Compendium of Mathematics & Physics

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Contents

Introduction

Goals

This compendium originated out of the necessity for a compact summary of important theorems and formulas during physics and mathematics classes at university. When the interest in more (and more exotic) subjects grew, this collection lost its compactness and became the chaos it now is. Although there should exist some kind of overall structure, it was not always possible to keep every section self-contained or respect the order of the chapters.

It should definitely not be used as a formal introduction to any subject. It is neither a complete work nor a fact-checked one, so the usefulness and correctness is not guaranteed. However, it can be used as a look-up table for theorems and formulas, and as a guide to the literature. To this end, each chapter begins with a list of useful references. At the same time, only a small number of statements are proven in the text (or appendices). This was done to keep the text as concise as possible (a failed endeavour). However, in some cases the major ideas underlying the proofs are provided.

Structure and conventions

Sections and statements that require more advanced concepts, in particular concepts from later chapters or (higher) category theory, will be labelled by the *clubs* symbol ♣. Some definitions, properties or formulas are given with a proof or an extended explanation whenever I felt like it. These are always contained in a blue frame to make it clear that they are not part of the general compendium. When a section uses notions or results from a different chapter at its core, this will be recalled in a green box at the beginning of the section.

Definitions in the body of the text will be indicated by the use of **bold font**. Notions that have not been defined in this summary but that are relevant or that will be defined further on in the compendium (in which case a reference will be provided) are indicated by *italic text*. Names of authors are also written in *italic*.

Objects from a general category will be denoted by a lower-case letter (depending on the context, upper-case might be used for clarity), functors will be denoted by upper-case letters and the categories themselves will be denoted by symbols in **bold font**. In the later chapters on physics, specific conventions for the different types of vectors will often be adopted. Vectors in Euclidean space will be denoted by a bold font letter with an arrow above, e.g. $\vec{\boldsymbol{a}}$, whereas vectors in Minkowski space (4-vectors) and differential forms will be written without the arrow, e.g. $\boldsymbol{a}$. Matrices and tensors will always be represented by capital letters and, dependent on the context, a specific font will be adopted.

Part I

Quantum Theory

Table of Contents

Chapter 1

Quantum Mechanics

The main reference for this chapter is Bransden and Joachain (2000). In the first two sections, the two basic formalisms of quantum mechanics are introduced: wave and matrix mechanics. The main reference for the mathematically rigorous treatment of quantum mechanics, in particular in the infinite-dimensional setting, is Moretti (2016). The main reference for the generalization to curved backgrounds is Schuller (2016). The section on the WKB approximation is based on Bates and Weinstein (1997). Relevant chapters in this compendium are, amongst others, **??**, **??** and **??**.

Chapter Contents

1.1 Introduction

This section will give both an introduction and formal treatment of the objects and notions used in quantum mechanics.

1.1.1 Dirac–von Neumann postulates

Axiom 1.1 (States). The states of a (closed) system are represented by vectors in a (complex) Hilbert space $\mathcal{H}$. In the infinite-dimensional setting, one often further restricts to separable spaces, i.e. the spaces are required to admit a countable Hilbert basis.

Notation 1.1.1 (Dirac notation). State vectors $|\psi\rangle$ are called **ket**'s and their duals $\langle\psi|$ are called **bra**'s. The inner product of a state $|\phi\rangle$ and a state $|\psi\rangle$ is denoted by $\langle\phi\,|\,\psi\rangle$. This notation is often called the **braket notation** (or Dirac notation).

Axiom 1.2 (Observables). Every physical property is represented by a bounded, self-adjoint operator. In the finite-dimensional case, this is equivalent to an operator that admits a complete set of eigenfunctions.

Definition 1.1.2 (Compatible observables). Two observables are said to be compatible if they share a complete set of eigenvectors.

Formula 1.1.3 (Closure relation). For a complete set of eigenvectors, the closure relation (also called the **resolution of the identity**) is given by (see also **??**)

$$\sum_n |\psi_n\rangle\langle\psi_n| + \int_X |x\rangle\langle x|\,dx = \mathbb{1}\,, \tag{1.1}$$

where the sum ranges over the discrete spectrum and the integral over the continuous spectrum. For simplicity, the summation will also be used for the continuous part.

Axiom 1.3 (Born rule). Let $\mathcal{H}$ be the Hilbert space of a physical system and consider an observable $\widehat{O}$. If $|\psi\rangle$ is a state vector and $\widehat{P}_\phi$ is the projection onto an eigenvector $|\phi\rangle$ of $\widehat{O}$, the probability of observing the state $|\phi\rangle$ is given by:

$$\frac{\langle\psi\mid\widehat{P}_\phi\mid\psi\rangle}{\langle\psi\mid\psi\rangle}=\frac{|\langle\psi\mid\phi\rangle|^2}{\langle\psi\mid\psi\rangle}\,. \tag{1.2}$$

Property 1.1.4 (Projectivization). In light of the Born rule, the dynamics of a system does not depend on the global phase or normalization, i.e. states are represented by rays in a projective Hilbert space $\mathcal{H}\mathbb{P}$ (**??**).

Combining Born's rule with **??**, gives the following definition.

Definition 1.1.5 (Expectation value). The expectation value of an observable $\widehat{A}$ in a (normalized) state $|\psi\rangle$ is defined as follows:

$$\langle\widehat{A}\rangle_\psi:=\langle\psi|\widehat{A}|\psi\rangle\,. \tag{1.3}$$

The subscript ψ is often left implicit. As in ordinary statistics (**??**), the uncertainty or variance is defined as follows:

$$\Delta A:=\langle\widehat{A}^2\rangle-\langle\widehat{A}\rangle^2\,. \tag{1.4}$$

Formula 1.1.6 (Uncertainty relation). Let $\widehat{A},\hat{B}$ be two observables and let $\Delta A,\Delta B$ be the corresponding uncertainties. The (**Robertson**) uncertainty relation reads as follows:

$$\Delta A\Delta B\geq\frac{1}{4}\Big|\Big\langle\big[\widehat{A},\hat{B}\big]\Big\rangle\Big|^2\,. \tag{1.5}$$

Axiom 1.4 (Projection[1]). Let $\mathcal{H}$ be the Hilbert space of a physical system and consider an observable $\widehat{O}$ with eigenvalues $\{o_i\}_{i\in I}$. After measuring the observable $\widehat{O}$ in the state $|\psi\rangle$, the outcome will be one of the eigenvalues o_i and system will 'collapse' to, i.e. get projected onto, the eigenstate $\widehat{P}_{o_i}|\psi\rangle\equiv|o_i\rangle$.

Axiom 1.5 (Unitary evolution). The evolution of a closed system is unitary, i.e. there exists a unitary operator $\widehat{U}(t,t')\in\mathrm{Aut}(\mathcal{H})$, for all times $t\leq t'$, such that

$$|\psi(t')\rangle=\widehat{U}(t,t')|\psi(t)\rangle\,. \tag{1.6}$$

1.2 Schrödinger picture

Since the energy is of paramount importance in physics, the associated eigenvalue equation deserves its own name.

[1] Also called the **measurement postulate**.

Formula 1.2.1 (Time-independent Schrödinger equation).

$$\widehat{H}|\psi\rangle = E|\psi\rangle \tag{1.7}$$

The operator $\widehat{H}$ is called the **Hamiltonian** of the system. The wave function ψ is an element of the vector space $L^2(\mathbb{R},\mathbb{C})\otimes\mathcal{H}$ with $\mathcal{H}$ the internal Hilbert space (describing, for example, the spin or charge of a particle). This is an eigenvalue equation for the energy levels of the system.

@@ INTRODUCE POSITION/CONFIGURATION REPRESENTATION @@

The time evolution of a wave function was governed by Axiom 1.5. By passing to generators, the following equation is obtained.

Formula 1.2.2 (Time-dependent Schrödinger equation).

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle\,. \tag{1.8}$$

In case $\widehat{H}$ is time independent, the TISE can be obtained from this equation by separation of variables (see below).

Proof (Derivation of TISE from TDSE). Starting from the one-dimensional TDSE in position space with a time-independent Hamiltonian, one can perform a separation of variables and assert a solution of the form $\psi(x,t) = X(x)T(t)$. Inserting this in the previous equation gives

$$i\hbar X(x)T'(t) = \big(\widehat{H}X(x)\big)T(t)\,.$$

Dividing both sides by $X(x)T(t)$ and rearranging the terms gives

$$i\hbar\frac{T'(t)}{T(t)} = \frac{\widehat{H}X(x)}{X(x)}\,.$$

Because the left side only depends on t and the right side only depends on x, one can conclude that they both have to equal a constant $E\in\mathbb{C}$. This leads to the following system of differential equations:

$$\begin{cases} i\hbar T'(t) = ET(t)\,,\\ \widehat{H}X(x) = EX(x)\,. \end{cases}$$

The first equation immediately gives a solution for T:

$$T(t) = C\exp\left(-\frac{iE}{\hbar}t\right). \tag{1.9}$$

The second equation is exactly the TISE (Formula 1.2.1). □

Example 1.2.3 (Massive particle in a stationary potential).

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right)\psi(x,t) \tag{1.10}$$

In this case, the TISE reads as follows:

$$\psi''(x) = -\frac{2m}{\hbar^2}\big(E - V(x)\big)\psi(x)\,. \tag{1.11}$$

Formula 1.2.4 (General solution). A general solution of the TDSE (for time-independent Hamiltonians) is given by the following formula (cf. **??**):

$$\psi(x,t) = \sum_E c_E\psi_E(x)e^{-\frac{i}{\hbar}Et}\,, \tag{1.12}$$

where the functions $\psi_E(x)$ are the eigenfunctions of the TISE. The coefficients c_E can be found using the orthogonality relations

$$c_E = \left(\int_{\mathbb{R}}\overline{\psi_E}(x)\psi(x,t_0)\,dx\right)e^{\frac{i}{\hbar}Et_0}\,. \tag{1.13}$$

1.3 Heisenberg–Born–Jordan picture

In the previous section, the central object was the wave function. It was this object that evolved in time and the operators acting on the Hilbert space of physical states were assumed to be fixed. However, it is also possible to transfer this dependence on time to the operators.

Formula 1.3.1 (Time-dependent observables).

$$\widehat{O}_H(t) := e^{\frac{i}{\hbar}\widehat{H}t}\widehat{O}_S(t)e^{-\frac{i}{\hbar}\widehat{H}t} \tag{1.14}$$

The equivalence between the Schödinger and Heisenberg pictures essentially come from the fact that the time-evolving expectation values of operators are given by the following formula:

$$\langle\widehat{O}(t)\rangle = \langle\psi|e^{\frac{i}{\hbar}\widehat{H}t}\widehat{O}(t)e^{-\frac{i}{\hbar}\widehat{H}t}|\psi\rangle\,. \tag{1.15}$$

The difference between the pictures is simply the choice of whether to include the evolution operator in the states or in the operators.

Using the above transformation, the Schödinger equation (Formula 1.2.2) can also be reexpressed.

Formula 1.3.2 (Time-dependent Schrödinger equation).

$$\frac{\partial \widehat{O}_H}{\partial t}(t) = \frac{i}{\hbar}\left[\widehat{H}_H(t), \widehat{O}_H(t)\right] + \left(\frac{\partial \widehat{O}}{\partial t}(t)\right)_H \tag{1.16}$$

Taking this expression for the Schrödinger equation and taking expectation values (using the linearity of the equation), gives the following (interaction-independent) result.

