Compendium of Mathematics & Physics

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Introduction

Goals

This compendium originated out of the necessity for a compact summary of important theorems and formulas during physics and mathematics classes at university. When the interest in more (and more exotic) subjects grew, this collection lost its compactness and became the chaos it now is. Although there should exist some kind of overall structure, it was not always possible to keep every section self-contained or respect the order of the chapters.

It should definitely not be used as a formal introduction to any subject. It is neither a complete work nor a fact-checked one, so the usefulness and correctness is not guaranteed. However, it can be used as a look-up table for theorems and formulas, and as a guide to the literature. To this end, each chapter begins with a list of useful references. At the same time, only a small number of statements are proven in the text (or appendices). This was done to keep the text as concise as possible (a failed endeavour). However, in some cases the major ideas underlying the proofs are provided.

Structure and conventions

Sections and statements that require more advanced concepts, in particular concepts from later chapters or (higher) category theory, will be labelled by the *clubs* symbol ♣. Some definitions, properties or formulas are given with a proof or an extended explanation whenever I felt like it. These are always contained in a blue frame to make it clear that they are not part of the general compendium. When a section uses notions or results from a different chapter at its core, this will be recalled in a green box at the beginning of the section.

Definitions in the body of the text will be indicated by the use of **bold font**. Notions that have not been defined in this summary but that are relevant or that will be defined further on in the compendium (in which case a reference will be provided) are indicated by *italic text*. Names of authors are also written in *italic*.

Objects from a general category will be denoted by a lower-case letter (depending on the context, upper-case might be used for clarity), functors will be denoted by upper-case letters and the categories themselves will be denoted by symbols in **bold font**. In the later chapters on physics, specific conventions for the different types of vectors will often be adopted. Vectors in Euclidean space will be denoted by a bold font letter with an arrow above, e.g. $\vec{\mathbf{a}}$, whereas vectors in Minkowski space (4-vectors) and differential forms will be written without the arrow, e.g. $\mathbf{a}$. Matrices and tensors will always be represented by capital letters and, dependent on the context, a specific font will be adopted.

Chapter 1

Calculus of Variations

The standard references for (global) variational calculus are Anderson (1992); Takens (1979).

Chapter Contents

1.1 Introduction

Definition 1.1.1 (Variational symmetry). Consider an integral quantity I defined by a Lagrangian function:

$$I_M = \int_M L(q, u, u_I)\, dq\,, \tag{1.1}$$

where u are (analytic) functions of the variables q. A transformation $(q, u) \longrightarrow (q', u')$ of the variables[1] is called a variational (or Noether) symmetry if it satisfies

$$\int_{M'} L(q', u', u'_I)\, dq' = \int_M L(q, u, u_I)\, dq \tag{1.2}$$

for arbitrary M.

Following *Lie*, the notion of a group of transformations is introduced.

Definition 1.1.2 (Finite continuous group). A collection of analytic functions, closed under inverses and composition, such that every function depends analytically on a finite number of parameters. In this chapter, these groups will be denoted by $\mathfrak{G}_k$, where $k \in \mathbb{N}$ is the number of independent parameters.

Remark 1.1.3. It should be clear that this is the same as a finite-dimensional Lie group (**??**).

Instead of structure parameters, one can also generalize to structure functions.

Definition 1.1.4 (Infinite continuous group). A collection of analytic functions, closed under inverses and composition, such that every function depends analytically on a finite number of arbitrary (analytic) functions. In this chapter, these groups will be denoted by $\mathfrak{G}_{\infty,k}$, where $k\mathbb{N}$ is the number of independent functions.

Remark 1.1.5. In physics terminology, the infinite groups would be obtained by 'gauging' a global symmetry group $\mathfrak{G}_k$.

Theorem 1.1.6 (Noether). *Consider an integral quantity I that is invariant under a continuous group $\mathfrak{G}$.*

I) If $\mathfrak{G}$ is finite continuous of the form $\mathfrak{G}_k$, there exist $k \in \mathbb{N}$ independent (linear) combinations among the Lagrangian expressions that are equal to divergences. Conversely, if there exist k independent combinations among the Lagrangian expressions that are divergences, then I is invariant under a group of the form $\mathfrak{G}_k$.

II) If $\mathfrak{G}$ is infinite continuous of the form $\mathfrak{G}_{\infty,k}$, there exist $k \in \mathbb{N}$ independent relations among the Lagrangian expressions and their derivatives[2]. Conversely, if k such relations exist,

[1] The transformations of the derivatives ∂u are induced by the ones for u.

[2] The order up to which the derivatives occur is equal to the order of derivatives up to which the transformations depend on the k arbitrary functions.

then I is invariant under a group of the form $\mathfrak{G}_{\infty,k}$.

Remark 1.1.7. In fact, the first theorem is also valid in the limit of an infinite number of parameters.

Property 1.1.8 (Improper relations). Divergence relations $\sum_i \psi_i \bar{\delta} u_i = \nabla \cdot B$ obtained in a variational problem with symmetry group $\mathfrak{G}_k$ can be classified into two groups:

- If the quantities B are linear combinations of Lagrangian expressions (and their derivatives) and divergence-free quantities, the divergence relations are said to be improper.
- In all other cases, the divergence relations are said to be proper.

Theorem 1.1.9 (Noether's third theorem). *A finite continuous symmetry group $\mathfrak{G}_k$ of an integral quantity is a subgroup of an infinite continuous symmetry group $\mathfrak{G}_{\infty,k}$ if and only if the divergence relations are improper.*

For Lagrangians describing 'point particles', where $M \subseteq \mathbb{R}$, the following result is obtained.

Example 1.1.10 (One dimension). Consider the following infinitesimal transformations

$$\begin{aligned} q^i &\longrightarrow q^i + \varepsilon\xi^i(q^k, t)\,, \\ t &\longrightarrow t + \varepsilon\tau(q^k, t)\,, \\ \dot{q}^i &\longrightarrow \dot{q}^i + \varepsilon(\dot{\xi}^i - \dot{q}^i\dot{\tau})\,, \end{aligned} \tag{1.3}$$

where the transformation of the 'velocities' on the last line, induced by the coordinate transformations, is called a **prolongation** (see also Definition 1.3.3 further below). These transformations generate Noether symmetries if they leave the Lagrangian invariant up to a total derivative (at first order) for every subinterval $[t_0, t_1] \subseteq [a, b]$ and for some function $f(q, t)$:

$$\int_{\tilde{t}_0}^{\tilde{t}_1} L(\tilde{q}, \dot{\tilde{q}}, \tilde{t})\, d\tilde{t} = \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt + \varepsilon \int_{t_0}^{t_1} \frac{\mathrm{d}f}{\mathrm{d}t}\, dt + O(\varepsilon^2)\,. \tag{1.4}$$

This is equivalent to requiring that the transformation is a solution of the following differential equation (sometimes called the **Rund–Trautman identity**):

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\xi^i + \frac{\partial L}{\partial \dot{q}^i}(\dot{\xi}^i - \dot{q}^i\dot{\tau}) + L\dot{\tau} = \dot{f}\,. \tag{1.5}$$

By Noether's (first) theorem one obtains, for every such symmetry, a conserved quantity of the following form

$$F := f - \left[L\tau + \frac{\partial L}{\partial \dot{q}^i}(\xi^i - \dot{q}^i\tau)\right]. \tag{1.6}$$

It is important to note that the left-hand side of Eq. (1.5) is simply the Lie derivative of the Lagrangian form Ldt with respect to the vector field that generates the transformations (1.3). (This will turn out to be an important concept in the calculus of variations. See Definition 1.3.40 further on.)

1.2 Jet bundles

1.2.1 Topology

Although the following constructions can be defined in the general context of fibred manifolds, they will only be considered in the case of smooth fibre bundles. Only the notion of a jet will be defined in general for functions between smooth manifolds.

Definition 1.2.1 (Jet). Consider two smooth manifolds M,N and let $r \in \mathbb{N}$. Smooth functions $f,g \in C^\infty(M,N)$ with local coordinates (f^i) and (g^i) are said to define the same r-jet at a point $p \in M$ if and only if

$$\left.\frac{\partial^\alpha f^i}{\partial x^\alpha}\right|_p = \left.\frac{\partial^\alpha g^i}{\partial x^\alpha}\right|_p \tag{1.7}$$

for all $0 \leq i \leq \dim(M)$ and every multi-index α with $0 \leq |\alpha| \leq r$. It is clear that this defines an equivalence relation. The r-jet at $p \in M$ of a representative f is denoted by $j^r_p f$. The integer r is called the **order** of the jet. The set of all r-jets of functions between manifolds M and N is denoted by $J^r(M,N)$.

Definition 1.2.2 (Jet projections). Let M,N be smooth manifolds and consider the jet space $J^r(M,N)$. The **source** and **target projections** are defined as follows:

$$\begin{aligned} \pi_r &: J^r(M,N) \rightarrow M : j^r_p f \mapsto p\,, \\ \pi_{r,0} &: J^r(M,N) \rightarrow N : j^r_p f \mapsto f(p)\,. \end{aligned} \tag{1.8}$$

One can also define a **k-jet projection** $\pi_{r,k}$ as the map

$$\pi_{r,k} : J^r(M,N) \rightarrow J^k(M,N) : j^r_p f \mapsto j^k_p f\,, \tag{1.9}$$

where $k \leq r$. The k-jet projections satisfy the transitivity property $j_{k,m} = j_{r,m} \circ j_{k,r}$.

Definition 1.2.3 (Prolongation). Let $f : M \rightarrow N$ be a smooth function. The r-jet prolongation $j^r f$ is defined as the following map:

$$j^r f : M \rightarrow J^r(M,N) : p \mapsto j^r_p f\,. \tag{1.10}$$

Property 1.2.4 (Topology). Every diffeomorphism $f : M \rightarrow M'$ induces a morphism of jet spaces $J^r(g,N) : J^r(M',N) \rightarrow J^r(M,N)$ by pullback. Similarly, every smooth function $g : N \rightarrow N'$ induces a morphism of jet spaces $J^r(M,g) : J^r(M,N) \rightarrow J^r(M,N')$ by pushforward.

Now, consider two charts (U,φ) and (V,ψ) of M and N, respectively. Construct the morphism

$$J^r(\varphi^{-1},\psi) := J^r(\varphi^{-1},V)\circ J^r(\varphi(U),\psi) : J^r(U,V)\to J^r\big(\varphi(U),\psi(V)\big)\,. \tag{1.11}$$

This morphism is invertible and, hence, can be used as a local chart. The topology on $J^r(M,N)$ is induced by the manifold topology corresponding to the maximal atlas of all such charts.

Definition 1.2.5 (Whitney C^k-topology). Let M,N be two smooth manifolds and consider the manifold of k-jets $J^k(M,N)$. A basis for the Whitney C^k-topology on $C^\infty(M,N)$ is given by the sets

$$S^k(U) := \{f\in C^\infty(M,N)\mid j^kf\in U\}\,, \tag{1.12}$$

where U is open in $J^k(M,N)$.

Property 1.2.6. When the manifold M is compact, the Whitney and compact-open topologies on $C^\infty(M,N)$ coincide. In general, the Whitney topology is the topology of global uniform convergence.

Remark 1.2.7 (Fibre bundles). The above definitions can also be used to define jet manifolds of (local) sections of bundles. One just sets $M=B$ and $N=E$ and restricts to the subset satisfying

$$j^r_{\sigma(p)}\pi\circ j^r_p\sigma = j^r_p\mathbb{1}_M\,. \tag{1.13}$$

This is just the prolongation of the definition of sections $\pi\circ\sigma=\mathbb{1}_M$.

The r-jet bundle corresponding to the projection π is defined as the triple $(J^r(E),B,\pi_r)$. The bundle charts (U_i,φ_i) on E define induced bundle charts on $J^r(E)$ in the following way:

$$U_i^r := \{j^r_p\sigma\mid\sigma(p)\in U_i\} \tag{1.14}$$

$$\varphi_i^r := \left(x^k,u^\alpha,\left.\frac{\partial^I u^\alpha}{\partial x^I}\right|_p\right), \tag{1.15}$$

where I is a multi-index such that $0\leq|I|\leq r$. The partial derivatives

$$\left.\frac{\partial^I u^\alpha}{\partial x^I}\right|_p$$

are called the **derivative coordinates** on $J^r(E)$.

Definition 1.2.8 (Holonomic section). Consider a fibre bundle $\pi:E\to M$. A (local) section σ of π gives rise to a (local) section of π_r given by the r-jet prolongation of σ. Sections of $J^r(E)$ that lie in the image of j^r are said to be holonomic.

Definition 1.2.9 (Infinite jet bundle). The inverse limit (**??**) of the projections $\pi_{k,k-1} : J^k(E) \to J^{k-1}(E)$. It can be shown (in an algebro-geometric fashion) that the smooth functions on the infinite jet bundle $J^\infty(E)$ are (at least locally) given by smooth functions on some finite jet bundle. This just means that the infinite jet bundle is defined by taking its algebra of smooth functions to be the direct limit of those on the finite jet bundles. By extension, it can be shown that any smooth morphism $J^\infty(E) \to E'$ into a finite-dimensional manifold factorizes through a finite jet bundle. Furthermore, a map $Q \to J^\infty(E)$ or $J^\infty(E) \to J^\infty(E')$ is said to be smooth if the composition with any smooth map is again smooth.

Property 1.2.10 (Jet comonad). The assignment of jet bundles constitutes a comonad: $J^\infty : \mathbf{Bundle}(X) \to \mathbf{Bundle}(X)$. The counit sends the prolongation of a section to the section evaluated at a point. The comultiplication reshuffles derivatives.

1.2.2 Contact structure

Every jet manifold J^k carries a natural constact structure (**??**).

Definition 1.2.11 (Contact form). Given a fibre bundle $\pi : E \to M$ and one of its jet bundles $J^k(E)$, a differential form $\omega \in \Omega^\bullet(J^k(E))$ is called a contact form if it is annihilated by all jet prolongations, i.e. $(j^{k+1}\sigma)^*\omega = 0$ for all sections $\sigma \in \Gamma(E)$.

The space of such contact forms generates a differential ideal and, in turn, defines a distribution, called the **Cartan distribution**. For finite-order jet spaces, this distribution is completely nonintegrable. However, for infinite jet bundles (where Frobenius's theorem need not hold), the distribution becomes involutive and integrable.

Locally, the differential ideal is generated by the following contact forms:

$$\theta^\alpha_I := \mathbf{d}u^\alpha_I - \sum_\mu u^\alpha_{I,\mu}\mathbf{d}x^\mu\,, \tag{1.16}$$

where the x^μ and u^α_I are the independent and dependent variables, respectively.

Property 1.2.12 (Cartan connection). There exists a connection on the infinite jet bundle $J^\infty(E)$ where the horizontal subbundle is exactly given by the Cartan distribution. This connection can be shown to be flat.

1.2.3 Differential operators

Alternative Definition 1.2.13 (Differential operator). Let E_1, E_2 be smooth fibre bundles over the same base manifold M. A differential operator $\widetilde{D} : E_1 \to E_2$ is a bundle morphism $J^\infty(E_1) \to E_2$. This induces a map of sections $D : \Gamma(E_1) \to \Gamma(E_2)$ such that $D = \widetilde{D} \circ j^\infty$. It is said to be of **order** $k \in \mathbb{N}$ if it factors through the k-jet bundle, i.e. if j^∞ can be replaced by j^k. If the bundle morphism $\widetilde{D}$ is a vector bundle morphism, the differential operator is said to be **linear**.

This definition can easily be phrased in terms of the jet comonad. The category of differential operators is equivalent to the (co)Kleisli category (**??**) of the jet comonad J^∞.

Property 1.2.14 (Formal adjoints). Consider two differential operators

$$D, D^\dagger : \Gamma(E) \to \Gamma\big(E^* \otimes \Lambda^{\dim(M)}(M)\big)\,. \tag{1.17}$$

These operators are said to be formally adjoint if there exists a bilinear differential operator $K : \Gamma(E) \otimes \Gamma(E) \to \Omega^{\dim(M)}(M)$ such that the following condition is satisfied for all sections σ_1, σ_2 of E:

$$\langle D(\sigma_1), \sigma_2\rangle - \langle \sigma_1, D^\dagger(\sigma_2)\rangle = \mathsf{d}K(\sigma_1, \sigma_2)\,. \tag{1.18}$$

This formula can be interpreted as a generalization of Green's identities. In the case where M is compact, Stokes' theorem **??** shows that D and $D^\dagger$ are related through integration by parts.

Definition 1.2.15 (Generalized vector field). Consider a vector bundle $\pi : E \to M$. A generalized vector field on M is a 'vector field' whose components are, locally, given by smooth functions on the infinite jet bundle $J^\infty(E)$, i.e. it is a smooth map $X : J^\infty(E) \to TM$ and, hence, a smooth map $X : J^k(E) \to TM$ for some $k \in \mathbb{N}$ such that $X(\sigma) \in T_{\pi_k(\sigma)}M$.

1.3 Variational bicomplex ♣

In this section, the language of jet bundles, as introduced in the previous section, will be used to rephrase the classical theory of variational calculus in more general geometric terms. A smooth function on the infinite jet bundle will be denoted by $f[u]$, i.e. the arguments will be written inside square brackets.

1.3.1 Differential structure

A smooth fibre bundle $\pi : E \to M$ over a base manifold M will be considered.[3] First, the de Rham operator $\mathbf{d}$ on the infinite jet bundle $J^\infty(E)$ is decomposed in a horizontal and a vertical part:

$$\mathbf{d} = \mathsf{d} + \delta\,. \tag{1.19}$$

The **horizontal derivative** d lifts the de Rham differential on M to E (hence the name). On smooth functions in $C^\infty(J^\infty(E))$, it acts as follows:

$$\mathsf{d}f := (D_\mu f)\mathsf{d}x^\mu\,, \tag{1.20}$$

[3] In fact, the fibre bundle can be replaced with any fibred manifold.

where the **total derivative**[4]

$$D_\mu f := \frac{\partial f}{\partial x^\mu} + \frac{\partial f}{\partial u^\alpha} u^\alpha_{,\mu} + \frac{\partial f}{\partial u^\alpha_{,\nu}} u^\alpha_{,\nu\mu} + \cdots \tag{1.21}$$

is introduced. The horizontal de Rham operator can be extended to all of $\Omega^\bullet(J^\infty(E))$ through the Leibniz property and the condition

$$\mathsf{d} \circ \delta = -\delta \circ \mathsf{d}\,, \tag{1.22}$$

which follows from the nilpotency of the differentials. The differentials d, δ turn $\Omega^\bullet(J^\infty(E))$ into a bigraded complex, called the **variational bicomplex**.

Remark 1.3.1. Some authors use the term variational bicomplex for the bicomplex of **local forms** $\Omega^\bullet_{\mathrm{loc}}(M \times \Gamma(E))$, which is defined as the image of $\Omega^\bullet(J^\infty(E))$ under the prolongation map $M \times \Gamma(E) \to J^\infty(E)$. This way, they can work with forms over the (trivial) field bundle, while maintaining the property that all objects only depend on finite-order jets. Furthermore, when working in full generality the de Rham complex over M is twisted by the orientation bundle (**??**).

