Compendium of Mathematics & Physics

Nicolas Dewolf

July 15, 2022

Contents

Chapter 1

Set Theory

1.1 Axiomatization

1.1.1 ZFC

The following set of axioms and axiom schemata gives a basis for axiomatic set theory that fixes a number of issues in naive set theory where one takes the notion of set for granted. This theory is called **Zermelo-Frenkel** set theory (ZF). When extended with the axiom of choice (see further) it is called ZFC, where the C stand for "choice".

Axiom 1.1 (Power set).

$$\forall x : \exists y : \forall z \big[z \in y \iff \forall w (w \in z \implies w \in x) \big] \tag{1.1}$$

The set y is called the power set $P(x)$ of x.

Axiom 1.2 (Extensionality).

$$\forall x, y : \forall z \big[z \in x \iff z \in y \big] \implies x = y \tag{1.2}$$

This axiom allows us to compare two sets based on their elements.

Axiom 1.3 (Regularity[1]).

$$\forall x : (\exists z \in x) \implies (\exists a \in x) \wedge \neg(\exists b \in a : b \in x) \tag{1.3}$$

This axiom says that for every non-empty set x one can find an element $a \in x$ such that x and a are disjoint. Among other things this axiom implies that no set can contain itself.

The following axiom is technically not an axiom but an axiom schema, i.e. for every predicate φ one obtains an axiom:

Axiom 1.4 (Specification).

$$\forall w_1, \ldots, w_n, A : \exists B : \forall x \big(x \in B \iff (x \in A \wedge \varphi(x, w_1, \ldots, w_n, A)) \big) \tag{1.4}$$

This axiom (schema) says that for every set x one can build another set of elements in x that satisfy a given predicate. By the axiom of extensionality this subset $B \subseteq A$ is unique.

[1] Also called the **axiom of foundation**.

1.1.2 Material set theory

ZF(C) is an instance of material set theory. Every element of a set is itself a set and, hence, has some kind of internal structure.

Definition 1.1.1 (Pure set). A set U such that for every sequence $x_n \in x_{n-1} \in \cdots \in x_1 \in U$ all the elements x_i are also sets.

Definition 1.1.2 (Urelement[2]). An object that is not a set.

1.1.3 Universes

?? TODO ??

To be able to talk about sets without running into problems such as *Russel's paradox*, where one needs (or wants) to talk about the collection of all things satisfying a certain condition, one can introduce the concept of a universal set or universe (of discourse). This set takes the place of the ‘‘collection of things’’ and all operations performed on its elements, i.e. the sets that one wants to work with, act within this universe.

Definition 1.1.3 (Grothendieck universe). A Grothendieck universe U is a pure set satisfying the following axioms:

1. **Transitivity**: If $x \in U$ and $y \in x$, then $y \in U$;
2. **Power set**: If $x \in U$, then $P(x) \in U$;
3. **Pairing**: If $x, y \in U$, then $\{x, y\} \in U$; and
4. **Unions**: If $I \in U$ and $\{x_i\}_{i\in I} \subset U$, then $\bigcup_{i\in I} x_i \in U$.

1.1.4 Structural set theory

In contrast to material set theory, the fundamental notions in this theory are sets and the relations between them. An element of a set does not have any internal structure and only becomes relevant if one specifies extra structure (or relations) on the sets. This implies that elements of sets are not sets themselves. In fact this would be a meaningless statement since, by default, they lack internal structure. Even stronger, it is meaningless to compare two elements if one does not provide relations or extra structure on the sets.

?? COMPLETE ??

1.1.5 ETCS ♣

?? COMPLETE ??

Remark. ETCS is the abbreviation of ‘‘Elementary Theory of the Category of Sets’’.

Axiom 1.5. The category of sets is a well-pointed (elementary) topos.

1.1.6 Real numbers

Axiom 1.6 (Ordering). The set of real numbers is an ordered field $(\mathbb{R}, +, \cdot, <)$.

Axiom 1.7 (Dedekind completeness). Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum.

Axiom 1.8. The rational numbers form a subset of the real numbers: $\mathbb{Q} \subset \mathbb{R}$.

---

[2]Sometimes called an **atom**.

Remark. There is only one way to extend the field of rational numbers to the field of reals such that it satisfies the previous axioms. This implies that for every two possible constructions, there exists a bijection between the two.

Definition 1.1.4 (Extended real line).

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\} \equiv [-\infty, \infty] \tag{1.5}$$

1.2 Set operations

Definition 1.2.1 (Symmetric difference).

$$A\Delta B := (A\backslash B) \cup (B\backslash A) \tag{1.6}$$

Definition 1.2.2 (Complement). Let Ω be the universe of discours (Section 1.1.3) and let $E \subseteq \Omega$. The complement of E is defined as follows:

$$E^c := \Omega\backslash E. \tag{1.7}$$

Formula 1.2.3 (de Morgan's laws).

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c \tag{1.8}$$

$$\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c \tag{1.9}$$

Definition 1.2.4 (Relation). A relation between sets X and Y is a subset of the Cartesian product $X \times Y$. A relation on X is then simply a subset of $X \times X$. This definition can easily be extended to n-ary relations by working with subsets of n-fold products.

Definition 1.2.5 (Converse relation). Consider a relation $R \subset X \times Y$ between two sets X, Y. The converse relation R^t is defined as follows:

$$R^t := \big\{(y, x) \in Y \times X \,\big|\, (x, y) \in R\big\}. \tag{1.10}$$

Definition 1.2.6 (Composition of relations). Consider two relations $R \subset X \times Y$ and $S \subset Y \times Z$ between three sets X, Y and Z. The composition $S \circ R$ is defined as follows:

$$S \circ R := \big\{(x, z) \in X \times Z \,\big|\, \exists y \in Y : (x, y) \in R \wedge (y, z) \in S\big\}. \tag{1.11}$$

1.3 Functions

1.3.1 (Co)domain

Definition 1.3.1 (Domain). Let $f : X \to Y$ be a function. The set X is called the domain of f.

Notation 1.3.2. The domain of f is denoted by $\mathrm{dom}(f)$.

Definition 1.3.3 (Support). Let $f : X \to \mathbb{R}$ be a function with an arbitrary domain X. The support of f is defined as the set of points where f is nonzero.

Notation 1.3.4. The support of f is denoted by $\mathrm{supp}(f)$.

Notation 1.3.5. Let X, Y be two sets. The set of functions $f : X \to Y$ is denoted by Y^X or $\mathrm{Map}(X, Y)$. (See also Definition 2.6.22 for a generalization.)

Definition 1.3.6 (Codomain). Let $f : X \to Y$ be a function. The set Y is called the codomain of f.

Definition 1.3.7 (Image). Let $f : X \to Y$ be a function. The following subset of Y is called the image of f:

$$\mathrm{im}(f) := \{y \in Y \mid \exists x \in X : f(x) = y\}. \tag{1.12}$$

Remark. Some authors use the notions of codomain and image interchangeably.

Definition 1.3.8 (Level set). Consider a function $f : X \to \mathbb{R}$. The following set is called the level set of f at $c \in \mathbb{R}$:

$$f^{-1}(c) := \{x \in X \mid f(x) = c\}. \tag{1.13}$$

For $X = \mathbb{R}^2$ the level sets are called **level curves** and for $X = \mathbb{R}^3$ they are called **level surfaces**.

1.3.2 Functions

Definition 1.3.9 (Injective). A function $f : A \to B$ is said to be injective or **one-to-one** if the following condition is satisfied:

$$\forall a, a' \in A : f(a) = f(a') \implies a = a'. \tag{1.14}$$

Notation 1.3.10 (Injective function).

$$f : A \hookrightarrow B$$

Definition 1.3.11 (Surjective). A function $f : A \to B$ is said to be surjective or **onto** if the following condition is satisfied:

$$\forall b \in B, \exists a \in A : f(a) = b. \tag{1.15}$$

Notation 1.3.12 (Surjective function).

$$f : A \twoheadrightarrow B$$

Definition 1.3.13 (Bijection). A function that has an inverse. Equivalently, a function that gives a one-to-one correspondence between the elements of the domain and those of the codomain.

Notation 1.3.14 (Isomorphic sets).

$$X \cong Y$$

Theorem 1.3.15 (Cantor-Bernstein-Schröder). *Consider two sets A, B. If there exist injections $A \hookrightarrow B$ and $B \hookrightarrow A$, there exists a bijection $A \cong B$.*

Definition 1.3.16 (Involution). A function $f : A \to A$ such that $f^2 = \mathrm{id}_A$, i.e. f is its own inverse. Every involution is in particular a bijection.

1.4 Collections

1.4.1 Families and filters

Definition 1.4.1 (Power set). Let S be a set. The power set is defined as the set of all subsets of S and is (often) denoted by $P(S)$ or 2^S. The existence of this set is enforced by the power set axiom 1.1.

Corollary 1.4.2. All sets are elements of their power set: $S \in P(S)$.

Definition 1.4.3 (Collection). Let A be a set. A collection of elements in A is a subset of A.

Definition 1.4.4 (Family). Let A, I be two sets. A family of elements of A with **index set** I is a function $f : I \to A$. A family with index set I is often denoted by $(x_i)_{i\in I}$. In contrast to collections, a family can "contain" multiple copies of the same element.

Definition 1.4.5 (Helly family). A Helly family of order k is a pair (X, F) with $F \subset P(X)$ such that for every finite $G \subset F$:

$$\bigcap_{V\in G} V = \emptyset \implies \exists H \subseteq G : \left(\bigcap_{V\in H} V = \emptyset\right) \wedge \Big(|H| \leq k\Big). \tag{1.16}$$

A Helly family of order 2 is sometimes said to have the **Helly property**.

Definition 1.4.6 (Diagonal). The diagonal of a set S is defined as follows:

$$\Delta_S := \big\{(a, a) \in S \times S \mid a \in S\big\}. \tag{1.17}$$

Definition 1.4.7 (Cover). A cover of a set S is a collection of sets $\mathcal{F} \subseteq P(S)$ such that

$$\bigcup_{V\in\mathcal{F}} V = S. \tag{1.18}$$

Definition 1.4.8 (Partition). A partition of X is a family of disjoint subsets $(A_i)_{i\in I} \subset P(X)$ such that $\bigcup_{i\in I} A_i = X$.

Definition 1.4.9 (Refinement). Let P be a partition of X. A refinement P' of P is a collection of subsets such that every $A \in P$ can be written as a disjoint union of elements in P'. It follows that every refinement is also a partition.

Definition 1.4.10 (Filter). Let X be a partially ordered set. A family $\mathcal{F} \subseteq P(X)$ is a filter on X if it satisfies the following conditions:

1. **Empty set**: $\emptyset \notin \mathcal{F}$;
2. **Closed under intersections**: $\forall A, B \in \mathcal{F} : A \cap B \in \mathcal{F}$; and
3. **Closed under inclusion**: if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Definition 1.4.11 (Filtration). Consider a set A together with a collection of subsets F_iA indexed by a totally ordered set I. The collection is said to be a filtration of A if

$$i \leq j \implies F_iA \subseteq F_jA. \tag{1.19}$$

A filtration is said to be **exhaustive** if $\bigcup_i F_iA = A$ and **separated** if $\bigcap_i F_iA = \emptyset$.

Definition 1.4.12 (Associated grading). In the case where one can define quotient objects every filtration $\{F_iA\}_{i\in\mathbb{N}}$ of A defines an associated graded object $\{G_iA := F_iA/F_{i-1}A\}$.

1.4.2 Algebra of sets

Definition 1.4.13 (Algebra of sets). A collection $\mathcal{F} \subset P(X)$ is a called an algebra over X if it is closed under finite unions, finite intersections and complements. The pair $(X, \mathcal{F})$ is also called a **field of sets**.

Definition 1.4.14 (σ-algebra). A collection $\Sigma \subset P(X)$ is called a σ-algebra over a set X if it satisfies the following axioms:

1. **Total space**: $X \in \Sigma$,
2. **Closed under complements**: $\forall E \in \Sigma : E^c \in \Sigma$, and
3. **Closed under countable unions**: $\forall \{E_i\}_{i=1}^n \subset \Sigma : \bigcup_{i=1}^n E_i \in \Sigma$.

Remark 1.4.15. Axioms (2) and (3) together with de Morgan's laws (1.8) and (1.9) imply that a σ-algebra is also closed under countable intersections.

Corollary 1.4.16 (Algebra of sets). Every algebra of sets is a σ-algebra.

Property 1.4.17 (Intersections). The intersection of a family of σ-algebras is again a σ-algebra.

Definition 1.4.18 (Generated σ-algebras). A σ-algebra $\mathcal{G}$ is said to be generated by a collection of sets $\mathcal{A}$ if

$$\mathcal{G} = \bigcap \{\mathcal{F} \mid \mathcal{F} \text{ is a } \sigma\text{-algebra that contains } \mathcal{A}\}. \tag{1.20}$$

Equivalently it is the smallest σ-algebra containing $\mathcal{A}$.

Notation 1.4.19. The σ-algebra generated by a collection of sets $\mathcal{A}$ is often denoted by $\mathcal{F}_\mathcal{A}$ or $\sigma(\mathcal{A})$.

Definition 1.4.20 (Product σ-algebras). The product σ-algebra $\mathcal{F}_1 \times \mathcal{F}_2$ on $X_1 \times X_2$ can be defined in the following equivalently ways:

- $\mathcal{F}$ is generated by the collection

$$\mathcal{C} = \{A_1 \times \Omega_2 \mid A_1 \in \mathcal{F}_1\} \cup \{\Omega_1 \times A_2 \mid A_2 \in \mathcal{F}_2\}.$$

- $\mathcal{F}$ is the smallest σ-algebra containing the products $A_1 \times A_2$ for all $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$.

Definition 1.4.21 (Monotone class). Let $\mathcal{A}$ be a collection of sets. $\mathcal{A}$ is called a monotone class if it has the following two properties:

1. For every increasing sequence $A_1 \subset A_2 \subset \cdots$:

$$\bigcup_{i=1}^\infty A_i \in \mathcal{A}.$$

2. For every decreasing sequence $A_1 \supset A_2 \supset \cdots$:

$$\bigcap_{i=1}^\infty A_i \in \mathcal{A}.$$

Theorem 1.4.22 (Monotone class theorem). *Let $\mathcal{A}$ be an algebra of sets 1.4.13. If $\mathcal{G}_\mathcal{A}$ is the smallest monotone class containing $\mathcal{A}$ then it coincides with the σ-algebra generated by $\mathcal{A}$.*

1.5 Ordered sets

1.5.1 Posets

Definition 1.5.1 (Preordered set). A set equipped with a reflexive and transitive binary relation.

Definition 1.5.2 (Partially ordered set). A set P equipped with a binary relation $\leq$ is called a partially ordered set (or **poset**) if the following 3 axioms are fulfilled for all elements $a, b, c \in P$:

1. **Reflexivity**: $a \leq a$,
2. **Antisymmetry**: $a \leq b \wedge b \leq a \implies a = b$, and
3. **Transitivity**: $a \leq b \wedge b \leq c \implies a \leq c$.

Equivalently, it is a preordered set for which the binary relation is also antisymmetric.

Definition 1.5.3 (Order-preserving map). A function $f : (P, \leq) \to (P', \preceq)$ of posets such that $p \leq q \implies f(p) \preceq f(q)$ for all $p, q \in P$.

Definition 1.5.4 (Hasse diagram). Consider a finite poset $(P, \leq)$. A Hasse diagram for P is a graph where the vertices are given by the elements of P and two vertices x, y are connected upwardly if and only if $x \leq y$ in P.

Example 1.5.5. Consider the power set of $\{0, 1\}$ with its canonical poset structure. A Hasse diagram of this poset is

$$\begin{tikzcd}[column sep=small] & \{0,1\} & \\ \{0\} \arrow[ur, dash] & & \{1\}. \arrow[ul, dash] \\ & \emptyset \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

Definition 1.5.6 (Totally ordered set). A poset P with the property that for all $a, b \in P$ either $a \leq b$ or $b \leq a$ is called a (nonstrict) totally ordered set. This property is called **totality**.

Definition 1.5.7 (Strict total order). A nonstrict order $\leq$ has an associated strict order $<$ that satisfies $a < b \iff a \leq b \wedge a \neq b$.

Definition 1.5.8 (Linear order). A binary relation $<$ on a set P satisfying the following conditions for all $x, y, z \in P$:

1. **Irreflexivity**: $x \not< x$,
2. **Asymmetry**: $x < y \implies y \not< x$,
3. **Transitivity**: $x < y \wedge y < z \implies x < z$,
4. **Comparison**: $x < z \implies x < y \vee y < z$, and
5. **Connectedness**: $x \not< y \wedge y \not< x \implies x = y$.

Remark 1.5.9. By negation one can freely pass between linear orders and total orders. However, without the law of the excluded middle, there exists no bijection between these two.

Definition 1.5.10 (Supremum). The supremum $\sup(X)$ of a poset X is the least upper bound of X.

Definition 1.5.11 (Infimum). The infimum $\inf(X)$ of a poset X is the greatest lower bound of X.

Definition 1.5.12 (Maximum). If $\sup(X) \in X$, the supremum is called the maximum of X. This is denoted by $\max(X)$.

Definition 1.5.13 (Minimum). If $\inf(X) \in X$, the supremum is called the minimum of X. This is denoted by $\min(X)$.

Definition 1.5.14 (Chain). A totally ordered subset of a poset.

Theorem 1.5.15 (Zorn's lemma[3]). *Let $(P, \leq)$ be a poset. If every chain in P has an upper bound in P, then P has a maximal element.*

Definition 1.5.16 (Directed[4] set). A set X equipped with a preorder $\leq$ with the additional property that every 2-element subset has an upper bound, i.e. for every two elements $a, b \in X$, there exists an element $c \in X$ such that $a \leq c \wedge b \leq c$.

Definition 1.5.17 (Net). A net on a set X is a subset of X indexed by a directed set I.

Definition 1.5.18 (Cut). Let $(P, \leq)$ be a totally ordered set. A cut or **decomposition** of P is a pair (A, B) of disjoint subsets such that $P = A \cup B$ in the ordered sense.

Cuts can be classified as follows:

- **Jumps**: A has a greatest element and B has a least element.
- **Dedekind cut**: Either A has a greatest element and B has no least element or A has no greatest element, but B has a least element.
- **Gap**: A has no greatest element and B has no least element.

1.5.2 Ordinals ♣

Definition 1.5.19 (Well-ordering). A **well-founded** linear order, i.e. a linear order $\leq$ such that every nonempty subset has a minimal element.

Definition 1.5.20 (Ordinal number). Consider the class of all well-ordered sets. An ordinal (type or rank) is an isomorphism class of well-ordered sets. The class of ordinals is itself well-ordered by inclusion of ''initial segments''.

However, this definition gives problems within the ZF(C) framework of set theory since these equivalence classes are proper classes and not sets. To overcome this problem one can use a different approach. By using a well-defined construction one can for every class select a particular representative and call this representative the **ordinal number** of all well-ordered sets isomorphic to it.

The most-used construction is the one by *Von Neumann*. For every well-ordered set W there exists a function $W \to P(W) : w \mapsto W_{\leq w} := \{w' \in W \mid w' \leq w\}$ that restricts to an order isomorphism on its image. This leads to the following definition:

Definition 1.5.21 (Von Neumann ordinal). A set that is strictly well-ordered by membership and such that every element is also a subset.

The first finite von Neumann ordinals are given as an example:

[3] This theorem is equivalent to the *axiom of choice*.

[4] Sometimes called a **filtered** set or **upward** directed set. **Downward** directed sets are analogously defined with a lower bound for every two elements.

- $0 := \emptyset$,
- $1 := \{0\} = \{\emptyset\}$,
- $2 := \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, and so on.

Property 1.5.22. Every ordinal type is uniquely order-preserving isomorphic to an ordinal number. Consequently, every order-preserving isomorphism between an order type and itself is the identity.

Definition 1.5.23 (Successor). Every ordinal number α has a **successor** α^+ (using the Von Neumann definition this is simply $\alpha^+ := \alpha \cup \{\alpha\}$). An ordinal that is not the successor of another ordinal number is called a **limit ordinal**.

Definition 1.5.24 (Regular ordinal). A limit ordinal α that is not the limit of a set of smaller ordinals with order type less than α.

Remark 1.5.25. The *Burali-Forti paradox* is the statement that the class of all ordinals (and by extension the class of all well-ordered sets) is not a set.

1.5.3 Cardinals ♣

There also exist numbers representing the sizes of sets, the **cardinal numbers**. These "numbers" should satisfy the following conditions:

- Every set has a well-defined cardinality.
- Every cardinal number is the cardinality of some set.
- Bijective sets have the same cardinality.

Guided by these conditions one could naively use the following definition:

Definition 1.5.26 (Cardinal number[5]). An isomorphism class of sets (under bijections).

However, similar to the problem encountered for ordinals above, these classes are not sets. To solve this, one can also use a similar trick and select a specific representative. For cardinals the following choice is made:

Alternative Definition 1.5.27 (Cardinal number). The cardinal number of a set is the smallest ordinal rank of any well-order on it, i.e. any ordinal number bijective to it.[6] The cardinal numbers inherit a well-ordering from the ordinal numbers.

Remark 1.5.28 (Ordering). The Cantor-Bernstein-Schröder theorem induces a partial ordering on cardinal numbers. However, without the axiom of choice this can never be a total ordering. This problem is also apparent in the above definition since the ordinal rank of sets is used together with the *well-ordering theorem*, which is equivalent to the axiom of choice.

Similar to ordinal numbers one can also define successors of cardinal numbers:

Definition 1.5.29 (Successor). Given a cardinal κ, its successor κ^+ is defined as the smallest cardinal larger than κ.

Remark. It should be noted that the successor of a cardinal number is not necessarily the same as its successor as an ordinal number (in fact this is only the case for finite cardinals).

[5] Also called the **cardinality** of a set.

[6] The well-ordering theorem (if assumed) assures that this definition coincides with the naive one above.

Definition 1.5.30 (Regular cardinal). An infinite cardinal κ such that there exist no set of cardinality κ that is the union of less than κ subsets of cardinality less than κ:

$$\kappa = \sum_{i\in I}\lambda_i \wedge \forall i\in I : \lambda_i < \kappa \implies |I| \geq \kappa. \tag{1.21}$$

The following theorem can easily be proven by a diagonal argument:

Theorem 1.5.31 (Cantor). *Let S be any set of cardinality κ. The power set $P(S)$ has cardinality greater than κ.*

1.5.4 Lattices

Definition 1.5.32 (Semilattice). A poset $(P,\leq)$ for which every 2-element subset has a supremum (also called a **join**) in P is called a join-semillatice. Similarly, a poset $(P,\leq)$ for which every 2-element subset has an infimum (also called a **meet**) in P is called a meet-semilattice.

Notation 1.5.33. The join of $\{a,b\}$ is denoted by $a\wedge b$. The meet of $\{a,b\}$ is denoted by $a\vee b$.

Definition 1.5.34 (Lattice). A poset $(P,\leq)$ is called a lattice if it is both a join- and a meet-semilattice.

The above definition also allows for a purely algebraic formulation (in this case some authors might speak about **lattice-ordered sets**):

Alternative Definition 1.5.35 (Lattice). A lattice is an algebraic structure that admits operations $\wedge,\vee$ and constants $\top,\bot$ that satisfy the following axioms:

1. Both $\wedge$ and $\vee$ are idempotent, commutative and associative.
2. The operations satisfy the **absorption laws**:

$$a\vee(a\wedge b) = a \qquad\qquad a\wedge(a\vee b) = a. \tag{1.22}$$

3. $\top$ and $\bot$ are the respective identities of $\wedge$ and $\vee$.

To go from this definition to the order-theoretic one, define the partial order

$$a\leq b \iff a\wedge b = a.$$

There exists an equivalent relation for the join.

Definition 1.5.36 (Bounded lattice). A lattice $(P,\leq)$ that contains a greatest element (denoted by $\top$ or 1) and a smallest element (denoted by $\bot$ or 0) such that

$$\bot \leq x \leq \top \tag{1.23}$$

for all $x\in P$. These elements are the identities for the join and meet operations:

$$x\wedge\top = x \qquad\qquad x\vee\bot = x. \tag{1.24}$$

Definition 1.5.37 (Frame). A complete lattice[7] $(P,\leq)$ for which the **infinite distributivity law** is satisfied:

$$y\wedge\left(\bigvee_{i\in I}x_i\right) = \bigvee_{i\in I}(y\wedge x_i). \tag{1.25}$$

[7]When working with categories this has to be restricted to "all small joins/meets" or, equivalently, the index category should be a set.

Definition 1.5.38 (Heyting algebra). A bounded lattice H such that for every two elements $a, b \in H$ there exists a greatest element $x \in H$ for which

$$a \wedge x \leq b. \tag{1.26}$$

This element is denoted by $a \rightarrow b$. The **pseudo-complement** $\neg a$ of an element $a \in H$ is then defined as $a \rightarrow \perp$.

Definition 1.5.39 (Boolean algebra). A Boolean algebra X is a Heyting algebra in which the **law of excluded middle** holds:

$$\forall x \in X : \neg\neg x = x. \tag{1.27}$$

This can be equivalently stated as

$$\forall x \in X : x \vee \neg x = \top. \tag{1.28}$$

1.6 Partitions

1.6.1 Partition

Definition 1.6.1 (Composition). Let $k, n \in \mathbb{N}$. A k-composition of n is a k-tuple $(t_1, \ldots, t_k)$ such that $\sum_{i=1}^k t_k = n$.

Definition 1.6.2 (Partition). Let $n \in \mathbb{N}$. A partition of n is an ordered composition of n. Hence multiple different composition can determine the same partition.

Definition 1.6.3 (Young diagram[8]). A Young diagram is a visual representation of the partition of an integer n. It is a left justified system of boxes, where every row corresponds to a part of the partition:

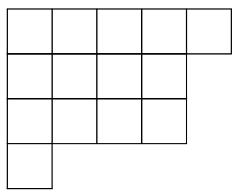

Figure 1.1: A Young diagram representing the partition $(5, 4, 4, 1)$ of 14.

Definition 1.6.4 (Conjugate partition). Let λ be a partition of n with associated Young diagram $\mathcal{D}$. The conjugate partition λ' is obtained by reflecting $\mathcal{D}$ across its main diagonal.

Example 1.6.5. Conjugating Diagram 1.1 gives Diagram 1.2 below. The associated partition is $(4, 3, 3, 3, 1)$.

Definition 1.6.6 (Young tableau). Consider a Young diagram of shape λ. A Young tableau of shape λ is a filling of the corresponding Young diagram by the elements of a totally ordered set (with n elements). This tableau is said to be **standard** if every row and every column is increasing.

Formula 1.6.7 (Hook length formula). The **hook** $H_{i,j}$ is defined as the part of a Young diagram given by the cell (i, j) together with all cells below and to the right of (i, j). Given a hook $H_{i,j}$, define the hook length $h_{i,j}$ as the sum of all elements in $H_{i,j}$.

[8]Sometimes called a **Ferrers diagram**.

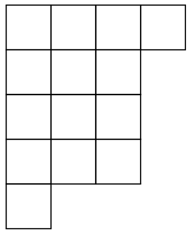

Figure 1.2: A Young diagram representing the partition $(4, 3, 3, 3, 1)$ of 14.

The number of all possible standard Young tableaux of shape λ (where λ defines a partition of n) is given by the following formula:

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}. \tag{1.29}$$

Definition 1.6.8 (Young tabloid). A Young tabloid of shape λ is defined as the equivalence class of Young tableaux that are connected by permuting the elements within a row. These are often drawn as in Figure 1.3.

1	2	3	5	8
4	6	9	10	
7	11	12	14	
15				

Figure 1.3: A Young tabloid associated to the Young diagram in Figure 1.1.

1.6.2 Superpartition

For the physical background of the notions introduced in this section, see Chapter **??**.

Definition 1.6.9 (Superpartition). Let $m, n \in \mathbb{N}$. A superpartition in the *m-fermion sector* is a sequence of integers of the following form:

$$\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_n), \tag{1.30}$$

where the first m numbers are strictly ordered, i.e. $\Lambda_i > \Lambda_{i+1}$ for all $i < m$, and the last $n - m$ numbers form a normal partition.

Both sequences, separated by a semicolon, form in fact distinct partitions themself. The first one represents the *antisymmetric fermionic* sector (this explains the strict order) and the second one represents the *symmetric bosonic* sector. This amounts to the following notation:

$$\Lambda \equiv (\lambda^a; \lambda^s).$$

The degree of the superpartition is given by $|\Lambda| = \sum_{i=1}^n \Lambda_i$.

Notation 1.6.10. A superpartition of degree n in the m-fermion sector is said to be a superpartition of $(n|m)$. To every superpartition Λ one can also associate a unique partition Λ^* by removing the semicolon and reordering the numbers such that they form a partition of n. The superpartition Λ can then be represented by the Young diagram belonging to Λ^* where the rows belonging to the fermionic sector are ended by a circle.

Chapter 2

Category theory

For the general theory of categories, the classical reference is [28]. The main reference for (co)end calculus is [115], while a thorough introduction to the theory of enrichment is given in [25]. For the theory of higher categories and its applications to topology and algebra, the reader is referred to the book by *Baez et al.* [42]. A good starting point for bicategories (and more) is the paper by *Leinster* [91].

2.1 Categories

Definition 2.1.1 (Category). A category $\mathbf{C}$ consists of two collections, the objects $\mathrm{ob}(\mathbf{C})$ and the morphisms $\hom(\mathbf{C})$ or $\mathrm{mor}(\mathbf{C})$, that satisfy the following conditions:

1. **Source and target**: For every morphism $f \in \hom(\mathbf{C})$ there exist two objects $s(f), t(f) \in \mathrm{ob}(\mathbf{C})$, the source and target. The collection of all morphisms with source x and target y is denoted by $\mathrm{Hom}_{\mathbf{C}}(x,y)$ or $\mathbf{C}(x,y)$.
2. **Existence of composition**: For every two morphisms $f \in \mathbf{C}(y,z)$ and $g \in \mathbf{C}(x,y)$, the composite $f \circ g$ is an element of $\mathbf{C}(x,z)$. Moreover, composition is required to be associative.
3. **Existence of identity**: For every $x \in \mathrm{ob}(\mathbf{C})$, there exists an identity morphism $\mathbb{1}_x \in \mathbf{C}(x,x)$. Identity morphisms are required to satisfy $f \circ \mathbb{1}_x = f = \mathbb{1}_y \circ f$ for every morphism $f \in \mathbf{C}(x,y)$.

Remark 2.1.2. One technically does not need to consider objects as a separate notion since every object can be identified with its identity morphism (which exists by definition) and, hence, one can work solely with morphisms. It should be noted that for higher categories this remark can be omitted since the objects are always regarded as 0-morphisms in that context.

Definition 2.1.3 (Subcategory). A subcategory is said to be **full** if for every two objects $x, y \in \mathrm{ob}(\mathbf{S})$:

$$\mathbf{S}(x,y) = \mathbf{C}(x,y). \tag{2.1}$$

A subcategory is said to be **wide** or **lluf** if it contains all objects, i.e. $\mathrm{ob}(\mathbf{S}) = \mathrm{ob}(\mathbf{C})$.

Definition 2.1.4 (Small category). A category $\mathbf{C}$ for which both $\mathrm{ob}(\mathbf{C})$ and $\hom(\mathbf{C})$ are sets. A category $\mathbf{C}$ is said to be locally small if for every two objects $x, y \in \mathrm{ob}(\mathbf{C})$ the collection of morphisms $\mathbf{C}(x,y)$ is a set. A category equivalent to a small category is said to be **essentially small**.

Definition 2.1.5 (Opposite category). Let $\mathbf{C}$ be a category. The opposite category $\mathbf{C}^{op}$ is constructed by reversing all arrows in $\mathbf{C}$, i.e. a morphism in $\mathbf{C}^{op}(x,y)$ is a morphism in $\mathbf{C}(y,x)$.

Property 2.1.6 (Involution). From the definition of the opposite category it readily follows that *op* is an involution:

$$(\mathbf{C}^{op})^{op} = \mathbf{C}. \tag{2.2}$$

2.2 Functors

Definition 2.2.1 (Covariant functor). Let $\mathbf{A}, \mathbf{B}$ be categories. A (covariant) functor is an assignment $F : \mathbf{A} \to \mathbf{B}$ satisfying the following conditions:

1. F maps every object $x \in \mathrm{ob}(\mathbf{A})$ to an object $Fx \in \mathrm{ob}(\mathbf{B})$.
2. F maps every morphism $\phi \in \mathbf{A}(x,y)$ to a morphism $F\phi \in \mathbf{B}(Fx, Fy)$.
3. F preserves identities, i.e. $F\mathbb{1}_x = \mathbb{1}_{Fx}$.
4. F preserves compositions, i.e. $F(\phi \circ \psi) = F\phi \circ F\psi$.

Property 2.2.2 (Category of categories). Small categories, together with (covariant) functors between them, form a category $\mathbf{Cat}$. The restriction to small categories is important since otherwise one would obtain an inconsistency similar to *Russell's paradox*. In certain foundations one can also consider the ''category'' $\mathbf{CAT}$ of all categories, but this would not be a large category anymore. It would be something like a ''very large'' category.

Definition 2.2.3 (Contravariant functor). Let $\mathbf{A}, \mathbf{B}$ be categories. A contravariant functor is an assignment $F : \mathbf{A} \to \mathbf{B}$ satisfying the following conditions:

1. F maps every object $x \in \mathrm{ob}(\mathbf{A})$ to an object $Fx \in \mathrm{ob}(\mathbf{B})$.
2. F maps every morphism $\phi \in \mathbf{A}(x,y)$ to a morphism $F\phi \in \mathbf{B}(Fy, Fx)$.
3. F preserves identities, i.e. $F\mathbb{1}_x = \mathbb{1}_{Fx}$.
4. F reverses compositions, i.e. $F(\phi \circ \psi) = F\psi \circ F\phi$.

A contravariant functor can also be defined as a covariant functor from the opposite category and, accordingly, from now on the word ''covariant'' will be dropped when talking about functors.

Definition 2.2.4 (Endofunctor). A functor of the form $F : \mathbf{C} \to \mathbf{C}$.

Definition 2.2.5 (Presheaf). A contravariant functor $G : \mathbf{C}^{op} \to \mathbf{Set}$. The collection of all presheaves on $\mathbf{C}$ forms a category $\mathbf{Psh}(\mathbf{C})$ (also denoted by $\widehat{\mathbf{C}}$).

Example 2.2.6 (Hom-functor). Let $\mathbf{C}$ be a locally small category. Every object $x \in \mathrm{ob}(\mathbf{C})$ induces a functor $h^x : \mathbf{C} \to \mathbf{Set}$ defined as follows:

- h^x maps every object $y \in \mathrm{ob}(\mathbf{C})$ to the set $\mathbf{C}(x,y)$.
- For all $y, z \in \mathrm{ob}(\mathbf{C})$, h^x maps every morphism $f \in \mathbf{C}(y,z)$ to the fucntion

$$f \circ - : \mathbf{C}(x,y) \to \mathbf{C}(x,z) : g \mapsto f \circ g.$$

Remark 2.2.7. The contravariant hom-functor h_x is defined by replacing $\mathbf{C}(x,-)$ with $\mathbf{C}(-,x)$ and replacing postcomposition with precomposition.

Definition 2.2.8 (Faithful functor). A functor $F : \mathbf{A} \to \mathbf{B}$ for which the map

$$\mathbf{A}(x,y) \to \mathbf{B}(Fx, Fy)$$

is injective for all objects $x, y \in \mathrm{ob}(\mathbf{A})$.

Definition 2.2.9 (Full functor). A functor $F : \mathbf{A} \to \mathbf{B}$ for which the map

$$\mathbf{A}(x, y) \to \mathbf{B}(Fx, Fy)$$

is surjective for all objects $x, y \in \mathrm{ob}(\mathbf{A})$.

Definition 2.2.10 (Embedding). A fully faithful functor.

Definition 2.2.11 (Essentially surjective functor). A functor $F : \mathbf{A} \to \mathbf{B}$ such that for every object $y \in \mathrm{ob}(\mathbf{B})$, there exists an object $x \in \mathrm{ob}(\mathbf{A})$ with $Fx \cong y$.

Definition 2.2.12 (Profunctor[1]). A functor of the form $F : \mathbf{B}^{op} \times \mathbf{A} \to \mathbf{Set}$. Such a functor is often denoted by $F : \mathbf{A} \nrightarrow \mathbf{B}$.[2] Elements of the set $F(x, y)$ are sometimes called **heteromorphisms** (between x and y).

It should be noted that presheafs on $\mathbf{C}$ are profunctors of the form $1 \nrightarrow \mathbf{C}$.

2.2.1 Natural transformations

Definition 2.2.13 (Natural transformation). Let $F, G : \mathbf{A} \to \mathbf{B}$ be two functors. A natural transformation $\psi : F \Rightarrow G$[3] consists of a collection of morphisms satisfying the following two conditions:

1. For every object $x \in \mathrm{ob}(\mathbf{A})$, there exists a morphism $\psi_x : Fx \to Gx$ in $\hom(\mathbf{B})$. This morphism is called the **component** of ψ at x. (It is often said that ψ_x **is natural in** x.)
2. For every morphism $f \in \mathbf{A}(x, y)$, the diagram below commutes:

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\psi_x"'] & Fy \arrow[d, "\psi_y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

Definition 2.2.14 (Functor category). Consider two categories $\mathbf{A}, \mathbf{B}$ where $\mathbf{A}$ is small. The functors $F : \mathbf{A} \to \mathbf{B}$ form the objects of a category with the natural transformations as morphisms. This category is denoted by $[\mathbf{A}, \mathbf{B}]$ or $\mathbf{B}^{\mathbf{A}}$ (the latter is a generalization of 1.3.5).

Definition 2.2.15 (Dinatural transformation). Consider two profunctors $F, G : \mathbf{A} \nrightarrow \mathbf{A}$ or, more generally, two functors $F, G : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{B}$. A dinatural transformation is a family of morphisms

$$\eta_x : F(x, x) \to G(x, x)$$

that make Diagram 2.1 commute for every morphism $f : y \to x$.

Definition 2.2.16 (Representable functor). Let $\mathbf{C}$ be a locally small category. A functor $F : \mathbf{C} \to \mathbf{Set}$ is said to be representable if there exists an object $x \in \mathrm{ob}(\mathbf{C})$ such that F is naturally isomorphic to h^x. The pair $(x, \psi : F \Rightarrow h^x)$ is called a **representation** of F.

Theorem 2.2.17 (Yoneda lemma). *Let $\mathbf{C}$ be a locally small category and let $F : \mathbf{C} \to \mathbf{Set}$ be a functor. For every object $x \in \mathrm{ob}(\mathbf{C})$, there exists a natural isomorphism*[4]

$$\eta_x : \mathrm{Nat}(h^x, F) \to Fx : \psi \mapsto \psi_x(\mathbb{1}_x). \tag{2.3}$$

[1] Sometimes called a **distributor**.
[2] This is the convention by *Borceux*. Some other authors, such as [11], use the opposite convention.
[3] This notation is in analogy with the general notation for 2-morphisms. See Section 2.9 for more information.
[4] Here, the fact that $\mathrm{Nat}(h^-, -)$ can be seen as a functor $\mathbf{Set}^{\mathbf{C}} \times \mathbf{C} \to \mathbf{Set}$ is used.

$$
\begin{tikzcd}
& F(x,y) \arrow[dl, "F(\mathbb{1}_x, f)"'] \arrow[dr, "F(f, \mathbb{1}_y)"] & \\
F(x,x) \arrow[d, "\eta_x"'] & & F(y,y) \arrow[d, "\eta_y"] \\
G(x,x) \arrow[dr, "G(f, \mathbb{1}_x)"'] & & G(y,y) \arrow[dl, "G(\mathbb{1}_y, f)"] \\
& G(y,x) &
\end{tikzcd}
$$

Figure 2.1: Dinatural transformation.

Corollary 2.2.18 (Yoneda embedding). When F is another hom-functor h^y, the following result is obtained:

$$\mathrm{Nat}(h^x, h^y) \cong \mathbf{C}(y, x). \tag{2.4}$$

Note that y appears in the first argument on the right-hand side.

Let $\mathbf{C}(f, -)$ denote the natural transformation corresponding to the morphism $f \in \mathbf{C}(y, x)$. The functor h^-, mapping an object $x \in \mathrm{ob}(\mathbf{C})$ to its hom-functor $\mathbf{C}(x, -)$ and a morphism $f \in \mathbf{C}(y, x)$ to the natural transformation $\mathbf{C}(f, -)$, can also be interpreted as a covariant functor $G : \mathbf{C}^{op} \to \mathbf{Set}^{\mathbf{C}}$. This way the Yoneda lemma can be seen to give rise to an embedding h^- of $\mathbf{C}^{op}$ in the functor category $\mathbf{Set}^{\mathbf{C}}$.

As usual, all of this can be done for contravariant functors. This gives an embedding

$$\mathcal{Y} := h_- : \mathbf{C} \hookrightarrow \widehat{\mathbf{C}}, \tag{2.5}$$

called the Yoneda embedding.

Definition 2.2.19 (Local object). Consider a collection of morphisms $S \subset \hom(\mathbf{C})$. An object $c \in \mathrm{ob}(\mathbf{C})$ is said to be S-local if the Yoneda embedding $\mathcal{Y}c$ maps morphisms in S to isomorphisms in **Set**. A morphism $f \in \hom(\mathbf{C})$ is said to be S-local if its image under the Yoneda embedding of every S-local object is an isomorphism in **Set**.

2.2.2 Equivalences

Definition 2.2.20 (Equivalence of categories). Two categories $\mathbf{A}, \mathbf{B}$ are said to be equivalent if there exist functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{A}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors.

