Compendium of Mathematics & Physics

Nicolas Dewolf

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Contents

Introduction

Goals

This compendium originated out of the necessity for a compact summary of important theorems and formulas during physics and mathematics classes at university. When the interest in more (and more exotic) subjects grew, this collection lost its compactness and became the chaos it now is. Although there should exist some kind of overall structure, it was not always possible to keep every section self-contained or respect the order of the chapters.

It should definitely not be used as a formal introduction to any subject. It is neither a complete work nor a fact-checked one, so the usefulness and correctness is not guaranteed. However, it can be used as a look-up table for theorems and formulas, and as a guide to the literature. To this end, each chapter begins with a list of useful references. At the same time, only a small number of statements are proven in the text (or appendices). This was done to keep the text as concise as possible (a failed endeavour). However, in some cases the major ideas underlying the proofs are provided.

Structure and conventions

Sections and statements that require more advanced concepts, in particular concepts from later chapters or (higher) category theory, will be labelled by the *clubs* symbol ♣. Some definitions, properties or formulas are given with a proof or an extended explanation whenever I felt like it. These are always contained in a blue frame to make it clear that they are not part of the general compendium. When a section uses notions or results from a different chapter at its core, this will be recalled in a green box at the beginning of the section.

Definitions in the body of the text will be indicated by the use of **bold font**. Notions that have not been defined in this summary but that are relevant or that will be defined further on in the compendium (in which case a reference will be provided) are indicated by *italic text*. Names of authors are also written in *italic*.

Objects from a general category will be denoted by a lower-case letter (depending on the context, upper-case might be used for clarity), functors will be denoted by upper-case letters and the categories themselves will be denoted by symbols in **bold font**. In the later chapters on physics, specific conventions for the different types of vectors will often be adopted. Vectors in Euclidean space will be denoted by a bold font letter with an arrow above, e.g. $\vec{\boldsymbol{a}}$, whereas vectors in Minkowski space (4-vectors) and differential forms will be written without the arrow, e.g. $\boldsymbol{a}$. Matrices and tensors will always be represented by capital letters and, dependent on the context, a specific font will be adopted.

Part I

Set Theory, Algebra & Category Theory

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Chapter 1

Set Theory

Chapter Contents

1.1 Axiomatization

1.1.1 ZFC

The following set of axioms and axiom schemata gives a foundation for axiomatic set theory whilst fixing a number of issues in naive set theory, where the notion of a set is taken for granted. This theory is called **Zermelo–Frenkel** set theory (ZF). When extended with the axiom of choice (see further on), it is called ZFC, where the C stand for 'choice'.

Axiom 1.1 (Power set).

$$\forall x : \exists y : \forall z\big(z \in y \iff \forall w(w \in z \implies w \in x)\big) \tag{1.1}$$

The set y is called the power set $P(x)$ or 2^x of x.

Axiom 1.2 (Extensionality).

$$\forall x,y : \forall z\big(z \in x \iff z \in y\big) \implies x = y \tag{1.2}$$

This axiom allows to compare two sets based on their elements, i.e. two sets are (strictly) equal if and only if they have the same elements.

Remark 1.1.1. As will be made clear throughout this document, strict equality as in the extensionality axiom is often not what one cares about in practice. *Equivalence* and *isomorphism* will be the practically relevant notions.

Axiom 1.3 (Regularity[1]).

$$\forall x : (\exists z \in x) \implies (\exists a \in x) \land \neg(\exists b \in a : b \in x) \tag{1.3}$$

This axiom says that for every nonempty set x, one can find an element $a \in x$ such that x and a are disjoint. Among other things, this axiom implies that no set can contain itself.

The following axiom is technically not an axiom but an axiom schema, i.e. for every predicate φ, one obtains an axiom.

[1] Also called the **axiom of foundation**.

Axiom 1.4 (Specification).

$$\forall w_1, \dots, w_n, A : \exists B : \forall x\big[x \in B \iff \big(x \in A \land \varphi(x, w_1, \dots, w_n, A)\big)\big] \tag{1.4}$$

This axiom (schema) says that, for every set x, one can build another set of elements in x that satisfy a given predicate. By the axiom of extensionality, this subset $B \subseteq A$ is unique.

Notation 1.1.2 (Set builder notation). The sets constructed through the axiom of specification are often denoted as follows:

$$B = \{x \in A \mid \phi(x)\}\,, \tag{1.5}$$

where the parameters $w_1, \dots, w_n$ were suppressed for conciseness.

1.1.2 Material set theory

ZF(C) is an instance of *material set theory*. Every element of a set is itself a set and, hence, possesses some kind of internal structure.

Definition 1.1.3 (Pure set). A set U such that, for every sequence

$$x_n \in x_{n-1} \in \cdots \in x_1 \in U\,, \tag{1.6}$$

all elements x_i are also sets.

Definition 1.1.4 (Urelement[2]). An object that is not a set.

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1.1.3 Structural set theory

In contrast to material set theory, the fundamental notions in this theory are the relations between sets. An element of a set does not have any internal structure and only becomes relevant if one specifies extra structure (given by relations) on the sets. This implies that elements of sets are not sets themselves. In fact, this would be a meaningless statement since, by default, they lack any internal structure. Even stronger, it is meaningless to compare two elements if one does not provide relations or extra structure on the sets.

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[2]Sometimes called an **atom**.

1.1.4 ETCS ♣

Remark. ETCS is the abbreviation of 'Elementary Theory of the Category of Sets'.

Axiom 1.5. The category of sets is a *well-pointed* (*elementary*) *topos* (see **??**).

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1.1.5 Universes

To be able to talk about sets without running into problems such as *Russel's paradox*, where one needs (or wants) to talk about the collection of all things satisfying a certain condition, one can introduce the concept of a 'universal set' or 'universe (of discourse)'. This set takes the place of the 'collection of things' and all operations act within this universe.

Definition 1.1.5 (Grothendieck universe). A pure set U satisfying the following conditions:

1. **Transitivity**: If $x \in U$ and $y \in x$, then $y \in U$.
2. **Power set**: If $x \in U$, then $P(x) \in U$.
3. **Pairing**: If $x, y \in U$, then $\{x, y\} \in U$.
4. **Unions**: If $I \in U$ and $\{x_i\}_{i\in I} \subset U$, then $\bigcup_{i\in I} x_i \in U$.

1.2 Finite sets

Definition 1.2.1 (Bishop finiteness). A set S is said to be (Bishop) finite if there exists a bijection $S \cong [n]$ for some $n \in \mathbb{N}$.

In ZFC, any other reasonable definition of finite sets will be equivalent to the one above. However, in constructive mathematics, this is not true anymore. Because the *internal logic* of *elementary topoi* (see **??**) is constructive, some more information is given here.

Definition 1.2.2 (Kuratowski finiteness). A set S is said to be Kuratowski finite or **finitely indexed set** if there exists a surjection $[n] \twoheadrightarrow S$ for some $n \in \mathbb{N}$.

Definition 1.2.3 (Dedekind finiteness). A set S is said to be Dedekind finite if any injection $S \hookrightarrow S$ is a bijection.

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1.3 Real numbers

Axiom 1.6 (Ordering). The set of real numbers is an *ordered field* (see **??**).

Axiom 1.7 (Dedekind completeness). Every nonempty subset of $\mathbb{R}$ that is bounded from above has a *supremum* (see Definition 1.7.9 and Definition 1.7.38).

Axiom 1.8. The rational numbers form a subset of the real numbers: $\mathbb{Q} \subset \mathbb{R}$.

Remark 1.3.1. There is only one way (up to isomorphisms) to extend the field of rational numbers to the field of reals such that it satisfies the previous axioms.

Definition 1.3.2 (Extended real line).

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\} \equiv [-\infty, \infty] \tag{1.7}$$

1.4 Set operations

Definition 1.4.1 (Cartesian product). Let X, Y be two sets. Their (Cartesian) product is defined as follows:

$$X \times Y := \{(x, y) \mid x \in X \wedge y \in Y\}\,. \tag{1.8}$$

Definition 1.4.2 (Symmetric difference).

$$A \Delta B := (A \backslash B) \cup (B \backslash A) \tag{1.9}$$

Definition 1.4.3 (Diagonal). The diagonal of a set X is defined as follows:

$$\Delta_X := \big\{(x, x) \in X \times X \mid x \in X\big\}\,. \tag{1.10}$$

Definition 1.4.4 (Complement). Let Ω be the universe of discours (Section 1.1.5) and consider $X \subseteq \Omega$. The complement of X is defined as follows:

$$X^c := \Omega \backslash X\,. \tag{1.11}$$

Formula 1.4.5 (de Morgan's laws).

$$\left(\bigcup_i X_i\right)^c = \bigcap_i X_i^c \tag{1.12}$$

$$\left(\bigcap_i X_i\right)^c = \bigcup_i X_i^c \tag{1.13}$$

Definition 1.4.6 (Relation). A relation between two sets X and Y is a subset of the Cartesian product $X \times Y$. A relation on X is then simply a subset of $X \times X$. This definition can easily be extended to n-ary relations by working with subsets of n-fold products.

Definition 1.4.7 (Converse relation). Consider a relation $R \subseteq X \times Y$. The converse relation R^t is defined as follows:

$$R^t := \left\{(y,x) \in Y \times X \mid (x,y) \in R\right\}. \tag{1.14}$$

Definition 1.4.8 (Composition of relations). Consider two relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ between three sets X, Y and Z. The composition $S \circ R$ is defined as follows:

$$S \circ R := \left\{(x,z) \in X \times Z \mid \exists y \in Y : (x,y) \in R \wedge (y,z) \in S\right\}. \tag{1.15}$$

1.5 Functions

1.5.1 (Co)domain

Definition 1.5.1 (Domain). Let $f : X \to Y$ be a function. The set X is called the domain of f.

Notation 1.5.2. The domain of f is denoted by $\mathsf{dom}(f)$.

Definition 1.5.3 (Support). Let $f : X \to \mathbb{R}$ be a function. The support of f is defined as the set of points where f is nonzero.

Notation 1.5.4. The support of f is denoted by $\mathsf{supp}(f)$.

Notation 1.5.5. Let X, Y be two sets. The set of functions $f : X \to Y$ is denoted by Y^X or $\mathsf{Map}(X, Y)$. (See also Definition 3.1.22 for a generalization.)

Definition 1.5.6 (Codomain). Let $f : X \to Y$ be a function. The set Y is called the codomain of f.

Definition 1.5.7 (Image). Let $f : X \to Y$ be a function. The following subset of Y is called the image of f:

$$\mathsf{im}(f) := \{y \in Y \mid \exists x \in X : f(x) = y\}. \tag{1.16}$$

Remark. Some authors use the notions of codomain and image interchangeably.

Definition 1.5.8 (Level set). Consider a function $f : X \to \mathbb{R}$. The following set is called the level set of f at $c \in \mathbb{R}$:

$$f^{-1}(c) := \{x \in X \mid f(x) = c\}. \tag{1.17}$$

For $X = \mathbb{R}^2$ the level sets are called **level curves** and for $X = \mathbb{R}^3$ they are called **level surfaces**. More generally, for functions $f : X \to Y$, one calls $f^{-1}(y)$ the **preimage** or **fibre** of f over y.

1.5.2 Functions

Definition 1.5.9 (Injective). A function $f : A \to B$ is said to be injective or **one-to-one** if the following condition is satisfied:

$$\forall a, a' \in A : f(a) = f(a') \implies a = a' . \tag{1.18}$$

Notation 1.5.10 (Injective function).

$$f : A \hookrightarrow B$$

Definition 1.5.11 (Surjective). A function $f : A \to B$ is said to be surjective or **onto** if the following condition is satisfied:

$$\forall b \in B : \exists a \in A : f(a) = b . \tag{1.19}$$

Notation 1.5.12 (Surjective function).

$$f : A \twoheadrightarrow B$$

Definition 1.5.13 (Bijection). A function that has an inverse. Equivalently, a function that gives a one-to-one correspondence between the elements of the domain and those of the codomain.

Notation 1.5.14 (Isomorphic sets).

$$X \cong Y$$

Theorem 1.5.15 (Cantor–Bernstein–Schröder). *Consider two sets A, B. If there exist injections $A \hookrightarrow B$ and $B \hookrightarrow A$, there exists a bijection $A \cong B$.*

Definition 1.5.16 (Involution). A function $f : A \to A$ such that $f^2 = \mathbb{1}_A$, i.e. f is its own inverse. Every involution is, in particular, a bijection.

1.6 Collections

1.6.1 Families and filters

Definition 1.6.1 (Power set). Let X be a set. The power set is defined as the set of all subsets of X and is (often) denoted by $P(X)$ or 2^X. The existence of this set is enforced by the power set axiom (Axiom 1.1).

Corollary 1.6.2. All sets are elements of their power set: $X \in P(X)$.

Definition 1.6.3 (Collection). A collection of elements of a set X is a subset of X. Often this collection is indexed by some **index set** I, in which case it is often denoted as $\{x_i\}_{i \in I}$.

Definition 1.6.4 (Family). Let X, I be two sets. A family of elements of X with **index set** I is a function $f : I \to X$. A family with index set I is often denoted by $(x_i)_{i\in I}$. In contrast to collections, a family can contain multiple copies of the same element.

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Definition 1.6.5 (Helly family). A Helly family of order $k \in \mathbb{N}$ is a pair (X, F) with $F \subset P(X)$ such that for every finite $G \subset F$:

$$\bigcap_{V\in G} V = \emptyset \implies \exists H \subseteq G : \left(\bigcap_{V\in H} V = \emptyset\right) \wedge \left(|H| \leq k\right). \tag{1.20}$$

A Helly family of order 2 is sometimes said to have the **Helly property**.

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Definition 1.6.6 (Cover). A cover of a set X is a collection of sets $\{V_i\}_{i\in I} \subseteq P(X)$ such that:

$$\bigcup_{i\in I} V_i = X. \tag{1.21}$$

Definition 1.6.7 (Partition). A partition of X is a collection of disjoint subsets $\{A_i\}_{i\in I} \subset P(X)$ such that $\bigcup_{i\in I} A_i = X$, i.e. it is a disjoint cover.

Definition 1.6.8 (Refinement). Let P be a partition of X. A refinement P' of P is a collection of subsets such that every $A \in P$ can be written as a disjoint union of elements in P'. It follows that every refinement is also a partition.

Definition 1.6.9 (Filter). Let X be a set. A family $\mathcal{F} \subseteq P(X)$ is a filter on X if it satisfies the following conditions:

1. **Nonemptiness**: $\emptyset \notin \mathcal{F}$.
2. **Downward closure**: $\forall A, B \in \mathcal{F} : A \cap B \in \mathcal{F}$.
3. **Isotony**: if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Definition 1.6.10 (Filtration). Consider a set X together with a collection of subsets $\{F_i\}_{i\in I}$ indexed by a *totally ordered set* I (see Definition 1.7.5). The collection is said to be a filtration of X if

$$i \leq j \implies F_i \subseteq F_j. \tag{1.22}$$

A filtration is said to be **exhaustive** if $\bigcup_{i\in I} F_i = X$ and **separated** if $\bigcap_{i\in I} F_i = \emptyset$.

Definition 1.6.11 (Associated grading). When one can define *quotient objects*, every filtration $\{F_i\}_{i\in I}$ of X defines an associated graded object $\{G_i := F_i/F_{i-1}\}_{i\in I}$.

1.6.2 Algebra of sets

Definition 1.6.12 (Algebra of sets). A collection $\mathcal{F} \subset P(X)$ is a called an algebra over X if it satisfies the following conditions:

1. **Totality**: $X \in \mathcal{F}$,
2. **Closed under complements**: $\forall E \in \mathcal{F} : E^c \in \mathcal{F}$, and
3. **Closed under finite unions**: $\forall E, E' \in \mathcal{F} : E \cup E' \in \mathcal{F}$.

The pair $(X, \mathcal{F})$ is also called a **field of sets** or **concrete Boolean algebra**.

Extending this definition to closure under countable unions gives another common notion.

Definition 1.6.13 (σ-algebra). A collection $\Sigma \subset P(X)$ is called a σ-algebra over a set X if it satisfies the following conditions:

1. **Totality**: $X \in \Sigma$,
2. **Closed under complements**: $\forall E \in \Sigma : E^c \in \Sigma$, and
3. **Closed under countable unions**: $\forall \{E_n\}_{n\in\mathbb{N}} \subset \Sigma : \bigcup_{n=1}^{+\infty} E_n \in \Sigma$.

The pair (X, Σ) is called a **measurable space**. The elements $E \in \Sigma$ are called **measurable sets**.

Remark 1.6.14. Conditions 1 and 2 imply that both σ-algebra and fields of sets always contain the empty set. Conditions 2 and 3, together with de Morgan's laws (1.12) and (1.13), imply that a σ-algebra is also closed under countable intersections and, analogously, that a field of sets is closed under finite intersections.

Property 1.6.15 (Intersections). The intersection of a collection of σ-algebras is again a σ-algebra.

Definition 1.6.16 (Generated σ-algebras). A σ-algebra $\mathcal{G}$ is said to be generated by a collection of sets $\mathcal{A}$ if

$$\mathcal{G} = \bigcap \{\mathcal{F} \mid \mathcal{F} \text{ is a } \sigma\text{-algebra that contains } \mathcal{A}\}\,. \tag{1.23}$$

Equivalently, it is the smallest σ-algebra containing $\mathcal{A}$.

Notation 1.6.17. The σ-algebra generated by a collection of sets $\mathcal{A}$ is often denoted by $\sigma(\mathcal{A})$.

Definition 1.6.18 (Product σ-algebras). The product σ-algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$ on $X_1 \times X_2$ can be defined in the following equivalent ways:

- $\mathcal{F}$ is generated by $\{A_1 \times \Omega_2 \mid A_1 \in \mathcal{F}_1\} \cup \{\Omega_1 \times A_2 \mid A_2 \in \mathcal{F}_2\}$.
- $\mathcal{F}$ is generated by $\{A_1 \times A_2 \mid A_1 \in \mathcal{F}_1 \wedge A_2 \in \mathcal{F}_2\}$.

Definition 1.6.19 (Monotone class). A collection of sets $\mathcal{A}$ that has the following two properties:

1. For every increasing sequence $A_1 \subset A_2 \subset \cdots : \bigcup_{n=1}^{+\infty} A_n \in \mathcal{A}$.
2. For every decreasing sequence $A_1 \supset A_2 \supset \cdots : \bigcap_{n=1}^{+\infty} A_n \in \mathcal{A}$.

Theorem 1.6.20 (Monotone class theorem). *Let $\mathcal{A}$ be an algebra of sets. If $\mathcal{G}_{\mathcal{A}}$ is the smallest monotone class containing $\mathcal{A}$, it coincides with the σ-algebra generated by $\mathcal{A}$.*

1.7 Ordered sets

1.7.1 Posets

Definition 1.7.1 (Partially ordered set). A set P equipped with a binary relation $\leq$ is called a partially ordered set (or **poset**) if the following 3 conditions are satisfied for all elements $x, y, z \in P$:

1. **Reflexivity**: $x \leq x$,
2. **Antisymmetry**: $x \leq y \wedge y \leq x \implies x = y$, and
3. **Transitivity**: $x \leq y \wedge y \leq z \implies x \leq z$.

If the antisymmetry condition is dropped, a **preorder** is obtained.

Definition 1.7.2 (Order-preserving map). A function $f : (P, \leq) \to (P', \preceq)$ of posets such that $x \leq y \implies f(x) \preceq f(y)$ for all $x, y \in P$. These functions are also said to be **monotone**.

Definition 1.7.3 (Hasse diagram). Consider a finite poset $(P, \leq)$. A Hasse diagram for P is a *graph* where the vertices are given by the elements of P and two vertices x, y are connected upwardly if and only if $x \leq y$ in P.

Example 1.7.4. Consider the power set of $\{0, 1\}$ with its canonical poset structure. A Hasse diagram of this poset is

$$
\begin{tikzcd}
 & \{0,1\} & \\
\{0\} \arrow[ur, dash] & & \{1\}. \arrow[ul, dash] \\
 & \emptyset \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
$$

Definition 1.7.5 (Totally ordered set). A poset P with the property that for all $x, y \in P$, either $x \leq y$ or $y \leq x$. This property is called **totality**.

Definition 1.7.6 (Strict total order). A nonstrict order $\leq$ has an associated strict order $<$ that satisfies $x < y \iff x \leq y \wedge x \neq y$.

Definition 1.7.7 (Linear order). A binary relation $<$ on a set P satisfying the following conditions for all $x, y, z \in P$:

1. **Irreflexivity**: $x \nless x$,
2. **Asymmetry**: $x < y \implies y \nless x$,
3. **Transitivity**: $x < y \wedge y < z \implies x < z$,
4. **Comparison**: $x < z \implies x < y \vee y < z$, and
5. **Connectedness**: $x \nless y \wedge y \nless x \implies x = y$.

Remark 1.7.8. By negation, one can freely pass between linear orders and total orders. However, without the law of the excluded middle, there exists no bijection between them.

Definition 1.7.9 (Supremum). The supremum $\mathsf{sup}(P)$ of a poset P is the least upper bound of P.

Definition 1.7.10 (Infimum). The infimum $\mathsf{inf}(P)$ of a poset P is the greatest lower bound of P.

Definition 1.7.11 (Maximum). If $\mathsf{sup}(P) \in P$, the supremum is called the maximum of P. This is denoted by $\mathsf{max}(P)$.

Definition 1.7.12 (Minimum). If $\mathsf{inf}(P) \in P$, the supremum is called the minimum of P. This is denoted by $\mathsf{min}(P)$.

Definition 1.7.13 (Chain). A totally ordered subset of a poset.

Theorem 1.7.14 (Zorn's lemma[3]). *Let $(P, \leq)$ be a poset. If every chain in P has an upper bound in P, then P has a maximal element.*

Definition 1.7.15 (Directed[4] set). A set P equipped with a preorder $\leq$ with the additional property that every 2-element subset has an upper bound, i.e. for every two elements $x, y \in P$, there exists an element $z \in P$ such that $x \leq z \wedge y \leq z$.

Definition 1.7.16 (Net). A net on a set X is a subset of X indexed by a directed set I.

[3]This theorem is equivalent to the *axiom of choice*.

[4]Sometimes called a **filtered** set or **upward** directed set. **Downward** directed sets are analogously defined with a lower bound for every two elements.

Definition 1.7.17 (Cut). Let $(P, \leq)$ be a totally ordered set. A cut or **decomposition** of P is a pair (A, B) of disjoint subsets such that $P = A \cup B$ in the ordered sense, i.e. every element of A is smaller than every element of B. Cuts can be classified as follows:

- **Jumps**: A has a greatest element and B has a least element.
- **Dedekind cut**: Either A has a greatest element and B has no least element or A has no greatest element, but B has a least element.
- **Gap**: A has no greatest element and B has no least element.

The notion of a filter on a set X (Definition 1.6.9) is a specific example of a more general notion, where the underlying poset is the power set of X.

Definition 1.7.18 (Filter). Let X be a poset. A filter on X is a subset $\mathcal{F} \subseteq X$ such that:

1. **Nonemptiness**: $\mathcal{F} \neq \emptyset$,
2. **Downward closure**: $\forall x, y \in \mathcal{F} : \exists z \in \mathcal{F} : z \leq x \wedge z \leq y$, and
3. **Isotony**: If $x \in \mathcal{F}$ and $x \leq y$, then $y \in \mathcal{F}$.

If there exists no strictly greater filter $\mathcal{F} \subset \mathcal{F}'$, then $\mathcal{F}$ is called an **ultrafilter**.

The following theorem is independent of the ZF axioms, but strictly weaker than the *axiom of choice*.

Theorem 1.7.19 (Ultrafilter lemma). *Every proper filter on a set is contained in an ultrafilter.*

1.7.2 Ordinals ♣

Definition 1.7.20 (Well-ordering). A **well-founded** linear order, i.e. a linear order such that every nonempty subset has a minimal element.

Definition 1.7.21 (Ordinal number). Consider the class of all well-ordered sets. An ordinal (type or rank) is an isomorphism class of well-ordered sets. The class of ordinals is itself well ordered by inclusion of 'initial segments'.

However, this definition gives problems within the ZF(C) framework of set theory, since these equivalence classes are proper classes and not sets. To overcome this problem, one can use a different approach. By using a well-defined construction, one can, for every class, select a particular representative and call this representative the **ordinal number** of all well-ordered sets isomorphic to it.

The most common construction is the one by *Von Neumann*. For every well-ordered set W, there exists a function

$$W \to P(W) : w \mapsto W_{\leq w} := \{w' \in W \mid w' \leq w\} \tag{1.24}$$

that restricts to an order isomorphism on its image. This leads to the following definition.

Definition 1.7.22 (Von Neumann ordinal). A set that is strictly well ordered by membership and such that every element is also a subset.

The first finite von Neumann ordinals are given as an example:

- $0 := \emptyset$,
- $1 := \{0\} = \{\emptyset\}$,
- $2 := \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, and so on.

Property 1.7.23. Every ordinal type is uniquely order-isomorphic to an ordinal number. Consequently, every order-preserving isomorphism between an order type and itself is the identity.

Definition 1.7.24 (Successor). Every ordinal number α has a **successor** α^+. Using the Von Neumann definition, this is simply $\alpha^+ := \alpha \cup \{\alpha\}$. An ordinal that is not the successor of another ordinal number is called a **limit ordinal**.

Definition 1.7.25 (Regular ordinal). A limit ordinal α that is not the limit of a set of smaller ordinals with order type less than α.

Property 1.7.26 (Burali–Forti paradox). The class of all ordinals (and, by extension, the class of all well-ordered sets) is not a set.

1.7.3 Cardinals ♣

There also exist numbers representing the sizes of sets, the **cardinal numbers**. These 'numbers' should satisfy the following conditions:

1. Every set has a well-defined cardinality.
2. Every cardinal number is the cardinality of some set.
3. Bijective sets have the same cardinality.

Guided by these conditions, one could naively use the following definition.

Definition 1.7.27 (Cardinal number[5]). An isomorphism class of sets (under bijections).

[5] Also called the **cardinality** of a set.

However, similar to the problem encountered for ordinals above, these classes are not sets. To solve this, one can also use a similar trick and select a specific representative. For cardinals, the following choice is common.

Alternative Definition 1.7.28 (Cardinal number). The cardinal number of a set is the smallest ordinal rank of any well-order on it, i.e. any ordinal number bijective to it.[6] The cardinal numbers inherit a well-ordering from the ordinal numbers.

Remark 1.7.29 (Ordering). The Cantor–Bernstein–Schröder theorem 1.5.15 induces a partial ordering on cardinal numbers. However, without the axiom of choice this can never be a total ordering. This problem is also apparent in the above definition since the ordinal rank of sets is used together with the *well-ordering theorem*, which is equivalent to the axiom of choice.

Similar to ordinal numbers, one can also define successors of cardinal numbers.

Definition 1.7.30 (Successor). Given a cardinal κ, its successor κ^+ is defined as the smallest cardinal greater than κ.

Remark 1.7.31. It should be noted that the successor of a cardinal number is not necessarily the same as its successor as an ordinal number (in fact, this is only the case for finite cardinals).

