - i\theta \tag{3.7}$$

with θ the symplectic potential. This operator can also be interpreted in terms of a Hamiltonian flow. The Hamiltonian flow of X_f can be lifted (up to a phase) to an automorphism $\psi_t[f]$ on L that preserves both the metric and the connection. The prequantum operator $\hat{f}$ is then simply given by

$$\hat{f}s = -i\frac{\mathsf{d}}{\mathsf{d}t}(\psi_t[f]s)\bigg|_{t=0}\,. \tag{3.8}$$

Example 3.3.5 (Spinning particle). Consider as phase space the 2-sphere $S^2(r)$ with radius $r \in \mathbb{R}$. In this case, the symplectic form can be written as $\omega = r^2 \sin\theta \mathsf{d}\theta \wedge \mathsf{d}\varphi$. This form is only integral for a discrete set of values of r, namely for $r \in \mathbb{Z}/2$. Up to a factor $\hbar$, this is exactly the quantization rule for the angular momentum. The reason is

that S^2 is a homogeneous space for SU(2), the group characterizing spinning particles. In fact, using the theory of coadjoint orbits (see further below), one can show that this quantization procedure coincides with the *KKS quantization* of the coadjoint orbit $S^2 = \mathsf{SU}(2)/\mathsf{U}(1)$.

At this point, it can easily be seen that there is a problem with the dimension of the prequantum state space. For the cotangent bundle $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, the resulting state space would be $L^2(\mathbb{R}^{2n}, \mathbb{C})$. However, from ordinary quantum theory it is well-known that the right Hilbert space is $L^2(\mathbb{R}^n, \mathbb{C})$. In general, the above procedure would give wave functions that depend on $2n$ variables instead of the n coordinates of configuration space that are normally found in quantum mechanics. A solution is obtained by making a choice of 'configuration space' or, in terms of ordinary quantum mechanics, to choose a 'representation' of the system.

Construction 3.3.6 (Geometric quantization). Let (M, ω) be a symplectic manifold. A (geometric) quantization of (M, ω) is given by a prequantum line bundle L together with a polarization $\mathcal{P}$ of M. The 'naive' modification of the quantum state space is given by the subspace of $\mathcal{H}_L$ of those sections that are covariantly constant along $\mathcal{P}$, i.e. those sections $s \in \Gamma(L)$ that satisfy $\nabla_X s = 0$ for all $X \in \mathcal{P}$. These sections are also called **polarized sections**.

The fact that a polarization is required, and not merely an n-dimensional involutive distribution, follows from an additional consistency condition imposed by the condition $\nabla_X s = 0$ for all $X \in \mathcal{P}$. Because ω also represents the curvature of the connection on L, one obtains

$$\omega(X, Y)s = [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s = 0 \tag{3.9}$$

for all $X, Y \in \mathcal{P}$. This implies that $\mathcal{P}$ defines an isotropic submanifold. For a completely integrable system, a natural choice would be given by the distribution spanned by the Hamiltonian vector fields.

Now, if the prequantum operators ought to represent genuine operators on the quantum state space, one should have

$$\nabla_X s = 0 \implies \nabla_X(\hat{f}s) = 0 \tag{3.10}$$

for all sections $s \in \Gamma(L)$ and $X \in \mathcal{P}$. Using the general formula for iterated covariant derivatives and the fact that the leaves of P are Lagrangian, one finds the following condition:

$$[X, X_f] = 0 \tag{3.11}$$

for all $X \in \mathcal{P}$. So, in general, one should restrict to the subspace of $C^\infty(M)$ on those functions whose Hamiltonian flow preserves $\mathcal{P}$.

There is, however, a problem with this construction. Nothing ensures that $\mathcal{H}_L$ contains any polarized sections. As an example, consider a cotangent bundle with its vertical polarization. In this case, the polarized sections are given by functions that only depend on the base coordinates q^i and not on the fibre (momentum) coordinates p_i. However, because the fibres are noncompact, the integral of such a section with respect to the Liouville measure will always diverge. This particular issue can be resolved by integrating over the leaf space $M \cong T^*M/D$, where D is the isotropic foliation associated to $\mathcal{P}$.

However, even though a possible divergence coming from noncompact fibres is resolved, another problem arises. To be able to integrate over a manifold one needs a volume form, but there is not always a canonical choice available. A solution is given by working with densities.

Method 3.3.7 (Half-form quantization). For this method, the polarization is assumed to be real and have simply connected leaves. Furthermore, the manifold M is assumed to admit a metaplectic structure or, by virtue of **??**, $\mathcal{P}$ is assumed to admit a metalinear structure. (A choice of such a metaplectic structure is called a **metaplectic correction**.) Using this structure, one can define the half-form bundle $\delta^{1/2}$. Given a prequantum line bundle L, one defines the twisted bundle of L-valued half-forms $L \otimes \delta^{1/2}$. A **wave function** is defined as a section $\psi \in \Gamma(L \otimes \delta^{1/2})$ such that, locally, $\psi = \lambda \otimes \mu$ with $\nabla_X \lambda = 0$ and $\mathcal{L}_X \mu = 0$ for all $X \in \overline{\mathcal{P}}$. By pairing two wave functions, one obtains a 1-density on M that can be integrated. The quantum state space is then defined as the L^2-completion of the space of wave functions.

To extend the definition of operators to half-form quantized manifolds, one simply needs to extend the definition to density bundles. Because $\hat{f}$ represents the Hamiltonian flow, a natural choice is the Lie derivative:

$$\hat{f}(s \otimes \mu) := (\hat{f}s) \otimes \mu - is \otimes \mathcal{L}_{X_f} \mu \,. \tag{3.12}$$

Example 3.3.8 (Kähler quantization). For this method, the polarization is assumed to be positive Kähler, i.e. P is the antiholomorphic tangent bundle of a Kähler manifold. By taking the (local) symplectic potential to be the holomorphic derivative of the Kähler potential, one obtains the space of holomorphic sections as the prequantum state space (a different choice of potential results in a phase transformation). It can be shown that a natural inner product is given by

$$\langle \psi_1 \mid \psi_2 \rangle = \int_{\mathbb{C}^n} \overline{\psi_1}(z) \psi_2(z) \exp(-|z|^2/2)\, dz^n \,. \tag{3.13}$$

This is often called the **Bargmann**, **Segal–Bargmann** or **Bargmann–Fock** representation. The coordinates $z, \bar{z}$ are represented by the operators z and ∂_z. These correspond to creation and annihilation operators.

Remark 3.3.9. If the positivity assumption would be dropped, the Kähler potential K, which was $-|z|^2/2$ above, would be indefinite. This would, in turn, imply that the integral diverges.

Remark 3.3.10. Recall **??**. If M is compact Kähler, the existence of a metaplectic correction is equivalent to the existence of a spin structure. This leads to the L-twisted Dolbeault–Dirac operator:

$$\overline{\partial}_\nabla + \overline{\partial}_\nabla^\dagger : \Omega^{0,\bullet}(M) \otimes L \otimes \delta^{1/2} \to \Omega^{0,\bullet}(M) \otimes L \otimes \delta^{1/2} \,. \tag{3.14}$$

From the perspective of the Dolbeault complex, the differential is twisted by a metaplectically corrected prequantum line bundle, but from the perspective of the Dirac operator, one only twists by the line bundle.

Method 3.3.11 (Bohr–Sommerfeld quantization). Here, the polarization $\mathcal{P}$ is again assumed to be real, but the leaves are not required to be simply connected. The connection ∇ along $\mathcal{P}$ is flat when restricted to a single leaf $\Lambda_m \subset M$. When Λ_m is not simply connected, the holonomy group can be nontrivial. However, the defining condition of $\mathcal{H}$ is that sections should be covariantly constant. This implies that either the section is zero or that the holonomy around any loop vanishes. The support of all sections in $\mathcal{H}$ is, therefore, given by the union S of all leaves on which ∇ is trivial. This space is called the **Bohr–Sommerfeld variety**.

Vanishing holonomy implies

$$\exp\left(i \oint_\gamma \theta \right) = 1 \tag{3.15}$$

for all loops γ, where θ is the symplectic potential. In terms of Darboux coordinates, this gives (up to a factor 2π)

$$\oint_\gamma p_i \, \mathrm{d}q^i \in \mathbb{Z} \,. \tag{3.16}$$

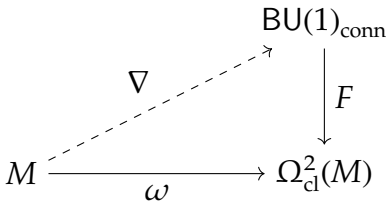
This is exactly the old Bohr–Sommerfeld quantization condition. When using half-density quantization, an additional contribution coming from the covariant derivative on densities would have to be added to the right-hand side:

$$\oint_\gamma p_i \, \mathrm{d}q^i = 2\pi(k_\gamma + d_\gamma) \,, \tag{3.17}$$

where $k_\gamma \in \mathbb{Z}$.

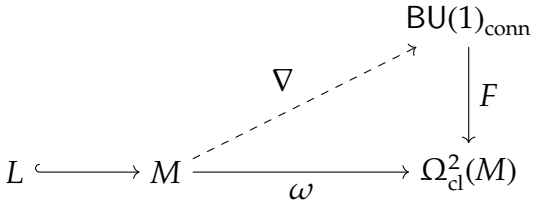
Definition 3.3.12 (Bohr–Sommerfeld leaf). More generally, a Lagrangian submanifold L of a prequantized symplectic manifold (M, ω) is said to be a **Bohr–Sommerfeld leaf** if $[\nabla_L]$ vanishes in $H^2_{\mathrm{conn}}(L)$, the differential cohomology of L (**??**). On simply connected manifolds, this is equivalent to having trivial holonomy.

This can be restated in terms of lifts to differential cohomology. The prequantization of a symplectic manifold is a lift as shown in the following diagram:



(3.18)

where $\mathsf{BU}(1)_{\text{conn}}$ is the Deligne complex (**??**). A submanifold $L \subset M$ is isotropic (Lagrangian if it is maximal) if the lift



(3.19)

is trivial in $H^2_{\text{conn}}(L)$.

3.3.2 Coadjoint orbits

@@ COMPLETE (SEE ALSO **??**) @@

3.4 Toeplitz quantization

Definition 3.4.1 (C^*-algebra bundle). A (continuous) bundle of C^*-algebras over a locally compact Hausdorff space X is a bundle $\pi : A \to X$ such that:

1. $\pi^{-1}(x)$ is a C^*-algebra for all $x \in X$.
2. $\Gamma(A)$ is a C^*-algebra.
3. $\mathsf{ev}_x(\Gamma(A))$ is dense in $\pi^{-1}(x)$ for all $x \in X$.
4. ev_x is a $*$-epimorphism for all $x \in X$.
5. For all $\nu \in \Gamma(A)$,
$$\|\nu\| = \sup_{x \in X} \|\nu(x)\| . \tag{3.20}$$
6. For all $f \in C_0(X)$ and $\nu \in \Gamma(A)$, there exists a $\nu_f \in \Gamma(A)$ such that $\nu_f(x) = f(x)\nu(x)$ for all $x \in X$.
7. **Continuity**: For all $\nu \in \Gamma(A)$, the map $x \mapsto \|\nu(x)\|$ is in $C_0(X)$.

@@ CHECK DEFINITION (difference between field and bundle of algebras) @@

In the remainder of this section, M will be assumed to be a compact, connected Kähler manifold. Moreover, consider a prequantization L and a Hermitian line bundle L_0. For all $n \in \mathbb{N}$, consider the line bundle $L \otimes L_0^n$, assumed to be positive (**??**) with its Hilbert space of holomorphic sections $\mathcal{H}_n$ (which is finite dimensional by **??**).

Definition 3.4.2 (Toeplitz quantization). The map $\mathfrak{T}_n : C^\infty(M) \to \mathsf{End}(\mathcal{H}_n)$ given by[3]

$$\mathfrak{T}_n(f) := \pi_n \circ f \,, \tag{3.21}$$

where $\pi_n : \Gamma(L \otimes L_0) \to \mathcal{H}_n$ is the canonical projection onto holomorphic sections. It can be shown that $\mathfrak{T}_n$ is unital and completely positive.

To generalize this procedure to vector bundles, note that the Kodaira vanishing theorem (**??**) implies that the kernel of the $L \otimes L_0^n$-twisted Dolbeault operator $\overline{\partial}$ is concentrated in degree 0 and, hence, that

$$\ker(\overline{\partial}) = \mathcal{H}_n \,. \tag{3.22}$$

The appropriate generalization to vector bundles will be the $E^* \otimes L \otimes L_0^n$-twisted Dolbeault–Dirac operator D_E from **??**:

$$\mathfrak{T}_n : \Gamma(E) \to \mathsf{Hom}\big(\ker(D_E), \mathcal{H}_n\big) : \sigma \mapsto \pi_n \circ \sigma \,, \tag{3.23}$$

where multiplication by σ is defined through the natural pairing of E and E^*.

Now, the Toeplitz maps $\mathfrak{T}_n$ piece together to give an operator $\mathfrak{T} : C^\infty(M) \to \prod_{n\in\mathbb{N}} \mathsf{End}(\mathcal{H}_n)$. This operator induces an isomorphism

$$\mathcal{T} : C^\infty(M) \cong \mathbb{A}/\mathbb{A}_0 \,, \tag{3.24}$$

where $\mathbb{A}_0 := \bigoplus_{n\in\mathbb{N}} \mathsf{End}(\mathcal{H}_n)$ and $\mathbb{A} := \mathbb{A}_0 \oplus \mathsf{im}(\mathfrak{T})$.

A general quantization, in the sense of Remark 3.1.2, is now given by the fact that there exists a bundle $A \to \widehat{\mathbb{N}}$, with $\widehat{\mathbb{N}}$ the one-point compactification$\{0, \dots, +\infty\}$, such that $\Gamma(A) = \mathbb{A}$, $A_n = \mathsf{End}(\mathcal{H}_n)$ for all $n \in \mathbb{N}$, and $A_{+\infty} = \mathcal{T}^{-1} \circ \pi_{\mathbb{A}_0}(\mathbb{A})$.

Remark 3.4.3. An equivalent quantization can be obtained from the operators

$$\mathfrak{Q}_n(f) := \mathfrak{T}_n\left(f + \frac{\Delta f}{2n}\right). \tag{3.25}$$

@@ COMPLETE (Hawkins (2000)) @@

[3] This particular form, first multiplying by a smooth function and then projecting onto holomorphic sections, is very similar to the definition of *Toeplitz operators*, where the projection is onto the *Hardy space*, hence the name.

3.5 Constrained systems

In this section, classical systems with constraints, i.e. dynamical systems (M, ω, H) with an algebra of first-class constraints $\{\phi_a\}_{a\in I}$, are considered as in **??**.