A weaker notion is that of a **weak equivalence**. Two categories $\mathbf{A}, \mathbf{B}$ are said to be weakly equivalent if there exist functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{A}$ that are fully faithful and essentially surjective. Assuming the axiom of choice, every weak equivalence is also a (strong) equivalence (in fact this statement is equivalent to the axiom of choice).

Definition 2.2.21 (Skeletal category). A category in which every isomorphism is necessarily an identity morphism. The **skeleton** of a category is an equivalent skeletal category (often taken to be a subcategory by choosing a representative from every isomorphism class).

If one does not assume the axiom of choice, the skeleton is merely a *weakly equivalent* skeletal category.

Definition 2.2.22 (Decategorification). Let $\mathbf{C}$ be an (essentially) small category. The set of isomorphism classes of $\mathbf{C}$ is called the decategorification of $\mathbf{C}$. This amounts to a functor $\mathrm{Decat} : \mathbf{Cat} \to \mathbf{Set}$.

2.2.3 Stuff, structure and property

To classify properties of objects and the *forgetfulness* of functors, it is interesting to make a distinction between stuff, structure and property. Consider for example a group. This is a set (*stuff*) equipped with a number of operations (*structure*) that obey some relations (*properties*).

Using these notions one can classify forgetful functors in the following way:

- A functor forgets nothing if it is an equivalence of categories.
- A functor forgets at most properties if it is fully faithful.
- A functor forgets at most structure if it is faithful.
- A functor forgets at most stuff if it is just a functor.

?? COMPLETE (see e.g. nLab or the paper "Why surplus structure is not superfluous" by Nicholas Teh et al.) ??

2.2.4 Adjunctions

Definition 2.2.23 (Hom-set adjunction). Let $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{A}$ be two functors. These functors form a (hom-set) adjunction $F \dashv G$ if the following isomorphism is natural in both x and y:

$$\Phi_{x,y} : \mathbf{B}(Fx, y) \cong \mathbf{A}(x, Gy). \tag{2.6}$$

The functor F (resp. G) is called the left (resp. right) adjoint and the image of a morphism under either of the natural isomoprhisms is called the adjunct of the other morphism.[5].

Notation 2.2.24. An adjunction $F \dashv G$ between categories $\mathbf{A}, \mathbf{B}$ is often denoted by

$$\begin{tikzcd} \mathbf{B} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathbf{A}. \arrow[l, bend right, "F"'] \end{tikzcd}$$

Definition 2.2.25 (Unit-counit adjunction). Let $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{A}$ be two functors. These functors form a unit-counit adjunction if there exist natural transformations

$$\varepsilon : F \circ G \Rightarrow \mathbb{1}_{\mathbf{B}} \tag{2.7}$$
$$\eta : \mathbb{1}_{\mathbf{A}} \Rightarrow G \circ F \tag{2.8}$$

such that the following compositions are identity morphisms:

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F \tag{2.9}$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G. \tag{2.10}$$

These identities are sometimes called the **triangle** or **zig-zag identities** (the latter results from the shape of the associated *string diagram*). The transformations η and ε are called the **unit** and **counit** respectively.

[5] The terms "adjunct" and "adjoint" are sometimes used interchangeably (cf. French versus Latin).

Property 2.2.26 (Equivalence of the above definitions). Every hom-set adjunction induces a unit-counit adjunction. Let Φ be the natural isomorphism associated to the hom-set adjunction $F \dashv G$. The counit ε_y is obtained as the adjunct $\Phi^{-1}_{Gy,y}(\mathbb{1}_{Gy})$ of the identity morphism on $Gy \in \text{ob}(\mathbf{A})$, and the unit η_x is analogously defined as the adjunct $\Phi_{c,Fc}(\mathbb{1}_{Fx})$ of the identity morphism at $Fx \in \text{ob}(\mathbf{B})$.

Conversely, every unit-counit adjunction induces a hom-set adjunction. Consider a morphism $f : Fx \to y$. The (right) adjunct is defined as the composition

$$\widetilde{f} := Gf \circ \eta_x : x \to (G \circ F)x \to Gy.$$

To construct a (left) adjunct, consider a morphism $\widetilde{g} : x \to Gy$:

$$g := \varepsilon_y \circ F\tilde{g} : Fx \to (F \circ G)y \to y.$$

Definition 2.2.27 (Reflective subcategory). A full subcategory is said to be reflective (resp. coreflective) if the inclusion functor admits a left (resp. right) adjoint.

Property 2.2.28 (Adjoint equivalence). Any equivalence of categories is part of an adjoint equivalence, i.e. an adjunction for which the unit and counit morphisms are invertible.

2.3 General constructions

Definition 2.3.1 (Dagger category). A category equipped with a contravariant involutive endofunctor, this functor is often denoted by $\dagger : \mathbf{C} \to \mathbf{C}$, similar to the adjoint operator for Hermitian matrices.

Remark 2.3.2. The concept of a dagger structure allows the usual definition of **unitary** and **self-adjoint** morphisms, i.e. morphism satisfying

$$f^\dagger = f^{-1} \qquad \text{or} \qquad f^\dagger = f. \tag{2.11}$$

Definition 2.3.3 (Comma category). Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be three categories and let $F : \mathbf{A} \to \mathbf{C}$ and $G : \mathbf{B} \to \mathbf{C}$ be two functors. The comma category $F \downarrow G$ is defined as follows:

- **Objects**: The triples (x, y, γ) where $x \in \text{ob}(\mathbf{A}), y \in \text{ob}(\mathbf{B})$ and $\gamma : Fx \to Gy$.
- **Morphisms**: The morphisms $(x, y, \gamma) \to (k, l, \sigma)$ are pairs (f, g) with $f : x \to k \in \hom(\mathbf{A})$ and $g : y \to l \in \hom(\mathbf{B})$ such that $\sigma \circ Ff = Gg \circ \gamma$.
- Composition of morphisms is defined componentwise.

Definition 2.3.4 (Arrow category). The comma category of the pair of functors $(\mathbb{1}_\mathbf{C}, \mathbb{1}_\mathbf{C})$. This is equivalently the functor category $[\mathbf{2}, \mathbf{C}]$ where $\mathbf{2}$ is the **interval category/walking arrow** $\{0 \to 1\}$.

Definition 2.3.5 (Functorial factorization). A *section* (see Definition 2.4.1) of the composition functor

$$\circ : [\mathbf{3}, \mathbf{C}] \to [\mathbf{2}, \mathbf{C}],$$

where $\mathbf{3}$ is the poset $\{0 \to 1 \to 2\}$.

Definition 2.3.6 (Slice category). Let $\mathbf{C}$ be a category and consider an object $x \in \text{ob}(\mathbf{C})$. The slice category $\mathbf{C}_{/x}$ of $\mathbf{C}$ over x is defined as follows:

- **Objects**: The morphisms in $\mathbf{C}$ with codomain x.
- **Morphisms**: The morphisms $f \to g$ are morphisms h in $\mathbf{C}$ such that $g \circ h = f$.

This category is also called the **over-category** of x. By dualizing one obtains the **under-category** of x.

2.3.1 Fibred categories ♣

Definition 2.3.7 (Fibre category). Let $\Pi : \mathbf{A} \to \mathbf{B}$ be a functor. The fibre category (of Π) over $y \in \mathrm{ob}(\mathbf{B})$ is the subcategory of $\mathbf{A}$ consisting of all objects $x \in \mathrm{ob}(\mathbf{A})$ such that $\Pi\, x = y$ and all morphisms $m \in \hom(\mathbf{A})$ such that $\Pi\, m = \mathbb{1}_y$. It will be denoted by $\mathbf{A}_y$.

Morphisms in $\mathbf{A}$ that are mapped to a morphism f in $\mathbf{B}$ are called **f-morphisms** and, in particular (using the identification of objects and their identity morphisms), morphisms in $\mathbf{A}_y$ are called **y-morphisms**. Similarly, **B-categories** are defined as the categories equipped with a (covariant) functor to $\mathbf{B}$. (It is not hard to see that these form a *2-category* under composition of functors that respects the $\mathbf{B}$-category structure.)

Definition 2.3.8 (Cartesian morphism). Consider a $\mathbf{B}$-category $\Pi : \mathbf{A} \to \mathbf{B}$. A morphism f in $\mathbf{A}$ is called Π-Cartesian if every Πf-morphism factors uniquely through a y-morphism, where y is the domain of Πf.

There also exists a notion of stronger notion. A **strongly Cartesian morphism** is a morphism $f \in \hom(\mathbf{A})$ such that for every morphism $\varphi \in \hom(\mathbf{A})$ with the same target and every factorization of $\Pi\varphi$ through Πf there exists a unique factorization of φ through f that maps to the given factorization of $\Pi\varphi$.

The following diagram (where the triangles commute) should clarify the above (technical) definitions:

$$\begin{tikzcd}
\forall x' \arrow[d, "\exists! g"'] \arrow[dr, "\forall\varphi"] & & \overset{\Pi}{\mapsto} & \Pi\, x' \arrow[d, "\forall\nu"'] \arrow[dr, "\Pi\varphi"] & \\
x_1 \arrow[r, "f"'] & x_2 & & \Pi\, x_1 \arrow[r, "\Pi f"'] & \Pi\, x_2
\end{tikzcd}$$

The diagram for (weak) Cartesian morphisms is obtained by identifying the objects $\Pi\, x'$ and $\Pi\, x_1$, i.e. by restricting to the case $\nu = \mathbb{1}_{\Pi x_1}$.

The Cartesian morphisms are said to be **inverse images** of their projections under Π and the object x_1 is called an **inverse image** of x_2 by Πf. The Cartesian morphisms of a fibre category are exactly the isomorphisms of that category.

Definition 2.3.9 (Fibred category). A $\mathbf{B}$-category $\Pi : \mathbf{A} \to \mathbf{B}$ is called a fibred category or **Grothendieck fibration** if the following conditions are satisfied:

1. For each morphism in $\mathbf{B}$ whose codomain lies in the range of Π and each lift of this codomain to $\mathbf{A}$, there exists at least one inverse image with the given codomain (in the weak sense).
2. The composition of two Cartesian morphisms is again Cartesian (in the weak sense).

If one instead works with strongly Cartesian morphisms, the second condition follows from the first one. However, it should be noted that in a fibred category a morphism is weakly Cartesian if and only if it is strongly Cartesian.

Definition 2.3.10 (Cleavage). Given a $\mathbf{B}$-category $\Pi : \mathbf{A} \to \mathbf{B}$, a cleavage is the choice of a Cartesian g-morphism $f : x \to y$ for every $y \in \mathrm{ob}(\mathbf{A})$ and morphism $g : b \to \Pi\, a'$. A $\mathbf{B}$-category equipped with a cleavage is said to be **cloven**.

It is clear that the existence of cleavage is sufficient for a category to be fibred and, conversely (assuming the axiom of choice), every fibred category admits a cleavage.

The following example can be obtained as a Grothendieck fibration with discrete fibres:

Example 2.3.11 (Discrete fibration). A functor $F : \mathbf{A} \to \mathbf{B}$ such that for every object $x \in \mathrm{ob}(\mathbf{A})$ and every morphism $f : y \to Fx$ in $\mathbf{B}$ there exists a unique morphism $g : z \to x$ in $\mathbf{A}$ such that $Fg = f$.

Example 2.3.12 (Groupoidal fibration). If every morphism is required to be Cartesian, the notion of a groupoid(al) fibration or a **category fibred in groupoids** is obtained. The reason for this name is that every fibre is a groupoid. An equivalent definition is that the associated pseudofunctor (see the construction below) factors through the embedding $\mathbf{Grpd} \hookrightarrow \mathbf{Cat}$.

Property 2.3.13 (Grothendieck construction ♣). Every fibred category $\Pi : \mathbf{A} \to \mathbf{B}$ defines a *pseudofunctor*[6] $F : \mathbf{B}^{op} \to \mathbf{Cat}$ which sends objects to fibre categories and arrows $f : c \to d$ to the pullback functor $f^* : \mathbf{A}_d \to \mathbf{A}_c$ constructed from a Cartesian lift of f. This pullback functor acts as follows:

- For every object $x \in \mathbf{A}_d$, f^*x is the domain of the Cartesian lift of f through x.
- For every morphism $(\alpha : x \to y) \in \mathbf{A}_d$ there exists a diagram of the form

$$\begin{tikzcd} f^*x \arrow[r] \arrow[d, dashed, "f^*\alpha"'] & x \arrow[d, "\alpha"] \\ f^*y \arrow[r] & y \end{tikzcd}$$

  Because the horizontal morphism are both projected to f and α is projected to the identity, there exists a unique factorization of the diagram through a morphism $f^*\alpha : f^*x \to f^*y$.

Conversely, every pseudofunctor gives rise to a fibred category through the Grothendieck construction $\int : [\mathbf{C}^{op}, \mathbf{Cat}] \to \mathbf{Cat}_{/\mathbf{C}}$ as follows. (These two constructions constitute a 2-equivalence of 2-categories.). Consider a pseudofunctor $F : \mathbf{C}^{op} \to \mathbf{Cat}$. The "bundle" $\int F$ consists of the following data:

- The objects are pairs (x, y) with $x \in \mathrm{ob}(\mathbf{C})$ and $y \in \mathrm{ob}(Fx)$.
- The morphisms $(x, y) \to (x', y')$ are pairs $(f : x \to x', \alpha : y \to Ff(y'))$.

Given a cleavage, the morphisms of the Grothendieck construction are exactly the factorizations of f-morphisms through the canonical lifting of f in the cleavage.

Property 2.3.14 (Functors). A pseudofunctor is a functor if and only if the cleavage of the associated fibred category is **split(ting)**, i.e. it contains all identities and is closed under composition.

Example 2.3.15 (Category of elements). The Grothendieck construction applied to an ordinary presheaf $F : \mathbf{C}^{op} \to \mathbf{Set}$.

2.3.2 Monads

Definition 2.3.16 (Monad). A monad is a triple (T, μ, η) where $T : \mathbf{C} \to \mathbf{C}$ is an endofunctor and $\mu : T^2 \to T, \eta : \mathbb{1}_\mathbf{C} \to T$ are natural transformations satisfying the following (coherence) conditions:

1. As natural transformations from T^3 to T:

$$\mu \circ T\mu = \mu \circ \mu_T. \tag{2.12}$$

[6] See Definition 2.9.9 towards the end of this chapter.

2. As natural transformations from T to itself:

$$\mu \circ T\eta = \mu \circ \eta_T = \mathbb{1}. \tag{2.13}$$

These conditions say that a monad is a monoid **??** in the category $\mathbf{End}_\mathbf{C}$ of endofunctors on $\mathbf{C}$. Accordingly, η and μ are often called the **unit** and **multiplication** maps.

Example 2.3.17 (Adjunction). Every adjunction $F \dashv G$, with unit ε and counit η, induces a monad of the form $(GF, G\varepsilon F, \eta)$.

Definition 2.3.18 (Algebra over a monad[7]). Consider a monad (T, μ, η) on a category $\mathbf{C}$. An algebra over T is a couple (x, κ), where $x \in \mathrm{ob}(\mathbf{C})$ and $\kappa : Tx \to x$, such that the following conditions are satisfied:

1. $\kappa \circ T\kappa = \kappa \circ \mu_x$, and
2. $\kappa \circ \eta_x = \mathbb{1}_x$.

Morphisms $(x, \kappa_x) \to (y, \kappa_y)$ of T-algebras are morphisms $f : x \to y$ in $\mathbf{C}$ such that $f \circ \kappa_x = \kappa_y \circ Tf$. An algebra of the form (Tx, μ_x) is said to be **free**.

Definition 2.3.19 (Eilenberg-Moore category). Given a monad T over a category $\mathbf{C}$, the Eilenberg-Moore category $\mathbf{C}^T$ is defined as the category of T-algebras.

Definition 2.3.20 (Kleisli category). Consider a monad T on a category $\mathbf{C}$. The Kleisli category $\mathbf{C}_T$ is defined as the full subcategory of $\mathbf{C}^T$ on the **free** T-algebras. This is equivalently the category with objects $\mathrm{ob}(\mathbf{C}_T) := \mathrm{ob}(\mathbf{C})$ and morphisms $\mathbf{C}_T(x, y) := \mathbf{C}(x, Ty)$.

Definition 2.3.21 (Monadic adjunction). An adjunction between categories $\mathbf{A}$ and $\mathbf{B}$ is said to be monadic if there exists an equivalence between B and the Eilenberg-Moore category of the induced monad.

Definition 2.3.22 (Monadic functor). A functor is said to be monadic if it admits a left adjoint such that the adjunction is monadic.

The following theorem characterizes monadic functors (for more information on some of the concepts, see Section 2.4 further below):

Theorem 2.3.23 (Beck's monadicity theorem). *Consider a functor $F : \mathbf{A} \to \mathbf{B}$. This functor is monadic if and only if the following conditions are satisfied:*

- *F admits a left adjoint.*
- *F reflects isomorphisms, i.e. all morphims in the preimage of an isomorphism are also isomorphisms.*
- *$\mathbf{A}$ has all coequalizers of F-split parallel pairs[8] and F preserves these coequalizers.*

Remark 2.3.24 (Crude monadicity theorem). A sufficient condition for monadicity is obtained by replacing the third condition above by the following weaker statement: "$\mathbf{A}$ has all coequalizers of reflexive pairs and F preserves these coequalizers."

[7] A more suitable name would be "module over a monad", since these are modules over a monoid if monads are regarded as monoids in $\mathbf{End}_\mathbf{C}$.

[8] These are parallel pairs f, g such that the images Ff, Fg under F admit a split coequalizer.

Definition 2.3.25 (Closure operator). Consider a monad $(T : \mathbf{C} \to \mathbf{C}, \eta, \mu)$. This monad is called a closure operator or **modal operator** if the multiplication map is a natural isomorphism, i.e. if the monad is idempotent.

Given a closure operator $T : \mathbf{C} \to \mathbf{C}$, the object Tx is called the closure of $x \in \mathrm{ob}(\mathbf{C})$ and the associated morphism η_x is called the **closing map**. $x \in \mathrm{ob}(\mathbf{C})$ itself is said to be T**-closed** exactly if its closing map is an isomorphism.

An object $x \in \mathrm{ob}(\mathbf{C})$ is called a **modal type** if the unit $\eta_x : x \to Tx$ is an isomorphism.

Remark 2.3.26 (Bicategories ♣). A monad can be defined in any bicategory as a 1-morphism $t : x \to x$ together with two 2-morphisms that satisfy conditions similar to the ones above. The above definition is then just a specific case of this more general definition in the 2-category **Cat**.

In the general setting one can then also define a **module** over a monad. First of all, one can regard any object $x \in \mathrm{ob}(\mathbf{C})$ as a functor from the terminal category 1. One can then replace 1 by any other category in the ordinary definition to obtain a general algebra (or module) over a given monad. It is this definition that readily generalizes to bicategories, i.e. a module is a 1-morphism $a : x \to y$ together with a 2-morphism that satisfies the same conditions as an algebra over a monad in **Cat**.

2.4 Morphisms and diagrams

2.4.1 Morphisms

Definition 2.4.1 (Section). A section of a morphism $f : x \to y$ is a right-inverse, i.e. a morphism $g : y \to x$ such that $f \circ g = \mathbb{1}_y$. f itself is called a **retraction** of g and y is called a **retract** of x.

Definition 2.4.2 (Monomorphism). Let $\mathbf{C}$ be a category. A morphism $\mu \in \mathbf{C}(x, y)$ is called a monomorphism, **mono** or **monic morphism** if for every object $z \in \mathrm{ob}(\mathbf{C})$ and every two morphisms $\alpha_1, \alpha_2 \in \mathbf{C}(z, x)$ such that $\mu \circ \alpha_1 = \mu \circ \alpha_2$ one can conclude that $\alpha_1 = \alpha_2$.

Definition 2.4.3 (Epimorphism). Let $\mathbf{C}$ be a category. A morphism $\varepsilon \in \mathbf{C}(x, y)$ is called an epimorphism, **epi** or **epic morphism** if for every object $z \in \mathrm{ob}(\mathbf{C})$ and every two morphisms $\alpha_1, \alpha_2 \in \mathbf{C}(y, z)$ such that $\alpha_1 \circ \varepsilon = \alpha_2 \circ \varepsilon$ one can conclude that $\alpha_1 = \alpha_2$.

Definition 2.4.4 (Split monomorphism). A morphism $f : x \to y$ that is a section of some other morphism $g : y \to x$. It can be shown that every split mono is in fact a mono and even an **absolute mono**, i.e. it is preserved by all functors.

The morphism g can be seen to satisfy the dual condition and hence is called a **split epimorphism**. It can be shown to be an absolute epi.

Definition 2.4.5 (Balanced category). A category in which every monic epi is an isomorphism.

Definition 2.4.6 (Reflexive pair). Two parallel morphisms $f, g : x \to y$ are said to form a reflexive pair if they have a common section, i.e. if there exists a morphism $\sigma : y \to x$ such that $f \circ \sigma = g \circ \sigma = \mathbb{1}_y$.

Definition 2.4.7 (Subobject). Let $\mathbf{C}$ be a category and let $x \in \mathrm{ob}(\mathbf{C})$ be any object. A subobject y of x is a mono $y \hookrightarrow x$.

In fact, one should work up to isomorphisms and, accordingly, the formal definition goes as follows. A subobject y of x in the category $\mathbf{C}$ is an isomorphism class of monos $i : y \hookrightarrow x$ in the slice category $\mathbf{C}/x$.

Definition 2.4.8 (Well-powered category). A category $\mathbf{C}$ is said to be well-powered if for every object $x \in \mathrm{ob}(\mathbf{C})$ the class of subobjects $\mathrm{Sub}(x)$ is small.

2.4.2 Initial and terminal objects

Definition 2.4.9 (Initial object). An object $\emptyset$ such that for every other object x there exists a unique morphism $\iota_x : \emptyset \to x$.

Definition 2.4.10 (Terminal object). An object 1 such that for every other object x there exists a unique morphism $\tau_x : x \to 1$.

Property 2.4.11 (Uniqueness). If an initial (or terminal) object exists, it is unique (up to isomorphisms).

Definition 2.4.12 (Zero object). An object that is both initial and terminal. The zero object is often denoted by 0.

Property 2.4.13 (Zero morphism). From the definition of the zero object it follows that for any two objects x, y there exists a unique morphism $0_{xy} : x \to 0 \to y$.

Definition 2.4.14 (Pointed category). A category containing a zero object.

Definition 2.4.15 (Global element). Let $\mathbf{C}$ be a category with a terminal object 1. A global element of an object $x \in \mathrm{ob}(\mathbf{C})$ is a morphism $1 \to x$.

Property 2.4.16. Every global element is monic.

Definition 2.4.17 (Pointed object). An object x equipped with a global element $1 \to x$. This morphism is sometimes called the **basepoint**.

Remark 2.4.18. In the category **Set** the elements of a set S are in one-to-one correspondence with the global elements of S. Furthermore, there is the the important property (*axiom of functional extensionality*) that two functions $f, g : S \to S'$ coincide if their values at every element $s \in S$ coincide or, equivalently, if their precompositions with global elements coincide.

However, this way of checking equality can fail in other categories. Consider for example **Grp**, the category of groups, with its zero object $0 = \{e\}$. The only morphism from this group to any other group G is the one mapping e to the unit in G. It is obvious that precomposition with this morphism says nothing about the equality of other morphisms. To recover the extensionality property from **Set**, the notion of an "element" should be generalized:

Definition 2.4.19 (Generalized element). Let $\mathbf{C}$ be category and consider an object $x \in \mathrm{ob}(\mathbf{C})$. For any object $y \in \mathrm{ob}(\mathbf{C})$, a morphism $y \to x$ is called a generalized element of x. They are also called y**-elements** in x or elements of **shape** y in x.

Definition 2.4.20 (Generator). Let $\mathbf{C}$ be a category. A collection of objects $\mathcal{O} \subset \mathrm{ob}(\mathbf{C})$ is called a collection of generators or **separators** for $\mathbf{C}$ if the generalized elements of shape $\mathcal{O}$ are sufficient to distinguish between all morphisms in $\mathbf{C}$:

$$\forall x, y \in \mathrm{ob}(\mathbf{C}), \forall f, g \in \mathbf{C}(x, y) : \Big(f \neq g \implies \exists o \in \mathcal{O}, \exists h \in \mathbf{C}(o, x) : f \circ h \neq g \circ h\Big). \tag{2.14}$$

Definition 2.4.21 (Well-pointed category). A category for which the terminal object is a generator.

2.4.3 Lifts

Definition 2.4.22 (Lifts and extensions). A lift of a morphism $f : x \to y$ along an epi $e : z \to y$ is a morphism $g : x \to z$ satisfying $f = e \circ g$. Dualizing this definition gives the notion of extensions. (The epi/mono condition is often dropped in the literature.)

Definition 2.4.23 (Lifting property). A morphism $f : x \to y$ has the left lifting property with respect to a morphism $g : x' \to y'$ (or g has the right lifting property with respect to f) if for every commutative diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] & x' \arrow[d, "g"] \\ y \arrow[r] \arrow[ur, dashed, "\exists\psi"] & y' \end{tikzcd}$$

there exists a morphism $\psi : y \to x'$ such that the triangles commute. If the morphism ψ is unique, then f and g are said to be **orthogonal**.

Definition 2.4.24 (Injective and projective morphisms). Consider a class of morphisms $I \subseteq \hom(\mathbf{C})$. A morphism $f \in \hom(\mathbf{C})$ is said to be I-injective (resp. I-projective) if it has the right (resp. left) lifting property with respect to all morphisms in I.

Given a set of morphisms I, the sets of I-injective and I-projective morphisms are denoted by $\mathrm{rlp}(I)$ and $\mathrm{llp}(I)$, respectively.

Definition 2.4.25 (Injective and projective objects). If $\mathbf{C}$ has a terminal object 1, an object x is called I-injective if its terminal morphism is I-injective. If $\mathbf{C}$ has an initial object, I-projective objects can be defined dually. (See Figure 2.2.)

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "\forall f \in I"'] & i \\ y \arrow[ur, dashed, "\exists\psi"'] & \end{tikzcd}$$

(a) Injective object i.

$$\begin{tikzcd} & x \arrow[d, "\forall f \in I"] \\ p \arrow[r, "g"'] \arrow[ur, dashed, "\exists\psi"] & y \end{tikzcd}$$

(b) Projective object p.

Figure 2.2: Injective and projective objects.

If I is the class of monomorphisms (resp. epimorphisms), the terminology is simplified to **injective** (resp. **projective**) objects. For projective objects this is also equivalent to requiring that the (covariant) hom-functor preserves epimorphisms.

A category $\mathbf{C}$ is said to **have enough injectives** if for every object there exists a monomorphism into an injective object. The category is said to **have enough projectives** if for every object there exists an epimorphism from a projective object.

Definition 2.4.26 (Fibrations and cofibrations). Consider a category $\mathbf{C}$ together with a class $I \subseteq \hom(\mathbf{C})$ of morphisms. A morphism $f \in \hom(\mathbf{C})$ is called an I-fibration (resp. I-cofibration) if it has the right (resp. left) lifting property with respect to all I-projective (resp. I-injective) morphisms.

2.4.4 Limits and colimits

Definition 2.4.27 (Diagram). A diagram in $\mathbf{C}$ with index category $\mathbf{I}$ is a (covariant) functor $D : \mathbf{I} \to \mathbf{C}$.

Definition 2.4.28 (Cone). Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram. A cone from $c \in \mathrm{ob}(\mathbf{C})$ to D consists of a family of morphisms $\psi_i : c \to Di$ indexed by $\mathbf{I}$ such that $\psi_j = Df \circ \psi_i$ for all morphisms $f : i \to j \in \hom(\mathbf{I})$. This is depicted in Figure 2.3a.

$$\begin{tikzcd} & c \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_{f(i)}"] & \\ Di \arrow[rr, "Df"'] & & Df(i) \end{tikzcd}$$

(a) Component of cone over D.

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "\psi_i"'] & & b \arrow[dl, "\phi_i"] \\ & Di & \end{tikzcd}$$

(b) Morphism of cones.

Figure 2.3: Category of cones.

Alternative Definition 2.4.29. The above definition can be reformulated by defining an additional functor $\Delta_x : \mathbf{I} \to \mathbf{C}$ that maps every element $i \in \text{ob}(\mathbf{I})$ to x and every morphism $g \in \text{hom}(\mathbf{I})$ to $\mathbb{1}_x$, i.e. $\Delta : C \to [\mathbf{I}, \mathbf{C}]$ is the **diagonal functor**. The morphisms ψ_i can then be seen to be the components of a natural transformation $\psi : \Delta_x \Rightarrow D$. Hence, a cone (x, ψ) is an element of $[\mathbf{I}, \mathbf{C}](\Delta_x, D)$.

Definition 2.4.30 (Morphism of cones). Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram and let (x, ψ) and (y, ϕ) be two cones over D. A morphism between these cones is a morphism of the apexes $f : x \to y$ such that the diagrams of the form 2.3b commute for all $i \in \text{ob}(\mathbf{I})$. The cones over D together with these morphisms form a category $\mathbf{Cone}(D)$, in fact this can easily be seen to be the comma category $\Delta \downarrow D$.

Definition 2.4.31 (Limit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$. The limit of this diagram, denoted by $\lim D$, is (if it exists) the terminal object of the category $\mathbf{Cone}(D)$.

Remark. In the older literature the name **projective limit** was sometimes used. The dual notion, a **colimit**, is often called an **inductive limit** in the older literature.

This definition leads to the following universal property:

Universal Property 2.4.32. Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram. For every cone $(x, \psi) \in \mathbf{Cone}(D)$, there exists a unique morphism $f : x \to \lim D$. This defines a bijection

$$[\mathbf{I}, \mathbf{C}](\Delta_x, D) \cong \mathbf{C}(x, \lim D).$$

If all (small) limits exist, the limit functor $\lim : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$ can be defined. The universal property of limits then implies that it is right adjoint to the constant functor Δ.

For diagrams in **Set** one can use the fully faithfulness of the Yoneda embedding to obtain the following expression:

$$\lim D \cong [\mathbf{I}, \mathbf{Set}](\Delta_*, D). \tag{2.15}$$

Remark 2.4.33. In Section 2.7 on enriched category theory, a generalization (the so-called *weighted limits*) of the above construction will be given that is better suited to the enriched setting and allows to express a wide variety of constructions as (weighted) limits.

Example 2.4.34 (Terminal object). The terminal object 1 is the limit of the empty diagram.

Definition 2.4.35 (Finitely complete category). A category is said to be finitely complete if it has all finite limits. If all (small) limits exist, the category is said to be **complete**. The dual notion for colimits is called **(finite) cocompleteness**.

Example 2.4.36 (Presheaf categories). All presheaf categories are both complete and cocomplete.

Definition 2.4.37 (Continuous functor). A functor that preserves all small limits.

Example 2.4.38 (Hom-functors). In a locally small category every hom-functor is continuous (in fact these functors even preserve limits that are not necessarily small). This implies for example that

$$\mathbf{C}(x, \lim D) \cong \lim \mathbf{C}(x, D). \tag{2.16}$$

In the case where $\mathbf{C}$ is small, one can characterize the Yoneda embedding through a universal property:

Universal Property 2.4.39 (Free cocompletion). The Yoneda embedding $\mathbf{C} \hookrightarrow \widehat{\mathbf{C}}$ turns the presheaf category $\widehat{\mathbf{C}}$ into the **free cocompletion** of $\mathbf{C}$, i.e. there exists an equivalence of categories between the functor category of cocontinuous functors $[\widehat{\mathbf{C}}, \mathbf{D}]_{\text{cont}}$ and the ordinary functor category $[\mathbf{C}, \mathbf{D}]$.

Definition 2.4.40 (Tiny object). An object in a locally small category for which the covariant hom-functor preserves small colimits. This is sometimes called a **small-projective** object since it is in particular projective[9].

Definition 2.4.41 (Cauchy completion). Let $\mathbf{C}$ be a small category. An important (small and full) subcategory of the free cocompletion of $\mathbf{C}$ is given by the Cauchy completion, i.e. the subcategory of $\widehat{\mathbf{C}}$ on the tiny objects.[10] It can be shown that the free cocompletion of the Cauchy completion coincides with the one on $\mathbf{C}$ (up to equivalence).

A category is said to be **Cauchy-complete** if it is equivalent to its Cauchy completion. It can be shown that a category is Cauchy-complete if and only if it has all small absolute colimits.

Definition 2.4.42 (Filtered category). A category in which every finite diagram admits a cocone. For regular cardinals κ, this notion can be generalized. A category is said to be κ-filtered if every diagram with less than κ arrows admits a cocone. (In this terminology filtered categories are the same as ω-filtered categories.)

Definition 2.4.43 (Directed limit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$. The limit (resp. colimit) of D is said to be codirected (resp. directed) if $\mathbf{I}$ is a downward (resp. upward) directed set 1.5.16.

The following definition is a categorification of the previous one:

Definition 2.4.44 (Filtered limit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$. The limit (resp. colimit) of D is said to be cofiltered (resp. filtered) if $\mathbf{I}$ is a cofiltered (resp. filtered) category.

Property 2.4.45. A category has all directed limits if and only if it has all filtered limits. (A dual statement holds for colimits.)

Definition 2.4.46 (Pro-object). A functor $F : \mathbf{I} \to \mathbf{C}$ where $\mathbf{I}$ is a small cofiltered category. The names stems from the fact that one can interpret pro-objects as formal cofiltered (projective) limits.

Definition 2.4.47 (Compact object). An object for which the covariant hom-functor preserves all filtered colimits. These objects are also said to be **finitely presentable**.[11]

[9] Epimorphisms are characterized by a *pushout* (see 2.4.61 further below).

[10] A generalization in the context of enriched categories is given by the *Karoubi envelope*.

[11] This name derives from the fact that modules are finitely presented if and only if their covariant hom-functor preserves direct limits (i.e. directed colimits in the context of algebra).

Definition 2.4.48 (Product). Let $\mathbf{I}$ be a discrete category. The (co)limit over a diagram $D : \mathbf{I} \to \mathbf{C}$ is called a (co)product in $\mathbf{C}$.

Definition 2.4.49 (Equalizer). Consider a diagram of the form

$$x \overset{f}{\underset{g}{\rightrightarrows}} y.$$

The limit of this diagram is called the equalizer of f and g. It consists of an object e and a morphism $\varepsilon : e \to x$ such that the following **fork** diagram

$$e \overset{\varepsilon}{\to} x \overset{f}{\underset{g}{\rightrightarrows}} y \tag{2.17}$$

is universal with respect to (e, ε). By dualizing one obtains **cofork** diagrams $x \rightrightarrows y \to z$ and their universal versions, the **coequalizers**.

Definition 2.4.50 (Split coequalizer). A cofork diagram

$$x \overset{f}{\underset{g}{\rightrightarrows}} y \overset{\tau}{\to} z$$

together with a section φ of f and a section σ of τ such that $\sigma \circ \tau = g \circ \varphi$.

Definition 2.4.51 (Regular morphisms). A mono (resp. epi) is said to be regular if it arises as an equalizer (resp. coequalizer) of two parallel morphisms.

Property 2.4.52 (Regular bimorphism). Both monic regular epimorphisms and epic regular monomorphisms are isomorphisms.

Alternative Definition 2.4.53 (Finitely complete category). A category is said to be finitely complete if it has a terminal object and if all binary equalizers and products exist.

Definition 2.4.54 (Span). A span in a category C is a diagram of the form 2.4a. By definition of a diagram, a span in C is equivalent to a functor $S : \mathbf{\Lambda} \to \mathbf{C}$, where $\mathbf{\Lambda}$ is the category with three objects $\{-1, 0, 1\}$ and two morphisms $i : 0 \to -1$ and $j : 0 \to 1$. For this reason $\mathbf{\Lambda}$ is sometimes called the walking or universal span.

$$\begin{tikzcd} & s \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x & & y \end{tikzcd}$$

(a) Span (category theory).

$$\begin{tikzcd} x \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

(b) Cospan.

Figure 2.4: (Co)span diagrams.

Definition 2.4.55 (Pullback). The pullback or **fibre product** of two morphisms $f : x \to z$ and $g : y \to z$ is defined as the limit of cospan 2.4b. The full diagram characterizing the pullback, which has the form of a square, is sometimes called a **Cartesian square**.

Notation 2.4.56 (Pullback). The pullback of two morphisms $f : x \to z$ and $g : y \to z$ is often denoted by $x \times_z y$. The associated pullback square is sometimes written as in Figure 2.5a.

Property 2.4.57 (Product). If a terminal object 1 exists, the pullback $x \times_1 y$ is equal to the product $x \times y$.

$$\begin{tikzcd} x \times_z y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pb}"] & y \arrow[d, "g"] \\ x \arrow[r, "f"'] & z \end{tikzcd}$$

(a) Pullback square.

$$\begin{tikzcd} z \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{po}"] & y \arrow[d] \\ x \arrow[r] & x \sqcup_z y \end{tikzcd}$$

(b) Pushout square.

Figure 2.5: Pullback and pushout diagrams.

Definition 2.4.58 (Kernel pair). Consider a morphism $f : x \to y$. Its kernel pair is defined as the pullback of f along itself.

Definition 2.4.59 (Pushout). The dual notion of a pullback, i.e. the colimit of a span. See Figure 2.5b.

Property 2.4.60. Pullbacks preserve monos and pushouts preserve epis.

Alternative Definition 2.4.61 (Epimorphism). A morphism whose cokernel pair is the identity.

Property 2.4.62 (Span category ♣). Consider a category $\mathbf{C}$ with pullbacks. The category $\mathbf{Span}(\mathbf{C})$ is defined as the category with the same objects as $\mathbf{C}$ but with spans as morphisms. Composition of spans is given by pullbacks. By including morphisms of spans, $\mathbf{Span}(\mathbf{C})$ can be refined to a bicategory.

Definition 2.4.63 (Wedge). Consider a profunctor $F : \mathbf{C} \nrightarrow \mathbf{C}$. A wedge $e : w \to F$ is an object $w \in \mathrm{ob}(\mathbf{Set})$ together with a collection of morphisms $e_x : w \to F(x, x)$ indexed by $\mathbf{C}$ such that for every morphism $f : x \to y$ the following diagram commutes:

$$\begin{tikzcd} & w \arrow[dl, "e_x"'] \arrow[dr, "e_y"] & \\ F(x,x) \arrow[dr, "F(\mathbb{1}_x, f)"'] & & F(y,y) \arrow[dl, "F(f, \mathbb{1}_y)"] \\ & F(x,y) & \end{tikzcd}$$

As was the case for cones, this can be reformulated in terms of (di)natural transformations. A wedge (w, e) of a profunctor $F : \mathbf{C} \nrightarrow \mathbf{C}$ is a dinatural transformation from the constant profunctor Δ_w to F.

Definition 2.4.64 (End). The end of a profunctor $F : \mathbf{C} \nrightarrow \mathbf{C}$ is defined as the universal wedge of F. The components of the wedge are called the **projection maps** of the end. This stems from the fact that for a discrete category the end coincides with the product $\prod_{x \in \mathrm{ob}(\mathbf{C})} F(x, x)$.

This is equivalent to a definition in terms of equalizers. Consider the two canonical maps

$$\prod_{x \in \mathrm{ob}(\mathbf{C})} \mathbf{C}(x, x) \rightrightarrows \prod_{f : x \to y} \mathbf{C}(x, y).$$

This diagram can be interpreted as the product of all lower halves of the wedge diagrams above. It is not hard to see that its equalizer (universally) satisfies the wedge condition for all $f \in \hom(\mathbf{C})$.

Notation 2.4.65 (End). The end of a profunctor $F : \mathbf{C} \nrightarrow \mathbf{C}$ is often denoted using an integral sign with subscript:

$$\int_{x\in\mathbf{C}} F(x,x).$$

For the dual construction, called a **coend**, an integral sign with superscript is used.

Example 2.4.66 (Natural transformations). Consider two functors $F, G : \mathbf{A} \to \mathbf{B}$. The map $(x,y) \mapsto \mathbf{B}(Fx, Gy)$ gives a profunctor $H : \mathbf{A} \nrightarrow \mathbf{A}$. By looking at the wedge condition for this profunctor, the following equality for all morphisms $f : x \to y$ can be derived:

$$\tau_y \circ Ff = Gf \circ \tau_x, \tag{2.18}$$

where τ is the wedge projection. Comparing this equality to Definition 2.2.13 gives

$$\mathrm{Nat}(F,G) = \int_{x\in\mathbf{A}} \mathbf{B}(Fx, Gx). \tag{2.19}$$

Property 2.4.67. Using the continuity 2.4.37 of the hom-functor, one can prove the following equality which can be used to turn ends into coends and vice versa:

$$\mathbf{Set}\left(\int^{x\in\mathbf{C}} F(x,x), y\right) = \int_{x\in\mathbf{C}} \mathbf{Set}(F(x,x), y). \tag{2.20}$$

Using the above properties and definitions, one obtains the following two statements, called the **Yoneda reduction** and **co-Yoneda lemma**:

Property 2.4.68 (Ninja Yoneda lemma). Let $F : \mathbf{A} \to \mathbf{B}$ be a covariant functor (similar statements hold for contravariant functors).

$$\int_{x\in\mathbf{A}} \mathbf{Set}(\mathbf{A}(-,x), Fx) \cong F \tag{2.21}$$

$$\int^{x\in\mathbf{A}} \mathbf{A}(x,-) \times Fx \cong F. \tag{2.22}$$

For a generalization to the enriched setting see Definition 2.7.16.

Remark 2.4.69. A common remark at this point is the comparison with the Dirac distribution (**??**):

$$\int \delta(x-y) f(x) = f(y). \tag{2.23}$$

By interpreting the functor F as a function, the representable functors can be seen to behave as Dirac distributions.