Theorem 1.3.3 (Ehrenfest). *Let $\widehat{H}$ be the Hamiltonian and consider an observable $\widehat{O}$. The expectation value of this operator evolves as follows:*

$$\frac{\mathrm{d}\langle\widehat{O}\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\langle[\widehat{O}, \widehat{H}]\rangle + \left\langle\frac{\partial \widehat{O}}{\partial t}\right\rangle. \tag{1.17}$$

Remark 1.3.4 (Equivalence). It is important to note that the Schrödinger equation could be replaced by Ehrenfest's theorem. They are entirely equivalent.

But, given the abstract state vectors $|\psi\rangle$ from Section 1.1.1, how does one recover the position (configuration) representation $\psi(x)$? This is simply the projection of the state vector $|\psi\rangle$ on the 'basis function' $\delta(x)$, i.e. $\psi(x)$ represents an expansion coefficient in terms of a 'basis' for the physical Hilbert space. In the same way, one can obtain the momentum representation $\psi(p)$ by projecting onto the plane waves e^{ipx}.

Remark 1.3.5. It should be noted that neither the 'basis states' $\delta(x)$, nor the plane waves e^{ipx} are square integrable and, hence, they are not elements of the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$. This issue can be resolved through the concept of *rigged Hilbert spaces*.

@@ COMPLETE @@

1.3.1 Hydrogen atom

Consider the hydrogen atom, i.e. a single proton (the nucleus) orbited by a single electron with only the electrostatic Coulomb force acting between them (gravity can safely be neglected):

$$\widehat{H} := \frac{\widehat{p}_p^2}{2m_p} + \frac{\widehat{p}_e^2}{2m_e} - \frac{e^2}{4\pi\varepsilon r^2}\,. \tag{1.18}$$

It is not hard to see that this is the quantum mechanical version of the Kepler problem (**??**). The special property of the Kepler problem was that it contained a 'hidden' symmetry that gave rise to the conserved Laplace–Runge–Lenz vector (**??**). As is the case for all conserved charges in quantum mechanics, this symmetry induces a degeneracy of the energy eigenvalues. Degeneracy of the magnetic quantum number $m \in \mathbb{N}$

follows from rotational symmetry, but the energy levels of the hydrogen atom only depend on the principal quantum number $n \in \mathbb{N}$. It is the degeneracy of the total angular quantum number $l \in \mathbb{N}$ that is due to this 'hidden' SO(4)-symmetry. It is often called an 'accidental degeneracy' for this reason.

@@ COMPLETE @@

1.3.2 Molecular dynamics

Consider the Hamiltonian of two interacting atoms:

$$\widehat{H} = \frac{\widehat{P}_1^2}{2M_1} + \frac{\widehat{P}_2^2}{2M_2} + \frac{\hat{q}_1\hat{q}_2}{4\pi\varepsilon R^2} + \sum_i \frac{\hat{p}_i^2}{2m} - \frac{e\hat{q}_1}{4\pi\varepsilon r_{i1}^2} - \frac{e\hat{q}_2}{4\pi\varepsilon r_{i2}^2} + \sum_{i\neq j} \frac{e^2}{4\pi\varepsilon r_{ij}^2}, \tag{1.19}$$

where the indices i, j indicate the electrons and uppercase symbols denote operators associated to the nuclei.

Except for the most simple situations, solving the Schrödinger equation for this Hamiltonian becomes intractable (both analytically and numerically). However, in general, one can approximate the situation. The masses of nuclei are much larger than those of the electrons and this influences their motion, they move much slower than the electrons. In essence, the nuclei and electrons live on different time scales and this allows to decouple their dynamics:

$$\widehat{H}_{\text{nucl}} = \frac{\widehat{P}_1^2}{2M_1} + \frac{\widehat{P}_2^2}{2M_2} + \frac{Q_1Q_2}{4\pi\varepsilon R^2} + V_{\text{eff}}(R_1, R_2). \tag{1.20}$$

The electrons generate an effective potential for the nuclei and the Schrödinger equation decouples as follows:

$$\begin{aligned} \widehat{H}_{\text{nucl}}(R)\psi(R) &= E\psi(R)\,, \\ \widehat{H}_{\text{el}}(r,R)\phi(r,R) &= E_{\text{el}}\phi(r,R)\,. \end{aligned} \tag{1.21}$$

This is the so-called **Born–Oppenheimer approximation**. From a more modern physical perspective, this approximation can also be seen to be a specific instance of renormalization theory, where the short time-scale (or, equivalently, the high energy-scale) degrees of freedom are integrated out of the theory.

1.4 Mathematical formalism

1.4.1 Weyl systems

Definition 1.4.1 (Canonical commutation relations). Two observables $\widehat{A}, \widehat{B}$ are said to obey a canonical commutation relation (CCR) if they satisfy (up to a constant factor

$\hbar$)

$$[\widehat{A}, \widehat{B}] = i\,. \tag{1.22}$$

The prime examples are the position and momentum operators $\hat{x}, \hat{p}$. Through functional calculus, one can also define the exponential operators $e^{is\widehat{A}}$ and $e^{it\widehat{B}}$. The above relation then induces the so-called **Weyl form** of the CCR:

$$e^{is\widehat{A}}e^{it\widehat{B}} = e^{ist}e^{it\widehat{B}}e^{is\widehat{A}}\,. \tag{1.23}$$

Theorem 1.4.2 (Stone–von Neumann). *All pairs of irreducible, unitary one-parameter subgroups satisfying the Weyl form of the CCRs are unitarily equivalent.*

Corollary 1.4.3. The Schrödinger and Heisenberg pictures are unitarily equivalent.

In fact, one can generalize the Weyl form of the CCRs.

Definition 1.4.4 (Weyl system). Let (L, ω) be a symplectic vector space and let K be a complex vector space. Consider a map W from L to the space of unitary operators on K. The pair (K, W) is called a Weyl system over (L, ω) if it satisfies

$$W(z)W(z') = e^{i/2\omega(z,z')}W(z + z') \tag{1.24}$$

for all $z, z' \in L$, i.e. W is a projective representation of the Abelian group L and ω is, up to rescaling, the group cocycle inducing it (**??**). The relation itself is called a **Weyl relation**.

Definition 1.4.5 (Heisenberg system). Let W be a Weyl system. The selfadjoint generators $\phi(z)$, which exist by Stone's theorem **??**, of the maps $t \mapsto W(tz)$ are said to form a Heisenberg system. These operators satisfy the following properties:

1. **Positive homogeneity**: $\lambda\phi(z) = \phi(\lambda z)$ for all $\lambda > 0$,
2. **Commutator**: $[\phi(z), \phi(z')] = -i\omega(z, z')$, and
3. **Weak additivity**: $\phi(z + z')$ is the closure (**??**) of $\phi(z) + \phi(z')$.

Remark 1.4.6. It should be noted that the Weyl relations are more fundamental than their infinitesimal counterparts. Only the Weyl relations are well defined on more general spaces and when passing to a relativistic setting.

Recall **??**, where the framework of measure theory and distributions was generalized to the noncommutative context.

Property 1.4.7 (Schrödinger representation). Consider a distribution d on a (real) TVS V. There exists a unique unitary representation U of the additive group V^* on $L^2(V,d)$ such that

$$U(\lambda)f = e^{id(\lambda)}f \tag{1.25}$$

for all bounded tame functions f and such that 1 is cyclic for U in $L^2(V,d)$. Moreover, this representation is continuous with respect to the finest locally convex topology on V (the one generated by all seminorms on V)[2].

@@ EXPLAIN RELEVANCE e.g. Baez, Segal, and Zhou (2014) @@

1.4.2 Dirac–von Neumann postulates: revisited

Section 1.1.1 presented the axioms of quantum mechanics in terms of Hilbert spaces and the operators thereon. However, the incredible insight of *von Neumann* was that one can do away with the Hilbert space. By **??**, the observables of a quantum-mechanical system form a C^*-algebra. Consequently, the idea is to rephrase the axioms in purely C^*-algebraic terms (**??**). By **??**, these two approaches are equivalent.

Axiom 1.6 (Observables). A physical system is characterized by a C^*-algebra, with the observables corresponding to the self-adjoint elements.

Axiom 1.7 (States). A state of a quantum-mechanical system is given by a state of the associated C^*-algebra (**??**).

Axiom 1.8 (Born rule). The expectation value of an observable a in a state ω is given by the evaluation $\omega(a)$.

Remark 1.4.8. Section 2.2 will link this axiom to traces and operator theory through **??** and **??**.

Axiom 1.9 (Projection).

Axiom 1.10 (Unitary evolution).

@@ CORRECT ALL AXIOMS @@

1.4.3 Symmetries

Property 1.4.9 (States). By the postulates of quantum mechanics, states are represented by rays in the projective Hilbert space $\mathcal{H}\mathbb{P}$. The probabilities, given by the Born rule (Axiom 1.3), can be expressed in terms of the *Fubini–Study metric* on $\mathcal{H}\mathbb{P}$ as follows:

$$\mathcal{P}(\psi,\phi) := \cos^2\big(d_{\text{FS}}(\psi,\phi)\big) = \frac{|\langle\psi\,|\,\phi\rangle|^2}{\langle\psi\,|\,\psi\rangle\langle\phi\,|\,\phi\rangle}, \tag{1.26}$$

where $|\psi\rangle, |\phi\rangle$ are representatives of the states ψ,ϕ in $\mathcal{H}\mathbb{P}$.

[2] This topology is also known as the **algebraic topology**

Definition 1.4.10 (Symmetry). A quantum symmetry (or **quantum automorphism**) is an isometric automorphism of $\mathcal{H}\mathbb{P}$. The group of these symmetries is denoted by $\mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P})$.

The following theorem due to *Wigner* gives a (linear) characterization of quantum symmetries.[3]

Theorem 1.4.11 (Wigner). *Every quantum automorphism of $\mathcal{H}\mathbb{P}$ is induced by a unitary or anti-unitary operator on $\mathcal{H}$.*

This is equivalent to saying that the group morphism

$$\pi : \mathsf{Aut}(\mathcal{H},\mathcal{P}) := \mathsf{U}(\mathcal{H}) \times \mathsf{AU}(\mathcal{H}) \to \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \tag{1.27}$$

is surjective. Together with the kernel $\mathsf{U}(1)$, given by phase shifts, this forms a short exact sequence:

$$1 \longrightarrow \mathsf{U}(1) \longrightarrow \mathsf{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \longrightarrow 1\,. \tag{1.28}$$

In the case of symmetry breaking (e.g. lattice systems), the full symmetry group is reduced to a subgroup $G \subset \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P})$. The group of operators acting on $\mathcal{H}$ is then given by the pullback $\widetilde{G}$ of the diagram

$$\mathsf{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \longleftarrow G\,. \tag{1.29}$$

It should also be noted that the kernel of the homomorphism $\widetilde{G} \to G$ is again $\mathsf{U}(1)$. This leads to the property that $\widetilde{G}$ is a $\mathbb{Z}_2$-twisted (hence noncentral) $\mathsf{U}(1)$-extension of G, where the twist is induced by the homomorphism $\phi : \mathsf{Aut}(\mathcal{H},\mathcal{P}) \to \mathbb{Z}_2$ that says whether an operator is implemented unitarily or anti-unitarily.