Definition 1.3.2 (Local Lagrangian). A top-degree horizontal form on $J^\infty(E)$. Because $\Omega^{\dim(M),0}(J^\infty(E))$ is one-dimensional, such forms are proportional to the volume form:

$$\mathbf{L} = L\,\mathsf{Vol}\,, \tag{1.23}$$

where L is a smooth function on the infinite jet bundle. The function L is called the **Lagrangian density**. By its very nature, this implies that L locally only depends on partial derivatives up to some finite order.

For any such horizontal form, one can define a functional on $\Gamma_c(M)$ as follows:

$$S : \phi \mapsto \int_M (j^\infty \phi)^* \mathbf{L}\,. \tag{1.24}$$

Functions of this form are called **local functionals**.

Now, consider a (generalized) vector field X on the total space E. This vector field can be lifted to $J^k(E)$ in a canonical way.

Definition 1.3.3 (Prolongation of vector fields). Given a generalized vector field X on a fibre bundle $\pi : E \to M$, there exists a unique vector field $j^k E$ on the jet bundle $J^k(E)$ defined by the following conditions:

1. X and $j^k X$ coincide on $C^\infty(E)$.
2. $j^k X$ preserves the contact ideal, i.e. if θ is a contact form, then $\mathcal{L}_{j^k X}\theta$ is also a contact form.

[4] In fact, this formula is virtually the same as the one for the true total derivative. However, partial derivatives are replaced by jet coordinates.

Locally, the prolongation of a vector field

$$X = X^\mu \partial_\mu + X^\alpha \partial_\alpha \tag{1.25}$$

is given by

$$(j^k X)^\alpha_I = D_I(X^\alpha - u^\alpha_\mu X^\mu) + u^\alpha_{\mu I} X^\mu \tag{1.26}$$

for all $|I| \leq k$.

A similar definitions exists for vector fields on the base manifold.

Definition 1.3.4 (Total vector field). Given a generalized vector field X on M, its total vector field tot$X$ is defined by the following conditions:

1. X and tot$X$ coincide on $C^\infty(M)$, and
2. tot$X \lrcorner\, \omega = 0$ if ω is a contact form.

An explicit formula can be obtained by replacing partial derivatives by total derivatives:

$$X = X^\mu \partial_\mu \longrightarrow \mathrm{tot}X = X^\mu D_\mu\,. \tag{1.27}$$

In particular, the total vector fields associated to a coordinate-induced basis ∂_μ are exactly the total derivatives D_μ.

Definition 1.3.5 (Evolutionary vector field). A generalized vector field that projects to 0 on the base manifold, i.e. a π-vertical (generalized) vector field. The space of evolutionary vector fields on E is denoted by $\mathsf{Ev}(J^\infty(E))$.

The prolongation of an evolutionary vector field to $J^\infty(E)$ still projects to 0 on M. By extension, all vector fields on $J^\infty(E)$ that preserve the contact ideal and project to 0 on M are called evolutionary vector fields. Such vector fields are of the form

$$X = X^\alpha_I[u]\partial^I_\alpha\,. \tag{1.28}$$

The name stems from the fact that these vector fields define PDEs that (locally) describe the evolution of the fibres.

The prolongation of an evolutionary vector field can be written as follows:

$$j^\infty X = \sum_{|I|=0}^{+\infty} (D_I X^\alpha)\partial^I_\alpha\,. \tag{1.29}$$

Property 1.3.6. By writing out Cartan's magic formula **??** with respect to **d**, one can prove that the prolongation of an evolutionary vector field also satisfies this formula with respect to δ and that $\iota_{j^\infty X}$ and d anticommute.

Property 1.3.7 (Evolutionary decomposition). Consider a generalized vector field

$$X = X^\mu \partial_\mu + X^\alpha \partial_\alpha \tag{1.30}$$

on E. By extending the tot-construction to generalized vector fields as

$$\mathsf{tot}X := \mathsf{tot}(\pi_* X), \tag{1.31}$$

one can define the evolutionary part of X as follows:

$$X_{\mathsf{ev}} := X - (\pi_{\infty,0})_*(\mathsf{tot}X). \tag{1.32}$$

Locally, this can be written as

$$X_{\mathsf{ev}} = \big(X^\alpha - u^\alpha_\mu X^\mu\big)\partial_\alpha. \tag{1.33}$$

Using this definition, the prolongation of X can be decomposed as follows:

$$j^\infty X = j^\infty X_{\mathsf{ev}} + \mathsf{tot}X. \tag{1.34}$$

The evolutionary part is sometimes also called the **characteristic** of X.

To close this section, the Cartan calculus of the variational bicomplex is summarized.

Formula 1.3.8 (Cartan calculus). Let X, Y denote a generalized vector field and an evolutionary vector field, respectively.

1. Exterior derivatives:

$$\mathsf{d}x^\mu = \mathbf{d}x^\mu, \tag{1.35}$$
$$\delta u^\alpha_I = \mathbf{d}u^\alpha_I - u^\alpha_{I\mu}\mathsf{d}x^\mu. \tag{1.36}$$

2. Interior product:

$$\{\mathsf{d}, \iota_{j^\infty Y}\}_+ = 0. \tag{1.37}$$

3. Lie derivatives:

$$\mathcal{L}_{j^\infty X} = \mathbf{d}\iota_{j^\infty X} + \iota_{j^\infty X}\mathbf{d}, \tag{1.38}$$
$$\mathcal{L}_{j^\infty Y} = \delta\iota_{j^\infty Y} + \iota_{j^\infty Y}\delta, \tag{1.39}$$
$$[\mathsf{d}, \mathcal{L}_{j^\infty Y}] = [\delta, \mathcal{L}_{j^\infty Y}] = 0. \tag{1.40}$$

1.3.2 Euler operators

Property 1.3.9 (Total differential operators). A differential operator $P : \mathsf{Ev}(J^\infty(E)) \rightarrow \Omega^\bullet(J^\infty(E))$ that can locally be written as

$$P(X) = \sum_{|I|=0}^{k} \big(D_I X^\alpha\big) P^I_\alpha \tag{1.41}$$

for some differential forms P^I_α. By (formally) integrating by parts, this can locally be written as

$$P(X) = \sum_{|I|=0}^{k} D_I\big(X^\alpha Q^I_\alpha\big)\,, \tag{1.42}$$

where the smooth forms Q^I_α can be expressed as follows:

$$Q^I_\alpha = \sum_{|J|=0}^{k-|I|} \binom{|I|+|J|}{|J|} (-1)^{|J|} D_J P^{IJ}_\alpha\,. \tag{1.43}$$

The zeroth-order part Q_α also defines a total differential operator:

$$E_P(X) := X^\alpha Q_\alpha\,. \tag{1.44}$$

This operator is called the **Euler operator** (associated to P).

Property 1.3.10 (Decomposition of total differential operators). Consider a total differential operator $P : \mathsf{Ev}(J^\infty(E)) \rightarrow \Omega^{\mathrm{dim}(M),k}(J^\infty(E))$. The Euler operator E_P is the unique globally defined (zeroth order) operator such that, on each chart U, one can find a total differential operator R_U that satisfies the following equation:

$$P(X) = E_P(X) + \mathrm{d}R_U(X)\,. \tag{1.45}$$

The operator R_U can locally be expressed as follows:

$$R_U(X) = \sum_{|I|=0}^{k-1} D_I\big(X^\alpha D_\mu \lrcorner\, Q^{\mu I}_\alpha\big)\,. \tag{1.46}$$

The decomposition above can, in fact, be shown to hold globally. However, in that case, the expression of R cannot be expressed easily in terms of the coefficients of P (only for P of order 2 does a canonical expression exist).

Example 1.3.11 (Euler–Lagrange operator). Consider a local Lagrangian $\mathbf{L}$. This form induces a total differential operator as follows:

$$P_{\mathbf{L}}(X) := \mathcal{L}_{j^\infty X}\mathbf{L} = \sum_{|I|=0}^{k} (D_I X^\alpha)(\partial^I_\alpha L)\,\mathsf{Vol}\,. \tag{1.47}$$

The coefficients (1.43) associated to this operator are given by the following formula (to turn these coefficients into true differential forms, one should multiple them by the volume form):

$$E^I_\alpha(L) := \sum_{|J|=0}^{k-|I|} \binom{|I|+|J|}{|J|} (-1)^{|J|} D_J\left(\partial^{IJ}_\alpha L\right). \tag{1.48}$$

The induced Euler operator is exactly the Euler–Lagrange operator associated to variational problems. For this reason, and the fact that they are induced by a Lie derivative, the coefficients E^I_α are called **Lie–Euler operators**.

Given a local Lagrangian $\mathbf{L}$, its **Euler–Lagrange form** is defined as

$$\delta_{\text{EL}}\mathbf{L} := E_\alpha(L)\delta u^\alpha \wedge \mathsf{Vol}\,. \tag{1.49}$$

An explicit formula for the Euler–Lagrange derivative is given by the following formula (this is just Eq. (1.48) for $|I| = 0$):

$$\delta_{\text{EL}}L := \left(\frac{\partial L}{\partial u^\alpha} - D_\mu \frac{\partial L}{\partial u^\alpha_\mu} + \cdots\right)\delta u^\alpha\,. \tag{1.50}$$

The set of functions that satisfy $\delta_{\text{EL}}L = 0$ is called the **shell**. The functions for which also all higher-order derivatives vanish, i.e. the elements of $\{x \in J^\infty(E) \mid \forall I : D_I\delta_{\text{EL}}L(x) = 0\}$, are said to be **on-shell**.

Property 1.3.12 (Naturality). The Euler–Lagrange operators are natural operators in the following sense:

- $\delta_{\text{EL}}|_p$ only depends on the germ at $p \in J^\infty(E)$.
- If $\phi : E \to E'$ is fibre-preserving, then

$$\delta_{\text{EL}}((j^\infty\phi)^*\mathbf{L}) = (j^\infty\phi)^*(\delta_{\text{EL}}(\mathbf{L}))\,. \tag{1.51}$$

It can be shown that δ_{EL} is the unique (up to scaling) linear, natural differential operator from $\Omega^{\dim(M),0}(J^\infty(E))$ to $\Omega^{\dim(M),1}(J^\infty(E))$.

The following property generalizes the property that the Euler–Lagrange equations remain invariant under addition of a divergence to the Lagrangian.

Property 1.3.13 (Divergences). If a smooth function is locally an order-k divergence, i.e. $f = D_IA^I$ for smooth functions A^I and $|I| = k$, the Lie–Euler operators E^J_α vanish on f for all $|J| < k$.

Property 1.3.14 (Local variational formula). Decomposition 1.3.10 shows that the Euler–Lagrange operator is the unique operator such that locally, for all local Lagrangians $\mathbf{L}$ and all evolutionary vector fields X, the following equation holds

$$\mathcal{L}_{j^\infty X}\mathbf{L} = j^\infty X \lrcorner\, \delta_{\text{EL}}(\mathbf{L}) + \mathrm{d}(j^\infty X \lrcorner\, \gamma) \tag{1.52}$$

for some $\gamma \in \Omega^{\dim(M)-1,1}(J^\infty(E))$.

1.3.3 Functional complex

Example 1.3.15 (Interior Euler operator). For every smooth form $\omega \in \Omega^{p,q}(J^\infty(E))$, one can define a total differential operator as follows:

$$P_\omega(X) := j^\infty X \lrcorner\, \omega\,. \tag{1.53}$$

As for the previous example, this operator induces (higher) Euler operators:

$$F^I_\alpha(\omega) := \sum_{|J|=0}^{k-|I|} \binom{|I|+|J|}{|J|} (-1)^{|J|} D_J\big(\partial^{IJ}_\alpha \lrcorner\, \omega\big)\,. \tag{1.54}$$

Since, in this case, they arise from interior multiplication, they are called interior Euler operators. For $p = \mathrm{dim}(M)$, one again obtains a globally defined Euler operator (also called the interior Euler operator):

$$I(\omega) := \frac{1}{q}\delta u^\alpha \wedge F_\alpha(\omega)\,. \tag{1.55}$$

The interior Euler operator defines a sequence of spaces, the so-called spaces of **functional forms**, as follows:

$$\mathcal{F}^q(J^\infty(E)) := \big\{\omega \in \Omega^{\mathrm{dim}(M),q}(J^\infty(E)) \,|\, I(\omega) = \omega\big\}\,. \tag{1.56}$$

Property 1.3.16. The interior Euler operator has the following important properties:

- I is a projection $I^2 = I$.
- I vanishes on (locally) d-exact forms.
- $\delta_V := I \circ \delta$ endows $\mathcal{F}^\bullet(J^\infty(E))$ with the structure of a cochain complex. It is sometimes called the **Helmholtz operator**.
- $\delta_{\mathrm{EL}}\mathbf{L} = \delta_V \mathbf{L}$ for all local Lagrangians $\mathbf{L}$.

Corollary 1.3.17. The Euler–Lagrange operator δ_{EL} vanishes on locally d-exact forms and commutes with the Lie derivative of projectable vector fields.

The degree-2 functional forms also admit a local characterization:

Property 1.3.18 (Local expression for $\mathcal{F}^2$). Consider a collection of smooth functions $A^I_{\alpha\beta} \in C^\infty(J^\infty(U))$ on some chart $U \subset E$ that satisfy

$$A^I_{\alpha\beta} = (-1)^{|I|+1} A^I_{\beta\alpha}\,. \tag{1.57}$$

The local $(\mathrm{dim}(M), 2)$-form

$$w^{|I|} := \delta u^\alpha \wedge \big(A^I_{\alpha\beta}\delta u^\beta_I + D_I(A^I_{\beta\alpha}\delta u^\beta)\big) \wedge \mathrm{Vol} \tag{1.58}$$

is an element of $\mathcal{F}^2(J^\infty(U))$. Furthermore, every degree-2 functional form can locally be expressed as a sum of the form

$$\omega = w^0 + w^1 + w^2 + \cdots\,. \tag{1.59}$$

Although local characterizations for functional forms $\omega \in \mathcal{F}^q(J^\infty(E))$ with $q \geq 2$ are still not well understood, there exists a more high-level characterization.

Property 1.3.19 (General characterization). A form $\omega \in \Omega^{\mathrm{dim}(M),q}(J^\infty(E))$ is functional if and only if there exists a linear, **formally skew-adjoint**[5] differential operator

$$P : \mathrm{Ev}(J^\infty(E)) \to \Omega^{\mathrm{dim}(M),q-1}(J^\infty(E))$$

such that

$$\omega = \delta u^\alpha \wedge P_\alpha\,. \tag{1.60}$$

This representation is unique if it exists.

The forms in $\mathcal{F}^q$ are said to be functional due to the following property.

Property 1.3.20 (Functionals). To every smooth k-form $\omega \in \Omega^\bullet(J^\infty(E))$, compact subset $K \subset E$ and $q := k - \mathrm{dim}(M)$ generalized vector fields on E, one can assign a functional on $\Gamma(U)$, where U is a chart containing K, by the following formula:

$$W_\omega(X_1,\dots,X_q)[\sigma] := \int_K (j^\infty\sigma)^*\omega(j^\infty X_1,\dots,j^\infty X_q)\,. \tag{1.62}$$

The integer q is called the **degree** of W_ω. In general, these functionals are invariant under the addition of a d-exact form. However, the assignment $\omega \mapsto W_\omega$ is a bijection for fixed K and X_i.

Construction 1.3.21. The differential δ_V on $\mathcal{F}^\bullet$ induces a differential on the space of functionals of the above form:

$$\delta W_\omega := W_{\delta_V\omega}\,. \tag{1.63}$$

An equivalent definition can be given by a formula similar to **??**, where an evolutionary vector field X acts on a degree-0 functional by Lie derivation:

$$X(W_\omega[\sigma]) := \int_V (j^\infty\sigma)^*(\mathcal{L}_{j^\infty X}\omega)\,. \tag{1.64}$$

1.3.4 Structure of the bicomplex

A well-defined morphism of variational bicomplexes should preserve the bigrading of forms, but this will clearly not be the case in general. Therefore, a 'projected pullback' can be introduced:

$$\Phi^\sharp : \Omega^{p,q}(J^\infty(E')) \to \Omega^{p,q}(J^\infty(E)) : \omega \mapsto \pi^{p,q}(\Phi^*\omega)\,. \tag{1.65}$$

[5] This is a generalization of Property 1.2.14. P is said to be formally skew-adjoint if there exists an $(m-1,q-1)$-form K such that

$$j^\infty X \lrcorner P(Y) + j^\infty Y \lrcorner P(X) = \mathrm{d}K \tag{1.61}$$

for all $X, Y \in \mathrm{Ev}(J^\infty(E))$.

Here, the projection $\pi^{p,q}$ is the projection of the variational bicomplex, not the one of jet bundles (which is denoted by subscripts).

Remark 1.3.22. The projection $\pi^{p,q} : \Omega^{p+q}(J^\infty(E)) \to \Omega^{p,q}(J^\infty(E))$ is defined by substituting $\mathbf{d}u^\alpha_I \longrightarrow \delta u^\alpha_I + u^\alpha_{\mu I}\mathrm{d}x^\mu$ and then projecting onto the correct horizontal and vertical degrees. Note that this does not preserve the order of forms due to the presence of the factor $u^\alpha_{\mu I}$.

The main argument for introducing the projected pullback is that functionals of the form of Property 1.3.20 only care about (m, q)-forms for a specific q (in particular **action functionals**, i.e. $q = 0$).

Formula 1.3.23 (Local Lagrangians). For local Lagrangians $\mathbf{L} \equiv L\,\mathrm{Vol}_{E'}$, the projected pullback acts as follows:

$$\Phi^\sharp \mathbf{L} = (L \circ \Phi)\det(D_\mu f^\mu)\,\mathrm{Vol}_E\,, \tag{1.66}$$

where $\pi'(\Phi[u]) = (f^\mu)$. So, one obtains the usual formula for pullbacks of top-dimensional forms, but with partial derivatives replaced by total derivatives.

An important property of the de Rham differential is its naturality (**??**). The following property states the 'naturality' of the different operators on the variational bicomplex with respect to the projected pullback.

Property 1.3.24. Consider a morphism $\Phi : J^\infty(E) \to J^\infty(E')$.

- $\Phi^\sharp$ and δ commute if and only if Φ covers a morphism of the base manifolds.
- $\Phi^\sharp$ and d commute if and only if Φ^* is a contact transformation.
- $\Phi^\sharp$ commutes with both differentials if and only if it coincides with Φ^*

Furthermore, the projected pullback defines a contravariant functor on the subcategory on morphisms that satisfy at least one of the above properties.

The following property gives an infinitesimal analogue of the above considerations.

Property 1.3.25. Consider a vector field X on $J^\infty(E)$. Its Lie derivative will, in general, not respect the bigrading of the variational bicomplex (unless X is evolutionary) and, therefore, the 'projected Lie derivative' is introduced:

$$\mathcal{L}^\sharp_X : \Omega^{p,q}(J^\infty(E)) \to \Omega^{p,q}(J^\infty(E)) : \omega \mapsto \pi^{p,q}(\mathcal{L}_X\omega)\,. \tag{1.67}$$

This operator satisfies the following properties:

- It commutes with δ if and only if X is π_∞-related to a vector field on M.
- It commutes with d if and only if X is the prolongation of a generalized vector field on E.