Property 2.4.70.

$$\int_{F\in\mathbf{coPsh}(\mathbf{C})} \mathbf{Set}(Fx, Fy) \cong \mathbf{C}(x,y) \tag{2.24}$$

Definition 2.4.71 (Category of elements). Consider a presheaf $F : \mathbf{C}^{op} \to \mathbf{Set}$. Its category of elements $\mathrm{El}(F)$ is defined as the comma category $(\mathcal{Y} \downarrow !_F)$, where $!_F : * \to [\mathbf{C}^{op}, \mathbf{Set}]$ sends the unique object to F itself. Equivalently, it is the category with objects the pairs $(c, x) \in \mathrm{ob}(\mathbf{C}) \times Fc$ and morphisms $f \in \mathbf{C}(c, c')$ such that $c = Ff(c')$.

This category comes equipped with a canonical forgetful functor

$$\mathbf{C}_F : \mathrm{El}(F) \to \mathbf{C} : (c,x) \mapsto c. \tag{2.25}$$

Remark 2.4.72. The category of elements is usually defined for covariant functors. To obtain that definition one should take the opposite of the category of elements (and also take the opposite of the forgetful functor).

Definition 2.4.73 (Kan extension). Consider two functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{A} \to \mathbf{C}$. The right Kan extension of F along G is given by the universal functor $\mathrm{Ran}_G F : \mathbf{C} \to \mathbf{B}$ and natural transformation $\eta : \mathrm{Ran}_G F \circ G \Rightarrow F$:

$$\begin{tikzcd} \mathbf{C} \arrow[dr, dashed, "\mathrm{Ran}_G F"] & \\ \mathbf{A} \arrow[u, "G"] \arrow[r, "F"'] \arrow[ur, phantom, "\Downarrow\eta"] & \mathbf{B} \end{tikzcd}$$

The left Kan extension $\mathrm{Lan}_G F$ is obtained by dualizing this construction.

Property 2.4.74 (Complete categories). Complete (resp. cocomplete) categories admit all right (resp. left) Kan extensions.

Definition 2.4.75 (Preservation of Kan extension). A Kan extension $\mathrm{Lan}_G F$ is said to be **absolute** if every functor with the same codomain as preserves the Kan extension, i.e. a Kan extension is absolute if right whiskering it by another functor defines the Kan extension of the composition. If it is only preserved by all representable functors, the Kan extension is said to be **pointwise**.

Alternative Definition 2.4.76 (Kan extension). The construction above gives a functor Ran_G from the functor category $[\mathbf{A}, \mathbf{B}]$ to the functor category $[\mathbf{C}, \mathbf{B}]$. The right Kan extension Ran_G can be defined as the right adjoint to the pullback functor $G^* : F \mapsto F \circ G$. Similarly, the left Kan extension can be defined as the left adjoint to the pullback functor.

In the spirit of partial adjoints or partial limits, this definition can be used to define **local Kan extensions**. Although the left (or right) Kan extension functors do not have to exists globally, the extension of a single functor could still exist. This local version is defined by the following natural isomorphism (here given for a left extension):

$$[\mathbf{A}, \mathbf{B}](F, G^* -) \cong [\mathbf{C}, \mathbf{B}](\mathrm{Lan}_G F, -). \tag{2.26}$$

Remark 2.4.77. Using this equivalence of hom-spaces, Kan extensions can be generalized from **Cat** to any *2-category.*

Example 2.4.78 (Limit). Denote the terminal category by $\mathbf{1}$. By choosing the functor G in the definition of a right Kan extension to be the unique functor $!_{\mathbf{C}} : \mathbf{C} \to \mathbf{1}$, one obtain the universal property characterizing limits 2.4.32:

$$\lim F \cong \mathrm{Ran}_{!_C} F. \tag{2.27}$$

Similarly, colimits can be obtained as left Kan extensions.

The existence of Kan extensions can also be used to determine the existence of adjoints:

Property 2.4.79 (Adjoint functors). A functor $F : \mathbf{A} \to \mathbf{B}$ admits a left (resp. right) adjoint if and only if the right (resp. left) Kan extension of the identity functor $\mathbb{1} : \mathbf{A} \to \mathbf{A}$ along F exists. If it exists as an absolute extension, the left adjoint is given exactly by this Kan extension.

Definition 2.4.80 (Codensity monad). Consider a general functor $F : \mathbf{A} \to \mathbf{B}$. If the right Kan extension $\mathrm{Ran}_F F$ exists, it defines a monad. Functors for which this monad is the identity are said to be **codense**.[12] Left Kan extensions give, by duality, rise to *density comonads*.

2.5 Internal structures

Property 2.5.1 (Eckmann-Hilton argument). A monoid internal to **Mon**, the category of monoids is the same as a commutative monoid. (See also Property **??**.)

Definition 2.5.2 (Internal category). Let $\mathcal{E}$ be a category with pullbacks. A category $\mathbf{C}$ internal to $\mathcal{E}$ consists of the following data:

- an object $C_0 \in \mathrm{ob}(\mathcal{E})$ of objects;
- an object $C_1 \in \mathrm{ob}(\mathcal{E})$ of morphisms;
- source and target morphisms $s, t \in \mathcal{E}(C_1, C_0)$;
- an "identity-assigning" morphism $e \in \mathcal{E}(C_0, C_1)$ such that

  $$s \circ e = \mathbb{1}_{C_0} \qquad\qquad t \circ e = \mathbb{1}_{C_0};$$

  and

- a composition morphism $c : C_1 \times_{C_0} C_1 \to C_1$ such that the following equations hold:

  $$\begin{aligned} s \circ c &= s \circ \pi_1 & t \circ c &= t \circ \pi_2 \\ \pi_1 &= c \circ (e \times_{C_0} \mathbb{1}) & c \circ (\mathbb{1} \times_{C_0} e) &= \pi_2 \end{aligned}$$
  $$c \circ (c \times_{C_0} \mathbb{1}) = c \circ (\mathbb{1} \times_{C_0} c),$$

  where π_1, π_2 are the canonical projections associated with the pullback $C_1 \times_{C_0} C_1$ of (s, t).

Morphisms between these categories, suitably called **internal functors**, are given by a pair of morphisms (in $\mathcal{E}$) between internal objects and morphisms, that preserve composition and identities. Internal natural transformations are defined in a similar way.

Notation 2.5.3. The *(bi)category* of internal categories in $\mathcal{E}$ is denoted by $\mathbf{Cat}(\mathcal{E})$. It should be noted that for $\mathcal{E} = \mathbf{Set}$, the ordinary category of small categories $\mathbf{Cat}(\mathbf{Set}) = \mathbf{Cat}$ is obtained.

Alternative Definition 2.5.4. The above definition can be reformulated in a very elegant way. An internal category in $\mathcal{E}$ is a monad in the bicategory $\mathbf{Span}(\mathcal{E})$ of spans in $\mathcal{E}$ as shown in Figure 2.6.

Functors between internal categories are not the only relevant morphisms. However, when defining (co)presheafs such as the hom-functor, a problem occurs. In **Cat** there exist, by definition, maps to the ambient category **Set** (ordinary category theory has a set-theoretic foundation). However, for internal categories there does not necessarily exist a morphism $\mathbf{C} \to \mathcal{E}$. To solve this problem one can consider a more general structure:

Definition 2.5.5 (Internal diagram). A left module over a monad in $\mathbf{Span}(\mathcal{E})$. The dual notion is better known as an **internal presheaf**.

In fact, this is a specific instance of an even more general concept (for more information on the definitions and applications see [11, 28]):

[12] Codense functors are usually defined in a different way, but one can show that this is an equivalent definition (hence the name).

Span gives source and target maps

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 & & C_0 \end{tikzcd}$$

Multiplication gives composition

$$\begin{tikzcd} & C_1 \times_{C_0} C_1 \arrow[dl, "s \circ \pi_0"'] \arrow[dr, "t \circ \pi_1"] \arrow[dd, "c"] & \\ C_0 & & C_0 \\ & C_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

Unit gives identity

$$\begin{tikzcd} & C_0 \arrow[dl, "\mathbb{1}_a"'] \arrow[dr, "\mathbb{1}_a"] \arrow[dd, "e"] & \\ C_0 & & C_0 \\ & C_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

Figure 2.6: Internal category as a monad in $\mathbf{Span}(\mathcal{E})$.

Definition 2.5.6 (Internal profunctor). A bimodule between monads in $\mathbf{Span}(\mathcal{E})$. Together with the above definitions this gives rise to an equivalence $\mathbf{Mod}(\mathbf{Span}(\mathcal{E})) \cong \mathbf{Prof}(\mathcal{E})$.

Construction 2.5.7 (Internal Yoneda profunctor). Consider an internal functor $F : \mathbf{A} \to \mathbf{B}$. This functor induces two internal profunctors $F_* : \mathbf{B} \nrightarrow \mathbf{A}$ and $F^* : \mathbf{A} \nrightarrow \mathbf{B}$:

For F_* the object span is defined as (the profunctor F^* is defined similarly)

$$A_0 \xleftarrow{\pi_0} A_0 \times_{B_0} B_1 \xrightarrow{t \circ \pi_1} B_0.$$

The action of $f \in B_1$ is given by postcomposition with f in the second factor, while the action of $g \in A_1$ is given by precomposition with Fg in the second factor and changing to the domain of g in the first factor.

It can easily be shown that the profunctors induced by an identity functor $\mathbb{1}_{\mathbf{C}}$ have an object span that corresponds to the internal category $\mathbf{C}$ with the actions given by (internal) composition. In the case of $\mathcal{E} = \mathbf{Set}$ this boils down to the hom-functor. The fact that the object span is equivalent to the category $\mathbf{C}$ is essentially the Yoneda embedding. For this reason this profunctor is in general called the (internal) Yoneda profunctor $\mathcal{Y}(\mathbf{C})$.

2.6 Monoidal categories

Definition 2.6.1 (Monoidal category). A category $\mathbf{C}$ equipped with a bifunctor

$$- \otimes - : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

called the **tensor product** or **monoidal product**, a distinct object $\mathbf{1}$ called the **unit object**, and the following three natural isomorphisms called the **coherence maps**:

- **Associator**: $\alpha_{x,y,z} : (x \otimes y) \otimes z \cong x \otimes (y \otimes z)$;
- **Left unitor**: $\lambda_x : \mathbf{1} \otimes x \cong x$; and
- **Right unitor**: $\rho_x : x \otimes \mathbf{1} \cong x$.

$$\begin{tikzcd}
(x \otimes \mathbf{1}) \otimes y \arrow[rr, "\alpha_{x,\mathbf{1},y}"] \arrow[dr, "\rho_x \otimes \mathbb{1}_y"'] & & x \otimes (\mathbf{1} \otimes y) \arrow[dl, "\mathbb{1}_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}$$

Figure 2.7: Triangle diagram.

$$\begin{tikzcd}
& ((w \otimes x) \otimes y) \otimes z \arrow[r, "\alpha_{w,x,y} \otimes \mathbb{1}_z"] \arrow[dl, "\alpha_{w\otimes x,y,z}"'] & (w \otimes (x \otimes y)) \otimes z \arrow[dr, "\alpha_{w,x\otimes y,z}"] & \\
(w \otimes x) \otimes (y \otimes z) \arrow[drr, "\alpha_{w,x,y\otimes z}"'] & & & w \otimes ((x \otimes y) \otimes z) \arrow[dl, "\mathbb{1}_w \otimes \alpha_{x,y,z}"] \\
& & w \otimes (x \otimes (y \otimes z)) &
\end{tikzcd}$$

Figure 2.8: Pentagon diagram.

These natural transformations are required make the **triangle** and **pentagon** diagrams 2.7 and 2.8 commute.

A monoidal category for which the associator and the unitors are identity transformations is often said to be **strict**.

Example 2.6.2 (Cartesian category). A monoidal category where the monoidal product is given by the ordinary product 2.4.48.

Definition 2.6.3 (Scalar). In a monoidal category the scalars are defined as the endomorphisms $\mathbf{1} \rightarrow \mathbf{1}$. The set of scalars forms a commutative monoid.

Property 2.6.4. Every scalar $s : \mathbf{1} \rightarrow \mathbf{1}$ induces a natural transformation $s : \mathbb{1}_{\mathbf{C}} \Rightarrow \mathbb{1}_{\mathbf{C}}$ with components

$$s_x : x \cong \mathbf{1} \otimes x \xrightarrow{s\otimes\mathbb{1}_x} \mathbf{1} \otimes x \cong x.$$

For every morphism $f \in \hom(\mathbf{C})$, the naturality square $f \circ s_x = s_y \circ f$ alo defines a morphism $s \diamond f$ that is equivalently given by $\rho_y \circ (f \otimes s) \circ \rho_x^{-1}$ (one could have used the left unitors as well). These morphisms satisfy the following well-known rules of scalar multiplication from linear algebra:

- $s \diamond (s' \diamond f) = (s \circ s') \diamond f$,
- $(s \diamond f) \circ (s' \diamond g) = (s \circ s') \diamond (f \circ g)$, and
- $(s \diamond f) \otimes (s' \diamond g) = (s \circ s') \diamond (f \otimes g)$.

Definition 2.6.5 (Weak inverse). Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a monoidal category and consider an object $x \in \mathrm{ob}(\mathbf{C})$. An object $y \in \mathrm{ob}(\mathbf{C})$ is called a weak inverse of x if it satisfies $x \otimes y \cong \mathbf{1}$.

Remark 2.6.6. One can show that the existence of a one-sided weak inverse (as in the definition above) is sufficient to prove that it is in fact a two-sided weak inverse, i.e. $y \otimes x \cong \mathbf{1}$ also holds.

Theorem 2.6.7 (MacLane's coherence theorem). *Consider two functors $F, G : \mathbf{A} \to \mathbf{B}$ between two monoidal categories $\mathbf{A}, \mathbf{B}$. Any two natural transformations $\eta, \varepsilon : F \Rightarrow G$, constructed solely from the associator and the unitors, coincide.*

2.6.1 Braided categories

Definition 2.6.8 (Braided monoidal category). A monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$ equipped with a natural isomorphism
$$\sigma_{x,y} : x \otimes y \cong y \otimes x$$
that makes the two **hexagon** diagrams 2.9a and 2.9b commute for all $x, y, z \in \mathrm{ob}(\mathbf{C})$. The isomorphism σ is called the **braiding** (morphism).

$$\begin{tikzcd}
& (x \otimes y) \otimes z \arrow[dl, "\sigma_{x,y} \otimes \mathbb{1}_z"'] \arrow[dr, "\alpha_{x,y,z}"] & \\
(y \otimes x) \otimes z \arrow[d, "\alpha_{y,x,z}"'] & & x \otimes (y \otimes z) \arrow[d, "\sigma_{x,y\otimes z}"] \\
y \otimes (x \otimes z) \arrow[dr, "\mathbb{1}_y \otimes \sigma_{x,z}"'] & & (y \otimes z) \otimes x \arrow[dl, "\alpha_{y,z,x}"] \\
& y \otimes (z \otimes x) &
\end{tikzcd}$$

(a) Hexagon diagram 1.

$$\begin{tikzcd}
& x \otimes (y \otimes z) \arrow[dl, "\mathbb{1}_x \otimes \sigma_{y,z}"'] \arrow[dr, "\alpha^{-1}_{x,y,z}"] & \\
x \otimes (z \otimes y) \arrow[d, "\alpha^{-1}_{x,z,y}"'] & & (x \otimes y) \otimes z \arrow[d, "\sigma_{x\otimes y,z}"] \\
(x \otimes z) \otimes y \arrow[dr, "\sigma_{x,z} \otimes \mathbb{1}_y"'] & & z \otimes (x \otimes y) \arrow[dl, "\alpha^{-1}_{z,x,y}"] \\
& (z \otimes x) \otimes y &
\end{tikzcd}$$

(b) Hexagon diagram 2.

Figure 2.9: Hexagon diagram.

Property 2.6.9 (Yang-Baxter equation). The components $\sigma_{x,x}$ of a braiding satisfy the *Yang-Baxter* equation. More generally, the braiding σ satisfies the following equation for all objects $x, y, z \in \mathrm{ob}(\mathbf{C})$:
$$(\sigma_{y,z} \otimes \mathbb{1}_x) \circ (\mathbb{1}_y \otimes \sigma_{x,z}) \circ (\sigma_{x,y} \otimes \mathbb{1}_z) = (\mathbb{1}_z \otimes \sigma_{x,y}) \circ (\sigma_{x,z} \otimes \mathbb{1}_y) \circ (\mathbb{1}_x \otimes \sigma_{y,z}). \tag{2.28}$$

Remark 2.6.10. When drawing the above equality using string diagrams, it can be seen that the Yang-Baxter equation corresponds to the invariance of string diagrams under a *Reidemeister III move.*

Definition 2.6.11 (Symmetric monoidal category). A braided monoidal category where the braiding σ satisfies
$$\sigma_{x,y} \circ \sigma_{y,x} = \mathbb{1}_{x \otimes y}\,. \tag{2.29}$$

In Chapter **??** the theory of monoidal categories is continued.

2.6.2 Monoidal functors

Definition 2.6.12 (Monoidal functor). Let $(\mathbf{A}, \otimes, \mathbf{1_A}), (\mathbf{B}, \circledast, \mathbf{1_B})$ be two monoidal categories. A functor $F : \mathbf{A} \to \mathbf{B}$ is said to be monoidal if there exists:

1. A natural isomorphism $\psi_{x,y} : Fx \circledast Fy \Rightarrow F(x \otimes y)$ that makes Diagram 2.10 commute.
2. An isomorphism $\phi : \mathbf{1_B} \to F\mathbf{1_A}$ that makes the two diagrams in Figure 2.11 commute.

$$\begin{tikzcd}
(Fx \circledast Fy) \circledast Fz \arrow[r, "\alpha_{\mathbf{B}}"] \arrow[d, "\psi_{x,y} \circledast \mathbb{1}_{Fz}"'] & Fx \circledast (Fy \circledast Fz) \arrow[d, "\mathbb{1}_{Fx} \circledast \psi_{y,z}"] \\
F(x \otimes y) \circledast Fz \arrow[d, "\psi_{x\otimes y,z}"'] & Fx \circledast F(y \otimes z) \arrow[d, "\psi_{ax,y\otimes z}"] \\
F\Big((x \otimes y) \otimes z\Big) \arrow[r, "F\alpha_{\mathbf{A}}"'] & F\Big(x \otimes (y \otimes z)\Big)
\end{tikzcd}$$

Figure 2.10: Monoidal functor.

$$\begin{tikzcd}
Fx \circledast \mathbf{1_B} \arrow[r, "\mathbb{1}_{Fx} \circledast \phi"] \arrow[d, "\rho_{\mathbf{B}}"'] & Fx \circledast F\mathbf{1_A} \arrow[d, "\psi_{x,\mathbf{1_A}}"] \\
Fx & F(x \otimes \mathbf{1_A}) \arrow[l, "F\rho_{\mathbf{A}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1_B} \circledast Fy \arrow[r, "\phi \circledast \mathbb{1}_{Fy}"] \arrow[d, "\lambda_{\mathbf{B}}"'] & F\mathbf{1_A} \circledast Fy \arrow[d, "\psi_{\mathbf{1_A},y}"] \\
Fy & F(\mathbf{1_A} \otimes y) \arrow[l, "F\lambda_{\mathbf{A}}"]
\end{tikzcd}$$

Figure 2.11: Unitality diagrams.

Remark 2.6.13. The maps ψ and ϕ are also called **coherence maps** or **structure morphisms**.

Property 2.6.14 (Canonical unit). For every monoidal functor F there exists a canonical isomorphism $\phi : \mathbf{1_B} \to F\mathbf{1_A}$ defined by the commutative Diagram 2.12.

$$\begin{tikzcd}
\mathbf{1_B} \circledast F\mathbf{1_A} \arrow[r, "\lambda_{\mathbf{B}}"] \arrow[d, "\phi \circledast \mathbb{1}_{F\mathbf{1_A}}"'] & F\mathbf{1_A} \arrow[d, "F\lambda_{\mathbf{A}}"] \\
F\mathbf{1_A} \circledast F\mathbf{1_A} \arrow[r, "\psi_{\mathbf{1_A},\mathbf{1_A}}"'] & F(\mathbf{1_A} \otimes \mathbf{1_A})
\end{tikzcd}$$

Figure 2.12: Canonical unit isomorphism.

Definition 2.6.15 (Lax monoidal functor). A monoidal functor for which the coherence maps are merely morphisms and not isomorphisms.

Definition 2.6.16 (Monoidal natural transformation). A natural transformation η between (lax) monoidal functors (F, ψ, ϕ_F) and $(G, \widetilde{\psi}, \phi_G)$ that makes the diagrams in Figure 2.13 commute.

Definition 2.6.17 (Monoidal equivalence). An equivalence of monoidal categories consisting of monoidal functors and monoidal natural isomorphisms.

Theorem 2.6.18 (MacLane's strictness theorem). *Every monoidal category is monoidally equivalent to a strict monoidal category.*

2.6.3 Closed categories

Definition 2.6.19 (Internal hom). Let $(\mathbf{M}, \otimes, \mathbf{1})$ be a monoidal category. In this setting one can generalize the "currying" procedure, i.e. the identification of maps $x \times y \to z$ with maps

$$\begin{tikzcd} & \mathbf{1_B} \arrow[dl, "\phi_F"'] \arrow[dr, "\phi_G"] & \\ F\mathbf{1_A} \arrow[rr, "\eta_{\mathbf{1_A}}"'] & & G\mathbf{1_A} \end{tikzcd} \qquad \begin{tikzcd} Fx \circledast Fy \arrow[r, "\psi_{a,b}"] \arrow[d, "\eta_a \circledast \eta_b"'] & F(x \otimes y) \arrow[d, "\eta_{a\otimes b}"] \\ Gx \circledast Gy \arrow[r, "\widetilde{\psi}_{a,b}"'] & G(x \otimes y) \end{tikzcd}$$

Figure 2.13: Monoidal natural transformation.

$x \to (y \to z)$. The internal hom-functor $\underline{\text{Hom}}$ is defined by the following natural isomorphism:

$$\text{Hom}(x \otimes y, z) \cong \text{Hom}(x, \underline{\text{Hom}}(y, z)). \tag{2.30}$$

The existence of all internal homs is equivalent to the existence of a right adjoint to the tensor functor.

Notation 2.6.20. The internal hom $\underline{\text{Hom}}(x, y)$ is also often denoted by $[x, y]$. From now on this convention will be followed (unless otherwise specified).

Definition 2.6.21 (Closed monoidal category). A monoidal category is said to be closed monoidal if it has all internal homs. If the monoidal structure is induced by a (Cartesian) product structure, the category is often said to be **Cartesian closed**.

A category for which all slice categories are Cartesian closed is said to be **locally Cartesian closed**. A locally Cartesian closed category with a terminal object is also Cartesian closed.

Definition 2.6.22 (Exponential object). In the case of Cartesian (monoidal) categories, the internal hom $\underline{\text{Hom}}(x, y)$ is called the exponential object. This object is often denoted by y^x.

In Cartesian closed categories a different, but frequently used, notation is $x \Rightarrow y$. However, this notation will not be used as it might be confusion with the notation for *2-morphisms*.

Definition 2.6.23 (Cartesian closed functor). A functor between Cartesian closed categories that preserves products and exponential objects. As such it is the natural notion of functor between Cartesian closed categories.

Property 2.6.24 (Frobenius reciprocity). A functor R between Cartesian closed categories that admits a left adjoint L is Cartesian closed if and only if the natural transformation

$$L(y \times Rx) \to Ly \times x \tag{2.31}$$

is a natural isomorphism.

Property 2.6.25 (Global elements). The following isomorphism is natural in both $x, y \in \text{ob}(\mathbf{M})$:

$$\mathbf{M}(\mathbf{1}, [x, y]) \cong \mathbf{M}(x, y). \tag{2.32}$$

It is this relation that gives the best explanation for the term "internal hom". One also immediately obtains the following natural isomorphism:

$$\mathbf{M}(x, [\mathbf{1}, y]) \cong \mathbf{M}(x, y). \tag{2.33}$$

Because the Yoneda embedding is fully faithful this implies that $[\mathbf{1}, y] \cong y$. Although the global elements $\mathbf{M}(\mathbf{1}, y)$ do not fully specify an object y, this does hold internally.

Property 2.6.26 (Symmetry). Let $\mathbf{M}$ be a closed monoidal category. The definition of an internal hom can also be internalized, i.e. there exists a natural isomorphism of the form

$$[x \otimes y, z] \cong [x, [y, z]]. \tag{2.34}$$

Furthermore, if $\mathbf{M}$ is also symmetric, there exists an internal isomorphism of the form

$$[x, [y, z]] \cong [y, [x, z]]. \tag{2.35}$$

Definition 2.6.27 (Strong adjunction). Consider a monoidal category $\mathbf{M}$ together with two endofunctors $L, R : \mathbf{M} \to \mathbf{M}$. These functors are said to form a strong adjunction if there exists a natural isomorphism

$$[Lx, y] \cong [x, Ry]. \tag{2.36}$$

Property 2.6.25 above implies that every strong adjunction is in particular an adjunction in the sense of Section 2.2.4.

2.7 Enriched category theory

The following definition is due to *Bénabou*. It should represent the "ideal place in which to do category theory".

Definition 2.7.1 (Cosmos). A complete and cocomplete closed symmetric monoidal category.

Definition 2.7.2 (Enriched category). Let $(\mathcal{V}, \otimes, \mathbf{1})$ be a monoidal category. A $\mathcal{V}$-enriched category, also called a $\mathcal{V}$-category[13], consists of the following elements:

- a collection of objects $\mathrm{ob}(\mathbf{C})$, and
- for every pair of objects $x, y \in \mathrm{ob}(\mathbf{C})$, an object $\mathbf{C}(x, y) \in \mathrm{ob}(\mathcal{V})$ for which the following morphisms exist:
    1. $\mathrm{id}_x : \mathbf{1} \to \mathbf{C}(x, x)$ giving the (enriched) identity morphism, and
    2. $\circ_{xyz} : \mathbf{C}(y, z) \otimes \mathbf{C}(x, y) \to \mathbf{C}(x, z)$ replacing the usual composition.

The associativity and unity properties are given by commutative diagrams for the id and $\circ$ morphisms together with the associators and unitors in $\mathcal{V}$.

Definition 2.7.3 (Change of base). Consider a monoidal functor $F : \mathcal{V} \to \mathcal{W}$. This induces a change of base functor $F_* : \mathcal{V}\mathbf{Cat} \to \mathcal{W}\mathbf{Cat}$ by applying F to every hom-object.

Definition 2.7.4 (Underlying category). Given a $\mathcal{V}$-enriched category $\mathbf{C}$, the underlying category $\mathbf{C}_0$ is defined as follows:

- **Objects**: $\mathrm{ob}(\mathbf{C})$
- **Morphisms**: $\mathcal{V}(\mathbf{1}, \mathbf{C}(x, y))$,

where $\mathbf{1}$ is the monoidal unit in $\mathcal{V}$. This construction can be obtained as the functor $\mathcal{V}\mathbf{Cat}(\mathcal{I}, -)$ where $\mathcal{I}$ is the one-object $\mathcal{V}$-category with $\mathcal{I}(*, *) \equiv \mathbf{1}$.

Property 2.7.5 ($\mathcal{V}$ as a $\mathcal{V}$-category). Consider a closed monoidal category $\mathcal{V}$. This category can be given the structure $\widetilde{\mathcal{V}}$ of a $\mathcal{V}$-category by taking the hom-objects to be the internal homs, i.e. $\widetilde{\mathcal{V}}(x, y) := [x, y]$ for all $x, y \in \mathcal{V}$. Property 2.6.25 then implies that there exists an isomorphism between the underlying category $\widetilde{\mathcal{V}}_0$ and the original category $\mathcal{V}$.

[13]Not to be confused with the notation for fibre categories 2.3.7.

$$
\begin{tikzcd}
 & \mathbf{A}(x,y) \arrow[dl, "\lambda^{-1}"'] \arrow[dr, "\rho^{-1}"] & \\
\mathbf{1} \otimes \mathbf{A}(x,y) \arrow[d, "\eta_y \otimes F_{x,y}"'] & & \mathbf{A}(x,y) \otimes \mathbf{1} \arrow[d, "G_{x,y} \otimes \eta_x"] \\
\mathbf{B}(Fy,Gy) \otimes \mathbf{B}(Fx,Fy) \arrow[dr, "\circ"'] & & \mathbf{B}(Gx,Gy) \otimes \mathbf{B}(Fx,Gx) \arrow[dl, "\circ"] \\
 & \mathbf{B}(Fx,Gy) &
\end{tikzcd}
$$

Figure 2.14: $\mathcal{V}$-naturality diagram.

Given two $\mathcal{V}$-enriched categories, one can define suitable functors between them:

Definition 2.7.6 (Enriched functor). A $\mathcal{V}$-enriched functor $F : \mathbf{A} \to \mathbf{B}$ consists of the following data:

- a function $F_0 : \mathrm{ob}(\mathbf{A}) \to \mathrm{ob}(\mathbf{B})$ (as for ordinary functors), and
- for every two objects $x, y \in \mathrm{ob}(\mathbf{A})$, a morphism $F_{x,y} : \mathbf{A}(x,y) \to \mathbf{B}(Fx, Fy)$ in $\mathcal{V}$.

These have to satisfy the “usual” composition and unit conditions.

By extending (2.19) using enriched ends, one obtains a definition of enriched natural transformations and, therefore, also a definition of enriched functor categories.:

$$[\mathbf{A}, \mathbf{B}](F, G) := \int_{x \in \mathbf{A}} \mathbf{B}(Fx, Gx). \tag{2.37}$$

Given two $\mathcal{V}$-enriched functors $F, G : \mathbf{A} \to \mathbf{B}$ one can also try to define $\mathcal{V}$-natural transformations by extending the usual definition of natural transformations 2.2.13:

Definition 2.7.7 (Enriched natural transformation). An ordinary natural transformation consists of an $\mathrm{ob}(\mathbf{A})$-indexed family of morphism $\eta_x : Fx \to Gx$. This can also be interpreted as an $\mathrm{ob}(\mathbf{A})$-indexed family of morphisms $\eta_x : 1 \to \mathbf{B}(Fx, Gx)$ from the initial object (one-element set). By analogy, a $\mathcal{V}$-natural transformation is defined as an $\mathrm{ob}(\mathbf{A})$-indexed family of morphisms $\eta_x : \mathbf{1} \to \mathbf{B}(Fx, Gx)$ from the monoidal unit. The usual naturality square is replaced by the naturality hexagon 2.14.

The question then becomes how these two definitions are related. The end (2.37) comes equipped with a projection $\varepsilon_x : [\mathbf{A}, \mathbf{B}](F, G) \to \mathbf{B}(Fx, Gx)$. Precomposing this morphism with a morphism in the underlying category, i.e. an element of $\mathcal{V}(\mathbf{1}, [\mathbf{A}, \mathbf{B}](F, G))$, exactly gives a $\mathcal{V}$-natural transformation. So the underlying category of $[\mathbf{A}, \mathbf{B}]$ is the ordinary category of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations.

2.7.1 Enriched constructions

Definition 2.7.8 (Functor tensor product). Consider a covariant functor $G : \mathbf{C} \to \mathcal{V}$ and a contravariant functor $F : \mathbf{C}^{op} \to \mathcal{V}$ into a monoidal category $\mathcal{V}$, where $\mathbf{C}$ does not have to be enriched over $\mathcal{V}$. The tensor product of F and G is defined as the following coend:

$$F \otimes_{\mathbf{C}} G := \int^{x \in \mathbf{C}} Fx \otimes Gx. \tag{2.38}$$

It should be noted that the above tensor product does not produce a new functor, instead it only gives an object in $\mathcal{V}$. A different type of tensor product, one that does give a functor, exists in the enriched setting (note that there is no relation between these two definitions):

Definition 2.7.9 (Day convolution). Consider a monoidally cocomplete category $\mathcal{V}$, i.e. cocomplete monoidal category for which the tensor product bifunctor is cocontinuous in each argument, together with a $\mathcal{V}$-enriched category $\mathbf{C}$. The convolution or tensor product (if it exists) of two $\mathcal{V}$-enriched functors $F, G : \mathbf{C} \to \mathcal{V}$ is defined as the following coend:

$$F \otimes_{\text{Day}} G := \iint^{x,y \in \mathbf{C}} \mathbf{C}(x \otimes y, -) \otimes Fx \otimes Gy. \tag{2.39}$$

Property 2.7.10 (Monoidal structure). In the case where $\mathbf{M}$ is a closed symmetric monoidal category, the Day convolution is associative and, hence, defines a monoidal structure on the functor category $[\mathbf{C}, \mathbf{M}]$. The tensor unit is given by the functor (co)represented by the tensor unit in $\mathbf{C}$.

Definition 2.7.11 (Copower). Consider a $\mathcal{V}$-enriched category $\mathbf{C}$. The copower (or tensor) functor $\cdot : \mathcal{V} \times \mathbf{C} \to \mathbf{C}$ is defined by the following natural isomorphism:

$$\mathbf{C}(v \cdot x, y) \cong [v, \mathbf{C}(x, y)], \tag{2.40}$$

where the bracket $[-, -]$ on the right-hand side denotes the internal hom in $\mathcal{V}$. Dually, the power (or cotensor) functor $[-, -] : \mathcal{V} \times \mathbf{C} \to \mathbf{C}$ is defined by the following natural isomorphism:

$$\mathbf{C}(x, [v, y]) \cong [v, \mathbf{C}(x, y)], \tag{2.41}$$

where the bracket $[-, -]$ on the right-hand side again denotes the internal hom in $\mathcal{V}$. If an enriched category admits all (co)powers, it is said to be **(co)powered** (over its enriching category).

Remark 2.7.12. Equation (2.35) says that every (closed) symmetric monoidal category $\mathbf{M}$ is powered over itself, the power just being the internal hom. The same holds for the copower, which is just the usual tensor product functor.

Example 2.7.13 (Disjoint unions). Every (co)complete (locally) small category $\mathbf{C}$ admits the structure of a $\mathbf{Set}$-(co)powered category:

$$x^S := \prod_{s \in S} x \tag{2.42}$$

$$S \cdot x := \bigsqcup_{s \in S} x. \tag{2.43}$$

The definition and properties of internal hom-functors and (co)powers can be formalized as follows:

Definition 2.7.14 (Two-variable adjunction). Consider three categories $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. A two-variable adjunction $\mathbf{A} \times \mathbf{B} \to \mathbf{C}$ consists of three bifunctors:

- $- \otimes - : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$,
- $\hom_L : \mathbf{A}^{op} \times \mathbf{C} \to \mathbf{B}$, and
- $\hom_R : \mathbf{B}^{op} \times \mathbf{C} \to \mathbf{A}$

admitting the following natural isomorphisms:

$$\mathbf{C}(x \otimes y, z) \cong \mathbf{A}(x, \hom_R(y, z)) \cong \mathbf{B}(y, \hom_L(x, z)). \tag{2.44}$$

It should be noted that fixing any of the variables gives rise to ordinary adjunctions in the sense of Section 2.2.4.

Property 2.7.15 (Powers and copowers). A category $\mathbf{C}$ enriched over a monoidal category $\mathcal{V}$ is powered and copowered over $\mathcal{V}$ exactly if the hom-functor $\mathbf{C}^{op} \times \mathbf{C} \to \mathcal{V}$ is the right adjoint in an enriched two-variable adjunction. The power and copower functors are then given by the other two adjoints.

The following definition constructs Kan extensions in the enriched setting (these can be shown to reduce to 2.4.73 when enriching over $\mathbf{Set}$):

Alternative Definition 2.7.16 (Kan extension). Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be categories enriched over a monoidal category $\mathcal{V}$. If $\mathbf{B}$ is assumed to be copowered over $\mathcal{V}$, one can define the left Kan extension of $F : \mathbf{A} \to \mathbf{B}$ along $G : \mathbf{A} \to \mathbf{C}$ as a coend:

$$\mathrm{Lan}_G F := \int^{x \in \mathbf{A}} \mathbf{C}(Gx, -) \cdot Fx. \tag{2.45}$$

If $\mathbf{B}$ is assumed to be powered over $\mathcal{V}$, one can define the right Kan extension as an end:

$$\mathrm{Ran}_G F := \int_{x \in \mathbf{A}} [\mathbf{C}(-, Gx), Fx]. \tag{2.46}$$

Remark 2.7.17. By choosing $\mathcal{V} = \mathbf{Set}, \mathbf{C} = \mathbf{A}$ and $G = \mathbb{1}_{\mathbf{A}}$ in the previous definition, one obtains the ninja Yoneda lemma 2.4.68.

Property 2.7.18. Kan extensions computed using (co)ends as above are pointwise in the sense of Definition 2.4.75.

Alternative Definition 2.7.19 (Functor tensor product). Let $\mathbf{B}$ be a $\mathcal{V}$-enriched category. Consider a covariant functor $G : \mathbf{A} \to \mathbf{B}$ and a contravariant functor $F : \mathbf{A}^{op} \to \mathcal{V}$. The tensor product 2.7.8 can be generalized whenever $\mathbf{B}$ is copowered over $\mathcal{V}$:

$$F \otimes_{\mathbf{A}} G := \int^{x \in \mathbf{A}} Fx \cdot Gx. \tag{2.47}$$

2.7.2 Weighted (co)limits

In this section the definition of ordinary limits and, in particular, the defining universal property 2.4.32 is revisited. In this construction the constant functor Δ_x was one of the main ingredients. This functor can be factorized as $\mathbf{I} \to 1 \to \mathbf{C}$, where 1 denotes the terminal category. On the level of morphisms this factorization takes the form $\mathbf{I}(i, j) \to * \to \mathbf{C}(x, x)$, where $*$ denotes the terminal one-element set. However, whenever the enriching context is not $\mathbf{Set}$, one does not necessarily have access to a terminal object.

To avoid this issue, limits will first be redefined as representing objects. To this end, consider a general diagram $D : \mathbf{I} \to \mathbf{C}$. By postcomposition with the Yoneda embedding one obtains the presheaf-valued diagram $\mathbf{C}(-, D-) : \mathbf{I} \to [\mathbf{C}^{op}, \mathbf{Set}]$. Since presheaf categories are complete (Example 2.4.36), the limit of this diagram exists:

$$\mathbf{Set}(S, \lim \mathbf{C}(x, D-)) \cong [\mathbf{I}, \mathbf{Set}](\Delta_S, \mathbf{C}(x, D-)).$$

By restricting to the terminal set $S = *$, one obtains

$$\lim \mathbf{C}(x, D-) \cong [\mathbf{I}, \mathbf{Set}](\Delta_*, \mathbf{C}(x, D-)).$$

If this presheaf is representable, one can use the continuity of the hom-functor, together with the fact that the Yoneda embedding is fully faithful, to show that the representing object is (isomorphic to) $\lim D$, i.e.

$$[\mathbf{I}, \mathbf{Set}](\Delta_*, \mathbf{C}(x, D-)) \cong \mathbf{C}(x, \lim D). \tag{2.48}$$

?? CLEAN THIS UP (note that continuity and pointwise definition was already mentioned for ordinary limits) ??

Definition 2.7.20 (Weighted limit). This definition can now be generalized by replacing the constant functor Δ_* by any functor $W : \mathbf{I} \to \mathbf{Set}$. A representing object is then called the W-weighted limit of D. This object is often denoted by $\lim^W D$ or $\{W, D\}$. To distinguish weighted limits from ordinary ones, the latter are sometimes called **conical limits**.

Remark 2.7.21. A motivation for this construction is the following. As was already pointed out in Remark 2.4.18, the mere knowledge of global elements $1 \to x$ is often not enough to characterize an object x. In general one should look at the collection of generalized elements. When applying this ideology to the case of cones, one sees that replacing the functor Δ_* by a more general functor is the same as replacing the global elements $* \to Di$ by generalized elements $Wi \to Di$.

The generalization to the enriched setting is now evident. There is no reference to the terminal object left, so one can replace **Set** by any enriching category. In the enriched setting, (co)end formulas for (weighted) limits will often be used:

Formula 2.7.22 (Enriched weighted limits). By expressing the natural transformations as an end as in Equation (2.19) and by using the canonical powering in **Set**, one can express ordinary weighted limits as follows:

$$\lim^W D \cong \int_{i \in \mathbf{I}} [Wi, Di]. \tag{2.49}$$

The generalization to other enriching categories is now straightforward. Consider a diagram $D : \mathbf{I} \to \mathbf{C}$ and a weight functor $W : \mathbf{I} \to \mathcal{V}$, where $\mathbf{C}$ is $\mathcal{V}$-enriched. If $\mathbf{C}$ is powered over $\mathcal{V}$, the W-weighted limit of D is defined by the same formula as above:

$$\lim^W D := \int_{i \in \mathbf{I}} [Wi, Di]. \tag{2.50}$$

In a similar way one can define weighted colimits in copowered $\mathcal{V}$-categories as coends:

$$\operatorname{colim}^W D := \int^{i \in \mathbf{I}} Wi \cdot Di. \tag{2.51}$$

Here, the weight functor W is required to be contravariant since colimits (and cocones in general) are natural transformations between contravariant functors.

Property 2.7.23 (Weighted limits are Homs). In the case $\mathbf{C} = \mathcal{V}$, the powering functor becomes the internal hom and, therefore, one sees that weighted limits are given by (enriched) natural transformations (as was the case for ordinary conical limits).

In the following example the weighted colimit is calculated with respect to the Yoneda embedding:

Example 2.7.24 (Hom-functor). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$. When using the Yoneda embedding $\mathcal{Y}i = \mathbf{I}(-, i)$ as the weight functor, one obtains the following property by virtue of the Yoneda lemma:

$$\mathrm{colim}^{\mathcal{Y}i} D \cong Di. \tag{2.52}$$

A similar statement for weighted limits can be obtained with the covariant Yoneda embedding.