@@ COMPLETE @@

1.4.4 Symmetric states

Axiom 1.11 (Symmetrization postulate). Let $\mathcal{H}$ be the single-particle Hilbert space. A system of $n \in \mathbb{N}$ identical particles is described by a state $|\Psi\rangle$ belonging to either $S^n\mathcal{H}$ or $\Lambda^n\mathcal{H}$. These **bosonic** and **fermionic** states are, respectively, of the form

$$|\Psi_B\rangle = \sum_{\sigma \in S_n} |\psi_{\sigma(1)}\rangle \cdots |\psi_{\sigma(n)}\rangle \tag{1.30}$$

and

$$|\Psi_F\rangle = \sum_{\sigma \in S_n} \mathsf{sgn}(\sigma) |\psi_{\sigma(1)}\rangle \cdots |\psi_{\sigma(n)}\rangle\,, \tag{1.31}$$

where the $|\psi_i\rangle$ are single-particle states and S_n is the permutation group on n elements.

[3] It is a particular case of a more general theorem in projective geometry.

Remark 1.4.12. In ordinary quantum mechanics, this is a postulate, but in quantum field theory, this is a consequence of the *spin-statistics theorem*. @@ ADD THIS THEOREM TO [QFT] @@

Definition 1.4.13 (Slater determinant). Let $\{\phi_i(\vec{q})\}_{i\leq n}$ be a set of wave functions, called **spin orbitals**, describing a system of n identical fermions. The totally antisymmetric wave function of the system is given by

$$\psi(\vec{q}_1,\dots,\vec{q}_n)=\frac{1}{\sqrt{n!}}\det\begin{pmatrix}\phi_1(\vec{q}_1) & \cdots & \phi_n(\vec{q}_1)\\ \vdots & \ddots & \vdots\\ \phi_1(\vec{q}_n) & \cdots & \phi_n(\vec{q}_n)\end{pmatrix}. \tag{1.32}$$

A similar function can be defined for bosonic systems using the concept of *permanents*.

1.5 Foundations ♣

1.5.1 Measurement problem

If one looks at the Schrödinger equation (Formula 1.2.2) or Ehrenfest's theorem (Theorem 1.3.3), it is easy to see that time evolution is entirely linear and deterministic. Superpositions are preserved under Hamiltonian flow (a crucial ingredient of quantum mechanics) and, given an initial state, time evolution will always lead to the same final state. However, the Born rule (Axiom 1.3), which governs 'measurements' is very nonlinear and nondeterministic. It is probabilistic and, once a 'measurement' has been performed, the state has 'collapsed' onto an eigenstate of the observable under consideration.

The issue of what constitutes a 'measurement' — Is it a conscious human doing an experiment? Is it a mouse interfering with an experiment? Is it two particles interacting? ...[4] — and why exactly the Born rule holds and what it entails, i.e. how probabilities arise, is known as the measurement problem. On a historical note, it should be noted that, after an initial surge of interest shortly after the 5th Solvay Conference (1927), where quantum mechanics was formally established, the study of the foundations of quantum mechanics (the measurement problem specifically) became an infamous topic due to the pragmatic mentality of nuclear physics during the 20th century.

@@ ADD (dynamical collapse, epistemic) @@

[4] This (perhaps artificial) boundary between classical and quantum is sometimes called the **Heisenberg cut**.

1.5.2 Copenhagen interpretation

The Copenhagen interpretation[5] takes the foundations of quantum mechanics as presented above very literally.

@@ COMPLETE (e.g. collapse) @@

1.5.3 Many-worlds interpretation

This interpretation, originating with *Everett*, posits a different idea, which does away with the need of the explicit Born rule axiom. In this interpretation, there is a kind of 'universal wave function', which governs both the observer and the experiment. A 'measurement' is then simply an entanglement-inducing interaction between these two subsystems.

The main implication of such an interpretation is, however, that the universal wave function branches every time such an interaction occurs. More precisely, assume that 'we', the observers, perform a measurement on some system (for simplicity, assume that the measurement has a binary outcome). The measurement process is then described as follows:

$$|\mathrm{in}\rangle_{\mathrm{obs}}|\mathrm{in}\rangle_{\mathrm{exp}} \longrightarrow \lambda_0|0\rangle_{\mathrm{obs}}|0\rangle_{\mathrm{exp}} + \lambda_1|1\rangle_{\mathrm{obs}}|1\rangle_{\mathrm{exp}}\,. \tag{1.33}$$

Taking this superposition as a physical reality, this means that if we had measured the state 0, a copy of us living on the other branch will have measured 1 (and the other way around).

@@ COMPLETE (e.g. origin of probabilities) @@

1.5.4 Relational quantum mechanics

An important notion in classical physics is that of a *reference frame*, i.e. a choice of axes and scales. Usually, this corresponds to choosing an observer, relative to which one expresses the motion of all other objects. In relativity, the relative treatment of physics was the grand breakthrough by Einstein. However, although this notion had been left aside for a long time in the treatment of quantum mechanics and a specific choice of reference frame was silently assumed, this assumption was not as innocuous as it appears. Superposition and complementarity make a definite choice of absolute reference frame impossible.

To understand the relevance of a relational approach to quantum mechanics, consider the following thought experiment.

[5] This name stems from the fact that its initial proponents were from the group of physicists centered around *Bohr*.

Definition 1.5.1 (Wigner's friend). Consider two observers, Wigner and his friend, performing an experiment as shown diagrammatically in Fig. 1.1. One envisions Wigner standing outside the laboratory, having no way to observe what happens inside the lab, and his friend who performs an experiment inside the lab. The paradox arises from the two ways one can describe the sequence of the friend performing a measurement and Wigner checking up on the results in the classical (Copenhagen) interpretation.

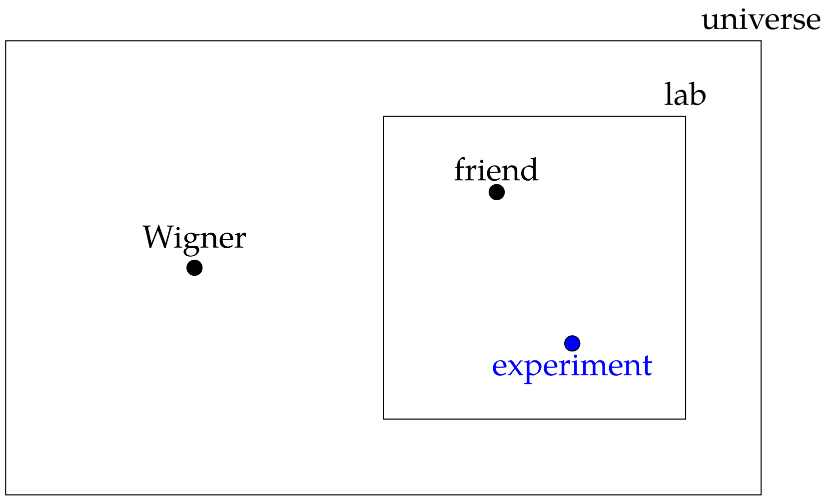

Figure 1.1: Wigner's friend thought experiment.

From the point of view of the friend, at the moment of measurement, the projection/collapse axiom states that the wave function describing friend + experiment 'collapses' to:

$$|\psi\rangle = |\uparrow\rangle_{\text{friend}}|\uparrow\rangle_{\text{exp}}\,. \tag{1.34}$$

However, from the point of view of Wigner, who has not observed the measurement, the state is described by

$$|\psi\rangle' = \alpha|\uparrow\rangle_{\text{friend}}|\uparrow\rangle_{\text{exp}} + \beta|\downarrow\rangle_{\text{friend}}|\downarrow\rangle_{\text{exp}}\,. \tag{1.35}$$

Whereas in the many-world approach one would simply take the branching approach, which is fully unitary and resolves this issue by avoiding collapse, the relational approach takes collapse at face value, but states that observations are relative, i.e. always with respect to some fixed observer (be it a person, a classical object or another quantum-mechanical system). From this point of view, textbook Copenhagen QM is simply quantum mechanics with respect to some god-given observer, and collapse and unitary evolution do not have to be reconciled.

The more general idea is that information and, hence, the values of observables are a relative notion, i.e. variables only attain their values when considered with respect to a certain observer. As such, RQM is an epistemic interpretation of quantum mechanics in that the wave function only captures 'our' information about the system (or universe) and not the 'true' physical state. When applied to the notion of instantaneity (or velocity), this line of thinking will give rise to (special) relativity as in **??** (and **??**).

@@ COMPLETE @@

For example, consider three observers: Alice, Bob and Charlie. Assume that each observer has a spin-$\frac{1}{2}$ particle and that, relative to Alice, the joint state is given by

$$|\psi\rangle^{A}_{ABC} = |\uparrow\rangle^{A}_{A}\Big(|\uparrow\rangle^{A}_{B} + |\downarrow\rangle^{A}_{B}\Big)|\downarrow\rangle^{A}_{C}\,. \tag{1.36}$$

Note that this state is separable. Now, what would the state be relative to Bob? If one supposes that changes of reference frame are *coherent* (to be formalized below), the joint state will be

$$|\psi\rangle^{B}_{ABC} = |\uparrow\rangle^{B}_{B}\Big(|\uparrow\rangle^{B}_{A}|\downarrow\rangle^{B}_{C} + |\downarrow\rangle^{B}_{A}|\uparrow\rangle^{B}_{C}\Big)\,. \tag{1.37}$$

A mere change of reference frame, an operation that would classically leave the physics invariant, has transformed a product state into an entangled state.

Axiom 1.12 (Relational physics). Given $n \in \mathbb{N}$ systems[6], any state is described relative to one of these systems. Given a choice of 'observing system', let it be system i, the state of system i is given by a fiducial state $|0\rangle^{i}_{i}$.

Axiom 1.13 (Coherent change). Consider a change of reference frame $0 \longrightarrow i$ such that

$$\begin{cases} |\psi\rangle^{0} \longrightarrow |\psi\rangle^{i} \\ |\phi\rangle^{0} \longrightarrow |\phi\rangle^{i}\,. \end{cases} \tag{1.38}$$

Then

$$\alpha|\psi\rangle^{0} + \beta|\phi\rangle^{0} \longrightarrow \alpha|\psi\rangle^{i} + \beta|\phi\rangle^{i} \tag{1.39}$$

for all $\alpha, \beta \in \mathbb{C}$.

Abstractly, a (classical) reference frame is defined as follows in the spirit of **??** and **??**.

Definition 1.5.2 (Reference frame). Let X be an object of interest. Whereas a coordinate chart on X, modeled on an object Y, is given by a morphism $Y \to X$, a **coordinate system** on X is given by an isomorphism $Y \cong X$, i.e. a global coordinate chart. A reference frame is coordinate system for which Y corresponds the a physical system.

[6] An abstraction of the notion of observer.

Let the system of interest X admit a group action that is both free and transitive, turning it into a G-torsor (**??**). At the level of sets, one has $X \cong G$ and a choice of origin, i.e. a specific choice of isomorphism, corresponds to a choice of reference frame (the identity element corresponding to the fiducial state above). A change of reference frames $s^0 \longrightarrow s^i$, frome system 0 to system i, is given by the right regular action of the relative coordinate of i on all relative coordinates:

$$\phi^{0\longrightarrow i}(e, g_1^0, \dots, g_n^0) \mapsto (g_0^i, g_1^0 g_0^i, \dots, e, \dots, g_n^0 g_0^i)\,, \tag{1.40}$$

where the relation $g_i^0 = (g_0^i)^{-1}$ was used. It should be noted that this boils down to a *passive transformation*. When passing to the quantization of these systems, one should assume that G is locally compact and comes equipped with the canonical Haar measure (**??**). In this case, a quantization is given by the space of square-integrable functions $L^2(G)$, where basis states are labeled by group elements.