Definition 1.7.32 (Regular cardinal). An infinite cardinal κ such that there exists no set of cardinality κ that is the union of less than κ subsets of cardinality less than κ:

$$\left(\kappa = \sum_{i\in I}\lambda_i\right) \wedge \left(\forall i \in I : \lambda_i < \kappa\right) \implies |I| \geq \kappa. \tag{1.25}$$

The following theorem can easily be proven by a diagonal argument.

Theorem 1.7.33 (Cantor). *Let S be a set of cardinality κ. The power set $P(S)$ has cardinality strictly greater than κ.*

1.7.4 Lattices

Definition 1.7.34 (Semilattice). A poset $(P, \leq)$ for which every 2-element subset has a supremum, called the **join**, is called a **join-semilattice**. Similarly, a poset $(P, \leq)$ for which every 2-element subset has an infimum, called the **meet**, is called a **meet-semilattice**.

Notation 1.7.35. The join of $\{x, y\}$ is denoted by $x \wedge y$. The meet of $\{x, y\}$ is denoted by $x \vee y$.

[6] The *well-ordering theorem* (if assumed) assures that this definition coincides with the naive one above.

Definition 1.7.36 (Lattice). A poset that is both a join- and a meet-semilattice.

The above definition also allows for a purely algebraic formulation (in this case, some authors might speak about **lattice-ordered sets**).

Alternative Definition 1.7.37 (Lattice). A lattice is an algebraic structure that admits operations $\wedge, \vee$ that satisfy the following conditions:

1. Both $\wedge$ and $\vee$ are idempotent, commutative and associative.
2. The operations satisfy the **absorption laws**:

$$x \vee (x \wedge y) = x \qquad\qquad x \wedge (x \vee y) = x. \tag{1.26}$$

To go from this definition to the order-theoretic one, define the partial order

$$x \leq y \iff x \wedge y = x\,. \tag{1.27}$$

There exists an equivalent relation for the join.

Definition 1.7.38 (Complete lattice). A lattice is said to be σ-complete (resp. complete) if it admits all[7] countable (resp. all) joins. It can be proven that complete lattices also admit all meets (as a special case of the *Adjoint Functor Theorem* **??**). If only bounded (from above) subsets admit a join, the lattice is said to be **Dedekind-complete**.

Definition 1.7.39 (Bounded lattice). A lattice $(L, \leq, \wedge, \vee)$ that contains a greatest element (denoted by $\top$ or 1) and a smallest element (denoted by $\perp$ or 0) such that

$$\perp \leq x \leq \top \tag{1.28}$$

for all $x \in P$. These elements are the identities for the join and meet operations:

$$x \wedge \top = x \qquad\qquad x \vee \perp = x\,. \tag{1.29}$$

Definition 1.7.40 (Frame). A complete lattice $(L, \leq, \wedge, \vee)$ for which the **infinite distributivity law** is satisfied:

$$y \wedge \left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I}(y \wedge x_i)\,. \tag{1.30}$$

Definition 1.7.41 (Distributive lattice). A lattice $(L, \leq, \wedge, \vee)$ such that

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{1.31}$$

[7] When working with *categories* (see Chapter 2), this has to be restricted to 'all small joins/meets' or, equivalently, the index category should be a set.

and[8]

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{1.32}$$

for all $a, b, c \in L$. If only

$$a \leq c \implies a \vee (b \wedge c) = (a \vee b) \wedge c \tag{1.33}$$

holds, the lattice is said to be **modular**.

Definition 1.7.42 (Complemented lattice). A bounded lattice $(L, \leq, \wedge, \vee, \top, \bot)$ such that for every $a \in L$, there exists at least one $b \in L$ such that

$$a \wedge b = \bot \qquad \text{and} \qquad a \vee b = \top\,. \tag{1.34}$$

If a consistent assignment of **(ortho)complements** exists, i.e. for all $a, b \in L$:

1. **Involutivity**: $a^{\perp\perp} = a$, and
2. **Order-reversing**: $a \leq b \iff b^\perp \leq a^\perp$,

the lattice is said to admit an **orthocomplementation**. When, furthermore,

$$a \leq b \implies a \vee (a^\perp \wedge b) = b \tag{1.35}$$

for all $a, b \in L$, then L is said to be **orthomodular**. (Comparing to the previous definition, this shows that orthomodularity is an even weaker form of distributivity than modularity, where distributivity only holds with respect to complements.)

Property 1.7.43 (de Morgan). All (ortho)complemented lattices satisfy de Morgan's laws (1.12) and (1.13).

Definition 1.7.44 (Heyting algebra). A bounded lattice $(L, \leq, \wedge, \vee, \top, \bot)$ such that, for every two elements $a, b \in L$, there exists a greatest element $x \in L$ such that

$$a \wedge x \leq b\,. \tag{1.36}$$

This element is denoted by $a \to b$. The **pseudocomplement** $\neg a$ of an element $a \in L$ is then defined as $a \to \bot$. Note that pseudocomplements do not define an orthocomplementation since

$$a \vee a^\perp = \top \tag{1.37}$$

does not have to hold.

Definition 1.7.45 (Boolean algebra). A Heyting algebra L in which the **law of excluded middle** holds:

$$\forall a \in L : \neg\neg a = a\,. \tag{1.38}$$

This can be equivalently stated as

$$\forall a \in L : a \vee \neg a = \top\,. \tag{1.39}$$

This is equivalent to a complemented, distributive lattice.

[8]This second condition is actually a consequence of the first.

1.8 Limits

Definition 1.8.1 (Direct system). Let $(I, \leq)$ be a directed set (Definition 1.7.15) and let $(A_i)_{i\in I}$ be a family of objects. Consider a family of morphisms $(f_{ij} : A_i \to A_j)_{i,j\in I}$ between these objects with the following properties:

1. for every $i \in I$: $f_{ii} = \mathbb{1}_{A_i}$, and
2. for every $i \leq j \leq k \in I$: $f_{ik} = f_{jk} \circ f_{ij}$.

The pair (A_i, f_{ij}) is called a direct system (over I).

Definition 1.8.2 (Direct limit). Consider a direct system (A_i, f_{ij}) over a directed set I. The direct (or **inductive**) limit A of this direct system is defined as follows:

$$\varinjlim A_i := \bigsqcup_{i\in I} A_i \Big/ \sim\,, \tag{1.40}$$

where the equivalence relation is given by

$$x \in A_i \sim y \in A_j \iff \exists k \in I : f_{ik}(x) = f_{jk}(y)\,. \tag{1.41}$$

Informally put: two elements are equivalent if they eventually become the same. The operations on A are defined such that the inclusion maps $\phi_i : A_i \to A$ are morphisms.

Definition 1.8.3 (Inverse system). Let $(I, \leq)$ be a directed set (Definition 1.7.15) and let $(A_i)_{i\in I}$ be a family of objects. Consider a family of morphisms $(f_{ij} : A_j \to A_i)_{i,j\in I}$ between these objects with the following properties:

1. for every $i \in I$: $f_{ii} = \mathbb{1}_{A_i}$, and
2. for every $i \leq j \leq k \in I$: $f_{ik} = f_{ij} \circ f_{jk}$.

The pair (A_i, f_{ij}) is called an inverse system (over I).

Definition 1.8.4 (Inverse limit). Consider an inverse system (A_i, f_{ij}) over a directed set I. The inverse (or **projective**) limit A of this inverse system is defined as follows:

$$\varprojlim A_i := \left\{ \vec{a} \in \prod_{i\in I} A_i \,\middle|\, a_i = f_{ij}(a_j), \forall i \leq j \right\}. \tag{1.42}$$

For all $k \in I$, there exists a natural projection $\pi_k : \varprojlim A_i \to A_k$.

Remark 1.8.5. The direct and inverse limit are each other's (*categorical*) dual. The former is a *colimit* while the latter is a *limit* in category theory. (See Section 2.4.4.)

1.9 Partitions

1.9.1 Partition

Definition 1.9.1 (Composition). Let $k, n \in \mathbb{N}$. A k-composition of n is a k-tuple $(t_1, \dots, t_k) \subset \mathbb{N}$ such that $\sum_{i=1}^{k} t_k = n$.

Definition 1.9.2 (Partition). Let $n \in \mathbb{N}$. A partition of n is an ordered composition of n (usually in decreasing order). Hence multiple different composition can determine the same partition.

Definition 1.9.3 (Young diagram[9]). A Young diagram is a visual representation of the partition of an integer n. It is a left justified system of boxes, where every row corresponds to an element of the partition:

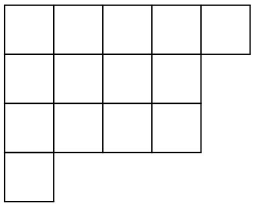

Figure 1.1: A Young diagram representing the partition $(5, 4, 4, 1)$ of 14.

Definition 1.9.4 (Conjugate partition). Let λ be a partition of n with associated Young diagram $\mathcal{D}$. The conjugate partition λ' is obtained by reflecting $\mathcal{D}$ across the main diagonal. Since the number of boxes is left invariant, conjugate partitions are partitions for the same integer.

Example 1.9.5. Conjugating Diagram 1.1 gives Diagram 1.2 below. The associated partition is $(4, 3, 3, 3, 1)$.

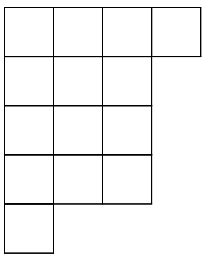

Figure 1.2: A Young diagram representing the partition $(4, 3, 3, 3, 1)$ of 14.

Definition 1.9.6 (Young tableau). Consider a Young diagram of shape λ. A Young tableau of shape λ is a filling of the corresponding Young diagram by the elements of a totally ordered set (with n elements). This tableau is said to be **standard** if every row and every column is increasing.

[9] Sometimes called a **Ferrers diagram**.

Formula 1.9.7 (Hook length formula). The **hook** $H_{i,j}$ is defined as the part of a Young diagram given by the cell (i, j) together with all cells below and to the right of (i, j). Given a hook $H_{i,j}$, define the **hook length** $h_{i,j}$ as the cardinality of $H_{i,j}$.

The number $f^\lambda \in \mathbb{N}$ of possible standard Young tableaux of shape λ, where λ defines a partition of n, is given by the following formula:

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}\,. \tag{1.43}$$

Definition 1.9.8 (Young tabloid). A Young tabloid of shape λ is defined as an equivalence class of Young tableaux that are related by permuting the elements within a row. These are often drawn as in Fig. 1.3. Note that every Young tabloid is represented by exactly one Young tableau.

$$\begin{array}{ccccc}
\hline
1 & 2 & 3 & 5 & 8 \\
\hline
4 & 6 & 9 & 10 & \\
\cline{1-4}
7 & 11 & 12 & 14 & \\
\cline{1-4}
15 & & & & \\
\cline{1-1}
\end{array}$$

Figure 1.3: A Young tabloid associated to the Young diagram in Figure 1.1.

1.9.2 Superpartition ♣

For the physical background of the notions introduced in this section, see Section 6.4.

Definition 1.9.9 (Superpartition). Let $m, n \in \mathbb{N}$. A superpartition in the m-**fermion sector** is a sequence of integers of the following form:

$$\Lambda = (\Lambda_1, \dots, \Lambda_m; \Lambda_{m+1}, \dots, \Lambda_n)\,, \tag{1.44}$$

where the first m numbers are strictly ordered, i.e. $\Lambda_i > \Lambda_{i+1}$ for all $i < m$, and the last $n - m$ numbers form a normal partition.

Both sequences, separated by a semicolon, form in fact distinct partitions themself. The first one represents the **antisymmetric, fermionic sector** (this explains the strict order) and the second one represents the **symmetric, bosonic sector**. This amounts to the following notation:

$$\Lambda \equiv (\lambda^a; \lambda^s)\,.$$

The **degree** of the superpartition is given by $|\Lambda| := \sum_{i=1}^n \Lambda_i$.

Notation 1.9.10. A superpartition of degree $n \in \mathbb{N}$ in the m-fermion sector is said to be a superpartition of $(n|m)$. To every superpartition Λ, one can also associate a unique partition Λ^* by removing the semicolon and reordering the numbers such that they form a partition of n. The superpartition Λ can then be represented by the Young diagram belonging to Λ^*, where the rows belonging to the fermionic sector are terminated by a circle.

Chapter 2

Category theory

For the general theory of categories, the classical reference is Mac Lane (2013) or the more modern account Riehl (2017). The main reference for (co)end calculus is Loregian (2021), while a thorough introduction to the theory of enrichment is given in Kelly (1982). For the theory of higher categories and its applications to topology and algebra, the reader is referred to the book by Baez and May (2009). A good starting point for bicategories (and more) is the paper by Leinster (1998).

Chapter Contents

2.1 Categories

Definition 2.1.1 (Category). A category $\mathbf{C}$ consists of two collections, the objects $\mathrm{ob}(\mathbf{C})$ and the morphisms $\mathrm{hom}(\mathbf{C})$ or $\mathrm{mor}(\mathbf{C})$, that satisfy the following conditions:

1. **Source and target**: For every morphism $f \in \mathrm{hom}(\mathbf{C})$, there exist two objects $s(f), t(f) \in \mathrm{ob}(\mathbf{C})$, the source and the target. The collection of all morphisms with source x and target y is denoted by $\mathrm{Hom}_{\mathbf{C}}(x, y)$ or $\mathbf{C}(x, y)$.

2. **Composition**: For every two morphisms $f \in \mathbf{C}(y, z)$ and $g \in \mathbf{C}(x, y)$, the composite $f \circ g$ is an element of $\mathbf{C}(x, z)$. Moreover, composition is required to be associative.

3. **Identity**: For every $x \in \mathrm{ob}(\mathbf{C})$, there exists an identity morphism $\mathbb{1}_x \in \mathbf{C}(x, x)$. Identity morphisms are required to satisfy $f \circ \mathbb{1}_x = f = \mathbb{1}_y \circ f$ for all morphisms $f \in \mathbf{C}(x, y)$.

Remark 2.1.2. One technically does not need to consider objects as a separate notion since every object can be identified with its identity morphism (which exists by definition) and, hence, one can work solely with morphisms. It should be noted that for higher categories this remark can be omitted since the objects are always regarded as 0-morphisms in that context.

Definition 2.1.3 (Subcategory). Consider two categories $\mathbf{C}$ and $\mathbf{S}$. $\mathbf{S}$ is called a subcategory of $\mathbf{C}$ if $\mathrm{ob}(\mathbf{S})$ and $\mathrm{hom}(\mathbf{S})$ are subcollections of $\mathrm{ob}(\mathbf{C})$ and $\mathrm{hom}(\mathbf{C})$, respectively.

A subcategory is said to be **full** if for every two objects $x, y \in \mathrm{ob}(\mathbf{S})$:

$$\mathbf{S}(x, y) = \mathbf{C}(x, y)\,. \tag{2.1}$$

A subcategory is said to be **wide** or **lluf** if it contains all objects:

$$\mathrm{ob}(\mathbf{S}) = \mathrm{ob}(\mathbf{C})\,. \tag{2.2}$$

Definition 2.1.4 (Replete subcategory). A subcategory $\mathbf{S} \subseteq \mathbf{C}$ such that if $x \in \mathrm{ob}(\mathbf{S})$ and $f : x \cong y \in \mathrm{hom}(\mathbf{C})$, then also $y \in \mathrm{ob}(\mathbf{S})$ and $f \in \mathrm{hom}(\mathbf{S})$.

Definition 2.1.5 (Small category). A category $\mathbf{C}$ for which both $\mathrm{ob}(\mathbf{C})$ and $\mathrm{hom}(\mathbf{C})$ are sets. A category $\mathbf{C}$ is said to be **locally small** if for every two objects $x, y \in \mathrm{ob}(\mathbf{C})$ the collection of morphisms $\mathbf{C}(x, y)$ is a set. A category *equivalent* (see further down below) to a small category is said to be **essentially small**.

Definition 2.1.6 (Opposite category). Let $\mathbf{C}$ be a category. The opposite category $\mathbf{C}^{\mathrm{op}}$ is constructed by reversing all arrows in $\mathbf{C}$, i.e. a morphism in $\mathbf{C}^{\mathrm{op}}(x, y)$ is a morphism in $\mathbf{C}(y, x)$.

Property 2.1.7 (Involution). From the definition of the opposite category it readily follows that *op* is an involution:

$$(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}\,. \tag{2.3}$$

2.2 Functors

Definition 2.2.1 (Covariant functor). Let $\mathbf{C}, \mathbf{D}$ be categories. A (covariant) functor is an assignment $F : \mathbf{C} \to \mathbf{D}$ satisfying the following conditions:

1. F maps every object $x \in \mathsf{ob}(\mathbf{C})$ to an object $Fx \in \mathsf{ob}(\mathbf{D})$.
2. F maps every morphism $\phi \in \mathbf{C}(x, y)$ to a morphism $F\phi \in \mathbf{D}(Fx, Fy)$.
3. F preserves identities, i.e. $F\mathbb{1}_x = \mathbb{1}_{Fx}$.
4. F preserves compositions, i.e. $F(\phi \circ \psi) = F\phi \circ F\psi$.

Remark 2.2.2 (Category of categories). Small categories, together with (covariant) functors between them, form a category $\mathbf{Cat}$. The restriction to small categories is important since otherwise one would obtain an inconsistency similar to *Russell's paradox*. In certain foundations one can also consider the 'category' $\mathbf{CAT}$ of all categories, but this would not be a large category anymore. It would be something like a 'very large' category.

Definition 2.2.3 (Contravariant functor). Let $\mathbf{C}, \mathbf{D}$ be categories. A contravariant functor is an assignment $F : \mathbf{C} \to \mathbf{D}$ satisfying the following conditions:

1. F maps every object $x \in \mathsf{ob}(\mathbf{C})$ to an object $Fx \in \mathsf{ob}(\mathbf{D})$.
2. F maps every morphism $\phi \in \mathbf{C}(x, y)$ to a morphism $F\phi \in \mathbf{D}(Fy, Fx)$.
3. F preserves identities, i.e. $F\mathbb{1}_x = \mathbb{1}_{Fx}$.
4. F reverses compositions, i.e. $F(\phi \circ \psi) = F\psi \circ F\phi$.

A contravariant functor can also be defined as a covariant functor from the opposite category and, accordingly, from now on the word 'covariant' will be dropped when talking about functors.

Definition 2.2.4 (Endofunctor). A functor of the form $F : \mathbf{C} \to \mathbf{C}$.

Definition 2.2.5 (Presheaf). A functor $G : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. The collection of all presheaves on a (small) category $\mathbf{C}$ forms a category $\mathbf{Psh}(\mathbf{C})$. This is sometimes also denoted by $\widehat{\mathbf{C}}$.

Example 2.2.6 (Hom-functor). Let $\mathbf{C}$ be a locally small category. Every object $x \in \mathrm{ob}(\mathbf{C})$ induces a functor $h^x : \mathbf{C} \to \mathbf{Set}$ defined as follows:

- h^x maps every object $y \in \mathrm{ob}(\mathbf{C})$ to the set $\mathbf{C}(x, y)$.
- For all $y, z \in \mathrm{ob}(\mathbf{C})$, h^x maps every morphism $f \in \mathbf{C}(y, z)$ to the fucntion

$$f \circ - : \mathbf{C}(x, y) \to \mathbf{C}(x, z) : g \mapsto f \circ g \,.$$

Remark 2.2.7. The contravariant hom-functor h_x is defined by replacing $\mathbf{C}(x, -)$ with $\mathbf{C}(-, x)$ and replacing postcomposition with precomposition.

Definition 2.2.8 (Faithful functor). A functor $F : \mathbf{C} \to \mathbf{D}$ for which the map

$$\mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$$

is injective for all objects $x, y \in \mathrm{ob}(\mathbf{C})$.

Definition 2.2.9 (Full functor). A functor $F : \mathbf{C} \to \mathbf{D}$ for which the map

$$\mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$$

is surjective for all objects $x, y \in \mathrm{ob}(\mathbf{C})$.

Definition 2.2.10 (Embedding). A fully faithful functor.

Definition 2.2.11 (Concrete category). A category equipped with an embedding into $\mathbf{Set}$. The objects of such categories can be interpreted as sets with additional structure.

Definition 2.2.12 (Essentially surjective functor). A functor $F : \mathbf{C} \to \mathbf{D}$ such that for every object $y \in \mathrm{ob}(\mathbf{D})$, there exists an object $x \in \mathrm{ob}(\mathbf{C})$ with $Fx \cong y$.

Definition 2.2.13 (Profunctor[1]). A functor of the form $F : \mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$. Such a functor is often denoted by $F : \mathbf{C} \nrightarrow \mathbf{D}$.[2] Elements of the set $F(x, y)$ are called **hetero-morphisms** (between x and y).

It should be noted that presheaves on $\mathbf{C}$ are profunctors of the form $1 \nrightarrow \mathbf{C}$.

Definition 2.2.14 (Reflection). A functor $F : \mathbf{C} \to \mathbf{D}$ is said to **reflect** a property if whenever the property holds for Fc, it also holds for $c \in \mathbf{C}$. (Here, c could also be a morphism.)

[1] Sometimes called a **distributor**.

[2] This is the convention by *Borceux*. Some other authors, such as Johnstone (2014), use the opposite convention.

2.2.1 Natural transformations

Definition 2.2.15 (Natural transformation). Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors. A natural transformation $\psi : F \Rightarrow G$[3] consists of a collection of morphisms satisfying the following two conditions:

1. For every object $x \in \mathrm{ob}(\mathbf{C})$, there exists a morphism $\psi_x : Fx \to Gx$ in $\mathrm{hom}(\mathbf{D})$. This morphism is called the **component** of ψ at x. (It is often said that ψ_x **is natural in** x.)

2. For every morphism $f \in \mathbf{C}(x, y)$, the diagram below commutes:

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\psi_x"'] & Fy \arrow[d, "\psi_y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

Definition 2.2.16 (Functor category). Consider two categories $\mathbf{C}$ and $\mathbf{D}$, where $\mathbf{C}$ is small. The functors $F : \mathbf{C} \to \mathbf{D}$ form the objects of a category with the natural transformations as morphisms. This category is denoted by $[\mathbf{C}, \mathbf{D}]$ or $\mathbf{D}^{\mathbf{C}}$ (the latter is a generalization of Notation 1.5.5).

Definition 2.2.17 (Dinatural transformation). Consider two profunctors $F, G : \mathbf{C} \nrightarrow \mathbf{C}$ or, more generally, two functors $F, G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$. A dinatural transformation is a family of morphisms

$$\eta_x : F(x, x) \to G(x, x)$$

that make Diagram 2.1 commute for every morphism $f : y \to x$.

Definition 2.2.18 (Representable functor). Let $\mathbf{C}$ be a locally small category. A functor $F : \mathbf{C} \to \mathbf{Set}$ is said to be representable if there exists an object $x \in \mathrm{ob}(\mathbf{C})$ such that F is naturally isomorphic to h^x. The pair $(x, \psi : F \Rightarrow h^x)$ is called a **representation** of F.

Theorem 2.2.19 (Yoneda lemma). *Let $\mathbf{C}$ be a locally small category and let $F : \mathbf{C} \to \mathbf{Set}$ be a functor. For every object $x \in \mathrm{ob}(\mathbf{C})$, there exists a natural isomorphism*[4]

$$\eta_x : \mathrm{Nat}(h^x, F) \to Fx : \psi \mapsto \psi_x(\mathbb{1}_x)\,. \tag{2.4}$$

Corollary 2.2.20 (Yoneda embedding). When F is another hom-functor h^y, the following result is obtained:

$$\mathrm{Nat}(h^x, h^y) \cong \mathbf{C}(y, x)\,. \tag{2.5}$$

[3]This notation is in analogy with the general notation for *2-morphisms*. See **??** for more information.

[4]Here, the fact that $\mathrm{Nat}(h^-, -)$ can be seen as a functor $\mathbf{Set}^{\mathbf{C}} \times \mathbf{C} \to \mathbf{Set}$ is used.

$$\begin{tikzcd}
& F(x,y) \arrow[dl, "F(\mathbb{1}_x, f)"'] \arrow[dr, "F(f, \mathbb{1}_y)"] & \\
F(x,x) \arrow[d, "\eta_x"'] & & F(y,y) \arrow[d, "\eta_y"] \\
G(x,x) \arrow[dr, "G(f, \mathbb{1}_x)"'] & & G(y,y) \arrow[dl, "G(\mathbb{1}_y, f)"] \\
& G(y,x) &
\end{tikzcd}$$

Figure 2.1: Dinatural transformation.

Note that y appears in the first argument on the right-hand side.

Let $\mathbf{C}(f,-)$ denote the natural transformation corresponding to the morphism $f \in \mathbf{C}(y,x)$. The functor h^-, mapping an object $x \in \mathsf{ob}(\mathbf{C})$ to its hom-functor $\mathbf{C}(x,-)$ and a morphism $f \in \mathbf{C}(y,x)$ to the natural transformation $\mathbf{C}(f,-)$, can also be interpreted as a covariant functor $G : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{C}}$. This way, the Yoneda lemma can be seen to give rise to an embedding h^- of $\mathbf{C}^{\mathrm{op}}$ in the functor category $\mathbf{Set}^{\mathbf{C}}$.

As usual, all of this can be done for contravariant functors. This gives an embedding

$$\mathcal{Y} := h_- : \mathbf{C} \hookrightarrow \mathbf{Psh}(\mathbf{C})\,, \tag{2.6}$$

called the Yoneda embedding.

Definition 2.2.21 (Local object). Consider a collection of morphisms $S \subseteq \mathsf{hom}(\mathbf{C})$. An object $c \in \mathsf{ob}(\mathbf{C})$ is said to be S-local if the Yoneda embedding $\mathcal{Y}c$ maps morphisms in S to isomorphisms in $\mathbf{Set}$. A morphism $f \in \mathsf{hom}(\mathbf{C})$ is said to be S-local if its image under the Yoneda embedding of every S-local object is an isomorphism in $\mathbf{Set}$.

2.2.2 Equivalences

Definition 2.2.22 (Equivalence of categories). Two categories $\mathbf{C}, \mathbf{D}$ are said to be equivalent if there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors.

A weaker notion is that of a **weak equivalence**. Two categories $\mathbf{C}, \mathbf{D}$ are said to be weakly equivalent if there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ that are fully faithful and essentially surjective. Assuming the axiom of choice, every weak equivalence is also a (strong) equivalence (in fact this statement is equivalent to the axiom of choice).

Definition 2.2.23 (Skeletal category). A category in which every isomorphism is necessarily an identity morphism. The **skeleton** of a category is an equivalent skeletal category (often taken to be a subcategory by choosing a representative from every isomorphism class).

If one does not assume the axiom of choice, the skeleton is merely a weakly equivalent skeletal category.

Definition 2.2.24 (Decategorification). Let $\mathbf{C}$ be an (essentially) small category. The set of isomorphism classes of $\mathbf{C}$ is called the decategorification of $\mathbf{C}$. This amounts to a functor $\mathrm{Decat} : \mathbf{Cat} \to \mathbf{Set}$.