Alternative Definition 2.7.25 (Weighted (co)limits). The above property can be used to axiomatize small weighted (co)limits in bicomplete categories:

1. **Yoneda**: For every object $i \in \mathrm{ob}(\mathbf{I})$ there exist isomorphisms
$$\lim{}^{\mathbf{I}(i,-)} D \cong Di \qquad \text{and} \qquad \mathrm{colim}^{\mathbf{I}(-,i)} D \cong Di. \tag{2.53}$$
2. **Cocontinuity**: The weighted (co)limit functors are cocontinuous in the weights.

One can also express Kan extensions as weighted limits (this simply follows from expression 2.7.16):

Property 2.7.26 (Kan extensions). Consider functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{A} \to \mathbf{C}$. If for every $x \in \mathrm{ob}(\mathbf{C})$ the weighted limit $\lim^{\mathbf{C}(x, G-)} F$ exists, these limits can be combined into a functor that can be shown to be the right Kan extension $\mathrm{Ran}_G F$. The left Kan extension can be obtained as a weighted colimit.

Property 2.7.27 (Category of elements). The weighted (co)limits of a functor (over **Set**) can also be expressed in terms of the category of elements 2.4.71 of the weight:

$$\lim{}^W F \cong \lim F \circ \mathbf{C}_W, \tag{2.54}$$

where the limit on the right-hand side is a conical limit.

2.8 Abelian categories

2.8.1 Additive and Abelian categories

Definition 2.8.1 (Pre-additive category). A (locally small) category enriched over **Ab**, i.e. a category in which every hom-set is an Abelian group and composition is bilinear.

Property 2.8.2. Let **A** be a pre-additive category. The following statements are equivalent for an object $x \in \mathrm{ob}(\mathbf{A})$:

- x is initial,
- x is final, or
- $\mathbb{1}_x = 0$.

It follows that every initial/terminal object in a pre-additive category is automatically a zero object 2.4.12.

Property 2.8.3 (Biproducts). In a pre-additive category the following isomorphism holds for all finitely indexed sets $\{x_i\}_{i \in I}$:

$$\prod_{i \in I} x_i \cong \coprod_{i \in I} x_i. \tag{2.55}$$

Finite (co)products in pre-additive categories are often called **direct sums**. In general, if a product and coproduct exist and are equal, one also speaks of a **biproduct**.

Definition 2.8.4 (Additive category). A pre-additive category in which all finite products exist.

When working with additive categories, it is generally assumed that the associated functors are of a specific type:

Definition 2.8.5 (Additive functor). Let $\mathbf{A}, \mathbf{A}'$ be additive categories. A functor $F : \mathbf{A} \to \mathbf{A}'$ is said to be additive if it preserves finite biproducts:

1. It preserves zero objects: $F\,0_{\mathbf{A}} \cong 0_{\mathbf{A}'}$.
2. There exists a natural isomorphism $F(x \oplus y) \cong Fx \oplus Fy$.

This notion can be generalized to pre-additive categories. A functor between pre-additive categories is said to be additive if it acts as a group morphism on hom-spaces.

Definition 2.8.6 (Grothendieck group). Let $\mathbf{A}$ be an additive category and consider its decategorification 2.2.22. This set carries the structure of an Abelian monoid and, hence, the Grothendieck construction **??** can be applied to obtain an Abelian group $K(\mathbf{A})$. This group is called the Grothendieck group of $\mathbf{A}$.

In a (pre-)additive category one can some classical notions from (homological) algebra such as images and kernels:

Definition 2.8.7 (Kernel). Let $f : x \to y$ be a morphism. A[14] kernel of f is a morphism $k : z \to x$ such that:

1. $f \circ k = 0$.
2. **Universal property**: Every morphism $k' : z' \to x$ such that $f \circ k' = 0$ factors uniquely through k.

This implies that a kernel of f could equivalently be defined as the equalizer of f and 0.

Notation 2.8.8 (Kernel). If the kernel of $f : x \to y$ exists, it is denoted by $\ker(f)$.

Definition 2.8.9 (Cokernel). Let $f : x \to y$ be a morphism. A cokernel of f is a morphism $p : y \to z$ such that:

1. $p \circ f = 0$.
2. **Universal property**: Every morphism $p' : y \to z'$ such that $p' \circ f = 0$ factors uniquely through p.

This implies that a cokernel of f could equivalently be defined as the coequalizer of f and 0.

Notation 2.8.10 (Cokernel). If the cokernel of $f : x \to y$ exists, it is denoted by $\mathrm{coker}(f)$.

Remark 2.8.11. The name and notation of the kernel and the cokernel (in the categorical sense) is explained by remarking that $\ker(f)$ represents the functor

$$F : z \mapsto \ker\Big(\mathbf{C}(z, x) \to \mathbf{C}(z, y)\Big),$$

where ker denotes the algebraic kernel **??**, and similarly for he cokernel.

Definition 2.8.12 (Pseudo-Abelian category). An additive category in which every projection/idempotent has a kernel.

[14]Note the word "*a*". The kernel of a morphism is only determined up to an isomorphism.

Definition 2.8.13 (Pre-Abelian category). An additive category in which every morphism has a kernel and cokernel.

Definition 2.8.14 (Abelian category). A pre-Abelian category in which every mono is a kernel and every epi is a cokernel or, equivalently, if for every morphism f there exists an isomorphism

$$\mathrm{coker}(\ker(f)) \cong \ker(\mathrm{coker}(f)). \tag{2.56}$$

Property 2.8.15 (Injectivity and surjectivity). In Abelian categories a morphism is monic if and only if it is injective, i.e. its kernel is 0. Analogously, a morphism is epic if and only if it is surjective, i.e. its cokernel is 0.

Example 2.8.16 (k-linear category). Let $\mathbf{Vect}_k$ denote the category of vector spaces over the base field k. A k-linear category is a category enriched over $\mathbf{Vect}_k$. (If the base field is clear, the subscript is often left implicit.)

2.8.2 Exact functors

Definition 2.8.17 (Exact functor). Let $F : \mathbf{A} \to \mathbf{A}'$ be an additive functor between additive categories.

- F is said to be left-exact if it preserves kernels.
- F is said to be right-exact if it preserves cokernels.
- F is said to be exact if it is both left- and right-exact.

Corollary 2.8.18. The previous definition implies the following properties (which can in fact be used as an alternative definition):

- If F is left-exact, it maps an exact sequence of the form

  $$0 \longrightarrow x \longrightarrow y \longrightarrow z$$

  to an exact sequence of the form

  $$0 \longrightarrow Fx \longrightarrow Fy \longrightarrow Fz.$$

- If F is right-exact, it maps an exact sequence of the form

  $$x \longrightarrow y \longrightarrow z \longrightarrow 0$$

  to an exact sequence of the form

  $$Fx \longrightarrow Fy \longrightarrow Fz \longrightarrow 0.$$

- If F is exact, it maps short exact sequences to short exact sequences.

Notation 2.8.19 (Left or right). The category of left modules ${}_R\mathbf{Mod}$ over a ring R is equivalent (as an Abelian category) to the category of right modules $\mathbf{Mod}_{R^{op}}$ over the opposite ring R. For this reason one often makes no difference between left and right modules (only bimodules are truly relevant) and "the category of R-modules" is just denoted by $R\mathbf{Mod}$.

Theorem 2.8.20 (Freyd-Mitchell embedding theorem). *Every small Abelian category admits a fully faithful, exact functor into a category of the form $R\mathbf{Mod}$ for some unital ring R.*

Theorem 2.8.21 (Eilenberg-Watts). *Let R, S be two (not necessarily unital) rings. The tensor product functor induces an equivalence between the category of R-S-bimodules and the category of cocontinuous functors $R\mathbf{Mod} \to S\mathbf{Mod}$.*

2.8.3 Finiteness

Definition 2.8.22 (Simple object). Let $\mathbf{A}$ be an Abelian category. An object $a \in \mathrm{ob}(\mathbf{A})$ is said to be simple if the only subobjects of a are 0 and a itself. An object is said to be semisimple if it is a direct sum of simple obejcts.

Definition 2.8.23 (Semisimple category). A category is said to be semisimple if every object is semisimple (where in general the direct sums are taken over finite index sets).

Definition 2.8.24 (Jordan-Hölder series). A filtration

$$0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_n = x$$

of an object x is said to be a Jordan-Hölder series if the quotient objects x_i/x_{i-1} are simple for all $i \leq n$. If the series has finite length, the object x is said to be **finite**.

Theorem 2.8.25 (Jordan-Hölder). *If an object in an Abelian category is finite, all of its Jordan-Hölder series have the same length. In particular, the multiplicities of simple objects are the same for all such series.*

Theorem 2.8.26 (Krull-Schmidt). *Any object in an Abelian category of finite length admits a unique decomposition as a direct sum of indecomposable objects*[15]*.*

Definition 2.8.27 (Locally finite). A k-linear Abelian category is said to be locally finite if it satisfies the following conditions:

1. every hom-space is finite-dimensional, and
2. every object has finite length.

Definition 2.8.28 (Finite). A k-linear Abelian category is said to be finite if it satisfies the following conditions:

1. It is locally finite.
2. It has enough projectives or, equivalently, every simple object has a *projective cover*.
3. The set of isomorphism classes of simple objects is finite.

Theorem 2.8.29 (Schur's lemma). *Let* $\mathbf{A}$ *be an Abelian category. For every two simple objects* x, y*, all nonzero morphisms* $x \to y$ *are isomorphisms. In particular, if* x, y *are two non-isomorphic simple objects, then* $\mathbf{A}(x, y) = 0$*. Furthermore,* $\mathbf{A}(x, x)$ *is a division ring for every simple object* x*.*

Corollary 2.8.30. If $\mathbf{A}$ is locally finite and k is algebraically closed, then $\mathbf{A}(x, x) \cong k$ for all simple objects x. This follows from the fact that the only finite-dimensional division algebra over an algebraically closed field is the field itself.

The Freyd-Mitchell theorem 2.8.20 can be adapted to the finite linear case as follows:

Theorem 2.8.31 (Deligne). *Every finite* k*-linear Abelian category is* k*-linearly equivalent to a category of the form* $A\mathbf{Mod}^{\mathrm{fin}}$ *for* A *a finite-dimensional* k*-algebra.*

Construction 2.8.32 (Deligne tensor product). Let $\mathbf{A}, \mathbf{B}$ be two Abelian categories. Their Deligne (tensor) product is defined (if it exists) as the category $\mathbf{A} \boxtimes \mathbf{B}$ for which there exists a bijection between right exact functors $\mathbf{A} \boxtimes \mathbf{B} \to \mathbf{C}$ and right exact functors $\mathbf{A} \times \mathbf{B} \to \mathbf{C}$ (the latter being right exact in each argument).

For finite Abelian categories it can be shown that their Deligne product always exists. By the Deligne embedding theorem one can find an explicit description. Consider two finite-dimensional k-algebras A, B. The category $A\mathbf{Mod}^{\mathrm{fin}} \boxtimes B\mathbf{Mod}^{\mathrm{fin}}$ is equivalent to the category $A \otimes_k B\mathbf{Mod}^{\mathrm{fin}}$.

[15] An object is **indecomposable** if it cannot be written as a direct sum of its subobjects.

2.9 Higher category theory ♣

2.9.1 n-categories

Definition 2.9.1 (n-category). A (strict) n-category consists of:

- objects (0-morphisms),
- 1-morphisms going between 0-morphisms,
- ...
- n-morphisms going between $(n-1)$-morphisms,

such that the composition of k-morphisms ($k \leq n$) is associative and satisfies the unit laws as required in an ordinary category. By generalizing this definition to arbitrary n one can define the notion of a (strict) ∞-category.

If one relaxes the associativity and unit laws up to higher coherent morphisms, one obtains the notion a weak n-category. Explicit definitions for such categories have been constructed up to tetracategories ($n = 4$). However, this construction by *Trimble* takes about 50 pages of diagrams.

Remark. n-morphisms are also called n**-cells**. This makes their relation to topological spaces (and in particular simplicial spaces) more visible.

Example 2.9.2. The classical examples of a 1-category and 2-category are **Set** and **Cat**, respectively.

Property 2.9.3 (Composition in 2-categories). 2-morphisms can be composed in two different ways:

- **Horizontal composition**: Consider two 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : f' \Rightarrow g'$ where $f' \circ f$ and $g' \circ g$ are well-defined. These 2-morphisms can be composed as
$$\beta \circ \alpha : f' \circ f \Rightarrow g' \circ g.$$
- **Vertical composition**: Consider two 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ where f, g and h have the same domain and codomain. These 2-morphisms can be composed as
$$\beta \cdot \alpha : f \Rightarrow h.$$

As a consistency condition the horizontal and vertical composition are required to satisfy the following **interchange law**:

$$(\alpha \cdot \beta) \circ (\gamma \cdot \delta) = (\alpha \circ \gamma) \cdot (\beta \circ \delta). \tag{2.57}$$

Definition 2.9.4 ((n, r)-category). A higher (∞-)category for which

- all parallel k-morphisms with $k > n$ are equivalent and, hence, trivial.
- all k-morphisms with $k > r$ are invertible (or equivalences in the fully weak ∞-sense).

Definition 2.9.5 (Weak inverse). Let $\mathbf{C}$ be a 2-category. A 1-morphism $f : x \to y$ is weakly invertible if there exist a 1-morphism $g : y \to x$ and 2-isomorphisms $g \circ f \Rightarrow \mathbb{1}_x$ and $f \circ g \Rightarrow \mathbb{1}_y$.

At this point it should be obvious that the definition of a unit-counit adjunction 2.2.25 can be generalized to general 2-categories:

Definition 2.9.6 (Adjunction in 2-category). Let $\mathbf{C}$ be a 2-category. An adjunction in $\mathbf{C}$ is a pair of 1-morphisms $F : x \to y$ and $G : y \to x$ together with 2-morphisms $\varepsilon : F \circ G \Rightarrow \mathbb{1}_y$ and $\eta : \mathbb{1}_x \Rightarrow G \circ F$ that satisfy the zig-zag identities.

Remark 2.9.7 (Duals and adjunctions). By looking at the defining relations of duals in a rigid monoidal category (see Section **??**), it should be clear that these are in fact the same as the defining relations of the unit and counit of an adjunction. This is a consequence of the fact that a 2-category with a single object can be regarded as a (strict) monoidal category where the composition in the 2-category becomes the tensor product in the monoidal category. Similarly, adjoint 1-morphisms in the 2-category become duals in the monoidal category.

Property 2.9.8 (Monoidal categories). Consider a monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$. From this monoidal category one can construct the so-called **delooping** bicategory $\mathbf{BC}$ in the following way:

- There is a single object $*$.
- The 1-morphisms in $\mathbf{BC}$ are the objects in $\mathbf{C}$.
- The 2-morphisms in $\mathbf{BC}$ are the morphisms in $\mathbf{C}$.
- Horizontal composition in $\mathbf{BC}$ is the tensor product in $\mathbf{C}$.
- Vertical composition in $\mathbf{BC}$ is composition in $\mathbf{C}$.

Conversely, every 2-category with a single object comes from a monoidal category. Hence, the 2-category of (pointed) 2-categories with a single object and the 2-category of monoidal categories are equivalent. (This property and its generalizations are the content of the *delooping hypothesis*.)

In the same way one can deloop a braided monoidal category twice and find an identification with a one-object tricategory with one 1-morphism. However, this identification is not a trivial one as it makes use of the Eckmann-Hilton argument to identify different monoidal structures on this tricategory. (See also Section **??**.)

2.9.2 n-functors

Definition 2.9.9 (2-functor). A 2-functor $F : \mathbf{A} \to \mathbf{B}$ (often called a **pseudofunctor**) is a morphism between bicategories. It consists of the following data:

- a function $F_0 : \mathrm{ob}(\mathbf{A}) \to \mathrm{ob}(\mathbf{B})$, and
- for every two objects $x, y \in \mathrm{ob}(\mathbf{A})$, a functor $F_{x,y} : \mathbf{A}(x, y) \to \mathbf{B}(Fx, Fy)$.

The function F_0 and the functors $F_{x,y}$ are also often denoted by F by abuse of notation. This data is required to satisfy some coherence conditions. These are specified by the following data:

1. **Associator**: For every pair of composable 1-morphisms $f \circ g$ in $\hom(\mathbf{A})$, a 2-isomorphism $\gamma_{f,g} : Ff \circ Fg \Rightarrow F(f \circ g)$ such that for every triple of composable morphisms $f \circ g \circ h$ in $\hom(\mathbf{A})$ the following identity holds:
$$\gamma_{f\circ g,h} \circ (\gamma_{f,g} \cdot \mathbb{1}_{Fh}) = \gamma_{f,g\circ h} \circ (\mathbb{1}_{Ff} \cdot \gamma_{g,h}). \tag{2.58}$$
2. **Unitor**: For every object $x \in \mathrm{ob}(\mathbf{A})$, a 2-isomorphism $\iota_x : \mathbb{1}_{Fx} \Rightarrow F\mathbb{1}_x$ such that for every morphism $f : x \to y$ in $\hom(\mathbf{A})$ the following identities hold:
$$\iota_y \cdot \mathbb{1}_{Ff} = \gamma_{\mathbb{1}_y,f} \tag{2.59}$$
$$\mathbb{1}_{Ff} \cdot \iota_x = \gamma_{f,\mathbb{1}_x}. \tag{2.60}$$

Note that to be completely formal one should have inserted the unitors and associators of the bicategories $\mathbf{A}, \mathbf{B}$.

Definition 2.9.10 (Lax natural transformation). Consider two 2-functors $F, G : \mathbf{A} \to \mathbf{B}$ between bicategories. A lax natural transformation $\eta : F \Rightarrow G$ consists of the following data:

1. for every object $x \in \mathrm{ob}(\mathbf{A})$, a 1-morphism $\eta_x : Fx \to Gx$, and
2. for every 1-morphism $f : x \to y$ in $\hom(\mathbf{A})$, a 2-morphism $\eta_f : Gf \circ \eta_x \Rightarrow \eta_y \circ Ff$ such that the η_f are the components of a natural transformation $(\eta_x)^* \circ G \Rightarrow (\eta_y)_* \circ F$ and such that the assignment $f \mapsto \eta_f$ satisfies the "obvious" identity and composition axioms.

Remark 2.9.11. As usual in the context of higher category theory one can speak of lax 2-functors if the associator and unitors are merely required to be 2-morphisms and of strict 2-functors if these morphisms are required to be identities. If the natural transformations between morphism categories in the definition of a lax natural transformation are all isomorphisms, this is called a **pseudonatural transformation**. If the 1-morphisms η are equivalences, they are called lax natural equivalences.

Definition 2.9.12 (Modification). Consider two bicategories $\mathbf{A}, \mathbf{B}$, two 2-functors $F, G : \mathbf{A} \to \mathbf{B}$ and two parallel (lax) natural transformations $\alpha, \beta : F \Rightarrow G$. A modification $\mathfrak{m} : \alpha \Rrightarrow \beta$ maps every object $x \in \mathrm{ob}(\mathbf{A})$ to a 2-morphism $\mathfrak{m}_x : \alpha_x \Rightarrow \beta_x$ such that $\beta_f \circ (\mathbb{1}_{Gf} \cdot \mathfrak{m}_x) = (\mathfrak{m}_y \cdot \mathbb{1}_{Ff}) \circ \alpha_f$.

This is generalized as follows:

Definition 2.9.13 (Transfor). A k-transfor[16] between two n-categories maps j-morphisms to $(j + k)$-morphisms (in a coherent way).

Example 2.9.14. The definitions for operations in bicategories above lead us to the following "explicit" expressions for k-transfors (for small k):

- $k = 0$: n-functors,
- $k = 1$: (n-)natural transformations,
- $k = 2$: modifications, and
- $k = 3$: **perturbations**.

The following definition generalizes the notion of essential surjectivity 2.2.11 to higher category theory:

Definition 2.9.15 (n-surjective functor). An ∞-functor $F : \mathbf{A} \to \mathbf{B}$ is said to be n-surjective if for any two parallel $(n-1)$-morphisms f, g in $\mathbf{A}$ and n-morphism $\alpha : Ff \to Fg$ in $\mathbf{B}$, there exists an n-morphism $\widetilde{\alpha}$ in $\mathbf{A}$ such that $F\widetilde{\alpha} \cong \alpha$.

Definition 2.9.16 (Indexed category). Consider a category $\mathbf{I}$. An $\mathbf{I}$-indexed category is a pseudofunctor $\mathbf{C} : \mathbf{I}^{op} \to \mathbf{Cat}$, i.e. a 2-presheaf on $\mathbf{I}$. Indexed functors and natural transformations are defined analogously.

[16]This name was first introduced by *Crans* in [90]. A different name that is sometimes used is (n, k)-**transformation**, but this should not be confused with the natural transformations in the context of (n, r)-categories.

2.9.3 Higher (co)limits

Definition 2.9.17 (Weighted 2-limit). Consider 2-categories $\mathbf{I}, \mathbf{C}$ together with 2-functors $W : \mathbf{I} \to \mathbf{Cat}$ and $F : \mathbf{I} \to \mathbf{C}$. By direct generalization of the ordinary definition of weighted limits, one says that $\lim^W F$ is the W-weighted (2-)limit of F if there exists a pseudonatural equivalence

$$\mathbf{C}(x, \lim^W F) \cong [\mathbf{I}, \mathbf{Cat}](W, \mathbf{C}(x, F-)). \tag{2.61}$$

By restricting to the 2-category of strict 2-categories, strict 2-functors and strict natural transformations the resulting notion of a weighted 2-limit coincides with that of an ordinary weighted limit enriched in **Cat** (since strict 2-categories are simply **Cat**-enriched 1-categories.)

?? COMPLETE ??

2.10 Groupoids

Definition 2.10.1 (Groupoid). A (small) groupoid $\mathcal{G}$ is a (small) category in which all morphisms are invertible.

Example 2.10.2 (Delooping). Consider a group G. Its delooping $\mathbf{B}G$ is defined as the one-object groupoid for which $\mathbf{B}G(*, *) = G$.

Property 2.10.3 (Representations). Consider a group G together with its delooping $\mathbf{B}G$. When considering *representations* as functors $\rho : \mathbf{B}G \to \mathbf{FinVect}$, one can see that the intertwiners **??** are exactly the natural transformations. More generally, all G-sets **??** can be obtained as functors $\mathbf{B}G \to \mathbf{Set}$.

Definition 2.10.4 (Core). Let $\mathbf{C}$ be a (small) category. The core $\mathrm{Core}(\mathbf{C}) \in \mathbf{Grpd}$ of $\mathbf{C}$ is defined as the maximal subgroupoid of $\mathbf{C}$.

Definition 2.10.5 (Orbit). Let $\mathcal{G}$ be a groupoid with O, M respectively the sets of objects and morphisms. On O one can define an equivalence $x \sim y \iff \exists \phi : x \to y$. The equivalence classes are called orbits and the set of orbits is denoted by O/M.

Definition 2.10.6 (Transitive component). Let $\mathcal{G}$ be a groupoid with O, M respectively the sets of objects and morphisms and let s, t denote the source and target maps on M. Given an orbit $o \in O/M$, the transitive component of M associated to o is defined as $s^{-1}(o)$, or equivalently, as $t^{-1}(o)$.

Property 2.10.7. Every groupoid is a (disjoint) union of its transitive components.

Definition 2.10.8 (Transitive groupoid). A groupoid $\mathcal{G}$ is said to be transitive if for all objects $x \neq y \in \mathrm{ob}(\mathcal{G})$, the set $\mathcal{G}(x, y)$ is not empty.

2.11 Lawvere theories ♣

Definition 2.11.1 (Lawvere theory). Let $\mathbf{F}$ denote the skeleton of **FinSet**. A Lawvere theory consists of a small category $\mathbf{L}$ and a strict (finite) product-preserving *identity-on-objects* functor $\mathcal{L} : \mathbf{F}^{op} \to \mathbf{L}$.

Equivalently, a Lawvere theory is a small category $\mathbf{L}$ with a **generic object** c_0 such that every object $c \in \mathrm{ob}(\mathbf{L})$ is a finite power of c_0.

Property 2.11.2. Lawvere theories $(\mathbf{L}, \mathcal{L})$ form a category **Law**. Morphisms between Lawvere theories are (finite) product-preserving functors.

Definition 2.11.3 (Model). A model or **algebra** over a Lawvere theory $\mathbf{L}$ is a (finite) product-preserving functor $A : \mathbf{L} \to \mathbf{Set}$.

?? COMPLETE ??

2.12 Operad theory ♣

2.12.1 Operads

Definition 2.12.1 (Plain operad[17]). Let $\mathcal{O} = \{P(n)\}_{n\in\mathbb{N}}$ be a collection of sets, called n-**ary operations** (where n is called the **arity**). The collection $\mathcal{O}$ is called a plain operad if it satisfies following axioms:

1. $P(1)$ contains an identity element $\mathbb{1}$.
2. For all positive integers $n, k_1, \dots, k_n$ there exists a composition map

$$\begin{aligned} \circ : P(n) \times P(k_1) \times \cdots \times P(k_n) &\to P(k_1 + \cdots + k_n) \\ : (\psi, \theta_1, \dots, \theta_n) &\mapsto \psi \circ (\theta_1, \dots, \theta_n) \end{aligned} \tag{2.62}$$

that satisfies two additional axioms:

- **identity**:

$$\theta \circ (\mathbb{1}, \dots, \mathbb{1}) = \mathbb{1} \circ \theta = \theta, \tag{2.63}$$

and

- **associativity**:

$$\begin{aligned} &\psi \circ \Big(\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n})\Big) \\ &\qquad = \Big(\psi \circ (\theta_1, \dots, \theta_n)\Big) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \theta_{2,1}, \dots, \theta_{n,k_n}). \end{aligned} \tag{2.64}$$

If the operad is represented using planar tree diagrams, the associativity obtains a nice intuitive form. When combining planar tree diagrams in three layers, the associativity axiom says that one can either first glue the first two layers together or one can first glue the last two layers together.

Remark 2.12.2. Plain operads can be defined in any monoidal category. In the same way symmetric operad can be defined in any symmetric monoidal category.

Example 2.12.3 (Endomorphism operad). Consider a vector space V. For every $n \in \mathbb{N}$, one can define the endomorphism algebra $\mathrm{End}(V^{\otimes n}, V)$. The endomorphism operad $\mathcal{E}\mathrm{nd}(V)$ is defined as $\{\mathrm{End}(V^{\otimes n}, V)\}_{n\in\mathbb{N}}$.

Definition 2.12.4 (O-algebra). An object X is called an algebra over an operad O if there exist morphisms

$$O(n) \times X^n \to X$$

for every $n \in \mathbb{N}$ satisfying the usual composition and identity laws. Alternatively, this can be rephrased as the existence of a (plain) operad morphism $O(n) \to \mathcal{E}\mathrm{nd}(X)$.

Example 2.12.5 (Categorical O-algebra). An O-algebra in the category $\mathbf{Cat}$.

[17] Also called a **nonsymmetric operad** or **non-Σ operad**.

2.12.2 Algebraic topology

Definition 2.12.6 (Stasheff operad). A topological operad $\mathcal{K}$ such that $\mathcal{K}(n)$ is given by the n^{th} *Stasheff polytope/associahedron*. Composition is given by the inclusion of faces.

Definition 2.12.7 (A_∞-space). An algebra over the Stasheff operad. This induces the structure of a multiplication that is associative up to a coherent homotopy.

Definition 2.12.8 (Little k-cubes operad). A topological operad for which every topological space $\mathcal{P}(n)$ consists of all possible configurations of n embedded k-cubes in a (unit) k-cube. Composition is given by the obvious way of inserting one unit k-cube in one of the smaller embedded k-cubes.

Property 2.12.9 (Recognition principle). If a connected topological space X forms an algebra over the little k-cubes operad, it is (weakly) homotopy equivalent to the k-fold loop space $\Omega^k Y$ of another pointed topological space Y. For $k = 1$, one should technically use the Stasheff operad, but it can be shown that this is related to the little interval operad.

?? COMPLETE ??

Chapter 3

Homological algebra

References for this chapter are [46, 47].

3.1 Chain complexes

Definition 3.1.1 (Chain complex). Let $\mathbf{A}$ be an additive category (often an Abelian category) and consider a collection $\{C_k\}_{k\in\mathbb{Z}}$ of objects and a collection $\{\partial_k : C_k \to C_{k-1}\}_{k\in\mathbb{Z}}$ of morphisms in $\mathbf{A}$ such that for all $k \in \mathbb{Z}$:

$$\partial_k \circ \partial_{k+1} = 0. \tag{3.1}$$

This structure is called a chain complex[1] and the morphisms ∂_k are called the **boundary operators** or **differentials**. Elements of $\mathrm{im}(\partial_k)$ are called **boundaries** or **exact elements** and elements of $\ker(\partial_k)$ are called **cycles** or **closed elements**. The chain complex $\{(C_k, \partial_k)\}_{k\in\mathbb{Z}}$ is often denoted by $(C_\bullet, \partial_\bullet)$ or simply by $C_\bullet$ if the choice of boundary operators is clear.

Morphisms between chain complexes are called **chain maps** and they are defined as a collection of morphisms $\{f_k : C_k \to D_k\}_{k\in\mathbb{Z}}$ such that for all $k \in \mathbb{Z}$ the following equation holds:

$$\partial'_k \circ f_k = f_{k-1} \circ \partial_k, \tag{3.2}$$

where ∂_k, ∂'_k are the boundary operators of $C_\bullet$ and $D_\bullet$, respectively. Given an additive category $\mathbf{A}$, one can define the category $\mathbf{Ch}(\mathbf{A})$ of chain complexes and chain maps in $\mathbf{A}$.

Remark 3.1.2 (Reversal). Given a chain (resp. cochain) complex C one can easily construct a cochain (resp. chain) complex $\widetilde{C}$ by setting $\widetilde{C}_k := C_{-k}$.

Definition 3.1.3 (Chain homology). Given a chain complex $C_\bullet$, one can define its homology groups $H_n(C_\bullet)$. Since $\partial^2 = 0$, the kernel $\ker(\partial_k)$ is a subgroup of the image $\mathrm{im}(\partial_{k+1})$ and it is even a normal subgroup. This way one can define the quotient group:

$$H_k(C_\bullet) := \frac{\ker(\partial_k)}{\mathrm{im}(\partial_{k+1})}. \tag{3.3}$$

The kernel in this definition, i.e. the group of k-cycles, is denoted by $Z_k(C_\bullet)$. The image in this definition, i.e. group of $(k+1)$-boundaries, is denoted by $B_k(C_\bullet)$. The homology groups themselves also form a chain complex $H_\bullet(C_\bullet)$, but with trivial differentials.

[1] A **cochain complex** is constructed similarly with an ascending order $\partial_k : C_k \to C_{k+1}$.

Definition 3.1.4 (Quasi-isomorphism). A chain map for which the induced morphisms on homology are isomorphisms.

Definition 3.1.5 (Chain homotopy). Two chain maps $f, g : C_\bullet \to D_\bullet$ are said to be chain-homotopic if there exists a chain map $s : C_\bullet \to D_\bullet$ such that the following equation is satisfied:

$$f - g = s \circ \partial_C + \partial_D \circ s. \tag{3.4}$$

A chain map homotopy-equivalent to the zero map is said to be **null-homotopic**. If there exist two chain maps $f : C_\bullet \leftrightarrows D_\bullet : g$ such that both $f \circ g$ and $g \circ f$ are chain-homotopic to the identity, $C_\bullet$ and $D_\bullet$ are said to be (chain-)homotopy equivalent.

Property 3.1.6. Chain-homotopic maps induce coinciding maps in homology. In particular, every (chain-)homotopy equivalence is a quasi-isomorphism.

Corollary 3.1.7 (Vanishing homology). If a null-homotopic chain map $f : C_\bullet \to C_\bullet$ exists, then $H_\bullet(C_\bullet)$ vanishes.

Definition 3.1.8 (Differential modulo differential). Consider a chain complex $(C_\bullet, \partial_\bullet)$ together with a chain endomorphism d. This endomorphism is said to be a differential modulo ∂ if it satisfies the following conditions:

1. $d\partial + \partial d = 0$, and
2. $d^2 = [\partial, D]$ for some other chain endomorphism D, i.e. d^2 is ∂-exact in $\mathrm{End}(C_\bullet)$.

The first condition states that d descends to a chain endomorphism on $H_\bullet(C_\bullet)$. The second condition states that d is actually a differential on $H_\bullet(C_\bullet)$. The resulting homology theory is denoted by $H_\bullet(d \mid H_\bullet(C_\bullet))$.

The following definitions use the language of Chapter **??**:

Definition 3.1.9 (Differential graded algebra). A differential graded algebra (often called a **dg-algebra**, **DGA** or **dga**) is a (co)chain complex that carries the structure of an algebra where the differential acts as a derivation. Equivalently, it is a graded algebra equipped with a nilpotent derivation of degree ± 1.

Definition 3.1.10 (Connective DGA). A DGA $A_\bullet$ with vanishing (co)homology in negative degree, i.e. $H_{<0}(A_\bullet) = 0$. For every connective DGA, one can find a quasi-isomorphic DGA concentrated in nonnegative degree.

Definition 3.1.11 (Semifree DGA). A DGA for which the underlying graded algebra is isomorphic (as a graded algebra) to the tensor algebra over a graded vector space.

Definition 3.1.12 (Differential graded-commutative algebra). A graded-commutative differential graded algebra. This is often abbreviated as **DGCA** or **dgca**.

Definition 3.1.13 (Semifree dgca). A DGCA for which the underlying graded-commutative algebra is isomorphic (as a graded-commutative algebra) to the exterior algebra over a graded vector space.

Definition 3.1.14 (Minimal model). Let $(C_\bullet, \partial_\bullet)$ be a (cohomological) DGCA of finite type. A model for $C_\bullet$ is a quasi-isomorphism $\rho : (A_\bullet, d_\bullet) \to (C_\bullet, \partial_\bullet)$ from a semifree DGCA $(A_\bullet, d_\bullet)$. This model is said to be **minimal** if $A_\bullet$ is freely generated in degrees ≥ 2 and satisfies $dA \subseteq \Lambda^{\geq 2}A$.

Remark 3.1.15 (Model structure on DGCAs ♣). By Property **??**, the (minimal) models of DGCAs are (minimal) Sullivan algebras. From a model theory point of view, the (minimal) Sullivan algebras are the cofibrant objects and the (minimal) models are the cofibrant replacements.

3.2 Exact sequences

Definition 3.2.1 (Exact sequence). Let **A** be an additive category and consider a sequence of objects and morphisms in **A**:

$$C_0 \xrightarrow{\Phi_1} C_1 \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_n} C_n. \tag{3.5}$$

This sequence is said to be exact if for every $k \in \mathbb{N}$:

$$\mathrm{im}(\Phi_k) = \ker(\Phi_{k+1}). \tag{3.6}$$

In particular this means that $\Phi_{k+1} \circ \Phi_k = 0$ for all $k \in \mathbb{N}$, which in turn implies that exact sequences are a special type of chain complexes 3.1.1.

Definition 3.2.2 (Short exact sequence). A short exact sequence is an exact sequence with exactly three nonzero terms:

$$0 \longrightarrow C_0 \xrightarrow{\Phi_1} C_1 \xrightarrow{\Phi_2} C_3 \longrightarrow 0. \tag{3.7}$$

Usually, all other exact sequences are said to be **long**.

Property 3.2.3 (Morphisms in exact sequences). By looking at some small examples, one can derive some important constraints for certain exact sequences. Consider the sequence

$$0 \longrightarrow C \xrightarrow{\Phi} D.$$

This sequence can only be exact if Φ is an injective morphism (monomorphism). This follows from the fact that the only element in the image of the map $0 \to C$ is 0 because the map is a morphism. It follows that the kernel of Φ is trivial and, hence, that Φ is injective.

Analogously, the sequence

$$C \xrightarrow{\Psi} D \longrightarrow 0$$

is exact if and only if Ψ is a surjective morphism (epimorphism). This follows from the fact that the kernel of the map $D \to 0$ is all of C, which implies that Ψ is surjective (by exactness).

Combining these two cases shows that

$$0 \longrightarrow C \xrightarrow{\Sigma} D \longrightarrow 0$$

is exact if and only if Σ is a **bimorphism** (if **A** is Abelian, Σ is even an isomorphism by Property 2.4.52).

3.3 Resolutions

Consider some Abelian category **A** and let **Ch**(**A**) denote the category of chain complexes with objects in **A**.

Definition 3.3.1 (Acyclic complex). A chain complex $C_\bullet \in \mathbf{Ch}(\mathbf{A})$ is said to be acyclic if the sequence

$$\cdots \longrightarrow C_{k+1} \longrightarrow C_k \longrightarrow C_{k-1} \longrightarrow \cdots$$

is exact or, equivalently, if the homology complex $H_\bullet(C_\bullet)$ vanishes.

Remark. Some references, especially the older ones, use a slightly different definition of acyclicity. In their definition, the sequence is exact except in degree 0, i.e. $H_0(C_\bullet) \neq 0$.

Definition 3.3.2 (Resolution). Consider an object X in $\mathbf{A}$. A resolution of X is given by an acyclic chain complex in $\mathbf{A}$ of the form

$$\cdots \longrightarrow C_1 \longrightarrow C_0 \xrightarrow{\varepsilon} X \longrightarrow 0. \tag{3.8}$$

This also implies that X is the zeroth homology group of the chain complex $C_\bullet := \{C_k\}_{k\geq 0}$. The morphism $\varepsilon : C_0 \to X$ is often called the **augmentation map** and the complex $C_\bullet \to X \to 0$ is called the **augmentation** of $C_\bullet$.

In practice it is often convenient to restrict to a specific type of resolution. For example, by considering chain complexes with only injective or projective objects (Figures 2.2a and 2.2b), one obtains injective or projective resolutions. If every object in A admits a projective (resp. injective) resolution, $\mathbf{A}$ is said to **have enough projectives** (resp. **injectives**).

Theorem 3.3.3 (Homological perturbation). *Consider a resolution $(C_\bullet, \partial_\bullet)$ and denote the grading in $C_\bullet$ by r. Furthermore, consider a differential d modulo ∂ of degree 0 and denote the associated grading by* $\deg$. *There exists a differential s satisfying the following properties:*

- $\deg(s) - r(s) = 1$, *and*
- $s = \delta + d + \sum_{i=1}^{\infty} s_{(i)}$ *where* $r(s_{(i)}) = i$ *and* $\deg(s_{(i)}) = i + 1$.

Moreover, any differential that satisfies these properties has the same homology as d:

$$H_\bullet(s) \cong H_\bullet(d \mid H_\bullet(C_\bullet)) \equiv H_\bullet(d \mid H_0(C_\bullet)). \tag{3.9}$$

3.4 Derived functors

Given an additive functor 2.8.5, one can define its **prolongation** on the category of chain complexes:

Definition 3.4.1 (Prolongation). Let $F : \mathbf{A} \to \mathbf{A}'$ be an additive functor. The prolongation of F is a functor $\overline{F} : \mathbf{Ch}(\mathbf{A}) \to \mathbf{Ch}(\mathbf{A}')$ obtained by applying F to every object in a chain complex and to every diagram in the definition of a chain map. As is common, by abuse of notation the prolongation will also often be denoted by F.

To understand and unify the various long exact sequences in (co)homology and to formulate general statements about these theories, one can introduce the concept of derived functors.

Definition 3.4.2 (Left derived functor). Let $\mathbf{A}$ be an Abelian category with enough projectives and consider a right-exact functor $F : \mathbf{A} \to \mathbf{A}'$. The left derived functors L_iF are defined in the following way.

> Pick an object X in A and construct a projective resolution $P_\bullet \xrightarrow{\varepsilon} X \to 0$. Apply the prolongation to this resolution and construct the homology of the resulting chain complex:
>
> $$L_iF(X) := H_i(FP_\bullet). \tag{3.10}$$
>
> In particular, $L_0F(X) = F(X)$.

Right derived functors of left-exact functors can be constructed dually by choosing an injective resolution, applying the prolongation and taking the cohomology of the resulting cochain complex. In the remainder of this section all statements will be given for right-exact functors and left derived functors.

Remark 3.4.3 (Contravariant functors). The above construction was given for covariant functors. For contravariant functors one defines the derived functors as those of the opposite functor. This is equivalent to starting with an injective (resp. projective) resolution for the calculation of left (resp. right) derived functors since injective objects are projective in the opposite category and similarly homology becomes cohomology in the opposite category.

Property 3.4.4 (Exact functors). If F is exact, the above construction immediately implies that the derived functors L_i vanish for $i \geq 1$.

Property 3.4.5 (Projective objects). Consider a right-exact functor F together with its left derived functors L_iF. If an object P is projective, then $L_iF(P) = 0$ for all $i \geq 1$. This can easily be shown by remarking that every projective object P admits a projective resolution of the form

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P \longrightarrow P \longrightarrow 0.$$

Now of course one could wonder why the resolutions used in the construction of derived functors are required to be projective or injective. This seems to be a very strong requirement. The reason is that, when using the above definitions, the result is independent of the resolution used in the sense that the derived functors are naturally isomorphic. However, in certain situations one might want to work with a more general resolution. For example, in the next section, when considering the tensor product, it would be useful if one could just work with *flat* modules.

Definition 3.4.6 (Acyclic resolution). Consider a right-exact functor F together with its left derived functors L_iF. An object X is said to be F**-acyclic** if $L_iF(X) = 0$ for all $i \geq 1$. A resolution of an object is said to be F-acyclic if all objects in the resolution are F-acyclic.

Property 3.4.7 (Derived functors for acyclic resolutions). Derived functors of a right-exact (resp. left-exact) functor F constructed using an F-acyclic resolution are isomorphic to those obtained using a projective (resp. injective) resolution.