Note that this does not simply follow from the previous property since X does not necessarily define a flow on $J^\infty(E)$.

Aside from the differentials on the variational bicomplex, one should also look at the structure of the functional complex $(\mathcal{F}^\bullet, \delta_V)$ in (1.56). It can be shown that requiring both $[I, \Phi^\sharp] = 0$ and $[\delta, \Phi^\sharp] = 0$ is very restrictive. Furthermore, a complete characterization of those morphisms Φ that satisfy only $[I, \Phi^\sharp] = 0$ is not fully understood. However, the infinitesimal version is easier to handle since it only involves linear equations:

Property 1.3.26. Let $n \in \mathbb{N}$ be the rank of E and consider a vector field X on $J^\infty(E)$.

- If $n = 1$, then $\mathcal{L}_X^\sharp$ commutes with I if and only if X is locally the prolongation of a generalized vector field on E of the form
$$Y = -\frac{\partial S}{\partial u_\mu}\partial_\mu + \left(S - u_\mu\frac{\partial S}{\partial u_\mu}\right)\partial_u\,, \tag{1.68}$$
where S is a function on the first jet bundle $J^1(U)$.
- If $n > 1$, then $\mathcal{L}_X^\sharp$ commutes with I if and only if X is the prolongation of a vector field on E.

The next step is to define operators that do preserve the functional complex. To this end, the projection property of I is used:

$$\begin{aligned} \Phi^\natural &: \Omega^{m,q}(J^\infty(E)) \to \mathcal{F}^q(J^\infty(E')) : \omega \mapsto (I \circ \Phi^\sharp)\omega\,,\\ \mathcal{L}_X^\natural &: \Omega^{m,q}(J^\infty(E)) \to \mathcal{F}^q(J^\infty(E)) : \omega \mapsto (I \circ \mathcal{L}_X^\sharp)\omega\,. \end{aligned} \tag{1.69}$$

Property 1.3.27. These operators satisfy the following properties:

- If Φ is a contact transformation, then $\Phi^\natural$ commutes with both I and δ_V.
- If X is a generalized vector field on E, then $\mathcal{L}_{j^\infty X}^\natural$ commutes with both I and δ_V.
- $\Phi^\natural$ preserves locally variational forms.

The projected and functionally projected operators also satisfy the following relations.

Property 1.3.28 (Euler–Lagrange operator). Consider a local Lagrangian $\mathbf{L}$. If Φ is a contact transformation, then
$$\delta_{\mathrm{EL}}(\Phi^\sharp\mathbf{L}) = \Phi^\natural(\delta_{\mathrm{EL}}\mathbf{L})\,. \tag{1.70}$$

If X is a generalized vector field, then
$$\delta_{\mathrm{EL}}(\mathcal{L}_{j^\infty X}^\sharp\mathbf{L}) = \mathcal{L}_{j^\infty X}^\natural(\delta_{\mathrm{EL}}\mathbf{L})\,. \tag{1.71}$$

Formula 1.3.29 (Functionally projected Lie derivative). Let X be a generalized vector field on E and let $\omega \in \mathcal{F}^\bullet(J^\infty(E))$ be a functional form.

$$\mathcal{L}^\natural_{j^\infty X}\omega = \delta_V(j^\infty X_{\mathrm{ev}} \lrcorner\, \omega) + I(j^\infty X_{\mathrm{ev}} \lrcorner\, \delta_V\omega) \tag{1.72}$$

For projectable vector fields, one can replace the left-hand side by the ordinary Lie derivative $\mathcal{L}_{j^\infty X}\omega$.

In the remainder of this section, the homological properties of the variational bicomplex over a local chart (or equivalently, over a trivial bundle) will be studied. To prove the acyclicity of the various subcomplexes, the usual approach of finding a null-homotopy (**??**) will be followed, i.e. a cochain map will be found $h : C_\bullet \rightarrow C_\bullet$ such that

$$\mathbb{1} = \mathrm{d} \circ h + h \circ \mathrm{d}\,. \tag{1.73}$$

Property 1.3.30 (Vertical complex is exact). Homotopy operators $h_V^{p,q} : \Omega^{p,q} \rightarrow \Omega^{p,q-1}$ for the vertical complex

$$0 \longrightarrow \Omega^p(M) \xrightarrow{\pi^*_\infty} \Omega^{p,0} \xrightarrow{\delta} \Omega^{p,1} \xrightarrow{\delta} \cdots \tag{1.74}$$

are given by the following formula

$$h_V^{p,q}(\omega) = \int_0^1 \frac{1}{t}\Phi^*_{\log t}(j^\infty R \lrcorner\, \omega)\,dt\,, \tag{1.75}$$

where $R := u^\alpha\partial_\alpha$ is the (vertically) radial vector field and $\Phi_\varepsilon : (x,u) \mapsto (x, e^\varepsilon u)$. It is not too difficult to check that, for source forms, this gives rise to Eq. (1.95).

The analogous statement for the horizontal complex is a bit more involved.

Property 1.3.31 (Augmented horizontal complex is exact). Homotopy operators $h_H^{p,q} : \Omega^{p,q} \rightarrow \Omega^{p-1,q}$ for the augmented horizontal complex

$$0 \longrightarrow \Omega^{0,q} \xrightarrow{\mathrm{d}} \Omega^{1,q} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{\dim(M),q} \xrightarrow{I} \mathcal{F}^q \xrightarrow{I} 0 \tag{1.76}$$

are given by the following formula

$$h_H^{p,q}(\omega) = \frac{1}{q}\sum_{|I|=0}^{k-1} \frac{|I|+1}{m-p+|I|+1} D_I\big(\delta u^\alpha \wedge F^{I\mu}_\alpha(D_\mu \lrcorner\, \omega)\big)\,, \tag{1.77}$$

where the F^I_α are the interior Euler operators (1.54).

Corollary 1.3.32 (Functional decomposition). The de Rham spaces on $J^\infty(E)$ admit the following decomposition:

$$\Omega^{p,q}(J^\infty(E)) = \mathrm{d}\Omega^{p-1,q}(J^\infty(E)) \oplus \mathcal{F}^q(J^\infty(E))\,, \tag{1.78}$$

where the functional part is obtained by applying I to a form.

Using the above properties, one can prove the acyclicity of the **Euler–Lagrange complex** $\mathcal{E}$ (again, over a local chart):

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^{0,0} \xrightarrow{\mathsf{d}} \Omega^{1,0} \xrightarrow{\mathsf{d}} \cdots \xrightarrow{\mathsf{d}} \Omega^{\dim(M),0} \xrightarrow{\delta_{\mathrm{EL}}} \mathcal{F}^1 \xrightarrow{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \cdots . \tag{1.79}$$

Explicit formulas are not shown since these are too complicated for the current objective. See Anderson (1992) for a full account.

Remark 1.3.33 (Minimal solutions). Although the (local) exactness of the variational bicomplex is stated, it should be noted that this is not an optimal solution. Consider the example of locally variational source forms (see the next two sections), i.e. differential forms of the form $\delta_{\mathrm{EL}}\mathbf{L}$. From the form of the homotopy operator $\mathcal{F}^1 \to \Omega^{m,0}$, it should be clear that an order-k source form is mapped to an order-k Lagrangian. However, the Euler–Lagrange operator δ_{EL} will, in general, map order-l Lagrangians to order-$2l$ source forms. Hence, it can be seen that the homotopy operator will, in general, not give Lagrangians of minimal order.

1.3.5 Variational problems

The forms in $\mathcal{F}^1$ admit the following characterization.

Definition 1.3.34 (Source form). A differential form $\omega \in \Omega^{\dim(M),1}(J^\infty(E))$ such that the evaluation on a vector field only depends on the projection $(\pi_{\infty,0})_*X \in TE$. After pulling back along the prolongation map j^∞, this can be written as follows:

$$\Omega^{\dim(M),1}_{\text{source}}(E) := \delta C^\infty(E) \wedge \Omega^{\dim(M),0}(E) . \tag{1.80}$$

Locally, it can be written as

$$\omega = \omega_\alpha(x, u, u_I)\delta u^\alpha \wedge \mathsf{Vol} . \tag{1.81}$$

The PDEs associated to the source form ω are called **source equations**.

Remark. In fact, one can extend the above definition to all of $\Omega^{\bullet,1}(J^\infty(E))$. So, in general, $\mathcal{F}^1$ is only a subspace of the space of source forms.

Property 1.3.35 (First variational formula). In Property 1.3.14, it was shown that the Euler–Lagrange operator locally satisfies

$$\mathcal{L}_{j^\infty X}\mathbf{L} = j^\infty X \lrcorner\, \delta_{\mathrm{EL}}\mathbf{L} + \mathsf{d}(j^\infty X \lrcorner\, \gamma) \tag{1.82}$$

for some locally defined form γ that can be constructed from the Lie–Euler operators of $\mathbf{L}$. Using Cartan's magic formula and the fact that $\mathbf{L}$ is horizontal, this can be rewritten in terms of differentials. Moreover, using Corollary 1.3.32, one can find a globally defined form that still satisfies the same formula (with the disadvantage that it does not admit a canonical expression):

$$\delta\mathbf{L} = \delta_{\mathrm{EL}}\mathbf{L} + \mathsf{d}\sigma . \tag{1.83}$$

This formula is often called the first variational formula for the following reason. Given a local Lagrangian $\mathbf{L}$, one can look at solutions of the associated variational principle obtained by extremizing over perturbations of a field configuration. Such perturbations are generated by evolutionary vector fields and, hence, one can write the extremality condition as

$$\forall X \in \mathsf{Ev}(J^\infty(E)) : \mathcal{L}_{j^\infty X}\mathbf{L} = 0 \,. \tag{1.84}$$

Equation (1.83) then becomes:

$$\delta_{\mathrm{EL}}\mathbf{L} + \mathrm{d}\sigma = 0 \,. \tag{1.85}$$

So, up to boundary contributions, the first variational formula says that extremality of the local Lagrangian (globally) corresponds to the vanishing of the Euler–Lagrange operator.

Corollary 1.3.36 (Global variational formula). The variational formula can also be extended to all generalized vector fields:

$$\mathcal{L}^\sharp_{j^\infty X}\mathbf{L} = X_{\mathrm{ev}} \lrcorner\, \delta_{\mathrm{EL}}(\mathbf{L}) + \mathrm{d}\sigma \tag{1.86}$$

for some $\sigma \in \Omega^{\dim(M),0}(J^\infty(E))$.

Definition 1.3.37 (Lepage form). A $\dim(M)$-form $\rho \in \Omega^\bullet(J^\infty(E))$ such that

$$\pi^{\dim(M),0}(X \lrcorner\, \mathbf{d}\rho) = 0 \tag{1.87}$$

for all $\pi_{\infty,0}$-vertical vector fields X. Given a local Lagrangian $\mathbf{L} \in \Omega^{\dim(M),0}(J^\infty(E))$, $\mathbf{L}$ and ρ are said to be Lepage equivalent if $\pi^{\dim(M),0}\rho = \mathbf{L}$.

Property 1.3.38 (Lepage equivalent). Consider the first variational formula for a local Lagrangian $\mathbf{L}$. A Lepage equivalent is given by the form $\mathbf{L} + \gamma$.

Definition 1.3.39 (Distinguished symmetry). A distinguished (generalized) symmetry of a source form $\Delta \in \mathcal{F}^1(J^\infty(E))$ is a generalized vector field X on E such that

$$\mathcal{L}^\natural_{j^\infty X}\Delta = 0 \,. \tag{1.88}$$

As with the formula above, one can replace the projected Lie derivative by an ordinary Lie derivative when X is projectable.

If the source form comes from a local Lagrangian, the above definition admits a specific case by Property 1.3.28 and the fact that δ_{EL} annihilates d-exact forms (Corollary 1.3.17).

Definition 1.3.40 (Bessel-Hagen symmetry). A Bessel-Hagen or **divergence** symmetry of a Euler–Lagrange form $\delta_{\mathrm{EL}}(\mathbf{L})$ is a generalized vector field X such that

$$\mathcal{L}^\sharp_{j^\infty X}\mathbf{L} = \mathrm{d}\eta \tag{1.89}$$

for some $\eta \in \Omega^{\dim(M),0}(J^\infty(E))$.

If this condition holds locally, distinguished symmetries and Bessel-Hagen symmetries coincide. However, if the Bessel-Hagen condition is required to hold globally, the Bessel-Hagen symmetries form only a subset of the distinguished symmetries.

Definition 1.3.41 (Local conservation law). A generator of a local conservation law of a source form $\Delta \in \mathcal{F}^1(J^\infty(E))$ is an evolutionary vector field X such that

$$\delta_{\text{EL}}(j^\infty X \lrcorner \Delta) = 0 \tag{1.90}$$

or, again by 1.3.17,

$$X \lrcorner \Delta = \mathrm{d}\eta \tag{1.91}$$

for some local $\eta \in \Omega^{\dim(M),0}(J^\infty(E))$. Corollary 1.3.36 shows that to every global (generalized) symmetry, there corresponds a global conservation law.

Remark 1.3.42. Note that the Bessel-Hagen/divergence symmetries are symmetries of the source form, while the local symmetries coming from the first variational formula exist on the level of local Lagrangians.

Property 1.3.43 (Lie algebra of symmetries). Given a source form $\Delta \in \mathcal{F}^1(J^\infty(E))$, one can equip the vector space of generalized vector fields satisfying the following two conditions with the structure of a Lie algebra:

- They are distinguished symmetries.
- Their evolutionary part is a generator of local conservation laws.

Theorem 1.3.44 (Noether). *If $\Delta \in \mathcal{F}^1(J^\infty(E))$ is locally variational, a generalized vector field on E is a distinguished symmetry if and only if its evolutionary part is a generator of local conservation laws.*

1.3.6 Inverse problem

The inverse problem in the calculus of variations consists of determining when a given system of PDEs can be obtained from a variational problem, i.e. when a source form Δ can be written in the form $\delta_{\text{EL}}\mathbf{L}$, these are said to be **locally variational**.

Helmholtz was the first to study the inverse problem, so the following sufficient conditions are named after him.

Property 1.3.45 (Helmholtz conditions). By applying the Euler–Lagrange operator δ_{EL} to its defining variational formula (Property 1.3.14) and using the (infinitesimal) naturality condition (1.51) and the fact that it annihilates d-exact forms (Corollary 1.3.17), it can be seen that a source form Δ can be obtained from a local Lagrangian if

$$\mathcal{L}_{j^\infty X}\Delta = \delta_{\text{EL}}(X \lrcorner \Delta) \tag{1.92}$$

is satisfied for all evolutionary vector fields X. This can also be rewritten in terms of the Helmholtz operator δ_V (hence its name):[6]

$$\delta_V\Delta = 0\,. \tag{1.93}$$

Formula 1.3.46 (Local expression). Consider a source form that admits the local expression

$$\Delta = P_\alpha \delta u^\alpha \wedge \mathsf{Vol}\ .$$

The Helmholtz conditions can locally be expressed as follows:

$$(-1)^{|I|}\partial^I_\alpha P_\beta = E^I_\beta(P_\alpha)\,, \tag{1.94}$$

where the E^I_β are the Lie–Euler operators (1.48).

Example 1.3.47 (Volterra–Vainberg formula). If E is trivial and $\Delta = F_\alpha\delta u^\alpha \wedge \mathsf{Vol}$ satisfies the Helmholtz conditions, then

$$L := \int_0^1 u^\alpha F_\alpha[tu]\,dt \tag{1.95}$$

satisfies $\Delta = \delta_{\mathsf{EL}}\mathbf{L}$. If Δ is homogeneous of degree $k \in \mathbb{N}$ in u, this can be expressed as

$$\mathbf{L} = \frac{1}{k+1}\iota_R\Delta\,, \tag{1.96}$$

where $R := u^\alpha\partial_\alpha$ is the (vertically) radial vector field on E.

1.3.7 Finite jet bundles

A last subject that will be considered in this section is the restriction of the variational bicomplex to finite jet bundles. However, as is clear from the definition of the horizontal differential, forms of order $k \in \mathbb{N}$ are mapped to forms of order $k+1$. Therefore, attention will have to be restricted to a specific subcomplex of $\Omega^\bullet(J^\infty(E))$:

$$\Omega^\bullet_k(E) \subset \Omega^\bullet(J^{k+1}(E)) := \delta\text{-closure of } \Omega^\bullet(J^k(E))\,. \tag{1.97}$$

From the basic definitions and properties of the two differentials, it follows that $\Omega^\bullet_k$ is generated by $C^\infty(J^k(E))$, the horizontal basis $\{\mathsf{d}x^\mu\}_{\mu\leq\dim(M)}$ and the contact basis $\{\delta u^\alpha_I\}^{|I|\leq k}_{\mu\leq\dim(M)}$. The next step is to further restrict to a horizontally closed subcomplex. To this end, consider forms $\omega \in \Omega^{p,q}_k(E)$ of the form

$$\omega = \left[\mathsf{d}u^{\alpha_1}_{I_1} \wedge \ldots \wedge \mathsf{d}u^{\alpha_r}_{I_r} \wedge \mathsf{d}\delta u^{\beta_1}_{J_1} \wedge \ldots \wedge \mathsf{d}\delta u^{\beta_s}_{J_s}\right] \wedge f\mathsf{d}x^{\kappa_1} \wedge \ldots \wedge \mathsf{d}x^{\kappa_{p-r-s}}\,, \tag{1.98}$$

[6] This expression also immediately follows from Property 1.3.16.

where $|I_i| = |J_i| = k-1$ and $f \in C^\infty(J^{k-1}(E))$. It is immediately clear that the subcomplex of such forms is also horizontally closed. The factor in between square brackets can also be rewritten as follows:

$$J = \frac{1}{(r+s)!}\frac{D(u^{\alpha_1}_{I_1},\dots,u^{\alpha_r}_{I_r},\delta u^{\beta_1}_{J_1},\dots,\delta u^{\beta_s}_{J_s})}{D(x^{\mu_1},\dots,x^{\mu_r},x^{\nu_1},\dots,x^{\nu_s})}\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_r}\wedge\mathrm{d}x^{\nu_1}\wedge\dots\wedge\mathrm{d}x^{\nu_s}\,. \tag{1.99}$$

This factor has the form of a Jacobian determinant (with respect to total derivatives) and, as such, the subcomplex spanned by the above forms is called the **Jacobian complex** $\mathcal{J}^\bullet_k(E)$.

Property 1.3.48 (Alternative characterizations). The Jacobian complex can also be characterized as follows:

- Consider the projection $\pi^{\bullet,0}:\Omega^\bullet\rightarrow\Omega^{\bullet,0}$ (note that this maps forms in $\Omega^r(J^k(E))$ to forms in $\Omega^{r,0}_{k+1}(E)$ due to Remark 1.3.22).

$$\mathcal{J}^{p,q}_k(E) = \Omega^{p,q}(J^\infty(E))\cap\delta\text{-closure of }\pi^{\bullet,0}\big(\Omega^\bullet(J^{k-1}(E))\big)\,. \tag{1.100}$$

- For $p<m$, the Jacobian complex contains those forms for which d does not increase the order:

$$\mathcal{J}^{p,q}_k(E) = \{\omega\in\Omega^{p,q}_k \mid \mathrm{d}\omega\in\Omega^{p+1,q}_k(E)\}\,. \tag{1.101}$$

- If $\omega\in\mathcal{J}^{p,q}_k(E)$, then ω is a polynomial in u^α_i's and δu^α_I's of degree $\leq r$ with $|I| = k$.