2.2.3 Stuff, structure and property

To classify properties of objects and the *forgetfulness* of functors, it is interesting to make a distinction between stuff, structure and property. Consider for example a group. This is a set (*stuff*) equipped with a number of operations (*structure*) that obey some relations (*properties*).

Using these notions one can classify forgetful functors in the following way:

- A functor forgets nothing if it is an equivalence of categories.
- A functor forgets at most properties if it is fully faithful.
- A functor forgets at most structure if it is faithful.
- A functor forgets at most stuff if it is just a functor.

@@ COMPLETE (see e.g. nLab or the paper "Why surplus structure is not superfluous" by Nicholas Teh et al.) @@

2.2.4 Adjunctions

Definition 2.2.25 (Hom-set adjunction). Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be two functors. These functors form a (hom-set) adjunction $F \dashv G$ if the following isomorphism is natural in both x and y:

$$\Phi_{x,y} : \mathbf{D}(Fx, y) \cong \mathbf{C}(x, Gy)\,. \tag{2.7}$$

The functor F (resp. G) is called the left (resp. right) adjoint and the image of a morphism under either of the natural isomorphisms is called the adjunct of the other morphism.[5].

[5] The terms 'adjunct' and 'adjoint' are sometimes used interchangeably (cf. French versus Latin).

Notation 2.2.26. An adjunction $F \dashv G$ between categories $\mathbf{C}, \mathbf{D}$ is often denoted by

$$\begin{tikzcd} \mathbf{D} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, bend right, "F"'] \end{tikzcd}.$$

Definition 2.2.27 (Unit-counit adjunction). Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be two functors. These functors form a unit-counit adjunction if there exist natural transformations

$$\varepsilon : F \circ G \Rightarrow \mathbb{1}_{\mathbf{D}} \tag{2.8}$$

$$\eta : \mathbb{1}_{\mathbf{C}} \Rightarrow G \circ F \tag{2.9}$$

such that the following compositions are identity morphisms:

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F\,, \tag{2.10}$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G\,. \tag{2.11}$$

These identities are sometimes called the **triangle** or **zig-zag identities** (the latter results from the shape of the associated *string diagram*). The transformations η and ε are called the **unit** and **counit**, respectively.

Property 2.2.28 (Equivalence). Every hom-set adjunctions induces a unit-counit adjunction where the counit ε_y is obtained as the adjunct $\Phi^{-1}_{Gy,y}(\mathbb{1}_{Gy})$ of the identity morphism on $Gy \in \mathrm{ob}(\mathbf{C})$ and the unit η_x is given by the adjunct $\Phi_{c,Fc}(\mathbb{1}_{Fx})$ of the identity morphism at $Fx \in \mathrm{ob}(\mathbf{D})$.

Conversely, every unit-counit adjunction induces a hom-set adjunction. The (right) adjunct of a morphism $f : Fx \to y$ is given by the composition

$$\tilde{f} := Gf \circ \eta_x : x \to (G \circ F)x \to Gy \tag{2.12}$$

and the (left) adjunct of a morphism $\tilde{g} : x \to Gy$ is given by:

$$g := \varepsilon_y \circ F\tilde{g} : Fx \to (F \circ G)y \to y\,. \tag{2.13}$$

Definition 2.2.29 (Reflective subcategory). A full subcategory is said to be reflective (resp. coreflective) if the inclusion functor admits a left (resp. right) adjoint.

Property 2.2.30 (Adjoint equivalence). Any equivalence of categories is part of an adjoint equivalence, i.e. an adjunction for which the unit and counit morphisms are invertible.

Property 2.2.31. Given an adjunction, one obtains an adjoint equivalence by restricting to the full subcategories on which the unit and counit become isomorphisms.

Adjunctions can also be defined through a third alternative. This links the definition to universal properties.

Definition 2.2.32 (Universal morphism). Consider a functor $F : \mathbf{C} \to \mathbf{D}$ and an object $d \in \mathsf{ob}(\mathbf{D})$. A universal morphism from d to F is a pair $(c, f : d \to Fc)$ such that all other morphisms $f' : d \to Fc'$ factor uniquely through f by the image of a morphism in $\mathbf{C}$.

Alternative Definition 2.2.33 (Adjoint functor). A functor $F : \mathbf{C} \to \mathbf{D}$ is a left adjoint if for each object $c \in \mathsf{ob}(\mathbf{C})$ there exists a universal morphism from F to c. A functor $G : \mathbf{C} \to \mathbf{D}$ is a right adjoint if for each object $d \in \mathsf{ob}(\mathbf{D})$ there exists a universal morphism from d to G.

The functor to which these functors are adjoint can be recovered as that mapping objects to the object-part of their universal morphism.

2.3 General constructions

Definition 2.3.1 (Dagger category). A category equipped with a contravariant involutive endofunctor, this functor is often denoted by $\dagger : \mathbf{C} \to \mathbf{C}$, similar to the adjoint operator for Hermitian matrices.

Remark 2.3.2. The concept of a dagger structure allows the usual definition of **unitary** and **self-adjoint** morphisms, i.e. morphism satisfying

$$f^\dagger = f^{-1} \qquad \text{or} \qquad f^\dagger = f\,. \tag{2.14}$$

Definition 2.3.3 (Comma category). Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be three categories and let $F : \mathbf{A} \to \mathbf{C}$ and $G : \mathbf{B} \to \mathbf{C}$ be two functors. The comma category $F \downarrow G$ is defined as follows:

- **Objects**: The triples (x, y, γ) where $x \in \mathsf{ob}(\mathbf{A}), y \in \mathsf{ob}(\mathbf{B})$ and $\gamma : Fx \to Gy$.
- **Morphisms**: The morphisms $(x, y, \gamma) \to (k, l, \sigma)$ are pairs (f, g) with $f : x \to k \in \mathsf{hom}(\mathbf{A})$ and $g : y \to l \in \mathsf{hom}(\mathbf{B})$ such that $\sigma \circ Ff = Gg \circ \gamma$.

Composition of morphisms is defined componentwise.

Definition 2.3.4 (Arrow category). The comma category of the pair of functors $(\mathbb{1}_{\mathbf{C}}, \mathbb{1}_{\mathbf{C}})$. This is equivalently the functor category $[\mathbf{2}, \mathbf{C}]$, where $\mathbf{2}$ is the **interval category/walking arrow** $\{0 \to 1\}$.

Definition 2.3.5 (Functorial factorization). A *section* (see Definition 2.4.1) of the composition functor

$$\circ : [\mathbf{3}, \mathbf{C}] \to [\mathbf{2}, \mathbf{C}]\,,$$

where $\mathbf{3}$ is the poset $\{0 \to 1 \to 2\}$.

If F is the identity functor and $G : \mathbf{1} \to \mathbf{C}$ picks out a single object, the notion of a slice category is obtained (by interchanging these choices, one can also define **coslice categories**).

Definition 2.3.6 (Slice category). Let $\mathbf{C}$ be a category and consider an object $x \in \mathrm{ob}(\mathbf{C})$. The slice category (or **overcategory**) $\mathbf{C}_{/x}$ of $\mathbf{C}$ over x is defined as follows:

- **Objects**: The morphisms in $\mathbf{C}$ with codomain x, and
- **Morphisms**: The morphisms $f \to g$ are morphisms h in $\mathbf{C}$ such that $g \circ h = f$.

By dualizing one obtains the **undercategory** of x.

2.3.1 Fibred categories ♣

Definition 2.3.7 (Fibre category). Let $\Pi : \mathbf{C} \to \mathbf{D}$ be a functor. The fibre category (of Π) over $y \in \mathrm{ob}(\mathbf{D})$ is the subcategory of $\mathbf{C}$ consisting of all objects $x \in \mathrm{ob}(\mathbf{C})$ such that $\Pi x = y$ and all morphisms $f \in \mathrm{hom}(\mathbf{C})$ such that $\Pi f = \mathbb{1}_y$. It will be denoted by $\mathbf{C}_y$.

Morphisms in $\mathbf{C}$ that are mapped to a morphism g in $\mathbf{D}$ are called **g-morphisms** and, in particular (using the identification of objects and their identity morphisms), morphisms in $\mathbf{C}_y$ are called **y-morphisms**. Similarly, **D-categories** are defined as the categories equipped with a (covariant) functor to $\mathbf{D}$. (It is not hard to see that these form a *2-category* under composition of functors that respects the $\mathbf{D}$-category structure.)

Definition 2.3.8 (Cartesian morphism). Consider a $\mathbf{D}$-category $\Pi : \mathbf{C} \to \mathbf{D}$. A morphism f in $\mathbf{C}$ is called Π-Cartesian if every Πf-morphism factors uniquely through a y-morphism, where y is the domain of Πf.

There also exists a stronger notion. A **strongly Cartesian morphism** is a morphism $f \in \mathrm{hom}(\mathbf{C})$ such that for every morphism $\varphi \in \mathrm{hom}(\mathbf{C})$ with the same target and every factorization of $\Pi\varphi$ through Πf, there exists a unique factorization of φ through f that maps to the given factorization of $\Pi\varphi$.

The following diagram, where the triangles commute, should clarify the above (technical) definitions:

$$\begin{tikzcd}
\forall x' \arrow[d, dashed, "\exists ! g"'] \arrow[dr, "\forall\varphi"] & & & \Pi x' \arrow[d, "\forall\nu"'] \arrow[dr, "\Pi\varphi"] & \\
x_1 \arrow[r, "f"'] & x_2 & \overset{\Pi}{\mapsto} & \Pi x_1 \arrow[r, "\Pi f"'] & \Pi x_2
\end{tikzcd}$$

The diagram for (weak) Cartesian morphisms is obtained by identifying the objects $\Pi x'$ and Πx_1, i.e. by restricting to the case $\nu = \mathbb{1}_{\Pi x_1}$.

The Cartesian morphisms are said to be **inverse images** of their projections under Π and the object x_1 is called an **inverse image** of x_2 by Πf. The Cartesian morphisms of a fibre category are exactly the isomorphisms of that category.

Definition 2.3.9 (Fibred category). A $\mathbf{D}$-category $\Pi : \mathbf{C} \to \mathbf{D}$ is called a fibred category or **Grothendieck fibration** if the following conditions are satisfied:

1. For each morphism in $\mathbf{D}$, whose codomain lies in the range of Π, and each lift of this codomain to $\mathbf{C}$, there exists at least one inverse image with the given codomain (in the weak sense).
2. The composition of two Cartesian morphisms is again Cartesian (in the weak sense).

If one instead works with strongly Cartesian morphisms, the second condition follows from the first one. However, it should be noted that, in a fibred category, a morphism is weakly Cartesian if and only if it is strongly Cartesian.

Definition 2.3.10 (Cleavage). Given a $\mathbf{D}$-category $\Pi : \mathbf{C} \to \mathbf{D}$, a cleavage is the choice of a Cartesian g-morphism $f : x \to y$ for every $y \in \mathrm{ob}(\mathbf{C})$ and morphism $g : d \to \Pi y$. A $\mathbf{D}$-category equipped with a cleavage is said to be **cloven**.

The existence of cleavage is sufficient for a category to be fibred and, conversely (assuming the axiom of choice), every fibred category admits a cleavage.

The following example can be obtained as a Grothendieck fibration with discrete fibres.

Example 2.3.11 (Discrete fibration). A functor $F : \mathbf{C} \to \mathbf{D}$ such that, for every object $x \in \mathrm{ob}(\mathbf{C})$ and every morphism $f : y \to Fx$ in $\mathbf{D}$, there exists a unique morphism $g : z \to x$ in $\mathbf{C}$ such that $Fg = f$.

Example 2.3.12 (Groupoidal fibration). If every morphism is required to be Cartesian, the notion of a groupoid(al) fibration or a **category fibred in groupoids** is obtained. The reason for this name is that every fibre is a groupoid. An equivalent definition is that the associated *pseudofunctor* (see the construction below) factors through the embedding $\mathbf{Grpd} \hookrightarrow \mathbf{Cat}$.

Property 2.3.13 (Grothendieck construction ♣). Every fibred category $\Pi : \mathbf{C} \to \mathbf{D}$ defines a *pseudofunctor*[6] $F : \mathbf{D}^{\mathrm{op}} \to \mathbf{Cat}$ that sends objects to fibre categories and arrows $f : c \to c'$ to the pullback functor $f^* : \mathbf{C}_{c'} \to \mathbf{C}_c$ constructed from a Cartesian lift of f. This pullback functor acts as follows:

- For every object $x \in \mathbf{C}_{c'}$, f^*x is the domain of the Cartesian lift of f through x.

[6] See **??**.

- For every morphism $(\alpha : x \to y) \in \mathbf{C}_{c'}$ there exists a diagram of the form

$$\begin{tikzcd} f^*x \arrow[r] \arrow[d, dashed, "f^*\alpha"'] & x \arrow[d, "\alpha"] \\ f^*y \arrow[r] & y \end{tikzcd}$$

  Because the horizontal morphism are both projected to f and α is projected to the identity, there exists a unique factorization of the diagram through a morphism $f^*\alpha : f^*x \to f^*y$.

Conversely, every *pseudofunctor* gives rise to a fibred category through the Grothendieck construction $\int : [\mathbf{C}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{Cat}/\mathbf{C}$ as follows. (These two constructions constitute a 2-equivalence of 2-categories.). Consider a *pseudofunctor* $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$. The 'bundle' $\int F$ consists of the following data:

- The objects are pairs (x, y) with $x \in \mathrm{ob}(\mathbf{C})$ and $y \in \mathrm{ob}(Fx)$.
- The morphisms $(x, y) \to (x', y')$ are pairs $\big(f : x \to x', \alpha : y \to Ff(y')\big)$.

Given a cleavage, the morphisms of the Grothendieck construction are exactly the factorizations of f-morphisms through the canonical lifting of f in the cleavage.

Property 2.3.14 (Functors). A *pseudofunctor* is a functor if and only if the cleavage of the associated fibred category is **split(ting)**, i.e. it contains all identities and is closed under composition.

Definition 2.3.15 (Category of elements). Consider a presheaf $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. Its category of elements $\mathrm{El}(F)$ or $\int_{\mathbf{C}} F$ is defined as the comma category $(\mathcal{Y} \downarrow !_F)$, where $!_F : * \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ sends the unique object to F itself. Equivalently, it is the category with objects the pairs $(c, x) \in \mathrm{ob}(\mathbf{C}) \times Fc$ and morphisms $f \in \mathbf{C}(c, c')$ such that $c = Ff(c')$, i.e. it is the Grothendieck construction applied to F.

This category comes equipped with a canonical forgetful functor

$$\mathbf{C}_F : \mathrm{El}(F) \to \mathbf{C} : (c, x) \mapsto c\,. \tag{2.15}$$

Remark 2.3.16. The category of elements is often defined for covariant functors. To obtain that definition, one should take the opposite of the category of elements (and also take the opposite of the forgetful functor).

2.3.2 Monads

Definition 2.3.17 (Monad). A monad is a triple (T, μ, η) where $T : \mathbf{C} \to \mathbf{C}$ is an endofunctor and $\mu : T^2 \to T, \eta : \mathbb{1}_{\mathbf{C}} \to T$ are natural transformations satisfying the following (coherence) conditions:

1. As natural transformations from T^3 to T:

$$\mu \circ T\mu = \mu \circ \mu_T\,. \tag{2.16}$$

2. As natural transformations from T to itself:

$$\mu \circ T\eta = \mu \circ \eta_T = \mathbb{1}\,. \tag{2.17}$$

These conditions say that a monad is a monoid (**??**) in the category $\mathbf{End_C}$ of endofunctors on $\mathbf{C}$. Accordingly, η and μ are often called the **unit** and **multiplication** maps.

Example 2.3.18 (Adjunction). Every adjunction $F \dashv G$, with unit ε and counit η, induces a monad of the form $(GF, G\varepsilon F, \eta)$.

Definition 2.3.19 (Algebra over a monad[7]). Consider a monad (T, μ, η) on a category $\mathbf{C}$. An algebra over T or T-algebra is a couple (x, κ), where $x \in \mathsf{ob}(\mathbf{C})$ and $\kappa : Tx \to x$, such that the following conditions are satisfied:

1. $\kappa \circ T\kappa = \kappa \circ \mu_x$, and
2. $\kappa \circ \eta_x = \mathbb{1}_x$.

Morphisms $(x, \kappa_x) \to (y, \kappa_y)$ of T-algebras are morphisms $f : x \to y$ in $\mathbf{C}$ such that $f \circ \kappa_x = \kappa_y \circ Tf$. An algebra of the form (Tx, μ_x) is said to be **free**. The object x is called the **carrier** of the algebra

Definition 2.3.20 (Eilenberg–Moore category). Given a monad T over a category $\mathbf{C}$, the Eilenberg–Moore category $\mathbf{C}^T$ is defined as the category of T-algebras.

Definition 2.3.21 (Kleisli category). Consider a monad T on a category $\mathbf{C}$. The Kleisli category $\mathbf{C}_T$ is defined as the full subcategory of $\mathbf{C}^T$ on the free T-algebras. This is equivalently the category with objects $\mathsf{ob}(\mathbf{C}_T) := \mathsf{ob}(\mathbf{C})$ and morphisms $\mathbf{C}_T(x, y) := \mathbf{C}(x, Ty)$.

Morphisms in the Kleisly category are composed in the 'obvious way':

$$f \circ_{\mathbf{C}_T} g := \mu_Z \circ Tf \circ g \tag{2.18}$$

for all $f \in \mathbf{C}_T(Y, TZ)$ and $g \in \mathbf{C}_T(X, TY)$.

Definition 2.3.22 (Monadic adjunction). Consider an adjunction $L \dashv R$ between categories $\mathbf{C}$ and $\mathbf{D}$ with the induced monad T. The natural morphism $R\varepsilon : T \circ R \Rightarrow R$ endows R with a T-algebra structure and, hence, induces a functor $\mathbf{C} \to \mathbf{D}^T$ between $\mathbf{C}$ and the Eilenberg-Moore category of T. $L \dashv R$ is said to be monadic if this functor is an equivalence.

[7] A more suitable name would be "module over a monad", since these are modules over a monoid if monads are regarded as monoids in $\mathbf{End_C}$.

Definition 2.3.23 (Monadic functor). A functor is said to be monadic if it admits a left adjoint such that the adjunction is monadic.

The converse of 2.3.18 is also true:

Property 2.3.24. Every monad $T : \mathbf{C} \to \mathbf{C}$ can be obtained from an adjunction. The canonical choice is the adjunction

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift right, "U_T"'] \arrow[r, phantom, "\perp"] & \mathbf{C}^T \arrow[l, shift right, "F_T"'] \end{tikzcd}, \tag{2.19}$$

where F_T is the forgetful functor and U_T sends an object to the free T-algebra on it.

The following theorem characterizes monadic functors (for more information on some of the concepts, see Section 2.4 further below).

Theorem 2.3.25 (Beck's monadicity theorem). *Consider a functor $F : \mathbf{C} \to \mathbf{D}$. This functor is monadic if and only if the following conditions are satisfied:*

- *F admits a left adjoint.*
- *F reflects isomorphisms.*
- *$\mathbf{C}$ has all coequalizers of F-split parallel pairs*[8] *and F preserves these coequalizers.*

Remark 2.3.26 (Crude monadicity theorem). A sufficient condition for monadicity is obtained by replacing the third condition above by the following weaker statement: "$\mathbf{C}$ has all coequalizers of reflexive pairs and F preserves these coequalizers."

Definition 2.3.27 (Closure operator). Consider a monad $(T : \mathbf{C} \to \mathbf{C}, \eta, \mu)$. This monad is called a closure operator or **modal operator** if the multiplication map is a natural isomorphism, i.e. if the monad is idempotent. Equivalently, it is idempotent if and only if $\eta \circ T$ is a natural isomorphism.

Given a closure operator $T : \mathbf{C} \to \mathbf{C}$, the object Tx is called the closure of $x \in \mathrm{ob}(\mathbf{C})$ and the associated morphism η_x is called the **closing map**. An object $x \in \mathrm{ob}(\mathbf{C})$ itself is said to be T**-closed** exactly if its closing map is an isomorphism.

An object $x \in \mathrm{ob}(\mathbf{C})$ is called a **modal type** if the unit $\eta_x : x \to Tx$ is an isomorphism.

Property 2.3.28. Every (co)reflective subcategory inclusion (Definition 2.2.29) induces a closure operator. Conversely, every closure operator is induced by a (co)reflective subcategory.

[8]These are parallel pairs f, g such that the images Ff, Fg under F admit a split coequalizer.

Remark 2.3.29 (Bicategories ♣). A monad can be defined in any bicategory as a 1-morphism $t : x \to x$ together with two 2-morphisms that satisfy conditions similar to the ones above. The above definition is then just a specific case of this more general definition in **Cat**.

In the general setting one can then also define a **module** over a monad. First of all, one can regard any object $x \in \mathrm{ob}(\mathbf{C})$ as a functor from the terminal category **1**. By replacing **1** by any other category in the ordinary definition one obtains a general algebra (or module). It is this definition that readily generalizes to bicategories, i.e. a module is a 1-morphism $a : x \to y$ together with a 2-morphism that satisfies the same conditions as an algebra over a monad in **Cat**.

2.4 Morphisms and diagrams

2.4.1 Morphisms

Definition 2.4.1 (Section). A section of a morphism $f : x \to y$ is a right inverse, i.e. a morphism $g : y \to x$ such that $f \circ g = \mathbb{1}_y$. f itself is called a **retraction** of g and y is called a **retract** of x.

Definition 2.4.2 (Monomorphism). Let $\mathbf{C}$ be a category. A morphism $\mu \in \mathbf{C}(x, y)$ is called a monomorphism, **mono** or **monic morphism** if for every object $z \in \mathrm{ob}(\mathbf{C})$ and every two morphisms $\alpha_1, \alpha_2 \in \mathbf{C}(z, x)$ such that $\mu \circ \alpha_1 = \mu \circ \alpha_2$, one can conclude that $\alpha_1 = \alpha_2$.

Definition 2.4.3 (Epimorphism). Let $\mathbf{C}$ be a category. A morphism $\varepsilon \in \mathbf{C}(x, y)$ is called an epimorphism, **epi** or **epic morphism** if for every object $z \in \mathrm{ob}(\mathbf{C})$ and every two morphisms $\alpha_1, \alpha_2 \in \mathbf{C}(y, z)$ such that $\alpha_1 \circ \varepsilon = \alpha_2 \circ \varepsilon$, one can conclude that $\alpha_1 = \alpha_2$.

A family of morphisms $\{f_i : x_i \to y\}_{i \in I}$ is called **jointly epimorphic** if

$$\alpha_1 \circ f_i = \alpha_2 \circ f_i \tag{2.20}$$

for all $i \in I$ implies that $\alpha_1 = \alpha_2$.

Definition 2.4.4 (Split monomorphism). A morphism $f : x \to y$ that is a section of some other morphism $g : y \to x$. It can be shown that every split mono is in fact a mono and even an **absolute mono**, i.e. it is preserved by all functors.

The morphism g can be seen to satisfy the dual condition and, hence, is called a **split epimorphism**. It can be shown to be an absolute epi.

Definition 2.4.5 (Balanced category). A category in which every monic epi is an isomorphism.

Definition 2.4.6 (Reflexive pair). Two parallel morphisms $f, g : x \to y$ are said to form a reflexive pair if they have a common section, i.e. if there exists a morphism $\sigma : y \to x$ such that $f \circ \sigma = g \circ \sigma = \mathbb{1}_y$.

Definition 2.4.7 (Subobject). Let $\mathbf{C}$ be a category and let $x \in \mathsf{ob}(\mathbf{C})$ be any object. A subobject y of x is a mono $y \hookrightarrow x$.

In fact, one should work up to isomorphisms and, accordingly, the formal definition goes as follows: a subobject y of x is an isomorphism class of monos $i : y \hookrightarrow x$ in the slice category $\mathbf{C}_{/x}$.

Definition 2.4.8 (Well-powered category). A category $\mathbf{C}$ such that for every object $x \in \mathsf{ob}(\mathbf{C})$ the class of subobjects $\mathsf{Sub}(x)$ is small.

2.4.2 Initial and terminal objects

Definition 2.4.9 (Initial object). An object $\emptyset$ such that for every other object x there exists a unique morphism $\iota_x : \emptyset \to x$. If one drops the uniqueness, the notion of a **weakly initial object** is obtained.

Definition 2.4.10 (Terminal object). An object 1 such that for every other object x there exists a unique morphism $\tau_x : x \to 1$.

Property 2.4.11 (Uniqueness). If an initial (or terminal) object exists, it is unique (up to isomorphisms).

Definition 2.4.12 (Zero object). An object that is both initial and terminal. The zero object is often denoted by 0.

Property 2.4.13 (Zero morphism). From the definition of the zero object it follows that for any two objects x, y there exists a unique morphism $0_{xy} : x \to 0 \to y$.

Definition 2.4.14 (Pointed category). A category containing a zero object.

Definition 2.4.15 (Global element). Let $\mathbf{C}$ be a category with a terminal object 1. A global element of an object $x \in \mathsf{ob}(\mathbf{C})$ is a morphism $1 \to x$.

Property 2.4.16. Every global element is monic.

Definition 2.4.17 (Pointed object). An object x equipped with a global element $1 \to x$. This morphism is sometimes called the **basepoint**.

Remark 2.4.18. In the category $\mathbf{Set}$, the elements of a set S are in one-to-one correspondence with the global elements of S. Furthermore, there is the the important property (*axiom of functional extensionality*) that two functions $f, g : S \to S'$ coincide if their values at every element $s \in S$ coincide or, equivalently, if their precompositions with global elements coincide.

However, this way of checking equality can fail in other categories. Consider for example **Grp**, the category of groups, with its zero object $0 = \{e\}$. The only morphism from this group to any other group G is the one mapping e to the unit in G. It is obvious that precomposition with this morphism says nothing about the equality of other morphisms. To recover the extensionality property from **Set**, the notion of an 'element' should be generalized:

Definition 2.4.19 (Generalized element). Let $\mathbf{C}$ be a category and consider an object $x \in \mathrm{ob}(\mathbf{C})$. For any object $y \in \mathrm{ob}(\mathbf{C})$, a morphism $y \to x$ is called a generalized element of x. These morphisms are also called y**-elements** in x or elements of **shape** y in x.

Definition 2.4.20 (Generator). Let $\mathbf{C}$ be a category. A collection of objects $\mathcal{O} \subset \mathrm{ob}(\mathbf{C})$ is called a collection of generators or **separators** for $\mathbf{C}$ if the generalized elements of shape $\mathcal{O}$ are sufficient to distinguish between all morphisms in $\mathbf{C}$:

$$\forall x, y \in \mathrm{ob}(\mathbf{C}) : \forall f, g \in \mathbf{C}(x, y) : \big(f \neq g \implies \exists o \in \mathcal{O} : \exists h \in \mathbf{C}(o, x) : f \circ h \neq g \circ h\big)\,. \tag{2.21}$$

Definition 2.4.21 (Well-pointed category). A category for which the terminal object is a generator.