3.5.1 Dirac procedure

The first approach to the quantization of constrained systems is due to *Dirac*. Instead of trying to pass to the reduced phase space or introducing additional gauge fixing conditions, *Dirac* simply worked with all variables and represented these as operators acting on an enlarged Hilbert space. The constraints, represented by operators $\widehat{G}_a$, satisfy

$$\widehat{G}_a|\psi\rangle = 0 \tag{3.26}$$

for all physical states $|\psi\rangle$. The constraint algebra $\{\phi_a, \phi_b\} = C^c_{ab}\phi_c$ gives rise to a quantum algebra

$$[\widehat{G}_a, \widehat{G}_b] = i\hbar\, C^c_{ab}\widehat{G}_c + \hbar^2\widehat{D}_{ab}\,, \tag{3.27}$$

where $\widehat{D}_{ab}$ represents a **quantum anomaly**, i.e. a correction term resulting from the quantization of the classical algebra (e.g. operator ordering). The issue here is that both the left-hand side and the first term on the right-hand side vanish exactly on physical states, so $\widehat{D}_{ab}$ should also vanish on these states for all $a, b \in I$. However, if this would be true, the physical Hilbert space would be heavily restricted (in certain cases, it even becomes trivial). The conclusion is that the quantum anomaly breaks the first-class structure of the constraints and, therefore, the constraints do not generate gauge transformations anymore (this is why the anomy is sometimes called a **gauge anomaly**). A similar issue appears when quantizing the classical evolution equation

$$\{H, \phi_a\} = V^b_a\phi_b\,. \tag{3.28}$$

@@ COMPLETE @@

3.5.2 BRST quantization

For this approach, one starts from the classical BRST construction from **??** and tries to find a Hilbert space representation of the extended algebra containing the classical functions, the ghosts and the ghost momenta. The BRST charge becomes a self-adjoint operator satisfying

$$[\widehat{\Omega}, \widehat{\Omega}] = 2\widehat{\Omega}^2 = 0\,. \tag{3.29}$$

However, in stark contrast to the classical situation, where a BRST charge always exists, the quantum case does not necessarily admit such a construction.

Property 3.5.1 (Ghost states). If the state space can be decomposed according to ghost number, the following statements hold:

- The ghost number of a homogeneous state is of the form

$$g = g_0 + z \in \mathbb{Z} \tag{3.30}$$

  for g_0 either 0 or $\frac{1}{2}$. Fractional ghost numbers occur when the number of constraints is odd.

- The inner product of two homogeneous states with ghost numbers g, g' vanishes if $g + g' \neq 0$. This implies that states with nonzero ghost number are null states.

Definition 3.5.2 (Physical state space). The physical states are defined similarly to the gauge-invariant functions in the classical setting, i.e. states are deemed physical if they are BRST-closed:

$$\widehat{\Omega}|\psi\rangle = 0\,. \tag{3.31}$$

This operation is linear and, hence, filters out a linear subspace as required. Furthermore, BRST-closed operators (the **physical observables**) preserve this subspace and the BRST-exact operators give vanishing transition elements (and are, therefore, not physically observable as desired). This also implies that acting with a BRST-exact operator on a state, leaves the physical state unchanged. It follows that the true physical state space is given by $H^0(\widehat{\Omega})$.

3.5.3 BFV quantization

The extended phase space M_{ext} is defined by introducing dynamical Lagrange multipliers and their momenta

$$\{\lambda^m, \pi_n\} := \delta^m_n \tag{3.32}$$

together with two collections of (homologically) odd-degree ghosts and their momenta:

$$\begin{aligned} \{C^i, \overline{P}_j\} &:= \delta^i_j\,, \\ \{P^i, \overline{C}_j\} &:= \delta^i_j\,. \end{aligned} \tag{3.33}$$

Note that the Poisson bracket for the ghosts is a graded Poisson bracket. The subalgebra on the variables $(q, p, C, \overline{P})$ is called the **minimal subalgebra**, while the subalgebra on the variables $(\lambda, \pi, P, \overline{C})$ is called the **auxiliary subalgebra**.

A quantized algebra of observables is obtained through the (graded) Dirac correspondence. This algebra is naturally graded with respect to the ghost number defined by the self-adjoint operator

$$\mathcal{G} := \frac{1}{2}\left(C^i\overline{P}_i - \overline{P}^i C_i + P^i\overline{C}_i - \overline{C}_i P^i \right). \tag{3.34}$$

This operator acts on observables as follows:

$$[\mathcal{G}, A] = i\hbar\, \mathsf{gh}(A) A\,. \tag{3.35}$$

BFV quantization is obtained by defining a nilpotent BRST operator $\widehat{\Omega} = \widehat{\Omega}_{\text{min}} + \widehat{\Omega}_{\text{aux}}$.

Chapter 4

Perturbation Theory

Chapter Contents

4.1 Interaction picture 57
4.2 Rayleigh–Schrödinger theory 58
4.3 Adiabatic approximation 59
4.3.1 Berry phase 59

4.1 Interaction picture

Let $\widehat{H}(t) = \widehat{H}_0 + \widehat{V}(t)$ be the total Hamiltonian of a system, where $\widehat{V}(t)$ denotes the (possibly time-dependent) interaction potential. Let $|\psi(t)\rangle$ and $\widehat{O}$ denote a state and operator in the Schrödinger picture, respectively.

Definition 4.1.1 (Interaction representation). In the interaction picture, the state vector is defined as follows:

$$|\psi(t)\rangle_I := e^{\frac{i}{\hbar}\widehat{H}_0 t}|\psi(t)\rangle\,. \tag{4.1}$$

It follows, that the operators in the interaction picture are given by

$$\widehat{O}_I(t) = e^{\frac{i}{\hbar}\widehat{H}_0 t}\,\widehat{O}\,e^{-\frac{i}{\hbar}\widehat{H}_0 t}\,. \tag{4.2}$$

Formula 4.1.2 (Schrödinger equation). Using the previous formulas, the Schrödinger equation can be rewritten as follows:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle_I = \widehat{V}_I(t)|\psi(t)\rangle_I\,. \tag{4.3}$$

The time evolution of operators in the interaction picture is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{O}_I(t) = \frac{i}{\hbar}\left[\widehat{H}_0, \widehat{O}_I(t)\right]. \tag{4.4}$$

States evolve solely based on the interaction terms and operators evolve according to the free (time-independent) Hamiltonian.

Formula 4.1.3 (Tomonaga–Schwinger equation). As in Formula 1.2.5, the evolution operator $\widehat{U}(t,t')$ satisfies the following Schrödinger-type equation in the interaction picture:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_I(t,t') = \widehat{V}_I(t)\widehat{U}_I(t,t')\,. \tag{4.5}$$

Although this equation is often called the Tomonaga–Schwinger equation in the literature, this is actually not correct. The Tomonaga–Schwinger equation generalizes the Schrödinger equation to allow for evolutions between arbitrary Cauchy surfaces (@@ add reference @@).

Definition 4.1.4 (Møller operator). The operators relating the (free) asymptotic states to the interacting state:

$$|\psi(0)\rangle = \widehat{\Omega}_{\pm}|\mathrm{in/out}\rangle\,. \tag{4.6}$$

If, for $t \longrightarrow \pm\infty$, the interaction term vanishes, there should exist a wave function $|\phi(t)\rangle$ evolving under the free Hamiltonian such that

$$\|\psi(t) - \phi(t)\| \xrightarrow{t\to\pm\infty} 0\,. \tag{4.7}$$

With $|\mathrm{in/out}\rangle := \lim_{t\to\pm\infty}|\phi(t)\rangle$, this gives

$$\widehat{\Omega}_{\pm} = \lim_{t\to\mp\infty}\exp\left(\tfrac{i}{\hbar}\widehat{H}(t)t\right)\exp\left(-\tfrac{i}{\hbar}\widehat{H}_0 t\right) \tag{4.8}$$

with respect to the strong topology.

Remark 4.1.5. The limit in the definition of the Møller operators will only result in a well-defined operator for infinite-dimensional spaces.

4.2 Rayleigh–Schrödinger theory

The basic assumptions of Rayleigh–Schrödinger perturbation theory are that the perturbing Hamiltonian is time independent and that the eigenfunctions of the unperturbed Hamiltonian $\widehat{H}_0$ form a complete set for the perturbed Hamiltonian.

Formula 4.2.1. The perturbed eigenfunctions and eigenvalues can be expanded in the following way, where $\lambda \in \mathbb{R}$ is assumed to be a small (perturbation) parameter:

$$\begin{aligned} |\psi_n\rangle &= \sum_{i=0}^{+\infty}\lambda^i|\psi_n^{(i)}\rangle\,,\\ E_n &= \sum_{i=0}^{+\infty}\lambda^i E_n^{(i)}\,, \end{aligned} \tag{4.9}$$

where $i \in \mathbb{N}$ is called the **order** of the perturbation.

@@ EXPAND @@

4.3 Adiabatic approximation

4.3.1 Berry phase

Theorem 4.3.1 (Adiabatic theorem). *If a perturbation is acting slowly enough such that the system can adapt its configuration at every single moment, the system will remain in the same eigenstate.*

Remark. The original formulation by *Born* and *Fock* also required the system to be *gapped*. Later, this was shown to be inessential.

Consider a system for which the adiabatic approximation is valid and assume that the state starts of as an eigenstate. The wave function is then of the form

$$|\psi(t)\rangle = C_a(t) \exp\left(-\frac{i}{\hbar} \int_{t_0}^{t} E_a(t')\, dt' \right) |\psi_a(t)\rangle\,, \tag{4.10}$$

where the $|\psi_a(t)\rangle$ is an instantaneous eigenstate of $\widehat{H}(t)$. It follows from the orthonormality of the eigenstates $|\psi_a(t)\rangle$ that the coefficient $C_a(t)$ is just a phase factor, so it can be written as

$$C_a(t) = e^{i\gamma_a(t)}\,. \tag{4.11}$$

By substituting this ansatz in the wave function, the Schödinger equation gives a differential equation for the phase factors $\gamma_a(t)$. Integration gives

$$\gamma_a(t) = i \int_{t_0}^{t} \left\langle \psi_a(t') \middle| \frac{\partial \psi_a(t')}{\partial t'} \right\rangle dt'\,. \tag{4.12}$$

Due to time evolution, the wave function accumulates a phase through the coefficient $C_a(t)$. This phase is called the **Berry phase**.

Now, try to apply a phase transformation to remove the Berry phase:

$$|\psi'_a(t)\rangle := e^{i\eta(t)} |\psi_a(t)\rangle\,. \tag{4.13}$$

Inserting this in Eq. (4.12), gives

$$\gamma'_a(t) = \gamma_a(t) - \eta(t) + \eta(t_0)\,. \tag{4.14}$$

If the system is cyclic with period $\tau \in \mathbb{R}$, then $\psi_a(t_0) = \psi_a(t_0 + \tau)$. Combining this with the equations above, gives

$$\eta(t_0 + \tau) - \eta(t_0) = 2k\pi \tag{4.15}$$

for some $k \in \mathbb{N}$. This implies that the Berry phase cannot be eliminated through a basis transformation and, hence, this phase is physically observable!

Definition 4.3.2 (Berry connection). The quantity

$$\mathsf{A}(\vec{x}) := i\langle \psi_a(\vec{x}) \big| \nabla_{\vec{x}} \psi_a(\vec{x}) \rangle \,, \tag{4.16}$$

where $\nabla_{\vec{x}}$ denotes the gradient in phase space, is called the Berry connection (or **Berry gauge potential**). Applying Stokes' theorem to (4.12) gives

$$\gamma_a(t_0 + \tau) = \int \mathcal{B} \cdot d\vec{S} \,, \tag{4.17}$$

where $\mathcal{B} = \nabla_{\vec{x}} \times \mathsf{A}(\vec{x})$ is called the **Berry curvature**. Although the Berry connection is gauge-dependent, the Berry curvature is gauge-invariant.

Remark 4.3.3. Using the language of differential geometry (**??**), one immediately finds that the accumulated phase γ_a is simply the holonomy (**??**) associated with the Berry connection along the considered trajectory through phase space.

Chapter 5

Scattering Theory

Chapter Contents

5.1 Cross sections 61

5.2 Lippman–Schwinger equations 61

5.1 Cross sections

Formula 5.1.1 (Differential cross section).

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{N(\theta,\varphi)}{F}, \tag{5.1}$$

where F is the incoming flux and N the detected flow rate. Because N is not defined as a flux but as a rate, the differential cross section has the dimension of area.

Formula 5.1.2 (Fermi's golden rule). The transition probability from an initial state to a final state is given by

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\langle f|\widehat{H}|i\rangle|^2 \frac{\mathrm{d}n}{\mathrm{d}E_f}. \tag{5.2}$$

5.2 Lippman–Schwinger equations

In this section Hamiltonians of the form $\widehat{H} = \widehat{H}_0 + \widehat{V}$ are considered, where $\widehat{H}_0$ is the free Hamiltonian and $\widehat{V}$ the scattering potential. It will also be assumed that both the total Hamiltonian and the free Hamiltonian have the same eigenvalues (as in the previous chapter).

Formula 5.2.1 (Lippman–Schwinger equation).

$$|\psi^{(\pm)}\rangle = |\varphi\rangle + \frac{1}{E - \widehat{H}_0 \pm i\varepsilon}\widehat{V}|\psi^{(\pm)}\rangle, \tag{5.3}$$

where $|\varphi\rangle$ is an eigenstate of the free Hamiltonian with the same energy as $|\psi\rangle$, i.e. $\widehat{H}_0|\varphi\rangle = E|\varphi\rangle$.

Remark 5.2.2. The term $\pm i\varepsilon$ is added to the denominator because otherwise it would be singular. The term has no real physical meaning.

Formula 5.2.3 (Born series). If the Lippman–Schwinger equation is rewritten as

$$|\psi\rangle = |\varphi\rangle + \widehat{G}_0\widehat{V}|\psi\rangle\,, \tag{5.4}$$

where $\widehat{G}_0$ is the Green's operator (**??**) associated to $\widehat{H}_0$, one can derive the following series expansion by iterating the above expression:

$$|\psi\rangle = |\varphi\rangle + \widehat{G}_0\widehat{V}|\varphi\rangle + \left(\widehat{G}_0\widehat{V}\right)^2|\varphi\rangle + \cdots\,. \tag{5.5}$$

Convergence issues of this series can be resolved through a Borel resummation procedure (**??**).