It can be shown that the Jacobian subcomplex is (locally) exact and that it respects the structure of the Euler–Lagrange complex.

Property 1.3.49 (Exactness). Let E be trivial. If $\delta_{\mathrm{EL}}\omega = 0$ for $\omega\in\Omega^{m,q=0}_k(E)$ or $I(\omega)=0$ for $\omega\in\Omega^{m,q\geq1}_k(E)$, then $\omega\in\mathcal{J}^{m,q}_k(E)$ and $\omega = \mathrm{d}\eta$ for $\eta\in\mathcal{J}^{m-1,q}_k(E)$.

Property 1.3.50 (Functional dependence of Lagrangians). If Δ is a locally variational source form of order $k\in\mathbb{N}$, it is polynomial of degree m in k^{th}-order derivatives of the u^α.

1.4 Partial differential equations

In this section, the content of **??** is generalized using the language of differential geometry.

1.4.1 Algebraic formulation

In this section, partial differential equations of the form

$$f(x,u,u_I) = 0 \tag{1.102}$$

are considered, where f is a smooth function. A partial differential equation is regarded as an algebraic equation involving derivatives and, hence, f can be interpreted as a function on the jet bundle $J^k(\mathbb{R}^m)$, where $k \in \mathbb{N}$ is the order of the PDE and $m \in \mathbb{N}$ is the number of independent variables.

In this framework, one can define a solution of the above PDE as a function ϕ satisfying $f \circ j^k\phi = 0$. This can be rephrased in a geometric way. Every PDE of order $k \in \mathbb{N}$ defines a subbundle Σ^0 of the finite jet bundle $J^k(E)$ and a solution is simply a section of Σ^0.

Remark 1.4.1. For every $l \in \mathbb{N}$, define the subspace

$$\Sigma^l := \{(x, u, u_I) \in J^{k+l}(E) \mid \forall |J| \leq l : D_J f(x, u, u_I) = 0\} = J^l(\Sigma^0) \cap J^{k+l}(E)\,, \tag{1.103}$$

i.e. the set of holonomic sections of $J^{k+l}(E)$ that are l^{th}-order tangent to Σ^0. A function $\phi \in \Gamma(E)$ is a solution if there exists some $l \in \mathbb{N}$ such that the image of $j^{k+l}\phi$ lies in Σ^l and, conversely, $j^{k+l}\phi$ lies in Σ^l for all $l \in \mathbb{N}$ if ϕ is a solution.

Definition 1.4.2 (Formal integrability). A PDE $f : \Sigma^0 \hookrightarrow J^k(E)$ is called formally integrable if

1. all prolongations Σ^l are smooth manifolds.
2. all projections $\Sigma^{l+1} \to \Sigma^l$ are smooth fibre bundles.

Property 1.4.3 (Diffiety). The leaves of the Cartan distribution on $J^\infty(E)$ are the graphs of infinity-prolongations $j^\infty\phi$ for local sections $\phi \in \Gamma(E)$. When restricting the distribution to the PDE Σ^0, the leaves are given by the graphs of the infinity-prolongations of (local) solutions.

A pair $(\Sigma, \mathcal{C}(\Sigma))$ consisting of a smooth manifold Σ and a finite-dimensional distribution $\mathcal{C}(\Sigma)$, such that Σ is locally isomorphic to the infinity-prolongation of a PDE and $\mathcal{C}(\Sigma)$ is locally isomorphic to the associated Cartan distribution, is called a **diffiety** (short for **differential variety**).

Remark 1.4.4. The reason for this terminology stems from the apparent similarity with algebraic varieties. A variety is (locally) defined by a set of algebraic equations together with all algebraic consequences, i.e. it is defined by the ideal generated by the algebraic equations. Similarly, a diffiety is (locally) defined by a set of differential equations together with all differential consequences, i.e. it is defined by the differential ideal generated by the differential equations.

Definition 1.4.5 (Exterior system). Consider a general PDE of the form (1.102). This PDE can be turned into a set of differential forms on the zero locus of f, i.e. Σ_0:

$$\theta := \mathrm{d}u - u_\mu \mathrm{d}x^\mu\,, \tag{1.104}$$
$$\vdots$$

These differential one-forms generate an ideal in $\Omega^\bullet(\Sigma^0)$ that represents the differential equation in that it relates the variables in the algebraic condition $f = 0$ by a set of differential relations. It is a Pfaffian system (**??**) or, combined with the PDE $f = 0$ and its differential consequences, it gives a characteristic system (**??**). The associated distribution becomes integrable (sometimes called **completely integrable**) if and only if the ideal is a differential ideal by Frobrenius's theorem **??**. The integral manifolds of the distribution then give a solution of the PDE.

Formula 1.4.6 (Lagrange–Charpit equations). Let $f = 0$ be a first-order PDE on a smooth manifold and consider the differential closure of the exterior system of $f = 0$:

$$\begin{cases} f &= 0\,, \\ \mathrm{d}f &= \left(\frac{\partial f}{\partial x^\mu} + u_\mu \frac{\partial f}{\partial u}\right)\mathrm{d}x^\mu + \frac{\partial f}{\partial u_\mu}\mathrm{d}u_\mu = 0\,, \\ \theta &= 0\,, \\ \mathrm{d}\theta &= 0\,. \end{cases} \tag{1.105}$$

The characteristic system of these equations is given by the PDE itself, together with the Lagrange–Charpit equations:

$$\begin{aligned} \frac{\mathrm{d}x^1}{\partial f/\partial u_1} = \cdots = \frac{\mathrm{d}x^n}{\partial f/\partial u_n} &= \frac{-\mathrm{d}u_1}{\partial f/\partial x^1 + u_1(\partial f/\partial u)} \\ &= \cdots = \frac{\mathrm{d}u_n}{\partial f/\partial x^n + u_n(\partial f/\partial u)} = \frac{\mathrm{d}u}{\sum_\mu u_\mu(\partial f/\partial u_\mu)}\,. \end{aligned} \tag{1.106}$$

Definition 1.4.7 (Monge cone). Consider the characteristic system above. The rank of the Pfaffian system is $2\dim(M)$. At every point $(x^\mu, u) \in M \times \mathbb{R}$, the solutions (x^μ, u, u_μ) of the PDE determine a tangent plane to M. The $\dim(M)$-parameter family of tangent planes obtained by varying the derivative coordinates u_μ gives rise to the Monge cone with apex (x^μ, u). A solution to the PDE is a hypersurface that is everywhere tangent to the Monge cones. This approach to finding a solution is called the **method of the characteristics**.

1.4.2 Symmetries

Property 1.4.8. A Lie group G with Lie algebra $\mathfrak{g}$ is the symmetry group of a (nondegenerate) system F of PDEs if and only if

$$F = 0 \implies j^\infty X(F) = 0 \tag{1.107}$$

for all generators $X \in \mathfrak{g}$.

@@ COMPLETE @@

1.4.3 Pseudogroups ♣

Example 1.4.9 (Diffeomorphism jet). Let M be a smooth manifold. Consider the set $\mathcal{D}^\omega(M)$ of local analytic diffeomorphisms $\phi : M \to M : z \mapsto \phi(z)$. The locality property turns this set into a (smooth) pseudogroup (**??**).

By the Inverse Function Theorem **??**, one can define the diffeomorphism jet bundle $\mathcal{D}^r(M)$ as the subbundle of $J^r(M, M)$ for which

$$\det\left(\frac{\partial\phi^\alpha}{\partial z^\beta}\right) \neq 0\,.$$

It is also possible to endow this jet bundle with the structure of a groupoid (**??**). Using the source and target projections, one can check that two elements $f, g \in \mathcal{D}^r(M)$ can be multiplied if and only if $\pi_r(g) = \pi_{r,0}(f)$. The derivative coordinates can be found using the *Faá di Bruno formula*.

Furthermore, every pseudogroup $\mathcal{G} \subset \mathcal{D}^\omega$ induces a subbundle $\mathcal{G}^{(r)} \subset \mathcal{D}^{(r)}$. This structure gives rise to the following notions.

Definition 1.4.10 (Regular pseudogroup). Consider a smooth manifold M. Let $\mathcal{D}^\omega(M)$ be its diffeomorphism pseudogroup and let $\mathcal{G} \subset \mathcal{D}^\omega$ be another pseudogroup. If there exists a $k \in \mathbb{N}_0$, called the **order**, such that, for all $n \geq k$, the jets $\pi_n : \mathcal{G}^{(n)} \to M$ form an embedded submanifold of $\Pi_n : \mathcal{D}^{(n)} \to M$ and such that the jet projections $\pi_{n+1,n} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ are fibrations, then $\mathcal{G}$ is called a regular pseudogroup.

Definition 1.4.11 (Lie pseudogroup). Let $\mathcal{G} \subset \mathcal{D}^\omega$ be a regular analytic pseudogroup of order $k \in \mathbb{N}$. If every local diffeomorphism $\phi \in \mathcal{D}^\omega$ satisfying $j^k\phi \in \mathcal{G}^{(k)}$ is also an element of $\mathcal{G}$, then $\mathcal{G}$ is called a Lie pseudogroup.

Property 1.4.12. Let $\mathcal{G}$ be a Lie pseudogroup of order $k \in \mathbb{N}$. The regularity condition implies that for all $n \geq k$ the jet bundle $\mathcal{G}^{(n)}$ is described by a set of n^{th}-order PDEs

$$F\big(z, \phi^{(n)}\big) = 0\,. \tag{1.108}$$

The (local) solutions to these equations are exactly the analytic functions that have $(z_0, \phi_0^{(n)})$ as local coordinates of their n-jet at $z_0 \in M$.

The Lie condition on $\mathcal{G}$ implies that every solution to the system is, in fact, an element of $\mathcal{G}$. This system of equations is called the **determining system** of the Lie pseudogroup.

Definition 1.4.13 (Lie completion). Let $\mathcal{G}$ be a regular pseudogroup. The Lie completion $\overline{\mathcal{G}}$ of $\mathcal{G}$ is defined as the set of all (local) analytic diffeomorphisms solving the determining system of $\mathcal{G}$. This completion is itself a Lie pseudogroup. If $\mathcal{H}$ is a Lie pseudogroup, then $\overline{\mathcal{H}} = \mathcal{H}$.

@@ COMPLETE (IS THIS EVEN USEFUL?) @@

Chapter 2

Classical Field Theory

Rigorous definitions and statements about the mathematical concepts used in this chapter can be found in **?? ??**, **??**, **??** and 1.

Chapter Contents

2.1 Lagrangian field theory

The physical space will be assumed to be a (pseudo-)Riemannian, n-dimensional manifold (M, g) with the fields being sections of a vector bundle $E \to M$. In general, an **action** is a function $S : \Gamma_c(E) \to \mathbb{R}$ from the space of (compactly supported) sections of E to the real numbers. This is often given by local functionals for a local Lagrangian (Definition 1.3.2):

$$S : \Gamma_c(E) \to \mathbb{R} : \phi \mapsto \int_M (j^\infty\phi)^* L\, \mathsf{Vol}_M\,. \tag{2.1}$$

Associated to the manifold M, one can construct a cochain complex similar to the de Rham complex $\Omega^\bullet(M)$. This structure takes two geometric features into account. On the one hand, one has the ordinary de Rham differential d on the base manifold M, whereas, on the other hand, one has a differential δ induced by the variation of fields along the jet fibres. The total differential will be the sum of these as is standard in the context of bicomplexes. This defines the variational bicomplex.

Some key concepts from the calculus of variations are recalled here. The variational derivative or Euler–Lagrange derivative (1.50) is defined as follows (partial derivatives are denoted by subscripted commas, e.g. $\partial_\mu\partial_\nu\phi \equiv \phi_{,\mu\nu}$):[1]

$$\frac{\delta L}{\delta\phi} := \frac{\partial L}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial\phi_{,\mu}}\right) + \frac{\partial^2}{\partial x^\mu\partial x^\nu}\left(\frac{\partial L}{\partial\phi_{,\mu\nu}}\right) - \cdots . \tag{2.2}$$

By comparing this formula to the formula for the variation of the Lagrangian density, one obtains the first variational formula (1.83):

$$\delta L = \frac{\delta L}{\delta\phi^I}\delta\phi^I - \mathrm{d}\Theta[\phi] . \tag{2.3}$$

The first term vanishes on-shell because it is proportional to the Euler–Lagrange equation associated to the field ϕ^I. The last term contains the boundary terms obtained after performing integration by parts. The $(n-1,1)$-form Θ is called the **presymplectic potential**.

The **presymplectic current** ω is obtained by taking the variation of the presymplectic potential:

$$\omega[\phi] := \delta\Theta[\phi] . \tag{2.4}$$

On-shell, this form is closed, i.e. $\mathbf{d}\omega \approx 0$. Off-shell, this does not necessarily hold and, hence, the form is not properly symplectic. It can be shown that, if the variations $\delta\phi^I$ satisfy the linearised equations of motion, then for every gauge transformation ξ, there exists an $(n-2,1)$-form $k_\xi[\phi]$ such that $\omega \approx \mathrm{d}k_\xi[\phi]$.[2]

@@ EXPLAIN this last statement better @@

2.1.1 Noether's theorem

Theorem 2.1.1 (Noether's first theorem). *Consider an infinitesimal field transformation*

$$\phi \longrightarrow \phi + \alpha\delta\phi , \tag{2.5}$$

where the Lagrangian L depends on the fields and their first-order derivatives.[3] *In case of a symmetry, one obtains a conservation law of the following form:*

$$\partial_\mu\left(\frac{\partial L}{\partial\phi_{,\mu}}\delta\phi - \mathcal{J}^\mu\right) = 0 . \tag{2.6}$$

The factor between parentheses can be interpreted as a conserved current $j^\mu(x)$.

[1] Total derivatives with respect to spatial coordinates are still denoted by ∂, in contrast to Eq. (1.50), to allign with most works in the literature.

[2] More details can be found in e.g. Compère (2019).

[3] An extension to higher-order derivatives can be obtained by including further boundary terms.

Proof. *The general transformation rule for the Lagrangian is*

$$L \longrightarrow L + \alpha\delta L\,. \tag{2.7}$$

To have a symmetry, i.e. to keep the action invariant, the deformation factor has to be a divergence:

$$L \longrightarrow L + \alpha\partial_\mu \mathcal{J}^\mu\,. \tag{2.8}$$

To obtain the conservation law (2.6)*, the Lagrangian is varied explicitly:*

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial\phi_{,\mu}}\delta\phi_{,\mu}\\ &= \frac{\partial L}{\partial\phi}\delta\phi + \partial_\mu\left(\frac{\partial L}{\partial\phi_{,\mu}}\delta\phi\right) - \partial_\mu\left(\frac{\partial L}{\partial\phi_{,\mu}}\right)\delta\phi\\ &= \partial_\mu\left(\frac{\partial L}{\partial\phi_{,\mu}}\delta\phi\right) + \left[\frac{\partial L}{\partial\phi} - \frac{\partial L}{\partial\phi_{,\mu}}\right]\delta\phi\,.\end{aligned}$$

The second term vanishes due to the Euler–Lagrange equation (**??**)*. Combining these formulas gives*

$$\partial_\mu\left(\frac{\partial L}{\partial\phi_{,\mu}}\delta\phi\right) - \partial_\mu\mathcal{J}^\mu(x) = 0\,. \tag{2.9}$$

From this equation, one can conclude that the current

$$j^\mu(x) = \frac{\partial L}{\partial\phi_{,\mu}}\delta\phi - \mathcal{J}^\mu(x) \tag{2.10}$$

is conserved. □

The above conservation law can also be expressed in terms of a charge (such a current and its associated charge are generally called the **Noether current** and **Noether charge**):

$$Q[\Sigma] := \int_\Sigma j^0\, d^{n-1}x\,, \tag{2.11}$$

where Σ is a spacelike hypersurface. The conservation law can then simply be restated as

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0\,. \tag{2.12}$$

Definition 2.1.2 (Stress-energy tensor). Consider the translation of a scalar field:

$$\phi(x) \longrightarrow \phi(x+a) = \phi(x) + a^\mu\partial_\mu\phi(x)\,. \tag{2.13}$$

Because the Lagrangian is a scalar quantity, it transforms in the same way as the fields:

$$L \longrightarrow L + a^{\mu}\partial_{\mu}L = L + a^{\nu}\partial_{\mu}(L\delta^{\mu}_{\ \nu})\,. \tag{2.14}$$

On an n-dimensional manifold, this leads to the existence of $n \in \mathbb{N}$ conserved currents. These can be used to define the stress-energy tensor:

$$T^{\mu}_{\ \nu} = \frac{\partial L}{\partial \phi_{,\mu}}\partial_{\nu}\phi - L\delta^{\mu}_{\ \nu}\,. \tag{2.15}$$

Chapter 3

BRST Theory

The foundations for this subject were laid down by Dirac (1950). By introducing constraints, the coordinates and their momenta become dependent. This implies, for example, that the Hamiltonian equations of motion have to be modified.

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3.1 Constrained systems

3.1.1 Constraints

Definition 3.1.1 (Holonomic constraint). A constraint $f(q,t) = 0$ that only depends on the coordinates q^i and t and not on the derivatives.

Method 3.1.2 (Holonomic constraints). The Euler–Lagrange equations of a system with $k \in \mathbb{N}$ holonomic constraints

$$\phi_k(q,t) = 0 \tag{3.1}$$

can be obtained from the generalized action functional

$$S_E[q,\lambda] := \int_{t_1}^{t_2} \left[L(q,\dot{q},t) + \sum_{i=1}^{k} \lambda_i(t)\phi_i(q,t) \right] dt\,, \tag{3.2}$$

where the λ_i are Lagrange multipliers (**??**). Extremizing with respect to these multipliers induces the constraints:

$$\frac{\delta S_E}{\delta \lambda_i} = \phi_i = 0\,. \tag{3.3}$$

First, recall the Lagrangian equations of motion (**??**). By expanding these equations, it can be shown that the accelerations $\ddot{q}$ are uniquely determined by the coordinates and velocities $(q,\dot{q})$ if and only if the Hessian of the Lagrangian is invertible. If the Hessian is not invertible, the definition of the conjugate momenta (**??**) cannot be inverted to express velocities in terms of momenta. Alternatively, the coordinates q and momenta p are not independent and there must exist relations of the form

$$\phi(q,p) = 0\,. \tag{3.4}$$

Constraints of this type are called **primary constraints**. They do not serve to constrain the range of the coordinates q, they only couple the coordinates and the momenta.

Axiom 3.1 (Regularity conditions). It will always be assumed that the independent constraints, i.e. the minimal generating set, satisfy the following (equivalent) conditions:

- The constraints can locally serve as the first coordinates of a (regular) coordinate system.
- The differentials (gradients) $\mathrm{d}\phi_m$ are locally linearly independent.
- The variations $\delta\phi_m$ are of the order ε whenever the variations $\delta q^i, \delta p_i$ are of the order ε. (This is the original condition due to *Dirac*.)