Definition 2.4.22 (Free object). Consider a forgetful functor $U : \mathbf{C} \to \mathbf{D}$ (whatever this may mean). An object $c \in \mathrm{ob}(\mathbf{C})$ is said to be free over an object $x \in \mathrm{ob}(\mathbf{D})$ if there exists a universal morphism $\eta_x : x \to Uc$. These are the initial objects of the comma categories $x/\mathbf{U}$.

Property 2.4.23 (Free functor). Note that if the forgetful functor admits a left adjoint $F : \mathbf{D} \to \mathbf{C}$, every object in the image of F is free according to the previous definition. Moreover, if U admits a free object for every $x \in \mathrm{ob}(\mathbf{D})$, it has a left adjoint. For this reason, left adjoints to fortgetful functors are often called free functors.

Definition 2.3.19 can be generalized to endofunctors as follows:

Definition 2.4.24 (Algebra over an endofunctor). Consider an endofunctor $F : \mathbf{C} \to \mathbf{C}$. An algebra over F is a pair $(x, f : Fx \to x)$, where $x \in \mathrm{ob}(\mathbf{C})$ is often called the **carrier**.

Property 2.4.25. The category of algebras over an endofunctor is equivalent to the Eilenberg–Moore category of the **(algebraically-)free monad** it generates (if it exists).

Construction 2.4.26. This property could actually be interpreted as the definition of the free monad generated by an endofunctor. If it exists, it can be obtained as the monad induced by the free-forgetful adjunction induced by $F\mathbf{Alg} \to \mathbf{C}$.

When the free functor exists, it can be constructed as follows. Consider an endofunctor $F : \mathbf{C} \to \mathbf{C}$. The term introduction of an inductive type corresponds to a morphism

$Fc \to c$, i.e. to an algebra over F. Now, algebras over F should correspond to the algebras over its free monad, the functor $F^* := U \circ \mathcal{F}$, where $U : F\mathbf{Alg} \to \mathbf{C}$ is the forgetful functor and $\mathcal{F} : \mathbf{C} \to F\mathbf{Alg}$ the free functor. The latter sends every object $d \in \mathrm{ob}(\mathbf{C})$ to the initial object of the comma category d/U, i.e. to an object $(c, \alpha : Fc \to c, \beta : d \to c)$. However, as long as $\mathbf{C}$ admits coproducts, such a triple is equivalent to a pair $(c, \gamma : d + Fc \to c)$. Since the latter is an algebra over $\mathbb{1}_\mathbf{C} + F$, one finds that algebras over F are equivalent to initial algebras over $\mathbb{1}_\mathbf{C} + F$.

Theorem 2.4.27 (Lambek). *If $F : \mathbf{C} \to \mathbf{C}$ has an initial algebra $f : Fx \to x$, then f is an isomorphism.*

2.4.3 Lifts

Definition 2.4.28 (Lifts and extensions). A lift of a morphism $f : x \to y$ along an epi $e : z \to y$ is a morphism $g : x \to z$ satisfying $f = e \circ g$. Dualizing this definition gives the notion of extensions. (The epi/mono condition is often dropped in the literature.)

Definition 2.4.29 (Lifting property). A morphism $f : x \to y$ has the left lifting property with respect to a morphism $g : x' \to y'$ (or g has the right lifting property with respect to f) if for every commutative diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] & x' \arrow[d, "g"] \\ y \arrow[r] \arrow[ur, dashed, "\exists\psi"] & y' \end{tikzcd}$$

there exists a morphism $\psi : y \to x'$ such that the triangles commute. If the morphism ψ is unique, then f and g are said to be **orthogonal**.

Definition 2.4.30 (Injective / projective morphisms). Consider a class of morphisms $I \subseteq \mathrm{hom}(\mathbf{C})$. A morphism $f \in \mathrm{hom}(\mathbf{C})$ is said to be I-injective (resp. I-projective) if it has the right (resp. left) lifting property with respect to all morphisms in I.

Given a set of morphisms I, the sets of I-injective and I-projective morphisms are denoted by $\mathrm{rlp}(I)$ and $\mathrm{llp}(I)$, respectively.

Definition 2.4.31 (Injective and projective objects). If $\mathbf{C}$ has a terminal object 1, an object x is called I-injective if its terminal morphism is I-injective. If $\mathbf{C}$ has an initial object, I-projective objects can be defined dually. (See Fig. 2.2.)

If I is the class of monomorphisms (resp. epimorphisms), the terminology is simplified to **injective** (resp. **projective**) objects. For projective objects, this is also equivalent to requiring that the (covariant) hom-functor preserves epimorphisms.

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "\forall f \in I"'] & i \\ y \arrow[ur, dashed, "\exists\psi"'] & \end{tikzcd}$$

(a) Injective object i.

$$\begin{tikzcd} & x \arrow[d, "\forall f \in I"] \\ p \arrow[ur, dashed, "\exists\psi"] \arrow[r, "g"'] & y \end{tikzcd}$$

(b) Projective object p.

Figure 2.2: Injective and projective objects.

A category **C** is said to **have enough injectives** if, for every object, there exists a monomorphism into an injective object. The category is said to **have enough projectives** if, for every object, there exists an epimorphism from a projective object.

Definition 2.4.32 (Fibrations and cofibrations). Consider a category **C** together with a class $I \subseteq \mathrm{hom}(\mathbf{C})$ of morphisms. A morphism $f \in \mathrm{hom}(\mathbf{C})$ is called an I-fibration (resp. I-cofibration) if it has the right (resp. left) lifting property with respect to all I-projective (resp. I-injective) morphisms.

2.4.4 Limits and colimits

Definition 2.4.33 (Diagram). A diagram in **C** with index category **I** is a (covariant) functor $D : \mathbf{I} \to \mathbf{C}$.

Definition 2.4.34 (Cone). Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram. A cone from $c \in \mathrm{ob}(\mathbf{C})$ to D consists of a family of morphisms $\psi_i : c \to Di$ indexed by **I** such that $\psi_j = Df \circ \psi_i$ for all morphisms $f : i \to j \in \mathrm{hom}(\mathbf{I})$. (This is depicted in Fig. 2.3a.)

$$\begin{tikzcd} & c \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_{f(i)}"] & \\ Di \arrow[rr, "Df"'] & & Df(i) \end{tikzcd}$$

(a) Component of cone over D.

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "\psi_i"'] & & b \arrow[dl, "\phi_i"] \\ & Di & \end{tikzcd}$$

(b) Morphism of cones.

Figure 2.3: Category of cones.

Alternative Definition 2.4.35. The above definition can be reformulated by defining an additional functor $\Delta_x : \mathbf{I} \to \mathbf{C}$ that maps every element $i \in \mathrm{ob}(\mathbf{I})$ to x and every morphism $g \in \mathrm{hom}(\mathbf{I})$ to $\mathbb{1}_x$, i.e. $\Delta : C \to [\mathbf{I}, \mathbf{C}]$ is the **diagonal functor**. The morphisms ψ_i can then be seen to be the components of a natural transformation $\psi : \Delta_x \Rightarrow D$. Hence, a cone (x, ψ) is an element of $[\mathbf{I}, \mathbf{C}](\Delta_x, D)$.

Definition 2.4.36 (Morphism of cones). Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram and let (x, ψ) and (y, ϕ) be two cones over D. A morphism between these cones is a morphism of the apexes $f : x \to y$ such that the diagrams of the form 2.3b commute for all $i \in \mathrm{ob}(\mathbf{I})$. The cones over D together with these morphisms form a category $\mathbf{Cone}(D)$. In fact this can easily be seen to be the comma category $\Delta \downarrow D$.

Definition 2.4.37 (Limit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$. The limit of this diagram, denoted by $\lim D$, is (if it exists) the terminal object of the category $\mathbf{Cone}(D)$.

Remark 2.4.38. In the older literature, the name **projective limit** was sometimes used. The dual notion, a **colimit**, was often called an **inductive limit** in the older literature.

This definition leads to the following universal property.

Universal Property 2.4.39. Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram. For every cone $(x, \psi) \in \mathbf{Cone}(D)$, there exists a unique morphism $f : x \to \lim D$. This defines a bijection

$$[\mathbf{I}, \mathbf{C}](\Delta_x, D) \cong \mathbf{C}(x, \lim D)\,. \tag{2.22}$$

If all (small) limits exist, the limit functor $\lim : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$ can be defined. The universal property of limits then implies that it is right adjoint to the constant functor Δ.

For diagrams in **Set**, one can use the fully faithfulness of the Yoneda embedding to obtain the following expression:

$$\lim D \cong [\mathbf{I}, \mathbf{Set}](\Delta_*, D)\,. \tag{2.23}$$

Remark 2.4.40. In Section 3.2 on enriched category theory, a generalization of the above construction (the so-called *weighted limits*) will be given that is better suited to the enriched setting and allows to express a wide variety of constructions as (weighted) limits.

Example 2.4.41 (Terminal object). The terminal object 1 is the limit of the empty diagram.

Definition 2.4.42 (Finitely complete category). A category is said to be finitely complete if it has all finite limits. If all (small) limits exist, the category is said to be **complete**. The dual notion for colimits is called **(finite) cocompleteness**.

Example 2.4.43 (Presheaf categories). All presheaf categories are both complete and cocomplete.

Definition 2.4.44 (Continuous functor). A functor that preserves all small limits.

A more restricted form is also common in the literature.

Definition 2.4.45 (Exact functor). A functor that preserves all finite limits is said to be **left exact**. Analogously, a functor that preserves all finite colimits is said to be **right exact**.

Example 2.4.46 (Hom-functors). In a locally small category every hom-functor is continuous (in fact these functors even preserve limits that are not necessarily small). This implies for example that

$$\mathbf{C}(x, \lim D) \cong \lim \mathbf{C}(x, D)\,. \tag{2.24}$$

In the case where $\mathbf{C}$ is small, one can characterize the Yoneda embedding through a universal property:

Universal Property 2.4.47 (Free cocompletion). The Yoneda embedding $\mathbf{C} \hookrightarrow \widehat{\mathbf{C}}$ turns the presheaf category $\widehat{\mathbf{C}}$ into the **free cocompletion** of $\mathbf{C}$, i.e. there exists an equivalence of categories between the functor category of cocontinuous functors $[\widehat{\mathbf{C}}, \mathbf{D}]_{\mathsf{cont}}$ and the ordinary functor category $[\mathbf{C}, \mathbf{D}]$.

Definition 2.4.48 (Tiny object). An object in a locally small category for which the covariant hom-functor preserves small colimits. This is sometimes called a **small-projective** object since it is in particular projective[9].

Definition 2.4.49 (Cauchy completion). Let $\mathbf{C}$ be a small category. An important (small and full) subcategory of the free cocompletion of $\mathbf{C}$ is given by the Cauchy completion, i.e. the subcategory of $\widehat{\mathbf{C}}$ on the tiny objects.[10] It can be shown that the free cocompletion of the Cauchy completion coincides with the one on $\mathbf{C}$ (up to equivalence).

A category is said to be **Cauchy-complete** if it is equivalent to its Cauchy completion. It can be shown that a category is Cauchy-complete if and only if it has all small absolute colimits.

Definition 2.4.50 (Filtered category). A category in which every finite diagram admits a cocone. For regular cardinals κ, this notion can be generalized. A category is said to be κ-filtered if every diagram with less than κ arrows admits a cocone. (In this terminology, filtered categories are the same as ω-filtered categories.)

Definition 2.4.51 (Directed limit). Consider a diagram $D : \mathbf{I} \rightarrow \mathbf{C}$. The limit (resp. colimit) of D is said to be (co)directed (resp. directed) if $\mathbf{I}$ is a downward (resp. upward) directed set 1.7.15.

The following definition is a categorification of the previous one.

Definition 2.4.52 (Filtered limit). Consider a diagram $D : \mathbf{I} \rightarrow \mathbf{C}$. The limit (resp. colimit) of D is said to be (co)filtered (resp. filtered) if $\mathbf{I}$ is a cofiltered (resp. filtered) category.

[9] Epimorphisms are characterized by a *pushout* (see Definition 2.4.72 further below).

[10] A generalization in the context of enriched categories is given by the *Karoubi envelope*.

Property 2.4.53. A category has all directed limits if and only if it has all filtered limits. (A dual statement holds for colimits.)

Definition 2.4.54 (Finitary functor). A functor that preserves all filtered colimits.

Definition 2.4.55 (Pro-object). A functor $F : \mathbf{I} \rightarrow \mathbf{C}$ from a cofiltered category. By composing these functors with the Yoneda embedding $\mathcal{Y} : \mathbf{C} \rightarrow [\mathbf{C}^{op}, \mathbf{Set}]$, pro-objects can also be identified with cofiltered limits of representable presheaves. In conjunction with Remark 2.4.38, this clarifies the terminology.

Universal Property 2.4.56. The **procategory** $\mathbf{Pro}(\mathbf{C})$ is the universal completion of $\mathbf{C}$ under cofiltered limits. $\mathbf{Pro}(\mathbf{C})$ satisfies (cf. Universal Property 2.4.47):

- it admits all cofiltered limits, and
- if $\mathbf{D}$ admits all cofiltered limits, there is an equivalence of functor categories

$$[\mathbf{C}, \mathbf{D}] \cong \mathbf{Fin}\big(\mathbf{Pro}(\mathbf{C}), \mathbf{D}\big)\,, \tag{2.25}$$

  where the category on the right-hand side is the category of finitary functors.

Remark 2.4.57 (Ind-objects). By dualizing the above definitions, i.e. by replacing cofiltered limits by filtered colimits, the category of ind-objects $\mathbf{Ind}(\mathbf{C})$ is obtained.

Definition 2.4.58 (Compact object). An object for which the covariant hom-functor preserves all filtered colimits. These objects are also said to be **finitely presentable**.[11]

Definition 2.4.59 (Product). Let $\mathbf{I}$ be a discrete category. The (co)limit over a diagram $D : \mathbf{I} \rightarrow \mathbf{C}$ is called a (co)product in $\mathbf{C}$.

Example 2.4.60 (Equalizer). Consider a diagram of the form

$$x \underset{g}{\overset{f}{\rightrightarrows}} y\,.$$

The limit of this diagram is called the equalizer of f and g. It consists of an object e and a morphism $\varepsilon : e \rightarrow x$ such that the following **fork** diagram

$$e \xrightarrow{\varepsilon} x \underset{g}{\overset{f}{\rightrightarrows}} y \tag{2.26}$$

is universal with respect to (e, ε). By dualizing one obtains **cofork** diagrams $x \rightrightarrows y \rightarrow z$ and their universal versions, the **coequalizers**.

Example 2.4.61 (Split coequalizer). A cofork diagram

$$x \underset{g}{\overset{f}{\rightrightarrows}} y \xrightarrow{\tau} z$$

together with a section φ of f and a section σ of τ such that $\sigma \circ \tau = g \circ \varphi$.

[11]This name derives from the fact that modules are finitely presented if and only if their covariant hom-functor preserves direct limits (i.e. directed colimits in the context of algebra).

Definition 2.4.62 (Regular morphisms). A mono (resp. epi) is said to be regular if it arises as an equalizer (resp. coequalizer) of two parallel morphisms.

Although not all categories are balanced (Definition 2.4.5), the following property does hold in any category.

Property 2.4.63 (Regular bimorphism). Both monic regular epimorphisms and epic regular monomorphisms are isomorphisms.

Alternative Definition 2.4.64 (Finitely complete category). A category is said to be finitely complete if it has a terminal object and if all binary equalizers and products exist.

Definition 2.4.65 (Span). A span in a category C is a diagram of the form 2.4a. By definition of a diagram, a span in C is equivalent to a functor $S : \boldsymbol{\Lambda} \to \mathbf{C}$, where $\boldsymbol{\Lambda}$ is the category with three objects $\{-1, 0, 1\}$ and two morphisms $i : 0 \to -1$ and $j : 0 \to 1$. For this reason $\boldsymbol{\Lambda}$ is sometimes called the walking or universal span.

$$\begin{tikzcd} & s \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x & & y \end{tikzcd}$$

(a) Span (category theory).

$$\begin{tikzcd} x \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

(b) Cospan.

Figure 2.4: (Co)span diagrams.

Definition 2.4.66 (Pullback). The pullback of two morphisms $f : x \to z$ and $g : y \to z$ is defined as the limit of cospan 2.4b. The full diagram characterizing the pullback, which has the form of a square, is sometimes called a **Cartesian square**.

Notation 2.4.67 (Pullback). The pullback of two morphisms $f : x \to z$ and $g : y \to z$ is often denoted by $x \times_z y$. The associated pullback square is sometimes written as in Fig. 2.5a.

$$\begin{tikzcd} x \times_z y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pb}"] & y \arrow[d, "g"] \\ x \arrow[r, "f"'] & z \end{tikzcd}$$

(a) Pullback square.

$$\begin{tikzcd} z \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{po}"] & y \arrow[d] \\ x \arrow[r] & x \sqcup_z y \end{tikzcd}$$

(b) Pushout square.

Figure 2.5: Pullback and pushout diagrams.

Example 2.4.68 (Product). If a terminal object 1 exists, the pullback $x \times_1 y$ is equal to the product $x \times y$.

In fact, pullbacks are sometimes also called **fibred products**. The reason for this terminology is not only that products reduce to a particular case, but also that in the case of **Set** the pullbacks have a fibrewise product structure:

$$x \times_y z \cong \bigsqcup_{a \in x} f^{-1}(a) \times g^{-1}(a)\,. \tag{2.27}$$

Example 2.4.69 (Kernel pair). Consider a morphism $f : x \to y$. Its kernel pair is defined as the pullback of f along itself.

Definition 2.4.70 (Pushout). The dual notion of a pullback, i.e. the colimit of a span. See Fig. 2.5b.

Property 2.4.71. Pullbacks preserve monos and pushouts preserve epis.

Alternative Definition 2.4.72 (Epimorphism). A morphism whose cokernel pair is the identity.

Property 2.4.73 (Pasting law). Consider a diagram of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

If the right square is a pullback diagram, the left square is a pullback diagram if and only if the total diagram is. Dually, if the left square is a pushout diagram, the right square is a pushout diagram if and only if the total diagram is.

Property 2.4.74 (Span category ♣). Consider a category $\mathbf{C}$ with pullbacks. The category $\mathbf{Span}(\mathbf{C})$ is defined as the category with the same objects as $\mathbf{C}$ but with spans as morphisms. Composition of spans is given by pullbacks. By including morphisms of spans, $\mathbf{Span}(\mathbf{C})$ can be refined to a bicategory.

Definition 2.4.75 (Wedge). Consider a profunctor $F : \mathbf{C} \nrightarrow \mathbf{C}$. A wedge $e : w \to F$ is an object $w \in \mathrm{ob}(\mathbf{Set})$ together with a collection of morphisms $e_x : w \to F(x,x)$ indexed by $\mathbf{C}$ such that for every morphism $f : x \to y$ the following diagram commutes:

$$\begin{tikzcd}
& w \arrow[dl, "e_x"'] \arrow[dr, "e_y"] & \\
F(x,x) \arrow[dr, "F(\mathbb{1}_x, f)"'] & & F(y,y) \arrow[dl, "F(f, \mathbb{1}_y)"] \\
& F(x,y) &
\end{tikzcd}$$

As was the case for cones, this can be reformulated in terms of (di)natural transformations. A wedge (w, e) of a profunctor $F : \mathbf{C} \nrightarrow \mathbf{C}$ is a dinatural transformation from the constant profunctor Δ_w to F.

Definition 2.4.76 (End). The end of a profunctor $F : \mathbf{C} \nrightarrow \mathbf{C}$ is defined as the universal wedge of F. The components of the wedge are called the **projection maps** of the end. This stems from the fact that for a discrete category the end coincides with the product $\prod_{x\in\mathrm{ob}(\mathbf{C})} F(x, x)$.

This is equivalent to a definition in terms of equalizers. Consider the two canonical maps

$$\prod_{x\in\mathrm{ob}(\mathbf{C})} \mathbf{C}(x, x) \rightrightarrows \prod_{f:x\to y} \mathbf{C}(x, y)\,. \tag{2.28}$$

This diagram can be interpreted as the product of all lower halves of the wedge diagrams above. It is not hard to see that its equalizer (universally) satisfies the wedge condition for all $f \in \mathrm{hom}(\mathbf{C})$.

Notation 2.4.77 (End). The end of a profunctor $F : \mathbf{C} \nrightarrow \mathbf{C}$ is often denoted using an integral sign with subscript:

$$\int_{x\in\mathbf{C}} F(x, x)\,.$$

For the dual construction, called a **coend**, an integral sign with superscript is used.

Example 2.4.78 (Natural transformations). Consider two functors $F, G : \mathbf{C} \to \mathbf{D}$. The map $(x, y) \mapsto \mathbf{D}(Fx, Gy)$ gives a profunctor $H : \mathbf{C} \nrightarrow \mathbf{C}$. By looking at the wedge condition for this profunctor, the following equality for all morphisms $f : x \to y$ can be derived:

$$\tau_y \circ Ff = Gf \circ \tau_x\,, \tag{2.29}$$

where τ is the wedge projection. Comparing this equality to Definition 2.2.15 gives

$$\mathrm{Nat}(F, G) = \int_{x\in\mathbf{C}} \mathbf{D}(Fx, Gx)\,. \tag{2.30}$$

Property 2.4.79. Using the continuity of the hom-functor (Definition 2.4.44), one can prove the following equality which can be used to turn ends into coends and vice versa:

$$\mathbf{Set}\left(\int^{x\in\mathbf{C}} F(x, x), y\right) = \int_{x\in\mathbf{C}} \mathbf{Set}\big(F(x, x), y\big)\,. \tag{2.31}$$

Using the above properties and definitions, one obtains the following two statements, called the **Yoneda reduction** and **co-Yoneda lemma**:

Property 2.4.80 (Ninja Yoneda lemma). Let $F : \mathbf{C} \to \mathbf{Set}$ be a covariant functor (similar statements hold for contravariant functors).

$$\int_{x\in\mathbf{C}} \mathbf{Set}\big(\mathbf{C}(-,x), Fx\big) \cong F \tag{2.32}$$

$$\int^{x\in\mathbf{C}} \mathbf{C}(x,-)\times Fx \cong F \tag{2.33}$$

For a generalization to the enriched setting see Property 3.2.16.

Remark 2.4.81. A common remark at this point is the comparison with the Dirac distribution (**??**):

$$\int \delta(x-y)f(x) = f(y)\,. \tag{2.34}$$

By interpreting the functor F as a function, the representable functors can be seen to behave as Dirac distributions.

Property 2.4.82.

$$\int_{F\in\mathbf{coPsh}(\mathbf{C})} \mathbf{Set}(Fx, Fy) \cong \mathbf{C}(x,y) \tag{2.35}$$

Definition 2.4.83 (Kan extension). Consider two functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{A} \to \mathbf{C}$. The right Kan extension of F along G is given by the universal functor $\mathrm{Ran}_G F : \mathbf{C} \to \mathbf{B}$ and natural transformation $\eta : \mathrm{Ran}_G F \circ G \Rightarrow F$:

$$\begin{tikzcd}
\mathbf{C} \arrow[dr, dashed, "\mathrm{Ran}_G F"] & \\
\mathbf{A} \arrow[u, "G"] \arrow[r, "F"'] \arrow[ur, phantom, "\Downarrow\eta"] & \mathbf{B}
\end{tikzcd}$$

The left Kan extension $\mathrm{Lan}_G F$ is obtained by dualizing this construction.

Property 2.4.84 (Complete categories). Complete (resp. cocomplete) categories admit all right (resp. left) Kan extensions.

Definition 2.4.85 (Preservation of Kan extension). A Kan extension $\mathrm{Lan}_G F$ is said to be **absolute** if every functor with the same codomain as preserves the Kan extension, i.e. a Kan extension is absolute if right whiskering it by another functor defines the Kan extension of the composition. If it is only preserved by all representable functors, the Kan extension is said to be **pointwise** or **strong**. Pointwise Kan extensions can be expressed as (co)limits, an expression is provided in Definition 3.2.17 in the *enriched setting* using (co)ends.

Alternative Definition 2.4.86 (Kan extension). The definition above gives a natural isomorphism (here given for left extension):

$$[\mathbf{A},\mathbf{B}](F, G^*-) \cong [\mathbf{C},\mathbf{B}](\mathrm{Lan}_G F, -)\,. \tag{2.36}$$

In the spirit of partial adjoints or partial limits, this construction defines so-called **local Kan extensions**. If local Kan extensions exist for all functors $F \in [\mathbf{A}, \mathbf{B}]$, a right adjoint $\mathrm{Ran}_G : [\mathbf{A}, \mathbf{B}] \to [\mathbf{C}, \mathbf{B}]$ to the pullback functor $G^* : F \mapsto F \circ G$ is obtained. Similarly, left Kan extension can be defined as the left adjoint to the pullback functor.

Remark 2.4.87. Using this equivalence of hom-spaces, Kan extensions can be generalized from **Cat** to any *2-category*.

Example 2.4.88 (Limit). Denote the terminal category by $\mathbf{1}$. By choosing the functor G in the definition of a right Kan extension to be the unique functor $!_{\mathbf{C}} : \mathbf{C} \to \mathbf{1}$, one obtain the universal property characterizing limits (Universal Property 2.4.39):

$$\lim F \cong \mathrm{Ran}_{!_C} F\,. \tag{2.37}$$

Similarly, colimits can be obtained as left Kan extensions.

The existence of Kan extensions can also be used to determine the existence of adjoints.

Property 2.4.89 (Adjoint functors). A functor $F : \mathbf{C} \to \mathbf{D}$ admits a left (resp. right) adjoint if and only if the right (resp. left) Kan extension of the identity functor $\mathbb{1}_{\mathbf{C}}$ along F exists. If it exists as an absolute extension, the left adjoint is given exactly by this Kan extension.

Definition 2.4.90 (Codensity monad). Consider a general functor $F : \mathbf{C} \to \mathbf{D}$. If the right Kan extension $\mathrm{Ran}_F F$ exists, it defines a monad. Functors for which this monad is the identity are said to be **codense**.[12] Left Kan extensions give, by duality, rise to *density comonads*.

Property 2.4.91 (Faithfulness). Kan extension along a fully faithful functor is itself a fully faithful functor.

Property 2.4.92 (Representability). (Left) Kan extension of a representable along a functor F is equivalent to the representable of the image of F.

Property 2.4.93 (Adjoint quadruple). Consider an adjunction $F \dashv G$ and consider Kan extension of presheaves taking values in a bicomplete category. In this case precomposition with (the opposite of) one of the adjoints coincides with Kan extension along (the opposite) of the other:

$$(F^{\mathrm{op}})^* \cong \mathrm{Lan}_{G^{\mathrm{op}}}\,, \tag{2.38}$$

$$(G^{\mathrm{op}})^* \cong \mathrm{Ran}_{F^{\mathrm{op}}}\,. \tag{2.39}$$

This implies that every adjunction induces an adjoint quadruple

$$\mathrm{Lan}_{F^{\mathrm{op}}} \dashv \mathrm{Lan}_{G^{\mathrm{op}}} \dashv \mathrm{Ran}_{F^{\mathrm{op}}} \dashv \mathrm{Ran}_{G^{\mathrm{op}}}\,. \tag{2.40}$$

[12] Codense functors are usually defined in a different way, but one can show that this is an equivalent definition (hence the name).