Formula 5.2.4 (Born approximation). If the Born series is truncated at the first-order term in $\widehat{V}$, the Born approximation is obtained:

$$|\psi\rangle = |\varphi\rangle + \widehat{G}_0\widehat{V}|\varphi\rangle\,. \tag{5.6}$$

Part II

Relativity & Quantum Field Theory

Table of Contents

6 Special Relativity **66**

6.1 Lorentz transformations 66
6.2 Energy and momentum 67
6.3 Action principle 67

7 General Relativity **70**

7.1 Causal structure 70
7.2 Einstein field equations 71
7.3 Black holes 72
7.4 Conserved quantities 73
7.5 Tetradic formulation 73

8 Quantum Field Theory **75**

8.1 Klein–Gordon field 76
8.1.1 Lagrangian and Hamiltonian 76
8.1.2 Raising and lowering operators 76
8.1.3 Scalar propagator 78
8.1.4 Normalization constant 78
8.1.5 Invariant integration measure 78
8.2 Contractions and Wick's theorem 78
8.2.1 Bosonic fields 78
8.2.2 Fermionic fields 80
8.3 Quantum chromodynamics 81
8.4 Feynman rules 81
8.4.1 Scalar theory 81
8.4.2 Graph theory 82
8.4.3 Degree of divergence 83
8.4.4 Amplituhedron ♣ 84
8.5 Renormalization 84
8.5.1 Introduction: Statistical physics 85
8.5.2 Dimensional regularization 85

8.5.3 Wilsonian renormalization 87
8.5.4 Connes–Kreimer renormalization 87
8.6 Entanglement in QFT 90
8.6.1 Lattice theories 91

Chapter 6

Special Relativity

In this chapter, as will be the case in the chapters on quantum field theory, the mostly-minuses convention for the Minkowski signature is adopted unless stated otherwise, i.e. the signature is $(+,-,-,-)$. Furthermore, natural units will be used unless stated otherwise, i.e. $\hbar = c = 1$. This follows the introductory literature such as Greiner and Reinhardt (1996); Peskin and Schroeder (1995).

Chapter Contents

6.1 Lorentz transformations 66
6.2 Energy and momentum 67
6.3 Action principle 67

6.1 Lorentz transformations

Notation 6.1.1. In the context of special relativity, it is often useful to introduce the following quantities:

$$\begin{aligned} \beta &:= \frac{v}{c}, \\ \gamma &:= \frac{1}{\sqrt{1-\beta^2}}. \end{aligned} \tag{6.1}$$

The latter quantity is called the **Lorentz factor**.

Formula 6.1.2 (Lorentz transformations). Let V be a 4-vector. A Lorentz boost along

the x^1-axis is given by the following transformation:

$$\begin{cases} V'^0 = \gamma\left(V^0 - \beta V^1\right) \\ V'^1 = \gamma\left(V^1 - \beta V^0\right) \\ V'^2 = V^2 \\ V'^3 = V^3. \end{cases} \tag{6.2}$$

Remark 6.1.3. Setting $c = +\infty$ in the previous formulas recovers the Galilei transformations from classical mechanics (cf. Inönü–Wigner contractions (**??**)).

6.2 Energy and momentum

Formula 6.2.1 (4-velocity). In analogy to the definition of velocity in classical mechanics, the 4-velocity is defined as follows:

$$\mathsf{U} := \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}, \frac{\mathrm{d}x^1}{\mathrm{d}\tau}, \frac{\mathrm{d}x^2}{\mathrm{d}\tau}, \frac{\mathrm{d}x^3}{\mathrm{d}\tau}\right). \tag{6.3}$$

By applying the formulas for proper time and time dilatation, one obtains:

$$\mathsf{U} = \left(\gamma c, \gamma \vec{\boldsymbol{u}}\right). \tag{6.4}$$

Formula 6.2.2 (4-momentum). The 4-momentum is defined as follows:

$$\mathsf{p} = m_0 \mathsf{U}, \tag{6.5}$$

or, after defining $E := cp^0$:

$$\mathsf{p} = \left(\frac{E}{c}, \gamma m_0 \vec{\boldsymbol{u}}\right). \tag{6.6}$$

Definition 6.2.3 (Relativistic mass). The factor $m := \gamma m_0$ in the momentum 4-vector is called the relativistic mass. By introducing this quantity, the classical formula $\vec{\boldsymbol{p}} = m\vec{\boldsymbol{u}}$ for the 3-momentum can be generalized to 4-momenta p.

Formula 6.2.4 (Relativistic energy relation).

$$E^2 = p^2c^2 + m^2c^4 \tag{6.7}$$

This formula is often called the **Einstein relation**.

6.3 Action principle

The main guiding principles for writing down a relativistic action for a point particle are locality and invariance. The latter means that one should only use geometric

quantities, i.e. diffeomorphism-invariant quantities, while the former means that these should only depend on local information. For a single particle, the most obvious choice of action would be one that is proportional to the proper time along the worldline of the particle or, more invariantly, the arc length of the worldline:

$$S_{\text{point}} \sim \int_\gamma ds\,. \tag{6.8}$$

To get the units right, one should multiply by suitable Lorentz-invariant constants:

$$S_{\text{point}} := mc \int_\gamma ds\,. \tag{6.9}$$

By reparametrization invariance, one can choose a specific time coordinate, e.g. $\tau = ct$. In this coordinate system, the action becomes

$$S_{\text{point}} = mc \int_\gamma \sqrt{1 - \frac{v^2}{c^2}}\, c\, dt\,, \tag{6.10}$$

with v the speed of the particle. The Lagrangian density can be Taylor expanded as

$$L_{\text{point}} = mc^2 - \frac{1}{2}mv^2 + \cdots\,, \tag{6.11}$$

which recovers (up to a constant) the classical Lagrangian of a massive point particle when the speed is small $v \ll c$.

When trying to quantize this action, however, a problem occurs. After a Legendre transformation, the Hamiltonian becomes $H = \sqrt{p^2 + m^2c^4}$ (this is just the Einstein relation 6.2.4). When applying the ordinary Dirac procedure $p_\mu \longrightarrow i\partial_\mu$, this becomes a nonlocal operator (the whole reason for why Dirac introduced spinors).

Instead of passing to a spinor framework, one can try to write down an equivalent action that gives rise to a local Hamiltonian. One possibility is to pass to **light cone coordinates**:

$$x^\pm := x^0 \pm x^1\,. \tag{6.12}$$

However, here one makes a specific split of coordinates, which ruins Lorentz invariance. A better idea is to introduce a dynamical Lagrange multiplier (from here on, natural units are used):

$$S_{\text{point}} := \frac{1}{2}\int_\gamma \left[\eta^{-1}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)^2 - m^2\eta\right] d\tau\,. \tag{6.13}$$

The equation of motion for η is algebraic and gives

$$\eta = \frac{1}{m}\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}\,. \tag{6.14}$$

To find an interpretation of this multiplier, it is useful to consider the case where a metric h is introduced on the worldline. This implies that the action has to be 'covariantized':

$$\int_\gamma \left(g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + m^2 \right) d\tau \longrightarrow \int_\gamma \sqrt{h} \left(h g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + m^2 \right) d\tau \,. \tag{6.15}$$

From this perspective, it is clear that the Lagrange multiplier can be viewed as the square root of a dynamical metric on the worldline. It is an example of a *vielbein* (in this case, an '*einbein*'). Furthermore, this action is a so-called *nonlinear σ-model*.

Chapter 7

General Relativity

References for this chapter are Misner, Thorne, and Wheeler (2017); Rovelli and Vidotto (2014). See **??** for an introduction to the theory of Riemannian geometry. The mathematical background for the section on the *tetradic formulation* of GR can be found in **??**.

In this chapter, the signature convention of the previous chapter is reversed. For general relativity, it is often more convenient to use the mostly-pluses convention (this simply reduces the number of minus signs).

7.1 Causal structure

Definition 7.1.1 (Null coordinate). Consider a vector $v \in T_pM$ on a Lorentzian manifold (M, g). This vector is said to be null or **lightlike** if it satisfies the following condition:

$$g_p(v, v) = 0\,. \tag{7.1}$$

One can also define **timelike** and **spacelike** vectors in a similar way as those vectors having negative and positive norm, respectively.[1] Spacelike, lightlike and timelike curves are defined as curves for which every tangent vector is respectively spacelike, lightlike or timelike. A curve is said to be **causal** if its tangent vectors are time- or lightlike.

Definition 7.1.2 (Time-orientability). A Lorentzian manifold is said to be time orientable if there exists a nowhere-vanishing, timelike vector field. It should be noted that, in contrast to ordinary orientability, this notion is not purely topological. Moreover, neither orientability nor time-orientability implies the other. They are independent notions.

[1] For a mostly-minuses signature, one should interchange these definitions.

The choice of a time-orienting vector field τ divides the set of timelike vectors at a point into two equivalence classes. A curve γ is said to be future directed (resp. past directed) if $g(\tau, \dot{\gamma}) < 0$ (resp. $g(\tau, \dot{\gamma}) > 0$).

Definition 7.1.3 (Causal cone). Let M be a Lorentzian manifold. The causal cone of a point $p \in M$ is defined as the set $J^-(p) \cup J^+(p) \subset M$ of points that are connected to p by a (smooth) causal curve. The past and future cones are respectively defined as the sets of points that can be connected to p by a future-directed or past-directed casual curve. The boundaries of these causal cones are called the causal **lightcones**, sometimes denoted by $V^{\pm}(p)$.

Definition 7.1.4 (Causal closure). Let S be a subset of a Lorentzian manifold. The **causal complement** of S consists of all points that cannot be causally connected to any point in S. The causal closure of S is defined as the causal complement of the causal complement of S. A **causally closed set** is then defined as a set which is equal to its causal closure.

Definition 7.1.5 (Globally hyperbolic manifold). A Lorentzian manifold M that does not contain closed causal curves and for which $J^+(p) \cap J^-(q)$ is compact for any two points $p, q \in M$.

Definition 7.1.6 (Stationary spacetime). A spacetime (M, g) is called stationary if there exists a timelike Killing vector. By the *flowbox theorem*, there always exists a coordinate chart such that locally one can choose the Killing vector field to be ∂_0 and, hence, a spacetime is stationary if one can find a coordinate system for which the metric coefficients are time-independent.

Definition 7.1.7 (Celestial sphere). Consider a 4-dimensional Lorentzian manifold M. The celestial sphere at a point $p \in M$ is given by the projectivization $\mathbb{P}\big(J^+(p)\big) \cong \mathbb{CP}^1 \cong S^2$, i.e. it consists of all rays in the future (or, equivalently, past) lightcone at p. As the *nLab* so nicely phrases it "*your celestial sphere (the one around the point where your head is) is the sphere of which you directly perceive a portion when you look.*".

Property 7.1.8. By the above definition and **??**, the celestial sphere is given by $\mathbb{CP}^1$ or, in the spinorial setting, by the Weyl spinors modulo rescalings.

7.2 Einstein field equations

Formula 7.2.1 (Einstein field equations). The Einstein field equations without a cosmological constant Λ read as follows (all fundamental constants are shown for completeness):

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\,, \tag{7.2}$$

where $G_{\mu\nu}$ is the Einstein tensor (**??**) and $T_{\mu\nu}$ is the stress-energy tensor (**??**).

By taking the trace of both sides one obtains $T = -R$ and, hence, the Einstein field

equations can be rewritten as

$$R_{\mu\nu} = \widehat{T}_{\mu\nu}\,, \tag{7.3}$$

where $\widehat{T}_{\mu\nu} := T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$ is the **reduced stress-energy tensor**.

Formula 7.2.2 (Einstein–Hilbert action). The (vacuum) field equations can be obtained by applying the variational principle to the following action:

$$S_{\mathrm{EH}}[g_{\mu\nu}] := \int_M \sqrt{-g}R \ \mathrm{Vol}\,. \tag{7.4}$$

For manifolds with boundary, one needs an extra term to make the boundary contributions vanish (as to obtain a well-defined variational problem). This term is due to *Gibbons*, *Hawking* and *York*:[2]

$$S_{\mathrm{GHY}}[g_{\mu\nu}] := \oint_{\partial M} \epsilon\sqrt{h}K \ \mathrm{Vol}, \tag{7.5}$$

where h_{ab} is the induced metric on the boundary, K_{ab} is the extrinsic curvature and $\epsilon = \pm 1$ is a 'function' depending on whether the boundary is timelike or spacelike.

7.3 Black holes

Formula 7.3.1 (Schwarzschild metric).

$$ds^2 := \left(1 - \frac{R_s}{r}\right)c^2dt^2 - \left(1 - \frac{R_s}{r}\right)^{-1}dr^2 - r^2d\Omega^2\,, \tag{7.6}$$

where R_s is the Schwarzschild radius

$$R_s := \frac{2GM}{c^2}\,. \tag{7.7}$$

Theorem 7.3.2 (Birkhoff). *The Schwarzschild metric is the unique solution of the vacuum field equation under the additional constraints of asymptotic flatness and staticity.*

Formula 7.3.3 (Reissner–Nordström metric). If the black hole is allowed to have an electric charge Q, the Schwarzschild metric must be modified in the following way:

$$ds^2 := \left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right)^{-1}dr^2 - r^2d\Omega^2\,. \tag{7.8}$$

Remark 7.3.4. The electric field generated by a Reissner–Nordström black hole is given by

$$E^r = \frac{Q}{4\pi r^2}\,. \tag{7.9}$$

Although the coordinate r is not the proper distance, it still acts as a parameter for the surface of a sphere (as it does in a Euclidean or Schwarzschild metric). This explains why the above formula is the same as the one in classical electromagnetism.

@@ ADD KERR–NEWMAN, ERGOSPHERE, PENROSE MECHANISM, KRUSKAL @@

[2] Einstein had in fact already introduced a variant, the ΓΓ-*Lagrangian*.

7.4 Conserved quantities

Before raising any hope that conserved quantities are to be found in GR, the following result is given.

Property 7.4.1. By Noether's third theorem **??**, there exist no proper (local) conservation laws such as those of momentum and energy in classical mechanics, since the translation group is a subgroup of the infinite-dimensional symmetry group $\mathrm{Diff}(M)$.

7.5 Tetradic formulation

This section starts from the geometric interpretation of the (weak) equivalence principle, i.e. spacetime is locally modelled on Minkowski space. The natural language for this kind of geometry is that of Cartan geometries. By the Erlangen program, it is known that the Minkowski spacetime M^4 can be described as the coset space $\mathsf{ISO}(3,1)/\mathsf{SO}(3,1)$. The natural generalization is given by a Cartan geometry with model geometry $\big(\mathfrak{iso}(3,1), \mathfrak{so}(3,1)\big)$.

Property 7.5.1 (Cartan connection). This way, a $\mathsf{SO}(3,1)$-structure on the spacetime manifold M is obtained, i.e. a choice of Lorentzian metric g. The Cartan connection $\widetilde{\nabla}$ can also be decomposed as $\nabla + \mathsf{e}$ where:

- ∇ defines a $\mathfrak{so}(3,1)$-valued principal connection, and
- e defines a $\mathcal{M}^4$-valued solder form.

The principal connection ∇ is called the **spin connection** and e is called the **vierbein** or **tetrad**. These objects are well known in general relativity. The connection ∇ is the ordinary Levi-Civita connection associated to the Lorentzian manifold M (in case of vanishing torsion) and e gives the isometry between local (flat) Minkowski coordinates and 'global' coordinates:

$$g := \mathsf{e}^* \eta \tag{7.10}$$

or, locally,

$$g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij} \,. \tag{7.11}$$

Using the tetrad field, one can rewrite the Einstein–Hilbert action in a very elegant way. To this end, a new curvature form is defined:

$$F^i{}_{j\mu\nu} := e^i_\rho e^\sigma_j R^\rho{}_{\sigma\mu\nu} \,, \tag{7.12}$$

where $R^\rho{}_{\sigma\mu\nu}$ is the ordinary Riemann curvature tensor. The Einstein–Hilbert action is then equivalent[3] to the following Yang–Mills-like action:

[3] At least in the case of pure gravity (Rovelli & Vidotto, 2014).