One of the motivating properties of derived functors are the long exact sequences in (co)homology. All of these are a result of the following property:

Property 3.4.8 (Long exact sequence). Let $F : \mathbf{A} \to \mathbf{A}'$ be a right-exact functor (the left-exact case proceeds in a similar way). Consider a short exact sequence in $\mathbf{A}$:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0. \tag{3.11}$$

Now, choose projective resolutions for A and C. By the *horseshoe lemma*, one obtains a projective resolution for B that fits in a short exact sequence of chain complexes:

$$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0. \tag{3.12}$$

Since F is additive and the above sequence is exact, the induced complex is also exact, i.e. the sequence

$$0 \longrightarrow FA_\bullet \longrightarrow FB_\bullet \longrightarrow FC_\bullet \longrightarrow 0 \tag{3.13}$$

is exact and so the *zig-zag lemma* is applicable. This theorem gives the following long exact sequence in homology:

$$\cdots \longrightarrow H_i(FB_\bullet) \longrightarrow H_i(FC_\bullet) \longrightarrow H_{i-1}(FA_\bullet) \longrightarrow H_{i-1}(FB_\bullet) \longrightarrow \cdots . \tag{3.14}$$

These homology groups are by definition the same as the left derived functors ($L_i = H_i \circ F$) and, accordingly, a long exact sequence relating the different derived functors is obtained.

Corollary 3.4.9. The above long exact sequence of derived functors shows that the first derived functor gives the obstruction to F being exact. Since exact functors have vanishing derived functors, one obtains the following result:

$$L_1F = 0 \implies L_iF = 0 \qquad \forall i \geq 1, \tag{3.15}$$

and, more generally:

$$L_iF = 0 \implies L_jF = 0 \qquad \forall j \geq i. \tag{3.16}$$

3.4.1 Modules

Consider the tensor and hom-bifunctors $- \otimes -$ and $\mathrm{Hom}(-,-)$ in the category **Mod** of modules over some ring. The tensor functor is right-exact in both arguments, while the hom-functor is left-exact in both arguments and, hence, one can construct the associated left and right derived functors. For simplicity, everything will be constructed with respect to the first argument of these bifunctors. A proof that the derived functors are *balanced*, i.e. that one can use a projective resolution for either argument and obtain isomorphic results, can be found in the references cited at the beginning of the chapter.

Definition 3.4.10 (Tor-functor). Consider a ring R and an R-module A. The Tor-functors $\mathrm{Tor}_n^R(-,A)$ are defined as the left derived functors of the tensor functor $- \otimes_R A$.

Definition 3.4.11 (Ext-functor). Consider a ring R and an R-module A. The Ext-functors $\mathrm{Ext}_R^n(-,A)$ are defined as the right derived functors of the hom-functor $\mathrm{Hom}_R(-,A)$.

Definition 3.4.12 (Flat module). An R-module M such that the induced tensor functor

$$- \otimes_R M : R\mathbf{Mod} \to R\mathbf{Mod} \tag{3.17}$$

is exact. By Property 3.4.4 this implies that a flat module is $\otimes$-acyclic 3.4.6, which in turn implies by Property 3.4.7 that these modules can be used to construct a good resolution for calculating Tor-functors.

Definition 3.4.13 (Koszul complex). Consider a commutative ring R together with a free rank-r module M over R. For every morphism $s : M \to R$ one defines the Koszul complex $K(s)$ as follows:

$$0 \longrightarrow \Lambda^r M \longrightarrow \Lambda^{r-1} M \longrightarrow \cdots \longrightarrow M \xrightarrow{s} R \longrightarrow 0, \tag{3.18}$$

where the exterior powers $\Lambda^k M$ are defined as in Section **??**, i.e. they are the free modules spanned by totally antisymmetric k-tuples in M. The differentials are defined as

$$d_k(m_1 \wedge \ldots \wedge m_k) := \sum_{i=1}^{k} (-1)^{k+1} s(m_i) m_1 \wedge \ldots \wedge \widehat{m_i} \wedge \ldots \wedge m_k, \tag{3.19}$$

where the caret $\widehat{\cdot}$ means that this element is omitted. It is clear that $d_1 = s$. The homology of this complex is called the **Koszul homology** of s.

Example 3.4.14. Every finite sequence $(x_1, \ldots, x_n)$ in R (interpreted as a choice of basis for R^n) defines a morphism $s : R^n \to R$ by

$$s : R^n \to R : (r_1, \ldots, r_n) \mapsto r_1 x_1 + \cdots + r_n x_n. \tag{3.20}$$

The associated Koszul complex is denoted by $K(x_1, \ldots, x_n)$.

Property 3.4.15 (Koszul resolution). Let R be a commutative ring. If $(x_1, \ldots, x_n)$ is a **regular sequence** on R, i.e. for every $i \leq n$ the element x_i is a nonzero divisor of $R/(x_1, \ldots, x_{i-1})$, the Koszul homology of $K(x_1, \ldots, x_n)$ satisfies:

$$H_{i\geq 1}(K(x_1, \ldots, x_n)) = 0, \tag{3.21}$$

i.e. $K(x_1, \ldots, x_n)$ is a resolution of $R/(x_1, \ldots, x_n)$, called the Koszul resolution. By the very construction of the Koszul complex, it is even a free resolution.

Property 3.4.16 (Koszul-Tate resolution). Consider a commutative ring R with an ideal I. For any element $x \in I$, one can construct the polynomial algebra $R[t]$ on a formal generator t and extend the differential by $\partial t := x$. Because of this definition, the homology class of x in $R[t]$ vanishes. This procedure is said to "kill" the homology of x.

In a similar way one can kill the higher homology of I. If[2] $I \equiv (x_1, \ldots, x_k)$, one can consider the Koszul complex $(X^0, d^0) := (K(x_1, \ldots, x_k), d)$. Its homology is exactly the quotient A/I. However, since the sequence is not necessarily regular, the higher homology groups need not vanish. To this end, choose a generating set $(x'_1, \ldots, x'_l)$ of $H_1(X^0, d^0)$. Now, consider the Koszul complex $(X^1, d^1) := (K(x_1, \ldots, x_k, x'_1, \ldots, x'_l), d')$ induced by the morphism

$$s' : R^{k+l} \to R : (r_1, \ldots r_{k+l}) \mapsto r_1x_1 + \cdots + r_kx_k + r_{k+1}x'_1 + \cdots + r_{k+l}x'_l, \tag{3.22}$$

where the generators x_i are of degree 1 and the generators x'_i are degree 2 (in the definition of the Koszul complex one thus needs to replace the Grassmann algebra by the graded-commutative algebra **??**). It should be clear that $H_0(X^1, d^1) \cong R/I$ and $H_1(X^1, d^1) = 0$. The direct limit of this construction is called the **Koszul-Tate resolution** of (R, I).

3.4.2 Group cohomology

In this section an important application of derived functors is given. In fact this was one of the motivating applications. In different areas of mathematics and physics, the concept of group cohomology pops up. Some examples are the obstruction to group extensions, the classification of projective representations and the application of these concepts to the study of symmetry-protected topological order in condensed matter physics. However, the literature on these applications often starts with an ad hoc construction based on maps from a group to a module (see Definition **??**).

For simplicity only finite groups and Abelian coefficients groups will be considered. Every G-module **??** can be regarded as a module over the group ring $\mathbb{Z}[G]$, i.e. there exists an equivalences of categories between **Ab** and $\mathbb{Z}[G]$**Mod**. Assuming the axiom of choice, every module category over a ring has enough projectives and, hence, it makes sense to define group (co)homology using derived functors in $\mathbb{Z}[G]$**Mod**. For groups an explicit construction of a resolution that is not just $\mathbb{Z}[G]$-projective but even $\mathbb{Z}[G]$-free will be given.

The homology and cohomology of a finite group G with coefficients in a G-module A is defined using the Ext- and Tor-functors defined above:

$$H^\bullet(G; A) := \mathrm{Ext}^\bullet_{\mathbb{Z}[G]}(\mathbb{Z}, A) \tag{3.23}$$

$$H_\bullet(G; A) := \mathrm{Tor}^{\mathbb{Z}[G]}_\bullet(A, \mathbb{Z}), \tag{3.24}$$

where $\mathbb{Z}$ carries the trivial G-module structure. To explicitly calculate the (co)homology groups, one has to find an acyclic resolution of $\mathbb{Z}$:

[2] If R is Noetherian, this is always possible.

Construction 3.4.17 (Normalized bar resolution). Let P'_k be a free rank-k G-module. The boundary maps are defined as follows:

$$\partial_k(g_1,\dots,g_k) = g_1(g_2,\dots,g_k) + \sum_{i=1}^{k}(-1)^i(g_1,\dots,g_ig_{i+1},\dots,g_k) + (-1)^k(g_1,\dots,g_{k-1}). \quad (3.25)$$

To obtain the normalized bar[3] resolution (in inhomogeneous form), one has to quotient out the submodule of P'_n generated by tuples $(g_1,\dots,g_n)$ where one of the g_i's is the identity. It can be shown that the resulting quotient modules P_n form a $\mathbb{Z}[G]$-free resolution of $\mathbb{Z}$.

To explicitly calculate the cohomology groups $H^k(G;A) = H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(P_\bullet, A))$, it is often easier to work with a more explicit description of the involved hom-sets. Since P'_k is a free $\mathbb{Z}[G]$-module on G^k, it is isomorphic (as a module) to $\mathbb{Z}[G^{k+1}]$. This can be seen as follows. The generating set consists of all k-tuples of elements in G:

$$S = \{(g_1,\dots,g_k) \mid \forall i \leq k : g_i \in G\}.$$

Since the module is free over $\mathbb{Z}[G]$, one can write every element as a formal linear combination of elements of the form

$$g_0(g_1,\dots,g_k).$$

One can now construct a morphism φ between this module and $\mathbb{Z}[G^{k+1}]$, which carries the diagonal G-action, in the following way. On the generating set S, define φ as follows:

$$\varphi(g_1,\dots,g_k) := (e, g_1, g_1g_2, \dots, g_1g_2\cdots g_k). \quad (3.26)$$

It is not hard to show that this morphism is in fact an isomorphism (of G-modules) and that

$$H^k(G;A) = H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{k+1}], A)). \quad (3.27)$$

By a little more algebra, it can also be shown that this hom-set is isomorphic to the (set-theoretic) mapping space $\mathrm{Map}(G^k, A)$. This space can be given an Abelian group structure induced by the group structure on A. Combining these facts, one gets the following construction for the cohomology of groups:

Construction 3.4.18 (Group cohomology). Let G be a finite group and let A be a G-module. Denote by C^k the free Abelian group generated by the set-theoretic functions $f : G^k \to A$ with the property that if any of their arguments is the identity, the result is 0. The boundary maps ∂^k, induced by the maps defined in Equation (3.25), are given by:

$$(\partial^k f)(g_1,\dots,g_{k+1}) = g_1 \cdot f(g_2,\dots,g_{k+1}) + \sum_{i=1}^{k}(-1)^i f(\dots,g_ig_{i+1},\dots) + (-1)^{k+1} f(g_1,\dots,g_k). \quad (3.28)$$

This is exactly the relation used to obtain group cohomology in Definition **??**.

Property 3.4.19 (Finiteness). Let G be a finite group and let A be a G-module such that the underlying group is finitely generated. Since in this case the hom-groups are themselves finitely generated, the cohomology groups $H^k(G;A)$ for $k \geq 1$ are also finitely generated. Furthermore, they are annihilated by the order of G, so in particular they are all torsion. It follows that all cohomology groups are finite.

[3] One of the possible explanations for this name is that the formal generating elements are often written as $[g_1|g_2|\dots|g_k]$.

Property 3.4.20 (Bockstein homomorphism). Let G be a group and consider a short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

This exact sequence induces a long exact sequence in group cohomology. The connecting homomorphism

$$H^\bullet(G;C) \to H^{\bullet+1}(G;A) \tag{3.29}$$

is called the Bockstein homomorphism.

3.5 Spectral sequences

Remark. In this section the homological convention is adopted, i.e. differentials lower the degree.

Definition 3.5.1 (Spectral sequence). Consider a collection $\{(E_i, d_i)\}_{i\in\mathbb{N}}$ of differential objects. This collection is called a spectral sequence if it satisfies

$$H(E_i, d_i) \cong E_{i+1} \tag{3.30}$$

for every $i \in \mathbb{N}$. A morphism of spectral sequences is a collection of morphisms $(\varphi_i)_{i\in\mathbb{N}}$ satisfying

1. $\varphi_i \circ d_i = d'_i \circ \varphi_i$, and
2. $\varphi_{i+1} = H(\varphi_i)$.

The objects E_i are often called the **pages** or **terms** of the spectral sequence.

3.5.1 Exact couples

Definition 3.5.2 (Exact couple). A tuple $(A, B, \alpha, \beta, \gamma)$ that fits in a commutative diagram of the form 3.1a. A morphism of exact couples is a pair of morphisms $(f, g) : (A, B) \to (A', B')$ that fit in a commutative diagram of the form 3.1b.

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] & & A \arrow[dl, "\beta"] \\
& B \arrow[ul, "\gamma"] &
\end{tikzcd}$$

(a) Exact couple.

$$\begin{tikzcd}
A' \arrow[rrrr, "\alpha'"] & & & & A' \arrow[dddll, "\beta'"] \\
& A \arrow[ul, "f"'] \arrow[rr, "\alpha"] & & A \arrow[ur, "f"] \arrow[dl, "\beta"] & \\
& & B \arrow[ul, "\gamma"] \arrow[dd, "g"] & & \\
& & & & \\
& & B \arrow[uuull, "\gamma'"] & &
\end{tikzcd}$$

(b) Morphism of exact couples.

Figure 3.1: Category of exact couples.

From any exact couple $(A, B, \alpha, \beta, \gamma)$ one can construct a spectral sequence using the following prescription:

$$E_0 := B \tag{3.31}$$
$$d_0 := \beta \circ \gamma \tag{3.32}$$

$$\vdots$$

$$E_n := \frac{\gamma^{-1}(\alpha^n(A))}{\beta(\alpha^{-n}(0))} \tag{3.33}$$
$$d_n := \beta \circ \alpha^{-n} \circ \gamma \tag{3.34}$$

It is not hard to see that $E_{n+1} = H(E_n, d_n)$, so this construction gives a functor from the category of exact couples to the category of spectral sequences. The higher exact couples $(\alpha^n D, E_n, \ldots)$ are sometimes called **derived couples**.

One can also define the term E_∞ using the following limit procedure. For every n, take the elements in E_n that are closed under d_n and call these $E_{n,n+1}$. Since there exists a canonical surjection $E_{n,n+1} \to E_{n+1}$, one can then look at all the elements in $E_{n,n+1}$ for which the image in E_{n+1} is closed under d_{n+1}. Call the set of these elements $E_{n,n+2}$. The elements that remain after taking the limit of this operation form the set $E_{n,\infty}$. Now, take the direct limit of the $E_{n,\infty}$ to obtain E_∞. This is equivalent to

$$E_\infty := \frac{\cap_i Z(E_i)}{\cup_i B(E_i)}, \tag{3.35}$$

i.e. E_∞ contains the equivalence classes of elements that are cycles for all d_n but boundaries for none. If E_∞ is the associated graded object of some filtered object G, one says that the spectral sequence **converges** to G.

Now, consider a differential object (C, d) together with a filtration $\{F_pC\}_{p\in\mathbb{N}}$. Definition 1.4.11 of a filtration immediately gives a short exact sequence for every $p \in \mathbb{N}$:

$$0 \longrightarrow F_{p-1}C \longrightarrow F_pC \longrightarrow F_pC/F_{p-1}C \longrightarrow 0. \tag{3.36}$$

This short sequence in turn gives rise to a long exact sequence in homology, which can be expressed as an exact triangle, and this triangle further leads to an exact couple:

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd} \tag{3.37}$$

where $D_p = H(F_pC)$ and $E_p = H(F_pC/F_{p-1}C)$. From a more abstract, yet more useful, point of view one can consider the object E as a functor from the category of filtered (differential) objects to the category of graded objects. As such it is constructed from the composition of the homology functor H and the *associated graded object*-functor

$$\mathrm{Gr} : C \mapsto \Big\{G_pC := F_pC/F_{p-1}C\Big\}_{p\in\mathbb{N}}.$$

On the other hand, one could of course also construct the composition $\mathrm{Gr}\circ H$ that first maps a differential object to its homology object and then builds the graded object associated to the filtration

$$F_pH(C) := \mathrm{im}(H(F_pC) \to H(C)).$$

Some straightforward questions can arise at this point: "*How are the functors $H\circ\mathrm{Gr}$ and $\mathrm{Gr}\circ H$ related?*", "*Do they coincide?*", ... The latter question is easy to answer: "*No, they do not.*" However, they can be related and this exactly happens through a spectral sequence that says how the homology of the graded object associated to C can be related to the homology of C itself.

3.5.2 Filtered complexes

For the remainder of this section only graded differential objects will be considered, i.e. $(C_\bullet, d_\bullet) = (\{C_p\}_{p\in\mathbb{Z}}, d)$ such that $dC_p \subseteq C_p$. In this case the exact couple consist of $D_{p,q} := H_q(F_pC_\bullet)$ and $E_{p,q} := H_q(G_pC_\bullet)$ and, consequently, the objects are bigraded. The filtration is also required to be compatible with the differential, i.e. $dF_iC_j \subseteq F_iC_{j-1}$.

Remark. In contrast to most of the literature, the *complementary convention*, i.e. the convention where $p+q$ denotes the total degree and hence $E_{p,q} = H_{p+q}(G_pC_\bullet)$, is not adopted.

Before introducing an expression for a general page E_r, the terms of degree zero and one are considered to get some intuition. The differential on E_0 is given by

$$d_0 : \frac{F_pC_q}{F_{p-1}C_q} \to \frac{F_pC_{q-1}}{F_{p-1}C_{q-1}} \tag{3.38}$$

and is induced by the differential d on $C_\bullet$. The kernel of this map is clearly given by all elements $x \in F_pC_q$ such that $dx = 0 \mod F_{p-1}C_{q-1}$ (with the additional remark that one also has to take the quotient by $F_{p-1}C_q$). As a result one finds that the homology $E_1 = H(E_0, d_0)$ is given by

$$E_1^{p,q} := \frac{\{x \in F_pC_q \mid dx \in F_{p-1}C_{q-1}\}}{F_{p-1}C_q + dF_pC_{q+1}}. \tag{3.39}$$

The first term in the denominator was already explained above. The second term comes from the $\mathrm{im}(d_0)$-part in the definition of $H(E_0, d_0)$. One might suspect that some data is missing since the relevant map $d_0^{p,q+1}$ goes from $\frac{F_pC_{q+1}}{F_{p-1}C_{q+1}}$ to $\frac{F_pC_q}{F_{p-1}C_q}$. However, the image of $F_{p-1}C_{q+1}$ is a subspace of $F_{p-1}C_q$ and this is already included in the first term, so one might as well work with all of F_pC_{q+1}.

For arbitrary $r > 0$, one defines the page E_r as follows:

$$E_r^{p,q} := \frac{\{x \in F_pC_q \mid dx \in F_{p-r}C_{q-1}\}}{F_{p-1}C_q + dF_{p+r-1}C_{q+1}}. \tag{3.40}$$

To relate this more to the usual notions of (co)homology, one can rephrase this in terms of (co)chains, (co)cycles and (co)boundaries. Consider again a filtered complex $F_\bullet C_\bullet$. The following definitions are used:

- The elements of G_pC_q are called the (p,q)-**chains** (in filtering degree p).
- The elements of

  $$Z^r_{p,q} := \{c \in G_pC_q \mid dc = 0 \bmod F_{p-r}C_{q-1}\}$$

  are called r-**almost** (p,q)-**cycles**.

- The elements of
$$B^r_{p,q} := dF_{p+r-1}C_{q+1}$$
are called r-**almost** (p,q)-**boundaries**.

It is then easy to see that the page E^r satisfies

$$E^r_{p,q} = Z^r_{p,q}/B^r_{p,q}, \tag{3.41}$$

i.e. the homology is given by the quotient of the cycles by the boundaries. All these objects fit in a nice sequence of inclusions:

$$B^0_{p,q} \hookrightarrow \cdots \hookrightarrow B^\infty_{p,q} \hookrightarrow Z^\infty_{p,q} \hookrightarrow \cdots \hookrightarrow Z^0_{p,q}. \tag{3.42}$$

3.5.3 Convergence

Definition 3.5.3 (Limit term). Consider a spectral sequence $\{E^r_{p,q}\}$. If there exists a for every two integers $p,q \in \mathbb{Z}$ an integer $r(p,q) \in \mathbb{N}$ such that for all $r \geq r(p,q)$:

$$E^r_{p,q} \cong E^{r(p,q)}_{p,q}, \tag{3.43}$$

the object $E^\infty := \{E^{r(p,q)}_{p,q}\}$ is called the **limit term** and the sequence is said to **abut** to E^∞.

Example 3.5.4 (Collapsing sequence). If there exists an integer $r \in \mathbb{N}$ such that for all $s \geq r : d_s = 0$, the sequence is said to **collapse** at r and E^r is a limit term. A common example is where the nonvanishing elements of a term are concentrated in a single row or column.

Definition 3.5.5 (Convergence). A spectral sequence $E^r_{p,q}$ is said to converge to a graded object $H_\bullet$ with filtering $F_\bullet H_\bullet$, denoted by

$$E^r_{p,q} \Rightarrow H_\bullet,$$

if

$$E^\infty_{p,q} \cong G_pH_q \qquad\qquad \forall p,q \in \mathbb{Z}. \tag{3.44}$$

Definition 3.5.6 (Bounded sequence). A spectral sequence is said to be bounded if for all numbers $n,r \in \mathbb{Z}$, there only exists a finite number of nonvanishing elements of the form $E^r_{k,n-k}$. A common example are the **first quadrant spectral sequences** where the only nonvanishing elements have $p,q \geq 0$.

Property 3.5.7. Every bounded spectral sequence abuts.

Property 3.5.8 (Filtered complex). If the spectral sequence of a filtered complex $F_\bullet C_\bullet$, it converges to the chain homology of the complex:

$$E^r_{p,q} \Rightarrow H_\bullet(C). \tag{3.45}$$

?? COMPLETE ??

Chapter 4

Model theory ♣

General references for this chapter are [3, 4]. For more on monoidal model categories see [5]. A good reference for the section on simplicial spaces and, in particular, the theory of Segal spaces is [6]. For more on Reedy model structures see [7]. A gentle introduction to the theory of homotopy (co)limits can be found in [8, 9].

4.1 Simplicial sets

Definition 4.1.1 (Simplex category). The simplex category $\mathbf{\Delta}$ has as objects the posets of the form $[n] := \{0 < \ldots < n\}$ and as morphisms the order-preserving maps.

Definition 4.1.2 (Simplicial set). The category **sSet** of simplicial sets is defined as the presheaf category $\mathbf{Psh}(\mathbf{\Delta})$. For all $n \in \mathbb{N}$, the set of **n-simplices in** X is defined as the set $X_n := X([n])$.

Definition 4.1.3 (Simplicial object). By internalizing the notion of a simplicial set, one obtains the definition of a simplicial object, i.e. a simplicial object in a category $\mathbf{C}$ is a $\mathbf{C}$-valued presheaf on $\mathbf{\Delta}$.

Remark 4.1.4. Note that the notion of **simplicial category** can mean two distinct things. In general it will mean a category enriched in **sSet**. However, following the previous definition, it can also mean a simplicial object in the (2-)category **Cat**. It can be shown that all simplicially enriched categories are a specific kind of degenerate simplicial object in **Cat**, where the face and degeneracy maps are identity-on-objects.

Definition 4.1.5 (Standard simplex). For every $n \in \mathbb{N}$, the standard n-simplex $\Delta[n]$ is defined as the Yoneda embedding $\mathbf{\Delta}(-, [n])$. One can also define a functor $\Delta_{\text{top}} : \mathbf{\Delta} \to \mathbf{Top}$ that maps $[n]$ to the standard topological n-simplex Δ^n (Definition **??**).

Property 4.1.6. By the Yoneda lemma there exists a natural bijection between the set of n-simplices of a simplicial set X and the set of maps $\Delta[n] \to X$.

Property 4.1.7 (Face and degeneracy maps). All morphisms in the simplex category $\mathbf{\Delta}$ are generated by morphisms of the following two types:

- For every n and $i < n$, the unique map $\delta_{n,i} : [n-1] \to [n]$ that misses the i^{th} element.
- For every n and $i \leq n$, the unique map $\sigma_{n,i} : [n+1] \to [n]$ that duplicates the i^{th} element.

Under the action of a presheaf this gives the **face** and **degeneracy** maps $d_{n,i}$ and $s_{n,i}$. (If the index n is clear, it is often omitted.)

These morphisms satisfy the fundamental **simplicial identities**:

- $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$,
- $d_i \circ s_j = s_{j-1} \circ d_i$ for $i < j$,
- $d_i \circ s_j = \mathrm{id}$ for $i = j$ or $i = j+1$,
- $d_i \circ s_j = s_j \circ d_{i-1}$ for $i > j+1$, and
- $s_i \circ s_j = s_{j+1} \circ s_i$ for $i \leq j$.

Definition 4.1.8 (Connected components). Consider a simplicial set X. Its set of connected components $\pi_0(X)$ is defined as the quotient of X_0 under the relation

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \times X_0.$$

This defines a fucntor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$. By base change 2.7.3, this also induces a functor on simplicial categories.

Construction 4.1.9 (Nerve and realization). Consider a general functor $F : \mathbf{S} \to \mathbf{C}$ into a cocomplete category ($\mathbf{S}$ will often be a category of geometric shapes such as the simplex category $\mathbf{\Delta}$ or the *cube category* $\square$). Every such functor induces an adjunction

$$\begin{tikzcd} \mathbf{C} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "N"'] & \mathbf{Psh}(\mathbf{S}). \arrow[l, bend right, "|-|"'] \end{tikzcd} \tag{4.1}$$

The **realization functor** $|-|$ is defined as the left Kan extension $\mathrm{Lan}_{\mathcal{Y}}F$. The **nerve functor** $N : \mathbf{C} \to \mathbf{Psh}(\mathbf{S})$ is defined as $Nx := \mathbf{C}(F-, x) = \mathcal{Y}x \circ F$.

This definition can easily be generalized to the enriched setting. Furthermore, if one assumes that $\mathbf{C}$ is copowered over $\mathcal{V}$, the realization functor can be expressed as a coend:

$$|X| = \int^{s \in \mathbf{S}} Xs \cdot Fs. \tag{4.2}$$

Example 4.1.10 (Nerve of a category). To every small category $\mathbf{C}$ one can associate a simplicial set $N\mathbf{C}$ in the following way. The set $N\mathbf{C}_0$ is given by the set of objects in $\mathbf{C}$ and the set $N\mathbf{C}_1$ is given by the set of morphisms in $\mathbf{C}$. Now, for every two composable morphisms f, g one obtains a canonical commuting triangle by composition. Let $N\mathbf{C}_2$ be the set of all these triangles. The higher simplices are defined analogously. Face maps act by composing morphisms or by dropping the exterior morphisms in a composable string. Degeneracy maps act by inserting an identity morphism.

Equivalently, one can define the **simplicial nerve functor** in the following way. Every poset $[n]$ admits a canonical category structure for which the order-preserving maps give rise to the associated functors. By the above construction this inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{Cat}$ induces the nerve functor

$$N : \mathbf{Cat} \to \mathbf{sSet} : \mathbf{C} \mapsto \mathbf{Cat}(-, \mathbf{C}). \tag{4.3}$$

This way one obtains $N\mathbf{C}_k = \mathbf{Cat}([k], \mathbf{C})$. This object is by definition equivalent to the collection of all strings of k composable morphisms in $\mathbf{C}$. It can be shown that the simplicial nerve functor is fully faithfull.

Example 4.1.11 (Geometric realization). Consider a simplicial set X. From this object one can construct a topological space as follows. First, take a point for every element in X_0.

Then, glue 1-simplices between these points using the face maps. The higher (nondegenerate) simplices are attached analogously.

More abstractly, the geometric realization functor $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ is defined as a (left) Kan extension:

$$|\cdot| := \mathrm{Lan}_{\mathcal{Y}}\Delta_{\mathrm{top}}. \tag{4.4}$$

An application of the Yoneda lemma shows that the geometric realization can be expressed as a functor tensor product 2.7.8:

$$|X| = X \otimes_{\mathbf{\Delta}} \Delta_{\mathrm{top}} = \int^{n\in\mathbf{\Delta}} X_n \cdot \Delta^n. \tag{4.5}$$

This formula can easily be generalized to the category of simplical topological spaces (**sSet** is a full subcategory obtained by endowing every set with the discrete topology). In **Top** the coend can be expressed as the quotient space

$$|X| := \bigsqcup_{n\in\mathbb{N}} X_n \times \Delta^n/\sim, \tag{4.6}$$

where the equivalence relation identifies the points (x, f_*y) and (f^*x, y) for all morphisms $f \in \hom(\mathbf{\Delta})$. The morphisms f^*, f_* are the ones induced by X and $\Delta_{\mathrm{top}} : \mathbf{\Delta} \hookrightarrow \mathbf{Top}$. As an immediate example one obtains

$$|\Delta[n]| = \Delta^n, \tag{4.7}$$

which shows that $\Delta[n]$ really deserves to be called the standard n-simplex.

Example 4.1.12 (Singular set). Given a topological space X one can define a simplicial set $\mathrm{Sing}(X)$. Its components are defined as the set of morphisms from the standard (topological) n-simplex to X:

$$\mathrm{Sing}(X)_n := \mathbf{Top}(\Delta^n, X). \tag{4.8}$$

This is the object of relevance in the definition of singular (co)homology as given in Section **??**.

Property 4.1.13 (Classifying space). For a (discrete) group G one can construct two important objects: the delooping $\mathbf{B}G$ and the classifying space BG (Definitions 2.10.2 and **??**). As their notations imply there exists a relation between these space. By first taking the nerve of $\mathbf{B}G$ and then passing to its geometric realization, one obtains BG. In fact this method can be applied to any monoid A to obtain the so-called (two-sided) *bar construction*.

4.1.1 Homological algebra

In this section simplicial sets are related to homological algebra (Chapter 3). A basic introduction is [10].

Construction 4.1.14 (Alternating face map complex). From a simplicial Abelian group A, one can construct a connective chain complex as follows. For every $n \in \mathbb{N}$:

$$(CA)_n := A_n. \tag{4.9}$$

The boundary maps δ_n are defined as the alternating sum of the face maps:

$$\delta_n := \sum_{i=1}^{n} (-1)^i d_i. \tag{4.10}$$

Every group A_{n+1} contains a subgroup $D(A_n)$ generated by the degeneracy maps:

$$D(A_n) := \left\langle \bigcup_{i=1}^{n} s_i(A_n) \right\rangle. \tag{4.11}$$

If these degenerate simplices are quotiented out, the **normalized complex** is obtained.

This construction can be generalized to any simplicial group:

Construction 4.1.15 (Moore complex). Let G be a simplicial group. For every $n \in \mathbb{N}$:

$$(NG)_n := \bigcap_{i=1}^{n} \ker(d_i^n). \tag{4.12}$$

The differential ∂_n is given by the zeroth face map d_0^n.

Property 4.1.16 (Equivalence). For simplicial Abelian groups, the Moore complex and normalized complex are isomorphic. Moreover, the inclusion of the normalized complex into the alternating face map complex is a quasi-isomorphism.

Theorem 4.1.17 (Dold-Kan correspondence). *The functor that maps simplicial Abelian groups to normalized chain complexes gives an equivalence of categories* $\mathbf{sAb} \cong \mathbf{Ch}^+(\mathbf{Ab})$.

4.2 Localization

Definition 4.2.1 (Category with weak equivalences). A category $\mathbf{C}$ with a subcategory $\mathbf{W}$ such that:

1. $\mathbf{W}$ contains all isomorphisms in $\mathbf{C}$ (in particular, $\mathbf{W}$ is wide).
2. $\mathbf{W}$ satisfies the "2-out-of-3 property": If any two of $\{f, g, f \circ g\}$ are in $\hom(\mathbf{W})$, so is the third.

Definition 4.2.2 (Weak factorization system). Consider a category $\mathbf{C}$. A pair (L, R) of classes of morphisms in $\mathbf{C}$ is called a weak factorization system (WFS) if it satisfies the following properties:

1. Every morphism in $\mathbf{C}$ factorizes as a composition $g \circ f$ where $f \in L$ and $g \in R$.
2. L consists of exactly those morphisms in $\mathbf{C}$ that have the left lifting property 2.4.23 with respect to morphisms in R.
3. R consists of exactly those morphisms in $\mathbf{C}$ that have the right lifting property with respect to morphisms in L.

Remark. The original definition by *Quillen* only required that L and R satisfied the lifting properties with respect to each other, not that they were closed under this condition.[1] This was then enforced by introducing the condition that both L and R are closed under retracts in the arrow categories. It can be proven that this is equivalent to the definition including closure as above.

Definition 4.2.3 (Homotopical category). A category $\mathbf{C}$ equipped with a subcategory $\mathbf{W}$ such that:

[1] Model categories defined using the "strong" notion of weak factorization system were called **closed model categories**.

1. **W** contains all identity morphisms in **C** (in particular, **W** is wide).
2. **W** satisfies the "2-out-of-6 property": If $f \circ g$ and $g \circ f$ are in $\hom(\mathbf{W})$, so are f, g, h and $f \circ g \circ h$.

It is not hard to see that every homotopical category is a category with weak equivalences.

Definition 4.2.4 (Homotopical functor). A functor of homotopical categories that preserves weak equivalences.

Definition 4.2.5 (Gabriel-Zisman localization). Consider a category **C** with a collection of morphisms $M \subset \hom(\mathbf{C})$. The localization of **C** with respect to M is constructed by adding for each morphism $f \in M$ a formal inverse to $\hom(\mathbf{C})$.

More specifically, the localization consists of a category $\mathbf{C}[M^{-1}]$ and a functor $F_M : \mathbf{C} \rightarrow \mathbf{C}[M^{-1}]$ inverting M, i.e. it maps all morphisms in M to isomorphisms, that is universal with respect to this property. Universality means that for every other category **D** and functor $F : \mathbf{C} \rightarrow \mathbf{D}$ that inverts all morphisms in M the following conditions are satisfied:

- There exists a functor $Z_F : \mathbf{C}[M^{-1}] \rightarrow \mathbf{D}$ such that $Z_F \circ F_M$ is naturally isomorphic to F.
- The precomposition functor $F_M^* : [\mathbf{C}[M^{-1}], \mathbf{D}] \rightarrow [\mathbf{C}, \mathbf{D}]$ is fully faithfull.

Definition 4.2.6 (Homotopy category I). When **C** is a category with weak equivalences W, the localization $\mathbf{C}[W^{-1}]$ is called the homotopy category $\mathbf{Ho}(\mathbf{C})$. In this context the functor $\mathbf{C} \rightarrow \mathbf{Ho}(\mathbf{C})$ is sometimes denoted by $\gamma_{\mathbf{C}}$.

Remark 4.2.7 (Size issues). When **C** is small, so is its localization. However, even in the case were **C** is locally small, its localization might be large.

Definition 4.2.8 (Reflective localization). Consider an adjunction $F \dashv G : \mathbf{C} \rightarrow \mathbf{D}$. G is fully faithful, i.e. defines a reflective subcategory 2.2.27, if and only if F realizes **D** as a localization of **C**. The essential image of G consists exactly of the local objects in C.

Definition 4.2.9 (Derived functor). Consider a homotopical functor $F : \mathbf{A} \rightarrow \mathbf{B}$ and consider the composition $\gamma = \gamma_{\mathbf{B}} \circ F$ with the localization functor $\gamma_{\mathbf{B}} : \mathbf{B} \rightarrow \mathbf{Ho}(\mathbf{B})$. The derived functor $\mathbf{Ho}(F) : \mathbf{Ho}(\mathbf{A}) \rightarrow \mathbf{Ho}(\mathbf{B})$ is the functor obtained by the universal property of $\mathbf{Ho}(\mathbf{A})$ applied to γ.

This definition can be rephrased in terms of Kan extensions. Consider a homotopical functor $F : \mathbf{A} \rightarrow \mathbf{B}$. The left and right derived functors are defined as the following Kan extensions:

$$LF := \mathrm{Ran}_{\gamma_{\mathbf{A}}}(\gamma_{\mathbf{B}} \circ F) \tag{4.13}$$

$$RF := \mathrm{Lan}_{\gamma_{\mathbf{A}}}(\gamma_{\mathbf{B}} \circ F). \tag{4.14}$$

In fact, one can drop the assumption that **B** has weak equivalences (here, F should map weak equivalences to isomorphisms). In this case one simply has to replace $\gamma_{\mathbf{B}} \circ F$ by F in the above formulas.

Example 4.2.10 (Derived category). Consider an Abelian category **A** together with its category of chain complexes $\mathbf{Ch}(\mathbf{A})$. The derived category $\mathcal{D}(\mathbf{A})$ is defined as the localization of $\mathbf{Ch}(\mathbf{A})$ at the quasi-isomorphisms.

Remark 4.2.11. In this case it can be shown that one can first restrict to the naive homotopy category $\mathbf{K}(\mathbf{A})$, consisting of chain complexes and chain maps up to chain-homotopy, and then localize at the collection of quasi-isomorphisms.

4.3 Model categories

Definition 4.3.1 (Model structure). Let $\mathbf{C}$ be a category. A (**Quillen-**)model structure on $\mathbf{C}$ consists of three classes of morphisms:

- **weak equivalences** W,
- **fibrations** Fib, and
- **cofibrations** Cof

that satisfy the following two conditions:

1. W turns $\mathbf{C}$ into a category with weak equivalences 4.2.1.
2. $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ and $(\mathrm{Cof} \cap W, \mathrm{Fib})$ are weak factorization systems 4.2.2.

The morphisms in $\mathrm{Fib} \cap W$ and $\mathrm{Cof} \cap W$ are said to be **acyclic** or **trivial**.

Remark. That W contains all isomorphisms in fact follows from the property that any class of morphisms satisfying a lifting property contains all isomorphisms.

Definition 4.3.2 (Model category). A bicomplete category equipped with a model structure.[2]

Definition 4.3.3 (Proper model category). A model category is said to be left proper (resp. right proper) if weak equivalences are preserved by pushouts along cofibrations (resp. pullbacks along fibrations).

Definition 4.3.4 (Fibrant object). An object in a model category for which the terminal morphism is a fibration. Dually, an object in a model category is said to be **cofibrant** if the initial morphism is a cofibration.

Property 4.3.5 (Model structure on functor categories). Consider a (small) category $\mathbf{A}$ and a model category $\mathbf{B}$. In some cases the functor category $[\mathbf{A}, \mathbf{B}]$ admits two canonical model structures:

- **Injective model structure**: The weak equivalences are the natural transformations that are objectwise weak equivalences and the cofibrations are the natural transformations that are objectwise cofibrations.
- **Projective model structure**: The weak equivalences are the natural transformations that are objectwise weak equivalences and the fibrations are the natural transformations that are objectwise fibrations.

Property 4.3.6 (Resolution). In a model category the bicompleteness property implies that the initial and terminal object always exist. The weak factorization property then implies that for every object x one can find a weakly equivalent fibrant replacement x^{f} and a weakly equivalent cofibrant replacement x^{c} by suitably factorizing the morphisms to the initial and terminal objects. These replacements are also sometimes called **resolutions** or **approximations**.

If the weak factorization system is functorial 2.3.5, (co)fibrant replacement defines an endofunctor that is weakly equivalent to the identity functor.

Definition 4.3.7 (Quillen adjunction). Let $\mathbf{A}, \mathbf{B}$ be two model categories. An adjunction

$$\mathbf{B} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{A}$$

[2] *Quillen*'s original definition only required the existence of finite limits and finite colimits.

is called a **Quillen adjunction** if the left adjoint preserves cofibrations and acyclic cofibrations. The model category axioms imply that this is equivalent to requippring that the right adjoint preserves fibrations and acyclic fibrations. The adjoint functors are called (left and right) **Quillen functors**.

If $F \dashv G$ is a Quillen adjunction such that for all cofibrant objects x and fibrant objects y the morphism $Fx \to y$ is a weak equivalence if and only if the adjunct $x \to Gy$ is a weak equivalence, then $F \dashv G$ is called a **Quillen equivalence**.

Quillen equivalences can also be characterized at the level of homotopy categories:

Property 4.3.8. Let $F \dashv G$ be a Quillen adjunction. This pair is a Quillen equivalence if and only if the left (resp. right) derived functor LF (resp. RG) is an equivalence.

Property 4.3.9 (Derived adjunction). If $F \dashv G$ is a Quillen adjunction, the derived functors (LF, RG) also form an adjunction.

Property 4.3.10 (Doubly categorical interpretation). The map that sends a model category to its homotopy category and a Quillen functor to its derived functor is a *double pseudofunctor*. Amongst other things this implies that the composition of derived functors is naturally weakly equivalent to the derived functor of the composition.

Example 4.3.11 (Topological spaces). A first example of model categories is the category **Top** of topological spaces (Chapters **??** and **??**). This category can be endowed with a model structure by taking the weak equivalences to be the weak homotopy equivalences **??** and by taking the fibrations to be the Serre fibrations **??**.

Example 4.3.12 (Simplicial sets). As a second example consider the category **sSet** of simplicial sets 4.1.2. This category can be turned into a model category by taking the weak equivalences to be the morphisms that induce weak homotopy equivalences between geometric realizations and by taking the fibrations to be Kan fibrations. With this structure the fibrant objects are the Kan complexes and the cofibrations are the levelwise injections, i.e. the cofibrations are the monomorphisms.

Notation 4.3.13 (Quillen's model structure). The model structure defined in the above example is generally called Quillen's model structure on simplicial sets and it is denoted by $\mathbf{sSet}_{\text{Quillen}}$.

Property 4.3.14 (Quillen). The adjoint pair of geometric realization and singular set functors (4.1.11 and 4.1.12) gives a Quillen equivalence between the above model categories. This result allows to regard simplicial sets as if they were spaces and vice versa. Consequently, most of homotopy theory can be done in either category.