2.5 Internal structures

Property 2.5.1 (Eckmann–Hilton argument). A monoid internal to $\mathbf{Mon}$, the category of monoids, is the same as a commutative monoid. (See also **??**.)

Definition 2.5.2 (Internal category). Let $\mathcal{E}$ be a category with pullbacks. A category $\mathbf{C}$ internal to $\mathcal{E}$ consists of the following data:

- an object $C_0 \in \mathrm{ob}(\mathcal{E})$ of objects;
- an object $C_1 \in \mathrm{ob}(\mathcal{E})$ of morphisms;
- source and target morphisms $s, t \in \mathcal{E}(C_1, C_0)$;
- an 'identity-assigning' morphism $e \in \mathcal{E}(C_0, C_1)$ such that

$$s \circ e = \mathbb{1}_{C_0} \qquad\qquad t \circ e = \mathbb{1}_{C_0}\,; \tag{2.41}$$

  and

- a composition morphism $c : C_1 \times_{C_0} C_1 \to C_1$ such that the following equations hold:

$$\begin{gathered} s \circ c = s \circ \pi_1 \qquad\qquad t \circ c = t \circ \pi_2 \\ \pi_1 = c \circ (e \times_{C_0} \mathbb{1}) \qquad\qquad c \circ (\mathbb{1} \times_{C_0} e) = \pi_2 \\ c \circ (c \times_{C_0} \mathbb{1}) = c \circ (\mathbb{1} \times_{C_0} c)\,, \end{gathered} \tag{2.42}$$

  where π_1, π_2 are the canonical projections associated with the pullback $C_1 \times_{C_0} C_1$ of (s, t).

Morphisms between these categories, suitably called **internal functors**, are given by a pair of morphisms (in $\mathcal{E}$) between internal objects and morphisms, that preserve composition and identities. Internal natural transformations are defined in a similar way.

Notation 2.5.3. The *(bi)category* of internal categories in $\mathcal{E}$ is denoted by $\mathbf{Cat}(\mathcal{E})$. It should be noted that for $\mathcal{E} = \mathbf{Set}$, the ordinary category of small categories $\mathbf{Cat}(\mathbf{Set}) = \mathbf{Cat}$ is obtained.

Alternative Definition 2.5.4. The above definition can be reformulated in a very elegant way. An internal category in $\mathcal{E}$ is a monad in the bicategory $\mathbf{Span}(\mathcal{E})$ as shown in Fig. 2.6.

Functors between internal categories are not the only relevant morphisms. However, when defining (co)presheaves such as the hom-functor, a problem occurs. In $\mathbf{Cat}$ there exist, by definition, maps to the ambient category $\mathbf{Set}$ (ordinary category theory has a

Span gives source and target maps

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 & & C_0 \end{tikzcd}$$

Multiplication gives composition

$$\begin{tikzcd} & C_1 \times_{C_0} C_1 \arrow[dl, "s \circ \pi_0"'] \arrow[dr, "t \circ \pi_1"] \arrow[dd, "c"] & \\ C_0 & & C_0 \\ & C_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

Unit gives identity

$$\begin{tikzcd} & C_0 \arrow[dl, "\mathbb{1}_a"'] \arrow[dr, "\mathbb{1}_a"] \arrow[dd, "e"] & \\ C_0 & & C_0 \\ & C_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

Figure 2.6: Internal category as a monad in $\mathbf{Span}(\mathcal{E})$.

set-theoretic foundation). However, for internal categories there does not necessarily exist a morphism $\mathbf{C} \to \mathcal{E}$. To solve this problem, one can consider a more general structure.

Definition 2.5.5 (Internal diagram). A left module over a monad in $\mathbf{Span}(\mathcal{E})$. The dual notion is better known as an **internal presheaf**. The category of internal diagrams on an internal category $\mathbf{C} \in \mathbf{Cat}(\mathcal{E})$ is denoted by $\mathcal{E}^{\mathbf{C}}$.

This can be spelled out more explicitly. An internal diagram in an internal category $\mathbf{C} \in \mathbf{Cat}(\mathcal{E})$ consists of:

1. a morphism $\gamma_0 : F_0 \to C_0$, and
2. a morphism $\mathsf{ap} : F_0 \times_{C_0} C_1 \to F_0$

satisfying:

$$\begin{aligned} \gamma_0 \circ \mathsf{ap} &= d_1 \circ \pi_{C_1} \\ \mathsf{ap} \circ (\mathbb{1}_{F_0} \times e) &= \pi_{F_0} \\ \mathsf{ap} \circ (\mathsf{ap} \times \mathbb{1}_{C_0}) &= \mathsf{ap} \circ (\mathbb{1}_{F_0} \times c) \end{aligned} \tag{2.43}$$

Alternative Definition 2.5.6 (Internal diagram). An object of the slice category $\mathbf{Cat}(\mathcal{E})_{/\mathbf{C}}$

satisfying the following pullback condition:

$$\begin{tikzcd} F_1 \arrow[r, "d_1"] \arrow[d, "\gamma_1 \equiv \pi_{C_1}"'] \arrow[dr, phantom, "\text{pb}"] & F_0 \arrow[d, "\gamma_0"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd} \tag{2.44}$$

In fact, this is a specific instance of an even more general concept. For more information on the definitions and applications, see Johnstone (2014); Mac Lane (2013).

Definition 2.5.7 (Internal profunctor). A bimodule between monads in $\mathbf{Span}(\mathcal{E})$. Together with the above definitions, this gives rise to an equivalence

$$\mathbf{Mod}(\mathbf{Span}(\mathcal{E})) \cong \mathbf{Prof}(\mathcal{E}) \,. \tag{2.45}$$

Construction 2.5.8 (Internal Yoneda profunctor). Consider an internal functor $F : \mathbf{C} \to \mathbf{D}$. This functor induces two internal profunctors $F_* : \mathbf{D} \nrightarrow \mathbf{C}$ and $F^* : \mathbf{C} \nrightarrow \mathbf{D}$. For F_* (the profunctor F^* is defined similarly) the object span is defined as

$$C_0 \xleftarrow{\pi_0} C_0 \times_{D_0} D_1 \xrightarrow{t \circ \pi_1} D_0 \,. \tag{2.46}$$

The action of $f \in D_1$ is given by postcomposition with f in the second factor, while the action of $g \in C_1$ is given by precomposition with Fg in the second factor and changing to the domain of g in the first factor.

It can easily be shown that the profunctors induced by an identity functor $\mathbb{1}_{\mathbf{C}}$ have an object span that corresponds to the internal category $\mathbf{C}$ with the actions given by (internal) composition. In the case of $\mathcal{E} = \mathbf{Set}$, this boils down to the hom-functor. The fact that the object span is equivalent to the category $\mathbf{C}$ is essentially the Yoneda embedding. For this reason, this profunctor is in general called the (internal) Yoneda profunctor $\mathcal{Y}(\mathbf{C})$.

2.5.1 Groupoids

Definition 2.5.9 (Groupoid). A (small) groupoid $\mathcal{G}$ is a (small) category in which all morphisms are invertible.

Example 2.5.10 (Action groupoid). Consider a set X with an action of a group G. The action groupoid $X/\!\!/G$ is defined as the following category:

1. **Objects**: X,
2. **Morphisms**: An arrow $x \to y$ for every $g \in G$ such that $g \cdot x = y$.

Example 2.5.11 (Delooping). Consider a group G. Its delooping $\mathbf{B}G$ is defined as the one-object groupoid for which $\mathbf{B}G(*, *) = G$.

Property 2.5.12 (Representations). Consider a group G together with its delooping $\mathbf{B}G$. When considering *representations* as functors $\rho : \mathbf{B}G \to \mathbf{FinVect}$, one can see that the intertwiners (**??**) are exactly the natural transformations. More generally, all G-sets (**??**) can be obtained as functors $\mathbf{B}G \to \mathbf{Set}$.

Definition 2.5.13 (Core). Let $\mathbf{C}$ be a (small) category. The core $\mathrm{Core}(\mathbf{C}) \in \mathbf{Grpd}$ of $\mathbf{C}$ is defined as the maximal subgroupoid of $\mathbf{C}$.

Definition 2.5.14 (Orbit). Let $\mathcal{G}$ be a groupoid with O, M respectively the sets of objects and morphisms. On O one can define an equivalence $x \sim y \iff \exists \phi : x \to y$. The equivalence classes are called orbits and the set of orbits is denoted by O/M.

Definition 2.5.15 (Transitive component). Let $\mathcal{G}$ be a groupoid with O, M respectively the sets of objects and morphisms and let s, t denote the source and target maps on M. Given an orbit $o \in O/M$, the transitive component of M associated to o is defined as $s^{-1}(o)$, or equivalently, as $t^{-1}(o)$.

Property 2.5.16. Every groupoid is a (disjoint) union of its transitive components.

Definition 2.5.17 (Transitive groupoid). A groupoid $\mathcal{G}$ is said to be transitive if for all objects $x \neq y \in \mathrm{ob}(\mathcal{G})$, the set $\mathcal{G}(x, y)$ is not empty.

2.6 Lawvere theories ♣

Definition 2.6.1 (Lawvere theory). Let $\mathbf{F}$ denote the skeleton of $\mathbf{FinSet}$. A Lawvere theory consists of a small category $\mathbf{L}$ and a strict (finite) product-preserving *identity-on-objects* functor $\mathcal{L} : \mathbf{F}^{\mathrm{op}} \to \mathbf{L}$.

Equivalently, a Lawvere theory is a small category $\mathbf{L}$ with a **generic object** c_0 such that every object $c \in \mathrm{ob}(\mathbf{L})$ is a finite power of c_0.

Property 2.6.2. Lawvere theories $(\mathbf{L}, \mathcal{L})$ form a category $\mathbf{Law}$. Morphisms between Lawvere theories are (finite) product-preserving functors.

Definition 2.6.3 (Model). A model or **algebra** over a Lawvere theory $\mathbf{L}$ is a (finite) product-preserving functor $A : \mathbf{L} \to \mathbf{Set}$.

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2.7 Operad theory ♣

2.7.1 Operads

Definition 2.7.1 (Plain operad[13]). Let $\mathcal{O} = \{P(n)\}_{n\in\mathbb{N}}$ be a collection of sets, called **n-ary operations** (where n is called the **arity**). The collection $\mathcal{O}$ is called a plain operad if it satisfies following axioms:

1. $P(1)$ contains an identity element $\mathbb{1}$.
2. For all positive integers $n, k_1, \dots, k_n$ there exists a composition map
$$\begin{aligned}\circ &: P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n)\\ &: (\psi, \theta_1, \dots, \theta_n) \mapsto \psi \circ (\theta_1, \dots, \theta_n)\end{aligned} \tag{2.47}$$
that satisfies two additional axioms:
   - **identity**:
   $$\theta \circ (\mathbb{1}, \dots, \mathbb{1}) = \mathbb{1} \circ \theta = \theta, \tag{2.48}$$
   and
   - **associativity**:
   $$\begin{aligned}&\psi \circ \Big(\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n})\Big)\\ &\qquad = \Big(\psi \circ (\theta_1, \dots, \theta_n)\Big) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \theta_{2,1}, \dots, \theta_{n,k_n}).\end{aligned} \tag{2.49}$$

If the operad is represented using planar tree diagrams, the associativity obtains a nice intuitive form. When combining planar tree diagrams in three layers, the associativity axiom says that one can either first glue the first two layers together or one can first glue the last two layers together.

Remark 2.7.2. Plain operads can be defined in any monoidal category. In the same way symmetric operad can be defined in any symmetric monoidal category.

Example 2.7.3 (Endomorphism operad). Consider a vector space V. For every $n \in \mathbb{N}$, one can define the endomorphism algebra $\mathsf{End}(V^{\otimes n}, V)$. The endomorphism operad $\mathcal{E}\mathrm{nd}(V)$ is defined as $\{\mathsf{End}(V^{\otimes n}, V)\}_{n\in\mathbb{N}}$.

Definition 2.7.4 (O-algebra). An object X is called an algebra over an operad O if there exist morphisms
$$O(n) \times X^n \to X$$
for every $n \in \mathbb{N}$ satisfying the usual composition and identity laws. Alternatively, this can be rephrased as the existence of a (plain) operad morphism $O(n) \to \mathcal{E}\mathrm{nd}(X)$.

Example 2.7.5 (Categorical O-algebra). An O-algebra in the category **Cat**.

[13] Also called a **nonsymmetric operad** or **non-Σ operad**.

2.7.2 Algebraic topology

Definition 2.7.6 (Stasheff operad). A topological operad $\mathcal{K}$ such that $\mathcal{K}(n)$ is given by the n^{th} *Stasheff polytope/associahedron*. Composition is given by the inclusion of faces.

Definition 2.7.7 (A_∞-space). An algebra over the Stasheff operad. This induces the structure of a multiplication that is associative up to a coherent homotopy.

Definition 2.7.8 (Little k-cubes operad). A topological operad for which every topological space $\mathcal{P}(n)$ consists of all possible configurations of n embedded k-cubes in a (unit) k-cube. Composition is given by the obvious way of inserting one unit k-cube in one of the smaller embedded k-cubes.

Property 2.7.9 (Recognition principle). If a connected topological space X forms an algebra over the little k-cubes operad, it is (weakly) homotopy equivalent to the k-fold loop space $\Omega^k Y$ of another pointed topological space Y. For $k = 1$, one should technically use the Stasheff operad, but it can be shown that this is related to the little interval operad.

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Chapter 3

Higher-dimensional Algebra ♣

The main reference for this chapter is a series of eponymous papers (Baez & Crans, 2003; Baez & Lauda, 2003). For Kapranov-Voevodsky 2-vector spaces, the reader is referred to the original paper by Kapranov and Voevodsky (1994). References for the section on Berezin calculus are Choquet-Bruhat and DeWitt-Morette (2000); Losev (2007). The section about higher Lie theory is mainly based on Fiorenza (2004). For fusion and modular categories, the main reference is Etingof, Gelaki, Nikshych, and Ostrik (2016).

Chapter Contents

3.1 Monoidal categories

Definition 3.1.1 (Monoidal category). A category $\mathbf{C}$ equipped with a bifunctor

$$- \otimes - : \mathbf{C} \times \mathbf{C} \to \mathbf{C}\,, \tag{3.1}$$

called the **tensor product** or **monoidal product**, a distinct object $\mathbf{1}$, called the (**monoidal**) **unit**, and the following three natural isomorphisms, called the **coherence maps**:

- **Associator**: $\alpha_{x,y,z} : (x \otimes y) \otimes z \cong x \otimes (y \otimes z)$,
- **Left unitor**: $\lambda_x : \mathbf{1} \otimes x \cong x$, and
- **Right unitor**: $\rho_x : x \otimes \mathbf{1} \cong x$.

These natural transformations are required make the **triangle** and **pentagon** diagrams in Fig. 3.1 and Fig. 3.2, respectively, commute. A monoidal category for which the associator and the unitors are identity transformations is often said to be **strict**.

$$\begin{tikzcd}
(x \otimes \mathbf{1}) \otimes y \arrow[rr, "\alpha_{x,\mathbf{1},y}"] \arrow[dr, "\rho_x \otimes \mathbb{1}_y"'] & & x \otimes (\mathbf{1} \otimes y) \arrow[dl, "\mathbb{1}_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}$$

Figure 3.1: Triangle diagram.

Example 3.1.2 (Cartesian category). A monoidal category where the monoidal product is given by the ordinary product (Definition 2.4.59). If the monoidal product is not the ordinary product, but the monoidal unit is still terminal, the category is said to be **semicartesian**.

Definition 3.1.3 (Scalar). In a monoidal category, the scalars are defined as the endomorphisms $\mathbf{1} \to \mathbf{1}$. The set of scalars forms a commutative monoid.

$$\begin{tikzcd}
 & \big((w \otimes x) \otimes y\big) \otimes z \arrow[rr, "\alpha_{w,x,y} \otimes \mathbb{1}_z"] \arrow[dl, "\alpha_{w\otimes x,y,z}"'] & & \big(w \otimes (x \otimes y)\big) \otimes z \arrow[dr, "\alpha_{w,x\otimes y,z}"] & \\
(w \otimes x) \otimes (y \otimes z) \arrow[drr, "\alpha_{w,x,y\otimes z}"'] & & & & w \otimes \big((x \otimes y) \otimes z\big) \arrow[dll, "\mathbb{1}_w \otimes \alpha_{x,y,z}"] \\
 & & w \otimes \big(x \otimes (y \otimes z)\big) & &
\end{tikzcd}$$

Figure 3.2: Pentagon diagram.

Property 3.1.4. Every scalar $s : \mathbf{1} \to \mathbf{1}$ induces a natural transformation $s : \mathbb{1}_{\mathbf{C}} \Rightarrow \mathbb{1}_{\mathbf{C}}$ with components

$$s_x : x \cong \mathbf{1} \otimes x \xrightarrow{s \otimes \mathbb{1}_x} \mathbf{1} \otimes x \cong x\,. \tag{3.2}$$

For every morphism $f \in \mathsf{hom}(\mathbf{C})$, the naturality square $f \circ s_x = s_y \circ f$ alo defines a morphism $s \diamond f$ that is equivalently given by $\rho_y \circ (f \otimes s) \circ \rho_x^{-1}$ (one could have used the left unitors as well). These morphisms satisfy the following well-known rules of scalar multiplication from linear algebra:

- $s \diamond (s' \diamond f) = (s \circ s') \diamond f$,
- $(s \diamond f) \circ (s' \diamond g) = (s \circ s') \diamond (f \circ g)$, and
- $(s \diamond f) \otimes (s' \diamond g) = (s \circ s') \diamond (f \otimes g)$.

Definition 3.1.5 (Weak inverse). Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a monoidal category. An object $y \in \mathsf{ob}(\mathbf{C})$ is called a weak inverse of an object $x \in \mathsf{ob}(\mathbf{C})$ if it satisfies $x \otimes y \cong \mathbf{1}$.

Remark 3.1.6. One can show that the existence of a one-sided weak inverse (as in the definition above) implies that it is in fact a two-sided weak inverse, i.e. $y \otimes x \cong \mathbf{1}$ also holds.

Theorem 3.1.7 (MacLane's coherence theorem). *Consider two functors $F, G : \mathbf{A} \to \mathbf{B}$ between two monoidal categories $\mathbf{A}, \mathbf{B}$. Any two natural transformations $\eta, \varepsilon : F \Rightarrow G$, constructed solely from the associator and the unitors, coincide.*

3.1.1 Braided categories

Definition 3.1.8 (Braided monoidal category). A monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$ equipped with a natural isomorphism

$$\sigma_{x,y} : x \otimes y \cong y \otimes x \tag{3.3}$$

that makes the two **hexagon** diagrams 3.3a and 3.3b commute for all $x, y, z \in \mathrm{ob}(\mathbf{C})$. The isomorphism σ is called the **braiding** (morphism).

$$\begin{tikzcd}
& (x \otimes y) \otimes z \arrow[dl, "\sigma_{x,y} \otimes \mathbb{1}_z"'] \arrow[dr, "\alpha_{x,y,z}"] & \\
(y \otimes x) \otimes z \arrow[d, "\alpha_{y,x,z}"'] & & x \otimes (y \otimes z) \arrow[d, "\sigma_{x,y\otimes z}"] \\
y \otimes (x \otimes z) \arrow[dr, "\mathbb{1}_y \otimes \sigma_{x,z}"'] & & (y \otimes z) \otimes x \arrow[dl, "\alpha_{y,z,x}"] \\
& y \otimes (z \otimes x) &
\end{tikzcd}$$

(a) Hexagon diagram 1.

$$\begin{tikzcd}
& x \otimes (y \otimes z) \arrow[dl, "\mathbb{1}_x \otimes \sigma_{y,z}"'] \arrow[dr, "\alpha^{-1}_{x,y,z}"] & \\
x \otimes (z \otimes y) \arrow[d, "\alpha^{-1}_{x,z,y}"'] & & (x \otimes y) \otimes z \arrow[d, "\sigma_{x\otimes y,z}"] \\
(x \otimes z) \otimes y \arrow[dr, "\sigma_{x,z} \otimes \mathbb{1}_y"'] & & z \otimes (x \otimes y) \arrow[dl, "\alpha^{-1}_{z,x,y}"] \\
& (z \otimes x) \otimes y &
\end{tikzcd}$$

(b) Hexagon diagram 2.

Figure 3.3: Hexagon diagram.

Property 3.1.9 (Yang–Baxter equation). The components $\sigma_{x,x}$ of a braiding satisfy the *Yang–Baxter equation*. More generally, the braiding σ satisfies the following equation for all objects $x, y, z \in \mathrm{ob}(\mathbf{C})$:

$$(\sigma_{y,z} \otimes \mathbb{1}_x) \circ (\mathbb{1}_y \otimes \sigma_{x,z}) \circ (\sigma_{x,y} \otimes \mathbb{1}_z) = (\mathbb{1}_z \otimes \sigma_{x,y}) \circ (\sigma_{x,z} \otimes \mathbb{1}_y) \circ (\mathbb{1}_x \otimes \sigma_{y,z}) . \tag{3.4}$$

Remark 3.1.10. When drawing the above equality using string diagrams, it can be seen that the Yang–Baxter equation corresponds to the invariance of string diagrams under a *Reidemeister-III move*.

Definition 3.1.11 (Symmetric monoidal category). A braided monoidal category where the braiding σ satisfies

$$\sigma_{x,y} \circ \sigma_{y,x} = \mathbb{1}_{x\otimes y} . \tag{3.5}$$

3.1.2 Monoidal functors

Definition 3.1.12 (Monoidal functor). Let $(\mathbf{A}, \otimes, \mathbf{1}_\mathbf{A}), (\mathbf{B}, \circledast, \mathbf{1}_\mathbf{B})$ be two monoidal categories. A functor $F : \mathbf{A} \to \mathbf{B}$ is said to be monoidal if there exists:

1. a natural isomorphism $\psi_{x,y} : Fx \circledast Fy \Rightarrow F(x \otimes y)$ that makes the diagram in Fig. 3.4 commute, and

$$\begin{tikzcd}
(Fx \circledast Fy) \circledast Fz \arrow[r, "\alpha_{\mathbf{B}}"] \arrow[d, "\psi_{x,y} \circledast \mathbb{1}_{Fz}"'] & Fx \circledast (Fy \circledast Fz) \arrow[d, "\mathbb{1}_{Fx} \circledast \psi_{y,z}"] \\
F(x \otimes y) \circledast Fz \arrow[d, "\psi_{x\otimes y,z}"'] & Fx \circledast F(y \otimes z) \arrow[d, "\psi_{ax,y\otimes z}"] \\
F\big((x \otimes y) \otimes z\big) \arrow[r, "F\alpha_{\mathbf{A}}"'] & F\big(x \otimes (y \otimes z)\big)
\end{tikzcd}$$

Figure 3.4: Monoidal functor.

2. an isomorphism $\phi : \mathbf{1}_{\mathbf{B}} \to F\mathbf{1}_{\mathbf{A}}$ that makes the two diagrams in Fig. 3.5 commute.

The maps ψ and ϕ are also called **coherence maps** or **structure morphisms**.

$$\begin{tikzcd}
Fx \circledast \mathbf{1}_{\mathbf{B}} \arrow[r, "\mathbb{1}_{Fx} \circledast \phi"] \arrow[d, "\rho_{\mathbf{B}}"'] & Fx \circledast F\mathbf{1}_{\mathbf{A}} \arrow[d, "\psi_{x,\mathbf{1}_{\mathbf{A}}}"] \\
Fx & F(x \otimes \mathbf{1}_{\mathbf{A}}) \arrow[l, "F\rho_{\mathbf{A}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1}_{\mathbf{B}} \circledast Fy \arrow[r, "\phi \circledast \mathbb{1}_{Fy}"] \arrow[d, "\lambda_{\mathbf{B}}"'] & F\mathbf{1}_{\mathbf{A}} \circledast Fy \arrow[d, "\psi_{\mathbf{1}_{\mathbf{A}},y}"] \\
Fy & F(\mathbf{1}_{\mathbf{A}} \otimes y) \arrow[l, "F\lambda_{\mathbf{A}}"]
\end{tikzcd}$$

Figure 3.5: Unitality diagrams.

Property 3.1.13 (Canonical unit). For every monoidal functor F, there exists a canonical isomorphism $\phi : \mathbf{1}_{\mathbf{B}} \to F\mathbf{1}_{\mathbf{A}}$ defined by the commutative diagram in Fig. 3.6.

$$\begin{tikzcd}
\mathbf{1}_{\mathbf{B}} \circledast F\mathbf{1}_{\mathbf{A}} \arrow[r, "\lambda_{\mathbf{B}}"] \arrow[d, "\phi \circledast \mathbb{1}_{F\mathbf{1}_{\mathbf{A}}}"'] & F\mathbf{1}_{\mathbf{A}} \arrow[d, "F\lambda_{\mathbf{A}}"] \\
F\mathbf{1}_{\mathbf{A}} \circledast F\mathbf{1}_{\mathbf{A}} \arrow[r, "\psi_{\mathbf{1}_{\mathbf{A}},\mathbf{1}_{\mathbf{A}}}"'] & F(\mathbf{1}_{\mathbf{A}} \otimes \mathbf{1}_{\mathbf{A}})
\end{tikzcd}$$

Figure 3.6: Canonical unit isomorphism.

Definition 3.1.14 (Lax monoidal functor). A monoidal functor for which the coherence maps are merely morphisms and not isomorphisms.

Definition 3.1.15 (Monoidal natural transformation). A natural transformation η between (lax) monoidal functors (F, ψ, ϕ_F) and $(G, \widetilde{\psi}, \phi_G)$ that makes the diagrams in Fig. 3.7 commute.

Definition 3.1.16 (Monoidal equivalence). An equivalence of monoidal categories consisting of monoidal functors and monoidal natural isomorphisms.

$$\begin{tikzcd} & \mathbf{1_B} \arrow[dl, "\phi_F"'] \arrow[dr, "\phi_G"] & \\ F\mathbf{1_A} \arrow[rr, "\eta_{\mathbf{1_A}}"'] & & G\mathbf{1_A} \end{tikzcd} \qquad \begin{tikzcd} Fx \circledast Fy \arrow[r, "\psi_{a,b}"] \arrow[d, "\eta_a \circledast \eta_b"'] & F(x \otimes y) \arrow[d, "\eta_{a\otimes b}"] \\ Gx \circledast Gy \arrow[r, "\widetilde{\psi}_{a,b}"'] & G(x \otimes y) \end{tikzcd}$$

Figure 3.7: Monoidal natural transformation.

Theorem 3.1.17 (MacLane's strictness theorem). *Every monoidal category is monoidally equivalent to a strict monoidal category.*

3.1.3 Closed categories

Definition 3.1.18 (Internal hom). Let $(\mathbf{M}, \otimes, \mathbf{1})$ be a monoidal category. In this setting, one can generalize the *currying* procedure, i.e. the identification of maps $x \times y \to z$ with maps $x \to (y \to z)$. The internal hom-functor $\underline{\mathsf{Hom}}$ is defined by the following natural isomorphism:

$$\mathbf{M}(x \otimes y, z) \cong \mathbf{M}\big(x, \underline{\mathsf{Hom}}(y, z)\big). \tag{3.6}$$

The existence of all internal homs is equivalent to the existence of a right adjoint to the tensor functor.