Formula 7.5.2 (Palatini action).

$$S[e, \nabla] := \int_M \mathsf{e} \wedge \mathsf{e} \wedge *F\,. \tag{7.13}$$

This action is sometimes called the **tetradic Palatini action** and the resulting formulation of general relativity is called the **first-order formulation**. If one considers the same action but only as a functional of the tetrad field, one obtains the **second-order formulation** of gravity.[4]

Variation of the Palatini action gives the following EOM:

- $\delta\nabla$: $T(\mathsf{e}) = 0$ or, equivalently, $\nabla(\mathsf{e}) \equiv \nabla$, i.e. the torsion vanishes and the connection ∇ is, on-shell, equal to the Levi-Civita connection on M.
- $\delta\mathsf{e}$: The metric g satisfies the Einstein field equations.

Because of its importance in general relativity, the first factor in the Palatini action deserves a name.

Definition 7.5.3 (Plebanski form).

$$\Sigma := \mathsf{e} \wedge \mathsf{e} \tag{7.14}$$

Because of its internal antisymmetric Lorentz indices, one can interpret this object as an $\mathfrak{so}(3,1)$-valued two-form.

As was the case for 4D Yang–Mills theory, one can introduce a topological term that leaves the EOM invariant (up to boundary terms).

Definition 7.5.4 (Holst action[5]).

$$\begin{aligned} S[\mathsf{e}, \nabla] :&= \int_M \mathsf{e} \wedge \mathsf{e} \wedge *F + \frac{1}{\gamma}\int_M \mathsf{e} \wedge \mathsf{e} \wedge F \\ &= \int_M \left(*\mathbf{e} \wedge \mathsf{e} + \frac{1}{\gamma}\mathsf{e} \wedge \mathsf{e} \right) \wedge F. \end{aligned} \tag{7.15}$$

The coupling constant γ is called the **Barbero–Immirzi** constant.

@@ COMPLETE @@

[4] These formulations are equivalent for pure gravity. However, when coupling the theory to fermions, they differ by a four-fermion vertex. This follows from the introduction of torsion due to the fermions.

[5] Holst was actually the second author to include this term.

Chapter 8

Quantum Field Theory

The main reference for this chapter is Peskin and Schroeder (1995). For the section on the Connes–Kreimer algebra, see the original paper by Connes and Kreimer (1998) or the paper by Ebrahimi-Fard and Kreimer (2005). Another good reference on the formal structure of Feynman diagrams is Yeats (2017). The main references for the section on entanglement are Rangamani and Takayanagi (2017); Tuybens (2017).

Chapter Contents

8.1 Klein–Gordon field **76**
8.1.1 Lagrangian and Hamiltonian 76
8.1.2 Raising and lowering operators 76
8.1.3 Scalar propagator 78
8.1.4 Normalization constant 78
8.1.5 Invariant integration measure 78
8.2 Contractions and Wick's theorem **78**
8.2.1 Bosonic fields 78
8.2.2 Fermionic fields 80
8.3 Quantum chromodynamics **81**
8.4 Feynman rules **81**
8.4.1 Scalar theory 81
8.4.2 Graph theory 82
8.4.3 Degree of divergence 83
8.4.4 Amplituhedron ♣ 84
8.5 Renormalization **84**
8.5.1 Introduction: Statistical physics 85

8.5.2 Dimensional regularization . . . 85
8.5.3 Wilsonian renormalization . . . 87
8.5.4 Connes–Kreimer renormalization . . . 87
8.6 Entanglement in QFT . . . **90**
8.6.1 Lattice theories . . . 91

8.1 Klein–Gordon field

8.1.1 Lagrangian and Hamiltonian

The 'simplest' Lagrangian (density) is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\,. \tag{8.1}$$

Using the principle of least action, the following Euler–Lagrange equation is obtained:

$$\left(\partial^\mu\partial_\mu + m^2\right)\phi = 0\,. \tag{8.2}$$

This can be rewritten using the **d'Alembertian** $\Box = \partial_\mu\partial^\mu$:

$$(\Box + m^2)\phi = 0\,. \tag{8.3}$$

This equation is called the **Klein–Gordon equation**. In the limit $m \longrightarrow 0$, this equation reduces to the well-known wave equation (**??**).

From the Lagrangian (8.1), one can also derive a Hamiltonian function using **??** and **??**:

$$H = \frac{1}{2}\int_{\mathbb{R}^3}\left[\pi^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)\right]d^3x\,. \tag{8.4}$$

8.1.2 Raising and lowering operators

Fourier transforming the scalar field $\phi(\mathrm{x})$ and inserting it into the Klein–Gordon equation gives

$$\left(\partial_t^2 + p^2 + m^2\right)\phi(\mathrm{p}) = 0\,. \tag{8.5}$$

This is the equation for a simple harmonic oscillator with frequency $\omega = \sqrt{p^2 + m^2}$.

Analogous to ordinary quantum mechanics, raising and lowering operators $a_{\vec{p}}^\dagger$ and $a_{\vec{p}}$ can be defined such that

$$\phi(\vec{x}) = \int_{\mathbb{R}^3}\frac{d^3p}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\right), \tag{8.6}$$

$$\pi(\vec{x}) = \int_{\mathbb{R}^3}\frac{d^3p}{(2\pi)^{3/2}}(-i)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\right). \tag{8.7}$$

An equivalent definition is obtained by performing the transformation $\vec{p} \longrightarrow -\vec{p}$ in the second term of $\phi(\vec{x})$ and $\pi(\vec{x})$:

$$\phi(\vec{x}) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} + a^\dagger_{-\vec{p}} \right) e^{i\vec{p}\cdot\vec{x}} \,, \tag{8.8}$$

$$\pi(\vec{x}) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2}} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} - a^\dagger_{-\vec{p}} \right) e^{i\vec{p}\cdot\vec{x}} \,. \tag{8.9}$$

When the commutation relation

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] := \delta(\vec{p} - \vec{q}) \tag{8.10}$$

is imposed, the following commutation relation for the scalar field and its conjugate momentum is obtained:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y}) \,. \tag{8.11}$$

Now, the Hamiltonian can be calculated explicitly:

$$H = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2} [a_{\vec{p}}, a^\dagger_{\vec{p}}] \right) . \tag{8.12}$$

However, it is clear from Eq. (8.10) that the second term in this integral diverges. There are two reasons for this divergence. First, space is infinite, i.e. the d^3x integral in Eq. (8.4) diverges. This problem can be resolved by restricting the system to a (finite) part of space or by considering the energy density instead of the energy itself. Second, by including very large values for p in the integral, a parameter range is explored where the theory is likely to break down. To resolve this problem a 'high p'-cut-off should be introduced. A more practical solution, however, is to note that only energy differences are physical and so one can simply drop the second term altogether as it is merely a 'constant' (albeit an infinite one).

A corollary of Eq. (8.12) together with the canonical commutation relations is

$$[H, a^\dagger_{\vec{p}}] = \omega_p a^\dagger_{\vec{p}} \,, \tag{8.13}$$

$$[H, a_{\vec{p}}] = -\omega_p a_{\vec{p}} \,. \tag{8.14}$$

As was the case for the quantum harmonic oscillator, the creation and annihilation operators deserve their names and one can write:

$$|\vec{k}_1, \dots, \vec{k}_n\rangle = a^\dagger(\vec{k}_1) \cdots a^\dagger(\vec{k}_n)|0\rangle \,. \tag{8.15}$$

Furthermore, this equation together with the canonical commutation relations imply that the Klein–Gordon fields are bosonic fields. This way of generating the full Hilbert (or, in fact, Fock) state space is axiomatized by the GNS construction (**??**), where the vacuum plays the role of cyclic vector.

8.1.3 Scalar propagator

Formula 8.1.1 (Pauli–Jordan function).

$$i\Delta(x-y) := i[\phi(x),\phi(y)] = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_x}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) \tag{8.16}$$

When $x^0 = y^0$ (ETCR) or $(x-y)^2 < 0$ (spacelike curves), the Pauli–Jordan function is identically 0. (See also the *axiom of microcausality* **??**).

8.1.4 Normalization constant

By **??**, the delta function $\delta^{(3)}(\vec{p}-\vec{q})$ transforms as $\delta^{(3)}(\Lambda\vec{p}-\Lambda\vec{q})\frac{\Lambda E}{E}$ under a general Lorentz boost Λ. Although this is clearly not Lorentz invariant, the quantity $E_x\delta^{(3)}(\vec{p}-\vec{q})$ can be seen to be invariant. From this observation it follows, that the correct normalization in the momentum representation is

$$|p\rangle = \sqrt{2E_x}a_p^\dagger|0\rangle \tag{8.17}$$

and, hence,

$$\langle p \mid q\rangle = 2E_x(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,, \tag{8.18}$$

where the constants are a matter of convention to cancel the constants in Eq. (8.6).

8.1.5 Invariant integration measure

The factor $2E_p$ does not only occur in the normalization conditions. To find a Lorentz-invariant integration measure in spacetime, the following integral can be studied:

$$\int_{\mathbb{R}^3}\frac{d^3p}{2E_p} = \int_{\mathbb{R}^+\times\mathbb{R}^3}\delta(p^2-m^2)\,d^4p\,, \tag{8.19}$$

where the timelike component p^0 is restricted to be (strictly) positive. By using this measure, it is ensured that the integral of any Lorentz-invariant function is again Lorentz invariant.

Example 8.1.2 (One-particle identity operator).

$$\mathbb{1} := \int_{\mathbb{R}^3}\frac{d^3p}{2E_p}|p\rangle\langle p| \tag{8.20}$$

8.2 Contractions and Wick's theorem

8.2.1 Bosonic fields

In the following definitions, (field) operators will be decomposed as

$$\phi = \phi^{(+)} + \phi^{(-)}\,,$$

where the + symbol denotes the 'positive-frequency' part, i.e. the part consisting of annihilation operators. The 'negative-frequency' part is defined analogously. This terminology stems from classic Fourier theory. By looking at Eq. (8.6) and remembering that the (1, 3)-signature is adopted, the annihilators can be seen to always occur together with a positive-frequency exponential.

Definition 8.2.1 (Contraction for neutral bosonic fields).

$$\overbracket{\phi(\mathsf{x})\phi(\mathsf{y})} := \begin{cases} [\phi(\mathsf{x})^{(+)}, \phi(\mathsf{y})^{(-)}] & \text{if } x^0 > y^0 \\ [\phi(\mathsf{y})^{(+)}, \phi(\mathsf{x})^{(-)}] & \text{if } y^0 > x^0 \end{cases} \tag{8.21}$$

Formula 8.2.2 (Feynman propagator).

$$i\Delta_F(\mathsf{x}-\mathsf{y}) := \overbracket{\phi(\mathsf{x})\phi(\mathsf{y})} := i \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(\mathsf{x}-\mathsf{y})}}{p^2 - m^2 + i\varepsilon} \tag{8.22}$$

Definition 8.2.3 (Contraction for charged bosonic fields).

$$\overbracket{\phi(\mathsf{x})\overline{\phi}(\mathsf{y})} := \begin{cases} [\phi(\mathsf{x})^{(+)}, \overline{\phi}(\mathsf{y})^{(-)}] & \text{if } x^0 > y^0 \\ [\phi(\mathsf{y})^{(+)}, \overline{\phi}(\mathsf{x})^{(-)}] & \text{if } y^0 > x^0 \end{cases} \tag{8.23}$$

Definition 8.2.4 (Normal ordering). The normal ordering $\mathcal{N}$, often denoted by colons $: \; :$, of a sequence of field operators is defined as the permuted sequence in which all annihilation operators appear on the right of the creation operators, e.g.:

$$\mathcal{N}\big(\phi(\mathsf{x})\phi^\dagger(\mathsf{y})\phi(\mathsf{z})\big) = \phi^\dagger(\mathsf{y})\phi(\mathsf{x})\phi(\mathsf{z})\,. \tag{8.24}$$

Note that the normal ordering operator is not a morphism between CCR-algebras since this would lead to a contradiction:

$$b_i b_j^\dagger = b_j^\dagger b_i + \delta_{ij} \implies \mathcal{N}(b_i b_j^\dagger) = \mathcal{N}(b_j^\dagger b_i + \delta_{ij}) = \mathcal{N}(b_j^\dagger b_i) + \delta_{ij} \implies \delta_{ij} = 0\,. \tag{8.25}$$

The solution is given by the fact that inside the normal ordering all operators commute and, hence, this ordering can be axiomatized as an algebra morphism $\mathcal{N} : \mathsf{Sym}^\bullet A \to A$, where A is the CCR-algebra of the theory (see Miwa, Jimbo, Jimbo, and Date (2000)).

@@ IS THIS REFERENCE CORRECT? @@

Property 8.2.5. From this definition, it immediately follows that the vacuum expectation value of a normal ordered sequence is 0.

Formula 8.2.6 (Wick's theorem for bosonic fields).

$$\mathcal{T}\big(\phi(\mathsf{x}_1)\cdots\phi(\mathsf{x}_n)\big) = \mathcal{N}\big(\phi(\mathsf{x}_1)\cdots\phi(\mathsf{x}_n) + \text{all possible contractions}\big) \tag{8.26}$$

When acting with a time-ordered product on the vacuum, this relation implies that only fully contracted terms will remain. Moreover, by Formula 8.2.2, every such action can be expressed solely in terms of propagators.

Remark 8.2.7. In the case of charged bosons, only contractions of the form $\overbrace{\phi(\mathsf{x})\phi(\mathsf{y})}$ will remain because $[a, b^+] = 0$.

Corollary 8.2.8.

$$\overbrace{\phi(\mathsf{x})\phi(\mathsf{y})} = \mathcal{T}\big(\phi(\mathsf{x})\phi(\mathsf{y})\big) - \mathcal{N}\big(\phi(\mathsf{x})\phi(\mathsf{y})\big) \tag{8.27}$$

Wick's theorem has an analogue in probability theory.

Theorem 8.2.9 (Isserlis). *Let $X_1, \dots, X_n$ be a set of random variables following a multinormal distribution with mean 0.*

$$\mathsf{E}\big[X_1 \cdots X_n\big] = \sum_{\sigma \in P_{n,2}} \prod_{\{i,j\} \in \sigma} \mathsf{E}\big[X_i X_j\big], \tag{8.28}$$

where $P_{n,2}$ denotes the set of binary partitions of $\{1, \dots, n\}$. Because the mean of the distribution is zero, this expression is equal to

$$\mathsf{E}\big[X_1 \cdots X_n\big] = \sum_{\sigma \in P_{n,2}} \prod_{\{i,j\} \in \sigma} \mathsf{cov}\big[X_i, X_j\big]. \tag{8.29}$$

8.2.2 Fermionic fields

Definition 8.2.10 (Contraction).

$$\overbrace{\psi(\mathsf{x})\overline{\psi}(\mathsf{y})} := \begin{cases} \{\psi(\mathsf{x})^{(+)}, \overline{\psi}(\mathsf{y})^{(-)}\}_+ & \text{if } x^0 > y^0 \\ -\{\psi(\mathsf{y})^{(+)}, \overline{\psi}(\mathsf{x})^{(-)}\}_+ & \text{if } y^0 > x^0 \end{cases} \tag{8.30}$$

Remark 8.2.11. Only contractions of the form $\overbrace{\psi(\mathsf{x})\overline{\psi}(\mathsf{y})}$ will remain because $\{a, b^\dagger\}_+ = 0$.