A constrained dynamical system consists of a dynamic system (M, ω, H) together with a finite collection of constraints $\phi_m(q,p) = 0, m \in I$. If this system is derived from a Lagrangian $L(q,\dot{q})$, the calculus of variations easily extends to these constrained systems, where it gives the following modified Hamiltonian equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \sum_{m\in I} u_m \frac{\partial \phi_m}{\partial p_i}, \tag{3.5}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - \sum_{m\in I} u_m \frac{\partial \phi_m}{\partial q^i}, \tag{3.6}$$

where the u_m are functions of the coordinates and velocities that play a role similar to ordinary Lagrange multipliers. In terms of Poisson brackets, the constrained time evolution of a (time-independent) function is given by:

$$\dot{f} = \{H, f\} + \sum_{m\in I} u_m \{\phi_m, f\}. \tag{3.7}$$

Remark 3.1.3. The above relations follow from the property that the general solutions to $\lambda_i \delta q^i + \mu_i \delta p^i = 0$ for variations $\delta q^i, \delta p_i$ tangent to the constraint surface are of the form

$$\begin{cases} \lambda_i = \sum_{m\in I} u_m \frac{\partial \phi_m}{\partial q^i}, \\ \mu^i = \sum_{m\in I} u_m \frac{\partial \phi_m}{\partial p_i}. \end{cases} \tag{3.8}$$

Combining this result with the usual derivation of Hamilton's equations from a Lagrangian action principle gives the above modified equations.

Method 3.1.4 (General Poisson brackets). Until now, Poisson brackets were only defined for functions depending on the canonical coordinates (q,p). This definition can be generalized to arbitrary functions through the Poisson algebra properties (**??**). Furthermore, after working out the Poisson brackets, one can use the constraint equations to drop all terms that are proportional to ϕ_m.

For example, Eq. (3.7) can be rewritten as

$$\dot{f} = \left\{ H + \sum_{m\in I} u_m \phi_m, f \right\}. \tag{3.9}$$

To prove the equivalence, one can use the linearity and Leibniz properties. This involves the following equality

$$\{u_m \phi_m, f\} = \{u_m, f\}\phi_m + u_m \{\phi_m, f\}. \tag{3.10}$$

The Poisson brackets in the second term only involve functions depending on (q,p) and can be calculated in the usual way. The first term, however, involves a Poisson bracket of the Lagrange multiplier u_m. In general, these do not simply depend on q and p. Luckily, this does not pose a problem because the term is proportional to the constraints and, as such, vanishes on-shell. It is important that the constraints are only applied after the Poisson brackets have been fully worked out.

Notation 3.1.5 (Weak equality). The constraints ϕ_m only vanish on shell. To distinguish between functional equalities, i.e. equalities that also hold off-shell, and on-shell equalities, also called **weak equalities**, the latter are often denoted by the $\approx$ symbol. For example, the condition $\phi_m \approx 0$ is only a weak equality.

Using the above definitions, one can write an arbitrary time derivative as

$$\dot{f} \approx \{H_T, f\}\,, \tag{3.11}$$

where $H_T := H + \sum_{m\in I} u_m\phi_m$ is the total Hamiltonian.

Remark 3.1.6 (Closure). An important insight regarding weak equalities can be obtained by calculating the Poisson bracket with a function f that is strongly zero, i.e. a function that vanishes on-shell and whose variation also vanishes. In this case, $\{f,g\} \approx 0$ for all functions g, i.e. the brackets only vanish weakly. Furthermore, if f is only weakly zero, then $\{f,g\}$ does not even have to vanish at all.

Property 3.1.7 (Weakly vanishing functions). Assume that the constraints satisfy Axiom 3.1. A function that vanishes on shell is equal to some combination of the constraints (the coefficients might be functions themselves) (Henneaux & Teitelboim, 1992, p. 70).

Property 3.1.8 (Consistency conditions). By taking $f = \phi_n$ for any $n \in I$ in Eq. (3.7), a set of consistency conditions is obtained:

$$\{H, \phi_n\} + \sum_{m\in I} u_m\{\phi_m, \phi_n\} \approx 0\,. \tag{3.12}$$

It is possible that this condition reduces to an inconsistency of the type $1 \approx 0$. In this case, the equations of motion are inconsistent and the theory is not physical. If this is not the case, multiple possibilities can arise:

- After imposing the primary constraints, a tautology $0 = 0$ is found. This gives no new information.
- The equation reduces to an equation not involving the Lagrange multipliers u_m. This gives an additional constraint

  $$\chi(q,p) = 0\,. \tag{3.13}$$

  These are called **secondary constraints**.
- A condition on the coefficients u_m is obtained.

After having found a set of secondary constraints, this procedure can be iterated until no new contraints or conditions are found. Because the consistency conditions are linear in the coefficients u_m, the general solution can be written as

$$u_m = U_m + v_a V^a_m\,, \tag{3.14}$$

where U_m is a solution of the inhomogeneous equation (3.12) and the V^a_m are linearly independent solutions of the homogeneous equation

$$\sum_{m\in I} u_m\{\phi_m, \phi_n\} = 0\,. \tag{3.15}$$

The resulting coefficients v_a are completely arbitrary functions of time and will be referred to as the **Lagrange multipliers**. Therefore, the total Hamiltonian can be written in the form

$$H_T = H'(q,p) + v^a(t)\phi_a(q,p)\,, \tag{3.16}$$

where $\phi_a := \sum_{m\in I} V^a_m\phi_m$. The occurence of arbitrary functions in the Hamiltonian implies that the evolution of the phase space variables is not unique and, accordingly, that the theory has a gauge freedom.

Definition 3.1.9 (First- and second-class). A function $f(q,p)$ is said to be first class if its Poisson bracket with every constraint (both primary and secondary) is weakly zero. The function is said to be second class otherwise. It can be shown that both the total Hamiltonian H_T and the primary constraints ϕ_a are first class. The number of Lagrange multipliers v^a is equal to the number of primary, first-class constraints.

For first-class constraints $\{\phi_a\}_{a\in I}$, this leads to the following structure:

$$\{\phi_a, \phi_b\} = C^c_{ab}\phi_c + T^i_{ab}\frac{\delta S}{\delta x^i}\,, \tag{3.17}$$

where $C^c_{ab} : M \to \mathbb{R}$ are the **structure functions** and the $T^i_{ab} : M \to \mathbb{R}$ are antisymmetric under $a \leftrightarrow b$. The structure of the constraint algebra depends on the properties of these coefficients:

- $T_{ab} = 0$ and C^c_{ab} constant: **closed algebra** (this corresponds to Lie or L_∞-algebras),
- $T_{ab} = 0$: **soft algebra** (this corresponds to Lie or L_∞- algebroids **??**), and
- in general: **open algebra** (this corresponds to *central extensions of L_∞-algebroids*) .

Notation 3.1.10. To distinguish between first- and second-class constraints, the latter are often denoted by a separate symbol χ.

Property 3.1.11 (Closure). The Poisson bracket of two primary, first-class functions is first-class. So is the Poisson bracket of the total Hamiltonian and a primary, first-class constraint.

Corollary 3.1.12. The first-class constraints form a Lie algebra with respect to the Poisson bracket and the associated gauge transformations define a submanifold in phase space by Frobenius' theorem **??**.

Property 3.1.13. If the constraint algebra is closed, every gauge-invariant function is gauge equivalent to a strongly gauge-invariant function.

Remark 3.1.14 (Dirac conjecture). The primary, first-class constraints generate gauge transformations in the sense that variations in the coefficients, which are arbitrary, give rise to phase space variations that leave the physical state invariant. Some secondary constraints might also generate gauge transformations and *Dirac* even conjectured that this was the case for all constraints. However, counterexamples have been found. A common workaround is simply to restrict to systems where the conjecture is true and, from here on, the distinction between primary and secondary will be dropped. From this point of view, it makes sense to define the extended Hamiltonian

$$H_E := H_T + v^b(t)\phi_b(q,p)\,, \tag{3.18}$$

where b ranges over all secondary, first-class constraints. For gauge-invariant functions, i.e. those functions whose Poisson bracket with all first-class constraints vanishes, evolution with all three Hamiltonians H, H_T and H_E is identical. For general functions, only H_E takes into account the full gauge freedom.[1]

Formula 3.1.15 (Degrees of freedom). The number of degrees of freedom is given by the following formula:

$$\begin{aligned} 2 \times \#\text{d.o.f.} = \#&\text{canonical coordinates} \\ &- \#\text{second-class constraints} \\ &- 2 \times \#\text{first-class constraints}. \end{aligned}$$

Definition 3.1.16 (Dirac bracket). To take care of second-class constraints, *Dirac* introduced a modification of the Poisson bracket:

$$\{f,g\}_D := \{f,g\} - \{f,\chi_a\}C^{ab}\{\chi_b,g\}\,, \tag{3.19}$$

where the χ_a are the second-class constraints and the (invertible) matrix C^{ab} is the inverse of the matrix $C_{ab} := \{\chi_a,\chi_b\}$.

The benefit of using the Dirac bracket (after the Poisson bracket has been used to separate constraints in first-class and second-class constraints) is that second-class constraints become strong equalities, i.e. they can be used even before evaluating further (Dirac) brackets. The Dirac bracket satisfies the same algebraic properties as the Poisson bracket, i.e. it also defines a Poisson algebra. From here on, all constraints will be assumed to be first class, i.e. the Poisson bracket will be assumed to be the one obtained after applying the Dirac procedure to all second-class constraints.

Remark 3.1.17. Instead of splitting the constraints in first- and second-class instances and having to work with the nontrivial Dirac bracket, one can also try to remove second-class constraints in a different way. In the above formula for the degrees of freedom, the

[1]Note that H_T is the Hamiltonian that corresponds to a Lagrangian approach. H_E gives a more general theory.

factor 2 on the right-hand side is obtained by the introduction of gauge-fixing conditions. What these actually do is turning first-class constraints into second-class ones. In fact, the converse is also possible. One can obtain all second-class constraints as gauge-fixed first-class constraints after enlarging the system (although this procedure is not unique). After doing this, there is no need for the Dirac bracket anymore and one can simply work with the Poisson bracket (with the added complexity that all constraints now only hold weakly).

Definition 3.1.18 (Gauge-invariant functions). Consider the algebra $C^\infty(M)$ of smooth functions on phase space. In the spirit of algebraic geometry, the space of functions on the constraint surface Σ is given by the quotient algebra $C^\infty(\Sigma) := C^\infty(M)/\mathcal{N}$, where $\mathcal{N}$ is the ideal having Σ as its zero locus, i.e. $\mathcal{N}$ is the ideal generated by the constraints. The elements of $C^\infty(\Sigma)$ that are gauge-invariant, i.e. first class with respect to first-class constraints, should be considered as the **classical observables**. Passing to this extension effectively amounts to solving a(n integrable) system through constants of motion.

The restriction to gauge-invariant functions is also imperative if one wants to extend the Dirac bracket to $C^\infty(\Sigma)$. In general, the ideal $\mathcal{N}$ is not an ideal of the Dirac bracket. The gauge-invariant subalgebra is, in fact, the maximal subalgebra of $C^\infty(\Sigma)$ for which $\mathcal{N}$ is again an ideal.

3.1.2 Field theories

Up until now, the theories were assumed to have a finite number of degrees of freedom. However, in field theory, the phase space is (a subspace of) the module of sections of some vector bundle and, as such, is infinite dimensional. To ensure the existence of a well-behaved BRST theory, one has to restrict to 'local theories', i.e. all relevant functions/functionals should only depend on the fields and their derivatives up to some finite order.

Axiom 3.2 (Locality). A Hamiltonian gauge theory is local if the following conditions are satisfied:

1. The Hamiltonian action is local:
$$S[z,\phi] := \int_{t_1}^{t_2}\int_M \left(\dot{\phi}^I\pi_I - h_0 - \lambda^a\phi_a\right) d^nx\,dt\,, \tag{3.20}$$
where the field momenta π_I, the Hamiltonian density h_0 and the constraints ϕ_a are functions of the fields ϕ^I and their derivatives up to a finite order $k \in \mathbb{N}$, i.e. they are elements of a finite jet bundle $J^k(M)$.

2. The bracket $\{\phi^I(x), \phi^J(y)\}$ is local for all I, J and $x, y \in M$.

3. Any reducibility relation is again local.

The second condition ensures that taking iterated brackets respects the locality of the theory.

Axiom 3.3 (Local completeness). Any function that vanishes weakly is zero on the constraint surface by means differential identities of the constraints, i.e. it does not depend on the bounary conditions.

Remark 3.1.19. Local completeness can always be ensured by adding constraints or by replacing constraints by their (integral) primitives.

Axiom 3.1 can be extended to local field theories as follows.

Axiom 3.4 (Regularity condition). Let $V := \{\phi_a\}$ be a set of k-local constraints, with $k \in \mathbb{N}$. For every $n \in \mathbb{N}$, the jet prolongation $j^n V$ cuts out a submanifold of $J^{n+k}(M)$. It is assumed that there exists a generating set of $j^n V$ such that these independent constraints form a regular coordinate system of $j^n V$.

If the above axioms are satisfied, Property 3.1.7 can be generalized to field theories.

Property 3.1.20. If a local function vanishes on the constraint surface, it is an element of some finite prolongation of the constraint algebra.

3.1.3 Constraint surface

Property 3.1.21 (Geometric characterization). When restricted to a first-class constraint surface, the 'symplectic' form becomes maximally degenerate with rank

$$\mathsf{rk}(\omega) = \mathsf{dim}(M) - 2\,\mathsf{dim}(\Sigma)\,. \tag{3.21}$$

This essentially says that the constraint surface is coisotropic and, as a consequence, that the Poisson bracket is ill-defined (since this would involve the inverse of the symplectic form). After passing to the **reduced phase space**, i.e. the leaf space of the Hamiltonian foliation generated by the constraints, one again obtains a well-defined Poisson bracket that coincides with the ordinary Poisson bracket without any constraints.

The opposite situation arises for constraint surfaces that only involve second-class constraints. Here, the induced symplectic form is of maximal rank

$$\mathsf{rk}(\omega) = \mathsf{dim}(M) - \mathsf{dim}(\Sigma)\,, \tag{3.22}$$

which implies that the surface is isotropic. Furthermore, the induced Poisson bracket coincides with the restriction of the Dirac bracket.

Remark 3.1.22 (Algebraic characterization). The fact that first-class constraints define a coisotropic submanifold is not a peculiarity. A multiplicative ideal of a Poisson algebra that is closed under the Poisson bracket, is often called a **coisotrope** (or **coisotropic ideal**). Coisotropic submanifolds of a Poisson manifold (**??**) are exactly the zero loci of such coisotropes. In fact, one can restate the above constructions in purely algebraic terms. Given a Poisson algebra P and a coisotrope $\mathcal{J}$, one can pass to the quotient $N(\mathcal{J})/\mathcal{J}$, where N denotes the normalizer in P. This quotient is again a Poisson algebra, called the **reduced Poisson algebra**. This construction is strictly more general than the symplectic case considered above. (The sections further on could also be generalized to this setting.)

Theorem 3.1.23 (Abelianization). *Locally, one can always find a gauge transformation such that the constraint algebra becomes Abelian:*

$$\{\phi, \phi'\} = 0\,. \tag{3.23}$$

3.1.4 Fermionic systems

In this section, the study of constrained systems with 'fermionic' or odd statistics is considered. For an introduction to Grassmann numbers, see **??**. In general, the phase space will be assumed to be a supermanifold (**??**).

First, the ordinary Poisson bracket is extended to Grassmann-odd coordinates as follows:

$$\{\theta^i, \theta^j\} = 0 = \{\pi_i, \pi_j\} \tag{3.24}$$

and

$$\{\theta^i, \pi_j\} = \delta^i_j = \{\pi_j, \theta^i\}\,. \tag{3.25}$$

By defining the matrix $\sigma^{ij} := \{z^i, z^j\}$, where z can be any of the q, p, θ or π, one can then succintly write the Poisson bracket of superfunctions as follows:

$$\{f, g\} := \frac{\partial^R f}{\partial z^i}\sigma^{ij}\frac{\partial^L g}{\partial z^j}\,. \tag{3.26}$$

Note that this is virtually the same expression as the ordinary Poisson bracket, where the matrix σ was the inverse of the symplectic matrix. Writing out all terms gives

$$\{f, g\} = \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}\right) + (-1)^{\deg(f)}\left(\frac{\partial f}{\partial \pi_i}\frac{\partial g}{\partial \theta^i} + \frac{\partial f}{\partial \theta^i}\frac{\partial g}{\partial \pi_i}\right). \tag{3.27}$$

The algebraic properties of this generalized Poisson bracket are graded generalizations of those of the ordinary one:

$$\{f, g\} = -(-1)^{\deg(f)\deg(g)}\{g, f\} \tag{3.28}$$

$$\begin{aligned} 0 = \{f, \{g, h\}\} &+ (-1)^{[\deg(f)+\deg(g)]\deg(h)}\{h, \{f, g\}\} \\ &+ (-1)^{\deg(f)[\deg(g)+\deg(h)]}\{g, \{h, f\}\} \end{aligned} \tag{3.29}$$

$$\{f, gh\} = \{f, g\}h + (-1)^{\deg(f)\deg(g)} f\{g, h\} \tag{3.30}$$

$$\deg(\{f, g\}) = \deg(f) + \deg(g)\,. \tag{3.31}$$

The first two properties state that the generalized Poisson bracket gives rise to a Lie superalgebra (**??**) and the third property states that it is a *Poisson superalgebra*. In fact, this is the example that lends its name to the algebraic structure. Geometrically, the matrix σ gives rise to a *supersymplectic structure*.

3.2 Hamiltonian BRST theory

3.2.1 Introduction

Consider a dynamical system (M, ω, H) together with a set of first-class constraints $\{\phi_a\}_{a\in I}$. As was shown before, these constraints generate an algebra under the Poisson bracket. However, even more structure is present. To explore this structure, the phase space can be enlarged by introducing, for every constraint and every relation between constraints, a Grassmann-odd[2] **ghost variable** η^a and its canonical conjugate $\mathcal{P}_a$, i.e.

$$\{\mathcal{P}_a, \eta^b\} := -\delta^b_a\,. \tag{3.32}$$

These come with two types of grading:

1. the **pure ghost number** generated by $\mathsf{pure\,gh}(\mathcal{P}_a) = 0$ and $\mathsf{pure\,gh}(\eta^a) = 1$, and
2. the **antighost number** generated by $\mathsf{antigh}(\mathcal{P}_a) = 1$ and $\mathsf{antigh}(\eta^a) = 0$.

The (total) **ghost number** is defined as the difference of these ghost numbers:

$$\mathsf{gh}(f) := \mathsf{pure\,gh}(f) - \mathsf{antigh}(f)\,. \tag{3.33}$$

Moreover, both the ghosts and ghost momenta have Grassmann parity $\varepsilon_a + 1$.

On this extended phase space, one can then construct a nilpotent function Ω, the **BRST generator**, that induces a cohomology theory (**??**) through the Poisson bracket:

$$s := \{\cdot, \Omega\}\,. \tag{3.34}$$

This cohomology theory characterizes the gauge structure (such as gauge-invariant functions).