Notation 3.1.19. The internal hom $\underline{\mathsf{Hom}}(x, y)$ is also often denoted by $[x, y]$ (or $x \multimap y$). From now on, this convention will be followed (unless otherwise specified).

Definition 3.1.20 (Closed monoidal category). A monoidal category that admits all internal homs. If the monoidal structure is induced by a (Cartesian) product structure, the category is often said to be **Cartesian closed**. A category for which all slice categories are Cartesian closed is said to be **locally Cartesian closed**.

Property 3.1.21. A locally Cartesian closed category with a terminal object is Cartesian closed. Moreover, it is finitely complete. More generally, a locally Cartesian closed category has all pullbacks.

Definition 3.1.22 (Exponential object). In the case of Cartesian (monoidal) categories, the internal hom $[x, y]$ is called the exponential object. This object is often denoted by y^x.

In Cartesian closed categories, a different, but frequently used, notation is $x \Rightarrow y$. However, this notation will not be used as it might be confused with the notation for 2-*morphisms*.

Definition 3.1.23 (Cartesian closed functor). A functor between Cartesian closed categories that preserves products and exponential objects. As such, it is the natural notion of functor between Cartesian closed categories.

Property 3.1.24 (Frobenius reciprocity). A functor R between Cartesian closed categories that admits a left adjoint L is Cartesian closed if and only if the natural transformation

$$L(y \times Rx) \to Ly \times x \tag{3.7}$$

is a natural isomorphism.

Property 3.1.25 (Global elements). The following isomorphism is natural in both $x, y \in \mathrm{ob}(\mathbf{M})$:

$$\mathbf{M}\big(\mathbf{1}, [x, y]\big) \cong \mathbf{M}(x, y)\,. \tag{3.8}$$

It is this relation that gives the best explanation for the term 'internal hom'. One also immediately obtains the following natural isomorphism:

$$\mathbf{M}\big(x, [\mathbf{1}, y]\big) \cong \mathbf{M}(x, y)\,. \tag{3.9}$$

Because the Yoneda embedding is fully faithful, this implies that $[\mathbf{1}, y] \cong y$. Although the global elements $\mathbf{M}(\mathbf{1}, y)$ do not fully specify an object y, this does hold internally.

Property 3.1.26 (Symmetry). Let $\mathbf{M}$ be a closed monoidal category. The definition of an internal hom can also be internalized, i.e. there exists a natural isomorphism of the form

$$[x \otimes y, z] \cong [x, [y, z]]\,. \tag{3.10}$$

Furthermore, if $\mathbf{M}$ is also symmetric, there exists an internal isomorphism of the form

$$[x, [y, z]] \cong [y, [x, z]]\,. \tag{3.11}$$

Definition 3.1.27 (Strong adjunction). Consider a monoidal category $\mathbf{M}$ together with two endofunctors $L, R : \mathbf{M} \to \mathbf{M}$. These functors are said to form a strong adjunction if there exists a natural isomorphism

$$[Lx, y] \cong [x, Ry]\,. \tag{3.12}$$

Property 3.1.25 above implies that every strong adjunction is, in particular, an ordinary adjunction in the sense of Section 2.2.4.

3.2 Enriched category theory

The following definition is due to *Bénabou*. It should represent the 'ideal place in which to do category theory'.

Definition 3.2.1 (Cosmos). A complete and cocomplete, closed symmetric monoidal category.

Definition 3.2.2 (Enriched category). Let $(\mathcal{V}, \otimes, \mathbf{1})$ be a monoidal category. A $\mathcal{V}$-enriched category, also called a $\mathcal{V}$-category[1], consists of the following elements:

- a collection of objects $\mathsf{ob}(\mathbf{C})$, and
- for every pair of objects $x, y \in \mathsf{ob}(\mathbf{C})$, an object $\mathbf{C}(x, y) \in \mathsf{ob}(\mathcal{V})$ for which the following morphisms exist:
  1. $\mathrm{id}_x : \mathbf{1} \to \mathbf{C}(x, x)$ giving the (enriched) identity morphism, and
  2. $\circ_{xyz} : \mathbf{C}(y, z) \otimes \mathbf{C}(x, y) \to \mathbf{C}(x, z)$ replacing the usual composition.

The associativity and unity properties are given by commutative diagrams for the id and $\circ$ morphisms together with the associators and unitors in $\mathcal{V}$.

Definition 3.2.3 (Change of base). Consider a monoidal functor $F : \mathcal{V} \to \mathcal{W}$. This induces a change-of-base functor $F_* : \mathcal{V}\mathbf{Cat} \to \mathcal{W}\mathbf{Cat}$ by applying F to every hom-object.

Definition 3.2.4 (Underlying category). Given a $\mathcal{V}$-enriched category $\mathbf{C}$, the underlying category $\mathbf{C}_0$ is defined as follows:

- **Objects**: $\mathsf{ob}(\mathbf{C})$, and
- **Morphisms**: $\mathcal{V}\big(\mathbf{1}, \mathbf{C}(x, y)\big)$,

where $\mathbf{1}$ is the monoidal unit in $\mathcal{V}$. This construction can be obtained as the functor $\mathcal{V}\mathbf{Cat}(\mathcal{I}, -)$, where $\mathcal{I}$ is the one-object $\mathcal{V}$-category with $\mathcal{I}(*, *) := \mathbf{1}$.

Property 3.2.5 ($\mathcal{V}$ as a $\mathcal{V}$-category). Consider a closed monoidal category $\mathcal{V}$. This category can be given the structure $\widetilde{\mathcal{V}}$ of a $\mathcal{V}$-category by taking the hom-objects to be the internal homs, i.e. $\widetilde{\mathcal{V}}(x, y) := [x, y]$ for all $x, y \in \mathcal{V}$. Property 3.1.25 then implies that there exists an isomorphism between the underlying category $\widetilde{\mathcal{V}}_0$ and the original category $\mathcal{V}$.

Given two $\mathcal{V}$-enriched categories, one can define suitable functors between them.

Definition 3.2.6 (Enriched functor). A $\mathcal{V}$-enriched functor $F : \mathbf{A} \to \mathbf{B}$ consists of the following data:

- a function $F_0 : \mathsf{ob}(\mathbf{A}) \to \mathsf{ob}(\mathbf{B})$ (as for ordinary functors), and
- for every two objects $x, y \in \mathsf{ob}(\mathbf{A})$, a morphism $F_{x,y} : \mathbf{A}(x, y) \to \mathbf{B}(Fx, Fy)$ in $\mathcal{V}$.

[1] Not to be confused with the notation for fibre categories (Definition 2.3.7).

$$\begin{tikzcd}
& \mathbf{A}(x,y) \arrow[dl, "\lambda^{-1}"'] \arrow[dr, "\rho^{-1}"] & \\
\mathbf{1} \otimes \mathbf{A}(x,y) \arrow[d, "\eta_y \otimes F_{x,y}"'] & & \mathbf{A}(x,y) \otimes \mathbf{1} \arrow[d, "G_{x,y} \otimes \eta_x"] \\
\mathbf{B}(Fy,Gy) \otimes \mathbf{B}(Fx,Fy) \arrow[dr, "\circ"'] & & \mathbf{B}(Gx,Gy) \otimes \mathbf{B}(Fx,Gx) \arrow[dl, "\circ"] \\
& \mathbf{B}(Fx,Gy) &
\end{tikzcd}$$

Figure 3.8: $\mathcal{V}$-naturality diagram.

These have to satisfy the 'usual' composition and unit conditions.

By extending Eq. (2.30) using enriched ends, one obtains a definition of enriched natural transformations and, therefore, also a definition of enriched functor categories:

$$[\mathbf{A},\mathbf{B}](F,G) := \int_{x\in\mathbf{A}} \mathbf{B}(Fx,Gx)\,. \tag{3.13}$$

Given two $\mathcal{V}$-enriched functors $F,G : \mathbf{A} \to \mathbf{B}$, one can also try to define $\mathcal{V}$-natural transformations by extending the usual definition of natural transformations (Definition 2.2.15).

Definition 3.2.7 (Enriched natural transformation). An ordinary natural transformation consists of an $\mathsf{ob}(\mathbf{A})$-indexed family of morphism $\eta_x : Fx \to Gx$. This can also be interpreted as an $\mathsf{ob}(\mathbf{A})$-indexed family of morphisms $\eta_x : 1 \to \mathbf{B}(Fx,Gx)$ from the initial object (one-element set). By analogy, a $\mathcal{V}$-natural transformation is defined as an $\mathsf{ob}(\mathbf{A})$-indexed family of morphisms $\eta_x : \mathbf{1} \to \mathbf{B}(Fx,Gx)$ from the monoidal unit. The usual naturality square is replaced by the naturality hexagon in Fig. 3.8.

The question then becomes how these two definitions are related. The end in Eq. (3.13) comes equipped with a projection $\varepsilon_x : [\mathbf{A},\mathbf{B}](F,G) \to \mathbf{B}(Fx,Gx)$. Precomposing this morphism with a morphism in the underlying category $[\mathbf{A},\mathbf{B}]_0$ exactly gives a $\mathcal{V}$-natural transformation. So, the underlying category of $[\mathbf{A},\mathbf{B}]$ is the ordinary category of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations.

3.2.1 Enriched constructions

Definition 3.2.8 (Functor tensor product). Consider a covariant functor $G : \mathbf{C} \to \mathcal{V}$ and a contravariant functor $F : \mathbf{C}^{\mathsf{op}} \to \mathcal{V}$ into a monoidal category $\mathcal{V}$, where $\mathbf{C}$ does not have to be enriched over $\mathcal{V}$. The tensor product of F and G is defined as the following coend:

$$F \otimes_{\mathbf{C}} G := \int^{x\in\mathbf{C}} Fx \otimes Gx\,. \tag{3.14}$$

It should be noted that the above tensor product does not produce a new functor, instead it only gives an object in $\mathcal{V}$. A different type of tensor product, one that does give a functor, exists in the enriched setting (note that there is no relation between these two definitions).

Definition 3.2.9 (Day convolution). Consider a monoidally cocomplete category $\mathcal{V}$, i.e. cocomplete monoidal category for which the tensor product bifunctor is cocontinuous in each argument, together with a $\mathcal{V}$-enriched category $\mathbf{C}$. The convolution or tensor product (if it exists) of two $\mathcal{V}$-enriched functors $F, G : \mathbf{C} \to \mathcal{V}$ is defined as the following coend:

$$F \otimes_{\mathsf{Day}} G := \iint^{x,y\in\mathbf{C}} \mathbf{C}(x \otimes y, -) \otimes Fx \otimes Gy\,. \tag{3.15}$$

Property 3.2.10 (Monoidal structure). When $\mathbf{M}$ is closed symmetric monoidal, Day convolution is associative and, hence, defines a monoidal structure on the functor category $[\mathbf{C}, \mathbf{M}]$. The tensor unit is given by the functor (co)represented by the tensor unit in $\mathbf{C}$.

Definition 3.2.11 (Copower). Consider a $\mathcal{V}$-enriched category $\mathbf{C}$. The copower (or **tensor**) functor $\cdot : \mathcal{V} \times \mathbf{C} \to \mathbf{C}$ is defined by the following natural isomorphism:

$$\mathbf{C}(v \cdot x, y) \cong \mathcal{V}\big(v, \mathbf{C}(x, y)\big)\,. \tag{3.16}$$

Dually, the **power** (or **cotensor**) functor $[-,-] : \mathcal{V} \times \mathbf{C} \to \mathbf{C}$ is defined by the following natural isomorphism:

$$\mathbf{C}\big(x, [v, y]\big) \cong \mathcal{V}\big(v, \mathbf{C}(x, y)\big)\,. \tag{3.17}$$

If an enriched category admits all (co)powers, it is said to be **(co)powered** (over its enriching category).

Remark 3.2.12. Property 3.1.26 says that every (closed) symmetric monoidal category $\mathbf{M}$ is powered over itself, the power just being the internal hom. The same holds for the copower, which is just the usual tensor product functor.

Example 3.2.13 (Disjoint unions). Every (co)complete (locally) small category $\mathbf{C}$ admits the structure of a $\mathbf{Set}$-(co)powered category:

$$\begin{aligned} x^S &:= \prod_{s\in S} x\,,\\ S \cdot x &:= \coprod_{s\in S} x\,. \end{aligned} \tag{3.18}$$

The definition and properties of internal hom-functors and (co)powers can be formalized as follows.

Definition 3.2.14 (Two-variable adjunction). Consider three categories $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. A two-variable adjunction $\mathbf{A} \times \mathbf{B} \to \mathbf{C}$ consists of three bifunctors:

- $- \otimes - : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$,
- $\mathrm{Hom}_L : \mathbf{A}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{B}$, and
- $\mathrm{Hom}_R : \mathbf{B}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{A}$

admitting the following natural isomorphisms:

$$\mathbf{C}(x \otimes y, z) \cong \mathbf{A}\big(x, \mathrm{Hom}_R(y,z)\big) \cong \mathbf{B}\big(y, \mathrm{Hom}_L(x,z)\big)\,. \tag{3.19}$$

It should be noted that fixing any of the variables gives rise to ordinary adjunctions in the sense of Section 2.2.4.

Property 3.2.15 (Powers and copowers). A category $\mathbf{C}$ enriched over a monoidal category $\mathcal{V}$ is powered and copowered over $\mathcal{V}$ exactly if the hom-functor $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathcal{V}$ is the right adjoint in an enriched two-variable adjunction. The power and copower functors are then given by the other two adjoints.

The co-Yoneda lemma 2.4.80 can be generalized to the enriched setting.

Property 3.2.16 (Ninja Yoneda lemma). Let $F : \mathbf{C}^{\mathrm{op}} \to \mathcal{V}$ be a contravariant functor (similar statements hold for covariant functors).

$$\begin{gathered}\int^{x\in\mathbf{C}} \mathbf{C}(-,x) \otimes Fx \cong F \\ \int_{x\in\mathbf{C}} \mathcal{V}\big(\mathbf{C}(x,-), Fx\big) \cong F\,.\end{gathered} \tag{3.20}$$

The following definition constructs Kan extensions in the enriched setting (these can be shown to reduce to Definition 2.4.83 when enriching over $\mathbf{Set}$).

Alternative Definition 3.2.17 (Kan extension). Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be categories enriched over a monoidal category $\mathcal{V}$. If $\mathbf{B}$ is assumed to be copowered over $\mathcal{V}$, left Kan extension of $F : \mathbf{A} \to \mathbf{B}$ along $G : \mathbf{A} \to \mathbf{C}$ can be defined through a coend:

$$\mathrm{Lan}_G F := \int^{x\in\mathbf{A}} \mathbf{C}(Gx, -) \cdot Fx\,. \tag{3.21}$$

If $\mathbf{B}$ is assumed to be powered over $\mathcal{V}$, right Kan extension can be defined through an end:

$$\mathrm{Ran}_G F := \int_{x\in\mathbf{A}} \big[\mathbf{C}(-, Gx), Fx\big]\,. \tag{3.22}$$

If F is contravariant, the arguments of the hom-functors should simply be interchanged.

Remark 3.2.18. By choosing $\mathcal{V} = \mathbf{Set}, \mathbf{C} = \mathbf{A}$ and $G = \mathbb{1}_{\mathbf{A}}$ in the previous definition, one obtains the ninja Yoneda lemma 2.4.80.

Property 3.2.19. As already remarked in Definition 2.4.85, Kan extensions computed using (co)ends as above are strong.

Alternative Definition 3.2.20 (Functor tensor product). Let $\mathbf{B}$ be a $\mathcal{V}$-enriched category. Consider a covariant functor $G : \mathbf{A} \to \mathbf{B}$ and a contravariant functor $F : \mathbf{A}^{\mathrm{op}} \to \mathcal{V}$. The tensor product (Definition 3.2.8) can be generalized whenever $\mathbf{B}$ is copowered over $\mathcal{V}$:

$$F \otimes_{\mathbf{A}} G := \int^{x \in \mathbf{A}} Fx \cdot Gx \,. \tag{3.23}$$

3.2.2 Weighted (co)limits

In this section, the definition of ordinary limits and, in particular, the defining universal property 2.4.39 is revisited. In this construction, the constant functor Δ_x was one of the main ingredients. This functor can be factorized as $\mathbf{I} \to 1 \to \mathbf{C}$, where 1 denotes the terminal category. At the level of morphisms, this factorization takes the form $\mathbf{I}(i,j) \to * \to \mathbf{C}(x,x)$, where $*$ denotes the terminal one-element set. However, whenever the enriching context is not $\mathbf{Set}$, one does not necessarily have access to a terminal object.

To avoid this issue, limits will first be redefined as representing objects. To this end, consider a general diagram $D : \mathbf{I} \to \mathbf{C}$. By postcomposition with the Yoneda embedding, one obtains the presheaf-valued diagram $\mathbf{C}(-, D-) : \mathbf{I} \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$. Since presheaf categories are complete (Example 2.4.43), the limit of this diagram exists:

$$\mathbf{Set}\big(S, \lim \mathbf{C}(x, D-)\big) \cong [\mathbf{I}, \mathbf{Set}]\big(\Delta_S, \mathbf{C}(x, D-)\big)\,. \tag{3.24}$$

By restricting to the terminal set $S = *$, one obtains

$$\lim \mathbf{C}(x, D-) \cong [\mathbf{I}, \mathbf{Set}]\big(\Delta_*, \mathbf{C}(x, D-)\big)\,. \tag{3.25}$$

If this presheaf is representable, one can use the continuity of the hom-functor, together with the fact that the Yoneda embedding is fully faithful, to show that the representing object is (isomorphic to) $\lim D$, i.e.

$$[\mathbf{I}, \mathbf{Set}]\big(\Delta_*, \mathbf{C}(x, D-)\big) \cong \mathbf{C}(x, \lim D)\,. \tag{3.26}$$

@@ CLEAN THIS UP (note that continuity and pointwise definition was already mentioned for ordinary limits) @@

Definition 3.2.21 (Weighted limit). The above construction can now be generalized by replacing the constant functor Δ_* by any functor $W : \mathbf{I} \to \mathbf{Set}$. A representing object is then called the W-weighted limit of D. This object is often denoted by $\lim^W D$ or $\{W, D\}$. To distinguish weighted limits from ordinary ones, the latter are sometimes called **conical limits**.

Remark 3.2.22. A motivation for this construction is the following. As was already pointed out in Remark 2.4.18, the mere knowledge of global elements $1 \to x$ is often not enough to characterize an object x. In general, one should look at the collection of generalized elements. When applying this ideology to the case of cones, one sees that replacing the functor Δ_* by a more general functor is the same as replacing the global elements $* \to Di$ by generalized elements $Wi \to Di$.

The generalization to the enriched setting is now evident. There is no reference to the terminal object left, so one can replace **Set** by any enriching category. In the enriched setting, (co)end formulas for (weighted) limits will often be used.

Formula 3.2.23 (Enriched weighted limits). By expressing the natural transformations as an end as in Eq. (2.30) and by using the canonical powering in **Set**, one can express ordinary weighted limits as follows:

$$\mathrm{lim}^W D \cong \int_{i \in \mathbf{I}} [Wi, Di]\,. \tag{3.27}$$

The generalization to other enriching categories is now straightforward. Consider a diagram $D : \mathbf{I} \to \mathbf{C}$ and a weight functor $W : \mathbf{I} \to \mathcal{V}$, where $\mathbf{C}$ is $\mathcal{V}$-enriched. If $\mathbf{C}$ is powered over $\mathcal{V}$, the W-weighted limit of D is defined by the same formula as above:

$$\mathrm{lim}^W D := \int_{i \in \mathbf{I}} [Wi, Di]\,. \tag{3.28}$$

In a similar way, one can define weighted colimits in copowered $\mathcal{V}$-categories as coends:

$$\mathrm{colim}^W D := \int^{i \in \mathbf{I}} Wi \cdot Di\,. \tag{3.29}$$

Here, the weight functor W is required to be contravariant since colimits (and cocones in general) are natural transformations between contravariant functors.

Property 3.2.24 (Weighted limits are Homs). In the case $\mathbf{C} = \mathcal{V}$, the powering becomes the internal hom and, therefore, one sees that weighted limits are given by (enriched) natural transformations (as was the case for ordinary conical limits).

In the following example, the weighted colimit is calculated with respect to the Yoneda embedding.

Example 3.2.25 (Hom-functor). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$. When using the Yoneda embedding $\mathcal{Y}i = \mathbf{I}(-, i)$ as the weight functor, one obtains the following property by virtue of the Yoneda lemma:

$$\mathrm{colim}^{\mathcal{Y}i} D \cong Di\,. \tag{3.30}$$

A similar statement for weighted limits can be obtained with the covariant Yoneda embedding.

Alternative Definition 3.2.26 (Weighted (co)limits). The above property can be used to axiomatize small weighted (co)limits in bicomplete categories:

1. **Yoneda**: For every object $i \in \mathrm{ob}(\mathbf{I})$, there exist isomorphisms
$$\lim^{\mathbf{I}(i,-)} D \cong Di \qquad \text{and} \qquad \mathrm{colim}^{\mathbf{I}(-,i)} D \cong Di\,. \tag{3.31}$$

2. **Cocontinuity**: The weighted (co)limit functors are cocontinuous in the weights.

One can also express Kan extensions as weighted limits (this simply follows from Definition 3.2.17).

Property 3.2.27 (Kan extensions). Consider functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{A} \to \mathbf{C}$. If, for every $x \in \mathrm{ob}(\mathbf{C})$, the weighted limit $\lim^{\mathbf{C}(x,G-)} F$ exists, these limits can be combined into a functor that can be shown to be the right Kan extension $\mathrm{Ran}_G F$. The left Kan extension can be obtained as a weighted colimit.

Property 3.2.28 (Category of elements). The weighted (co)limits of a functor (over **Set**) can also be expressed in terms of the category of elements (Definition 2.3.15) of the weight:
$$\lim^W F \cong \lim F \circ \mathbf{C}_W\,, \tag{3.32}$$
where the limit on the right-hand side is a conical limit.

The reason for the (co)end notation for categories of elements can also be explained by noting that these can actually be obtained as weighted (co)limits or (co)ends (here given for covariants functors):
$$\mathrm{El}(F) \cong \int^{c\in\mathbf{C}} Jc \times \mathrm{Disc}\,Fc\,, \tag{3.33}$$
where $J : \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat} : c \mapsto c/\mathbf{C}$ assigns undercategories and $\mathrm{Disc} : \mathbf{Set} \to \mathbf{Cat}$ sends sets to discrete categories.

3.3 Abelian categories

3.3.1 Additive categories

Definition 3.3.1 (Pre-additive category). A (locally small) category enriched over **Ab**, i.e. a category in which every hom-set is an Abelian group and composition is bilinear.

Property 3.3.2. Let $\mathbf{A}$ be a pre-additive category. The following statements are equivalent for an object $x \in \mathrm{ob}(\mathbf{A})$:

- x is initial,

- x is final, or
- $\mathbb{1}_x = 0$.

It follows that every initial/terminal object in a pre-additive category is automatically a zero object (Definition 2.4.12).

Property 3.3.3 (Biproducts). In a pre-additive category, the following isomorphism holds for all finitely indexed sets $\{x_i\}_{i\leq n}$:

$$\prod_{i\leq n} x_i \cong \bigsqcup_{i\leq n} x_i\,. \tag{3.34}$$

Finite (co)products in pre-additive categories are often called **direct sums**. In general, if a product and coproduct exist and are isomorphic, one also speaks of a **biproduct**.

Definition 3.3.4 (Additive category). A pre-additive category admitting all finite products (and, hence, biproducts).

When working with additive categories, it is generally assumed that the associated functors are of a specific type.

Definition 3.3.5 (Additive functor). Let $\mathbf{A}, \mathbf{A}'$ be additive categories. A functor $F : \mathbf{A} \rightarrow \mathbf{A}'$ is said to be additive if it preserves finite biproducts:

1. It preserves zero objects: $F\,0_{\mathbf{A}} \cong 0_{\mathbf{A}'}$.
2. There exists a natural isomorphism $F(x \oplus y) \cong Fx \oplus Fy$.

This notion can be generalized to pre-additive categories. A functor between pre-additive categories is said to be additive if it acts by group morphisms on hom-spaces.

Definition 3.3.6 (Grothendieck group). Let $\mathbf{A}$ be an additive category and consider its decategorification (Definition 2.2.24). This set carries the structure of an Abelian monoid and, hence, the Grothendieck construction (**??**) can be applied to obtain an Abelian group $K(\mathbf{A})$. This group is called the Grothendieck group of $\mathbf{A}$.

In a (pre-)additive category, one can define some classical notions from (homological) algebra such as images and kernels.

Definition 3.3.7 (Kernel). Let $f : x \rightarrow y$ be a morphism. A[2] kernel of f is a morphism $k : z \rightarrow x$ such that:

$$f \circ k = 0 \tag{3.35}$$

and such that it is universal with respect to this property. This implies that a kernel of f could equivalently be defined as the equalizer of f and 0.

[2] 'A' since the kernel of a morphism is only determined up to an isomorphism.

Notation 3.3.8 (Kernel). If the kernel of $f : x \to y$ exists, it is denoted by $\mathsf{ker}(f)$.

Definition 3.3.9 (Cokernel). Let $f : x \to y$ be a morphism. A cokernel of f is a morphism $p : y \to z$ such that:

$$p \circ f = 0 \tag{3.36}$$

and such that it is universal with respect to this property. This implies that a cokernel of f could equivalently be defined as the coequalizer of f and 0.

Notation 3.3.10 (Cokernel). If the cokernel of $f : x \to y$ exists, it is denoted by $\mathsf{coker}(f)$.

Remark 3.3.11. The name and notation of the kernel and the cokernel (in the categorical sense) are explained by remarking that $\mathsf{ker}(f)$ represents the functor

$$F : z \mapsto \mathsf{ker}\big(\mathbf{C}(z, x) \to \mathbf{C}(z, y)\big)\,, \tag{3.37}$$

where ker denotes the algebraic kernel (**??**), and similarly for the cokernel.

Definition 3.3.12 (Pseudo-Abelian category). An additive category in which every projection/idempotent has a kernel.

Definition 3.3.13 (Pre-Abelian category). An additive category in which every morphism has a kernel and cokernel.

Definition 3.3.14 (Abelian category). A pre-Abelian category in which every mono is a kernel and every epi is a cokernel or, equivalently, if for every morphism f there exists an isomorphism

$$\mathsf{coker}\big(\mathsf{ker}(f)\big) \cong \mathsf{ker}\big(\mathsf{coker}(f)\big)\,. \tag{3.38}$$

Property 3.3.15. Every Abelian category is balanced (Definition 2.4.5).

Property 3.3.16 (Injectivity and surjectivity). In Abelian categories, a morphism is monic if and only if it is injective, i.e. its kernel is 0. Analogously, a morphism is epic if and only if it is surjective, i.e. its cokernel is 0.