Formula 8.2.12 (Feynman propagator).

$$i\Delta_F(\mathsf{x} - \mathsf{y}) := \overbrace{\psi(\mathsf{x})\overline{\psi}(\mathsf{y})} := i \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^4} \frac{d^4\mathsf{p}}{(2\pi)^4} \frac{\not{\mathsf{p}} + m}{p^2 - m^2 + i\varepsilon} e^{-i\mathsf{p}\cdot(\mathsf{x}-\mathsf{y})} \tag{8.31}$$

Remark 8.2.13 (Normal ordering). One should take into account the Fermi–Dirac statistics when permuting fermionic field operators under a normal ordering. A general factor $\mathsf{sgn}(\sigma)$, where σ is the permutation of the operators, will arise in every term, e.g.:

$$\mathcal{N}\big(\psi(\mathsf{x})\overline{\psi}(\mathsf{y})\psi(\mathsf{z})\big) = -\overline{\psi}(\mathsf{y})\psi(\mathsf{x})\psi(\mathsf{z}). \tag{8.32}$$

A similar remark should be made for the time-ordering operator $\mathcal{T}$. As was the case for bosonic theories, one should pay attention to the nature of the normal ordering. It is not a morphism between CAR-algebras, but instead it is an algebra morphism between a free (odd) algebra and a CAR-algebra.

8.3 Quantum chromodynamics

Property 8.3.1 (OZI rule[1]). Decay processes for which the corresponding Feynman diagrams become disconnected (initial states and final states are disconnected) when removing internal gluon lines, are suppressed with respect to other processes.

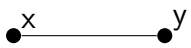
@@ COMPLETE or REMOVE @@

8.4 Feynman rules

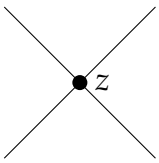
8.4.1 Scalar theory

By expanding the correlation functions in perturbation theory and applying Wick's theorem, one can rewrite every term using the following dictionary for Feynman diagrams (it is assumed that every coupling constant in the Lagrangian is divided by the necessary permutation factors, e.g. in ϕ^4-theory it is assumed that the constant is of the form $\lambda/4!$):

- Propagator $\Delta_F(x-y)$:



- Interaction vertex[2] $-i\lambda \int d^4z$:



The main idea behind these rules is to draw all possible diagrams consistent with the given interaction Lagrangian and translate them into analytic expressions. However, to obtain the correct normalization, one should take the following remark into account.
Remark 8.4.1. Symmetry factors of diagrams should be accounted for in analytic expressions. As an example, consider the following vacuum bubble:

Because the two legs can be interchanged, this diagram has **symmetry factor** 2 and, hence, it gives the analytic expression $-\frac{i\lambda}{2}\int d^4z\,\Delta_F(z-z)$.

[1] *Okubo*, *Zweig* and *Iizuka*
[2] Four legs were drawn as an example but this can be generalized to any order of interaction term.

8.4.2 Graph theory

Given a decorated graph Γ, representing a Feynman diagram, collapsing all vertices leads to its residue $\mathrm{res}(\Gamma)$. Such a one-vertex graph $\underline{r}$, where $r \in \mathbb{N}$ is the number or remaining edges, is sometimes also called the **external leg structure** of all graphs Γ with $\mathrm{res}(\Gamma) = \underline{r}$. Two other important properties of a graph Γ are its degree and its symmetry factor.

Definition 8.4.2 (Degree). Consider a graph Γ. Its degree $|\Gamma| \in \mathbb{N}$ is given by the number of independent loops in Γ or, equivalently, the rank of its first homology group $H_1(\Gamma; \mathbb{Z})$ (**??**), i.e. its first Betti number.

Definition 8.4.3 (1PI graph). A graph that remains connected when an internal edge is removed.

Several different ways exist to determine the number of relevant permutations of the Feynman diagram. One is to directly use Wick's theorem and get the combinatorics out of that result. However, two other useful approaches exist.

Definition 8.4.4 (Symmetry factor). Consider a Feynman diagram Γ consisting of $n \in \mathbb{N}$ internal vertices of valence $v \in \mathbb{N}_{\geq 3}$.[3] The symmetry factor is given by:

$$\mathrm{sym}(\Gamma) := \frac{1}{n!}\left(\frac{1}{v!}\right)^n C\,, \tag{8.33}$$

where $C \in \mathbb{N}$ denotes the number of different ways that the vertices can be contracted while resulting in the same diagram.

A more structured way is to consider the suitable notion of graph automorphism for Feynman diagrams. For ordinary graphs, two common definitions are either through an *adjacency matrix* or through a relation on edges. A third, equivalent, way, however, is through the notion of 'half-edge'.

Alternative Definition 8.4.5 (Graph). Consider a set of **half-edges** Γ. A graph structure on Γ consists of a partition $\mathcal{E}$ by subsets of cardinality 2, called the **edge** set, and a partition $\mathcal{V}$, the **vertex** set.

By slightly modifying this definition, a Feynman diagram is obtained.

Alternative Definition 8.4.6 (Symmetry factor). Consider a set of **half-edges** Γ. A Feynman diagram structure on Γ consists of a disjoint collection $\mathcal{E}$ of cardinality-2 subsets, called the **internal edge** set, and a partition $\mathcal{V}$ by subsets of cardinality at least 3, the **vertex** set. Half-edges not part of the internal edge set are called external edges. These determine the residue $\mathrm{res}(\Gamma)$.

An automorphism of a Feynman diagram consists of a bijection $\Gamma \to \Gamma$ that fixes the

[3] It is here assumed that all vertices are of the same valence and type. The definition is easily extended to more complex diagrams.

external edges and $\mathcal{E}, \mathcal{V}$. The symmetry factor of Γ is then given by:

$$\mathsf{sym}(\Gamma) = |\,\mathsf{Aut}(\Gamma)|\,. \tag{8.34}$$

Aside from the obvious algebra structure on Feynman diagrams, given by formal linear combinations and juxtaposition, there also exists a pre-Lie algebra structure (**??**):

$$(\Gamma_1, \Gamma_2) = \sum_{\Gamma} n(\Gamma_1, \Gamma_2; \Gamma)\Gamma\,, \tag{8.35}$$

where $n(\Gamma_1, \Gamma_2; \Gamma) \in \mathbb{N}$ counts the number of ways that, if Γ_2 is a subgraph of Γ, $\Gamma/\Gamma_2 \cong \Gamma_1$ and Γ ranges over 1PI diagrams.

8.4.3 Degree of divergence

Definition 8.4.7 (Mass dimension). Through the Compton wavelength

$$\lambda = \frac{h}{mc}\,, \tag{8.36}$$

the dimensions of length and mass are related. In particular, in natural units $\hbar = c = 1$, one obtains $[m] = [\lambda]^{-1}$. This dimension is called the mass dimension. @@ ADD Compton and de Broglie wavelength to [QM] @@

Now, consider a general Lagrangian density $\mathcal{L}$. Since this density is integrated against a volume form to obtain the dimensionless action, the mass dimension of $\mathcal{L}$ has to equal the spacetime dimension:

$$[\mathcal{L}] = d\,. \tag{8.37}$$

To obtain the mass dimension of a field ϕ, the general procedure, is to look at the mass term in the Lagrangian. For example, for scalar theories:

$$\begin{aligned}[\mathcal{L}_{\text{mass}}] &= \left[\frac{1}{2}m^2\phi^2\right] \\ &= 2 + 2[\phi]\end{aligned}$$

and, hence

$$[\phi] = \frac{d-2}{2}\,. \tag{8.38}$$

Definition 8.4.8 (Degree of divergence). Consider a general Feynman diagram Γ and its associated integral I_Γ. The (superficial) degree of divergence of Γ is defined as

$$D_\Gamma := \text{power of momenta in numerator} - \text{power of momenta in denominator}\,. \tag{8.39}$$

This can also be calculated at the level of the graph Γ:

$$D_\Gamma = dL - 2I_b - I_f - \sum_{i \in I} n_i d_i \,, \tag{8.40}$$

where L is the number of loops in Γ, n_i is the number of vertices of type $i \in I$, d_i is the number of derivatives in the interaction term corresponding to the vertex type $i \in I$ and I_b, I_f are the number of internal boson and fermion lines, respectively.

A relation between the mass dimension $[\Gamma]$ and the degree of divergence D_Γ is given by the following relation:

$$D_\Gamma = [\Gamma] - \sum_{n=3}^{+\infty} k_n [g_n] \,, \tag{8.41}$$

where k_n denotes the number of vertices of valence $n \in \mathbb{N}$.

@@ CHECK THIS (e.g. Peskin) @@

Property 8.4.9 (Renormalizability). A Feynman diagram Γ is UV divergent if $D_\Gamma \geq 0$, with $D_\Gamma = 0$ corresponding to logarithmic divergences, and, hence, $D_\Gamma < 0$ implies a UV-convergent diagram. If the maximal degree of divergence that occurs for a given theory is finite, the theory is said to be **renormalizable**. By Eq. (8.41), a necessary condition for renormalizability is that the mass dimension of all coupling constants is positive (or zero).

8.4.4 Amplituhedron ♣

In the 80s, *Parke* and *Taylor* discovered that, in QCD (Section 8.3), when scattering gluons, certain tree-level amplitudes consistently vanished. For $n \in \mathbb{N}$ gluons, if either all n or $n-1$ of them have the same helicity, the scattering amplitudes vanish. The **maximally helicity-violating** (MHV) amplitudes arise for $n-2$ gluons of the same helicity and can, in fact, be expressed in a single term, the **Parke–Taylor formula**:

$$\mathcal{A}(p_1^+ \cdots p_i^- \cdots p_j^- \cdots p_n^+) = i(-g)^{n-2} \frac{\langle p_i\, p_j \rangle^4}{\langle p_1\, p_2 \rangle \langle p_2\, p_3 \rangle \cdots \langle p_{n-1}\, p_n \rangle \langle p_n\, p_1 \rangle} \,, \tag{8.42}$$

where the convention of two gluons with negative helicity was chosen. Note that the bilinear operations in this formula are not inner products, but spinor bilinears (**??**) in the spinor-helicity formalism.

8.5 Renormalization

One of the biggest issues in (quantum) field theory are the divergences that arise everywhere in calculations involving loop diagrams. Renormalization theory tries to find a way around these (nonphysical) divergences.

8.5.1 Introduction: Statistical physics

Before introducing renormalization theory in the context of quatum field theory, it is helpful to study some applications in statistical physics, in particular in the study of lattice systems. To this end, this section will be focused on the study of the Ising model **??** on a lattice Λ:

$$\widehat{H} := -\sum_{\langle i,j\rangle\in\Lambda} J_{ij}\hat{S}_i\hat{S}_j - h\sum_{i\in\Lambda}\hat{S}_i\,. \tag{8.43}$$

8.5.2 Dimensional regularization

When calculating transition amplitudes and scattering cross-sections, three situations can arise:

1. Convergence: Only low-energy/low-momentum contributions matter.
2. UV divergence: These are sensivity to high-energy/high-momentum contributions, but only to these. Given a momentum cut-off, the integrals are fine.
3. Logarithmic divergence: These get contributions from both low- and high-energy scales and, hence, a cut-off does not solve the issue.

Consider the one-dimensional integral[4]

$$f_1(\theta) := \int_0^{+\infty} \frac{1}{x+\theta}\,dx\,. \tag{8.44}$$

For nonzero θ, this integral has no IR divergence, but the upper bound still leads to trouble since $\ln(x) \xrightarrow{x\longrightarrow\infty} +\infty$. To this end, one can (formally) analytically continue the integral into a space '$\mathbb{R}^{1-\varepsilon}$'. This leads to the following regularization scheme:

$$f_{1,\varepsilon}(\theta) := \int_0^{+\infty} \frac{x^{-\varepsilon}}{x+\theta}\,dx\,. \tag{8.45}$$

Looking at **??**, this integral can be seen to be

$$f_{1,\varepsilon}(\theta) = \frac{B(\varepsilon,1-\varepsilon)}{\theta^{\varepsilon}}\,. \tag{8.46}$$

Although this formula is finite for all $\theta\in\mathbb{R}$, it is now a function of ε and actually diverges for $\varepsilon\longrightarrow 0$ (which recovers the initial divergence).

This is where renormalization comes in, where counterms are subtracted to remove the dependence on ε and return a physical value. Here, the *on-shell renormalization scheme* is adopted:

$$f_1^R(\theta) := \lim_{\varepsilon\to 0}\big(f_{1,\varepsilon}(\theta) - f_{1,\varepsilon}(1)\big) = \lim_{\varepsilon\to 0} B(\varepsilon,1-\varepsilon)(\theta^{-\varepsilon}-1)\,. \tag{8.47}$$

[4] Any integrand $g(x)\sim O(\frac{1}{x})$ will do.

??, together with a Taylor expansion around $x = 0$, show that this is finite for $\varepsilon \longrightarrow 0$. If this were all, one could simply add counterterms for all such integrals. However, one also encounters interated integrals, corresponding to higher-order diagrams, and these should be regularized and renormalized as well. For a double integral, dimensional regularization gives:

$$f_{2,\varepsilon}(\theta) := \int_0^{+\infty} \int_0^{+\infty} \frac{x_1^{-\varepsilon}}{x_1 + \theta} \frac{x_2^{-\varepsilon}}{x_2 + x_1} \, dx_2 \, dx_1 = \int_0^{+\infty} \frac{x_1^{-\varepsilon}}{x_1 + \theta} f_{1,\varepsilon}(x_1) \, dx_1 \,. \tag{8.48}$$

Naively subtracting a counterterm as for f_1, will, however, not work in this case since there are two (logarithmic) divergences. One for fixed x_1, when $x_2 \longrightarrow +\infty$ and one when both $x_1, x_2 \longrightarrow +\infty$. The former is called a **subdivergence**.

Remark 8.5.1 (Nonlocality). The naive subtraction scheme $f_{2,\varepsilon}^R(\theta) := f_{2,\varepsilon}(\theta) - f_{2,\varepsilon}(1)$ can be found to be proportional to $\ln(\theta)$. In practice, the scale parameter θ is often related to the external momentum q^2. A term $\ln(q^2)$, however, points towards nonlocal behaviour since this term involves arbitrary high powers in q^2 or, after Fourier transforming back to position space, arbitrary high powers of a differential operator.