4.3.1 Monoidal structures

Definition 4.3.15 (Pushout-product). Let $(\mathbf{M}, \otimes)$ be a closed symmetric monoidal category and consider two morphisms $f : a \to b$ and $g : x \to y$ in $\mathbf{M}$. By taking suitable tensor products, one can form the span $a \otimes y \leftarrow a \otimes x \to b \otimes x$. If the pushout of this diagram exists, the pushout-product $f \square g$ is defined as the unique morphism from this pushout to $b \otimes y$ defined by the obvious diagram

$$\begin{tikzcd}
a \otimes x \arrow[r] \arrow[d] & b \otimes x \arrow[d] \\
a \otimes y \arrow[r] & b \otimes y.
\end{tikzcd}$$

The dual concept, where one of the arguments is contravariant, is sometimes called a **pullback-hom**, **pullback-exponential** or **pullback-power** depending on which bifunctor is used in the definition. In fact, the requirement that $\mathbf{M}$ carries a monoidal structure can be dropped and any bifunctor $\circledast : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ can be used. This more general construction is sometimes called the **Leibniz construction** and the case where the bifunctor is a tensor bifunctor is then called the **Leibniz tensor**.

If $\mathbf{M}$ is also a model category, it is said to satisfy the **pushout-product axiom** if the pushout-product of two cofibrations is again a cofibration, acyclic if any of the input cofibrations was.

Definition 4.3.16 (Quillen bifunctor). A bifunctor satisfying the pushout-product axiom that preserves colimits in both variables is called a **(left) Quillen bifunctor**. It should be noted that the tensor product automatically satisfies this last property in the case of closed monoidal categories.

In fact, the natural setting for defining Quillen bifunctors is that of two-variable adjunctions 2.7.14. Consider such a triple of bifunctors $(\otimes, \mathrm{hom}_L, \mathrm{hom}_R)$.

- The bifunctor $-\otimes- : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is said to be **left Quillen** if its pushout-product of cofibrations is again a cofibration and if the acyclicity of any of the domain morphisms implies the acyclicity of the result (i.e. it satisfies the pushout-product axiom).
- The bifunctors $\mathrm{hom}_L : \mathbf{C}^{op} \times \mathbf{E} \to \mathbf{D}$ and $\mathrm{hom}_R : \mathbf{D}^{op} \times \mathbf{E} \to \mathbf{C}$ are said to be **right Quillen** if the Leibniz product of a cofibration and a fibration is a fibration and if the acyclicity of any of the domain morphisms implies the acyclicity of the result.

It can be shown that if one of these bifunctors is Quillen, the others are too.

Remark 4.3.17. The fact that for two-variable adjunctions one does not mention preservation of (co)limits follows from the property that left (resp. right) adjoints preserve colimits (resp. limits).

Definition 4.3.18 (Monoidal model category). A model category $\mathbf{M}$ that carries the structure of a closed symmetric monoidal category $(\mathbf{M}, \otimes, \mathbf{1})$ such that the following compatibility conditions are satisfied:

1. **Pushout-product**: The tensor bifunctor and internal homs define a Quillen two-variable adjunction.
2. **Unit**: For every cofibrant object x and every cofibrant replacement $\mathbf{1}^c$ of $\mathbf{1}$, the induced morphism $\mathbf{1}^c \otimes x \to x$ is a weak equivalence.

Definition 4.3.19 (Enriched model category). Let $\mathcal{V}$ be a monoidal model category. A category $\mathbf{M}$ is called a $\mathcal{V}$-enriched model category if is satisfies the following conditions:

1. $\mathbf{M}$ is a $\mathcal{V}$-enriched category that is both powered and copowered over $\mathcal{V}$.
2. The underlying category $\mathbf{M}_0$ is a model category.
3. The copower is a left Quillen bifunctor or, equivalently, the power is a right Quillen bifunctor.

Example 4.3.20 (Simplicial model category). A model category enriched over $\mathbf{sSet}_{\mathrm{Quillen}}$. It can be shown that the full subcategory of a simplicial model category on fibrant-cofibrant objects is enriched over Kan-complexes.

$$\begin{tikzcd}
x \arrow[r] \arrow[dr, "f"'] & \mathrm{Cyl}(x) \arrow[d, "\eta"] & x \arrow[l] \arrow[dl, "g"] \\
& y &
\end{tikzcd}$$

(a) A left homotopy.

$$\begin{tikzcd}
& x \arrow[dl, "f"'] \arrow[d, "\lambda"] \arrow[dr, "g"] & \\
y & \mathrm{Path}(y) \arrow[l] \arrow[r] & y
\end{tikzcd}$$

(b) A right homotopy.

Figure 4.1: Homotopies in a model category.

4.3.2 Homotopy category

Definition 4.3.21 (Homotopy). Before being able to construct homotopies in general model categories, thereby generalizing the ideas from Section **??**, one has to define the counterpart of the unit interval in a model category $\mathbf{C}$. To this end, consider an object $x \in \mathrm{ob}(\mathbf{C})$. By taking the product and coproduct of two copies of x, one can construct the unique diagonal and codiagonal morphisms $\Delta : x \to x \times x$ and $\nabla : x \sqcup x \to x$. By factorizing these morphisms in $\mathbf{C}$, two weak equivalences are obtained:

$$x \to \mathrm{Path}(x) \qquad \text{and} \qquad \mathrm{Cyl}(x) \to x.$$

These objects are called the **path object** and **cylinder object**, respectively. By choosing the morphism to be a fibration (resp. cofibration), the notion of **good** cylinder (resp. path) object is obtained.

Using these objects one can define left and right homotopies between parallel morphisms $f, g : x \to y$. A left homotopy between f and g is a morphism $\eta : \mathrm{Cyl}(x) \to y$ such that Diagram 4.1a commutes. Analogously, a right homotopy between f and g is a morphism $\lambda : x \to \mathrm{Path}(y)$ such that Diagram 4.1b commutes. The existence of homotopies induces an equivalence relation on morphisms.

Example 4.3.22 (Topological spaces). Consider the category **Top**. A cylinder object $\mathrm{Cyl}(X)$ for a topological space X is given by the product $X \times [0,1]$. The codiagonal map is factorized as the endpoint inclusion $X \sqcup X \rightrightarrows X \times [0,1]$ followed by the collapse $X \times [0,1] \overset{\pi_X}{\to} X$, where it is not hard to show that the collapse is a homotopy equivalence. A left homotopy with respect to $X \times [0,1]$ is exactly a homotopy in the sense of Definition **??**.

A path object $\mathrm{Path}(X)$ is given by the mapping space $X^{[0,1]}$. The diagonal map is factorized by the basepoint inclusion $X \overset{\iota_0}{\to} X^{[0,1]}$ followed by the endpoint projections $X^{[0,1]} \rightrightarrows X \times X$. That the right homotopies with respect to $X^{[0,1]}$ give the same equivalence classes as the left homotopies is the content of Property **??**.

Property 4.3.23. If x is cofibrant, every left homotopy induces a right homotopy. Dually, if y is fibrant, every right homotopy induces a left homotopy.

Corollary 4.3.24. Whenever x is cofibrant and y is fibrant, the relations of being left-homotopic (or, equivalently, right-homotopic) coincide on $\mathbf{C}(x, y)$ and, in particular, define equivalence relations. The equivalence classes are denoted by $[x, y]$.

Property 4.3.25 (Stability under composition). Homotopies are preserved under both precomposition and postcomposition by arbitrary morphisms.

Property 4.3.26 (Weak equivalences). A morphism is a weak equivalence if and only if it is a homotopy inverse.

Definition 4.3.27 (Homotopy equivalence). Two objects in a model category are said to be homotopy equivalent if there exists morphisms $f : x \leftrightarrows y : g$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity. The morphisms f, g are then said to be homotopy equivalences.

In Definition 4.2.5 it was shown that one can assign to every category with weak equivalences a homotopy category. When the category has the additional structure of a model category, one can construct an equivalent category:

Alternative Definition 4.3.28 (Homotopy category II). Let $\mathbf{C}$ be a model category. The homotopy category $\mathbf{Ho}(\mathbf{C})$ is the category defined by the following data:

- **Objects**: $\mathrm{ob}(\mathbf{C})$, and
- **Morphisms**: $[x^{\mathrm{cf}}, y^{\mathrm{cf}}]$.

In fact it is easier to restrict to the subcategory $\mathbf{C_{cf}}$ of $\mathbf{Ho}(\mathbf{C})$ on the fibrant-cofibrant objects due to the following property (when restricting to a subcategory, the resulting homotopy category is only equivalent and not isomorphic to the localization):

Property 4.3.29. The homotopy category of a model category is equivalent to those of the full subcategories on (co)fibrant objects:

$$\mathbf{Ho}(\mathbf{C}) \cong \mathbf{Ho}(\mathbf{C_f}) \cong \mathbf{Ho}(\mathbf{C_c}) \cong \mathbf{Ho}(\mathbf{C_{cf}}). \tag{4.15}$$

Theorem 4.3.30 (Whitehead). *A weak equivalence between objects that are both fibrant and cofibrant is a homotopy equivalence.*

Property 4.3.31. A Quillen equivalence between model categories induces an equivalence of homotopy categories.

Property 4.3.32 (Monoidal model categories). The homotopy category of a monoidal model category has a closed monoidal structure defined by the induced derived adjunction. The homotopy category of an enriched model category is enriched, powered and copowered over the homotopy category of its enriching category, where the enriched structure is again given by the induced derived adjunction.

In some cases it is useful to consider categories that are strictly weaker than model categories but stronger than categories with weak equivalences. The prime example being the full subcategories of a model category on the (co)fibrant objects. These are often easier to handle in the setting of homotopy theory.

Definition 4.3.33 (Category of fibrant objects). A category $\mathbf{C}$ with weak equivalences $\mathbf{W} \hookrightarrow \mathbf{C}$ equipped with another subcategory $\mathbf{F} \hookrightarrow \mathbf{C}$ for which the morphisms are called **fibrations** such that the following conditions are satisfied:

1. $\mathbf{C}$ has finite products.
2. Fibrations and acyclic fibrations are preserved under arbitrary pullbacks.
3. Every object admits a **good path object**, i.e. a factorization of the product map $x \to x \times x$ as the composition of a weak equivalence and a fibration.
4. All objects are **fibrant**, i.e. the terminal map $x \to 1$ is a fibration for all objects $x \in \mathrm{ob}(\mathbf{C})$.

Theorem 4.3.34 (Factorization lemma). *Let $\mathbf{C}$ be a category of fibrant objects. Any morphism $f : x \to y$ can be factorized as the right inverse of an acyclic fibration followed by a fibration.*

The following theorem is important for characterizing functors that preserve weak equivalences:

Theorem 4.3.35 (Ken Brown's lemma). *Let* $\mathbf{A}$ *be a category of fibrant objects and let* $\mathbf{B}$ *be a category with weak equivalences. If a functor* $F : \mathbf{A} \to \mathbf{B}$ *maps acyclic fibrations to weak equivalences, it preserves all weak equivalences.*

Remark 4.3.36. An analogous theorem exists for *categories of cofibrant objects*.

The above lemma allows to define derived functors for Quillen functors between model categories (the following definition is also better suited for working in the ∞-setting than Definition 4.2.9):

Alternative Definition 4.3.37 (Derived functor). Let $F : \mathbf{A} \to \mathbf{B}$ be a left (resp. right) Quillen functor. The left (resp. right) derived functors are obtained by precomposition with the cofibrant (resp. fibrant) replacement functors.

Property 4.3.38 (Derived functors are absolute). It can be shown that derived functors built using (co)fibrant replacement are given by absolute Kan extensions. So, even though homotopy categories often do not admit all (co)limits, the resulting Kan extensions do exist.

4.3.3 Reedy model structure

Consider a bicomplete category $\mathbf{M}$ (this will often be $\mathbf{sSet}$). For any full subcategory inclusion $\mathbf{B} \hookrightarrow \mathbf{A}$ one obtains an induced **truncation** (or **restriction**) functor $\mathrm{tr} : \mathbf{M}^{\mathbf{A}} \to \mathbf{M}^{\mathbf{B}}$. The left and right adjoints of this functor are respectively called the **skeleton** and **coskeleton** functors.

Formula 4.3.39. The adjoint functors are defined by Kan extensions and, hence, can be expressed in terms of (co)ends and weighted (co)limits:

$$\mathrm{sk}(X)x := \int^{y\in\mathbf{B}} \mathbf{A}(y,x)\cdot Xy = \mathrm{colim}^{\mathbf{A}(-,x)} \tag{4.16}$$

$$\mathrm{cosk}(X)x := \int_{y\in\mathbf{B}} \big[\mathbf{A}(x,y), Xy\big] = \lim{}^{\mathbf{A}(x,-)}X. \tag{4.17}$$

Definition 4.3.40 (Skeletal sets). Let $\mathbf{M} = \mathbf{sSet}$ and consider the inclusion $\mathbf{\Delta}_{\leq n} \hookrightarrow \mathbf{\Delta}$ of the full subcategory on the objects $\{[0],\dots,[n]\}$. The n-truncation functor tr_n discards all simplices of degree higher than n or, in other words, it "truncates" a simplicial set at degree n.

The n-skeleton functor sk_n takes a simplicial set X of degree $\leq n$ and freely adds degenerate simplices in degrees $> n$, i.e. it is the smallest simplicial set containing X as a simplicial subset. The n-coskeleton functor cosk_n adds a simplex in degree $> n$ whenever all of its faces are present, i.e. the m-simplices in cosk_nX are given by the collection of all $(m+1)$-tuples of $(m-1)$-simplices that are compatible (along lower simplices).

Property 4.3.41 (Simplicial nerve). The nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ from Definition 4.1.10 is a fully faithful functor to the category of 2-coskeletal simplicial sets. This follows from the fact that in ordinary categories compositions of morphisms are unique and, hence, all higher-order tuples of composable morphisms are determined by composable pairs. In fact, this is just the characterization of (small) categories as categories internal to $\mathbf{Set}$ (under the isomorphism $C_2 \cong C_1 \times_{C_0} C_1$ which will be called the first *Segal condition* in Definition 4.4.19).

Definition 4.3.42 (Reedy category). A category $\mathbf{C}$ equipped with a **degree** function $\mathrm{ob}(\mathbf{C}) \to \alpha$, where α is an ordinal 1.5.20, and two wide subcategories $\mathbf{C}^{\pm}$ that satisfy the following conditions:

1. Nontrivial morphisms in $\mathbf{C}^{+}$ (strictly) increase the degree.
2. Nontrivial morphisms in $\mathbf{C}^{-}$ (strictly) decrease the degree.

3. All morphisms admit a unique factorization as a morphism in $\mathbf{C}^-$ followed by a morphism in $\mathbf{C}^+$. This factorization is sometimes called the **(canonical) Reedy factorization**.

Property 4.3.43 (Minimality). The Reedy factorization is the (unique) factorization of minimal degree, where the **degree** of a factorization $x \xrightarrow{f} y \xrightarrow{g} z$ is defined as the degree of y.

Property 4.3.44 (Isomorphisms are trivial). A morphism in a Reedy category is an isomorphism if and only if it is trivial.

Example 4.3.45. Some common examples of Reedy categories are discrete categories, finite posets, the simplex category $\mathbf{\Delta}$ and opposites of Reedy categories.

For Reedy categories one can also define n-truncation, n-skeleton and n-coskeleton functors by restricting to the full subcategories on elements of degree $\leq n$.

Definition 4.3.46 (Matching and latching objects). Let $\mathbf{R}$ be a Reedy category and consider a diagram $X \in [\mathbf{R}, \mathbf{C}]$ with $\mathbf{C}$ small. Consider the skeleton monad and coskeleton comonad (often just called the skeleton and coskeleton functors) $\mathbf{sk}_n := \mathrm{sk}_n \circ \mathrm{tr}_n$ and $\mathbf{cosk}_n := \mathrm{cosk}_n \circ \mathrm{tr}_n$. The latching and matching objects of X are defined as the restrictions of $\mathbf{sk}_{n-1}$ and $\mathbf{cosk}_{n-1}$ to the degree n subcategory of $\mathbf{R}$. The counits of these adjunctions give rise to the **latching** and **matching** maps.

One can also define the latching and matching objects through (co)limits. Define the subcategory $\mathbf{R}^+(r)$ as the subcategory of $\mathbf{R}^+/r$ on all objects except the identity. The latching object L_rX can be shown to be isomorphic to the colimit of X over $\mathbf{R}^+(r)$.

Example 4.3.47 (Simplicial objects). The above property allows to give a nice interpretation to latching objects in the case of $\mathbf{R} = \mathbf{\Delta}^{op}$. Using the *Eilenberg-Zilber lemma* it can be shown that the n^{th} latching object of a simplicial object is given by its collection of degenerate n-simplices.

Definition 4.3.48 (Boundary). The boundary $\partial\mathbf{R}(-, r)$ of a representable presheaf $\mathbf{R}(-, r)$ is defined as its latching object. It can be shown that $\partial\mathbf{R}(-, r)$ consists of exactly those morphisms that are not in $\mathbf{R}^-$ or, equivalently, as $\mathbf{sk}_{n-1}\mathbf{R}(-, r)$. The latching map coincides with the canonical inclusion $\partial\mathbf{R}(-, r) \hookrightarrow \mathbf{R}(-, r)$ if r is of degree n.

Formula 4.3.49. One can show that the latching and matching objects can be obtained through (co)limits weighted by boundaries:

$$M_rX \cong \lim^{\partial\mathbf{R}(r,-)} X \tag{4.18}$$

$$L_rX \cong \mathrm{colim}^{\partial\mathbf{R}(-,r)} X. \tag{4.19}$$

From here on $\mathbf{M}$ will be assumed to be a model category. In this case a canonical model structure on the functor category $[\mathbf{R}, \mathbf{M}]$ for Reedy R can be defined.

Definition 4.3.50 (Relative matching and latching objects). Consider the (weighted) colimit bifunctor colim : $\mathbf{Psh}(\mathbf{R}) \times [\mathbf{R}, \mathbf{M}] \to \mathbf{M}$. The Leibniz construction 4.3.15 allows to define the relative latching object of $f : X \to Y$ at $r \in \mathrm{ob}(\mathbf{R})$ as the Leibniz product of the boundary inclusion $\partial\mathbf{R}(-, r) \hookrightarrow \mathbf{R}(-, r)$ and f.

By Equations (2.52) and (4.19), the relative latching map is of the form $Xr \sqcup_{L_rX} L_rY \to Yr$ and the relative matching map is of the form $Xr \to Yr \times_{M_rY} M_rX$.

Property 4.3.51 (Reedy model structure). Let $\mathbf{R}$ be a Reedy category and let $\mathbf{M}$ be a model category. The functor category $[\mathbf{R}, \mathbf{M}]$ admits the following model structure:

- **Weak equivalences**: the objectwise weak equivalences.
- **Fibrations**: morphisms for which the relative matching map is a fibration (in $\mathbf{M}$) for all $r \in \mathrm{ob}(\mathbf{R})$
- **Cofibrations**: morphisms for which the relative latching map is a cofibration (in $\mathbf{M}$) for all $r \in \mathrm{ob}(\mathbf{R})$

Property 4.3.52 (Quillen (co)limit functors). Consider a $\mathcal{V}$-enriched model category $\mathbf{M}$ and a Reedy category $\mathbf{R}$. For every Reedy cofibrant functor W in $[\mathbf{R}, \mathcal{V}]$, the weighted limit and colimit functors are right and left Quillen, respectively.

?? COMPLETE (this section is too abstract and difficult at this point) ??

4.4 Simplicial spaces

4.4.1 Kan complexes

Definition 4.4.1 (Horn). Consider the standard simplex $\Delta[n]$. For all $n \geq 1$ and $0 \leq k \leq n$ the (n,k)-horn $\Lambda^k[n]$ is defined as the subsimplicial set obtained by removing the k^{th} face from $\partial\Delta[n]$. When $k = 0$ or $k = n$, the horn is said to be **outer**, otherwise it is said to be **inner**.

Definition 4.4.2 (Kan fibration). A morphism of simplicial sets that has the right lifting property with respect to all horn inclusions $\Lambda^k[n] \hookrightarrow \Delta[n]$.

Definition 4.4.3 (Kan complex). A simplicial set that has all horn fillers or, equivalently, a simplicial set for which the terminal morphism is a Kan fibration. The full subcategory of **sSet** on Kan complexes is denoted by **Kan**.

Property 4.4.4 (Horn filler condition). A simplicial set is the nerve of a (small) category if and only if all of its inner horns admit a unique filler (compositions are unique). If one requires all horns to admit a unique filler, the nerve of a groupoid is obtained.

By relaxing the above requirements one can generalize the notion of a category (this is due to *Boardman* and *Vogt*):

Definition 4.4.5 (Quasicategory[3]). A simplicial set for which all inner horns have (not necessarily unique) fillers. This condition is sometimes called the **Boardman condition**.

At this point a third instance of a ‘‘homotopy category’’ can be defined:

Definition 4.4.6 (Homotopy category III). Let X be a quasicategory. The homotopy category $\mathbf{Ho}(X)$ consists of the following data:

- **Objects**: X_0, and
- **Morphisms**: The quotient of X_1 under the relation $f \circ g \sim h$ if there exists a 2-simplex with edges f, g and h.

Property 4.4.7 (Fundamental category). If X is a quasicategory, its homotopy category is equivalent to its **fundamental category** $\pi_1(X)$, i.e. the image under the left adjoint of the (simplicial) nerve functor.

The following theorem is a restatement of Property 4.4.4:

Theorem 4.4.8 (Joyal). *A quasicategory is a Kan complex if and only if its homotopy category is a groupoid.*

[3] Some authors such as *Joyal* call these **logoi** (singular: **logos**).

4.4.2 Simplicial localization

Definition 4.4.9 (Homotopy category IV). Consider a simplicially enriched category $\mathbf{C}$. Its homotopy category $\pi_0(\mathbf{C})$ is defined as follows:

- **Objects**: $\mathrm{ob}(\mathbf{C})$, and
- **Morphisms**: $\pi_0\mathbf{C}(x,y)$.

Two morphism $f,g \in \mathbf{C}(x,y)$ are said to be **homotopic** if they are identified in $\pi_0(\mathbf{C})$. A morphism $f \in \mathbf{C}(x,y)$ is called a **homotopy equivalence** if it admits both a left and a right inverse in $\pi_0(\mathbf{C})$, i.e. if it becomes an isomorphism in the homotopy category.

Definition 4.4.10 (Dwyer-Kan equivalence I). Consider a simplicial functor $F : \mathbf{C} \to \mathbf{D}$ between two simplicially enriched categories. This functor is called a Dwyer-Kan equivalence if:

- F is **∞-fully faithful**, i.e. the induced map on hom-objects is a weak equivalence.[4]
- The induced map on connected components $\pi_0(F) : \pi_0(\mathbf{C}) \to \pi_0(\mathbf{D})$ is an equivalence (of categories). In fact, this condition can be relaxed to $\pi_0(F)$ being essentially surjective, since together with the previous condition this implies that $\pi_0(F)$ is an equivalence.

Construction 4.4.11 (Hammock localization). Let $(\mathbf{C}, W)$ be a category with weak equivalences. Its hammock localization (or **simplicial localization**) $L^H\mathbf{C}$ is the simplicially enriched category constructed as follows:

- **Objects**: $\mathrm{ob}(\mathbf{C})$, and
- **Morphisms**: $L^H\mathbf{C}(x,y)$ is the simplicial set defined as follows:

  First, for every $n \in \mathbb{N}$ one constructs a category with as objects the zigzags of morphisms in $\mathbf{C}$ relating x and y such that all left-pointing morphisms are in W, and as morphisms the endpoint-preserving "natural transformations" (in the sense that all triangles/squares in the resulting diagram commute). Then, the coproduct is taken over the (simplicial) nerves of the categories for all $n \in \mathbb{N}$. Finally, the quotient is taken by the equivalence relations generated by:

  1. inserting or removing identity morphisms, and
  2. composing morphisms.

The following property relates all constructions of what could be called homotopy categories:

Property 4.4.12. Let $(\mathbf{C}, W)$ be a category with weak equivalences. It can be shown that

$$\mathrm{Ho}(L^H\mathbf{C}) \cong \mathbf{C}[W^{-1}]. \tag{4.20}$$

This construction gives a more explicit description of the homotopy category. Furthermore, if $\mathbf{M}$ is a simplicial model category, the categories $\mathbf{M_{cf}}$ and $L^H\mathbf{M}$ are Dwyer-Kan equivalent.

Property 4.4.13. Quillen equivalent model categories have Dwyer-Kan equivalent simplicial localizations.

Property 4.4.14. Up to Dwyer-Kan equivalence, every simplicially enriched category can be obtained as the simplicial localization of a category with weak equivalences.

[4] This is a generalization of the ordinary definition for categories. (Simplicial categories will give models for ∞-*categories*.)

Definition 4.4.15 (Simplicial resolution). Consider an object x in a model category $\mathbf{M}$. A (co)simplicial resolution of x is a Reedy (co)fibrant (co)simplicial object X together with a weak equivalence $x \simeq X_0$.

Every object in a model category admits a (co)simplicial resolution by taking a (co)fibrant replacement of the constant (co)simplicial object.

Definition 4.4.16 (Bergner model structure). The catgory of simplicially enriched categories admits the following model structure:

- **Weak equivalences**: Dwyer-Kan equivalences,
- **Cofibrant objects**: *simplicial computads*, and
- **Fibrant objects**: **Kan**-enriched categories.

Construction 4.4.17 (Free resolution). Consider a (small) category $\mathbf{C}$. From this category we can construct a simplicial category $\mathfrak{C}\mathbf{C}$ consisting of the following data:

- **Objects**: $\mathrm{ob}(\mathbf{C})$, and
- **Morphisms**: $\hom(\mathfrak{C}\mathbf{C})_n := \hom(F^{n+1}\mathbf{C})$, where F is identity-on-objects and $F\mathbf{C}$ has as morphisms strings of composable morphisms in $\mathbf{C}$.

It can be shown that $\mathfrak{C}\mathbf{C}$ is a *simplicial computad* for any simplicially enriched category $\mathbf{C}$, $\mathfrak{C}$ acts as the cofibrant replacement functor in the Bergner model structure.

4.4.3 Segal spaces

Example 4.4.18 (Bisimplicial sets). One can also look at simplicial objects in **sSet** (these are in particular **sSet**-enriched). Such objects are often called bisimplicial sets or **simplicial spaces**[5]. Since **sSet** is a model category, Property 4.3.5 above says that **ssSet** itself admits a model structure. It can furthermore be shown that the injective model structure on **ssSet** coincides with the Reedy model structure (a rather nontrivial statement).

Definition 4.4.19 (Segal space). Consider a fibrant object X in the injective (or Reedy) model structure on **ssSet**. This bisimplicial set is called a Segal space if it satisfies the following weak form of the *Segal condition* for all $n \geq 1$:[6]

$$X_n \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1 \quad (n \text{ factors}). \tag{4.21}$$

These maps are called **Segal maps** (even when they are not weak equivalences). They are the morphisms induced by **spine** inclusions, i.e. inclusions of the union of edge 1-cells.

Definition 4.4.20 (Segal category). A bisimplicial set X is called a **Segal precategory** if X_0 is discrete. It is called a Segal category if in addition all its Segal maps are weak equivalences, i.e. if it is a Segal space with a discrete set of "objects".

Definition 4.4.21 (Mapping space). Consider a Segal space X. For every two points $x, y \in X_0$, the mapping space $\mathrm{Map}(x, y)$ is defined as the fibre of $(d_1, d_0) : X_1 \rightrightarrows X_0 \times X_0$ over the point (x, y). The identity element for $x \in X_0$ is given by $s_0(x)$.

The following two definitions should be compared to Definitions 4.4.9 and 4.4.10:

Definition 4.4.22 (Homotopy category V). Consider a Segal category X. Its homotopy category $\mathbf{Ho}(X)$ is defined as follows:

[5] The latter name follows from the fact that topological spaces and simplicial sets are (Quillen-)equivalent.

[6] If the Reedy condition is omitted, the limit on the right-hand side has to be replaced by a *homotopy limit*.

- **Objects**: X_0, and
- **Morphisms**: $\pi_0\mathrm{Map}(x, y)$.

Two points $f, g \in \mathrm{Map}(x, y)$ are said to be **homotopic** if they are identified in $\mathbf{Ho}(X)$. A point $f \in \mathrm{Map}(x, y)$ is called a **homotopy equivalence** if it admits both a left and a right inverse in $\mathbf{Ho}(X)$, i.e. if it becomes an isomorphism. The subspace $X_{\mathrm{hoequiv}} \subset X_1$ consists of the components that contain homotopy equivalences.[7]

Definition 4.4.23 (Dwyer-Kan equivalence II). A map F of Segal spaces such that:

1. the induced map on mapping spaces is a weak equivalence.
2. the induced map between homotopy categories is an equivalence (of categories).

Definition 4.4.24 (Complete Segal space). A Segal space X for which the map $s_0 : X_0 \to X_{\mathrm{hoequiv}}$ is a weak equivalence.

Property 4.4.25 (Dwyer-Kan equivalence). A map of Segal spaces is a Dwyer-Kan equivalence if and only if it is a weak equivalence in the *complete Segal space model structure*. A map of complete Segal spaces is a Dwyer-Kan equivalence if and only if it is a levelwise weak equivalence.

4.4.4 Coherent nerve

The nerve and realization functors 4.1.9 can also be modified to incorporate the higher homotopical data present in a simplicially enriched category.

First, define a cosimplicial simplicially enriched category $C : \mathbf{\Delta} \to \mathbf{sSetCat}$ that assigns to every finite ordinal $[n]$ the category consisting of the following data:

- **Objects**: $[n] \equiv \{0, 1, \dots, n\}$
- **Morphisms**: $C[n](i, j) := N(P_{ij})$, where N is the ordinary nerve functor on $\mathbf{Cat}$ and P_{ij} is the poset consisting of all subsets of $\{i, \dots, j\}$ that contain both i and j.

The (homotopy) **coherent nerve functor** (or **simplicial nerve functor**[8]) is the nerve functor induced by this cosimplicial object. It not only knowns about all possible morphisms, it also knows about all possible ways how one can obtain this morphism.

Remark 4.4.26. The cosimplicial object $C : \mathbf{\Delta} \to \mathbf{sSetCat}$ is in fact just the free resolution functor $\mathfrak{C}$ in the Bergner model structure 4.4.17 restricted to the subcategory $\mathbf{\Delta}$.

Property 4.4.27. The homotopy coherent nerve functor N_Δ is uniquely determined by the following relation:

$$\mathbf{sSet}(\Delta[n], N_\Delta\mathbf{C}) = \mathbf{sSetCat}(C[n], \mathbf{C}). \tag{4.22}$$

Property 4.4.28 (Simplicial realization). The left adjoint of the homotopy coherent nerve functor satisfies the following relation for all finite ordinals $[n]$:

$$|\Delta[n]| \cong C[n]. \tag{4.23}$$

Property 4.4.29 (Quasicategories). If $\mathbf{C}$ is $\mathbf{Kan}$-enriched, its coherent nerve is a quasicategory. By Example 4.3.20 this allows to associate a quasicategory to any simplicial model category by passing to the coherent nerve of its fibrant-cofibrant subcategory.

[7]It should be noted that if any point in a component is a homotopy equivalence, all points in that component are homotopy equivalences.

[8]This terminology was also used for the ordinary nerve functor taking values in **sSet**. In general it should be clear from the context which one is meant.

At this point it is interesting to reconsider the simplicial nerve construction 4.1.10. Although the functor $N : \mathbf{Cat} \to \mathbf{sSet}$ is fully faithfull, there is a problem in that weak equivalences of simplicial sets do not necessarily correspond to (weak) equivalences of categories. The issue comes from the definition of the canonical model structure 4.3.12 on **sSet**. Here, the weak equivalences are those morphisms that become weak equivalences after geometric realization. During this step information about the direction of morphisms is lost and it becomes impossible to distinguish categories from groupoids. Only when all morphisms are assumed to be invertible, i.e. the category is assumed to be groupoid, does one recover the converse statement.

A first solution is to use a different model structure with less weak equivalences:

Definition 4.4.30 (Joyal model structure). The category **sSet** of simplicial sets 4.1.2 admits a model structure defined by the following data:

- **Weak equivalences**: *weak categorical equivalences*, i.e. maps $f : X \to Y$ such that the induced simplicial functor $C(f) : C(X) \to C(Y)$ is a Dwyer-Kan equivalence.
- **Cofibrations**: monomorphisms.
- **Fibrant objects**: quasicategories 4.4.5.

When characterizing those simplicial sets that are nerves of categories, groupoids and categories are distinguished by whether all horns admit a (unique) filler or only inner horns admit a (unique) filler. However, when the fibrant objects are Kan complexes (those simplicial sets that admit all horn fillers), the homotopical structure only knows about nerves that come from a groupoid. By passing to the larger class of quasicategories, this problem is solved.

Instead of changing the model structure on **sSet** to overcome the issues with taking nerves of categories or groupoids, one can also change the construction of the nerve functor. An alternative approach was introduced by *Rezk*:

Definition 4.4.31 (Classifying diagram). Consider a (small) category $\mathbf{C}$ together with the functor category $\mathbf{C}^{[n]}$ where $[n]$ is the totally ordered set on $n+1$ elements. The classifying diagram of $\mathbf{C}$ is the bisimplicial set $\widetilde{N}\mathbf{C}$ defined levelwise as follows:

$$\widetilde{N}\mathbf{C}_k := N(\mathrm{Core}(\mathbf{C}))\,, \tag{4.24}$$

where N is the ordinary nerve functor and Core denotes the core functor 2.10.4.

The reason for why this construction is better for distinguishing categories and groupoids comes from the fact that information about isomorphisms is already captured at degree 0, while information about noninvertible morphisms is only captured from degree 1 onwards.

Property 4.4.32. If $\mathbf{C}$ is small, then $\widetilde{N}\mathbf{C}$ is a complete Segal space.

?? COMPLETE ??

4.5 Cofibrantly generated categories

4.5.1 Transfinite constructions

Before proceeding, some notions from ordinary category need to be specialized to the context of regular cardinals 1.5.30. In this section the symbol κ will always denote such a regular cardinal.

Definition 4.5.1 (κ-filtered category). A category in which every diagram with less than κ arrows admits a cocone.

Definition 4.5.2 (κ-directed limit). Consider a poset I such that every subposet of cardinality less than κ has a lower bound (upper bound for directed colimits). Such a set is said to be κ-(co)directed. A limit of a diagram over this poset is called a κ-(co)directed (co)limit.

The following definition is a categorification of the previous one:

Definition 4.5.3 (κ-filtered limit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$. The limit (resp. colimit) of D is said to be κ-cofiltered (resp. κ-filtered) if $\mathbf{I}$ is a κ-cofiltered (resp. κ-filtered) category.

It should be noted that an analogue of Property 2.4.45 also holds in the κ-context, i.e. a category has all κ-directed limits if and only if it has all κ-filtered limits (and analogously for colimits).

Definition 4.5.4 (Small object). An object for which there exists a regular cardinal κ such that its covariant hom-functor preserves all κ-filtered colimits. These objects are also said to be **κ-compact** or **κ-presentable**.

Definition 4.5.5 (Accessible category). A locally small category $\mathbf{C}$ for which there exists a regular cardinal κ such that $\mathbf{C}$ has all κ-filtered colimits and such that $\mathbf{C}$ contains a small subcategory $\mathbf{D}$ of κ-small objects that generates all objects by κ-filtered colimits, i.e. $\mathbf{C} \cong \mathrm{Ind}_\kappa(\mathbf{D})$.

Definition 4.5.6 (Locally presentable category). A cocomplete, accessible category. It can be shown that such categories are also complete and, hence, bicomplete.

Property 4.5.7. Every locally presentable category can be obtained as a full reflective subcategory of a presheaf category under an **accessible embedding**, i.e. an embedding that preserves filtered colimits.

Property 4.5.8 (Gabriel-Ulmer duality). **LFP** consists of the following data:

- **Objects**: locally finitely presentable categories.
- **1-Morphisms**: right adjoint functors preserving filtered colimits.
- **2-Morphisms**: natural transformations.

Lex consists of the following data:

- **Objects**: small, finitely complete categories.
- **1-Morphisms**: left exact functors.
- **2-Morphisms**: natural transformations.

There exists an equivalence $\mathbf{Lex}^{op} \cong \mathbf{LPF}$ that sends a small, finitely complete category to its category of left exact copresheaves.

Theorem 4.5.9 (Adjoint functor theorem[9]). *Consider a functor $F : \mathbf{C} \to \mathbf{D}$ between locally presentable categories. F admits a right adjoint if and only if it preserves all colimits. F admits a left adjoint if and only if it is accessible and if it preserves all limits.*

For ordinary categories the axioms guarantee the existence of a (unique) composition of any finite number of (composable) morphisms. However, in some cases it is useful, or even necessary, to talk about the "composite" of an infinite number of morphisms:

Definition 4.5.10 (Transfinite composition). Consider a category $\mathbf{C}$ with a collection of morphisms $I \subseteq \hom(\mathbf{C})$ and let α be an infinite ordinal 1.5.20. A (α-indexed) **transfinite sequence** of morphisms in I is a diagram of the form $D : \alpha \to \mathbf{C}$ such that:

1. Successor morphisms in α are mapped to elements of I.
2. D is *continuous* in the sense that for every limit ordinal $\beta < \alpha$: $D\beta \cong \operatorname*{colim}_{\gamma<\beta} D\gamma$.

$D\lambda$ denotes the restriction of D to the (full) subdiagram $\{\gamma \mid \gamma < \lambda\}$ of α. The transfinite composition of this sequence is the induced morphism $D_0 \to D\alpha \equiv \mathrm{colim}\ D$.

Definition 4.5.11 (Cell complex). Consider a cocomplete category $\mathbf{C}$ with a collection of morphisms $I \subseteq \mathrm{hom}(\mathbf{C})$. A **relative** I-cell complex is a transfinite composition of pushouts (of coproducts[10]) of morphisms in I. An I-cell complex is an object such that the unique morphism from the initial object is a relative I-cell complex.

Notation 4.5.12 (Relative cell complexes). The set of all relative I-cell complexes is often denoted by $\mathrm{cell}(I)$.

4.5.2 Cofibrant generation

The following is a famous result by *Quillen*:

Theorem 4.5.13 (Small object argument). *Let $\mathbf{C}$ be a locally presentable category with a collection of morphisms $I \subseteq \mathrm{hom}(\mathbf{C})$. Every morphism in $\mathbf{C}$ can be factorized as the composition of a morphism in $\mathrm{rlp}(I)$ followed by a morphism in $\mathrm{cell}(I)$.*

Remark 4.5.14. This theorem can be generalized to cocomplete categories where the morphisms in I are small relative[11] to $\mathrm{cell}(I)$. Sets of morphisms with this property are said to **admit a small object argument**.

Definition 4.5.15 (Cofibrantly generated model category). Consider a model category $\mathbf{C}$. This category is said to be cofibrantly generated by two sets of morphisms $I, J \subseteq \mathrm{hom}(\mathbf{C})$ if it satisfies the following conditions:

1. I and J both admit the small object argument.
2. The fibrations are given by $\mathrm{rlp}(J)$.
3. The acyclic fibrations are given by $\mathrm{rlp}(I)$.

It can be shown that the last two conditions are equivalent to the following ones:

2*. The cofibrations are the retracts (in the arrow category) of $\mathrm{cell}(I)$.

3*. The acyclic fibrations are the retracts (in the arrow category) of $\mathrm{cell}(J)$.

The morphisms in I and J are called the **generating cofibrations** and **generating acyclic cofibrations**, respectively.

Sometimes it is desirable to replace the model structure on a category $\mathbf{C}$ by one that has more weak equivalences (denote the new one by $\mathbf{C}_0$). If the cofibrations remain the same, this new structure has some nice properties:

- The fibrations are a subclass of the original ones.
- The acyclic fibrations remain the same.
- The identity functors $\mathrm{Id} : \mathbf{C}_0 \leftrightarrows \mathbf{C} : \mathrm{Id}$ form a Quillen adjunction.
- Every object in $\mathbf{C}$ is weakly equivalent (in $\mathbf{C}_0$) to one in $\mathbf{C}_0$.

[10] It can be shown that closure under coproducts follows automatically.

[11] *Small relative to a set of morphisms* is defined just as ordinary smallness, but with general κ-filtered colimits replaced by those that start from morphisms in the given set.

This procedure can be made explicit for a specific class of model categories. Let $\mathbf{C}$ be a left proper, cofibrantly generated, simplicial model category and consider a collection $S \subset \mathrm{hom}(\mathbf{C})$ of cofibrations with cofibrant domain. First, the notion of "S-local objects" is introduced:

Definition 4.5.16 (Local object). A fibrant object x is said to be S-local if for all morphisms in S the image under the Yoneda embedding of x is an acyclic Kan fibration. Analogously, a morphism is called an S-local weak equivalence if for all S-local fibrant objects its image under their Yoneda embeddings is an acyclic Kan fibration.

Property 4.5.17. Every weak equivalence is an S-local weak equivalence: $W \subset W_S$.

Construction 4.5.18 (Left Bousfield localization). Given a model category $\mathbf{C}$ with the same assumptions as before, the (left) Bousfield localization $L_S\mathbf{C}$ is defined as as the same category but with the following model structure:

- **Cofibrations**: $\mathrm{cof}(\mathbf{C})$
- **Acyclic cofibrations**: cofibrations that are S-local weak equivalences

If it exists, the Bousfield localization $L_S\mathbf{M}$ of $\mathbf{M}$ at S is the universal left Quillen functor $\gamma : \mathbf{M} \to L_S\mathbf{M}$ such that its left derived functor sends the image of S to isomorphisms in $\mathbf{Ho}(\mathbf{M})$. The identity functor gives a Quillen adjunction between $\mathbf{M}$ and $L_S\mathbf{M}$ and the induced adjunction on homotopy categories defines a reflective localization.