@@ VERIFY THIS STATEMENT @@

The change-of-reference-frame operator is given as follows:

$$\widehat{U}^{0\longrightarrow i} := \mathsf{SWAP}_{0,i} \circ \int_G \mathbb{1}_{L^2(G)} \otimes \widehat{U}_R(g_i^0)^{\otimes i-2} \otimes |g_0^i\rangle\langle g_i^0| \otimes \widehat{U}_R(g_i^0)^{\otimes n-i-2}\, dg_i^0\,, \tag{1.41}$$

where

$$\widehat{U}_R(g) : |x\rangle \mapsto |xg^{-1}\rangle \tag{1.42}$$

is the unitary implementation of the right regular action and dg denotes integration with respect to the Haar measure on G. It can be shown that $\widehat{U}^{0\longrightarrow i}$ is unitary, its inverse being given by $\widehat{U}^{i\longrightarrow 0}$ and composition is transitive. It can be shown that this procedure can be extended to any one-particle Hilbert space $\mathcal{H}$ as long as the inclusion $G \to \mathcal{H}$ is injective and maps G to an orthonormal basis of (a subset of) $\mathcal{H}$.

1.6 Angular Momentum

1.6.1 Angular momentum operator

Property 1.6.1 (Lie algebra). The angular momentum operators generate a Lie algebra (**??**). The Lie bracket is defined by the following commutation relation:

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk}\hat{J}_k\,. \tag{1.43}$$

Since rotations correspond to actions of the orthogonal group $\mathsf{SO}(3)$, it should not come as a surprise that the above relation is exactly the defining relation of the Lie algebra $\mathfrak{so}(3)$ from **??**.

Property 1.6.2. The mutual eigenbasis of $\hat{J}^2$ and $\hat{J}_z$ is defined by the following two eigenvalue equations:

$$\hat{J}^2|j,m\rangle = j(j+1)\hbar^2\,|j,m\rangle\,, \tag{1.44}$$

$$\hat{J}_z|j,m\rangle = m\hbar\,|j,m\rangle\,. \tag{1.45}$$

Definition 1.6.3 (Ladder operators[7]). The raising and lowering operators $\hat{J}_+$ and $\hat{J}_-$ are defined as follows:

$$\hat{J}_+ := \hat{J}_x + i\hat{J}_y \qquad \text{and} \qquad \hat{J}_- := \hat{J}_x - i\hat{J}_y\,. \tag{1.46}$$

These operators only change the quantum number $m_z \in \mathbb{N}$, not the total angular momentum.

Corollary 1.6.4. From the commutation relations of the angular momentum operators, one can derive the commutation relations of the ladder operators:

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z\,. \tag{1.47}$$

Formula 1.6.5. The total angular momentum operator $\hat{J}^2$ can now be expressed in terms of $\hat{J}_z$ and the ladder operators using the commutation relation (1.43):

$$\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z\,. \tag{1.48}$$

Remark 1.6.6 (Casimir operator). From the definition of $\hat{J}^2$, it follows that this operator is a Casimir invariant (**??**) of $\mathfrak{so}(3)$.

1.6.2 Rotations

Formula 1.6.7. An infinitesimal rotation $\hat{R}(\delta\vec{\boldsymbol{\varphi}})$ is given by the following formula:

$$\hat{R}(\delta\vec{\boldsymbol{\varphi}}) = \mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\delta\vec{\boldsymbol{\varphi}}\,. \tag{1.49}$$

A finite rotation can be generated by applying this infinitesimal rotation repeatedly:

$$\hat{R}(\vec{\boldsymbol{\varphi}}) = \left(\mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\frac{\vec{\boldsymbol{\varphi}}}{n}\right)^n = \exp\left(-\frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{\varphi}}\right). \tag{1.50}$$

Formula 1.6.8 (Matrix elements). Applying a rotation over an angle φ about the z-axis to a state $|j,m\rangle$ gives

$$\hat{R}(\varphi\vec{\boldsymbol{e}}_z)|j,m\rangle = \exp\left(-\frac{i}{\hbar}\hat{J}_z\varphi\right)|j,m\rangle = \exp\left(-\frac{i}{\hbar}m\varphi\right)|j,m\rangle\,. \tag{1.51}$$

[7] Also called the **creation** and **annihilation** operators (especially in quantum field theory).

Multiplying these states with a bra $\langle j', m'|$ and using the orthonormality of the eigenstates, gives the matrix elements of the rotation operator:

$$\widehat{R}_{ij}(\varphi\vec{e}_z) = \exp\left(-\frac{i}{\hbar}m\varphi\right)\delta_{jj'}\delta_{mm'}\,. \tag{1.52}$$

From the expression of the angular momentum operators and the rotation operator, it is clear that a general rotation has no effect on the total angular momentum number $j \in \mathbb{N}$. This means that the rotation matrix will be block diagonal with respect to j. This amounts to the following reduction of the representation of the rotation group:

$$\langle j, m'|\widehat{R}(\varphi\vec{n})|j, m\rangle = \mathcal{D}^{(j)}_{m,m'}(\widehat{R})\,, \tag{1.53}$$

where the functions $\mathcal{D}^{(j)}_{m,m'}(\widehat{R})$ are called the **Wigner D-functions**. For every value of j, there are $(2j+1)$ values for m. This implies that the matrix $\mathcal{D}^{(j)}(\widehat{R})$ is a $(2j+1)\times(2j+1)$-matrix.

1.6.3 Spinor representation

Definition 1.6.9 (Pauli matrices).

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.54}$$

From this definition, it is clear that the Pauli matrices are Hermitian and unitary. Together with the 2×2 identity matrix, they form a basis for the space of 2×2 Hermitian matrices. For this reason, the identity matrix is often denoted by σ_0 (especially in the context of relativistic QM).

Formula 1.6.10. In the spinor representation ($J = \frac{1}{2}$), the Wigner-D matrix reads as follows:

$$\mathcal{D}^{(1/2)}(\varphi\vec{e}_z) = \begin{pmatrix} e^{-i/2\varphi} & 0 \\ 0 & e^{i/2\varphi} \end{pmatrix}. \tag{1.55}$$

1.6.4 Coupling of angular momenta

Due to the tensor product structure of a coupled Hilbert space, the angular momentum operator $\hat{J}_i$ should now be interpreted as $\mathbb{1}\otimes\cdots\otimes\hat{J}_i\otimes\cdots\otimes\mathbb{1}$ (cf. **??**). Because the angular momentum operators $\hat{J}_{k\neq i}$ do not act on the space $\mathcal{H}_i$, one can pull these operators through the tensor product:

$$\hat{J}_i|j_1\rangle\otimes\cdots\otimes|j_n\rangle = |j_1\rangle\otimes\cdots\otimes\hat{J}_i|j_i\rangle\otimes\cdots\otimes|j_n\rangle\,. \tag{1.56}$$

The basis used above is called the **uncoupled basis**.

For simplicity, the total Hilbert space is, from here on, assumed to be that of a two-particle system. Let $\hat{J}$ denote the total angular momentum:

$$\hat{J} = \hat{J}_1 + \hat{J}_2 . \tag{1.57}$$

With this operator, one can define a **coupled** state $|J, M\rangle$, where M is the total magnetic quantum number which ranges from $-J$ to J.

Formula 1.6.11 (Clebsch–Gordan coefficients). Because both bases (coupled and uncoupled) span the total Hilbert space $\mathcal{H}$, there exists an invertible transformation between them. The transformation coefficients can be found by using the resolution of the identity:

$$|J, M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2 \,|\, J, M\rangle . \tag{1.58}$$

These coefficients are called the Clebsch–Gordan coefficients.

Property 1.6.12. By acting with the operator $\hat{J}_z$ on both sides of Formula 1.6.11, it is possible to prove that the Clebsch–Gordan coefficients are nonzero if and only if $M = m_1 + m_2$.

1.7 Approximation methods

1.7.1 WKB approximation

The Wentzel–Kramers–Brillouin (WKB) approximation[8] starts from the ansatz

$$\psi(\vec{q}) := \exp\big(iS(\vec{q})/\hbar\big) , \tag{1.59}$$

with $S : \mathbb{R}^n \to \mathbb{R}$ a phase function that is to be determined. Inserting this in the TDSE (in configuration representation) gives:

$$\left[\frac{\|\vec{\nabla}S(\vec{q})\|^2}{2m} + \big(V(\vec{q}) - E\big) - \frac{i\hbar\Delta S(\vec{q})}{2m}\right] \exp\big(iS(\vec{q})/\hbar\big) = 0 . \tag{1.60}$$

To first order, i.e. for slowly varying potentials, the last term can be ignored. In this case, the phase function satisfies the Hamilton–Jacobi equation (**??**):

$$H\big(\vec{q}, S'(\vec{q})\big) = \frac{\|\vec{\nabla}S(\vec{q})\|^2}{2m} + \big(V(x) - E\big) = 0 . \tag{1.61}$$

In physics, the Hamilton–Jacobi equation without time derivative is often called the **eikonal equation**[9]. This leads to the following result.

[8]This approach to solving second-order ODEs was essentially introduced a century earlier by *Green* and *Liouville*.

[9]This name stems from optics.

Property 1.7.1. A function $S : \mathbb{R}^n \to \mathbb{R}$ is a phase function for a first-order solution to the Schrödinger equation if its differential lies in a level set of the classical Hamiltonian $H : T^*\mathbb{R}^n \to \mathbb{R}$. These solutions are said to be **admissible**.

To obtain higher-order approximations, the solution has to be generalized beyond a pure phase function:

$$\psi(\vec{q}) = a(\vec{q}) \exp\big(iS(\vec{q})/\hbar\big) . \tag{1.62}$$

Assuming S is admissible, the factor $a : \mathbb{R}^n \to \mathbb{R}$ satisfies the **homogeneous transport equation**:

$$a\Delta S + 2\vec{\nabla}a \cdot \vec{\nabla}S = 0 . \tag{1.63}$$

If a satisfies this equation, ψ is called a **semiclassical state**. Note that this equation is equivalent to $a^2\vec{\nabla}S$ being divergence free or, equivalently:

$$\mathcal{L}_{\pi_* X^H}(a^2\,\mathrm{Vol}) = 0 . \tag{1.64}$$

Since Lie derivatives pull back under diffeomorphisms (**??**) and the image $\mathrm{im}(\mathrm{d}S)$ gives a trivial subbundle of $T^*\mathbb{R}^n$, this is also equivalent to

$$\mathcal{L}_{X^H}(a^2\pi^*\,\mathrm{Vol}) = 0 . \tag{1.65}$$

This quadratic behaviour in a leads to the idea that the correct object for representing quantum states is a half-density (see also **??**). This leads to the following statement:

> A second-order solution to the Schrödinger equation is given by a pair (S, a), where S is an admissible phase function and $a \in \Omega^{1/2}(\mathrm{im}(\mathrm{d}S))$ is a half-form that is invariant under the (classical) Hamiltonian flow.

The generalization to curved spaces, i.e. replacing $\mathbb{R}^{2n}$ by a symplectic manifold M, will be covered in Section 1.8.2.