Definition 3.3.17 (Linear category). Let $\mathbf{Vect}_k$ denote the category of vector spaces over the base field k. A k-linear category is a category enriched over $\mathbf{Vect}_k$. (If the base field is clear, it is often left implicit.)

3.3.2 Exact functors

In the setting of additive categories, Definition 2.4.45 can be characterized more simply.

Definition 3.3.18 (Exact functor). Let $F : \mathbf{A} \to \mathbf{A}'$ be an additive functor between additive categories.

- F is left exact if it preserves kernels.
- F is right exact if it preserves cokernels.
- F is exact if it is both left and right exact.

Corollary 3.3.19. The previous definition implies the following properties (which can, in fact, be used as an alternative definition):

- If F is left exact, it maps an exact sequence of the form
$$0 \longrightarrow x \longrightarrow y \longrightarrow z$$
to an exact sequence of the form
$$0 \longrightarrow Fx \longrightarrow Fy \longrightarrow Fz\,.$$
- If F is right exact, it maps an exact sequence of the form
$$x \longrightarrow y \longrightarrow z \longrightarrow 0$$
to an exact sequence of the form
$$Fx \longrightarrow Fy \longrightarrow Fz \longrightarrow 0\,.$$
- If F is exact, it maps short exact sequences to short exact sequences.

Notation 3.3.20 (Left or right). The category of left modules ${}_R\mathbf{Mod}$ over a ring R is equivalent (as an Abelian category) to the category of right modules $\mathbf{Mod}_{R^{\mathrm{op}}}$ over the opposite ring R. For this reason, one often makes no difference between left and right modules (only bimodules are truly relevant) and 'the category of R-modules' is just denoted by $R\mathbf{Mod}$.

Theorem 3.3.21 (Freyd–Mitchell embedding theorem). *Every small Abelian category admits a fully faithful, exact functor into a category of the form $R\mathbf{Mod}$ for some unital ring R.*

Theorem 3.3.22 (Eilenberg–Watts). *Let R, S be two (not necessarily unital) rings. The tensor product functor induces an equivalence between the category of R-S-bimodules and the category of cocontinuous functors $R\mathbf{Mod} \to S\mathbf{Mod}$.*

3.3.3 Finiteness

Definition 3.3.23 (Simple object). Let $\mathbf{A}$ be an Abelian category. An object $a \in \mathrm{ob}(\mathbf{A})$ is said to be simple if the only subobjects of a are 0 and a itself. An object is said to be semisimple if it is a direct sum of simple obejcts.

Definition 3.3.24 (Semisimple category). A category is said to be semisimple if every object is semisimple, where, in general, the direct sums are taken over finite index sets.

Definition 3.3.25 (Jordan–Hölder series). A filtration (Definition 1.6.10)

$$0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_n = x \tag{3.39}$$

of an object x is said to be a Jordan–Hölder series if the quotient objects x_i/x_{i-1} are simple for all $i \leq n$. If the series has finite length, the object x is said to be **finite**.

Theorem 3.3.26 (Jordan–Hölder). *If an object in an Abelian category is finite, all of its Jordan–Hölder series have the same length. In particular, the multiplicities of simple objects are the same for all such series.*

Theorem 3.3.27 (Krull–Schmidt). *Any object in an Abelian category of finite length admits a unique decomposition as a direct sum of indecomposable objects*[3]*.*

Definition 3.3.28 (Locally finite). A $\mathfrak{K}$-linear Abelian category is said to be locally finite if it satisfies the following conditions:

1. every hom-space is finite-dimensional, and
2. every object has finite length.

Definition 3.3.29 (Finite). A $\mathfrak{K}$-linear Abelian category is said to be finite if it satisfies the following conditions:

1. it is locally finite,
2. it has enough projectives or, equivalently, every simple object has a *projective cover*, and
3. the set of isomorphism classes of simple objects is finite.

Theorem 3.3.30 (Schur's lemma). *Let $\mathbf{A}$ be an Abelian category. For every two simple objects $x, y \in \mathrm{ob}(\mathbf{A})$, all nonzero morphisms $x \to y$ are isomorphisms. In particular, if x, y are two non-isomorphic simple objects, then $\mathbf{A}(x, y) = 0$. Furthermore, $\mathbf{A}(x, x)$ is a division ring for every simple object $x \in \mathrm{ob}(\mathbf{A})$.*

Corollary 3.3.31. If $\mathbf{A}$ is locally finite and $\mathfrak{K}$ is algebraically closed, then $\mathbf{A}(x, x) \cong \mathfrak{K}$ for all simple objects $x \in \mathrm{ob}(\mathbf{A})$. This follows from the fact that the only finite-dimensional division algebra over an algebraically closed field is the field itself.

The Freyd–Mitchell theorem 3.3.21 can be adapted to the finite linear case as follows.

Theorem 3.3.32 (Deligne). *Every finite $\mathfrak{K}$-linear Abelian category is $\mathfrak{K}$-linearly equivalent to a category of the form $A\mathbf{Mod}^{\mathrm{fin}}$ for A a finite-dimensional $\mathfrak{K}$-algebra.*

[3] An object is **indecomposable** if it cannot be written as a direct sum of its subobjects.

Construction 3.3.33 (Deligne tensor product). Let $\mathbf{A}, \mathbf{B}$ be two Abelian categories. Their Deligne (tensor) product is defined (if it exists) as the category $\mathbf{A} \boxtimes \mathbf{B}$ for which there exists a bijection between right exact functors $\mathbf{A} \boxtimes \mathbf{B} \to \mathbf{C}$ and right exact functors $\mathbf{A} \times \mathbf{B} \to \mathbf{C}$ (the latter being right exact in each argument).

For finite Abelian categories, it can be shown that their Deligne product always exists. By the Deligne embedding theorem, one can find an explicit description. Consider two finite-dimensional $\mathfrak{K}$-algebras A, B. The category $A\mathbf{Mod}^{\mathrm{fin}} \boxtimes B\mathbf{Mod}^{\mathrm{fin}}$ is equivalent to the category $(A \otimes_{\mathfrak{K}} B)\mathbf{Mod}^{\mathrm{fin}}$.

3.4 Duality

Definition 3.4.1 (Dual object). Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a monoidal category and consider an object $x \in \mathrm{ob}(\mathbf{C})$. A left dual[4] of x is an object $x^* \in \mathrm{ob}(\mathbf{C})$ together with two morphisms $\eta : \mathbf{1} \to x \otimes x^*$ and $\varepsilon : x^* \otimes x \to \mathbf{1}$, called the **unit** and **counit** morphisms[5], such that the diagrams in Fig. 3.9 commute. x is said to be **dualizable** if it admits a left dual.

$$\begin{tikzcd} & x & \\ \mathbf{1} \otimes x \arrow[ur, "\lambda_x"] \arrow[d, "\eta \otimes \mathbb{1}_x"'] & & x \otimes \mathbf{1} \arrow[ul, "\rho_x"'] \\ (x \otimes x^*) \otimes x \arrow[rr, "\alpha_{x,x^*,x}"] & & x \otimes (x^* \otimes x) \arrow[u, "\mathbb{1}_x \otimes \varepsilon"'] \end{tikzcd}$$

(a) Dual object I.

$$\begin{tikzcd} & x^* & \\ x^* \otimes \mathbf{1} \arrow[ur, "\rho_{x^*}"] \arrow[d, "\mathbb{1}_{x^*} \otimes \eta"'] & & \mathbf{1} \otimes x^* \arrow[ul, "\lambda_{x^*}"'] \\ x^* \otimes (x \otimes x^*) \arrow[rr, "\alpha^{-1}_{x^*,x,x^*}"] & & (x^* \otimes x) \otimes x^* \arrow[u, "\varepsilon \otimes \mathbb{1}_{x^*}"'] \end{tikzcd}$$

(b) Dual object II.

Figure 3.9: Dualizable objects.

Definition 3.4.2 (Rigid category). A monoidal category that has all duals. These categories are also said to be **autonomous**. If only left (resp. right) duals exist, the category is said to be left (resp. right) rigid.

Property 3.4.3 (Braided categories). In general, it is not true that left and right duals coincide. However, in a braided monoidal category this is the case.

Definition 3.4.4 (Compact closed category). A symmetric rigid category.

Example 3.4.5 (FinVect). Consider the category **FinVect** of finite-dimensional vector spaces (the ground field is assumed to be $\mathbb{R}$). The categorical dual of a vector space V is the algebraic dual V^*. The unit morphism is given by the 'resolution of the identity':

$$\eta : \mathbf{1} \to V \otimes V^* : 1 \mapsto \sum_{i=1}^{\dim(V)} e_i \otimes \phi^i \,, \tag{3.40}$$

[4] x is called the **right dual** of x^*. The right dual of y is often denoted by ${}^*y$.
[5] Also called the **coevaluation** and **evaluation** morphisms.

where $\{e_i\}$ and $\{\phi^i\}$ are dual bases of V and V^*, respectively.

It should be noted that the category **Vect** of all vector spaces is not rigid. By Property 3.4.3 above, left and right duals coincide in any braided monoidal category (such as **Vect**), but for infinite-dimensional vector spaces it is known that $A \cong (A^*)^*$ never holds and, as such, rigidity cannot be extended to **Vect**.

Property 3.4.6 (Tannaka duality). Consider the category $\mathcal{V} = \mathbf{FinVect}_k$. Using coends, one can reconstruct the base field from its modules, i.e. the objects in $\mathcal{V}$:

$$\int^{V\in\mathcal{V}} V^* \otimes V \cong k\,. \tag{3.41}$$

This result can be shown to hold for all compact closed categories $\mathcal{V}$. In this context, it is known as **Tannaka reconstruction**. A more general statement goes as follows:

$$\int^{V\in\mathcal{V}} \mathcal{V}(V,-) \otimes V \cong \mathrm{id}_{\mathcal{V}}\,. \tag{3.42}$$

For $\mathcal{V} = \mathbf{FinVect}_k$, the components $\eta_V : \mathcal{V}(V,V) \to k$ of the coend can be shown to coincide with the trace and, as such, the trace obtains a universal property.

Remark 3.4.7. This property can also be generalized by replacing $\mathcal{V}$ by a category of modules $A\mathbf{Mod}$ for some finite-dimensional algebra A. The end and coend give the algebra A and its dual A^*, respectively.

For certain purposes, it is useful to slightly weaken the notion of compact closed categories (e.g. Barr (1991); Kissinger and Uijlen (2019)).

Definition 3.4.8 ($*$-autonomous category). A symmetric monoidal category $(\mathbf{C},\otimes,\mathbf{1})$ with a fully faithful dualization functor $* : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ such that

$$\mathbf{C}(x\otimes y, z^*) \cong \mathbf{C}\big(x,(y\otimes z)^*\big) \tag{3.43}$$

for all $x,y,z \in \mathrm{ob}(\mathbf{C})$.

Property 3.4.9 (de Morgan duality). If $(\mathbf{C},\otimes,\mathbf{1},*)$ is a $*$-autonomous category, it admits a second $*$-autonomous structure $(\mathbf{C},\parr,\perp,*)$ given by

$$x \parr y := (x^* \otimes y^*)^* \tag{3.44}$$

for all $x,y \in \mathrm{ob}(\mathbf{C})$ and

$$\perp := \mathbf{1}^*\,. \tag{3.45}$$

Property 3.4.10 (Involution). Note that the definition of $*$-autonomous categories implies that

$$x \cong x^{**} \tag{3.46}$$

for all $x \in \mathrm{ob}(\mathbf{C})$. Moreover, $*$-autonomous categories are closed with the internal hom given by:

$$[x, y] = x^* \parr y \cong (x \otimes y^*)^* . \tag{3.47}$$

A $*$-autonomous category is compact closed if and only if duality is compatible with tensor products:

$$(x \otimes y)^* \cong x^* \otimes y^* . \tag{3.48}$$

Equivalently, a $*$-autonomous category is compact closed if and only if the de Morgan-dual monoidal structures coincide.

Definition 3.4.11 (Symmetric monoidal dagger category). A symmetric monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$ that also carries the structure of a dagger category (Definition 2.3.1) such that

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger \tag{3.49}$$

and such that the coherence and braiding morphisms are unitary.

Definition 3.4.12 (Dagger-compact category). A symmetric monoidal dagger category that is also a compact closed category such that the following diagram commutes for all objects:

$$\begin{tikzcd}
& \mathbf{1} \arrow[dl, "\eta"'] \arrow[dr, "\varepsilon^\dagger"] & \\
x^* \otimes x & & x \otimes x^* \arrow[ll, "\sigma_{x,x^*}"]
\end{tikzcd}$$

The trace on **FinVect** can be generalized as follows.

Definition 3.4.13 (Trace). Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a rigid category and consider $f \in \mathbf{C}(x, x^{**})$. The left (**categorical** or **quantum**) trace of f is defined as the following morphism in $\mathrm{End}_{\mathbf{C}}(\mathbf{1})$:

$$\mathrm{tr}^L(f) := \varepsilon_{x^*} \circ (f \otimes \mathbb{1}_{x^*}) \circ \eta_x . \tag{3.50}$$

For $f \in \mathbf{C}(x, {}^{**}x)$, the right trace is defined similarly:

$$\mathrm{tr}^R(f) := \varepsilon_{^{**}x} \circ (\mathbb{1}_x \otimes f) \circ \eta_{^*x} . \tag{3.51}$$

Property 3.4.14. The following linear algebra-like properties hold for the categorical trace:

- $\mathrm{tr}^L(f) = \mathrm{tr}^R(f^*)$,
- $\mathrm{tr}^L(f \otimes g) = \mathrm{tr}^L(f)\,\mathrm{tr}^L(g)$, and
- for additive categories: $\mathrm{tr}^L(f \oplus g) = \mathrm{tr}^L(f) + \mathrm{tr}^L(g)$.

The second and third property can be stated analogously for the right trace.

Definition 3.4.15 (Pivotal category). Let $\mathbf{C}$ be a rigid monoidal category. A pivotal structure on $\mathbf{C}$ is a monoidal natural isomorphism $\psi : \mathrm{id}_{\mathbf{C}} \Rightarrow **$.

Definition 3.4.16 (Dimension). Let $(\mathbf{C}, \psi)$ be a pivotal category and consider an object $x \in \mathrm{ob}(\mathbf{C})$. The dimension of x is defined as follows:

$$\mathrm{dim}_\psi(x) := \mathrm{tr}^L(\psi_x)\,. \tag{3.52}$$

Definition 3.4.17 (Spherical category). A pivotal category $(\mathbf{C}, \psi)$ in which the left and right dimensions with respect to ψ coincide: $\mathrm{dim}_\psi(x) = \mathrm{dim}_\psi(x^*)$ for all $x \in \mathrm{ob}(\mathbf{C})$.

Definition 3.4.18 (Calabi–Yau category). A $\mathbf{Vect}_k$-enriched category $\mathbf{C}$ equipped with a **trace** functional

$$\mathrm{tr}_x : C(x, x) \to k \tag{3.53}$$

for each object $x \in \mathrm{ob}(\mathbf{C})$ such that the induced pairing

$$\langle\cdot,\cdot\rangle : C(x, y) \otimes C(y, x) \to k : f \otimes g \mapsto \mathrm{tr}_x(g \circ f) \tag{3.54}$$

is symmetric and nondegenerate.

Example 3.4.19 (Frobenius algebra). A one-object Calabi–Yau category is the (pointed) monoid delooping of a *Frobenius algebra* (see **??**).

3.5 Tensor and fusion categories

Some definitions might slightly differ from the ones in the main references and some properties might be stated less generally. K denotes an algebraically closed field (this will often be $\mathbb{C}$).

Definition 3.5.1 (Tensor category). A monoidal category with the following properties:

1. it is rigid,
2. it is Abelian,
3. it is K-linear in a way compatible with the Abelian structure,

4. $\mathrm{End}(\mathbf{1}) \cong K$, and

5. $- \otimes -$ is bilinear on morphisms.

Some authors (such as Etingof et al. (2016)) also add 'locally finite' as a condition (Definition 3.3.28).

Remark 3.5.2. If K is not algebraically closed, one should replace the fourth condition by the condition that $\mathbf{1}$ is a simple object. However, if K is algebraically closed, these statements are equivalent.

Definition 3.5.3 (Tannakian category). A tensor category C over a field K such that there exists a field extension L/K (**??**) and a K-linear, exact and faithful monoidal functor $F : \mathsf{C} \rightarrow \mathsf{FinVect}_L$. If such a structure exists for $L = K$, the category is said to be **neutral**.

Definition 3.5.4 (Pointed tensor category). A tensor category where all simple objects are (weakly) invertible.

Definition 3.5.5 (Fusion category). A semisimple, finite tensor category.

Property 3.5.6. Let C be a fusion category. There exists a natural isomorphism $\mathbb{1}_{\mathsf{C}} \cong **$.

Remark 3.5.7. Although any fusion category admits a natural isomorphism between an object and its double dual, this morphism does not need to be monoidal. The fact that all fusion categories are pivotal was conjectured by *Etingof*, *Ostrik* and *Nikshych*. Currently, the best one can do for a general fusion category is a monoidal natural transformation between the identity functor and the fourth dualization functor $\mathbb{1}_{\mathsf{C}} \cong ****$.

Definition 3.5.8 (Categorical dimension). Consider a fusion category C and choose a natural isomorphism $\psi : \mathbb{1}_{\mathsf{C}} \cong **$. For every simple object $x \in \mathrm{ob}(\mathsf{C})$, one can define a dimension function, sometimes called the **norm squared**, in the following way:

$$|x|^2 := \mathrm{tr}(\psi_x)\,\mathrm{tr}\big((\psi_x^{-1})^*\big)\,. \tag{3.55}$$

If C is pivotal, this becomes $|x|^2 = \dim_\psi(x)\dim_\psi(x^*)$. In particular, when C is spherical, this becomes $|x|^2 = \dim_\psi(x)^2$. The categorical dimension, sometimes called the **Müger dimension**, is defined as follows:

$$\dim(\mathsf{C}) := \sum_{x\in\mathcal{O}(\mathsf{C})} |x|^2\,, \tag{3.56}$$

where $\mathcal{O}(\mathsf{C})$ denotes the set of isomorphism classes of simple objects.

Remark 3.5.9. It should be noted that the above quantities do not depend on the choice of isomorphism $\psi_x : x \cong x^{**}$ since all of them only differ by a scale factor.

Property 3.5.10 (Nonzero dimension). For any fusion category $\mathbf{C}$, one has that $\mathrm{dim}(\mathbf{C}) \neq 0$. In particular, if $K = \mathbb{C}$, then $\mathrm{dim}(\mathbf{C}) \geq 1$ (since the norm squared of any simple object is then also positive).

Definition 3.5.11 (G-graded fusion category). A semisimple linear category $\mathbf{C}$ is said to have a G**-grading**, where G is a finite group, if it can be decomposed as follows:

$$\mathbf{C} \cong \bigoplus_{g \in G} \mathbf{C}_g\,, \tag{3.57}$$

where every $\mathbf{C}_g$ is linear and semisimple. A fusion category $\mathbf{C}$ is said to be a (G-)graded fusion category if it admits a G-grading such that $\mathbf{C}_g \otimes \mathbf{C}_h \subseteq \mathbf{C}_{gh}$ for all $g, h \in G$.

Example 3.5.12 (G-graded vector spaces). Define the category $\mathbf{Vect}_G^\omega$ as having the same objects and morphisms as $\mathbf{Vect}_G$, the category of G-graded vector spaces, but with the associator given by the the 3-cocycle $\omega \in H^3(G; K^\times)$.

Property 3.5.13. Any pointed fusion category is equivalent to a category of the form $\mathbf{Vect}_G^\omega$ for some G and $\omega \in H^3(G; K^\times)$.

Theorem 3.5.14 (Tannaka duality). *The category of modules of a weak Hopf algebra*[6] *has the structure of a fusion category. Conversely, any fusion category can be obtained as the category of modules of a weak Hopf algebra.*

3.6 Ribbon and modular categories

Definition 3.6.1 (Ribbon structure). Consider a braided monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$ with braiding σ. A **twist** or **balancing** is a natural endomorphism θ such that the following equation is satisfied for all $x, y \in \mathrm{ob}(\mathbf{C})$:

$$\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ \sigma_{y,x} \circ \sigma_{x,y}\,. \tag{3.58}$$

If, in addition, $\mathbf{C}$ is rigid and the twist satisfies $\theta_{x^*} = (\theta_x)^*$ for all $x \in \mathrm{ob}(\mathbf{C})$, one speaks of a ribbon or **tortile** category.

Definition 3.6.2 (Drinfel'd morphism). Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a rigid braided monoidal category with braiding σ. This structure admits a canonical natural automorphism $\mathrm{id}_{\mathbf{C}} \cong **$ defined as follows:

$$x \xrightarrow{\mathbb{1}_x \otimes \eta_{x^*}} x \otimes x^* \otimes x^{**} \xrightarrow{\sigma_{x,x^*} \otimes \mathbb{1}_{x^{**}}} x^* \otimes x \otimes x^{**} \xrightarrow{\varepsilon_x \otimes \mathbb{1}_{x^{**}}} x^{**}\,. \tag{3.59}$$

Property 3.6.3. Let $\mathbf{C}$ be a braided monoidal category. Consider the Drinfel'd morphism $u : \mathrm{id}_{\mathbf{C}} \cong **$ defined above. Any natural isomorphism $\psi : \mathrm{id}_{\mathbf{C}} \cong **$ can be written as $u \circ \theta$ where $\theta \in \mathrm{Aut}(\mathbb{1}_{\mathbf{C}})$. It is not hard to see that this natural isomorphism is monoidal (and, hence, pivotal) exactly when θ is a twist. If $\mathbf{C}$ is a fusion category, the pivotal structure is spherical if and only if θ determines a ribbon structure.

[6] A weak version of **??**.

Definition 3.6.4 (Premodular category). A ribbon fusion category. Equivalently, a spherical braided fusion category.

Definition 3.6.5 (S-matrix). Given a premodular category $\mathbf{M}$ with braiding σ, the S-matrix is defined as follows:

$$S_{xy} := \mathrm{tr}(\sigma_{y,x} \circ \sigma_{x,y}) , \tag{3.60}$$

where $x, y \in \mathcal{O}(\mathbf{M})$ are (isomorphism classes of) simple objects. Since in a premodular category there are only finitely many isomorphism classes of simple objects (denote this number by $\mathcal{I}$), one can see that S is a $\mathcal{I} \times \mathcal{I}$-matrix.

Definition 3.6.6 (Modular category[7]). A premodular category for which the S-matrix is invertible.

Property 3.6.7. Let $\mathbf{M}$ be a modular category with S-matrix S and define the following matrix:

$$E_{xy} := \begin{cases} 1 & \text{if } x = y^* , \\ 0 & \text{otherwise} . \end{cases} \tag{3.61}$$

The following relation with the categorical dimension of $\mathbf{M}$ is obtained:

$$S^2 = \mathrm{dim}(\mathbf{M})E . \tag{3.62}$$

Formula 3.6.8 (Verlinde). Consider a modular category $\mathbf{M}$ with S-matrix S. Let $\mathcal{O}(\mathbf{M})$ denote the set of isomorphism classes of simple objects and let dim denote the (pivotal) dimension associated to the spherical structure on $\mathbf{M}$. Using the formula

$$S_{xy}S_{xz} = \mathrm{dim}(x) \sum_{w \in \mathcal{O}(\mathbf{M})} N_{yz}^{w} S_{xw} \tag{3.63}$$

for all $x, y, z \in \mathcal{O}(\mathbf{M})$, one obtains the following important relation:

$$\sum_{w \in \mathcal{O}(\mathbf{M})} \frac{S_{wy}S_{wz}S_{wx^*}}{\mathrm{dim}(w)} = \mathrm{dim}(\mathbf{M})N_{yz}^{x} . \tag{3.64}$$

This property implies that the S-matrix of a modular category determines the fusion coefficients of the underlying fusion category.

3.7 Module categories

By categorifying the definition of a module over a ring (**??**), one obtains the notion of a module category.

[7]'Modular tensor category' is often abbreviated as **MTC**.

Definition 3.7.1 (Module category). Let $\mathbf{M}$ be a monoidal category. A left $\mathbf{M}$-module (category) is a category $\mathbf{C}$ equipped with a bilinear functor $\triangleright : \mathbf{M} \times \mathbf{C} \to \mathbf{C}$ together with natural isomorphisms that categorify the associativity and unit conditions of modules (these are also required to be compatible with the associator and unitors of $\mathbf{M}$).

Remark 3.7.2. Similar to how a G-set can be defined as a functor $\mathbf{B}G \to \mathbf{Set}$ (Property 2.5.12), one can define a module category as a 2-functor $\mathbf{BM} \to \mathbf{Cat}$.

Analogous to Definition 3.1.18, one can also define internal homs for module categories.

Definition 3.7.3 (Internal hom). Consider a left $\mathbf{M}$-module $\mathbf{C}$. Given two objects $x, y \in \mathrm{ob}(\mathbf{C})$, one defines their internal hom (if it exists) as the object $\underline{\mathrm{Hom}}(x,y) \in \mathrm{ob}(\mathbf{M})$ satisfying the following condition

$$\mathbf{C}(m \triangleright x, y) \cong \mathbf{M}\big(m, \underline{\mathrm{Hom}}(x,y)\big) \tag{3.65}$$

for all $m \in \mathrm{ob}(\mathbf{M})$.

Property 3.7.4. It should be noted that for the case $\mathbf{C} \equiv \mathbf{M}$, where the action is given by the tensor product in $\mathbf{M}$, one obtains Definition 3.1.18 as a particular case.

3.8 Higher vector spaces

3.8.1 Kapranov–Voevodksy 2-vector spaces

The guiding principle for the definition of 2-vector spaces in this section will be the generalization of certain observations from studying the category $\mathbf{Vect}$ of ordinary vector spaces. Linear maps between vector spaces can (at least in finite dimensions) be represented as matrices with coefficients in the base field K. Coincidentally, this base field is also the tensor unit in $\mathbf{Vect}$. At the same time, all finite-dimensional vector spaces are isomorphic to spaces of the form K^n, where n is given by the dimension of the vector space.

Definition 3.8.1 (2-vector space). To define 2-vector spaces, *Kapranov* and *Voevodsky* lifted these observations to categories by replacing the base field K by the category $\mathbf{Vect}_K$. To wit, $\mathbf{2Vect}_K$ is defined as the 2-category consisting of the following data:

- **Objects**: Finite products of the form $\mathbf{Vect}_K^n$.
- **1-morphisms**: Collections $\|A_{ij}\|$ of finite-dimensional K-vector spaces, called **2-matrices**.
- **2-morphisms**: Collections (f_{ij}) of linear maps between finite-dimensional K-vector spaces.

The multiplication (or composition) of 1-morphisms is defined in analogy to the multiplication of ordinary matrices, but where the usual sum and product are replaced by the direct sum and tensor product.

A seemingly more formal definition uses the concepts of *ring* and module categories.