Remark 4.5.19. The notions of local object/morphisms can be slightly generalized (they give equivalent localizations). A **S-local morphism** $f : a \to b$ is a morphism such that for all objects $x \in \mathrm{ob}(\mathbf{M})$ for which the map

$$\mathbf{M}(g^{\mathrm{c}}, x^{\mathrm{f}}) : \mathbf{M}(d^{\mathrm{c}}, x^{\mathrm{f}}) \to \mathbf{M}(c^{\mathrm{c}}, x^{\mathrm{f}}) \tag{4.25}$$

is a weak equivalence for all $g \in S$ (x is said to be **S-local**), the map

$$\mathbf{M}(f^{\mathrm{c}}, x^{\mathrm{f}}) : \mathbf{M}(b^{\mathrm{c}}, x^{\mathrm{f}}) \to \mathbf{M}(a^{\mathrm{c}}, x^{\mathrm{f}}) \tag{4.26}$$

is also a weak equivalence. This is the homotopical version of Definition 2.2.19, after cofibrant (resp. fibrant) replacement of the domain (resp. codomain), the Yoneda embedding of every S-local object is weak equivalence, where S-local objects are those objects whose Yoneda embedding maps morphisms in S to weak equivalences.

Definition 4.5.20 (Combinatorial model category). A locally presentable, cofibrantly generated model category.

Theorem 4.5.21 (Dugger). *Every combinatorial model category is Quillen equivalent to a (left) Bousfield localization of a category of simplicial presheaves on a small category (with the global projective model structure).*

?? FINISH ??

4.6 Homotopy (co)limits

Consider a category $\mathbf{C}$ with weak equivalences together with diagrams $D, D' : \mathbf{I} \to \mathbf{C}$. Assume that there exists a weak equivalence between D and D', i.e. a natural transformation that consists of componentwise weak equivalences. Clearly this induces a morphism between (co)limits, but it would be nice if the construction of (co)limits would also preserve the homotopical structure, i.e. weakly equivalent diagrams should have weakly equivalent (co)limits.

The main purpose of this section is to introduce a modification of the ordinary (co)limit functors that takes into account the homotopical structure of the underlying categories.

A first step is the modification of (co)products:

Definition 4.6.1 (Homotopy (co)products). In ordinary categories the universal property of a product (and dually for coproducts) characterizes it as an object with an isomorphism

$$\mathbf{C}(-,c) \cong \prod_{i\in I} \mathbf{C}(-,c_i) \tag{4.27}$$

such that every I-indexed collection of morphisms $f_i : x \to c_i$ can be factorized as follows

$$\begin{tikzcd} x \arrow[d, dashed, "\exists!"'] \arrow[dr, "f_i"] & \\ c \arrow[r, "\pi_i"'] & c_i. \end{tikzcd}$$

To obtain a homotopical version, the commutativity of this diagram is relaxed up to a homotopy/path in the hom-space $\mathbf{M}(x, c_i)$. This leads to the definition of a homotopy product as an object c together with a natural weak (homotopy) equivalence

$$\psi_x : \mathbf{C}(x,c) \simeq \prod_{i\in I} \mathbf{C}(x,c_i). \tag{4.28}$$

The homotopy (co)products are unusual in the sense that they can be obtained as (co)limits in the homotopy category $\mathbf{Ho}(\mathbf{C})$.

More generally, one can define homotopy (co)limits in any homotopical category by passing to derived functors:

Definition 4.6.2 (Homotopy (co)limits). Let $\mathbf{A}$ be a homotopical category and let $\mathbf{B}$ be a small category. The homotopy limit and colimit functors are defined as the derived functors of the limit and colimit functors $(\mathrm{co})\lim : [\mathbf{B}, \mathbf{A}] \to \mathbf{A}$.

Remark 4.6.3. The reason for why (co)products can be obtained as ordinary (co)limits is related to the fact that their indexing categories are discrete. In this case the adjoint of the derived functor is equivalent to the diagonal functor on the homotopy category.

Property 4.6.4 (Model categories). Consider the specific case where $\mathbf{C}$ is a model category. If $[\mathbf{D}, \mathbf{C}]$ admits an injective (resp. projective) model structure, the homotopy limit (resp. colimit) always exists and can be obtained through fibrant (resp. cofibrant) replacement as in Definition 4.3.37.

Example 4.6.5 (Reedy categories). Consider the general case of diagrams $D : \mathbf{R} \to \mathbf{C}$ with $\mathbf{R}$ Reedy. First, note that the constant functor $\Delta : \mathbf{C} \to [\mathbf{R}, \mathbf{C}]$ maps weak equivalences to (pointwise) weak equivalences.[12] If the Reedy structure is such that the constant functor preserves cofibrations, then this functor is left Quillen and Ken Brown's lemma 4.3.35 implies that its right Quillen adjoint, the limit functor, preserves weak equivalences. In this case one can define the **homotopy limit** $\mathrm{holim}\, D$ as the functor $\lim(D \circ Q_{\mathrm{f}})$, where Q_{f} is the fibrant replacement-functor (which in this case acts pointwise). A dual construction gives rise to **homotopy colimits**.

[12] This is also true when $\mathbf{R}$ is not Reedy.

4.6.1 Simplicially enriched diagrams

In the setting where diagrams are enriched over **sSet**, one can define homotopy (co)limits in a more sophisticated way.

Definition 4.6.6 (Homotopy colimit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$ with $\mathbf{C}$ copowered over **sSet**. The homotopy colimit of D is defined as the following tensor product 2.7.19:

$$\operatorname{hocolim} D := N(-/\mathbf{I}) \otimes_{\mathbf{I}} D = \int^{i\in\mathbf{I}} N(i/\mathbf{I}) \cdot Di, \tag{4.29}$$

where N is the nerve functor 4.1.10.

A similar definition for homotopy limits can be used when the category is powered over **sSet**:

Definition 4.6.7 (Homotopy limit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$ with $\mathbf{C}$ powered over **sSet**. The homotopy limit of D is defined as the following hom-like object:

$$\operatorname{holim} D := \int_{i\in\mathbf{I}} [N(\mathbf{I}/i), Di]. \tag{4.30}$$

This is exactly the characterization of the homotopy limit as the **sSet**-natural transformations between $N(\mathbf{I}/-)$ and D.

Remark 4.6.8 (Bousfield-Kan map). The expressions from the above formulas are also known as the **Bousfield-Kan formulas**. It should be noted that the above definitions are not strictly equivalent to the ones from the previous section. To be precise, the Bousfield-Kan formulas are only weakly equivalent to the general definitions if the objects in $\mathbf{C}$ are replaced by their (co)fibrant replacements, i.e. if in the above expressions D is postcomposed by a (co)fibrant replacement-functor.

?? COMPLETE (CHECK THESE STATEMENTS) ??

By Definition 4.3.28 one can construct morphisms in a homotopy category as morphisms from a cofibrant replacement to a fibrant replacement. This allows to define diagrams-up-to-homotopy (in two settings):

Definition 4.6.9 (Homotopy coherent diagram). A morphism in the homotopy category of the Bergner model category. When considering diagrams taking values in a **Kan**-enriched category, one can use the free resolution functor to characterize them as functors $D : \mathfrak{C}\mathbf{I} \to \mathbf{C}$. Since the simplicial realization functor extends the free resolution functor $\mathfrak{C}$ to simplicial sets, one can also define homotopy coherent diagrams on simplicial sets.

Natural transformations between such diagrams are given by homotopy coherent diagrams $\mathfrak{C}(\mathbf{I} \times \Delta[1]) \to \mathbf{C}$ in analogy with ordinary homotopies. However, these do no compose uniquely in the sense that one does not obtain a well-defined diagram on $\mathbf{I} \times \Delta[2]$. Therefore, one does not obtain a category of homotopy coherent diagrams. For simplicial sets I it can be shown that the natural structure is that of a quasicategory:

$$\mathbf{CohDgrm}(\mathbf{I}, \mathbf{C}) \cong (N\mathbf{C})^{\mathbf{I}}, \tag{4.31}$$

where N is the simplicial nerve functor.

?? COMPLETE ??

4.7 ∞-categories

4.7.1 Simplicial approach

The first approach to ∞-category theory is the simplicial one. The motivation is Property 4.4.4, which relates the categorical structure to the existence of certain horn fillers. The generalization is then given by the notion of quasicategories 4.4.5.

Theorem 4.7.1 (Lurie). *An ∞-category is presentable if and only if it is equivalent to the coherent nerve of the fibrant-cofibrant subcategory of a combinatorial model category and, hence by Dugger's theorem 4.5.21, can be presented by simplicial presheaves.*

Chapter 5

Topos theory ♣

The main reference for this chapter is [11, 12]. For an introduction to stacks and descent theory, see [13].

5.1 Elementary topoi

Definition 5.1.1 (Subobject classifier). Consider a finitely complete category (in fact, the existence of a terminal object suffices). A subobject classifier is a mono[1] $\texttt{true} : 1 \hookrightarrow \Omega$ from the terminal object such that for every mono $\phi : x \hookrightarrow y$ there exists a unique morphism $\chi : y \to \Omega$ that fits in the following pullback square:

$$\begin{tikzcd}
x \arrow[r] \arrow[d, hook, "\phi"'] \arrow[dr, phantom, "\text{pb}"] & 1 \arrow[d, hook, "\texttt{true}"] \\
y \arrow[r, "\exists!\chi"'] & \Omega
\end{tikzcd}$$

Figure 5.1: Subobject classifier.

Alternative Definition 5.1.2. Consider a well-powered category $\mathbf{C}$. The assignment of subobjects $\mathrm{Sub}(x)$ to an object $x \in \mathrm{ob}(\mathbf{C})$ defines functor $\mathrm{Sub} : \mathbf{C}^{op} \to \mathbf{Set}$. A subobject classifier Ω is a representation of this functor, i.e. the following isomorphism is natural in x:

$$\mathrm{Sub}(x) \cong \mathbf{C}(x, \Omega). \tag{5.1}$$

Example 5.1.3 (Indicator function). The category **Set** has a subobject classifier: the 2-element set $\{\texttt{true}, \texttt{false}\}$. The morphism $\chi : S \to \Omega$ is the indicator function

$$\chi_S(x) = \begin{cases} \texttt{true} & x \in S \\ \texttt{false} & x \notin S. \end{cases} \tag{5.2}$$

Definition 5.1.4 (Elementary topos). An elementary topos is a finitely complete, Cartesian closed category admitting a subobject classifier. Equivalently, one can define an elementary topos as a finitely complete category in which all power objects exist.

[1] The symbol for this morphism will become clear in Section 5.2.

The power object Px of $x \in \mathrm{ob}(\mathcal{E})$ is related to the subobject classifier Ω by the following relation:

$$Px = \Omega^x. \tag{5.3}$$

Remark 5.1.5 (Finite colimits). The original definition by *Lawvere* also required the existence of finite colimits. However, it can be proven that finite cocompleteness follows from the other axioms.

Theorem 5.1.6 (Fundamental theorem of topos theory). *Let $\mathcal{E}$ be an elementary topos. For every object $x \in \mathrm{ob}(\mathcal{E})$ the slice category $\mathcal{E}_{/x}$ is also a topos. The subobject classifier is given by $\pi_2 : \Omega \times x \to x$.*

Property 5.1.7 (Balanced). All monos in a topos are regular. Hence, every mono arises as an equalizer. Since every epic equalizer is necessarily an isomorphism 2.4.52, it follows that every topos is balanced 2.4.5.

Property 5.1.8 (Epi-mono factorization). Every morphism $f : x \to y$ in a topos factorizes uniquely as an epi followed by a mono:

$$x \overset{e}{\twoheadrightarrow} z \overset{m}{\hookrightarrow} y. \tag{5.4}$$

The mono is called the **image** of f.

5.2 Internal logic

In this subsection finitely complete categories that admit a subobject classifier are considered (they do not have to be elementary topoi).

Definition 5.2.1 (Truth value). A global element of the subobject classifier, i.e. a morphism $1 \to \Omega$. The subobject classifier Ω is also sometimes called the **object of truth values**.

Property 5.2.2 (Internal Heyting algebra). For all objects x in an elementary topos, the poset of subobjects $\mathrm{Sub}(x)$ has the structure of a Heyting algebra 1.5.38. Hence, every topos canonically gives an external Heyting algebra, namely $\mathrm{Sub}(1)$. Furthermore, every power object is an internal Heyting algebra. This in particular includes the subobject classifier $\Omega = P1$.

?? COMPLETE ??

5.3 Geometric morphisms

Definition 5.3.1 (Base change). Consider a category $\mathbf{C}$ with pullbacks. For every morphism $f : x \to y$ one can define a functor $f^* : \mathbf{C}_{/y} \to \mathbf{C}_{/x}$. This functor acts by pullback along f.

Definition 5.3.2 (Logical morphism). Let $\mathcal{E}, \mathcal{F}$ be (elementary) topoi. A morphism $f : \mathcal{E} \to \mathcal{F}$ is said to be logical if it preserves finite limits, exponential objects and subobject classifiers.

Property 5.3.3. If a logical morphism has a left adjoint, it also has a right adjoint.

Definition 5.3.4 (Geometric morphism). Let $\mathcal{E}, \mathcal{F}$ be (elementary) topoi. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of an adjunction

$$\mathcal{E} \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{F}$$

where the left adjoint is left exact. The right adjoint f_* is called the **direct image** part of f and the left adjoint is called the **inverse image** part. If f^* itself has a left adjoint, then f is said to be **essential**.

Definition 5.3.5 (Geometric embedding). A geometric morphism for which the direct image part is fully faithful.

Property 5.3.6 (Characterization of geometric embeddings). Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric embedding and let $W \subset \hom(\mathcal{F})$ be the collection of morphisms that are mapped to isomorphisms under f^*. $\mathcal{E}$ is both equivalent to the full subcategory of $\mathcal{F}$ on W-local objects and the localization $\mathcal{F}[W^{-1}]$ at W (Definition 4.2.5).

Property 5.3.7 (Base change). The base change functors on a topos are logical and admit a left adjoint, the postcomposition functor. This implies that these functors can be refined to essential geometric morphisms.

Example 5.3.8 (Topological spaces). Every continuous function $f : X \to Y$ induces a geometric morphism

$$\begin{tikzcd} \mathbf{Sh}(X) \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathbf{Sh}(Y) \arrow[l, bend right, "f^*"'] \end{tikzcd}, \tag{5.5}$$

where the direct image functor f_* is defined as

$$f_*F(U) := F(f^{-1}U) \tag{5.6}$$

for any sheaf $F \in \mathbf{Sh}(X)$ and any open subset $U \in \mathbf{Open}(Y)$. The inverse image functor f^* is defined using the equivalence between sheaves on topological spaces and étalé spaces. Consider a sheaf $E \in \mathbf{Sh}(Y)$ as an étalé space $\pi : E \to Y$. The inverse image of E along a continuous function $f : X \to Y$ is the pullback of π along f.

By the previous example the global elements $* \to X$ of a topological space induce geometric morphisms of the form $\mathbf{Sh}(*) \to \mathbf{Sh}(X)$. By noting that $\mathbf{Sh}(*) = \mathbf{Set}$, one obtains the following generalization:

Definition 5.3.9 (Point). A point of a topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

Notation 5.3.10 (Category of topoi). The category of elementary topoi and geometric morphisms is a 2-category. It is denoted by **Topos**.

In fact, to obtain the structure of a 2-category, one needs to define an appropriate notion of 2-morphism. Because a geometric morphism consists of an adjunction, one can consider two distinct conventions. Either one can choose the 2-morphisms in **Topos** to be the natural transformations $f^* \Rightarrow g^*$ (with associated transformations $g_* \Rightarrow f_*$) or one can choose them to be the natural transformations $f_* \Rightarrow g_*$ (and associated transformations $g^* \Rightarrow f^*$). This chapter follows [11] and the "inverse image convention" is used, i.e. a 2-morphism $f \Rightarrow g$ consists of natural transformations $f^* \Rightarrow g^*$ and $g_* \Rightarrow f_*$.

5.4 Grothendieck topos

Definition 5.4.1 (Sieve). Let $\mathbf{C}$ be a small category. A sieve S on $\mathbf{C}$ is a fully faithfull discrete fibration $S \hookrightarrow \mathbf{C}$.

A sieve S on an object $x \in \mathbf{C}$ is a sieve in the slice category $\mathbf{C}_{/x}$. This means that S is a subset of $\mathrm{ob}(\mathbf{C}_{/x})$ that is closed under precomposition, i.e. if $y \to x \in S$ and $z \to y \in \hom(\mathbf{C})$, $z \to y \to x \in S$.

All of this can be summarized by saying that a sieve on an object $x \in \mathrm{ob}(\mathbf{C})$ is a subfunctor of the hom-functor $\mathbf{C}(-, x)$.

Example 5.4.2 (Maximal sieve). Let $\mathbf{C}$ be a category. The maximal sieve on $x \in \mathrm{ob}(\mathbf{C})$ is the collection of all morphisms $\{f \in \hom(\mathbf{C}) \mid \mathrm{cod}(f) = x\}$ or, equivalently, all of $\mathrm{ob}(\mathbf{C}_{/x})$.

Example 5.4.3 (Pullback sieve). Consider a morphism $f : x \to y$. Given a sieve S on y, one can construct the pullback sieve f^*S on x as the sieve of morphisms in S that factor through f:

$$f^*S(x) = \big\{g \,\big|\, f \circ g \in S(y)\big\}. \tag{5.7}$$

Property 5.4.4 (Presheaf topos). Consider the presheaf category $\mathbf{Psh}(\mathbf{C})$ for an arbitrary (small) category $\mathbf{C}$. This category is an elementary topos where the subobject classifier is defined on each object in the following way:

$$\underline{\Omega}(x) := \{S \mid S \text{ is a sieve on } x\}. \tag{5.8}$$

The action on a morphism $f : x \to y$ gives the morphism $\underline{\Omega}(f)$ that sends a sieve S to its pullback sieve f^*S.

The morphism $\texttt{true} : \underline{1} \hookrightarrow \underline{\Omega}$ is defined as the natural transformation assigning to every object its maximal sieve. For every subobject $\underline{K} \hookrightarrow \underline{X}$ the characteristic morphism χ_K is defined as follows. Consider an object $c \in \mathrm{ob}(\mathbf{C})$ and element $x \in \underline{X}(c)$. The component $\chi_K|_c$ is then given by

$$\chi_K|_c(x) := \{f \in \mathbf{C}(d, c) \mid \underline{X}(f)(x) \in \underline{K}(d)\}. \tag{5.9}$$

The following definition is due to *Giraud* (for the original definition using the notion of a *cover*, see the end of this section):

Definition 5.4.5 (Grothendieck topology). A Grothendieck topology on a category is a map J assigning to every object a collection of sieves satisfying the following conditions:

1. **Identity**[2]: For every object x the maximal sieve M_x is an element of $J(x)$ or, equivalently, all sieves generated by isomorphisms are in $J(x)$.
2. **Base change**: If $S \in J(x)$, then $f^*S \in J(y)$ for every morphism $f : y \to x$.
3. **Locality**: Consider a sieve S on x. If there exists a sieve $R \in J(x)$ such that for every morphism $(f : y \to x) \in R$ the pullback sieve $f^*S \in J(y)$, then $S \in J(x)$.

The sieves in J are called (J-)**covering sieves**. A collection of morphisms with codomain $x \in \mathrm{ob}(\mathbf{C})$ is called a **cover**[3] of x if the sieve generated by these morphisms is a covering sieve on x.

Example 5.4.6 (Topological spaces). These conditions have the following interpretation in the case of topological spaces:

- The collection of all open subsets covers a space U.
- If $\{U_i\}_{i\in I}$ covers U, then $\{U_i \cap V\}_{i\in I}$ covers $U \cap V$.
- If $\{U_i\}_{i\in I}$ covers U and if for every $i \in I$ the collection $\{U_{ij}\}_{j\in J_i}$ covers U_i, then $\{U_{ij}\}_{i\in I, j\in J_i}$ covers U.

The canonical Grothendieck topology on $\mathbf{Open}(X)$ is given by the sieves $S = \{U_i \hookrightarrow U\}_{i\in I}$, where $\bigcup_{i\in I} U_i = U$. This topology is denoted by $J_{\mathbf{Open}(X)}$.

Definition 5.4.7 (Site). A (small) category equipped with a Grothendieck topology J.

[2] The name itself stems from the fact that the maximal sieve is generated from the identity morphism.

[3] Sometimes this term is also used to denote any collection of morphism with common codomain x, i.e. without reference to a covering sieve.

Definition 5.4.8 (Matching family). Consider a presheaf $F \in \mathbf{Psh}(\mathbf{C})$ together with a sieve S on $x \in \mathrm{ob}(\mathbf{C})$. A matching family for S with respect to F is a natural transformation $\alpha : S \Rightarrow F$ between S, regarded as a subfunctor of $\mathbf{C}(-, x)$, and F.

More explicitly, it is an assignment of an element $x_f \in Fd$ to every morphism $(f : y \to x) \in S$ such that

$$F(g)(x_f) = x_{f\circ g} \tag{5.10}$$

for all morphisms $g : z \to y$. Equivalently, a matching family for S with respect to F is a set of elements $\{x_f\}_{f\in S}$ such that for all covering morphisms $f : y \to x, g : z \to x \in S$ and all morphisms $f' : c \to y, g' : c \to z$ such that $f \circ f' = g \circ g'$ the following equations holds:

$$F(f')(x_f) = F(g')(x_g). \tag{5.11}$$

Given such a matching family, one calls an element $a \in Fx$ an **amalgamation** if it satisfies

$$F(f)(a) = x_f \tag{5.12}$$

for all morphisms $f \in S(y)$. The existence of such an element can also be stated in terms of natural transformations. Consider the obvious inclusion ι_S of S into the the hom-functor $\mathbf{C}(-, x)$. Every morphism with codomain x can be obtained from the identity morphism by precomposition and, hence, a natural transformation $\mathbf{C}(-, x) \Rightarrow F$ is determined by its action on the identity morphisms $\mathbb{1}_x$. The existence of an amalgamation is thus equivalent to the existence of an extension of S along ι_S.

Remark 5.4.9. If the base category has all pullbacks, for example if it is a topos on its own, one can restrict the above commuting diagrams to the pullback diagrams of morphisms in the sieve S.

Definition 5.4.10 (Sheaf). Consider a site $(\mathbf{C}, J)$. A presheaf F on $\mathbf{C}$ is called a J-sheaf if every matching family for every covering sieve in J admits a unique amalgamation[4] or, equivalently, if all sieves admit a unique extension to representable presheaves.

The category $\mathbf{Sh}(\mathbf{C}, J)$ of J-sheaves on the site $(\mathbf{C}, J)$ is the full subcategory of $\mathbf{Psh}(\mathbf{C})$ on the presheaves that satisfy the above condition.

This definition can also be restated in terms of local objects 2.2.19:

Alternative Definition 5.4.11 (Sheaf). By definition every covering sieve admits a morphism into the Yoneda embedding: $\eta : S \hookrightarrow \mathcal{Y}x$. If the collection of all these morphisms is denoted by $\mathcal{S}$, a presheaf is a sheaf if and only if it is $\mathcal{S}$-local, i.e. if the following morphism is an isomorphism for all $\eta \in \mathcal{S}$:

$$Fx \cong \mathbf{Psh}(\mathcal{Y}x, F) \xrightarrow{\mathbf{Psh}(\eta,F)} \mathbf{Psh}(S, F). \tag{5.13}$$

This is also called the **descent condition** for sheaves. In this context the collection of matching families $\mathrm{Match}(S, F) := \mathbf{Psh}(S, F)$ for a sieve S with respect to a presheaf F is often called the **descent object** of S with respect to F.

Example 5.4.12 (Topological spaces). The usual category of sheaves $\mathbf{Sh}(X)$ on a topological space X is obtained as the category of sheaves on the site $(\mathbf{Open}(X), J_{\mathbf{Open}(X)})$. Since the morphisms in the covering sieves are exactly the inclusion maps $U_i \hookrightarrow U$, the pullback of two such morphisms is given by the intersection $U_i \cap U_j$. Hence, the condition for a matching

[4] If there exists at most one amalgamation, the presheaf is said to be **separated**.

family, as formulated in 5.4.8 above, gives the second part of Definition **??**. The uniqueness of an amalgamation is equivalent to the first part of that definition.

For topological spaces, sheaves are easily represented visually. A matching family assigns to every set U_i of an open cover $\mathcal{U} \equiv \{U_i\}_{i\in I}$ of U an element $x_i \in FU_i$, such that the restrictions coincide on double overlaps, as in step (1) in the figure below.

Cover — Matching family — Unique amalgamation

$$\begin{tikzcd}
& U_i \cap U_j \arrow[dl] \arrow[dr] & & & x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j} \arrow[dl] \arrow[dr] & & & x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j} \arrow[dl] \arrow[dr] & \\
U_i \arrow[dr] & & U_j \arrow[dl] \arrow[r, "(1)"] & x_i & & x_j \arrow[r, "(2)"] & x_i = x|_{U_i} \arrow[dr] & & x_j = x|_{U_j} \arrow[dl] \\
& U & & & & & & x &
\end{tikzcd}$$

The descent condition then states that for every such matching family, there exists a unique element x on U, such that the elements of the matching family are restrictions of x as in step (2) of the figure above.

The classical example would be the assignment of the set of continuous functions to open subsets of a topological space. When two functions, defined on two open sets, coincide on the intersection, there exists a unique continuous function defined on the union, such that it restricts to the given functions.

Example 5.4.13 (Canonical topology). The canonical topology on a category is the largest Grothendieck topology for which all representable presheaves are sheaves. A subcanonical topology is then defined as a subtopology of the canonical one, i.e. any Grothendieck topology for which all representable presheaves are sheaves.

Example 5.4.14 (Minimal and maximal topologies). The minimal Grothendieck topology on a category is the one for which only the maximal sieves are covering sieves. In this topology all presheaves are sheaves. The maximal Grothendieck topology is the one for which all sieves are covering sieves. In this topology only the terminal element of the associated presheaf category is a sheaf.

Definition 5.4.15 (Grothendieck topos). A category equivalent to the category of sheaves on a (small) site. This site is often called the **site of definition** for the given topos.

Property 5.4.16. Every Grothendieck topos is an elementary topos.

Property 5.4.17. For every Grothendieck topos there exists a site of definition for which the Grothendieck topology is (sub)canonical.

Construction 5.4.18 (Sheafification). Given a presheaf $\mathcal{F}$, one can construct a sheaf $\overline{\mathcal{F}}$ along the same lines of Construction **??**.

Definition 5.4.19 (Global sections functor). Every Grothendieck topos $\mathcal{E}$ admits a geometric morphism to **Set**, where the right adjoint assigns to an object x its set of global elements:

$$\Gamma : \mathcal{E} \to \mathbf{Set} : x \mapsto \mathcal{E}(1, x). \tag{5.14}$$

When $\mathcal{E}$ is the sheaf topos over a topological space, this is exactly the global sections functor **??**. The left adjoint assigns to every set S the copower $S \cdot 1 \equiv \bigsqcup_{s\in S} 1$. When $\mathcal{E}$ is a sheaf topos, this adjoint is exactly the constant sheaf functor. It is sometimes denoted by LConst.

A different approach for defining sheaf topoi is through an embedding of sheaves into presheaves.

Definition 5.4.20 (Local isomorphism). A system of local isomorphisms in $\mathbf{Psh}(\mathbf{C})$ is a class of morphisms in $\mathbf{Psh}(\mathbf{C})$ forming a system of weak equivalences 4.2.1 closed under pullbacks along morphisms out of representable presheaves.

Property 5.4.21 (Local isomorphisms and Grothendieck topologies). A system of local isos induces a *system of local epis* in the following way. $f : X \to Y$ is a local epi if $\mathrm{im}(f) \to Y$ is a local iso. A Grothendieck topology is defined by declaring a presheaf $F \in \mathbf{Psh}(\mathbf{C})$ to be a covering sieve at $X \in \mathrm{ob}(\mathbf{C})$ if $F \hookrightarrow \mathcal{Y}X$ is a local epi.

Alternative Definition 5.4.22 (Sheaf topos). A category $\mathbf{Sh}(\mathbf{C})$ equipped with a geometric embedding into $\mathbf{Psh}(\mathbf{C})$.

Proof of equivalence. By Property 5.3.6 such a category is equivalent to the full subcategory on S-local presheaves for some system of local isomorphisms S and, therefore, also to a sheaf topos in the sense of Grothendieck by the property above.

Remark 5.4.23 (Descent condition). This is essentially a restatement of the descent condition 5.4.11. Covering sieves, regarded as subfunctors, are in particular local isomorphisms. Stability of sieves under pullback together with the co-Yoneda lemma 2.4.68, which says that every presheaf is a colimit of representables, generates the full collection of local isomorphisms.

By Property 4.2.8, the construction of Grothendieck topoi as localizations of presheaf categories is equivalent to a definition in terms of reflections:

Corollary 5.4.24. Given a small category $\mathbf{C}$, there exists a bijection between the Grothendieck topologies on $\mathbf{C}$ and the equivalence classes of left exact reflective subcategories of $\mathbf{Psh}(\mathbf{C})$.

As a last point the weaker notion of coverages is introduced:

Definition 5.4.25 (Coverage). Let $\mathbf{C}$ be a category. A coverage on $\mathbf{C}$ is a map that assigns to every object $x \in \mathrm{ob}(\mathbf{C})$ a collection of families $\{f : y \to x\} \subset \hom(\mathbf{C})$ satisfying the following condition. If $\{f : y \to x\}$ is a **covering family** on x, then for every morphism $g : x' \to x$ there exists a covering family $\{f' : y' \to x'\}$ on x' such that every composite $g \circ f'$ factors through some f.

It should be clear that every coverage generates a sieve. Furthermore, although coverages are weaker and easier to handle, they are in fact equivalent for the purpose of sheaf theory:

Property 5.4.26. Consider a covering family C and let S_C be the sieve it generates. A presheaf is a sheaf for C if and only if it is a sheaf for S_C.

5.4.1 Topological sheaves

See Chapter **??** for the application of sheaves to topology.

Property 5.4.27 (Presheaf topos). Consider the presheaf category

$$\mathbf{Psh}(X) := \mathbf{Psh}(\mathbf{Open}(X))$$

over a topological space (X, τ). Unpacking Property 5.4.4 shows that this category is an elementary topos where the subobject classifier is given by

$$\Omega(U) = \{V \in \tau \mid V \subseteq U\}. \tag{5.15}$$

Construction 5.4.28 (Sheaves and étalé bundles). Let X be a topological space. The functor

$$I : \mathbf{Open}(X) \to \mathbf{Top}_{/X} : U \mapsto (U \hookrightarrow X)$$

induces the following adjunction:

$$\begin{tikzcd} \mathbf{Top}_{/X} \arrow[r, shift right, "\Gamma"'] \arrow[r, phantom, "\perp"] & \mathbf{Psh}(X). \arrow[l, shift right, "E"'] \end{tikzcd} \tag{5.16}$$

The slice category on the left-hand side is the category of (topological) bundles (Chapter **??**) over X. Both directions of the adjunction have a clear interpretation. The right adjoint assigns to every bundle its sheaf of local sections and the left adjoint assigns to every presheaf its bundle of germs.

By restricting to the subcategories on which this adjunction becomes an adjoint equivalence, one obtains the **étalé space** and **sheaf categories** respectively:

$$\mathbf{Et}(X) \cong \mathbf{Sh}(X). \tag{5.17}$$

The category on the right-hand side is the category of sheaves on a topological space X. The category on the left is the full subcategory on local homeomorphisms, i.e. the étalé spaces **??**.

Property 5.4.29 (Associated sheaf). The inclusion functor $\mathbf{Sh}(X) \hookrightarrow \mathbf{Psh}(X)$ admits a left adjoint, the sheafification functor that assigns to every presheaf its associated sheaf. This functor is given by the composition $\Gamma \circ E$, which is just Construction **??**.

The fact that the counit of the adjunction 5.4.28 restricts to an isomorphism on the full subcategory $\mathbf{Sh}(X)$ is equivalent to the fact that the sheafification of a sheaf Γ is again Γ.

Definition 5.4.30 (Petit and gros topoi[5]). Consider a topological space X together with its category of opens $\mathbf{Open}(X)$. The petit topos over X is defined as the sheaf topos $\mathbf{Sh}(X)$. It represents X as a "generalized space". (By Construction 5.4.28 the objects in a petit topos are the étalé spaces over the given base space.) However, one can also build a topos whose objects are themselves generalized spaces. To this end, choose a site S of "probes" and call the sheaf topos $\mathbf{Sh}(S)$ a gros topos. See Section **??** for more information.

Property 5.4.31 (Localic reflection). Mapping a topological space to its sheaf of continuous sections defines a functor $\mathbf{Sh} : \mathbf{Top} \to \mathbf{Topos}$ by Example 5.3.8. When restricted to the full subcategory of sober spaces **??**, this functor becomes fully faithful. Generalizing to sober locales even gives a reflective inclusion 2.2.27.

This property states that no information is lost when regarding (sober) topological spaces as sheaf topoi. This also explains the name "petit topos".

5.4.2 Lawvere-Tierney topology

Definition 5.4.32 (Lawvere-Tierney topology). As noted in Section 5.2 on the internal logic of elementary topoi, the subobject classifier Ω has the structure of an internal Heyting algebra and, in particular, that of a meet-semilattice, where the meet is given by the pullback of morphisms. This internal poset, viewed as an internal category, admits the construction of a closure operator $j : \Omega \to \Omega$ (Definition 2.3.25) satisfying the following condition:

$$j \circ \wedge = \wedge \circ (j \times j). \tag{5.18}$$

This condition states (in a nontrivial way) that j is (internally) order-preserving.

[5] For those that do not master French, petit and gros mean small and big, respctively.

Remark 5.4.33. The condition satisfied by the unit morphism in the definition of a closure operator can also be reformulated in this context as follows:

$$j \circ \texttt{true} = \texttt{true}. \tag{5.19}$$

The Lawvere-Tierney operator also induces a "closure operator" on all posets $\mathrm{Sub}(x)$ in the topos. Given an object x and a subobject $u \in \mathrm{Sub}(x)$, one defines the closure $j_*(u) \in \mathrm{Sub}(x)$ as the subobject classified by the characteristic morphism $j \circ \chi_u : x \to \Omega$.

Definition 5.4.34 (Dense object). Given a Lawvere-Tierney topology $j : \Omega \to \Omega$, a subobject $u \in \mathrm{Sub}(x)$ is said to be dense (in x) if it satisfies $j_*(u) = x$.

Alternative Definition 5.4.35 (Sheaf). Given a Lawvere-Tierney topology $j : \Omega \to \Omega$ on a topos $\mathcal{E}$, one calls an object $s \in \mathrm{ob}(\mathcal{E})$ a j-sheaf if for all dense morphisms $u \hookrightarrow x$ the induced map

$$\mathcal{E}(x, s) \to \mathcal{E}(u, s)$$

is a bijection.

Property 5.4.36. For the presheaf topos on a small category $\mathbf{C}$, the Grothendieck topologies on $\mathbf{C}$ are equivalent to Lawvere-Tierney topologies on $\mathbf{Psh}(\mathbf{C})$.

Sketch of proof. Since a Grothendieck topology assigns to every object a collection of sieves, Property 5.4.4 implies that $J(x) \subseteq \Omega_{\mathbf{Psh}}(x)$ for all $x \in \mathrm{ob}(\mathbf{C})$. By the base change condition of Grothendieck topologies, this relation is natural in x and, hence, J is a subobject of $\Omega_{\mathbf{Psh}}$. One thus finds a characteristic morphism $j : \Omega_{\mathbf{Psh}} \to \Omega_{\mathbf{Psh}}$ that can be proven (by the other conditions of Grothendieck topologies) to define a Lawvere-Tierney topology on $\mathbf{Psh}(\mathbf{C})$. Conversely, a Lawvere-Tierney topology is a morphism $j : \Omega \to \Omega$ and, hence, determines a unique subobject of $\Omega_{\mathbf{Psh}}$, i.e. a unique collection of sieves for every object $x \in \mathrm{ob}(\mathbf{C})$. From the conditions of Lawvere-Tierney topologies one can then prove that this collection satisfies the conditions of a Grothendieck topology.

Remark 5.4.37. It follows that Lawvere-Tierney topologies generalize Grothendieck topologies from presheaf topoi to arbitrary elementary topoi.

5.5 Stacks

5.5.1 2-sheafs

An important subject, especially in the context of gauge theories in physics, is that of groupoid-valued (pre)sheaves. To this end, sites are generalized to higher category theory.

Definition 5.5.1 (2-presheaf). Consider a 2-category $\mathbf{C}$. A 2-presheaf on $\mathbf{C}$ is a pseudofunctor $F : \mathbf{C}^{op} \to \mathbf{Cat}$. When $\mathbf{C}$ is the categorification of a 1-category, i.e. when it has discrete hom-categories, 2-presheaves are better known as **prestacks**.

Definition 5.5.2 (2-coverage). Virtually the same as an ordinary coverage 5.4.25, but factorization is only required to exist up to an isomorphism. A 2-category equipped with a 2-coverage is called a **2-site**.

As for 1-sites, every coverage generates a unique sieve. It is the full subcategory on those morphisms that factor through a covering map in the given coverage (again up to isomorphism).

As in the case of ordinary categories (Definition 5.4.11), one can define 2-sheaves through a descent condition:

Definition 5.5.3 (2-sheaf). A 2-presheaf $F : \mathbf{C}^{op} \to \mathbf{Cat}$ on a 2-site $(\mathbf{C}, J)$ is said to be a 2-sheaf with respect to J if for all sieves $S \in J$ the following functor is an equivalence:

$$Fc \cong \mathbf{Psh}_2(\mathcal{Y}c, F) \to \mathbf{Psh}_2(S, F), \tag{5.20}$$

where the fist equivalence is just the 2-Yoneda lemma.

Remark 5.5.4. It should be noted that 2-(pre)sheaves can also be defined on ordinary (1-)sites. Sieves, regarded as subfunctors of the Yoneda embedding, take values in **Set**. By composing these with the embedding $\mathbf{Set} \hookrightarrow \mathbf{Cat}$ of sets as (discrete) categories, one obtains 2-subfunctors of the 2-Yoneda embedding. Often 2-sheaves over 1-sites are called **stacks** (although this terminology is also used for general 2-sites).

Definition 5.5.5 (Prestack of groupoids). Consider a category $\mathbf{C}$. A prestack of groupoids on $\mathbf{C}$ is a **Grpd**-valued prestack on $\mathbf{C}$.

The category of (groupoid-valued) prestacks becomes **Grpd**-enriched if one takes the hom-category between two prestacks F, G to consist of the following data:

- **Objects**: The natural transformations $\alpha : F \Rightarrow G$ (note that the components are themselves functors).
- **Morphisms**: The "strict modifications" in the sense that they map objects in $\mathbf{C}$ to natural transformations satisfying the whiskering condition (see also Definition 2.9.12)

$$\mathbb{1}_{Ff} \cdot \mathfrak{m}_b = \mathfrak{m}_a \cdot \mathbb{1}_{Gf}. \tag{5.21}$$

For ordinary sites and presheaves, descent was defined in terms of matching families. Since presheaves are now taking values in a 2-category, the matching families are a bit more complex. However, this structure is already familiar from differential geometry and algebraic topology, where it is known under the name of the *Čech nerve*:

Definition 5.5.6 (Čech groupoid). Consider a site $(\mathbf{C}, J)$. To every covering family $\mathcal{U} \equiv \{f_i : x_i \to x\}_{i \in I}$ one can assign an internal groupoid in presheaves $C(\mathcal{U})$ consisting of the following data:

- **Objects**: $\bigsqcup_i \mathcal{Y}x_i$
- **Morphisms**: $\bigsqcup_{i,j} \mathcal{Y}x_i \times_{\mathcal{Y}x} \mathcal{Y}x_j$

This is equivalent to the (**Grpd**-valued) presheaf that assigns to every object $y \in \mathrm{ob}(\mathbf{C})$ the groupoid consisting of the following data:

- **Objects**: The pairs $(i, g_i : y \to x_i)$ where $x_i \in \mathcal{U}$.
- **Morphisms**: A unique arrow $(i, g_i) \to (j, g_j)$ if and only if $f_i \circ g_i = f_j \circ g_j$.

Comparing the definition of morphisms in the Čech groupoid to the condition for matching families in Definition 5.4.8, shows that one could presume that the Čech groupoid is related to the matching families. This intuition is indeed correct:

Property 5.5.7 (Matching families). Any ordinary presheaf F can be considered to be **Grpd**-valued by postcomposing with the embedding $\mathbf{Set} \hookrightarrow \mathbf{Grpd}$. For any covering family $\mathcal{U}$, there exists an isomorphism

$$[\mathbf{C}^{op}, \mathbf{Grpd}](C(\mathcal{U}), F) \cong \mathbf{Psh}_2(\mathcal{U}, F). \tag{5.22}$$

Because the Čech groupoid (co)represents a descent object, it is sometimes called a **codescent object**.

It is exactly this (co)descent property of the Čech groupoid that will be used in Chapter **??** to define (higher) smooth groupoids.

People with some experience in algebraic topology will also notice that the Čech groupoid only contains the first degrees of the Čech complex. The full Čech complex can be obtained from the following construction:

Definition 5.5.8 (Čech nerve). Consider a morphism $f : y \to x$ in a category $\mathbf{C}$. The Čech nerve $C_\bullet(f)$ is the simplicial object 4.1.3 that is defined as the $(k+1)$-fold pullback of f along itself in degree k. For a covering family $\mathcal{U} \equiv \{f_i : x_i \to x\}$, the Čech nerve is defined as $C_\bullet(\mathcal{U}) := C_\bullet(\bigsqcup_i x_i \to x)$.