1.8 Curved backgrounds ♣

Using the tools of distribution theory and differential geometry (**?? ??**, **??** and onwards), one can introduce quantum mechanics on curved backgrounds (in the sense of 'space', not 'spacetime').

1.8.1 Extending quantum mechanics

Remark 1.8.1 (Rigged Hilbert spaces). A first important remark to be made is that the classical definition of the wave function as an element of $L^2(\mathbb{R}^d, \mathbb{C})$ is not sufficient, even in flat Cartesian space. A complete description requires the introduction of so-called *Gel'fand triples* or *rigged Hilbert spaces*, where the space of square-integrable functions is replaced by the Schwartz space (**??**) of rapidly decreasing functions. The linear functionals on this space are then given by the tempered distributions.

When working on curved spaces or even in non-Cartesian coordinates on flat space, one can encounter problems with the definition of the self-adjoint operators $\widehat{q}^i$ and $\widehat{p}_i$. The naive definition $\widehat{q}^i = q^i, \widehat{p}_i = -i\partial_i$ gives rise to extra terms that break the canonical commutation relations and the selfadjointness of the operators (e.g. the angular position operator $\widehat{\varphi}$ on the circle together with its conjugate $\widehat{L}$) when calculating inner products.

An elegant solution to this problem is obtained by giving up the definition of the wave function as a well-defined function $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$. Assume that the physical space has the structure of a Riemannian manifold (M, g) and that the 'naive' wave functions take values in a vector space V. Then, construct a vector bundle E with typical fibre V over M. By **??**, an invariant description of the 'true' wave function is a map $\Psi : F(E) \rightarrow V$ or, locally, the pullback $\psi := \varphi^*\Psi$ for some local section $\varphi : U \subseteq M \rightarrow F(E)$. The Levi-Civita connection on M also induces a covariant derivative ∇ on E that can be used to define differential operators.

Now, a general inner product can be introduced:

$$\langle \psi, \phi \rangle := \int_M \overline{\psi(x)}\phi(x)\,\mathrm{Vol}_M\ . \tag{1.66}$$

Because the factor $\sqrt{\det(g)}$ transforms in the inverse manner of the measure dx, the integrand is invariant under coordinate transforms (something that is generally required of physical laws). Using this new inner product, one can for example check the selfadjointness of the momentum operator $\widehat{P}_i := -i\nabla_i$:

$$
\begin{aligned}
\langle\psi, \widehat{P}_i\phi\rangle &= \int_M \overline{\psi(x)}(-i\nabla_i)\phi(x)\sqrt{\det(g)}\,dx \\
&\overset{??}{=} \int_M \overline{\psi(x)}(-i\partial_i - i\omega_i)\phi(x)\sqrt{\det(g)}\,dx \\
&= \int_M \overline{(-i\partial_i\psi)(x)}\phi(x)\sqrt{\det(g)}\,dx + i\int_M \overline{\psi(x)}\phi(x)\Big(\partial_i\sqrt{\det(g)}\Big)\,dx \\
&\qquad - i\int_M \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx \\
&= \langle\widehat{P}_i\psi, \phi\rangle - i\int_M \overline{\psi(x)}\overline{\omega_i}\phi(x)\sqrt{\det(g)}\,dx \\
&\qquad + i\int_M \overline{\psi(x)}\phi(x)\Big(\partial_i\sqrt{\det(g)}\Big)\,dx \\
&\qquad - i\int_M \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx\,.
\end{aligned}
$$

Selfadjointness then requires that

$$\sqrt{\det(g)}(\omega_i + \overline{\omega_i}) = \partial_i\sqrt{\det(g)} \tag{1.67}$$

or

$$2\mathrm{Re}(\omega_i) = \partial_i \ln\Big(\sqrt{\det(g)}\Big)\,. \tag{1.68}$$

@@ COMPLETE (rewrite in global terms) @@

1.8.2 WKB approximation

Property 1.7.1 is generalized quite trivially after replacing $\mathbb{R}^n$ by a configuration manifold Q. A further step is provided by also generalizing **??**.

Property 1.8.2. A Lagrangian submanifold $\iota : L \hookrightarrow T^*Q$ will be called an admissible phase function for a first-order solution to the Schrödinger equation if it satisfies the classical Hamilton–Jacobi equation, i.e. lies in a level set of the classical Hamiltonian $H : T^*Q \to \mathbb{R}$, for a regular value.

To obtain a second-order solution, one also needs prefactor for the semiclassical states. The homogeneous transport equation (1.63) is generalized as follows:

$$a\Delta S + 2\mathcal{L}_{\nabla S}a = 0\,, \tag{1.69}$$

where Δ is the Laplace–Beltrami operator on Q. As before, a general second-order solution, assuming S is admissible, is given by a half-form $a \in \Omega^{1/2}(L)$ satisfying

$$\mathcal{L}_Y a = 0\,, \tag{1.70}$$

where Y is the (nonsingular) vector field on L induced by X^H. This then gives a second-order solution on Q by pulling back along the inverse $(\pi \circ \iota)^{-1}$, which is a diffeomorphism since L is projectable. Moreover, if L is exact (**??**), then S is induced by a primitive of the induced Liouville form $\iota^*\alpha$. If both the exactness and projectability conditions are dropped, the notion of a **geometric solution** are obtained.

To pass to this more general situation, some more structure is needed. If L is not exact, the Liouville form does not admit a global primitive. However, L does admit a (good) cover $\{U_k\}_{k\in I}$ such that on every patch, a second-order solution can be found, and then the problem becomes how to glue these together. The gluing condition is the following integrality condition:

$$\phi_k(x) - \phi_l(x) \in 2\pi\hbar\mathbb{Z}\,, \tag{1.71}$$

where ϕ_k is the phase function on U_k, for all $x \in U_k \cap U_l$. Note that this condition can only be satisfied for all $\hbar \in \mathbb{R}^+$ if $[\alpha] = 0$. However, this is exactly the condition that should be relaxed. Luckily, $\hbar$ should be a fixed value.

Definition 1.8.3 (Quantizable Lagrangian). A projectable Lagrangian submanifold $L \subset T^*M$ is said to be quantizable if there exists an $\hbar \in \mathbb{R}^+$ such that the restriction of the Liouville class to L is $\hbar$-integral, i.e. the integrality condition (1.71) is satisfied. All values $\hbar$ for which the integrality condition is satisfied, are said to be **admissible**.

Remark 1.8.4. Note that the admissible values for $\hbar$ will form a decreasing sequence of the form

$$\hbar_0, \frac{\hbar_0}{2}, \dots\,, \tag{1.72}$$

where $\hbar_0$ is the greatest admissible value.

For the weakening of the projectability condition, see Bates and Weinstein (1997). However, even without weakening that condition, there is still a remaining issue to the quantization of classical solutions. This will involve Maslov indices (**??**) and Morse theory (**??**).

Chapter 2

Quantum Information Theory

The section on (quantum) reference frames is based on De La Hamette and Galley (2020).

Chapter Contents

2.1 Entanglement

2.1.1 Introduction

Construction 2.1.1 (Schmidt decomposition). Consider a bipartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. For any such state, there exist orthonormal sets $\{|e_i\rangle, |f_j\rangle\}_{i,j\leq\kappa}$ such that

$$|\psi\rangle = \sum_{i=1}^{\kappa} \lambda_i |e_i\rangle \otimes |f_i\rangle\,, \tag{2.1}$$

where the coefficients λ_i are nonnegative real numbers. All objects in this expression can be obtained from a singular value decomposition of the coefficient matrix $\mathbf{C}$ of $|\psi\rangle$ in some bases of $\mathcal{H}_1$ and $\mathcal{H}_2$. The number $\kappa \in \mathbb{N}$ is called the **Schmidt rank** of $|\psi\rangle$.

Definition 2.1.2 (Entangled states). Consider a state $|\psi\rangle$ and consider its Schmidt decomposition. If the Schmidt rank is 1, i.e. the state can be written as $|\psi\rangle = |v\rangle \otimes |w\rangle$, the state is said to be **separable**. Otherwise, the state is said to be entangled.

The following theorem follow from the linearity of quantum mechanics.

Theorem 2.1.3 (No-cloning). *There is no unitary operator $\widehat{U}$ on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that*

$$\widehat{U}|\psi\rangle_1|\phi\rangle_2 = e^{i\alpha(\psi,\phi)}|\psi\rangle_1|\psi\rangle_2 \tag{2.2}$$

for all (normalized) $|\psi\rangle_1 \in \mathcal{H}_1$ and $|\phi\rangle_1 \in \mathcal{H}_2$.

Theorem 2.1.4 (No-deleting). *Consider a tripartite system $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ such that $\mathcal{H}_1 \cong \mathcal{H}_2$. If $\widehat{U}$ is a unitary operator on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ such that*

$$\widehat{U}|\psi\rangle_1|\psi\rangle_2|\phi\rangle_3 = |\psi\rangle_1|0\rangle_2\big|\phi_\psi\big\rangle_3 \tag{2.3}$$

for all $|\psi\rangle_1 \in \mathcal{H}_1$, where the final ancilla state $\big|\phi_\psi\big\rangle_3$ might depend on the initial state $|\psi\rangle_1$, then $\widehat{U}$ is simply a swap, i.e. $|\psi\rangle_1 \mapsto \big|\phi_\psi\big\rangle_3$ is an isometric embedding.

2.1.2 Bell states

Definition 2.1.5 (Bell state). A (binary) Bell state (also called a **cat state** or **Einstein–Podolsky–Rosen pair**) is defined as the following entangled state:

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)\,. \tag{2.4}$$

In fact, this state can be extended to a full maximally entangled basis for the 2-qubit Hilbert space:

$$\begin{aligned} |\Phi^-\rangle &:= \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big)\,,\\ |\Psi^+\rangle &:= \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)\,,\\ |\Psi^-\rangle &:= \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)\,. \end{aligned} \tag{2.5}$$

[1]Sometimes called **superdense coding**.

Method 2.1.6 (Dense coding[1]). Consider the Bell state $|\Phi^+\rangle$. By acting with one of the (unitary) spin-flip operators $\widehat{X}, \widehat{Y}, \widehat{Z}$, one can obtain any of the other three Bell states:

$$\begin{aligned} \widehat{X}|\Phi^+\rangle &= |\Phi^-\rangle\,, \\ \widehat{Y}|\Phi^+\rangle &= |\Psi^+\rangle\,, \\ \widehat{Z}|\Phi^+\rangle &= |\Psi^-\rangle\,. \end{aligned} \tag{2.6}$$

In a typical Alice-and-Bob-style experiment, one can ask whether this observation allows to achieve a better-than-classical communication channel. If Alice performs a spin flip on her qubit, although the resulting state has instantly 'changed' (cf. *spooky action at a distance*), Bob still cannot uniquely determine what this state is (since the resulting state is still maximally entangled). However, if Alice sends her qubit to Bob, the latter can perform a measurement on the composite system to find out what the state is and in this way determine which operation Alice performed $(\mathbb{1}, \widehat{X}, \widehat{Y}, \widehat{Z})$. Alice has thus effectively sent 2 classical bits of information through 1 qubit. Note that due to the fact that Alice still has to send her qubit through classical means, no faster-than-light communication is achieved.