Remark 3.2.1 (Nonminimal sectors). In certain situations, it is useful to extend the phase space even further by introducing additional conjugate pairs, e.g. the Lagrange multipliers in an action principle. Such descriptions are said to belong to the nonminimal sector. An example would be the *Nakanishi–Lautrup field B*, introduced when quantizing Yang–Mills theory (see **??**), which is conjugate to the *Faddeev–Popov antighost field* $\bar{c}$, i.e. $\{\bar{c}, \Omega\} = B$. This will be covered further down in Section 3.2.6.

[2]In fact, one can generalize this section to phase spaces that already contain odd variables. In that case, the ghost variables should have the opposite parity of the associated constraints.

3.2.2 Irreducible constraints

In this section, only systems with irreducible constraints are considered, i.e. it will be assumed that no relations between the constraints exist. To formulate the BRST complex in terms of invariant geometric notions, a homological and differential-geometric approach will be adopted.

A first step is to express the algebra of on-shell functions $C^\infty(\Sigma)$ in an invariant way. The idea is to rewrite the quotient $C^\infty(M)/\mathcal{N}$ as a homology group (which is invariant by its very nature). To this end, one passes to the Koszul complex (**??**) associated to the first-class constraints ϕ_a: $C^\infty(M) \otimes \mathbb{C}[\mathcal{P}_a]$. (Independence of the constraints exactly says that they form a regular sequence and, hence, the complex gives a homological resolution.) One then finds that $H_0(\delta) \cong C^\infty(M)/\mathcal{N} \cong C^\infty(\Sigma)$, where δ is the Koszul differential.

Definition 3.2.2 (Antighost number). The degree corresponding to δ is exactly the antighost number and the Koszul generators are the ghost momenta $\mathcal{P}_a$ associated to the constraints ϕ_a. This also implies that $\mathrm{antigh}(\delta) = -1$.

A second step is to characterize the gauge structure of the constraint surface Σ. To this end, a modified exterior derivative on the phase space is introduced. Although the gauge algebra spanned by the constraints ϕ_a does not necessarily generate a closed gauge group on the full phase space, it does so when restricted to the constraint surface. The $|I|$-dimensional leaves of the foliation generated by the constraints are called the **gauge orbits** and the Hamiltonian vector fields associated to the first-class constraints are tangent to these orbits. Furthermore, by the irreducibility of the constraints, the vector fields form a frame field for the tangent bundle of the gauge orbits.

Definition 3.2.3 (Longitudinal complex). Vector fields that are tangent to the gauge orbits are said to be **longitudinal** or **vertical** (not to be confused with the vertical vector fields from **??**). A frame for these vector fields is given by the Hamiltonian vector fields

$$X_a := \{\phi_a, \cdot\}\,. \tag{3.35}$$

The tangent bundle of Σ in M can be decomposed as follows:

$$TM|_\Sigma = T\mathcal{F} \oplus N\Sigma\,, \tag{3.36}$$

where $T\mathcal{F}$ denotes the tangent bundle of the foliation generated by the constraints and $N\Sigma$ denotes the normal bundle to the foliation.

This decomposition also turns the de Rham complex $\Omega(\Sigma) := \Omega(M)|_\Sigma$ into a bicomplex $\Omega^{\bullet,\bullet}(\Sigma) = \Lambda^\bullet T^*\mathcal{F} \otimes \Lambda^\bullet N^*\Sigma$. The longitudinal (or vertical) complex is given by the subcomplex $\Omega^{\bullet,0}(\Sigma)$, i.e. the longitudinal forms η^a are the multilinear duals of the longitudinal vector fields. The **longitudinal derivative** is the projection of the total de Rham differential on the longitudinal subcomplex, i.e. the operator $\mathbf{d} : \Omega^{\bullet,0}(\Sigma) \to \Omega^{\bullet+1,0}(\Sigma)$ given by

$$\mathbf{d}f := \mathrm{d}f = \{\phi_a, f\}\,\eta^a\,, \tag{3.37}$$

$$\mathbf{d}\eta^a := -\frac{1}{2}C^a_{bc}(q,p)\eta^b \wedge \eta^c\,, \tag{3.38}$$

where the $C^a_{bc}(q,p)$ are the structure functions of the constraint algebra. The Grassmann parity of the forms η^a is taken to be $\varepsilon_a + 1$. Note that the action of $\mathbf{d}$ is exactly the action of the Chevalley–Eilenberg differential from **??**. (This is not a coincidence as will be explained further on.)

The longitudinal complex can be extended to all of M by taking its elements to be the forms

$$A \equiv A_{i_1\ldots i_k}(q,p)\eta^{i_1} \wedge \cdots \wedge \eta^{i_k}\,, \tag{3.39}$$

where the coefficients are equivalence classes of weakly equal functions in $C^\infty(M)$.

Definition 3.2.4 (Pure ghost number). The longitudinal differential forms correspond to the ghosts in Section 3.2.1 and, accordingly, their degree corresponds to the (pure) ghost number. This also implies that $\mathsf{pure\ gh}(\mathbf{d}) = 1$.

Note that the number of Koszul generators is the same as the number of ghost fields, since both are induced by the constraints ϕ^a (as foretold in Section 3.2.1). To extend the Poisson bracket to the **extended phase space** containing phase space functions, ghost fields and ghost momenta, the following convention is introduced:

$$\{\mathcal{P}_a, \eta^b\} = -(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}\{\eta^b, \mathcal{P}_a\} := -\delta^b_a\,. \tag{3.40}$$

Definition 3.2.5 (Ghost number). The total ghost number of an element in the coordinate superalgebra $C^\infty(M) \otimes \mathbb{C}[\eta^a] \otimes \mathbb{C}[\mathcal{P}_a]$ on the extended phase space is defined as follows:

$$\mathsf{gh}(A) := \mathsf{pure\ gh}(A) - \mathsf{antigh}(A)\,. \tag{3.41}$$

It satisfies

$$\mathsf{gh}(AB) = \mathsf{gh}(A) + \mathsf{gh}(B)\,. \tag{3.42}$$

Property 3.2.6. The ghost number is equal to the eigenvalue of the operator

$$\mathcal{G} := i\eta^a\mathcal{P}_a\,. \tag{3.43}$$

When passing to longitudinal forms on all of M, the operator $\mathbf{d}$ fails to be a differential, i.e. $\mathbf{d}^2 \neq 0$ on M. It is only weakly zero outside Σ. Furthermore, when extending the longitudinal derivative $\mathbf{d}$ to the extended phase space, one has the freedom to choose the action on the ghost momenta under the constraint that $\mathrm{gh}(\mathbf{d}) = 1$ and $\mathrm{antigh}(\mathbf{d}) = 0$.[3] By making the choice

$$\mathbf{d}\mathcal{P}_a := (-1)^{\varepsilon_a}\eta^c C^b_{ca}\mathcal{P}_b\,, \tag{3.44}$$

the Koszul differential and longitudinal derivative satisfy $[\delta, \mathbf{d}] = 0$. This also turns $\mathbf{d}$ into a differential modulo δ (**??**). The homology of δ can be generalized to the full extended phase space by tensoring the Koszul resolution of $C^\infty(\Sigma)$ by $\mathbb{C}[\eta^a]$, since the latter is a free and, in particular, projective module. The cohomology of $\mathbf{d}$ modulo δ on M can be shown to coincide with the cohomology of $\mathbf{d}$ on Σ. This is exactly the **BRST cohomology** from the introduction.

Homological perturbation theory (**??**) now also says that there exists a true differential s on the extended phase space generating BRST cohomology.

Definition 3.2.7 (BRST operator). To any dynamical system governed by first-class constraints $\{\phi_a\}_{a\in I}$, one can associate a BRST differential[4]

$$\begin{aligned} s &= \delta + \mathbf{d} + \cdots\,, \\ \varepsilon(s) &= 1\,, \\ \mathrm{gh}(s) &= 1\,. \end{aligned} \tag{3.45}$$

given by the Poisson bracket with a **BRST function** Ω

$$sA = \{A, \Omega\} \tag{3.46}$$

that satisfies the following conditions:

1. It is of ghost degree 1:

$$\mathrm{gh}(\Omega) = 1\,. \tag{3.47}$$

2. It is nilpotent with respect to the Poisson bracket:

$$\{\Omega, \Omega\} = 0\,. \tag{3.48}$$

3. It is real/Hermitian:

$$\Omega^* = \Omega\,. \tag{3.49}$$

4. It is proportional to the constraints to lowest order:

$$\Omega = \eta^a\phi_a + \text{terms of nonzero pure ghost number}^{5}\,. \tag{3.50}$$

[3] Because δ is a boundary operator, i.e. it decreases the degree, there is less freedom in defining $\delta\eta^a$. Only $\delta\eta^a = 0$ is allowed.

[4] BRST stands for *Becchi*, *Rouet*, *Stora* and *Tyutin*.

Remark 3.2.8 (Off-shell closure). It should be noted that the nilpotency of the BRST operator not only holds on-shell, but everywhere on M. In this sense, the ghost momenta appearing in its definition are the fields necessary to close the algebra outside the constraint surface. In fact, it is important to work in the Hamiltonian formalism if one wants to achieve this off-shell nilpotency. It has been shown that in the Lagrangian formalism, this property cannot hold for gauge transformations that only close on-shell. (This latter property is related to the fact that the structure coefficients are generally functions of the canonical variables. Only when they are constants does the algebra of canonical transformations generated by the constraints close off-shell.) In fact, 'the BRST complex' historically only refers to the longitudinal complex. It only gives a resolution for the quotient by the constraint algebra. The full complex as considered here is called the **Batalin–Fradkin–Vilkovisky complex** (BFV theory).

It can be shown that the BRST operator only depends on the constraint surface and not on the choice of a local description.

Property 3.2.9 (Uniqueness). Any two BRST generators Ω, Ω' for the same constraint surface are related by a canonical transformation in the extended phase space.

Example 3.2.10 (Abelian constraint algebra). When the constraints form an Abelian algebra, i.e. $\{\phi_a, \phi_b\} = 0$, the terms involving higher ghost momenta vanish:

$$\Omega = \eta^a \phi_a \,. \tag{3.51}$$

Example 3.2.11 (Closed constraint algebra). When the constraints form a closed algebra, i.e. $\{\phi_a, \phi_b\} = C^c_{ab}\phi_c$ with C^c_{ab} constant, the BRST operator has a slightly more complex expression since the zeroth order term in the BRST expansion is not nilpotent on its own. However, since the structure functions C^c_{ab} are constants, all higher order terms still vanish:

$$\Omega = \eta^a \phi_a - \frac{(-1)^{\varepsilon_b}}{2}\eta^b\eta^c C^a_{bc}\mathcal{P}_a \,. \tag{3.52}$$

More generally, the BRST generator has this form in lowest degree:

$$\Omega = \eta^a \phi_a - \frac{(-1)^{\varepsilon_b}}{2}\eta^b\eta^c C^a_{cb}\mathcal{P}_a + \text{higher-order terms} \,. \tag{3.53}$$

Property 3.2.12 (BRST cohomology). For all functions f on the extended phase space, one has the following equality:

$$s^2 f = \{\{f, \Omega\}, \Omega\} = 0 \,, \tag{3.54}$$

i.e. s is a proper differential. In view of this structure, one says that a function is BRST closed function if

$$\{f, \Omega\} = 0 \tag{3.55}$$

[5] To ensure that the ghost number of Ω is 1, this means that the extra terms are at least quadratic in the ghost fields.

and that it is BRST exactwhen it can be written as

$$f = \{g, \Omega\} \tag{3.56}$$

for some function g. It is clear that the resulting BRST cohomology theory is gauge-invariant, since Ω is gauge-invariant.

3.2.3 Reducible constraints

In this section, the irreducibility requirement for the first-class constraints is relaxed. To recover the BRST complex as a homological object, one has to modify the construction from the previous section. First of all, the Koszul complex is not a resolution of $C^\infty(\Sigma)$ anymore. Because higher-order relations exist among the constraints, the higher-degree homology groups do not vanish. Mathematically, the issue is that the constraints do not form a regular sequence anymore. However, they still generate a module and so a Koszul–Tate resolution exists (**??**). Instead of only introducing ghost momenta corresponding to constraints, one also has to introduce ghost-of-ghosts.

A second problem occurs when trying to define the longitudinal complex and trying to combine it with the Koszul–Tate complex. The number of ghost momenta is now greater than the number of (longitudinal) ghost fields and, furthermore, the longitudinal algebra is not a tensor product $C^\infty(\Sigma) \otimes \mathbb{C}[\eta^a]$ due to the existence of relations among the constraints. The solution here is again to pass to a larger structure that has the correct 'homotopical' structure. In this case, this will be a Sullivan model (**??**), i.e. the Chevalley–Eilenberg algebra associated to an L_∞-algebroid (**??**).

Formula 3.2.13 (Ghost number). Due to the introduction of ghost-of-ghosts, Eq. (3.43) has to be modified. Let η^{a_1} denote the ghost fields, i.e. fields of (pure) ghost number 1, η^{a_2} the ghost-of-ghosts, i.e. fields of (pure) ghost number 2, etc. The total ghost number operator is then given by

$$\mathcal{G} := i\sum_{n=1}^{+\infty}(n+1)\eta^{a_n}\mathcal{P}_{a_n} + \text{constants due to operator ordering}\,, \tag{3.57}$$

where $\mathcal{P}_{a_n}$ are the ghost-of-ghost momenta of antighost number i.

Construction 3.2.14 (Longitudinal complex). Whereas there were as many Koszul generators as longitudinal basis forms in the case of irreducible algebras, this is not the case anymore for reducible algebras. Moreover, for reducible theories, the extended phase space can only locally be given a tensor product space. Sullivan models will allow to consider an equivalent dgca, with a global tensor product structure and as many longitudinal generators as Koszul–Tate generators, such that the associated cohomology coincides with the longitudinal cohomology.

In reducible theories, the constraints are related as follows

$$Z^{a_0}_{a_1}\phi_{a_0} = 0 \tag{3.58}$$

for some functions $Z^{a_0}_{a_1} : M \to \mathbb{R}$. The coefficients in the overcomplete 'basis' X_a of longitudinal vector fields (3.35) then satisfy the following relations:

$$(-1)^{\varepsilon_{a_0}(\varepsilon_{a_1}+1)}Z^{a_0}_{a_1}X_{a_0} = 0\,. \tag{3.59}$$

Of course, higher-order reducibility then corresponds to further relations $Z^{a_1}_{a_2}$ among these relations. For every reducibility identity of order $k \in \mathbb{N}$, one then adds a ghost(-of-ghost) generator η^{a_k} satisfying

$$\Delta\eta^{a_k} := (-1)^{\varepsilon_{a_k}+k+1}\eta^{a_k+1}Z^{a_k}_{a_k+1}\,. \tag{3.60}$$

and

$$\text{pure gh}(\eta^{a_k}) = k+1 \qquad \text{and} \qquad \varepsilon(\eta^{a_k}) = \varepsilon_{a_k}+k+1\,. \tag{3.61}$$

Note that the η^{a_k} are not given by differential forms as they were in the irreducible case!

Equation (3.53) can be generalized to the reducible setting by including the reducibiliy relations.

Formula 3.2.15. The BRST generator can be expanded as follows:

$$\Omega = \eta^{a_0}\phi_{a_0} + \eta^{a_k}Z^{a_k-1}_{a_k}\mathcal{P}_{a_k-1} + \text{higher-order terms}\,. \tag{3.62}$$

Remark 3.2.16 (Chevalley–Eilenberg complex). As has been noted before, there are some relations between BRST complexes and Chevalley–Eilenberg algebras. In fact, these relations are no mere coincidences. The constraint algebra

$$\{\phi_a, \phi_b\} = C^c_{ab}\phi_c \tag{3.63}$$

defines a Lie algebroid in the case of irreducible constraints and a (potentially truncated) L_∞-algebroid in the case of reducible constraints. Similar to **??** in Lie algebra cohomology, one can characterize invariants of Lie algebroids in terms of their Chevalley–Eilenberg cohomology. From this point of view, gauge-invariant functions do not just resemble elements of the zeroth cohomology group of a Chevalley–Eilenberg differential, they are exactly that.

3.2.4 Observables

By construction of the BRST complex, the s-cohomology coincides with the cohomology of the longitudinal differential. In negative ghost degree, it can be shown that BRST cohomology vanishes. In degree 0, one finds the physical observables.

Property 3.2.17 (Gauge-invariant functions). $H^0(s)$ is isomorphic to the set of equivalence classes of weakly equal, gauge-invariant functions.

Given a BRST-closed function f of ghost degree 0, the associated **classical observable** is obtained as the term of antighost number 0 in its BRST expansion. Conversely, any BRST-closed function of ghost number 0 is called a **BRST-invariant extension** of its term of antighost number 0.

Remark. The interpretation of higher cohomology groups will be adressed in **??**.

Property 3.2.18 (Poisson algebra). The Poisson bracket descends to $H^0(s)$ and defines a (graded) Poisson algebra structure. Furthermore, if f and g are BRST-invariant extensions of f_0 and g_0 respectively, the functions fg and $\{f,g\}$ are BRST-invariant extensions of f_0g_0 and $\{f_0,g_0\}$, respectively.

Example 3.2.19 (Extension of constraints). Consider a first-class constraint ϕ_a. A BRST-invariant extension is given by the Poisson bracket

$$G_a := \{-\mathcal{P}_a, \Omega\}\,. \tag{3.64}$$

This immediately shows that the extension G_a corresponds to the observable 0, since it is s-exact and, hence, vanishes in cohomology. If the constraints form a closed algebra, so do their extensions (due to the property above). However, in general, the BRST extensions do not obey any kind of algebra-like condition.

Remark 3.2.20 (Higher cohomology). The higher cohomology groups $H^{\bullet\geq 1}(s)$ do not have a straightforward physical explanation. $H^1(s)$ and $H^2(s)$ are related to symmetry breaking and anomalies.

3.2.5 Actions and gauge fixing

Consider a classical Hamiltonian H_0 and its BRST extension H. Because of Property 3.2.18, Hamiltonian dynamics can be defined on the entire extended phase space:

$$\frac{\mathrm{d}F}{\mathrm{d}t} := \{F, H\}\,. \tag{3.65}$$

For BRST-invariant functions, this is equivalent to the ordinary equations of motion for their associated classical observables. Because these equations do not have any gauge redundancies, they are said to be **gauge fixed**.

Definition 3.2.21 (Gauge fixing). One can change the Hamiltonian H by a BRST-exact term $\{K,\Omega\}$ without changing the cohomology, i.e. without changing the dynamics of BRST-invariant functions. However, the dynamics of noninvariant functions is modified. For this reason, the function K is called the **gauge-fixing fermion** (since it has to be odd for $\{K,\Omega\}$ to be even).

If the gauge-fixing fermion has the form

$$K = K_0 + k^{a_1}(z,\eta)\mathcal{P}_{a_1} + \text{higher antighost terms}\,, \tag{3.66}$$

the transformation is given by $H \to H + k^{a_1}(z,\eta)\mathcal{P}_{a_1}$. Note that the gauge-fixed Hamiltonian might involve multighost terms.