For ∞-sheaves the full Čech nerve will be used. However, for 2-sheaves and, in particular, stacks, only its 3-coskeleton is necessary. The extra information will encode the *cocycle condition* (**??**) known for example from the study of fibre bundles.

5.5.2 Stacks on a 1-site

For the definition of stacks, one needs the notions of fibred categories or, equivalently, pseudofunctors as defined in Section 2.3.1. The definitions are recalled here:

Consider a functor $\Pi : \mathbf{A} \to \mathbf{B}$. A morphism f in $\mathbf{A}$ is said to be Π-Cartesian if for every morphism φ in $\mathbf{A}$ and factorization of $\Pi\varphi$ through Πf in $\mathbf{B}$, there exists a unique factorization of φ through f. f is called the inverse image of Πf.

A fibred category consists of a functor $\Pi : \mathbf{A} \to \mathbf{B}$ such that for each morphism in $(f : c \to d)\mathbf{B}$ with $d \in \mathrm{im}(\Pi)$ and each lift $y \in \mathbf{A}_d$ there exists at least one inverse image in $(\widetilde{f} : x \to y) \in \mathbf{A}$ of f. By the Grothendieck construction every fibred category gives rise to a pseudofunctor $F : \mathbf{B}^{op} \to \mathbf{Cat}$ by sending objects to their fibres under Π and sending morphisms f to their pullback functors f^*.

Definition 5.5.9 (Descent datum). Consider a category $\mathbf{C}$ with a covering family $\mathcal{U} \equiv \{f_i : x_i \to x\}$ and a pseudofunctor $F : \mathbf{C}^{op} \to \mathbf{Cat}$. The projections associated to the pullback $x_i \cap x_j := x_i \times_x x_j$ will be denoted by π_i and π_j (and analogously for iterated pullbacks). A descent datum for $\mathcal{U}$ with respect to F is a pair of families $(\{g_i\}, \{f_{ij}\})_{i,j \in I}$, where $\{g_i\}$ is a matching family for $\mathcal{U}$ with respect to F and every f_{ij} is an isomorphism $\pi_i^* x_i \cong \pi_j^* x_j$. This data is required to satisify the following **cocycle condition**:

$$\pi_{ik}^* f_{ik} = \pi_{ij}^* f_{ij} \circ \pi_{jk}^* f_{jk}. \tag{5.23}$$

Morphisms $(\{g_i\}, \{f_{ij}\}) \to (\{g_i'\}, \{f_{ij}'\})$ between descent data are families of morphisms $\{\phi_i : g_i \to g_i'\}$ that satisfy

$$\pi_i^* \phi_i \circ f_{ij} = f_{ij}' \circ \pi_j^* \phi_j. \tag{5.24}$$

The category of descent data for $\mathcal{U}$ with respect to F will be denoted by $\mathrm{Descent}(\mathcal{U}, F)$.

Construction 5.5.10. Consider an object ξ in Fx. From this object one can construct a descent datum as follows. The objects g_i are the pullbacks $f_i^*\xi$ and the isomorphisms $f_{ij} : \pi_i^* f_i^* \xi \cong \pi_j^* f_j^* \xi$ are obtained from the fact that both these objects are (Cartesian) pullbacks of the same morphisms. Arrows in Fx induce morphisms of descent data by (Cartesian) pullbacks along the covering maps. This construction defines a functor $Fx \to \mathrm{Descent}(\mathcal{U}, F)$. It can be shown that this construction is independent of a choice of cleavage up to equivalence.

Definition 5.5.11 (Stack). Consider a fibred category F over a site $(\mathbf{C}, J)$.

- F is called a **separated prestack** if for each covering family $\mathcal{U}$ on $x \in \mathrm{ob}(\mathbf{C})$, the functor $Fx \to \mathrm{Descent}(\mathcal{U}, F)$ is fully faithful.
- F is called a **stack** if for each covering family $\mathcal{U}$ on $x \in \mathrm{ob}(\mathbf{C})$ the functor $Fx \to \mathrm{Descent}(\mathcal{U}, F)$ is an equivalence.

This is a generalization of the descent condition 5.4.11. This can be seen by observing that $\mathrm{Descent}(\mathcal{U}, F) \cong \mathbf{Psh}_2(S(\mathcal{U}), F)$, where $S(\mathcal{U})$ is the sieve generated by $\mathcal{U}$ regarded as a fibred category.

A more conceptual (although completely equivalent) generalization from (1-)sheaves to 2-sheaves can be obtained by starting from Property 5.5.7. There it was shown that matching families for (1-)presheaves can be obtained as natural transformations from the Čech groupoid.

Property 5.5.12 (Descent data and Čech nerve). Let $C(\mathcal{U})$ denote the 3-coskeleton of the Čech nerve $C_\bullet(\mathcal{U})$. Pseudonatural transformations $C(\mathcal{U}) \Rightarrow F$ can be shown to be equivalent to tuples $(c, \{c_i\}, \{c_{ij}\}, \{c_{ijk}\})$, where $c_i \in Fx_i$, that fit into cubes lying in the image of $C_2(\mathcal{U})$ in which all edges consist of Cartesian morphisms. Arrows between such cubes are given by arrows between the vertices that make the "obvious" diagrams commute.

By comparing these cubes to the previous definition of descent data, one obtains the following equivalence:

$$\mathrm{Descent}(\mathcal{U}, F) \cong [\mathbf{C}^{op}, \mathbf{Cat}](C(\mathcal{U}), F). \tag{5.25}$$

?? FINISH THIS ??

Remark 5.5.13 (1-sheaves). Although most of the above seems very abstract and complex compared to ordinary sheaves, it is not quite so. In fact, when restricting to pseudofunctors of the form $\mathbf{C}^{op} \to \mathbf{Set}$, where the embedding $\mathbf{Set} \hookrightarrow \mathbf{Cat}$ sends sets to discrete categories, one obtains ordinary sheaves as a subcategory of stacks. For example, by the equivalence between pseudofunctors and Grothendieck fibrations, it is known that the Cartesian pullbacks f^* are in fact just the images of morphism f under the pseudofunctor F. This way the condition $\pi_1^* c_i \cong \pi_2^* c_j$ can be rewritten as $Ff'_i(c_i) = Ff'_j(c_j)$, which is nothing but the matching family condition (5.11).

5.6 Higher topos theory

In this section the notion of topos is generalized from ordinary category theory to higher category theory. In particular, ∞-sheaves will be defined. This will require a suitable foundation for ∞-category theory. To this end the language of (simplicial) model categories as introduced in Chapter 4 will be used.

Definition 5.6.1 (∞-groupoid). Objects of the full simplicial subcategory of $\mathbf{sSet}_{Quillen}$ on Kan complexes. From Property 4.4.4, it is immediately clear how this generalizes the definition of ordinary groupoids. For groupoids one needs unique horn fillers (composition in ordinary categories is unique), while for ∞-groupoids this is allowed to be unique up to higher coherence.

Definition 5.6.2 ($(\infty,1)$-category). An ∞**Grpd**-enriched category or, equivalently, a simplicially enriched category for which all hom-objects are Kan complexes. The functor category between $(\infty,1)$-categories is defined through the (simplicial) nerve and realization functors 4.1.10:

$$[\mathbf{C}, \mathbf{D}] := |\mathbf{sSet}(N\mathbf{C}, N\mathbf{D})|. \tag{5.26}$$

Property 5.6.3 (Čech model structure). For any small category $\mathbf{C}$, the ∞-category of $\infty\mathbf{Grpd}$-valued ∞-sheaves can be represented by the category $[\mathbf{C}^{op}, \mathbf{sSet}]$ of simplicial presheaves on $\mathbf{C}$ by a theorem of *Lurie* 4.7.1, i.e. there exists an ∞-equivalence between $\mathbf{Sh}_{(\infty,1)}(\mathbf{C})$ and the full subcategory on fibrant-cofibrant objects of the (left Bousfield) localization of $[\mathbf{C}^{op}, \mathbf{sSet}]$ at the Čech nerve projections. The resulting model structure is called the **Čech model structure**.

A presheaf X is fibrant in this model structure if the map

$$\mathrm{Hom}(M, X) \to \mathrm{Hom}(\mathcal{C}(\mathcal{U}), X) \tag{5.27}$$

is a weak equivalence for all open covers $\mathcal{U}$, i.e. exactly if X satisfies the descent condition and, hence, is an ∞-stack.

The most straightforward definition of an ∞-sheaf generalizes Definition 5.4.11:

Definition 5.6.4 (∞-sheaf). Consider an ∞-site $(\mathbf{C}, J)$ and let S denote the collection of monomorphisms in $\mathbf{Psh}_\infty(\mathbf{C})$ induced by the covering sieves. An ∞-presheaf on $\mathbf{C}$ is called a J-sheaf if it is S-local. A presheaf with values in an ∞-category $\mathbf{D}$ is called a sheaf if the representable presheaf $\mathbf{D}(x, F-)$ is a J-sheaf for all $x \in \mathrm{ob}(\mathbf{D})$.

In terms of the Čech nerve $\mathcal{C}$, the descent condition can be written as follows:

$$Fx \simeq \mathbf{Psh}_\infty(\mathcal{C}(\mathcal{U}), F) \tag{5.28}$$

for all covers $\mathcal{U}$ of x, where $\simeq$ denotes a weak equivalence.

Definition 5.6.5 (∞-stack). An $(\infty, 1)$-sheaf taking values in $\infty\mathbf{Grpd}$.

Property 5.4.19 can be generalized as follows:

Property 5.6.6. For every ∞-topos $\mathbf{H}$ there exists a geometric morphism $(\mathrm{Disc} \dashv \Gamma) : \mathbf{H} \leftrightarrows \infty\mathbf{Grpd}$. Any morphism into a discrete object $\mathrm{Disc}(X)$ is constant.

The left adjoint is sometimes called the **discrete object functor**. This terminology stems from the case of the forgetful functor $\Gamma : \mathbf{Top} \to \mathbf{Set}$, where the (fully faithful) left adjoint equips a set with the discrete topology.

Example 5.6.7 (Sheaves on manifolds). One of the archetypal examples of ∞-topoi is the topos of sheaves over smooth manifolds. By the Yoneda embedding one can regard a manifold as a sheaf and the global sections functor maps this representable sheaf to the manifold itself: $\Gamma(M) = M$. For a Lie group one can construct the classifying stack $\mathbf{B}G$. The global sections functor maps this stack to the delooping groupoid BG.

Definition 5.6.8 (Mapping stack). Consider two ∞-stacks $X, Y \in \mathbf{Sh}_{(\infty,1)}(\mathbf{C})$. The mapping stack is defined as follows:

$$[X, Y](U) := \mathbf{Sh}_{(\infty,1)}(\mathbf{C})(X \times U, Y), \tag{5.29}$$

where on the right-hand side, U denotes the representable ∞-stack.

?? FINISH (PERHAPS MOVE infinity-CATEGORY STUFF TO CHAPTER ON MODEL THEORY) ??

5.7 Cohomology

In this section, cohomology will be generalized to the ∞-categorical setting.

First, take a topological space X and an ∞-groupoid G. Geometric realization 4.1.11 gives an equivalence $\infty\mathbf{Grpd} \cong \mathbf{Top}$ and, therefore, one can define the intrinsic cohomology of X with coefficients in G as follows:

$$H(X;G) := \pi_0\mathbf{Top}(X,|G|). \tag{5.30}$$

X can also be identified with its petit (∞-)topos $\mathbf{Sh}_{(\infty,1)}(X)$, in which X sits as the terminal object. From this point of view the intrinsic cohomology of X with coefficients in G is

$$\overline{H}(X;G) := \pi_0\mathbf{Sh}_{(\infty,1)}(X)(X,\mathrm{LConst}\,G) \cong \pi_0 \circ \Gamma \circ \mathrm{LConst}(G). \tag{5.31}$$

This is the **cohomology with constant coefficients** of X with coefficients in G. If X is paracompact, the two cohomologies coincide: $H(X;G) \cong \overline{H}(X;G)$.

Now, it is time to pass to general cohomology:

Definition 5.7.1 (Intrinsic cohomology). Consider a $(\infty,1)$-category $\mathbf{H}$. For every two objects $X, A \in \mathbf{H}$, the hom-space $\mathbf{H}(X,A)$ is an ∞-groupoid. Define the following notions:

- The objects in $\mathbf{H}(X,A)$ are called **cocycles**.
- The morphism in $\mathbf{H}(X,A)$ are called **coboundaries**.
- The set of connected components

  $$H(X;A) := \pi_0\mathbf{H}(X,A) = \mathrm{Hom}_{\mathbf{Ho(H)}}(X,A)\,, \tag{5.32}$$

  where $\mathbf{Ho(H)}$ is the homotopy category 4.3.28 of $\mathbf{H}$, is called the intrinsic cohomology of X with coefficients in A.

If the object A admits an n-delooping $\mathbf{B}^nA$, the n^{th} cohomology group of X is defined as

$$H^n(X;A) := H(X;\mathbf{B}^nA). \tag{5.33}$$

Example 5.7.2 (Singular cohomology). Consider a topological space X. For every group G one can define the first delooping 2.10.2, so one can also define the zeroth and first cohomology groups $H^{0,1}(X;G)$. Only when G is Abelian do higher deloopings exists (in fact, if G is Abelian all higher deloopings exist), and so in this case higher cohomology groups $H^{\geq 2}(X;G)$ can be defined. It can be shown that these coincide with the singular cohomology groups of X.

Example 5.7.3 (Group cohomology). Consider a (discrete) group G together with its delooping groupoid $\mathbf{B}G$. The group cohomology **??** of a group with coefficients in an Abelian group A is given by the intrinsic cohomology of $\infty\mathbf{Grpd}$ of the delooping groupoids:

$$H(G;A) \cong \pi_0\infty\mathbf{Grpd}(\mathbf{B}G,\mathbf{B}A). \tag{5.34}$$

By replacing the topos $\mathbf{H}$ by a slice topos $\mathbf{H}_{/X}$ one obtains twisted cohomology:

Definition 5.7.4 (Twisted cohomology). Consider a $(\infty,1)$-topos $\mathbf{H}$ with some object $X \in \mathrm{ob}(\mathbf{H})$. The mapping space $\mathbf{H}(X,A)$, the cocycles of X with coefficients in A, is easily seen to be isomorphic to the mapping space $\mathbf{H}_{/X}(X, X \times A)$, where the second argument is equipped with the canonical projection morphism. Morphisms in this space are just sections of the trivial A-∞-bundle over X. General twisted cohomology can then be defined as the space of sections of an arbitrary A-∞-bundle over X.

By passing to classifying morphisms of bundles one obtains the twist $\chi : X \to \mathbf{BAut}(A)$ and the universal bundle $\rho_A : A/\!/\mathbf{Aut}(A) \to \mathbf{BAut}(A)$. χ-twisted cohomology is then given by (hte connected components of) the following mapping space:

$$\mathbf{H}_{/\mathbf{BAut}(A)}(\chi, \rho_A). \tag{5.35}$$

5.8 Cohesion

In this section the terminology "(Grothendieck) topos **over** a topos $\mathcal{S}$" will mean a topos equipped with a geometric morphism to $\mathcal{S}$.

Definition 5.8.1 (Local topos). Consider a topos $\mathcal{E}$ over a base topos $\mathcal{S}$. $\mathcal{E}$ is said to be ($\mathcal{S}$-)local if the geometric morphism $(f^* \dashv f_*) : \mathcal{E} \leftrightarrows \mathcal{S}$ admits a right adjoint $f^!$ such that one of the following equivalent statements holds:

- $f^!$ is fully faithful.
- f^* is fully faithful.
- $f^!$ is an $\mathcal{S}$-indexed functor 2.9.16.
- $f^!$ is Cartesian closed 2.6.23.

If one takes $\mathcal{S} = \mathbf{Set}$, the conditions are automatically satisfied since all functors are **Set**-indexed.

The right adjoint is sometimes called the **codiscrete object functor** coDisc (in fact, this terminology is applied more generally when $\mathcal{E}$ is just any category). If this functor exists, $\mathcal{E}$ is said to have **codiscrete objects**.

Property 5.8.2. A topos is local if and only if 1 is tiny 2.4.40.

Definition 5.8.3 (Locally connected topos). An object in a category is said to be **connected** if its representable functor preserves finite coproducts. A topos is said to be **locally connected** if all objects can be written as coproducts of connected objects. This defines a functor

$$\Pi_0 : \mathcal{E} \to \mathbf{Set} : \bigsqcup_{i \in I} X_i \mapsto I \tag{5.36}$$

left adjoint to the discrete object functor (which is itself left adjoint to the global section functor). This functor is suitably called the **connected components functor**.

A topos is locally connected if and only if its global section geometric morphism is essential. More generally, a topos over some base topos $\mathcal{S}$ is said to be **locally connected** if its associated geometric morphism is essential and the left adjoint is $\mathcal{S}$-indexed. In the case of $(\infty, 1)$-topoi, the image of the functor Π_0 is called the **fundamental ∞-groupoid**.

Definition 5.8.4 (Connected topos). A topos over a base topos is said to be **connected** if the inverse image part of the associated geometric morphism is fully faithful. For sheaf topoi over a topological space X this is exactly the requirement that X is connected.

For locally connected topoi this amounts to the property that the left adjoint in its adjoint triple preserves the terminal object. Furthermore, a locally connected topos is said to be **strongly connected** if the left adjoint in its adjoint triple preserves finite products (in particular turning it into a connected topos).

Property 5.8.5. Every local topos is connected.

Definition 5.8.6 (Cohesive topos). A local, strongly connected topos. This implies the existence of an adjoint quadruple $(\Pi_0, \mathrm{Disc}, \Gamma, \mathrm{coDisc})$ where both Disc and coDisc are fully faithful.

Property 5.8.7 (Cohesive modalities). The adjoint quadruple on a cohesive topos induces an adjoint triple of modalities 2.3.25, i.e. idempotent (co)monads (see Section **??** for a formal introduction in the context of type theory):

$$(\int \dashv \flat \dashv \sharp) := (\mathrm{Disc} \circ \Pi_0 \dashv \mathrm{Disc} \circ \Gamma \dashv \mathrm{coDisc} \circ \Gamma). \tag{5.37}$$

These are respectively called the **shape**, **flat** and **sharp** modalities. The modal types of the flat and sharp modalities are called the **discrete** and **codiscrete objects**, respectively.

?? COMPLETE (e.g. work by Schreiber) ??

List of Symbols

The following symbols are used throughout the summary:

Abbreviations

AIC	Akaike information criterion
ARMA	autoregressive moving-average model
BCH	Baker-Campbell-Hausdorff
CCR	canonical commutation relation
CDF	cumulative distribution function
CFT	conformal field theory
CIS	completely integrable system
CP	completely positive
CPTP	completely positive, trace-preserving
CR	Cauchy-Riemann
DGA	differential graded algebra
DGCA	differential graded-commutative algebra
EPR	Einstein-Podolsky-Rosen
ETCS	Elementary Theory of the Category of Sets
FWHM	full width at half maximum
GA	geometric algebra
GHZ	Greenberger-Horne-Zeilinger
GNS	Gel'fand-Naimark-Segal
HoTT	Homotopy Type Theory
KKT	Karush-Kuhn-Tucker
LIVF	left-invariant vector field
MPO	matrix product operator
MPS	matrix product state
MTC	modular tensor category
NDR	neighbourhood deformation retract
OPE	operator product expansion
OZI	Okubo-Zweig-Iizuka
PAC	probably approximately correct
PL manifold	piecewise-linear manifold
PVM	projection-valued measure

RKHS	reproducing kernel Hilbert space
SVM	support-vector machine
TQFT	topological quantum field theory
VIF	variance inflation factor
ZFC	Zermelo-Frenkel set theory with the axiom of choice
TVS	topological vector space
Operations	
Ad_g	adjoint representation of a Lie group G
ad_X	adjoint representation of a Lie algebra $\mathfrak{g}$
arg	argument of a complex number
$\Box$	d'Alembert operator
$\deg(f)$	degree of the polynomial f
e	identity element of a group
$\Gamma(E)$	set of global sections of a fibre bundle E
Im	imaginary part of a complex number
$\mathrm{Ind}_f(z)$	index of a point $z \in \mathbb{C}$ with respect to a function f
$\hookrightarrow$	injective function
$\cong$	is isomorphic to
Par_t^γ	parallel transport map with respect to the curve γ
Re	real part of a complex number
Res	residue of a complex function
$\twoheadrightarrow$	surjective function
$\{\cdot,\cdot\}$	Poisson bracket
∂X	boundary of a topological space X
$\overline{X}$	closure of a topological space X
$X^\circ, \mathring{X}$	interior of a topological space X
$\sphericalangle(\cdot,\cdot)$	angle between two vectors
$X \times Y$	cartesian product of the sets X and Y
$X + Y$	sum of the vector spaces X and Y
$X \oplus Y$	direct sum of the vector spaces X and Y
$V \otimes W$	tensor product of the vector spaces V and W
$\mathbb{1}_X$	identity morphism on the object X
$\approx$	is approximately equal to
$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto
Collections	
Ab	category of Abelian groups
$\mathrm{Aut}(X)$	automorphism group of an object X
$\mathcal{B}_0(V,W)$	space of compact bounded operators between the Banach spaces V and W

$\mathcal{B}(V, W)$	space of bounded linear maps from the space V to the space W
CartSp	the category of Euclidean spaces and "suitable" homomorphisms (e.g. linear maps, smooth maps, ...)
$C_\bullet$	chain complex
$\mathbf{Ch}(\mathbf{A})$	category of chain complexes with objects in the additive category $\mathbf{A}$
$\mathbf{C}^\infty$	category of smooth spaces
$C_p^\infty(M)$	ring of smooth functions $f : M \to \mathbb{R}$ on a neighbourhood of $p \in M$
$C^\omega(V)$	the set of all analytic functions defined on the set V
$\mathrm{Conf}(M)$	conformal group of (pseudo-)Riemannian manifold M
$C(X, Y)$	set of continuous functions between two topological spaces X and Y
$\mathbf{C}^\infty\mathbf{Ring}, \mathbf{C}^\infty\mathbf{Alg}$	category of smooth algebras
Diff	category of smooth manifolds
DiffSp	category of diffeological spaces and smooth maps
D^n	standard n-disk
$\mathrm{dom}(f)$	domain of a function f
$\mathrm{End}(X)$	endomorphism monoid of a an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad
$\mathbf{FormalCartSp}_{\mathbf{diff}}$	category of infinitesimally thickened Euclidean spaces
$\mathrm{GL}(V)$	general linear group, the group of automorphisms of a vector space V
$\mathrm{GL}(n, K)$	general linear group: the group of all invertible $n \times n$-matrices over the field K
Grp	category of groups and group homomorphisms
Grpd	category of groupoids
$\mathrm{Hol}_p(\omega)$	holonomy group at the point p with respect to the principal connection ω
$\mathrm{Hom}_{\mathbf{C}}(V, W)$	set of homomorphisms from an object V to an object W in a category $\mathbf{C}$
hTop	homotopy category
$\mathrm{im}(f)$	image of a function f
$K^0(X)$	K-theory over a (compact Hausdorff) space X
Kan	category of Kan complexes
$\mathcal{K}_n(A, v)$	Krylov subspace of dimension n generated by the matrix A and the vector v
L^1	space of integrable functions
Law	category of Lawvere theories
Lie	category of Lie groups
$\mathfrak{Lie}$	category of Lie algebras
$\mathfrak{X}^L$	space of left-invariant vector fields on a Lie group
LX	free loop space on X
$\mathbf{Man}^p$	category of C^p-manifolds
Meas	category of measure spaces and measure-preserving functions
$N\mathbf{C}$	the simplicial nerve of a small category $\mathbf{C}$
$\mathbf{Open}(X)$	category of open subsets of a topological space X
$\mathrm{O}(n, K)$	group of $n \times n$ orthogonal matrices over a field K
$P(S), 2^S$	power set of S

$\mathrm{Pin}(V)$	pin group of the Clifford algebra $C\ell(V,Q)$
$\mathbf{Psh}(\mathbf{C}), \widehat{\mathbf{C}}$	category of presheaves on a (small) category $\mathbf{C}$
$\mathbf{Sh}(X)$	category of sheaves on a topological space X
$\mathbf{Sh}(\mathbf{C}, J)$	category of J-sheaves on a site $(\mathbf{C}, J)$
$\mathbf{\Delta}$	simplex category
$\mathrm{SL}_n(K)$	special linear group: group of all invertible n-dimensional matrices with unit determinant over the field K
S^n	standard n-sphere
$S^n(V)$	space of symmetric rank n tensors over a vector space V
$W^{m,p}(U)$	the Sobolov space in L^p of order m
$\mathbf{Span}(\mathbf{C})$	span category over $\mathbf{C}$
$\mathrm{Spec}(R)$	spectrum of a commutative ring R
$\mathrm{supp}(f)$	support of a function f
$\mathrm{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
S_n	symmetric group of degree n
$\mathrm{Sym}(X)$	symmetric group on the set X
$\mathrm{Sp}(n, K)$	group of matrices preserving a canonical symplectic form over the field K
$\mathrm{Sp}(n)$	compact symplectic group
$\mathrm{TL}_n(\delta)$	Temperley-Lieb algebra with $n-1$ generators and parameter δ.
T^n	standard n-torus (the n-fold Cartesian product of S^1)
Top	category of topological spaces
Topos	the 2-category of (elementary) topoi and geometric morphisms
$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\mathrm{U}(n, K)$	group of $n \times n$ unitary matrices over a field K
$\mathbf{Vect}(X)$	category of vector bundles over a manifold X
$\mathbf{Vect}_K$	category of vector spaces and linear maps over a field K
Y^X	set of functions from a set X to a set Y
$\emptyset$	empty set
$\pi_n(X, x_0)$	n^{th} homotopy space over X with basepoint x_0
$[a, b]$	closed interval
$]a, b[$	open interval
$\Lambda^n(V)$	space of antisymmetric rank n tensors over a vector space V
ΩX	(based) loop space on X
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on the manifold M
$\rho(A)$	resolvent set of a bounded linear operator A
$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on the manifold M
Units	
C	coulomb
T	tesla

Bibliography

[1] Andreas Gathmann. Algebraic geometry. `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf`.

[2] David Mumford. *The Red Book of Varieties and Schemes: Includes the Michigan Lectures (1974) on Curves and Their Jacobians*, volume 1358. Springer Science & Business Media, 1999.

[3] Emily Riehl. Homotopical categories: From model categories to $(\infty, 1)$-categories. 2019. arXiv:1904.00886.

[4] Mark Hovey. *Model Categories*. Number 63. American Mathematical Soc., 2007.

[5] Emily Riehl. Monoidal algebraic model structures. *Journal of Pure and Applied Algebra*, 217(6):1069--1104, 2013.

[6] Charles Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3):973--1007, 2001.

[7] Emily Riehl and Dominic Verity. The theory and practice of Reedy categories. *Theory and Applications of Categories*, 29, 2013.

[8] Pascal Lambrechts.

[9] Emily Riehl. Homotopy (limits and) colimits. `http://www.math.jhu.edu/~eriehl/hocolimits.pdf`.

[10] Jade Master. Why is homology so powerful? 2020. arXiv:2001.00314.

[11] Peter T. Johnstone. *Topos Theory*. Dover Publications, 2014.

[12] Olivia Caramello. Lectures on topos theory at the university of Insubria. `https://www.oliviacaramello.com/Teaching/Teaching.htm`.

[13] Angelo Vistoli. Notes on Grothendieck topologies, fibered categories and descent theory. *arXiv:math/0412512*, 2004.

[14] Nima Moshayedi. 4-manifold topology, donaldson-witten theory, floer homology and higher gauge theory methods in the BV-BFV formalism. 2021.

[15] John C. Baez and Urs Schreiber. Higher gauge theory. 2005. arXiv:math/0511710.

[16] Urs Schreiber. *From Loop Space Mechanics to Nonabelian Strings*. PhD thesis, 2005.

[17] Richard Sanders. Commutative spectral triples & the spectral reconstruction theorem.

[18] John Baez and Alexander Hoffnung. Convenient categories of smooth spaces. *Transactions of the American Mathematical Society*, 363(11):5789--5825, 2011.

[19] John C. Baez and Aaron D. Lauda. Higher-dimensional algebra v: 2-groups. 2003. arXiv:math/0307200.

[20] Thomas Augustin, Frank PA Coolen, Gert De Cooman, and Matthias CM Troffaes. *Introduction to imprecise probabilities.* John Wiley & Sons, 2014.

[21] Tetsuji Miwa, Michio Jimbo, Michio Jimbo, and E Date. *Solitons: Differential Equations, Symmetries and Infinite-dimensional Algebras*, volume 135. Cambridge University Press, 2000.

[22] Vladimir I. Arnol'd. *Mathematical Methods of Classical Mechanics*, volume 60. Springer Science & Business Media, 2013.

[23] Edwin J. Beggs and Shahn Majid. *Quantum Riemannian Geometry.* Springer, 2020.

[24] Marc Henneaux and Claudio Teitelboim. *Quantization of Gauge Systems.* Princeton university press, 1992.

[25] Gregory M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64. CUP Archive, 1982.

[26] Vladimir Vovk, Alex Gammerman, and Glenn Shafer. *Algorithmic Learning in a Random World.* Springer Science & Business Media, 2005.

[27] Mukund Rangamani and Tadashi Takayanagi. *Holographic Entanglement Entropy.* Springer, 2017.

[28] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5. Springer Science & Business Media, 2013.

[29] Shun-ichi Amari. *Information Geometry and Its Applications.* Springer Publishing Company, Incorporated, 2016.

[30] Charles W. Misner, Kip S. Thorne, and John A. Wheeler. *Gravitation.* Princeton University Press, 2017.

[31] Carlo Rovelli and Francesca Vidotto. *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory.* Cambridge University Press, 2014.

[32] Richard W. Sharpe. *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, volume 166. Springer Science & Business Media, 2000.

[33] John C. Baez, Irving E. Segal, and Zhengfang Zhou. *Introduction to Algebraic and Constructive Quantum Field Theory.* Princeton University Press, 2014.

[34] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology.* Graduate Texts in Mathematics. Springer New York, 1995.

[35] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[36] Bruce Blackadar. *Operator Algebras: Theory of C∗-Algebras and von Neumann Algebras.* Springer, 2013.

[37] Marek Capinski and Peter E. Kopp. *Measure, Integral and Probability.* Springer Science & Business Media, 2013.

[38] Georgiev Svetlin. *Theory of Distributions.* Springer, 2015.

[39] Gerd Rudolph and Matthias Schmidt. *Differential Geometry and Mathematical Physics: Part II. Fibre Bundles, Topology and Gauge Fields*. Springer, 2017.

[40] Martin Schottenloher. *A Mathematical Introduction to Conformal Field Theory*, volume 759. 2008.

[41] Dusa McDuff and Deitmar Salamon. *Introduction to Symplectic Topology*. Oxford Graduate Texts in Mathematics. Oxford University Press, 2017.

[42] John C. Baez and Peter May. *Towards Higher Categories*, volume 152 of *IMA Volumes in Mathematics and its Applications*. Springer, 2009.

[43] Mikhail. M. Kapranov and Vladimir A. Voevodsky. *2-categories and Zamolodchikov Tetrahedra Equations*, volume 56 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1994.

[44] Geoffrey Compère. *Advanced Lectures on General Relativity*, volume 952. Springer, 2019.

[45] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor Categories*, volume 205. American Mathematical Soc., 2016.

[46] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[47] Peter J. Hilton and Urs Stammbach. *A Course in Homological Algebra*. Springer.

[48] Jean-Luc Brylinski. *Loop Spaces, Characteristic Classes and Geometric Quantization*. Birkhauser.

[49] Antoine Van Proeyen and Daniel Freedman. *Supergravity*. Cambridge University Press.

[50] William S. Massey. *A Basic Course in Algebraic Topology*. Springer.

[51] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Westview Press.

[52] Nadir Jeevanjee. *An Introduction to Tensors and Group Theory for Physicists*. Birkhauser.

[53] Yvonne Choquet-Bruhat, Cecile DeWitt-Morette, and Margaret Dillard-Bleick. *Analysis, Manifolds and Physics, Part 1: Basics*. North-Holland.

[54] Yvonne Choquet-Bruhat and Cecile DeWitt-Morette. *Analysis, Manifolds and Physics, Part 2*. North-Holland.

[55] Herbet Goldstein, John L. Safko, and Charles P. Poole. *Classical Mechanics*. Pearson.

[56] Franco Cardin. *Elementary Symplectic Topology and Mechanics*. Springer.

[57] Walter Greiner and Joachim Reinhardt. *Field Quantization*. Springer.

[58] Walter Greiner. *Quantum Mechanics*. Springer.

[59] B. H. Bransden and Charles J. Joachain. *Quantum Mechanics*. Prentice Hall.

[60] Heydar Radjavi and Peter Rosenthal. *Invariant Subspaces*. Dover Publications.

[61] Max Karoubi. *K-Theory: An Introduction*. Springer.

[62] Damien Calaque and Thomas Strobl. *Mathematical Aspects of Quantum Field Theories*. Springer, 2015.

[63] Ivan Kolar, Peter W. Michor, and Jan Slovak. *Normal Operations in Differential Geometry*. Springer.

[64] Stephen B. Sontz. *Principal Bundles: The Classical Case.* Springer.

[65] Stephen B. Sontz. *Principal Bundles: The Quantum Case.* Springer.

[66] William Fulton and Joe Harris. *Representation Theory: A First Course.* Springer.

[67] Peter Petersen. *Riemannian Geometry.* Springer.

[68] Charles Nash and Siddharta Sen. *Topology and Geometry for Physicists.* Dover Publications.

[69] Ian M. Anderson. *The Variational Bicomplex.*

[70] Joel Robbin and Dietmar Salamon. The maslov index for paths. *Topology*, 32(4):827--844, 1993.

[71] Edward Witten. Global anomalies in string theory. In *Symposium on Anomalies, Geometry, Topology*, 6 1985.

[72] F.A. Berezin and M.S. Marinov.

[73] Paul A. M. Dirac. Generalized Hamiltonian dynamics. *Canadian Journal of Mathematics*, 2:129–148, 1950.

[74] Floris Takens. A global version of the inverse problem of the calculus of variations. *Journal of Differential Geometry*, 14(4):543--562, 1979.

[75] John C. Baez and Alissa S. Crans. Higher-dimensional algebra vi: Lie 2-algebras. 2003. arXiv:math/0307263.

[76] Edward Witten. Supersymmetry and Morse theory. *J. Diff. Geom*, 17(4):661--692, 1982.

[77] Marcus Berg, Cécile DeWitt-Morette, Shangjr Gwo, and Eric Kramer. The Pin groups in physics: C, P and T. *Reviews in Mathematical Physics*, 13(08):953--1034, 2001.

[78] Arindam Banerjee, Srujana Merugu, Inderjit S Dhillon, and Joydeep Ghosh. Clustering with Bregman divergences. *Journal of Machine Learning Research*, 6(Oct):1705--1749, 2005.

[79] Jean-Daniel Boissonnat, Frank Nielsen, and Richard Nock. Bregman Voronoi diagrams. *Discrete & Computational Geometry*, 44(2):281--307, 2010.

[80] Richard Palais. The symmetries of solitons. *Bulletin of the American Mathematical Society*, 34(4):339--403, 1997.

[81] Michael F. Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175--186, 1988.

[82] Jens Eisert, Christoph Simon, and Martin B Plenio. On the quantification of entanglement in infinite-dimensional quantum systems. *Journal of Physics A: Mathematical and General*, 35(17):3911--3923, 2002.

[83] Benoît Tuybens. Entanglement entropy of gauge theories. 2017.

[84] Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338--353, 1965.

[85] John C. Baez, Alexander E. Hoffnung, and Christopher Rogers. Categorified symplectic geometry and the classical string. *Communications in Mathematical Physics*, 293:701--725, 2010.

[86] Glenn Shafer and Vladimir Vovk. A tutorial on conformal prediction. *J. Mach. Learn. Res.*, 9:371-–421, 2008.

[87] Victor Chernozhukov, Kaspar Wüthrich, and Zhu Yinchu. Exact and robust conformal inference methods for predictive machine learning with dependent data. In *Proceedings of the 31st Conference On Learning Theory*, volume 75 of *Proceedings of Machine Learning Research*, pages 732--749. PMLR, 2018.

[88] Peter May. A note on the splitting principle. *Topology and Its Applications*, 153(4):605--609, 2005.

[89] Irina Markina. Group of diffeomorphisms of the unit circle as a principal $U(1)$-bundle.

[90] Sjoerd E. Crans. Localizations of transfors. 1998.

[91] Tom Leinster. Basic bicategories. 1998. arXiv:math/9810017.

[92] Alexander E. Hoffnung. Spans in 2-categories: A monoidal tricategory. 2011. arXiv:1112.0560.

[93] Eugenia Cheng and Nick Gurski. The periodic table of n-categories for low dimensions ii: Degenerate tricategories. 2007. arXiv:0706.2307.

[94] Mehmet B. Şahinoğlu, Dominic J. Williamson, Nick Bultinck, Michael Mariën, Jutho Haegeman, Norbert Schuch, and Frank Verstraete. Characterizing topological order with matrix product operators. 2014. arXiv:1409.2150.

[95] Dominic J. Williamson, Nick Bultinck, Michael Mariën, Mehmet B. Şahinoğlu, Jutho Haegeman, and Frank Verstraete. Matrix product operators for symmetry-protected topological phases: Gauging and edge theories. *Phys. Rev. B*, 94, 2016.

[96] Guifré Vidal. Efficient classical simulation of slightly entangled quantum computations. *Phys. Rev. Lett.*, 91, 2003.

[97] Aaron D. Lauda and Hendryk Pfeiffer. Open–closed strings: Two-dimensional extended TQFTs and Frobenius algebras. *Topology and its Applications*, 155(7):623--666, 2008.

[98] Domenico Fiorenza. An introduction to the Batalin-Vilkovisky formalism. 2004. arXiv:math/0402057v2.

[99] Stefan Cordes, Gregory Moore, and Sanjaye Ramgoolam. Lectures on 2d Yang-Mills theory, equivariant cohomology and topological field theories. arXiv:hep-th/9411210v2.

[100] Donald C. Ferguson. A theorem of Looman-Menchoff. `http://digitool.library.mcgill.ca/thesisfile111406.pdf`.

[101] Holger Lyre. Berry phase and quantum structure. arXiv:1408.6867.

[102] Florin Belgun. Gauge theory. `http://www.math.uni-hamburg.de/home/belgun/Gauge4.pdf`.

[103] Vladimir Itskov, Peter J. Olver, and Francis Valiquette. Lie completion of pseudogroups. *Transformation Groups*, 16:161--173, 2011.

[104] Richard Borcherds. Lie groups. `https://math.berkeley.edu/~reb/courses/261/`.

[105] Andrei Losev. From Berezin integral to Batalin-Vilkovisky formalism: A mathematical physicist's point of view. 2007.

[106] Edward Witten. Coadjoint orbits of the Virasoro group. *Comm. Math. Phys.*, 114(1):1--53, 1988.

[107] Sidney R. Coleman and Jeffrey E. Mandula. All possible symmetries of the S-matrix. *Phys. Rev.*, 159, 1967.

[108] Valter Moretti. Mathematical foundations of quantum mechanics: An advanced short course. *International Journal of Geometric Methods in Modern Physics*, 13, 2016.

[109] Antonio Michele Miti. Homotopy comomentum maps in multisymplectic geometry, 2021.

[110] Niclas Sandgren and Petre Stoica. On moving average parameter estimation. Technical Report 2006-022, Department of Information Technology, Uppsala University, 2006.

[111] John E. Roberts. Spontaneously broken gauge symmetries and superselection rules. 1974.

[112] Jean Gallier. Clifford algebras, Clifford groups, and a generalization of the quaternions, 2008. arXiv:0805.0311.

[113] Bozhidar Z. Iliev. Normal frames for general connections on differentiable fibre bundles. arXiv:math/0405004.

[114] Piotr Stachura. Short and biased introduction to groupoids. arXiv:1311.3866.

[115] Fosco Loregian. Coend calculus. arXiv:1501.02503.

[116] Frederic Schuller. Lectures on the geometric anatomy of theoretical physics. `https://www.youtube.com/channel/UC6SaWe7xeOp31Vo8cQG1oXw`.

[117] Nima Amini. Infinite-dimensional Lie algebras. `https://people.kth.se/~namini/PartIIIEssay.pdf`.

[118] Peter Selinger. Lecture notes on lambda calculus.

[119] Nigel Hitchin. Lectures on special Lagrangian submanifolds. `https://arxiv.org/abs/math/9907034v1`, 1999.

[120] Derek Sorensen. An introduction to characteristic classes. `http://derekhsorensen.com/docs/sorensen-characteristic-classes.pdf`, 2017.

[121] Arun Debray. Characteristic classes. `https://web.ma.utexas.edu/users/a.debray/lecture_notes/u17_characteristic_classes.pdf`.

[122] Jonathan R. Shewchuk. An introduction to the conjugate gradient method without the agonizing pain. Technical report, 1994.

[123] Chris Tiee. Contravariance, covariance, densities, and all that: An informal discussion on tensor calculus. `https://ccom.ucsd.edu/~ctiee/notes/tensors.pdf`, 2006.

[124] Will J. Merry. Algebraic topology. `https://www.merry.io/algebraic-topology`.

[125] Stacks project. `https://stacks.math.columbia.edu/`.

[126] The nlab. `https://ncatlab.org/nlab`.

[127] Wikipedia. `https://www.wikipedia.org/`.

[128] Joost Nuiten. Cohomological quantization of local prequantum boundary field theory. Master's thesis, 2013.

Index

D

E