First, one has to remove subdivergences. For f_2, this goes as follows:

$$\overline{f}_{2,\varepsilon}(\theta) := f_{2,\varepsilon}(\theta) - f_{1,\varepsilon}(\theta) f_{1,\varepsilon}(1) \,. \tag{8.49}$$

A term consisting of the original function with the subdivergent factor set to 1, times that same contribution evaluated at $\theta = 1$ is subtracted. In the next section, it will be shown how this procedure is generalized to arbitrary (logarithmically) divergent integrals. In the next step, one applies the 'naive' subtraction scheme as for f_1:

$$f_2^R(\theta) := \lim_{\varepsilon \to 0} \left(\overline{f}_{2,\varepsilon}(\theta) - \overline{f}_{2,\varepsilon}(1) \right) . \tag{8.50}$$

To keep track of subdivergences, a diagrammatic method can be adopted. General integrals can be represented by rooted trees. Some examples are shown in Fig. 8.1. These trees correspond to the following divergent integrals:

- f_{t_1}:

$$\int_0^{+\infty} \frac{x^{-\varepsilon}}{x + \theta} \, dx \,. \tag{8.51}$$

- f_{t_2}:

$$\int_0^{+\infty} \frac{x_1^{-\varepsilon} f_{t_1}(x_1)}{x_1 + \theta} \, dx_1 \,. \tag{8.52}$$

- $f_{t_{3,1}}$:

$$\int_0^{+\infty} \frac{x_1^{-\varepsilon} f_{t_2}(x_1)}{x_1 + \theta} \, dx_1 \,. \tag{8.53}$$

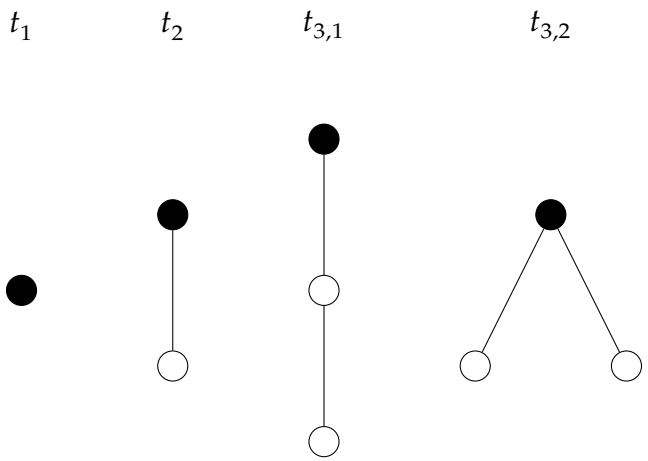


Figure 8.1: First four rooted trees.

- $f_{t_{3,2}}$:

$$\int_0^{+\infty} \frac{x_1^{-\varepsilon} f_{t_1}(x_1) f_{t_1}(x_1)}{x_1 + \theta} \, dx_1 \,. \tag{8.54}$$

8.5.3 Wilsonian renormalization

@@ COMPLETE @@

8.5.4 Connes–Kreimer renormalization

The trees by which divergences and subdivergences in Feynman diagrams can be organized, admit an interesting algebraic structure. Consider the free commutative algebra $\mathcal{H}_R$ on rooted trees, where multiplication corresponds to juxtaposition (creating a 'forest') with the empty tree as unit.

$\mathcal{H}_R$ admits more structure then simply the addition and multiplication. It will be equipped with a Hopf algebra structure (**??**). The operations are defined as follows:

- Counit:

$$\varepsilon(1) := 1 \qquad\qquad \varepsilon(T) := 0 \,. \tag{8.55}$$

- Coproduct:

$$\Delta(T) := T \otimes 1 + 1 \otimes T + \sum_c P_c(T) \otimes R_c(T) \,, \tag{8.56}$$

  where the last term is a sum over all possible combinations of cuts for which any path from a vertex to the root only passes through at most one cut, i.e. the **admissible cuts**.[5] Given a cut c, the **trunk** $R_c(T)$ is the part of the tree that was attached

[5] Although, in theory, the two trivial cuts, where either $R_c(T) = \emptyset$ or $R_c(T) = T$, are also admissible, these are not included.

to the root and the **branches** $P_c(T)$ is the forest given by all remaining components (this forest will consist of more than one tree if and only if c contained multiples cuts).

- Antipode (defined recursively):

$$S(1) := 1 \qquad S(T) := -T - \sum_c S\big(P_c(T)\big) R_c(T)\,. \tag{8.57}$$

Remark 8.5.2 (Sweedler notation). If the set of admissible cuts would be enlarged with the trivial options, the coproduct could be rewritten in Sweedler notation (**??**) as

$$\Delta(T) = \sum_{(T)} T_1 \otimes T_2\,. \tag{8.58}$$

For clarity's sake, this will not be done in this section unless explicitly noted.

Now, consider the operator $B_- : \mathcal{H}_R \to \mathcal{H}_R$ that removes the root from a tree (resulting in a forest) and the operator $B_+ : \mathcal{H}_R \to \mathcal{H}_R$ that takes a forest of trees and grafts all of them to a new root. These can easily seen to be inverses. With these operators, the coproduct can be rewritten as follows:

$$\Delta(T) = T \otimes 1 + (\mathbb{1} \otimes B_+) \circ \Delta \circ B_+(T)\,. \tag{8.59}$$

So, now, how to proceed? A Hopf algebra structure on Feynman graphs has been provided, but this does not say how the resulting Feynman amplitudes should be handled. These amplitudes, usually expressed through *Feynman rules*, are given by **characters** of $\mathcal{H}_R$, i.e. elements of $\mathbb{C}\mathsf{Alg}(\mathcal{H}_R, \mathbb{C})$. These form a group under the convolution product (**??**):

$$\phi_1 * \phi_2 = \mu_{\mathbb{C}} \circ (\phi_1 \otimes \phi_2) \circ \Delta\,. \tag{8.60}$$

Now, when regularizing these amplitudes, the characters do not take values anymore in $\mathbb{C}$ but in some $\mathbb{C}$-algebra A, such as the Laurent series $\mathbb{C}[[\varepsilon, \varepsilon^{-1}]]$ in the case of dimensional regularization, and, hence, one obtains the group of regularized characters $\mathbb{C}\mathsf{Alg}\big(\mathcal{H}_R, A\big)$.

The recursive expression for the antipode is the equation that ties in the Connes–Kreimer algebra with the substraction of subdivergent integrals from the previous section. The renormalized integrals are obtained through a recursive formula, akin to that of the antipode.

$$\phi^R(T) := S^R_\Phi * \phi(T)\,, \tag{8.61}$$

where S^R_ϕ is a deformation of $\phi \circ S$ obtained recursively as follows:

$$S^R_\phi(T) := -R\big[m \circ (S^R_\phi \otimes \pi) \circ \Delta\big]\,, \tag{8.62}$$

where $R : A \to A$ is a linear operator, m is the multiplication on A and $\pi := \mathbb{1}_{\mathcal{H}_R} - \eta \circ \varepsilon$ is the augmentation projection (the projection that kills nontrivial forests), or, more explicitly:

$$S_\phi^R(T) = -R\big[\phi(T)\big] - R\left[\sum_c S_\phi^R\big(P_c(T)\big)\phi\big(R_c(T)\big)\right]. \tag{8.63}$$

The assignment $\mathcal{R} : \phi \mapsto m \circ (S_\phi^R \otimes \pi) \circ \Delta$ is **Bogoliubov's R-operation**. Using the bialgebra laws, this can be rewritten as follows:

$$\phi^R(T) = \mathcal{R}[\phi](\Gamma) + S_\phi^R(\Gamma)\,. \tag{8.64}$$

Now, consider a monomial in the Lagrangian or, equivalently, an external leg structure $\underline{r}$. The total (combinatorial) Green's function corresponding to $\underline{r}$ is given by the formal sum of Feynman diagrams

$$\Gamma^{\underline{r}} := 1 + \sum_{\mathsf{res}(\Gamma)=\underline{r}} \alpha^{|\Gamma|} \frac{\Gamma}{\mathsf{sym}(\Gamma)}\,. \tag{8.65}$$

The renormalized term in the Lagrangian is then given by

$$Z^{\underline{r}} = S_\phi^R(\Gamma^{\underline{r}})\,. \tag{8.66}$$

The R-operation corresponds to the choice of renormalization scheme and isolates the singular part of the Feynman diagram, e.g.:

- On-shell renormalization:

$$R[f(\theta)] := f(1)\,. \tag{8.67}$$

- BPHZ[6]/MS[7] renormalization:@@ IS THIS CORRECT (or is this for DimReg?) @@

$$R[f(\theta)] := \mathsf{PolePart}_\varepsilon[f(1)]\,. \tag{8.68}$$

Property 8.5.3 (Butcher group). The Lie group $\mathbb{C}\mathsf{Alg}(\mathcal{H}_R, \mathbb{C})$ of characters has a Lie algebra the space of Feynman diagrams with the bracket induced by the pre-Lie algebra operation (8.35). This group is known as the (complex) Butcher group.[8]

Theorem 8.5.4 (Birkhoff decomposition). *Let $\mathcal{H}_R$ be the Hopf algebra of Feynman diagrams for a renormalizable pQFT and consider a commutative, unital Rota–Baxter algebra (A, R) of weight 1 (**??**) such that R is idempotent. Denote the space of A-regularized characters by $\mathcal{G}_A := \mathsf{Char}(\mathcal{H}_R, A)$ with unit $e := \eta_A \circ \varepsilon_{\mathcal{H}_R}$. The following statements hold:*

[6] *Bogoliubov, Parasiuk, Hepp* and *Zimmermann*

[7] minimal subtraction

[8] This group first arose in the combinatorial study of *Runga–Kutta method* in numerical analysis.

- $\big(\mathsf{Vect}_{\mathbb{C}}(\mathcal{H}_R, A), \mathcal{R}\big)$ *is a (complete filtered) Rota–Baxter algebra of weight 1, where* $\mathcal{R}(\phi) := R \circ \phi$.

- *For every* $\phi \in e + \mathcal{G}_A$, *there exists a unique decomposition*

$$\phi = \phi_-^{-1} * \phi_+ \,, \tag{8.69}$$

where $\phi_- \in e + \mathcal{R}(\mathcal{A}_1)$ *and* $\phi_+ \in e + \widetilde{\mathcal{R}}(\mathcal{A}_1)$, *with* $\widetilde{\mathcal{R}} := \mathbb{1} - \mathcal{R}$ *as in* **??**.

- *The Birkoff decomposition of* ϕ *is obtained recursively as*

$$\phi_- = e - \mathcal{R}\big(\phi_- * (\phi - e)\big) \qquad \phi_+ = e - \widetilde{\mathcal{R}}\big(\phi_+ * (\phi^{-1} - e)\big)\,. \tag{8.70}$$

These can be rewritten as follows when acting on a nontrivial Feynman diagram Γ*:*

$$\begin{aligned} \phi_-(\Gamma) &= -R\bigg(\phi(\Gamma) + \sum_{(\Gamma)} \phi_-(\Gamma_1)\phi(\Gamma_2)\bigg)\,, \\ \phi_+(\Gamma) &= \widetilde{R}\bigg(\phi(\Gamma) + \sum_{(\Gamma)} \phi_-(\Gamma_1)\phi(\Gamma_2)\bigg)\,, \end{aligned} \tag{8.71}$$

where Sweedler's notation (Remark 8.5.2) was used.

- *Spitzer's identity for noncommutative Rota–Baxter algebra (cf.* **??**) *implies that* ϕ_-, ϕ_+ *are algebra morphisms and that*

$$\phi_- = \exp^*\big(-\mathcal{R}\big(\chi(Z)\big)\big) \qquad \phi_+ = \exp^*\big(\widetilde{\mathcal{R}}\big(\chi(Z)\big)\big)\,, \tag{8.72}$$

where $Z \in \mathfrak{g}_A$ *is the generator of* ϕ *and* $\chi : \mathcal{A}_1 \to \mathcal{A}_1$ *is recursively defined as follows:*

$$\chi(Z) := Z - \mathsf{BCH}\big(\mathcal{R}(\chi(Z)), \widetilde{\mathcal{R}}(\chi(Z))\big)\,, \tag{8.73}$$

with BCH *the Baker–Campbell–Hausdorff operator (cf.* **??**)

$$\exp(x)\exp(y) = \exp\big(x + y + \mathsf{BCH}(x, y)\big)\,. \tag{8.74}$$

This theorem implies that the form of the regularized character $\phi_+ \equiv \phi^R$ and the counterterm $\phi_- \equiv S_\phi^R$ are completely fixed by the decomposable nature of (noncommutative) idempotent Rota–Baxter algebras of weight 1. The freedom in the choice of renormalization scheme only corresponds to the freedom in the exact choice of Rota–Baxter algebra A.

@@ COMPLETE @@

8.6 Entanglement in QFT

This section should be seen as a generalization of Chapter 2 to the continuum setting and, in particular, of the characterization and computation of entanglement.

8.6.1 Lattice theories

In this section, the most important definitions and constructions in ordinary quantum information theory are recalled and applied to a lattice theory. Taking the lattice spacing to zero will (formally) allow to extend the definitions to continuum field theories (up to some technicalities that will be explained when necessary). For simplicity, it will be assumed that the local Hilbert space is finite-dimensional.

Consider a bipartite subdivision $A \cup A^c$ of the lattice, given by a codimension-1 hypersurface ∂A, called the **entangling surface**. This induces a binary factorization of the total Hilbert space (all degrees of freedom are assumed to be confined to individual vertices) and, hence, one can compute the reduced density matrix for both A and its complement A^c. The eigenvalues, which solely depend on the entangling surface ∂A, allow to calculate the von Neumann entropy:[9]

$$S(\rho_A) := -\operatorname{tr}(\rho_A \ln \rho_A) = -\sum_i \rho_i \ln \rho_i \,. \tag{8.75}$$

In the same way, one can also introduce the Rényi q-entropy:

$$S_q(\rho_A) := \frac{1}{1-q} \ln\Big(\sum_i \rho_i^q\Big) \,. \tag{8.76}$$

Property 8.6.1 (Limiting case). First of all, one can analytically continue the definition of the q-entropy to arbitrary positive real numbers. The limit $q \longrightarrow 1$ coincides with the von Neumann entropy.

@@ COMPLETE (Ryu–Takayanagi, MERAs, ...) @@

[9] Certain assumptions ought to be made as to keep the entropy finite whenever the state-space is infinite-dimensional, since it can be shown that the set of states with infinite von Neumann entropy is trace norm-dense (see Eisert, Simon, and Plenio (2002)).

Part III

Appendices

Table of Contents

Appendix A ***G*-Structures** **94**

Appendix B Notes **97**

B.1 Noether's Theorem and Gauge-Gravity Duality by S. De Haro 97

B.1.1 Pseudotensors 97

B.1.2 AdS-CFT duality 99

B.2 A generalization of Noether's theorem and the information-theoretic approach to the study of symmetric dynamics by R. Spekkens 99

List of Symbols **101**

References **108**

Index **110**

Appendix A

G-Structures

In the following table, an overview of the most common G-structures on a smooth (simply-connected) manifold M^n is given.