Given a BRST invariant Hamiltonian and gauge-fixing fermion K, a BRST invariant action is given by

$$S_K[z,\eta,\mathcal{P}] := \int_{t_1}^{t_2}\left(\dot{z}^\mu a_\mu(z) + \sum_{k=0}^{+\infty}\dot{\eta}^{a_k}\mathcal{P}_{a_k} - H - \{K,\Omega\}\right)dt\,. \tag{3.67}$$

3.2.6 Nonminimal sectors

In certain situations, it is useful (or even necessary) to enlarge the extended phase space even more, without modifying $H^0(s)$. An example would be generally covariant gauges for relativistic (field) theories.

In their most basic form, nonminimal sectors arise from including cohomologically trivial canonical pairs $(\alpha,\beta,\pi_\alpha,\pi_\beta)$:

$$\begin{aligned} s\alpha &= \beta \\ s\pi_\beta &= \pi_\alpha\,. \end{aligned} \tag{3.68}$$

The BRST generator in this nonminimal sector is then given by $\pi_\alpha\beta$.

Remark 3.2.22. The distinction between minimal and nonminimal is mostly artificial. For example, the trivial condition $\phi \equiv 0$, characterizing an unconstrained system, can be implemented by either $\Omega = 0$ or $\Omega = \mathcal{P}\pi$. When ϕ is not included as an explicit constraint, the latter choice is nonminimal, whereas it becomes minimal when ϕ is included.

The nonminimal sectors can also be chosen in such a way that they contain the 'Lagrange multipliers' of the constraints in the extended Hamiltonian (3.18). The function v^a has as canonical momentum a weakly vanishing function b_a. Including these pairs does, accordingly, not change the dynamics of the theory. To preserve BRST invariance, however, one should also include appropriate ghost and antighost fields (and momenta) $(\rho^a,\overline{C}_a)$.

3.2.7 Faddeev–Popov action

Consider a closed constraint algebra:

$$\{\phi_a,\phi_b\} = C^c_{ab}\phi_c\,. \tag{3.69}$$

In this case, the minimal BRST generator was given by

$$\Omega_0 = \eta^a \phi_a - \frac{1}{2}\eta^a \eta^b C^c_{ba} \mathcal{P}_c\,. \tag{3.70}$$

For scalar functions, the BRST differential acts through gauge variations:

$$\{f, \Omega_0\} = (-1)^{\varepsilon_a} \delta_\eta f\,. \tag{3.71}$$

Note that, whereas the gauge variations usually have an infinitesimal parameter, this is here replaced by the ghosts η. A common (nonminimal) gauge-fixing fermion for Lagrangian field theory is

$$K = i\overline{C}_a \chi^a - \mathcal{P}_a v^a = \frac{i}{2}\overline{C}_a g^{ab} b_b\,, \tag{3.72}$$

where g^{ab} is a 'metric' (it is a proper metric for bosonic theories) and the χ^a's are real functions. This way, the gauge-fixed Hamiltonian contains a quadratic (kinetic) term in the momenta b_a which allows to recover a Lagrangian formulation.[6] Such gauges are called **propagating gauges**, i.e. the ghosts admit a well-defined propagator.

Definition 3.2.23 (Propagating gauge). Propagating gauges, which are quadratic in the conjugate momenta, allow to eliminate these momenta through their own equations of motion. The resulting gauge-fixed action only has BRST invariance on shell. This type of BRST transformations are called **Lagrangian BRST transformations.** The Lagrangian BRST invariance has an associated Noether current: the BRST generator Ω.

The equations of motion derived from the nonminimally extended action for the antighosts $\mathcal{P}_a$ and ρ^a are:

$$\dot{\overline{C}}_a - i\mathcal{P}_a i^{\varepsilon_a} = 0 \tag{3.73}$$

and

$$\dot{\eta}^a + (-1)^{\varepsilon_c} v^c \eta^b C^a_{bc} + i\rho^a (-i)^{\varepsilon_a} = 0\,. \tag{3.74}$$

These can be substituted back into the action to obtain the following gauge-fixed expression:

$$\begin{aligned} S'_K[z, v, b, \eta, \rho] &= S_E + S_{\text{gauge breaking}} + S_{\text{ghost}} \\ &= \int_{t_1}^{t_2} \left[\dot{z}^\mu a_\mu(z) - H_0 - v^a \phi_a\right] dt \\ &\quad + \int_{t_1}^{t_2} \left[\dot{v}^a + i^{\varepsilon_a}\chi^a + \frac{(-i)^{\varepsilon_a}}{2} b^a\right] b_a\, dt \\ &\quad - i^{\varepsilon_a + 1} \int_{t_1}^{t_2} \overline{C}_a \delta_\eta \left[\dot{v}^a + i^{\varepsilon_a}\chi^a\right] dt\,. \end{aligned} \tag{3.75}$$

[6] If the last term were note present, the gauge-fixed Hamiltonian would be linear in b_a and, hence, the momenta can not be eliminated.

This action, in turn, gives the following equations of motion for the momenta b_a conjugate to the Lagrange multipliers v^a:

$$\dot{v}^a + i^{\varepsilon_a}\chi^a + (-i)^{\varepsilon_a} b^a = 0\,. \tag{3.76}$$

Substituting this again, gives the following gauge-fixed ghost action:

$$S''_{\text{ghost}} = -\frac{i^{\varepsilon_a}}{2}\int_{t_1}^{t_2} \left[\left(\dot{v}^a + i^{\varepsilon_a}\chi^a\right)\left(\dot{v}_a + i^{\varepsilon_a}\chi_a\right)\right] dt. \tag{3.77}$$

The above expressions contain three important ingredients:

- the original gauge-invariant action,
- a gauge-symmetry breaking term that determines the Lagrange multipliers and, accordingly, freezes the gauge invariance of the action, and
- a Faddeev–Popov term that is quadratic in the ghosts.

The gauge symmetry-breaking term leads to a derivative gauge:

$$\dot{v}^a + i^{\varepsilon_a}\chi_a = 0\,. \tag{3.78}$$

Property 3.2.24 (Derivative gauges). Derivative gauge can be taken into account in the BRST-BV complex by including the Lagrange multipliers and their momenta as canonical coordinates.

Comparing Eq. (3.75) and Eq. (3.78), one can see that the ghost action contains the 'ghost variation' of the gauge-fixing condition.

Formula 3.2.25 (Ghost action). When the constraint algebra is closed, the Faddeev–Popov ghost Lagangian is of the form

$$L_{\text{ghost}} = \overline{\mathrm{C}}_a \delta_\eta F^a\,, \tag{3.79}$$

where F^a are the gauge-fixing conditions. For soft and open algebras, nonquadratic ghost vertices ruin this form. In fact, this is also the case for closed algebras for nonlinear gauges, i.e. when the gauge-fixing fermion is nonlinear in $\mathcal{P}$ and $\overline{\mathrm{C}}$.

3.3 Lagrangian BRST theory

When considering (classical) constrained Hamiltonian systems (Section 3.2) and their *quantization* (see **??**), the phase space is extended by both ghosts and antighosts. The former corresponded to differential forms along the gauge orbits (giving a resolution of the quotient by the gauge group) and the antighosts correspond to the Koszul–Tate generators characterizing the zero locus of the field equations (i.e. taking the intersection with the equations of motion). Now, what about Lagrangian field theories, where the solutions are not functions in $C^\infty(\mathbb{R}, M)$, but sections of a vector bundle $E \to M$ that solve the field equations $\frac{\delta S}{\delta \phi} = 0$? As shown below, such theories admit symmetries generated by the equations of motion themselves: the zilch symmetries. One can then play the same game as before, with the ordinary phase space replaced by the covariant phase space Σ and the symmetry group replaced by the full gauge group (including true gauge symmetries and zilch symmetries). As a start, the formalism for finite-dimensional systems will be introduced, i.e. the fields reduce to functions of time. Afterwards, the formalism, originally developed by *Peierls*, will be introduced in the absence of local gauge symmetries.

3.3.1 Zero Hamiltonian

In the previous sections, dynamical systems with constraints were considered. Using these tools, one can turn any system evolving under a physical, but nondynamical or external, time parameter t into a system having time as a canonical coordinate. In this setting, the time variable is treated on the same footing as the other coordinates. Such a system is said to be **generally covariant**.

If one starts from the single-particle action

$$S[q,p] = \int \left(p_i \dot{q}^i - H_0\right) dt, \tag{3.80}$$

one can introduce time as a generalized coordinate with momentum p_0 by modifying the action as follows:

$$S[q,p,t,p_0,u] = \int \left[p_0 t' + p_i q'^i - u(p_0 + H_0)\right] d\tau\,, \tag{3.81}$$

where the primes indicate derivatives with respect to the parameter τ and u acts as a Lagrange multiplier. It is easily checked that the resulting equations of motion are the same as for the original action.

The system now involves a single constraint $H_0 = -p_0$, which is first class. It is often called the **Hamiltonian constraint**. Aside from this constraint, the extended action contains no first-class Hamiltonian. Evolution is solely determined by a (first-class) constraint and, therefore, is given by a gauge transformation.

Remark 3.3.1 (Nonholonomicity). Up until now, all constraints were assumed to be holonomic, i.e. they did not explicitly depend on time. The presence of time derivatives is not compatible with the Poisson/Dirac bracket. However, when passing to a generally covariant system as above, the time variable loses its peculiar character and all constraints can be handled in the same way.

Property 3.3.2 (Vanishing Hamiltonian). If the canonical coordinates (q, p) transform as scalars under τ-reparametrizations, the Hamiltonian is weakly zero. Conversely, zero Hamiltonians give rise to reparametrization-invariant systems.

Although in many cases, it is reasonable to introduce a certain split of the coordinates and break general covariance — gauge-invariant observables and constants of motion are usually treated differently — this is not required. For a (weakly) zero Hamiltonian, first class functions are exactly constant of motion.

3.3.2 Field theory

Since the classical notion of phase space as the set of (q, p)-points in coordinate-momentum space, at a given time t, is clearly not covariant (the choice of a time slice ruins any form of relativistic invariance). In the previous section, one possible solution was considered: including time as a canonical coordinate. Here, another approach is embraced which is more convenient in field theory.

Definition 3.3.3 (Covariant phase space). Let S be a local action functional. The covariant phase space Σ associated to S is the set of solutions of the equations of motion

$$\frac{\delta S}{\delta \phi^I} = 0\,, \tag{3.82}$$

i.e. it is exactly the constraint surface.

As before, the physical observables are defined as the smooth functions on this new phase space Σ. These can be described as before. Let $\mathcal{E} \supseteq \Sigma$ be the set of all field histories, e.g. the space of sections $\Gamma(E)$ of a vector bundle. The ring of physical observables $C^\infty(\Sigma)$ is obtained as the quotient of $C^\infty(\mathcal{E})$ by the ideal of functions vanishing on-shell and by the gauge transformations.

3.3.3 Gauge algebra

Consider a local action S on a space M. A **gauge transformation** of S is a general field transformation, depending on M, that leaves the action invariant, i.e. it is a vertical automorphism (**??**) of the field bundle. The most general form of such transformations is

$$\delta_\varepsilon \phi^I = \overline{R}^I_{(0),\alpha}\varepsilon^\alpha + \overline{R}^{I,\mu}_{(1),\alpha}\partial_\mu\varepsilon^\alpha + \cdots + \overline{R}^{I,\mu_1\ldots\mu_s}_{(s),\alpha}\partial_{\mu_1\ldots\mu_s}\varepsilon^\alpha \equiv R^I_\alpha\varepsilon^\alpha\,, \tag{3.83}$$

where the coefficients $\overline{R}_{(i)}$ are general functions of the coordinates and, in the last step, a new shorthand was introduced where the summation over α also includes an integral over x (the **DeWitt convention**):

$$R^I_\alpha \varepsilon^\alpha := \int R^I_\alpha(x,x')\varepsilon^\alpha(x')\,dx' \tag{3.84}$$
$$= \int \sum_j \left(\overline{R}^i_{(0),j}(x)\delta(x-x') + \overline{R}^i_{(1),j}(x)\delta'(x-x') + \overline{R}^i_{(2),j}(x)\delta^{(2)}(x-x') + \cdots\right)\varepsilon^j(x')\,dx'\,.$$

Invariance of the action implies that

$$\delta_\varepsilon S = \frac{\delta S}{\delta\phi^I}\delta_\varepsilon\phi^I = \frac{\delta S}{\delta\phi^I}R^I_\alpha\varepsilon^\alpha = 0\,. \tag{3.85}$$

Because this should hold for every value of the transformation parameters ε^α, one immediately obtains the variational Noether identities.

Property 3.3.4 (Noether identities). If a local action is invariant under the transformation (3.83), then

$$\frac{\delta S}{\delta\phi^I}R^I_\alpha = 0 \tag{3.86}$$

for all 'indices' α. In contrast to Noether's theorem **??**, these identities do not imply conserved quantities. Instead, they imply that the equations of motion are not independent (cf. Noether's second theorem 1.1.6).

The structure of the infinitesimal gauge transformations is easily seen to be that of a (real) Lie algebra $\overline{\mathcal{G}}$, whilst that of finite (exponentiated) transformations is a Lie group. However, the gauge algebra is very large (in fact, it is infinite-dimensional) and contains a lot of physically irrelevant information. The simplest example is that of the **zilch symmetries** as referred to by Van Proeyen and Freedman (2012).

Definition 3.3.5 (Zilch symmetry). All transformations of the form

$$\delta_\varepsilon\phi^I = \varepsilon^{IJ}\frac{\delta S}{\delta\phi^J}\,, \tag{3.87}$$

where ε^{IJ} is antisymmetric, are physically irrelevant since they vanish on-shell by the equations of motion. The trivial gauge transformations form an ideal $\mathcal{N}$ of the gauge algebra and the physically relevant algebra is the quotient $\mathcal{G} := \overline{\mathcal{G}}/\mathcal{N}$. However, for some reasons it might be convenient to retain the full gauge algebra.

Another problem with the gauge algebra is that independent transformations might lead to dependent Noether identities, wich implies that there is still some redundancy. To analyze this issue, one first finds a minimal set of generating transformations.

Definition 3.3.6 (Generating set). A generating set of the gauge algebra (or **complete set of transformations**) is a set of transformations $\delta_\varepsilon\phi^I = R^I_\alpha\varepsilon^\alpha$ such that every gauge transformation can be written as follows:

$$\delta\phi^I = R^I_\alpha\mu^\alpha + M^{IJ}\frac{\delta S}{\delta\phi^J}\,, \tag{3.88}$$

where $M^{IJ} = -M^{JI}$. Because the coefficients might be functions of the fields and their derivatives, the generating set is, in general, not a basis for the gauge algebra. However, due to the Lie algebra structure, there must exist structure functions $C^{\gamma}_{\alpha\beta}$ and $M^{IJ}_{\alpha\beta}$ such that

$$R^J_\alpha \frac{\delta R^I_\beta}{\delta\phi^J} - R^J_\beta \frac{\delta R^I_\alpha}{\delta\phi^J} = C^{\gamma}_{\alpha\beta} R^I_\gamma + M^{IJ}_{\alpha\beta}\frac{\delta S}{\delta\phi^J}\,, \tag{3.89}$$

where $M^{IJ}_{\alpha\beta} = -M^{JI}_{\alpha\beta}$.

In Definition 3.1.9, the structure of the contraint algebra was considered. Comparing Eq. (3.17) to Eq. (3.89), it is clear that the constraint algebra and the generating algebra of the gauge transformations play a similar role:

- If all M are zero, the algebra is said to be **closed**. (Even though the generating set itself might not be closed as a Lie algebra because the C's generally are functions of the fields. The algebra is said to be **soft** in this case.)
- Otherwise, it is said to be **open**.

Similarly, A generating set is said to be **irreducible** if there exist no nontrivial combinations of elements:

$$R^I_\alpha \varepsilon^\alpha = M^{IJ}\frac{\delta S}{\delta\phi^J} \implies \varepsilon^\alpha = N^{\alpha I}\frac{\delta S}{\delta\phi^I}\,. \tag{3.90}$$

The following remark is the Lagrangian counterpart of Remark 3.2.16 in the Hamiltonian treatment of constrained systems.

Remark 3.3.7 (Lie algebroids). If one restricts to closed gauge algebras, i.e. ignores zilch symmetries, Eq. (3.89) is exactly the closure condition for a Lie algebroid (**??**). Higher relations between the generators, i.e. a reducible theory, turn the gauge algebra into a Lie n-algebroid or even a L_∞-algebroid.

3.3.4 Antifields

In the Hamiltonian BRST formalism of Section 3.2, two (co)homological constructions were combined. On the one hand, the quotient of phase space by the constraint algebra, which was modeled by a Chevalley–Eilenberg complex, and, on the other hand, the critical locus of the constraints, modeled by a Koszul–Tate resolution. This led to the extended phase space containing three types of objects:

1. Canonical coordinates: (q, p),
2. Ghosts: η (the Chevalley–Eilenberg generators), and
3. Antighosts: $\mathcal{P}$ (the Koszul–Tate generators).

In the Lagrangian setting, a very similar structure is obtained. However, now, the critical locus is not just the one induced by the Noether identities. One also has to model the equations of motion themselves. This implies that one also has to add antifields, i.e. Koszul–Tate generators corresponding to the field equation $\frac{\delta S}{\delta\phi^I} = 0$. This gives the **BV-BRST complex**.[7]

Definition 3.3.8 (Antifield). For every gauge symmetry $\widehat{G}_a$ in a generating set of the (proper) gauge group, one also introduces a set of antifields $\mathcal{P}_I$. These carry the following cohomological grading:

$$\begin{aligned} \varepsilon(\mathcal{P}_I) &= \varepsilon_I + 1\,, \\ \mathsf{antigh}(\mathcal{P}_I) &= \mathsf{antigh}(\phi^I) + 1 = 1\,. \end{aligned} \tag{3.91}$$

The antifields $\mathcal{P}_I$ are the Koszul–Tate generators associated to the zilch symmetries:

$$\delta\mathcal{P}_I := -\frac{\delta S}{\delta\phi^I}\,. \tag{3.92}$$

The ghost antifields (i.e. the antighosts from Section 3.2) arise as the Koszul–Tate generators induced by the Noether identities (Property 3.3.4), where the higher antifields correspond to reducibility relations among the symmetries, since the Noether identities $\delta(R^I_a\mathcal{P}_I) = 0$ induce elements of $H^1(\delta)$.

If one extends the above grading properties to the antighost fields, one obtains

$$\mathsf{antigh}(\mathcal{P}_a) = 2\,, \tag{3.93}$$

so the antifields are shifted in degree by 1 when compared to the Hamiltonian setting. The Grassmann parity is also shifted compared to the Hamiltonian case:

$$\varepsilon(\mathcal{P}_I) = \varepsilon_I + 1 \qquad\qquad \varepsilon(\mathcal{P}_a) = \varepsilon_a + 1 \qquad\qquad \cdots \tag{3.94}$$

Remark 3.3.9 (Interpretation). An interpretation for this parity shift and the extra antifields is that, in the Lagrangian framework, the field equations can be seen as the fundamental constraints and the Noether identities as reducibility relations. Hence, even without true gauge symmetry, the Lagrangian Koszul–Tate resolution is nontrivial even though the Hamiltonian one was trivial without constraints.