Definition 2.1.7 (GHZ state). The Greenberger–Horne–Zeilinger state is defined as the multiparticle qudit $(d, N > 2)$ version of the Bell state and is, therefore, also referenced to as a cat state:

$$|\mathsf{GHZ}\rangle = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}|i\rangle^{\otimes N}\,. \tag{2.7}$$

2.1.3 SRE states

@@ ADD @@

2.2 Density operators

Definition 2.2.1 (Density operator). Consider a (finite-dimensional) Hilbert space $\mathcal{H}$. A density operator on $\mathcal{H}$ is a linear operator $\rho \in \mathsf{End}(\mathcal{H})$ satisfying the following properties:

1. **Positivity**: $\langle v\,|\,\rho v\rangle \geq 0$ for all $v \in \mathcal{H}$,
2. **Hermiticity**: $\rho^\dagger = \rho$, and
3. **Unit trace**: $\mathsf{tr}(\rho) = 1$.

More concisely, density operators are the representing objects of normal states (**??**) on $\mathcal{B}(\mathcal{H})$.

Example 2.2.2 (Classical probability). A diagonal density matrix corresponds to a discrete probability distribution.

Definition 2.2.3 (Pure state). A state is said to be pure if it is described by an outer product of a state vector or, equivalently, by an idempotent density matrix:

$$\rho = |\psi\rangle\langle\psi|\,. \tag{2.8}$$

A density matrix that is not of this form gives rise to a **mixed state**.

Definition 2.2.4 (Reduced density operator). Let $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be the state of a bipartite system. The reduced density operator ρ_A of A is defined as follows:

$$\rho_A := \mathrm{tr}_B|\Psi\rangle\langle\Psi|\,. \tag{2.9}$$

Definition 2.2.5 (Purification). Let ρ_A be the density operator of a system A. A purification of ρ_A is a pure state $|\Psi\rangle$ of some composite system $A \otimes B$ such that

$$\rho_A = \mathrm{tr}_B|\Psi\rangle\langle\Psi|\,. \tag{2.10}$$

Property 2.2.6. Any two purifications of the same density operator ρ_A are related by a transformation $\mathbb{1}_A \otimes \widehat{V}$ with $\widehat{V}$ an isometry.

2.3 Channels

The following definition generalizes the content of **??** to the setting of partial information. When generalizing the projections in a PVM (spectral measure), one obtains a POVM.

Definition 2.3.1 (Positive operator-valued measure). First, let $\mathcal{H}$ be a finite-dimensional Hilbert space. A POVM on $\mathcal{H}$ consists of a finite set of positive (semi)definite operators $\{P_i\}_{i\leq n}$ such that

$$\sum_{i=1}^{n} P_i = \mathbb{1}_{\mathcal{H}}\,. \tag{2.11}$$

The probability to obtain state i, given a general state $\hat{\rho}$, is given by $\mathrm{tr}(\hat{\rho}P_i)$. Note that the operators are not necessarily orthogonal projectors, so n can be greater than $\dim(\mathcal{H})$.

Now, consider a measurable space (X, Σ) and a (possibly infinite-dimensional) Hilbert space $\mathcal{H}$. A POVM on X consists of a function $P : \Sigma \to \mathcal{B}(\mathcal{H})$ satisfying the following conditions:

1. P_E is positive and self-adjoint for all $E \in \Sigma$,

2. $P_X = \mathbb{1}_{\mathcal{H}}$, and

3. for all disjoint $(E_n)_{n\in\mathbb{N}} \subset \Sigma$:

$$\sum_{n\in\mathbb{N}} P_{E_n} = P_{\cup_{n\in\mathbb{N}} E_n} . \tag{2.12}$$

The following theorem can be derived from Stinespring's theorem **??**.

Theorem 2.3.2 (Naimark dilation theorem). *Every POVM P on $\mathcal{H}$ can be realized as a PVM Π on a, possibly larger, Hilbert space $\mathcal{K}$, i.e. there exists a bounded operator $V : \mathcal{K} \to \mathcal{H}$ such that*

$$P(\cdot) = V\Pi(\cdot)V^\dagger . \tag{2.13}$$

In the finite-dimensional setting, V can be chosen to be an isometry.

Recall the content of **??**.

Definition 2.3.3 (Completely positive trace-preserving). Consider a map $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ between bounded operators on two (finite-dimensional) Hilbert spaces. This map preserves density matrices if it positive (**??**) and if it is trace-preserving (**??**). Furthermore, to ensure that an operation applied to a subsystem does not interfere with the positivity of the complete system, they are also required to be completely positive (**??**).

Completely positive, trace-preserving (CPTP) maps are often called **quantum channels** or **superoperators**.

Theorem 2.3.4 (Choi–Jamiołkowski). *The following map between quantum channels $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ and density operators $\rho \in \mathrm{End}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is an isomorphism:*

$$\Phi \mapsto (\mathbb{1}_{\mathcal{H}_1} \otimes \Phi)|\mathsf{GHZ}\rangle\langle\mathsf{GHZ}| , \tag{2.14}$$

where the GHZ state was introduced in Definition 2.1.7.

2.4 Quantum logic

2.4.1 Birkhoff–von Neumann logic

Consider classical propositional logic. This is governed by the Boolean property, i.e. the set of all propositions admits the structure of a complete Boolean algebra (**??**). Now, the question becomes how to treat propositions in quantum mechanics (as needed in, for example, quantum computing). In the sense of *von Neumann*, the propositions should be characterized by $\{0, 1\}$-valued observables or, equivalently, by projection operators. As such, the natural lattice to consider logic is that of closed subspaces of the state space $\mathcal{H}$ (which is a Hilbert space). Birkhoff–von Neumann logic is the study of such lattices.

Note that, in contrast to classical logic, the lattices of closed subspaces are not Boolean. The lattices are merely complete, orthomodular lattices (**??**).[2] Now, although these lattices are themselves very interesting, there relevance for quantum logic are heavily discussed for several reasons (e.g. lack of distributivity, lack of a clear implication operator, lack of an extension to predicate logic). In the next section it will be explained how this issue can be avoided by embedding Birkhoff–von Neumann logic into the more general framework of linear logic.

2.4.2 Linear logic

A basic component of standard logic are the *structural inference rules*. These are inference rules that do not involve any logical operations. The following two inference rules for first-order logic control context extension (for an introduction to sequent calculus, see **??**):

- **Contraction**:

$$\frac{\Gamma, p_1 : P, p_2 : P \vdash t_{p_1,p_2} : T}{\Gamma, p : P \vdash t_{p,p} : T\,.} \tag{2.15}$$

  This rule states that, in a valid judgement, premises might be used more than once.

- **Weakening**:

$$\frac{\Gamma \vdash P : \mathsf{Type} \qquad \Gamma \vdash t : T}{\Gamma, P \vdash t : T\,.} \tag{2.16}$$

  This rule states that, any premise can be added to (the premises of) a valid judgement.

In terms of categorical semantics, these two rules correspond (in the independent setting) to the diagonal and projection morphisms in Cartesian categories (**??**).

Now, when considering quantum mechanics, two important results are the no-cloning and no-deleting theorem (Theorem 2.1.3 and Theorem 2.1.4). These correspond to the fact that the categories **FinVect** and **Hilb** are monoidal, but not Cartesian monoidal, i.e. the tensor product does not admit diagonal and projection morphisms. The natural type of logic in this setting is than a *substructural* one where the contraction and weakening rules are not valid.

[2] It should be noted that complete, orthomodular lattices are, in general, very different from those originating from Hilbert spaces (cf. *Piron's theorem*).

In linear logic, the following propositions exist:

1. **Variables**: Every propositional variable is a proposition.
2. **Negation**: If P is a proposition, so is $P^{\perp}$.
3. **Connectives**: If P, Q are propositions, then
   - **Additive conjunction**: $P\&Q$ is a proposition. (Read: P with Q.)
   - **Additive disjunction**: $P \oplus Q$ is a proposition. (Read: P plus Q.)
   - **Multiplicative conjunction**: $P \otimes Q$ is a proposition. (Read: P times Q.)
   - **Multiplicative disjunction**: $P \parr Q$ is a proposition. (Read: P par Q.)
4. **Constants**:
   - **Additive truth**: $\top$,
   - **Additive falsity**: $\mathbf{0}$,
   - **Multiplicative truth**: $\mathbf{1}$, and
   - **Multiplicative falsity**: $\perp$.
5. **Exponential connectives**: If P is a proposition, then
   - **Exponential conjunction**: $!P$ is a proposition. (Read: of course P.)
   - **Exponential disjunction**: $?P$ is a proposition. (Read: why not P.)

Given a context, the following inference rules are valid:[3]

1. **Identity***: If P is a propositional variable, then $P \vdash P$.
2. **Exchange***: Sequents remain valid under permutations.
3. **Restricted weakening***: If P is a proposition, then

$$\frac{\Gamma \vdash \Theta}{\Gamma, !P \vdash \Theta} \tag{2.17}$$

   and, dually,

$$\frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, ?P}. \tag{2.18}$$

[3]The form of these rules heavily depends on the exchange rule (the second item). Care must be taken if this rule is weakened.

4. **Restricted contraction***: If P is a proposition, then

$$\frac{!P, !P \vdash \Theta}{!P \vdash \Theta} \tag{2.19}$$

and, dually,

$$\frac{\Gamma \vdash ?P, ?P}{\Gamma \vdash ?P}. \tag{2.20}$$

5. **Negation**: If P is a proposition, then

$$\frac{\Gamma \vdash \Theta, P}{\Gamma, P^{\perp} \vdash \Theta} \tag{2.21}$$

and, conversely,

$$\frac{\Gamma, P \vdash \Theta}{\Gamma \vdash \Theta, P^{\perp}}. \tag{2.22}$$

Note that these rules allow to write any sequent in right form, i.e. $\vdash \Gamma^{\perp}, P$.

6. **Additive conjunction**: If P, Q are propositions, then

$$\frac{P \vdash \Theta}{P\&Q \vdash \Theta} \qquad \frac{Q \vdash \Theta}{P\&Q \vdash \Theta} \tag{2.23}$$

and, conversely,

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P\&Q}. \tag{2.24}$$

7. **Additive disjunction**: If P, Q are propositions, then

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \oplus Q} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \oplus Q} \tag{2.25}$$

and, conversely,

$$\frac{P \vdash \Theta \qquad Q \vdash \Theta}{P \oplus Q \vdash \Theta}. \tag{2.26}$$

8. **Multiplicative conjunction**: If P, Q are propositions, then

$$\frac{P, Q \vdash \Theta}{P \otimes Q \vdash \Theta} \tag{2.27}$$

and, conversely,

$$\frac{\Gamma \vdash P \qquad \Lambda \vdash Q}{\Gamma, \Lambda \vdash P \otimes Q}. \tag{2.28}$$

9. **Multiplicative disjunction**: If P, Q are propositions, then

$$\frac{\Gamma \vdash P, Q}{\Gamma \vdash P \parr Q} \tag{2.29}$$

and, conversely,

$$\frac{P \vdash \Delta \qquad Q \vdash \Theta}{P \parr Q \vdash \Delta, \Theta}. \tag{2.30}$$

10. **Truth and falsity**:

$$\begin{array}{cc} \Gamma \vdash \top & \mathbf{0} \vdash \Theta \\ \dfrac{\Gamma \vdash \Theta}{\Gamma, \mathbf{1} \vdash \Theta} & \vdash \mathbf{1} \\ \dfrac{\vdash \Theta}{\vdash \Theta, \bot} & \bot \vdash \;. \end{array} \tag{2.31}$$

11. **Exponential conjunction**: If P is a proposition, then

$$\frac{P \vdash \Theta}{!P \vdash \Theta} \tag{2.32}$$

and, conversely, whenever Γ consists solely of !-propositions and Θ consists solely of ?-propositions,

$$\frac{\Gamma \vdash P}{\Gamma \vdash !P}. \tag{2.33}$$

12. **Exponential disjunction**: If P is a proposition, then

$$\frac{\Gamma \vdash P}{\Gamma \vdash ?P} \tag{2.34}$$

and, conversely, whenever Γ consists solely of !-propositions and Θ consists solely of ?-propositions,

$$\frac{P \vdash \Theta}{?P \vdash \Theta}. \tag{2.35}$$

The inference rules with an asterisk are the structural rules. Note that a *cut-elimination theorem* holds and, hence, the identity and cut rules for general propositions can be derived from the rules above.