Geometric structure	Structure group	Remarks
Orientation	$\mathsf{SL}(n,\mathbb{R})$	$\mathsf{GL}^+(n,\mathbb{R})$ is sufficient for orientability. The special linear group gives rise to a volume form.
Riemannian metric	$\mathsf{O}(n)$	
Almost-symplectic structure*	$\mathsf{Sp}(n,\mathbb{R})$	Integrability (in the form of a closed form) gives a symplectic manifold.
Almost-complex structure*	$\mathsf{GL}(k,\mathbb{C})$	Integrability (in the sense of Newlander–Nirenberg) gives a complex manifold.
Almost-Hermitian structure*	$\mathsf{U}(k)$	Integrability gives a Kähler manifold.
Calabi–Yau*	$\mathsf{SU}(k)$	
Hyper-Kähler**	$\mathsf{Sp}(k)$	Hyper-Kähler implies Calabi–Yau.
Almost quaternionic**	$(\mathsf{GL}(k,\mathbb{H})\times\mathbb{H}^\times)/\mathbb{R}^\times$	Integrability gives a quaternionic manifold. $k \geq 2$ is required because for $k = 1$ one would obtain that every orientable 4-manifold is quaternionic (amongst other things).
Quaternionic Kähler**	$(\mathsf{Sp}(k)\times\mathsf{Sp}(1))/\mathbb{Z}_2$	These manifolds are not strictly Kähler since the structure group is not a subgroup of $\mathsf{U}(2k)$.

Structures marked with $*$ require the real dimension $n = 2k$ to be even. Structures marked with $**$ require the real dimension $n = 4k$ to be a multiple of 4.

Remark. This table is strongly related to the classification of (*irreducible*, simply-connected and *nonsymmetric*) Riemannian manifolds by *Berger*. (The $\mathsf{SL}(n,\mathbb{R})$-structure is technically not part of the original classification since it is not a subgroup of $\mathsf{O}(n)$ and, hence, the manifold is not necessarily Riemannian.) A more general classification for mani-

folds that are not necessarily Riemannian was initiated by *Berger* and finished by others. This extension will only be mentioned here. For references, see Rudolph and Schmidt (2017).

Since not all concepts from this classification were defined throughout the compendium they are explained here:

- **Irreducible**: A Riemannian manifold is said to be irreducible if it is not locally isomorphic to a product of Riemannian manifolds.
- **Symmetric**: A smooth manifold, locally modelled on $V \cong \mathbb{R}^n$, is said to be symmetric (or **locally symmetric**) if the curvature mapping $FM \to \Lambda^2 V^* \otimes \mathfrak{g}$ is covariantly constant.

Remark A.1.1. Although most manifolds from the above list admit an explicit definition, the quaternionic Kähler manifolds are exactly defined by their structure group/holonomy group.

It is also clear that hyper-Kähler manifolds are a specific class of quaternionic Kähler manifolds since $\mathsf{Sp}(k)$ can be embedded in $\mathsf{Sp}(k) \cdot \mathsf{Sp}(1)$. To exclude this class, one can just require the holonomy group to be all of $\mathsf{Sp}(k) \cdot \mathsf{Sp}(1)$. This is equivalent to requiring that quaternionic Kähler manifolds should have a nonvanishing scalar curvature.

This remark is related to the following property.

Property A.1.2 (Einstein). Every quaternionic Kähler manifold is also Einstein (**??**). The hyper-Kähler manifolds are exactly the quaternionic Kähler manifolds with vanishing scalar curvature (which is constant since the manifold is Einstein).

Appendix B

Notes

This chapter contains notes written down during lunch talks, courses and conferences.

B.1 Noether's Theorem and Gauge-Gravity Duality by S. De Haro

Date & location: October 6 2018, London
Conference: *The Philosophy and Physics of Noether's Theorems*

B.1.1 Pseudotensors

Maxwell theory has a Noether stress-energy tensor of the form

$$T^{\mu}_{\ \nu} := F^{\mu\lambda}\partial A_{\lambda} - \frac{1}{4}\delta^{\mu}_{\nu}F^{\lambda\kappa}F_{\lambda\kappa} \tag{B.1}$$

and an associated weak conservation law

$$\partial_{\mu}T^{\mu\nu} \approx 0. \tag{B.2}$$

Through Schwarz's theorem **??**, this tensor can be enlarged to a conserved *Belinfante tensor*

$$\overline{T}^{\mu}_{\ \nu} := T^{\mu}_{\ \nu} + \partial_{\lambda}U^{[\mu\lambda]}_{\ \ \ \nu}. \tag{B.3}$$

Using such an extension, the standard form

$$T^{\mu}_{\ \nu} := F^{\mu\lambda}F_{\nu\lambda} - \frac{1}{4}\delta^{\mu}_{\nu}F^{\lambda\kappa}F_{\lambda\kappa} + \text{eom} \tag{B.4}$$

can be obtained.

Now, consider the theory coupled to gravity. In this case partial derivatives have to be replaced by covariant derivatives and the (weak) conservation law becomes:

$$\nabla_\mu T^{\mu\nu}\,. \tag{B.5}$$

However, this identity only contains the stress-energy tensor of matter, not of gravity itself. A possible solution, due to *Einstein*, was to construct a stress-energy (pseudo)tensor[1] from the Christoffel symbols, since these are already responsible for the coupling to gravity in the conservation law above:

$$\partial_\mu(\sqrt{g}T^\mu_{\ \nu} + t^\mu_{\ \nu}) = 0\,. \tag{B.6}$$

As above this leads to a **superpotential**:

$$\sqrt{g}T^\mu_{\ \nu} + t^\mu_{\ \nu} = \partial_\lambda s^{\mu\lambda}_{\ \ \ \nu}\,, \tag{B.7}$$

where $\partial_\mu\partial_\lambda s^{\mu\lambda}_{\ \ \ \nu} = 0$. All conservation laws will be preserved if $s^{\mu\lambda}_{\ \ \ \nu} = s^{[\mu\lambda]}_{\ \ \ \nu}$.

The main issues with this new 'stress-energy tensor' are:

- it is not a tensor, and
- there are an infinite number of possibilities.

However, the superpotential can be related to boundary conditions and, therefore, has a physical intepretation. Consider the Hamiltonian

$$\overline{H}(n) := \int_\Sigma n^\mu H_\mu + \oint_{\partial\Sigma} B(n)\,, \tag{B.8}$$

where n is the ADM-like shift vector that generates tangential motion along the spacelike hypersurface Σ and $B(n) \sim n^\nu s^{\mu\lambda}_{\ \ \ \nu}$. Noether's theorem implies that H_μ is proportional to the EOM, which implies that $H(n)$ is a boundary term determined by the superpotential. It determines the quasilocal energy.

A different approach is the **Brown–York pseudotensor**. Here, a bounded spacetime region M is considered with two spacelike boundaries, the initial and final slices $\Sigma_\pm$, and a timelike hypersurface N. The matter-coupled gravitational action is given by:

$$S = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}R + \frac{1}{\kappa}\int_{\Sigma_\pm} d^3x\sqrt{h}K - \frac{1}{\kappa}\int_N d^3x\sqrt{-\gamma}\Theta + S_{\text{matter}} + S_{\text{ref}}\,, \tag{B.9}$$

where S_{ref} is a reference action to regularize the action. Variation with respect to the metric gives

$$\delta S = -\int_M d^4x E^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2}\int_{\Sigma_\pm} d^3x\sqrt{h}P_{ij}\delta h_{ij} + \frac{1}{2}\int_N d^3x\sqrt{\gamma}\tau^{ij}_{\text{BY}}\delta\gamma_{ij} + \text{matter terms}\,. \tag{B.10}$$

[1] Pseudotensor here just means an object that does not transform tensorially.

The first term gives the Einstein field equations and vanishes on-shell. The other terms define the conjugate momenta. As the notation implied, the object τ^{ij}_{BY} is the (quasilocal) Brown–York stress-energy tensor. For pure gravity this can be written as follows:

$$\tau^{ij}_{\text{BY}} = -\frac{1}{\kappa\sqrt{\gamma}}(\Theta\gamma^{ij} - \Theta^{ij}) + \text{regularizing terms}\,. \tag{B.11}$$

B.1.2 AdS-CFT duality

Now, consider an anti-de Sitter spacetime M. In the CFT description, the stress-energy tensor can be obtained by a functional derivative of the partition function with respect to some fixed background metric $g_{(0)}$:

$$\langle T_{ij}(x)\rangle = \frac{2}{\sqrt{g_{(0)}}}\frac{\delta W[g_{(0)}]}{\delta g^{ij}_{(0)}(x)}\,. \tag{B.12}$$

In the gravitational description the fixed metric rerepsents the asymptotic metric (up to conformal factors). A theorem by *Fefferman–Graham* says that near the boundary, the metric admits a local Poincaré form:

$$ds^2 = \frac{l^2}{r^2}(dr^2 + g_{ij}(r,x)dx^i dx^j)\,, \tag{B.13}$$

with $g_{ij}(0,x) = g_{(0)ij}(x)$. Holographic duality then shows that the qausilocal Brown–York tensor is equal to the holographic stress-energy tensor.

B.2 A generalization of Noether's theorem and the information-theoretic approach to the study of symmetric dynamics by R. Spekkens

Date & location: October 6 2018, London
Conference: *The Philosophy and Physics of Noether's Theorems*

The general idea of this talk is that (quantum) entanglement can be considered from a *resource theory* perspective. In this approach quantum entanglement is a **resource** and LOCC operations are the free operations, i.e. given sufficient entanglement they can be used to perform any quantum operation.

To characterize asymmetry measures, the following principle is adopted.
Axiom B.1 (Curie's principle). *The symmetries of the cause are to be found in the effect*, i.e. the effect is at least as symmetric as the cause.

Now, consider an **asymmetry measure**, i.e. a function $\mu : S(A) \to \mathbb{R}$ such that $\mu(\rho) \geq \mu(\Phi(\rho))$ for all symmetric quantum channels Φ (quantum channels that commute with

symmetry operations). For general dynamics (including open systems), where Noether's theorem does not apply, asymmetry measures give constraints on allowed evolutions. When restricting to closed systems, these measures turn into conserved quantities and for mixed states they are independent of Noether charges.

List of Symbols

The following abbreviations and symbols are used throughout the compendium.

Abbreviations

AIC	Akaike information criterion
ARMA	autoregressive moving-average model
BCH	Baker–Campbell–Hausdorff
BPS	Bogomol'nyi–Prasad–Sommerfield
BPST	Belavin–Polyakov–Schwarz–Tyupkin
BRST	Becchi–Rouet–Stora–Tyutin
CCR	canonical commutation relation
CDF	cumulative distribution function
CFT	conformal field theory
CIS	completely integrable system
CP	completely positive
CPTP	completely positive, trace-preserving
CR	Cauchy–Riemann
dga	differential graded algebra
dgca	differential graded-commutative algebra
EMM	equivalent martingale measure
EPR	Einstein–Podolsky–Rosen
ESM	equivalent separating measure
ETCS	Elementary Theory of the Category of Sets
FIP	finite intersection property
FWHM	full width at half maximum
GA	geometric algebra
GHZ	Greenberger–Horne–Zeilinger

GNS	Gel'fand–Naimark–Segal
HJE	Hamilton–Jacobi equation
HoTT	Homotopy Type Theory
KKT	Karush–Kuhn–Tucker
LIVF	left-invariant vector field
MCG	mapping class group
MPO	matrix-product operator
MPS	matrix-product state
MTC	modular tensor category
NDR	neighbourhood deformation retract
OPE	operator product expansion
OTC	over the counter
OZI	Okubo–Zweig–Iizuka
PAC	probably approximately correct
PDF	probability density function
PID	principal ideal domain
PL	piecewise-linear
PMF	probability mass function
POVM	positive operator-valued measure
PRP	predictable representation property
PVM	projection-valued measure
RKHS	reproducing kernel Hilbert space
SVM	support-vector machine
TDSE	time-dependent Schrödinger equation
TISE	time-independent Schrödinger equation
TQFT	topological quantum field theory
TVS	topological vector space
UFD	unique factorization domain
VC	Vapnik–Chervonenkis
VIF	variance inflation factor
VOA	vertex operator algebra
WKB	Wentzel–Kramers–Brillouin

ZFC	Zermelo–Frenkel set theory with the axiom of choice

Operations

Ad_g	adjoint representation of a Lie group G
ad_X	adjoint representation of a Lie algebra $\mathfrak{g}$
arg	argument of a complex number
$\Box$	d'Alembert operator
$\mathsf{deg}(f)$	degree of a polynomial f
e	identity element of a group
$\Gamma(E)$	set of global sections of a fibre bundle E
Im, $\Im$	imaginary part of a complex number
$\mathsf{Ind}_f(z)$	index of a point $z \in \mathbb{C}$ with respect to a function f
$\hookrightarrow$	injective function
$\cong$	is isomorphic to
$A \multimap B$	linear implication
$N \lhd G$	N is a normal subgroup of G
Par_t^γ	parallel transport map along a curve γ
Re, $\Re$	real part of a complex number
Res	residue of a complex function
$\twoheadrightarrow$	surjective function
$\{\cdot,\cdot\}$	Poisson bracket
$X \pitchfork Y$	transversally intersecting manifolds X, Y
∂X	boundary of a topological space X
$\overline{X}$	closure of a topological space X
X°, $\mathring{X}$	interior of a topological space X
$\sphericalangle(\cdot,\cdot)$	angle between two vectors
$X \times Y$	cartesian product of two sets X, Y
$X + Y$	sum of two vector spaces X, Y
$X \oplus Y$	direct sum of two vector spaces X, Y
$V \otimes W$	tensor product of two vector spaces V, W
$\mathbb{1}_X$	identity morphism on an object X
$\approx$	is approximately equal to

$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto

Objects

Ab	category of Abelian groups
$\mathsf{Aut}(X)$	automorphism group of an object X
$\mathcal{B}_0(V,W)$	space of compact bounded operators between two Banach spaces V, W
$\mathcal{B}_1(\mathcal{H})$	space of trace-class operators on a Hilbert space
$\mathcal{B}(V,W)$	space of bounded linear maps between two vector spaces V, W
CartSp	category of Euclidean spaces and 'suitable' morphisms (e.g. linear maps, smooth maps, ...)
$C(X,Y)$	set of continuous functions between two topological spaces X, Y
S'	centralizer of a subset (of a ring)
$C_\bullet$	chain complex
$\mathsf{Ch}(\mathsf{A})$	category of chain complexes with objects in an additive category A
C^∞, $\mathsf{SmoothSet}$	category of smooth sets
$C^\infty_p(M)$	ring of smooth functions $f : M \to \mathbb{R}$ on a neighbourhood of $p \in M$
$C\ell(A,Q)$	Clifford algebra over an algebra A induced by a quadratic form Q
$C^\omega(V)$	set of all analytic functions defined on a set V
$\mathsf{Conf}(M)$	conformal group of a (pseudo-)Riemannian manifold M
$\mathsf{C}^\infty\mathsf{Ring}$, $\mathsf{C}^\infty\mathsf{Alg}$	category of smooth algebras
$S_k(\Gamma)$	space of cusp forms of weight $k \in \mathbb{R}$
Δ_X	diagonal of a set X
Diff	category of smooth manifolds
DiffSp	category of diffeological spaces and smooth maps
$\mathcal{D}_M$	sheaf of differential operators
D^n	standard n-disk
$\mathsf{dom}(f)$	domain of a function f
$\mathsf{End}(X)$	endomorphism monoid of an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad
$\mathsf{FormalCartSp}_{\mathsf{diff}}$	category of infinitesimally thickened Euclidean spaces