Definition 3.3.10 (Antifield bracket). The antifield bracket (**antibracket**) of two functionals on $C^\infty(\Sigma)\otimes\mathbb{C}[\eta^a]\otimes\mathbb{C}[\mathcal{P}_I]\otimes\mathbb{C}[\mathcal{P}_a]$ is defined as follows:

$$\{f,g\} := \left(\frac{\partial^R f}{\partial\phi^I}\frac{\partial^L g}{\partial\mathcal{P}_I} - \frac{\partial^R f}{\partial\mathcal{P}_I}\frac{\partial^L g}{\partial\phi^I}\right) + \left(\frac{\partial^R f}{\partial\eta^a}\frac{\partial^L g}{\partial\mathcal{P}_a} - \frac{\partial^R f}{\partial\mathcal{P}_a}\frac{\partial^L g}{\partial\eta^a}\right), \tag{3.95}$$

where the index a denotes ghost (anti)fields of arbitrary degree.

[7] BV stands for Batalin–Vilkovisky as it did in the case of BFV theory.

Property 3.3.11. The BV-antibracket has the following algebraic properties:

- It is BRST-odd: $\mathsf{gh}(\{f,g\}) = \mathsf{gh}(f) + \mathsf{gh}(g) + 1$.
- It induces the structure of a Gerstenhaber algebra (**??**), where the degree is the Grassmann parity.

As before, if one considers the extended state space with the fields as 'coordinates' and the antifields as 'momenta', the antibracket gives the structure of an odd symplectic manifold and, in particular, that of a 'BV manifold' as defined below:

$$\{f,g\} = \frac{\partial^R f}{\partial z^\mu}\omega^{\mu\nu}\frac{\partial^L g}{z^\nu}\,, \tag{3.96}$$

where $(z^\mu) \equiv (\phi^I, \eta^a, \mathcal{P}_I, \mathcal{P}_a)$ and

$$(\omega^{\mu\nu}) := \begin{pmatrix} 0 & \delta^I_J + \delta^a_b \\ -\delta^I_J + \delta^a_b & 0 \end{pmatrix}. \tag{3.97}$$

Definition 3.3.12 (BV manifold). A **Batalin–Vilkovisky manifold** is a triple (M,ω,S) where M is a graded manifold, ω is a degree-1 symplectic form and S is a degree-0 function such that the classical master equation (**??**) is satisfied:

$$\{S,S\} = 0\,, \tag{3.98}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket induced by ω (**??**).

The most straightforward example of a BV manifold is the BV-BRST complex associated to a field theory. For this reason, the Poisson bracket is generally called the **antibracket**, while the grading is generally called the **ghost number** and denoted by gh. The function S is, for field theories, given by the action (functional).

Definition 3.3.13 (BV Laplacian). Consider a BV manifold (M,ω,S). A BV Laplacian is an operator Δ on the space of half-densities $|\Omega|^{1/2}(M)$ that satisfies the following conditions:

1. **Nilpotency**: $\Delta^2 = 0$, and
2. **Product rule**: $\Delta(fg) = (\Delta f)g + (-1)^{\varepsilon(f)}f(\Delta g) + (-1)^{\varepsilon(f)}\{f,g\}$.

This implies that the BV Laplacian acts as a graded derivation:

$$\Delta\{f,g\} = \{\Delta f,g\} + (-1)^{\varepsilon(f)+1}\{f,\Delta g\}\,. \tag{3.99}$$

The canonical BV Laplacian in field-antifield coordinates is given by

$$\Delta_{\mathrm{BV}} := \sum_I (-1)^{\varepsilon_I+1}\frac{\partial^R}{\partial\phi^I}\frac{\partial^R}{\partial\mathcal{P}_I} + \sum_a (-1)^{\varepsilon_a+1}\frac{\partial^R}{\partial\eta^a}\frac{\partial^R}{\partial\mathcal{P}_a}\,. \tag{3.100}$$

This operator satisfies

$$\Delta_{\mathrm{BV}} f = -\frac{1}{2}\mathsf{div} X_f\,, \tag{3.101}$$

for all $f \in C^\infty(M)$, where div denotes the divergence (**??**) with respect to the standard Berezinian volume form.

Definition 3.3.14 (BV integral). Consider the shifted cotangent bundle ΠT^*M on an m-dimensional smooth manifold. Let ψ, to be called the **gauge-fixing fermion**, be an odd function of the even coordinates of a Darboux basis (denote these by q). This function determines a projectable Lagrangian submanifold $L_\psi \subset \Pi T^*M$ by the Maslow–Hörmander theorem **??** given by

$$L_\psi = \left\{ (x^i, p_i) \in \Pi T^*M \,\middle|\, p_i = \frac{\partial \psi}{\partial x^i} \right\}. \tag{3.102}$$

The Batalin–Vilkovisky integral of a function $f \in C^\infty(M)$ with respect to ψ is defined as

$$\int_{L_\psi} f \sqrt{\mathsf{Vol}_\omega}\,, \tag{3.103}$$

where $\sqrt{\mathsf{Vol}_\omega}$ is the volume form on Lagrangian submanifolds induced by ω. The BV integral and BV Laplacian interact in the following way (for BV integrable f):

1. If $f = \Delta_{\mathrm{BV}} g$, then $\int_{L_\psi} f = 0$ for all gauge fixing fermions ψ.
2. If $\Delta_{\mathrm{BV}} f = 0$, then $\frac{\mathrm{d}}{\mathrm{d}t}\int_{L_{\psi_t}} f = 0$, where $\{\psi_t\}_{t\in\mathbb{R}}$ is a continuous family of gauge-fixing fermions.

The second property implies that the BV integral is invariant under deformations of the domain of integration.

Formula 3.3.15 (Quantum master equation). Consider the function $f := e^{iS/\hbar}$ on a BV manifold (M, ω, S). Because

$$\Delta_{\mathrm{BV}} f = \frac{i}{\hbar}\Delta S e^{iS/\hbar} + \left(\frac{i}{\hbar}\right)^2 \frac{1}{2}\{S, S\} e^{iS/\hbar}\,, \tag{3.104}$$

the condition that f is BV-harmonic is equivalent to S satisfying

$$\frac{1}{2}\{S, S\} - i\hbar\Delta_{\mathrm{BV}} S = 0\,. \tag{3.105}$$

This equation is called the quantum master equation. Expanding S as a power series in $\hbar$ shows that the order-0 term satisfies the classical master equation (**??**).

Example 3.3.16 (AKSZ model). The Alexandrov–Kontsevich–Schwarz–Zabronsky model considers the mapping space between a dg-manifold (M, Q) and a dg-symplectic manifold (N, ω, X_H), where X_H is Hamiltonian.

For any graded manifold Σ, one can construct the source manifold by taking $M := \Pi T\Sigma$ and $Q := d$. A symplectic form on $C^\infty(M, N)$ is then given by

$$\Omega := \int_{\Pi T\Sigma} \omega_{\mu\nu} \delta\Phi^\mu \delta\Phi^\nu \ \mathsf{Vol} \ . \tag{3.106}$$

The BV action is defined as follows:

$$S := \int_{\Pi T\Sigma} \left(\alpha_\mu d\Phi^\mu + \Theta \right) \ \mathsf{Vol} \,, \tag{3.107}$$

where α is a symplectic potential for ω, which necessarily exists globally by **??** if $\mathsf{gh}(\omega) \neq 0$.

In general, the symplectic form on $C^\infty(M, N)$ is induced from that on N by a pull-push operation. First one pulls back this form along the evaluation map $\mathsf{ev} : C^\infty(M, N) \times M \to N$ and then one pushes it forward along the projection on the first factor (cf. fibre integration (**??**)).

@@ COMPLETE (check S for example) @@

List of Symbols

The following abbreviations and symbols are used throughout the compendium.

Abbreviations

AIC	Akaike information criterion
ARMA	autoregressive moving-average model
BCH	Baker–Campbell–Hausdorff
BPS	Bogomol'nyi–Prasad–Sommerfield
BPST	Belavin–Polyakov–Schwarz–Tyupkin
BRST	Becchi–Rouet–Stora–Tyutin
CCR	canonical commutation relation
CDF	cumulative distribution function
CFT	conformal field theory
CIS	completely integrable system
CP	completely positive
CPTP	completely positive, trace-preserving
CR	Cauchy–Riemann
dga	differential graded algebra
dgca	differential graded-commutative algebra
EMM	equivalent martingale measure
EPR	Einstein–Podolsky–Rosen
ESM	equivalent separating measure
ETCS	Elementary Theory of the Category of Sets
FIP	finite intersection property
FWHM	full width at half maximum
GA	geometric algebra
GHZ	Greenberger–Horne–Zeilinger

GNS	Gel′fand–Naimark–Segal
HJE	Hamilton–Jacobi equation
HoTT	Homotopy Type Theory
KKT	Karush–Kuhn–Tucker
LIVF	left-invariant vector field
MCG	mapping class group
MPO	matrix-product operator
MPS	matrix-product state
MTC	modular tensor category
NDR	neighbourhood deformation retract
OPE	operator product expansion
OTC	over the counter
OZI	Okubo–Zweig–Iizuka
PAC	probably approximately correct
PDF	probability density function
PID	principal ideal domain
PL	piecewise-linear
PMF	probability mass function
POVM	positive operator-valued measure
PRP	predictable representation property
PVM	projection-valued measure
RKHS	reproducing kernel Hilbert space
SVM	support-vector machine
TDSE	time-dependent Schrödinger equation
TISE	time-independent Schrödinger equation
TQFT	topological quantum field theory
TVS	topological vector space
UFD	unique factorization domain
VC	Vapnik–Chervonenkis
VIF	variance inflation factor
VOA	vertex operator algebra
WKB	Wentzel–Kramers–Brillouin

ZFC	Zermelo–Frenkel set theory with the axiom of choice

Operations

Ad_g	adjoint representation of a Lie group G
ad_X	adjoint representation of a Lie algebra $\mathfrak{g}$
arg	argument of a complex number
$\Box$	d'Alembert operator
$\mathsf{deg}(f)$	degree of a polynomial f
e	identity element of a group
$\Gamma(E)$	set of global sections of a fibre bundle E
Im, $\mathfrak{I}$	imaginary part of a complex number
$\mathsf{Ind}_f(z)$	index of a point $z \in \mathbb{C}$ with respect to a function f
$\hookrightarrow$	injective function
$\cong$	is isomorphic to
$A \multimap B$	linear implication
$N \lhd G$	N is a normal subgroup of G
Par_t^γ	parallel transport map along a curve γ
Re, $\mathfrak{R}$	real part of a complex number
Res	residue of a complex function
$\twoheadrightarrow$	surjective function
$\{\cdot,\cdot\}$	Poisson bracket
$X \pitchfork Y$	transversally intersecting manifolds X, Y
∂X	boundary of a topological space X
$\overline{X}$	closure of a topological space X
$X^\circ, \mathring{X}$	interior of a topological space X
$\sphericalangle(\cdot,\cdot)$	angle between two vectors
$X \times Y$	cartesian product of two sets X, Y
$X + Y$	sum of two vector spaces X, Y
$X \oplus Y$	direct sum of two vector spaces X, Y
$V \otimes W$	tensor product of two vector spaces V, W
$\mathbb{1}_X$	identity morphism on an object X
$\approx$	is approximately equal to

$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto

Objects

$\mathbf{Ab}$	category of Abelian groups
$\mathsf{Aut}(X)$	automorphism group of an object X
$\mathcal{B}_0(V, W)$	space of compact bounded operators between two Banach spaces V, W
$\mathcal{B}_1(\mathcal{H})$	space of trace-class operators on a Hilbert space
$\mathcal{B}(V, W)$	space of bounded linear maps between two vector spaces V, W
CartSp	category of Euclidean spaces and 'suitable' morphisms (e.g. linear maps, smooth maps, ...)
$C(X, Y)$	set of continuous functions between two topological spaces X, Y
S'	centralizer of a subset (of a ring)
$C_\bullet$	chain complex
$\mathbf{Ch}(\mathbf{A})$	category of chain complexes with objects in an additive category $\mathbf{A}$
$\mathbf{C}^\infty$, $\mathbf{SmoothSet}$	category of smooth sets
$C^\infty_p(M)$	ring of smooth functions $f : M \to \mathbb{R}$ on a neighbourhood of $p \in M$
$C\ell(A, Q)$	Clifford algebra over an algebra A induced by a quadratic form Q
$C^\omega(V)$	set of all analytic functions defined on a set V
$\mathsf{Conf}(M)$	conformal group of a (pseudo-)Riemannian manifold M
$\mathbf{C}^\infty\mathbf{Ring}$, $\mathbf{C}^\infty\mathbf{Alg}$	category of smooth algebras
$S_k(\Gamma)$	space of cusp forms of weight $k \in \mathbb{R}$
Δ_X	diagonal of a set X
$\mathbf{Diff}$	category of smooth manifolds
$\mathbf{DiffSp}$	category of diffeological spaces and smooth maps
$\mathcal{D}_M$	sheaf of differential operators
D^n	standard n-disk
$\mathsf{dom}(f)$	domain of a function f
$\mathsf{End}(X)$	endomorphism monoid of an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad
$\mathbf{FormalCartSp}_{\mathbf{diff}}$	category of infinitesimally thickened Euclidean spaces

$\mathsf{Frac}(I)$	field of fractions of an integral domain I
$\mathfrak{F}(V)$	space of Fredholm operators on a Banach space V
$\mathbb{G}_a$	additive group (scheme)
$\mathsf{GL}(V)$	general linear group: group of automorphisms of a vector space V
$\mathsf{GL}(n, \mathfrak{K})$	general linear group: group of invertible $n \times n$-matrices over a field $\mathfrak{K}$
Grp	category of groups and group homomorphisms
Grpd	category of groupoids
$\mathsf{Hol}_p(\omega)$	holonomy group at a point p with respect to a principal connection ω
$\mathsf{Hom}_{\mathbf{C}}(V, W)$, $\mathbf{C}(V, W)$	collection of morphisms between two objects V, W in a category $\mathbf{C}$
hTop	homotopy category
$I(S)$	vanishing ideal on an algebraic set S
$I(x)$	rational fractions over an integral domain I
$\mathsf{im}(f)$	image of a function f
$K^0(X)$	K-theory over a (compact Hausdorff) space X
Kan	category of Kan complexes
$K(A)$	Grothendieck completion of a monoid A
$\mathcal{K}_n(A, v)$	Krylov subspace of dimension n generated by a matrix A and a vector v
L^1	space of integrable functions
Law	category of Lawvere theories
$\mathfrak{Lie}$	category of Lie algebras
Lie	category of Lie groups
$\mathfrak{X}^L$	space of left-invariant vector fields on a Lie group
$\mathsf{llp}(I)$	set of morphisms having the left lifting property with respect to I
LX	free loop space on a topological space X
$\mathbf{Man}^p$	category of C^p-manifolds
Meas	• category of measurable spaces and measurable functions, or • category of measure spaces and measure-preserving functions
M^4	four-dimensional Minkowski space
$M_k(\Gamma)$	space of modular forms of weight $k \in \mathbb{R}$
$\mathbb{F}^X$	natural filtration of a stochastic process $(X_t)_{t \in T}$
$N\mathbf{C}$	simplicial nerve of a small category $\mathbf{C}$

$\mathsf{O}(n, \mathfrak{K})$	group of $n \times n$ orthogonal matrices over a field $\mathfrak{K}$
$\mathbf{Open}(X)$	category of open subsets of a topological space X
$P(X), 2^X$	power set of a set X
$\mathsf{Pin}(V)$	pin group of the Clifford algebra $C\ell(V, Q)$
$\mathbf{Psh}(\mathbf{C}), \widehat{\mathbf{C}}$	category of presheaves on a (small) category $\mathbf{C}$
$R((x))$	ring of (formal) Laurent series in x with coefficients in R
$\mathsf{rlp}(I)$	set of morphisms having the right lifting property with respect to I
$R[[x]]$	ring of (formal) power series in x with coefficients in R
S^n	standard n-sphere
$S^n(V)$	space of symmetric rank n tensors over a vector space V
$\mathbf{Sh}(X)$	category of sheaves on a topological space X
$\mathbf{Sh}(\mathbf{C}, J)$	category of J-sheaves on a site $(\mathbf{C}, J)$
$\mathbf{\Delta}$	simplex category
$\mathsf{sing\ supp}(\phi)$	singular support of a distribution ϕ
$\mathsf{SL}_n(\mathfrak{K})$	special linear group: group of all $n \times n$-matrices with unit determinant over a field $\mathfrak{K}$
$W^{m,p}(U)$	Sobolev space in L^p of order m
$\mathbf{Span}(\mathbf{C})$	span category over a category $\mathbf{C}$
$\mathsf{Spec}(R)$	spectrum of a commutative ring R
$\mathbf{sSet}_{\mathrm{Quillen}}$	Quillen's model structure on simplicial sets
$\mathsf{supp}(f)$	support of a function f
$\mathsf{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
$\mathsf{Sym}(X)$	symmetric group of a set X
S_n	symmetric group of degree n
$\mathsf{Sym}(X)$	symmetric group on a set X
$\mathsf{Sp}(n, \mathfrak{K})$	group of matrices preserving a canonical symplectic form over a field $\mathfrak{K}$
$\mathsf{Sp}(n)$	compact symplectic group
$\mathbb{T}^n$	standard n-torus (n-fold Cartesian product of S^1)
$T_{\leq t}$	set of all elements smaller than (or equal to) $t \in T$ for a partial order T
$\mathsf{TL}_n(\delta)$	Temperley–Lieb algebra with $n-1$ generators and parameter δ
$\mathbf{Top}$	category of topological spaces and continuous functions
$\mathbf{Topos}$	(2-)category of (elementary) topoi and geometric morphisms

$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\mathsf{U}(n, \mathfrak{K})$	group of $n \times n$ unitary matrices over a field $\mathfrak{K}$
$V(I)$	algebraic set corresponding to an ideal I
$\mathbf{Vect}(X)$	category of vector bundles over a manifold X
$\mathbf{Vect}_{\mathfrak{K}}$	category of vector spaces and linear maps over a field $\mathfrak{K}$
Y^X	set of functions between two sets X, Y
$\mathbb{Z}_p$	group of p-adic integers
$\emptyset$	empty set
$\pi_n(X, x_0)$	n^{th} homotopy space over X with basepoint x_0
$[a, b]$	closed interval
$]a, b[$	open interval
$\Lambda^n(V)$	space of antisymmetric rank-n tensors over a vector space V
ΩX	(based) loop space on a topological space X
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on a manifold M
$\rho(A)$	resolvent set of a bounded linear operator A
$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on a manifold M

Units

C	Coulomb
T	Tesla

References

Anderson, I. M. (1992). *The Variational Bicomplex*.

Compère, G. (2019). *Advanced Lectures on General Relativity* (Vol. 952). Springer.

Dirac, P. A. M. (1950). Generalized Hamiltonian Dynamics. *Canadian Journal of Mathematics*, *2*, 129–148.

Henneaux, M., & Teitelboim, C. (1992). *Quantization of Gauge Systems*. Princeton university press.

Takens, F. (1979). A Global Version of the Inverse Problem of the Calculus of Variations. *Journal of Differential Geometry*, *14*(4), 543–562.

Van Proeyen, A., & Freedman, D. (2012). *Supergravity*. Cambridge University Press.

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