Linear implication is characterized as follows:

$$P \vdash Q \iff \vdash P^{\perp} \parr Q \iff \vdash P \multimap Q\,. \tag{2.36}$$

Remark 2.4.1 (Resource theory). Before passing to the properties that follow from the basic rules and the categorical semantics of linear logic (eventually passing to linear type theory), it is useful to rephrase the connectives and their inference rules in terms of 'resources'.

In this interpretation, an implication $A \implies B$ would mean that the resources A can be used to obtain the resources B. However, in ordinary logic, if A and $A \implies B$ hold, one can derive that B holds, but A also still holds. This is something that does not work with resources. If you can use resources A to construct B, the resources A are (usually[4]) used up.[5] One, hence, needs a more subtle and nuanced framework to capture these notions: linear logic. The implication that will be used, where resources are spent, is denoted by $A \multimap B$ for clarity. The two conjunctives, $\otimes$ and &, mean that two resources are available concurrently and separately, respectively. So, if $A \multimap B$ holds, then, since the contraction rule is not valid, one does not have $A \multimap B \otimes B$. However, one does have $A \otimes A \multimap B \otimes B$.

The interpretation of the connectives is as follows:

- $A \otimes B$: A and B are both available for use at the same time, e.g. one has a warehouse with A and a warehouse with B at the same time.
- $A \& B$: Either A or B are available for use, but not both.
- $A \parr B$:
- $A \oplus B$:

The connectives in linear logic satisfy (or generalize) many of the properties of ordinary logic.

Property 2.4.2 (Distributivity).

$$
\begin{aligned}
P \otimes (Q \oplus R) &= (P \otimes Q) \oplus (P \otimes R) \\
P \parr (Q \& R) &= (P \parr Q) \& (P \parr R) \\
P \otimes \mathbf{0} &= \mathbf{0} \\
P \parr \top &= \top
\end{aligned}
\tag{2.37}
$$

The exponential connectives can be used to turn additive connectives into multiplicative ones (and the other way around) as with the ordinary exponential function in calculus.

[4]This is not the case with catalysts.

[5]One could also give this a causal flavour (Girard, 1995).

Property 2.4.3.

$$
\begin{aligned}
!(P\&Q) &= !P \otimes !Q \\
?(P \oplus Q) &= !P \parr !Q \\
!\top &= \mathbf{1} \\
?\mathbf{0} &= \perp
\end{aligned}
\tag{2.38}
$$

Moreover, due to the apparent similarity with the operators in (S4) modal logic (**??**), the exponential connectives are sometimes also called **modalities**.

Linear negation can also be defined alternatively.

Property 2.4.4 (Negation). Linear negation admits the following recursive definition:

- $P^{\perp\perp} = P$,
- $(P\&Q)^\perp = P^\perp \oplus Q^\perp$,
- $(P \otimes Q)^\perp = P^\perp \parr Q^\perp$,
- $\top^\perp = \mathbf{0}$,
- $\mathbf{1}^\perp = \perp$, and
- $(!P)^\perp = ?P^\perp$.

Property 2.4.5 (Categorical semantics). Whereas standard Boolean logic is the internal logic of Cartesian closed categories — where conjunction, disjunction and implication correspond, respectively to products, coproducts and internal homs — linear logic is the internal logic of (a subclass of) $*$-autonomous categories (**??**).

The multiplicative conjunction $\otimes$ corresponds to the tensor product, hence the notation. Similar to ordinary logic, the linear implication $\multimap$ corresponds to taking internal homs. The important part, now, is that negation comes as a separate entity, in this case given by taking duals: $x^\perp \equiv x^*$. The multiplicative disjunction $\parr$ is then constructed through Property 2.4.4 (which corresponds to de Morgan duality as in **??**).

For the additive connectives, one needs the existence of finite products. The $*$-autonomy then also implies the existence of finite coproducts (again through de Morgan duality).

For the exponential connectives, some more structure is needed. As with modal logic, the structure is given by the existence of a suitable (co)monad.

2.5 Topos theory ♣

2.5.1 Bohr topos

Definition 2.5.1 (Bohr topos). Consider a C^*-algebra A and denote by $\mathsf{ComSub}(A)$ the poset (**??**) of commutative C^*-subalgebras. This set can be equipped with the **Alexandrov topology**[6], i.e. the topology for which the open sets are the upward closed subsets. The topological space $(\mathsf{ComSub}(A), \tau_{\text{Alex}})$ is called the Bohr site of A.

The sheaf topos over the Bohr site is called the Bohr topos $\mathbf{Bohr}(A)$. It can be turned into a ringed topos, where the ring object (which is even an internal commutative C^*-algebra) is given by the tautological functor

$$\underline{A} : \mathsf{ComSub}(A) \rightarrow \mathbf{Set} : C \mapsto C\,. \tag{2.39}$$

Property 2.5.2. A morphism in $\mathbf{C^*Alg}$ is commutativity reflecting if and only if the induced morphism on posets admits a right adjoint. Moreover, there exists a bijection between the following two classes of morphisms:

- Geometric morphisms $f : \mathbf{Bohr}(B) \rightarrow \mathbf{Bohr}(A)$ admitting a right adjoint together with epimorphisms of internal algebras $\underline{A} \rightarrow f^*\underline{B}$.
- Commutativity-reflecting functions $f : A \rightarrow B$ that restrict to algebra morphisms on all commutative subalgebras.

Definition 2.5.3 (Spectral presheaf). The presheaf Σ on a Bohr site assigning to every commutative subalgebra its Gel'fand spectrum.

The idea behind the Bohr topos is that, given a general C^*-algebra A, the Bohr topos $\mathbf{Bohr}(A)$ is interpreted as its quantum phase space. This is similar to **??**, where smooth spaces are also reinterpreted in terms of sheaf topoi.

Theorem 2.5.4 (Kochen–Specker). *For $A = \mathcal{B}(\mathcal{H})$, the spectral presheaf has no global elements if $\dim(\mathcal{H}) > 2$.*

Property 2.5.5 (Gleason's theorem). There exists a natural bijection between the quantum states of a C^*-algebra A and the classical states of $\underline{A}$ internal to $\mathbf{Bohr}(A)$.

Definition 2.5.6 (Bohrification). Consider a C^*-algebra A together with its Bohr topos $\mathbf{Bohr}(A)$. To its internal C^*-algebra $\underline{A}$, one can assign an internal locale $\underline{\Sigma}_A$ by (internal) Gel'fand duality (**??**). Under the equivalence **??**, one then obtains a locale Σ_A. The functor

$$\Sigma : \mathbf{C^*Alg} \rightarrow \mathbf{Loc} : A \mapsto \Sigma_A \tag{2.40}$$

[6]There exist an equivalences $\mathbf{Pre} \cong \mathbf{AlexTop}$ and $\mathbf{Pos} \cong \mathbf{AlexTop}_{T_0}$.

is called Bohrification. This locale can be constructed as the disjoint union

$$\Sigma_A = \bigsqcup_{C \in \mathsf{ComSub}(A)} \Phi_C \,, \tag{2.41}$$

the étale locale corresponding to the spectral presheaf (i.e. the spectral presheaf is the internal Gel'fand spectrum of $\underline{A}$). Its open sets are given by those subsets whose restrictions to commutative subalgebras are open in such a way that these restrictions are compatible with subalgebra inclusions.

Example 2.5.7 (Gel'fand spectrum). If A is a commutative C^*-algebra, its Bohrification is not isomorphic to its ordinary Gel'fand spectrum Φ_A. However, after replacing the topology on $\mathbf{Bohr}(A)$ by the double negation topology (**??**) and repeating the above construction, one obtains

$$\Phi_A \cong \Sigma_A^{\neg\neg} \,. \tag{2.42}$$

This locale can also be obtained in another way. Double negation $\neg\neg$ defines an (internal) **nucleus** on the (internal) locale Σ_A, i.e. a left-exact monad. $\Sigma_A^{\neg\neg}$ is then given by the fixed points of $\neg\neg : \Sigma_A \to \Sigma_A$.

By Property 2.5.2 above, the following relation is obtained.

Property 2.5.8 (Observables). Morphisms $\mathbf{Bohr}(A) \to \mathbf{Bohr}(C(\mathbb{R})_0)$ admitting a right adjoint together with an epimorphism $\underline{C_0(\mathbb{R})} \to f^*\underline{A}$ correspond to observables on A.

The topological bundle $\Sigma_A \to \mathsf{Alex}(\mathsf{ComSub}(A))$ also admits a topos-theoretic incarnation. There exists a (canonical) morphism of ringed topoi

$$\pi : \mathbf{Bohr}(A) \to \Big(\mathbf{Sh}\big(\mathsf{Alex}(\mathsf{ComSub}(A))\big), \underline{\mathbb{R}}\Big) \,, \tag{2.43}$$

whose underlying geometric morphism is simply the identity.

Property 2.5.9 (States). A positive and normalized section of the morphism $\pi : \mathbf{Bohr}(A) \to \Big(\mathbf{Sh}\big(\mathsf{Alex}(\mathsf{ComSub}(A))\big), \underline{\mathbb{R}}\Big)$ in the category of $\underline{\mathbb{R}}$*-module topoi*.

2.5.2 Internal logic

Whereas Section 2.4 covers quantum logic from the external point of view, the spectral presheaf (Definition 2.5.3) allows to treat it internally. By functional calculus (**??**), every proposition $A \in \Delta$ about a self-adjoint operator on a Hilbert space $\mathcal{H}$, i.e. every measurable subset of the spectrum $\sigma(A)$, corresponds to a projection $P_{A,\Delta}$. Now, the goal will be to assign to every such projection a 'classical interpretation' in the classical

context given by a commutative subalgebra of A. This is achieved by the subobject of the spectral presheaf Σ constructed as follows:

$$\delta_P(V) := \{\omega \in \Sigma_v \mid \omega(P|_V) = 1\}\,,$$

where $P|_V$ is the smallest projection in V such that $\mathsf{im}(P) \subseteq \mathsf{im}(P|_V)$. The morphism

$$\delta : \mathcal{P}(\mathcal{H}) \to \mathsf{Sub}(\Sigma)$$

is called the **daseinization** map. Note that this map extends to all self-adjoint operators through functional calculus.

A 'pure state' ψ in classical mechanics corresponds to a map from subsets of phase space to the subobject classifier (saying whether the point lies in the given subset or not):

$$T^{\psi} : \mathsf{Sub}(P) \to \{0, 1\}\,.$$

By analogy, a pure state $|\psi\rangle$ in quantum mechanics will correspond to a morphism from subobjects of the spectral presheaf:

$$T^{|\psi\rangle} : \mathsf{Sub}(\Sigma) \to \mathsf{Hom}_{\mathbf{Psh}(\mathsf{ComSub}(A))}(1, \Omega)\,,$$

where Ω is the subobject classifier from **??**.