$\mathsf{Frac}(I)$	field of fractions of an integral domain I
$\mathfrak{F}(V)$	space of Fredholm operators on a Banach space V
$\mathbb{G}_a$	additive group (scheme)
$\mathsf{GL}(V)$	general linear group: group of automorphisms of a vector space V
$\mathsf{GL}(n, \mathfrak{K})$	general linear group: group of invertible $n \times n$-matrices over a field $\mathfrak{K}$
Grp	category of groups and group homomorphisms
Grpd	category of groupoids
$\mathsf{Hol}_p(\omega)$	holonomy group at a point p with respect to a principal connection ω
$\mathsf{Hom}_{\mathsf{C}}(V, W)$, $\mathsf{C}(V, W)$	collection of morphisms between two objects V, W in a category C
hTop	homotopy category
$I(S)$	vanishing ideal on an algebraic set S
$I(x)$	rational fractions over an integral domain I
$\mathsf{im}(f)$	image of a function f
$K^0(X)$	K-theory over a (compact Hausdorff) space X
Kan	category of Kan complexes
$K(A)$	Grothendieck completion of a monoid A
$\mathcal{K}_n(A, v)$	Krylov subspace of dimension n generated by a matrix A and a vector v
L^1	space of integrable functions
Law	category of Lawvere theories
Lie	category of Lie groups
$\mathfrak{Lie}$	category of Lie algebras
$\mathfrak{X}^L$	space of left-invariant vector fields on a Lie group
$\mathsf{llp}(I)$	set of morphisms having the left lifting property with respect to I
LX	free loop space on a topological space X
Man^p	category of C^p-manifolds
Meas	• category of measurable spaces and measurable functions, or • category of measure spaces and measure-preserving functions
M^4	four-dimensional Minkowski space
$M_k(\Gamma)$	space of modular forms of weight $k \in \mathbb{R}$
$\mathbb{F}^X$	natural filtration of a stochastic process $(X_t)_{t \in T}$
$N\mathsf{C}$	simplicial nerve of a small category C

$\mathsf{O}(n, \mathfrak{K})$	group of $n \times n$ orthogonal matrices over a field $\mathfrak{K}$
$\mathsf{Open}(X)$	category of open subsets of a topological space X
$P(X), 2^X$	power set of a set X
$\mathsf{Pin}(V)$	pin group of the Clifford algebra $C\ell(V, Q)$
$\mathsf{Psh}(\mathsf{C}), \widehat{\mathsf{C}}$	category of presheaves on a (small) category C
$R((x))$	ring of (formal) Laurent series in x with coefficients in R
$\mathsf{rlp}(I)$	set of morphisms having the right lifting property with respect to I
$R[[x]]$	ring of (formal) power series in x with coefficients in R
S^n	standard n-sphere
$S^n(V)$	space of symmetric rank n tensors over a vector space V
$\mathsf{Sh}(X)$	category of sheaves on a topological space X
$\mathsf{Sh}(\mathsf{C}, J)$	category of J-sheaves on a site (C, J)
$\boldsymbol{\Delta}$	simplex category
$\mathsf{sing\ supp}(\phi)$	singular support of a distribution ϕ
$\mathsf{SL}_n(\mathfrak{K})$	special linear group: group of all $n \times n$-matrices with unit determinant over a field $\mathfrak{K}$
$W^{m,p}(U)$	Sobolov space in L^p of order m
$\mathsf{Span}(\mathsf{C})$	span category over a category C
$\mathsf{Spec}(R)$	spectrum of a commutative ring R
$\mathsf{sSet}_{\mathrm{Quillen}}$	Quillen's model structure on simplicial sets
$\mathsf{supp}(f)$	support of a function f
$\mathsf{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
$\mathsf{Sym}(X)$	symmetric group of a set X
S_n	symmetric group of degree n
$\mathsf{Sym}(X)$	symmetric group on a set X
$\mathsf{Sp}(n, \mathfrak{K})$	group of matrices preserving a canonical symplectic form over a field $\mathfrak{K}$
$\mathsf{Sp}(n)$	compact symplectic group
$\mathbb{T}^n$	standard n-torus (n-fold Cartesian product of S^1)
$T_{\leq t}$	set of all elements smaller than (or equal to) $t \in T$ for a partial order T
$\mathsf{TL}_n(\delta)$	Temperley–Lieb algebra with $n-1$ generators and parameter δ
Top	category of topological spaces and continuous functions
Topos	(2-)category of (elementary) topoi and geometric morphisms

$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\mathsf{U}(n, \mathfrak{K})$	group of $n \times n$ unitary matrices over a field $\mathfrak{K}$
$V(I)$	algebraic set corresponding to an ideal I
$\mathsf{Vect}(X)$	category of vector bundles over a manifold X
$\mathsf{Vect}_{\mathfrak{K}}$	category of vector spaces and linear maps over a field $\mathfrak{K}$
Y^X	set of functions between two sets X, Y
$\mathbb{Z}_p$	group of p-adic integers
$\emptyset$	empty set
$\pi_n(X, x_0)$	n^{th} homotopy space over X with basepoint x_0
$[a, b]$	closed interval
$]a, b[$	open interval
$\Lambda^n(V)$	space of antisymmetric rank-n tensors over a vector space V
ΩX	(based) loop space on a topological space X
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on a manifold M
$\rho(A)$	resolvent set of a bounded linear operator A
$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on a manifold M

Units

C	Coulomb
T	Tesla

References

Baez, J. C., Segal, I. E., & Zhou, Z. (2014). *Introduction to Algebraic and Constructive Quantum Field Theory*. Princeton University Press.

Bates, S., & Weinstein, A. (1997). *Lectures on the Geometry of Quantization* (Vol. 8). American Mathematical Soc.

Bransden, B. H., & Joachain, C. J. (2000). *Quantum Mechanics*. Prentice Hall.

Brylinski, J.-L. (1993). *Loop Spaces, Characteristic Classes and Geometric Quantization*. Birkhauser.

Camosso, S. (2021). Prequantization, Geometric Quantization, Corrected Geometric Quantization. *Journal of Applied Mathematics and Physics*, *09*(09), 2290–2320.

Connes, A., & Kreimer, D. (1998). Hopf Algebras, Renormalization and Noncommutative Geometry. *Communications in Mathematical Physics*, *199*(1), 203–242.

De La Hamette, A.-C., & Galley, T. D. (2020). Quantum reference frames for general symmetry groups. *Quantum*, *4*, 367.

Ebrahimi-Fard, K., & Kreimer, D. (2005). The Hopf algebra approach to Feynman diagram calculations. *Journal of Physics A: Mathematical and General*, *38*(50), R385.

Eisert, J., Simon, C., & Plenio, M. B. (2002). On the Quantification of Entanglement in Infinite-dimensional Quantum Systems. *Journal of Physics A: Mathematical and General*, *35*(17), 3911–3923.

Girard, J.-Y. (1995). Linear Logic: its syntax and semantics. In *Advances in Linear Logic* (1st ed., pp. 1–42). Cambridge University Press.

Greiner, W., & Reinhardt, J. (1996). *Field Quantization*. Springer.

Hawkins, E. (1998). *The Correspondence between Geometric Quantization and Formal Deformation Quantization.* arXiv. http://arxiv.org/abs/math/9811049

Hawkins, E. (2000). Geometric Quantization of Vector Bundles and the Correspondence with Deformation Quantization. *Communications in Mathematical Physics*, *215*(2), 409–432.

Misner, C. W., Thorne, K. S., & Wheeler, J. A. (2017). *Gravitation*. Princeton University Press.

Miwa, T., Jimbo, M., Jimbo, M., & Date, E. (2000). *Solitons: Differential Equations, Symmetries and Infinite-dimensional Algebras* (Vol. 135). Cambridge University Press.

Moretti, V. (2016). Mathematical Foundations of Quantum Mechanics: An Advanced

Short Course. *International Journal of Geometric Methods in Modern Physics*, *13*.

Peskin, M. E., & Schroeder, D. V. (1995). *An Introduction to Quantum Field Theory*. Westview Press.

Rangamani, M., & Takayanagi, T. (2017). *Holographic Entanglement Entropy*. Springer.

Rovelli, C., & Vidotto, F. (2014). *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory*. Cambridge University Press.

Rudolph, G., & Schmidt, M. (2017). *Differential Geometry and Mathematical Physics: Part II. Fibre Bundles, Topology and Gauge Fields*. Springer.

Schuller, F. (2016). *Lectures on the Geometric Anatomy of Theoretical Physics*. https://www.youtube.com/channel/UC6SaWe7xeOp31Vo8cQG1oXw

Stone, A. D. (2005). Einstein's unknown insight and the problem of quantizing chaos. *Physics Today*, *58*(8), 37–43.

Tuybens, B. (2017). *Entanglement Entropy of Gauge Theories* (PhD Thesis). Ghent University.

Tuynman, G. M. (2016). The metaplectic correction in geometric quantization. *Journal of Geometry and Physics*, *106*, 401–426.

Yeats, K. (2017). *A Combinatorial Perspective on Quantum Field Theory* (Vol. 15). Springer.

Index

action
  Einstein–Hilbert, 72
  Holst, 74
  spooky, 33
adiabatic
  approximation, 59
  theorem, 59
admissible
  solution, 28, 30
Alexandrov
  topology, 42
amplituhedron, 84
anomaly, 54
automorphism
  quantum, 19

Baker–Campbell–Hausdorff formula, 90
Barbero–Immirzi constant, 74
Bargmann, *see* Segal–Bargmann
Belinfante tensor, 97
Bell state, 33
Berry
  connection, 60
  phase, 59
Birkhoff, 72
  decomposition, 89
Bogoliubov
  R-operation, 89
Bohr
  topos, 42
Bohr–Sommerfeld condition, 47, 51
Bohr–Sommerfeld leaf, 51
Bohr–Sommerfeld variety, 51
Bohrification, 43
Born
  approximation, 62
  rule, 12, 18
  series, 62
Born–Oppenheimer approximation, 17
branch, 88
Brown–York pseudotensor, 98
Butcher group, 89

C*-algebra
  bundle, 52
canonical
  commutation relation, 17
Casimir
  invariant, 25
causal, 70
character, 88
Choi–Jamiołkowski, 36
Clebsch–Gordan coefficient, 27
closure
  relation, 11
coherence, 23
collapse, 12, 19
Connes–Kreimer renormalization, 87
context, 44
contraction, 37, 79
cross

section, 61
Curie
principle, 99

d'Alembert
operator, 76
daseinization, 44
deformation quantization
Fedosov, 47
Moyal, 47
degree, 82
of divergence, 83
dense
coding, 33
density
operator, 34
dimension
mass, 83
Dirac
condition, 47
correspondence, 45
Dirac–von Neumann postulates, 11, 18
divergence, 85
Dyson series, 14

edge, 82
Ehrenfest, 15
Einstein
energy relation, 67
field equations, 71
manifold, 96
Einstein–Podolsky–Rosen, *see* Bell state
energy, 67
entanglement, 33
entangling surface, 91
entropy
Rényi, 91
evolution, 12, 19
expectation, 12

Fermi
golden rule, 61
Feynman
diagram, 82
propagator, 79, 80
forest, 87
Fubini–Study metric, 19

Gel'fand
spectrum, 43
triple, 28
geometric
solution, 31
Gleason, 43
grafting, 87
graph, 82
1PI, 82
Green–Liouville method, *see* WKB approximation
Greenberger–Horne–Zeilinger state, 34
Groenewold–Van Hove, 46

half-
edge, 82
Hamilton
function, 13
Hardy space, 53
Heisenberg
cut, 20
group, 46
system, 18
uncertainty relation, 12
Hilbert
rigged Hilbert space, 16, 28
holonomy, 60
hyperbolic
manifold, 71

identity
operator, 78
inference rule, 37
interaction
picture, 57
irreducible

graph, 82
Isserlis, 80

Klein–Gordon equation, 76
Kochen–Specker, 43
Kontsevich quantization, 47

ladder operators, 25
light cone
coordinates, 68
lightlike, 70
Lippman–Schwinger equation, 61
Lorentz
factor, 66

Møller operator, 58
manifold
linear Poisson, 47
mass, 67
matrix
adjacency, 82
measurement, 12, 19
positive-operator valued, 35
metaplectic
correction, 50
modality, 41
momentum, 67

Naimark dilation theorem, 36
no-cloning, 33
no-deleting, 33
normal
ordering, 79
notation
braket, 11
Dirac, 11
nucleus, 43
null
vector, 70

observable, 11, 18, 44, 55
compatibility, 11
orientation
time, 70
OZI rule, 81

Palatini action, 74
Parke–Taylor formula, 84
Pauli
matrix, 26
Pauli–Jordan function, 78
permanent, 20
perturbation, 57
phase space, 43
Piron, 36
Plebanski form, 74
Poisson
vector space, 47
polarization, 49
pre-
quantum line bundle, 47
pure, 34
purification, 35

quantization, *see also* deformation
geometric, 49
half-form, 50
Kähler, 50
quantum
channel, 36

rank
Schmidt, 32
Rayleigh–Schrödinger theory, 58
reference frame, 24
quantum, 22
Reissner–Nordström metric, 72
renormalizability, 84
renormalization
BPHZ, 89
minimal subtraction, 89
on-shell, 85, 89
resolution
of the identity, 11

Robertson, *see* Heisenberg uncertainty relation
Runge–Kutta method, 89

Schmidt
  decomposition, 32
Schrödinger
  equation, 13, 57
  representation, 18
Schwarzschild metric, 72
Segal–Bargmann representation, 50
Segal–Kostant–Souriau procedure, 48
semiclassical, 28
separable
  state, 33
Slater
  determinant, 20
spacelike, 70
spectral
  presheaf, 43
sphere
  celestial, 71
spin, 48
  connection, 73
  orbitals, 20
spin-statistics theorem, 20
Spitzer's identity, 90
star product, 46
state, 11, 18, 19, 44
Stone–von Neumann, 17
stress-energy tensor, 71
subdivergence, 86
super-
  operator, 36
symmetrization postulate, 20
symmetry
  factor, 81, 82
  quantum, 19

tetrad, 73
time ordering, 14
timelike, 70
Toeplitz
  operator, 53
  quantization, 53
Tomonaga–Schwinger equation, 58
topology
  algebraic, 18
  double negation, 43
transport equation, 28
trunk, 88

velocity, 67
vertex, 82
vielbein, 73

Weil
  integrality, 47
Weyl
  relations, 17
  system, 17
Wick, 79
Wigner, 19
  D-function, 25
  friend, 22
WKB approximation, 27

Authors

Bohr, 21

Everett, 21

Gleason, 43
Green, 27

Kochen, 43

Liouville, 27

Parke, 84

Specker, 43

Taylor, 84

von Neumann, 18, 36

Wigner, 19