List of Symbols

The following abbreviations and symbols are used throughout the compendium.

Abbreviations

AIC	Akaike information criterion
ARMA	autoregressive moving-average model
BCH	Baker–Campbell–Hausdorff
BPS	Bogomol'nyi–Prasad–Sommerfield
BPST	Belavin–Polyakov–Schwarz–Tyupkin
BRST	Becchi–Rouet–Stora–Tyutin
CCR	canonical commutation relation
CDF	cumulative distribution function
CFT	conformal field theory
CIS	completely integrable system
CP	completely positive
CPTP	completely positive, trace-preserving
CR	Cauchy–Riemann
dga	differential graded algebra
dgca	differential graded-commutative algebra
EMM	equivalent martingale measure
EPR	Einstein–Podolsky–Rosen
ESM	equivalent separating measure
ETCS	Elementary Theory of the Category of Sets
FIP	finite intersection property
FWHM	full width at half maximum
GA	geometric algebra
GHZ	Greenberger–Horne–Zeilinger

GNS	Gel′fand–Naimark–Segal
HJE	Hamilton–Jacobi equation
HoTT	Homotopy Type Theory
KKT	Karush–Kuhn–Tucker
LIVF	left-invariant vector field
MCG	mapping class group
MPO	matrix-product operator
MPS	matrix-product state
MTC	modular tensor category
NDR	neighbourhood deformation retract
OPE	operator product expansion
OTC	over the counter
OZI	Okubo–Zweig–Iizuka
PAC	probably approximately correct
PDF	probability density function
PID	principal ideal domain
PL	piecewise-linear
PMF	probability mass function
POVM	positive operator-valued measure
PRP	predictable representation property
PVM	projection-valued measure
RKHS	reproducing kernel Hilbert space
SVM	support-vector machine
TDSE	time-dependent Schrödinger equation
TISE	time-independent Schrödinger equation
TQFT	topological quantum field theory
TVS	topological vector space
UFD	unique factorization domain
VC	Vapnik–Chervonenkis
VIF	variance inflation factor
VOA	vertex operator algebra
WKB	Wentzel–Kramers–Brillouin

ZFC	Zermelo–Frenkel set theory with the axiom of choice

Operations

Ad_g	adjoint representation of a Lie group G
ad_X	adjoint representation of a Lie algebra $\mathfrak{g}$
arg	argument of a complex number
$\square$	d'Alembert operator
$\mathsf{deg}(f)$	degree of a polynomial f
e	identity element of a group
$\Gamma(E)$	set of global sections of a fibre bundle E
Im, $\mathfrak{I}$	imaginary part of a complex number
$\mathsf{Ind}_f(z)$	index of a point $z \in \mathbb{C}$ with respect to a function f
$\hookrightarrow$	injective function
$\cong$	is isomorphic to
$A \multimap B$	linear implication
$N \lhd G$	N is a normal subgroup of G
Par_t^γ	parallel transport map along a curve γ
Re, $\mathfrak{R}$	real part of a complex number
Res	residue of a complex function
$\twoheadrightarrow$	surjective function
$\{\cdot,\cdot\}$	Poisson bracket
$X \pitchfork Y$	transversally intersecting manifolds X, Y
∂X	boundary of a topological space X
$\overline{X}$	closure of a topological space X
X°, $\mathring{X}$	interior of a topological space X
$\sphericalangle(\cdot,\cdot)$	angle between two vectors
$X \times Y$	cartesian product of two sets X, Y
$X + Y$	sum of two vector spaces X, Y
$X \oplus Y$	direct sum of two vector spaces X, Y
$V \otimes W$	tensor product of two vector spaces V, W
$\mathbb{1}_X$	identity morphism on an object X
$\approx$	is approximately equal to

$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto

Objects

Ab	category of Abelian groups
$\mathsf{Aut}(X)$	automorphism group of an object X
$\mathcal{B}_0(V, W)$	space of compact bounded operators between two Banach spaces V, W
$\mathcal{B}_1(\mathcal{H})$	space of trace-class operators on a Hilbert space
$\mathcal{B}(V, W)$	space of bounded linear maps between two vector spaces V, W
CartSp	category of Euclidean spaces and 'suitable' morphisms (e.g. linear maps, smooth maps, ...)
$C(X, Y)$	set of continuous functions between two topological spaces X, Y
S'	centralizer of a subset (of a ring)
$C_\bullet$	chain complex
$\mathbf{Ch}(\mathbf{A})$	category of chain complexes with objects in an additive category $\mathbf{A}$
$\mathbf{C}^\infty$, **SmoothSet**	category of smooth sets
$C^\infty_p(M)$	ring of smooth functions $f : M \to \mathbb{R}$ on a neighbourhood of $p \in M$
$C\ell(A, Q)$	Clifford algebra over an algebra A induced by a quadratic form Q
$C^\omega(V)$	set of all analytic functions defined on a set V
$\mathsf{Conf}(M)$	conformal group of a (pseudo-)Riemannian manifold M
$\mathbf{C}^\infty\mathbf{Ring}$, $\mathbf{C}^\infty\mathbf{Alg}$	category of smooth algebras
$S_k(\Gamma)$	space of cusp forms of weight $k \in \mathbb{R}$
Δ_X	diagonal of a set X
Diff	category of smooth manifolds
DiffSp	category of diffeological spaces and smooth maps
$\mathcal{D}_M$	sheaf of differential operators
D^n	standard n-disk
$\mathsf{dom}(f)$	domain of a function f
$\mathsf{End}(X)$	endomorphism monoid of an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad
$\mathbf{FormalCartSp}_{\mathbf{diff}}$	category of infinitesimally thickened Euclidean spaces

$\mathsf{Frac}(I)$	field of fractions of an integral domain I
$\mathfrak{F}(V)$	space of Fredholm operators on a Banach space V
$\mathbb{G}_a$	additive group (scheme)
$\mathsf{GL}(V)$	general linear group: group of automorphisms of a vector space V
$\mathsf{GL}(n, \mathfrak{K})$	general linear group: group of invertible $n \times n$-matrices over a field $\mathfrak{K}$
Grp	category of groups and group homomorphisms
Grpd	category of groupoids
$\mathsf{Hol}_p(\omega)$	holonomy group at a point p with respect to a principal connection ω
$\mathsf{Hom}_{\mathbf{C}}(V, W)$, $\mathbf{C}(V, W)$	collection of morphisms between two objects V, W in a category $\mathbf{C}$
hTop	homotopy category
$I(S)$	vanishing ideal on an algebraic set S
$I(x)$	rational fractions over an integral domain I
$\mathsf{im}(f)$	image of a function f
$K^0(X)$	K-theory over a (compact Hausdorff) space X
Kan	category of Kan complexes
$K(A)$	Grothendieck completion of a monoid A
$\mathcal{K}_n(A, v)$	Krylov subspace of dimension n generated by a matrix A and a vector v
L^1	space of integrable functions
Law	category of Lawvere theories
Lie	category of Lie groups
$\mathfrak{Lie}$	category of Lie algebras
$\mathfrak{X}^L$	space of left-invariant vector fields on a Lie group
$\mathsf{llp}(I)$	set of morphisms having the left lifting property with respect to I
LX	free loop space on a topological space X
$\mathbf{Man}^p$	category of C^p-manifolds
Meas	• category of measurable spaces and measurable functions, or • category of measure spaces and measure-preserving functions
M^4	four-dimensional Minkowski space
$M_k(\Gamma)$	space of modular forms of weight $k \in \mathbb{R}$
$\mathbb{F}^X$	natural filtration of a stochastic process $(X_t)_{t \in T}$
$N\mathbf{C}$	simplicial nerve of a small category $\mathbf{C}$

$\mathsf{O}(n, \mathfrak{K})$	group of $n \times n$ orthogonal matrices over a field $\mathfrak{K}$
$\mathbf{Open}(X)$	category of open subsets of a topological space X
$P(X), 2^X$	power set of a set X
$\mathsf{Pin}(V)$	pin group of the Clifford algebra $C\ell(V, Q)$
$\mathbf{Psh}(\mathbf{C}), \widehat{\mathbf{C}}$	category of presheaves on a (small) category $\mathbf{C}$
$R((x))$	ring of (formal) Laurent series in x with coefficients in R
$\mathsf{rlp}(I)$	set of morphisms having the right lifting property with respect to I
$R[[x]]$	ring of (formal) power series in x with coefficients in R
S^n	standard n-sphere
$S^n(V)$	space of symmetric rank n tensors over a vector space V
$\mathbf{Sh}(X)$	category of sheaves on a topological space X
$\mathbf{Sh}(\mathbf{C}, J)$	category of J-sheaves on a site $(\mathbf{C}, J)$
$\boldsymbol{\Delta}$	simplex category
$\mathsf{sing\ supp}(\phi)$	singular support of a distribution ϕ
$\mathsf{SL}_n(\mathfrak{K})$	special linear group: group of all $n \times n$-matrices with unit determinant over a field $\mathfrak{K}$
$W^{m,p}(U)$	Sobolev space in L^p of order m
$\mathbf{Span}(\mathbf{C})$	span category over a category $\mathbf{C}$
$\mathsf{Spec}(R)$	spectrum of a commutative ring R
$\mathbf{sSet}_{\text{Quillen}}$	Quillen's model structure on simplicial sets
$\mathsf{supp}(f)$	support of a function f
$\mathsf{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
$\mathsf{Sym}(X)$	symmetric group of a set X
S_n	symmetric group of degree n
$\mathsf{Sym}(X)$	symmetric group on a set X
$\mathsf{Sp}(n, \mathfrak{K})$	group of matrices preserving a canonical symplectic form over a field $\mathfrak{K}$
$\mathsf{Sp}(n)$	compact symplectic group
$\mathbb{T}^n$	standard n-torus (n-fold Cartesian product of S^1)
$T_{\leq t}$	set of all elements smaller than (or equal to) $t \in T$ for a partial order T
$\mathsf{TL}_n(\delta)$	Temperley–Lieb algebra with $n-1$ generators and parameter δ
$\mathbf{Top}$	category of topological spaces and continuous functions
$\mathbf{Topos}$	(2-)category of (elementary) topoi and geometric morphisms

$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\mathsf{U}(n, \mathfrak{K})$	group of $n \times n$ unitary matrices over a field $\mathfrak{K}$
$V(I)$	algebraic set corresponding to an ideal I
$\mathbf{Vect}(X)$	category of vector bundles over a manifold X
$\mathbf{Vect}_{\mathfrak{K}}$	category of vector spaces and linear maps over a field $\mathfrak{K}$
Y^X	set of functions between two sets X, Y
$\mathbb{Z}_p$	group of p-adic integers
$\emptyset$	empty set
$\pi_n(X, x_0)$	n^{th} homotopy space over X with basepoint x_0
$[a, b]$	closed interval
$]a, b[$	open interval
$\Lambda^n(V)$	space of antisymmetric rank-n tensors over a vector space V
ΩX	(based) loop space on a topological space X
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on a manifold M
$\rho(A)$	resolvent set of a bounded linear operator A
$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on a manifold M

Units

C	Coulomb
T	Tesla

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