Alternative Definition 3.8.2. A 2-vector space is a lax module category over **Vect** that is module-equivalent to $\mathbf{Vect}^n$ for some $n \in \mathbb{N}$. The 2-category **2Vect** is then defined as the 2-category with objects these 2-vector spaces, as 1-morphisms the associated **Vect**-module functors and as 2-morphisms the module natural transformations.

3.8.2 Baez–Crans 2-vector spaces

Definition 3.8.3 (2-vector space). A category internal to **Vect**. The morphism are **linear functors**, i.e. functors internal to **Vect**.

Remark 3.8.4. The above definition should not be confused with that of categories and functors enriched over **Vect**.

Example 3.8.5 (Base field). The base field K can be categorified to a 2-vector spaces by taking $K_0 = K_1 := K$ and $s = t = e := \mathbb{1}_K$. This object serves as a unit for the tensor product on $\mathbf{2Vect}_K$.

Property 3.8.6 (Chain complexes). There exists an equivalence between the (2-)categories of 2-vector spaces and 2-term chain complexes.

Proof (Sketch of construction). Given a 2-vector space (V_0, V_1), one can build a chain complex $C_\bullet$ as follows:

- $C_0 := V_0$,
- $C_1 := \ker(s)$, and
- $d := t|_{C_1}$,

where s, t are the source and target morphisms. □

Remark 3.8.7. The equivalence (ar the level of ordinary categories) is an instance of the *Dold–Kan correspondence* (see **??**).

Definition 3.8.8 (Arrow part). Consider a 2-vector space $V = (V_0, V_1)$. For any morphism $f \in V_1$, one defines the arrow part as follows:

$$\vec{f} := f - e\big(s(f)\big)\,, \tag{3.66}$$

where e, s are the identity and source morphisms in V. Any map can thus be recovered from its arrow part and its source. This allows to identify a map $f \in V_1$ with the pair

$\big(s(f), \vec{f}\,\big)$. Using arrow parts, one can rewrite the composition law of morphisms in an intuitive way:

$$g \circ f = \big(s(f), \vec{f} + \vec{g}\,\big)\,. \tag{3.67}$$

Definition 3.8.9 (Antisymmetric morphism). A natural morphism between n-linear functors in **2Vect** is said to be **completely antisymmetric** if its arrow part is completely antisymmetric.

3.9 Higher Lie theory

3.9.1 Lie superalgebras

Definition 3.9.1 (Internal Lie algebra). Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a linear symmetric monoidal category with braiding σ. A Lie algebra internal to $\mathbf{C}$ is an object $L \in \mathrm{ob}(\mathbf{C})$ and a morphism

$$[\cdot,\cdot] : L \otimes L \to L$$

satisyfing the following conditions:

1. **Antisymmetry**: $[\cdot,\cdot] + [\cdot,\cdot] \circ \sigma_{L,L} = 0$, and
2. **Jacobi identity**: $[\cdot,[\cdot,\cdot]] + [\cdot,[\cdot,\cdot]] \circ \tau + [\cdot,[\cdot,\cdot]] \circ \tau^2 = 0\,,$

where $\tau = (\mathbb{1}_L \otimes \sigma_{L,L}) \circ (\sigma_{L,L} \otimes \mathbb{1}_L)$ denotes cyclic permutation.

Example 3.9.2 (Lie superalgebra). When using the braiding

$$\sigma(x \otimes y) = (-1)^{\deg(x)\deg(y)} y \otimes x \tag{3.68}$$

in **sVect**, a Lie superalgebra (also called a super Lie algebra) is obtained. More generally, in $\mathbb{Z}$-**Vect**, a Lie bracket of degree k is induced by the braiding

$$\sigma(x \otimes y) = (-1)^{(\deg(x)-k)(\deg(y)-k)} y \otimes x\,. \tag{3.69}$$

It is simply a Lie bracket on the k-suspension $\Pi^k V$.

Example 3.9.3 (dg-Lie algebras). Lie algebras internal to $\mathbf{Ch}_\bullet(\mathbf{Vect})$ or its generalization to graded vector spaces. Sometimes these are also called strict L_∞-algebras (see further below).

The following notion is a slight modification of the idea of a (graded) Poisson algebra (**??**).

Definition 3.9.4 (Gerstenhaber algebra). A graded-commutative algebra equipped with a degree-1 Lie bracket that acts as a graded derivation:

$$[x, yz] = [x,y]z + (-1)^{\deg(x)(\deg(y)-1)} y[x,z]\,. \tag{3.70}$$

Definition 3.9.5 (Semistrict Lie 2-algebra). A (Baez–Crans) 2-vector space $L \equiv (L_0, L_1)$ equipped with the following morphisms:

- an antisymmetric bilinear functor $[\cdot,\cdot] : L \times L \to L$ (the **bracket**), and
- a completely antisymmetric trilinear natural isomorphism

$$J_{x,y,z} : [[x,y],z] \to [x,[y,z]] + [[x,z],y]\,, \tag{3.71}$$

  called the **Jacobiator**.

These structures are required to satisfy the *Jacobiator identity* (which is just the *Zamolodchikov tetrahedron equation*). If the Jacobiator is trivial, a **strict** Lie 2-algebra is obtained. By further relaxing the antisymmetry, one can obtain the fully weak version of Lie 2-algebras (see for example the work by *Roytenberg*).

From the previous section, it follows that one can define (weak) Lie 2-algebras as 2-term chain complexes equipped with a coherent Lie bracket.

Alternative Definition 3.9.6 (Lie 2-algebra). Consider a 2-term chain complex in the category **FinVect**:

$$0 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow 0\,. \tag{3.72}$$

This complex L is called a Lie 2-algebra if it comes equipped with the following structures:

- a chain map $[\cdot,\cdot] : L \otimes L \to L$ called the **bracket**,
- a chain homotopy $S : [\cdot,\cdot] \Rightarrow -[\cdot,\cdot] \circ \sigma$ called the **alternator**, and
- a chain homotopy

$$J : [\cdot,[\cdot,\cdot]] \Rightarrow [[\cdot,\cdot],\cdot] + [\cdot,[\cdot,\cdot]] \circ (\sigma \otimes \mathbb{1})\,, \tag{3.73}$$

  called the **Jacobiator**.

These chain homotopies are again required to satisfy higher coherence relations. From the previous definition, it follows that the vanishing of the alternator implies that L is semistrict. Analogously, a Lie 2-algebra for which the Jacobiator vanishes is said to be **hemistrict**. Note that this definition of weak Lie 2-algebras, when translated to the 2-vector space setting, would imply that the alternator and Jacobiator are merely natural transformations (and not isomorphisms)!

3.9.2 Lie n-algebras

Definition 3.9.7 (Semistrict Lie ω-algebra). By replacing internal categories by internal *ω-categories* and by relaxing the Jacobiator identity up to coherent homotopy, i.e. up to a completely antisymmetric quadrilinear modification which in turn satisfies an identity up to higher multilinear transfors, one obtains the definition of L_∞-algebras. Similar to A_∞-algebras, these too can be obtained as algebras over a suitable operad (however, in this case the operad is 'slightly' more complex: the cofibrant replacement of the *Lie operad*).

It can be shown that these structures are equivalent to the L_∞-algebras of *Stasheff* defined below.

Definition 3.9.8 (L_∞-algebra[8]). A graded vector space V equipped with a collection of morphisms $l_n : V^{\otimes n} \to V, n \in \mathbb{N}_0$ of degree $n-2$ subject to the relations

$$l_n(v_{\sigma(1)} \dots v_{\sigma(n)}) = \chi(\sigma; v_1, \dots, v_n) l_n(v_1 \dots v_n) \tag{3.74}$$

and

$$\sum_{\substack{i+j=n+1 \\ \sigma\in\mathsf{Unshuff}(i,j-1)}} (-1)^{i(j-1)} \chi(\sigma; v_1, \dots, v_n) l_i\big(l_j(v_{\sigma(1)} \cdots v_{\sigma(j)}) v_{\sigma(j+1)} \cdots v_{\sigma(n)}\big) = 0\,, \tag{3.75}$$

where $\mathsf{Unshuff}$ denotes the collection of unshuffles (**??**).

The l_1 map turns the L_∞-algebra into a chain complex. The l_2 map is a generalized Lie bracket since it is (graded-)antisymmetric. Higher l_n's can be identified with the Jacobiator and its generalizations. In the next section, a bottom-up approach will be given.

Remark 3.9.9. The definition can be rephrased in terms of graded maps $\hat{l}_n : \mathsf{Alt}^\bullet V \to V$.

Remark 3.9.10 (Curvature). The above definition can be generalized by including a nullary bracket l_0. Such L_∞-algebras are often said to be **curved**. The reason for this is that the coherence condition for l_0 says that

$$l_1 \circ l_1 = l_2(l_0, -)\,. \tag{3.76}$$

This terminology stems from the situation where l_1 is identified with the *exterior covariant derivative* on an *associated vector bundle* (see **??**).

Example 3.9.11 (Lie algebra). It can easily be checked that the L_∞-algebra with V concentrated in degree 1 is equivalent to the structure of an ordinary Lie algebra. Similarly, one obtains the notion of a Lie n-algebra by truncating an L_∞-algebra at degree n.

[8] Also called a **strong(ly) homotopy Lie algebra** (abbreviated to **sh Lie algebra**).

Property 3.9.12. 2-term L_∞-algebras or, equivalently, semistrict Lie 2-algebras are in correspondence with isomorphism classes of tuples $(\mathfrak{g}, V, \rho, l_3)$ where $\mathfrak{g}$ is a Lie algebra, (V, ρ) is Lie algebra representation of $\mathfrak{g}$ and l_3 is a V-valued *Lie algebra 3-cocycle* (see **??**).

Proof (Sketch of construction). Using the representation ρ, one can extend the Lie bracket from $\mathfrak{g}$ to the complex $0 \to V \to \mathfrak{g} \to 0$ through the formulas $[g, v] := \rho(g)v$ and $[v, g] := -[g, v]$. The cocycle condition for l_3 gives rise to the Jacobiator. □

Example 3.9.13. If one chooses a finite-dimensional Lie algebra $\mathfrak{g}$ with the trivial representation on $\mathbb{R}$ (or, more generally, the underlying field of $\mathfrak{g}$), one obtains

$$H^3(\mathfrak{g}; \mathbb{R}) \cong \mathbb{R}\,. \tag{3.77}$$

The different classes can be represented by scalar multiples of the Killing cocycle (see **??**). For every such scalar $\lambda \in \mathbb{R}$, one denotes the resulting Lie 2-algebra by $\mathfrak{g}_\lambda$.

Lie algebras and L_∞-algebras can also be dually characterized in terms of their *Chevalley–Eilenberg algebra* (see **??**).

Alternative Definition 3.9.14 (Lie algebra). Consider a finite-dimensional Lie algebra $\mathfrak{g}$. The transpose/dual of the Lie bracket $[\cdot,\cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ is a morphism $\delta : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$:

$$\delta\omega(g, h) := \omega([g, h])\,. \tag{3.78}$$

In fact, it is not hard to see that this is exactly the Chevalley–Eilenberg differential of $\mathsf{CE}(\mathfrak{g})$ (see **??**). Conversely, given a semifree dgca $(\Lambda^\bullet V^*, \mathrm{d})$, for some finite-dimensional vector space V, one obtains a finite-dimensional Lie algebra by restricting the differential to V^* and taking the transpose. In fact, the nilpotency condition $\mathrm{d}^2 = 0$ is equivalent to the Jacobi identity.

More generally, by passing to graded vector spaces of finite type concentrated in positive degree, one can characterize L_∞-algebras as semifree DGCAs.

Alternative Definition 3.9.15 (L_∞-algebra). The (graded) Leibniz rule implies that the differential δ is completely defined by its restriction to the generators $V^* \leq \Lambda^\bullet V^*$. The differential can be decomposed as follows:

$$\delta t^a := -\sum_{k=1}^{+\infty} \frac{1}{k!}[t_{a_1}, \ldots, t_{a_k}]_k^a\, t^{a_1} \wedge \cdots \wedge t^{a_k}\,, \tag{3.79}$$

where the basis t^a of V^* is dual to the basis t_a of V. Because δ is of degree 1, the coefficients $[\cdots]_k^a$ define a multilinear operator $[\cdots]_k : \Lambda^k V \to V$ of degree $n-1$. Some sources rephrase these brackets as morphism from the symmetric algebra $\mathrm{Sym}^\bullet V$, in which case their degree is just -1, cf. décalage (**??**).

The nilpotency condition $\delta^2 = 0$ implies a list of (quadratic) relations on the brackets $[\cdots]_k$ (with $\mathsf{d} := [\cdot]_1$):

$$\begin{aligned} \mathsf{d}^2 &= 0 \\ \mathsf{d}[\cdot,\cdot]_2 &= [\mathsf{d}\cdot,\cdot]_2 + [\cdot,\mathsf{d}\cdot]_2 \\ [[v_1,v_2],v_3]_2 + \text{cyc. perm.} &= \mathsf{d}[v_1,v_2,v_3]_3 - [\mathsf{d}v_1,v_2,v_3]_3 - [v_1,\mathsf{d}v_2,v_3]_3 - [v_1,v_2,\mathsf{d}v_3]_3 \\ &\vdots \end{aligned}$$

These relations can be interpreted as follows:

- d is a differential.
- d acts as a derivation with respect to the binary bracket.
- The Jacobi identity holds up to a chain homotopy (given by the ternary bracket).
- The higher relations are similar to the chain homotopy for the Jacobi identity.

When written out it in full detail, it can be checked that this is exactly the definition of an L_∞-algebra.

Definition 3.9.16 (Maurer–Cartan element). An element a of an L_∞-algebra V that satisfies the equation

$$\sum_{k=0}^{+\infty} \frac{1}{k!}[a,\dots,a]_k = 0\,. \tag{3.80}$$

For dg-Lie algebras, this reduces to the ordinary *Maurer–Cartan equation* (see **??**):

$$\mathsf{d}a + \frac{1}{2}[a,a] = 0\,. \tag{3.81}$$

This is no coincidence since the complex $\Omega^\bullet(M) \otimes \mathfrak{g}$ of Lie algebra-valued differential forms on a smooth manifold M carries a canonical dg-Lie algebra structure.

3.10 Monoidal n-categories

Definition 3.10.1 (Monoidal n-category). In general, one can define a monoidal n-category as a one-object $(n+1)$-category, similar to how monoidal categories give one-object bicategories by delooping (**??**). For the explicit definitions of monoidal bi- and tricategories, see the papers by Cheng and Gurski (2007) and Hoffnung (2011), respectively.

If one would put multiple compatible monoidal products on an n-category, by a version of the Eckmann–Hilton argument 2.5.1, all of these structures will be equivalent to a 'commutative' monoidal structure. By increasing the number of compatible structures, the 'commutativity' can be increased. This gives rise to the following definition which is stated in different terms (based on the *delooping hypothesis*).

Definition 3.10.2 (k-tuply monoidal n-categories). A pointed $(n+k)$-category (strict or weak) in which all parallel j-arrows for $j < k$ are equivalent. These categories form an $(n+k+1)$-category $\boldsymbol{k}\mathbf{Mon}\boldsymbol{n}\mathbf{Cat}$.

Example 3.10.3. For small values of k and n, the resulting structures coincide with some well-known constructions:

- $n = 0$:
  - $k = 0$: pointed set,
  - $k = 1$: monoid, and
  - $k \geq 2$: Abelian monoid.
- $n = 1$:
  - $k = 0$: 'pointed' category[9],
  - $k = 1$: monoidal category,
  - $k = 2$: braided monoidal category, and
  - $k \geq 3$: symmetric monoidal category.

The stabilization occurring for higher values of k is the content of the following hypothesis[10] by Baez and Dolan (1995).

Theorem 3.10.4 (Stabilization hypothesis). *For values $k \geq n+2$, the structure of a k-tuply monoidal n-category becomes maximally symmetric. Formally, this means that the inclusion* $\boldsymbol{k}\mathbf{Mon}\boldsymbol{n}\mathbf{Cat} \hookrightarrow (\boldsymbol{n}+\mathbf{2})\mathbf{Mon}\boldsymbol{n}\mathbf{Cat}$ *becomes an equivalence.*

3.10.1 Relation with group cohomology

See **??** for more information on group cohomology.

Consider a finite group G. As a first step, construct the group algebra $\mathbb{C}[G]$. As a monoid, one can consider this object as a G-graded monoidal 0-category. The ordinary

[9] As in category with a specified element not as in category with a zero object (Definition 2.4.14).
[10] For certain definitions of higher categories this has been proven in full generality.

multiplication $g * h = gh$ can be twisted to obtain a monoid $\mathbb{C}[G]^{\omega}$ with multiplication

$$g * h := e^{i\omega(g,h)} gh\,. \tag{3.82}$$

If associativity is still required to hold on the nose, one is led to the property that ω is in fact a group 2-cocycle. The equivalence classes of such twisted group algebras are then in correspondence with the second cohomology class $H^2(G; \mathrm{U}(1))$.

Before really going to higher category theory, one should first reflect on the different structures in the previous paragraph. Since the monoid is regarded as a monoidal category (call it M for convenience), one has a bifunctor $\mu : M \otimes M \to M$ (given by the twisted multiplication) that differs from the ordinary group multiplication by a phase. This phase can be viewed categorically as a natural isomorphism between the 'tensor products' in $\mathbb{C}[G]$ and M. At the same time, all the higher coherence conditions[11] (associativity, ...) are required to hold identically.

Now, drop the restriction on the product and take this to be a more general monoidal product bifunctor. To this end, replace the monoid $\mathbb{C}[G]$ by the G-graded monoidal category $\mathbf{Vect}_G$ and relax the associativity constraint up to a natural isomorphism α. When restricted to the simple objects of $\mathbf{Vect}_G$ this is given by a phase factor $e^{i\omega(g,h,k)}$. The pentagon condition for monoidal categories then implies that the function ω is a group 3-cocycle. In analogy with the case of monoids above, the equivalence classes of (twisted) monoidal structures on $\mathbf{Vect}_G$ is in correspondence with the third cohomology group $H^3(G; \mathrm{U}(1))$.

To go yet another step higher, move up a level in the chain of coherence conditions and relax the associativity constraint even more (for simplicity the one-object n-category point of view is adopted here). Instead of a natural isomorphism it only has to be an adjoint equivalence and at the same time the pentagon condition is replaced by an invertible modification. The coherence condition of this **pentagonator** then implies a classification of (twisted) monoidal bicategories, equivalent to $\mathbf{2Vect}_G^{\omega}$, by the fourth group cohomology $H^4(G; \mathrm{U}(1))$.

In a completely analogous way one can define more and more general structures. e.g. for monoidal tricategories one can translate the K_6-*associahedron* into an equation for an invertible perturbation which by the G-graded structure is equivalent to a group 5-cocycle.

Remark 3.10.5. This section is strongly related to the twisting procedure in n-dimensional *Dijkgraaf–Witten theories*.

[11] These can be parametrized by the *Stasheff polytopes/associahedra*.

Part II

Higher Set Theory

Table of Contents

Chapter 4

Logic and Type Theory

The main reference for this chapter is The Univalent Foundations Program (2013). For a formal introduction to λ-calculus, see Selinger (2008).

In almost every section of this chapter (at least the ones about type theory), some cross-references to analogous definitions and propositions in other parts of this compendium could have been inserted (Chapter 2 on category theory in particular). However, to reduce the number of references, these relations will only be mentioned and the reader is encouraged to take a look at the relevant chapters whilst or after reading this chapter.

Chapter Contents

4.1 Logic

4.1.1 Languages

Definition 4.1.1 (Language). An **alphabet** is a set of symbols. A **word** in the language is a string of symbols in the alphabet.

Consider an alphabet A. From this alphabet one can construct the free monoid A^* (the multiplication $*$ is sometimes called the **Kleene star**). This monoid represents the set of all words in A and a (formal) language is a subset $L \subseteq A^*$.

Definition 4.1.2 (Signature). Consider an alphabet A and a language L. A signature is a tuple (F, R, ar) that assigns a syntactic meaning to the symbols in A. F and R are respectively the sets of function symbols and relation symbols ($A = F \sqcup R$). The function $\mathsf{ar} : A \to \mathbb{N}$ assigns to every symbol its arity **arity**. Nullary function symbols are also called **constants**.

To give meaning to a language, some extra structure needs to be introduced.

Definition 4.1.3 (L-structure). Consider a (formal) language L. An L-structure consists of the following data:

1. A nonempty set U called the **universe**.
2. For each function symbol f, a function $\mathsf{ap}_f : U^{\mathsf{ar}(f)} \to U$. In particular, for each constant c, an element $u_c \in U$.
3. For each relation symbol $\in$, a set $R_\in \subseteq U^{\mathsf{ar}(\in)}$.

Definition 4.1.4 (L-term). A word in L, possibly containing new symbols (called **variables**), defined recursively as follows:

1. Every variable and every constant is a term.

2. For every n-ary function symbol f and terms $x_1, \ldots, x_n$, $f(x_1, \ldots, x_n)$ is also a term.

Definition 4.1.5 (L-formula). Consider a (formal) language L. An L-formula is a sentence consisting of terms in L together with parentheses and the following logical symbols (also called **logical connectives**):

- **Equality**: $=$,
- **Negation**: $\neg$,
- **Conjunction**: $\wedge$, and
- **Existential quantification**: $\exists$.

A variable is said to be **free** if it does not first appear next to a quantifier, otherwise it is said to be **bound**.

4.1.2 Propositional logic

Definition 4.1.6 (Proposition). A statement that is either *true* or *false* (not both).

Definition 4.1.7 (Paradox). A statement that cannot (consistently) be assigned a truth value.

Definition 4.1.8 (Contradiction). A statement that is always *false*.

Definition 4.1.9 (Tautology). A statement that is always *true*.

Notation 4.1.10 (Truth values). The truth values *true* and *false* are denoted by $\top$ and $\bot$ respectively.

Definition 4.1.11 (Logical connectives). The following logical operators are used in propositional logic:

- logical 'and' (**conjunction**): $P \wedge Q$,
- logical 'or' (**disjunction**): $P \vee Q$, and
- logical 'then' (**implication**): $P \implies Q$.

Using implication, one can also define the logical 'not' (**negation**): $\neg P :\equiv P \implies \bot$.

The basic inference rule is given by **modus ponens**:

$$\text{If } P \text{ and } P \implies Q\text{, then } Q. \tag{4.1}$$

One could also use negation as a primitive connective and introduce implication as

$$P \implies Q :\equiv \neg P \wedge Q\,. \tag{4.2}$$

The general deductive system for propositional logic is obtained by combining this rule with the following axioms:

1. If P, then $Q \implies P$.
2. If $P \implies Q \implies R$, then $P \implies Q$ implies $P \implies R$.
3. If $P \wedge Q$, then both P and Q.
4. If P, then $P \vee Q$.
5. If Q, then $P \vee Q$.
6. If P, then Q implies $P \wedge Q$.
7. If $P \implies Q$, then $R \implies Q$ implies $P \vee R \implies Q$.
8. If $\perp$, then P. This principle is often called ***ex falso quodlibet***.

Property 4.1.12 (Boolean algebra). The set of propositions in classical logic admits the structure of a complete Boolean algebra (**??**).

Remark 4.1.13 (Intuitionistic logic). The above axioms (together with modus ponens) define a specific type of propositional logic, called intuitionistic or **constructive** (propositional) logic. The main difference with classic logic is that the *law of the excluded middle* or, equivalently, the *double negation elimination* principle was not added. The reason why this makes the logic *constructive* is that to prove a statement it is not sufficient anymore to exclude the possibility of the statement being false. One has to explicitly construct evidence for the truth of the statement.

As was remarked in the chapter on topoi, intuitionistic logic can be defined internal to any elementary topos. All one needs is a Heyting algebra (Definition 1.7.44).

@@ EXPLAIN THIS @@

4.1.3 Sequent calculus

Definition 4.1.14 (Sequent). A general sequent is of the form

$$P_1, \dots, P_m \vdash Q_1, \dots, Q_n \,. \tag{4.3}$$

In such expressions, the commas on the left-hand side indicate conjunction, whereas those on the right-hand side indicate disjunction, i.e. this sequent states "when every P_i holds, then at least one of the Q_j hold as well". The above sequent if (strongly) equivalent to

$$\vdash (P_1 \wedge \cdots \wedge P_m) \implies (Q_1 \vee \cdots \vee Q_n) \,. \tag{4.4}$$

@@ COMPLETE @@

4.1.4 Predicate logic

@@ ADD @@

4.2 Introduction to type theory

In ordinary set theory, the main objects are sets and their elements (and derived concepts such as functions). The framework in which to state and prove propositions is (in general) given by first-order logic. (See Section 1.1 for more on this.) In type theory, however, one puts all these notions on the same footing. That is, one considers all concepts such as functions, propositions, sets, etc. as specific instances of the general notion of *type*. A specific function, proof or element can then be seen as an *inhabitant* of a given type.

Definition 4.2.1 (Type judgement). A statement of the form $a : A$, saying that a has the type A, is called a type judgement. Objects having a certain type are in general called **terms** (of that type).

Method 4.2.2 (Type definition). The general method for defining a new type consists of 4 steps/rules:

1. **Formation rule**: This rule says when the new type can be introduced, given a collection of pre-existing types.

2. **Introduction rule**: This rule gives a **constructor** of the new type, i.e. a way to construct a term of the new type[1], in terms of the types required by the formation rule. The pre-existing terms from which a new term can be constructed is often called the **context**.

3. **Elimination rule**: This rule says how the new type can be used.

4. **Computation rule**: This rule says how the elimination and introduction rules interact, i.e. how the elimination rules can actually be applied to a term of the given type.

As in The Univalent Foundations Program (2013), a universe hierarchy à la Russell will be adopted, i.e. a sequence of universes $(\mathcal{U}_n)_{n\in\mathbb{N}}$ will be used where the terms of every universe are types and every universe is cumulative in the sense that $A : \mathcal{U}_n \implies A : \mathcal{U}_{n+1}$. In general, the subscripts will be omitted. However, one should take into account that every well-typed judgement should admit a formulation in which subscripts can be assigned in a consistent way.

[1] As in object-oriented programming languages.

In contrast to ordinary set theory, two kinds of equality will be considered. First, there is the **judgemental** or **definitional equality**. This says, as the name implies, that two judgements are equal by definition and, as such, its validity lives in the metatheory (it is not a proposition and, hence, cannot be proven). For example, if $f(x)$ is defined as x^2, then $f(5)$ is, by definition, equal to 5^2. Equalities of this sort will be denoted by the $\equiv$ symbol (and, in definitions, $:\equiv$ will be used instead of $:=$). The second equality is the **propositional equality**. This states that two judgements are provably equal. Again, consider the function $f(x) :\equiv x^2$. In this case, the proposition $f(5) = 25$ is not true by definition and can be proven (it would, however, depend on the definition of the natural numbers). This sort of equality will be denoted by an ordinary equals sign $=$.

4.3 Basic constructions

4.3.1 Functions

Functions can be introduced in two ways. Either through a direct definition, such as in the case of the default example $f(x) :\equiv x^2$, or through *λ-abstraction*. Although the former one is clearly more useful during explicit calculations, the latter will often be used when working with abstract proofs. (For an introduction to λ-calculus, see the next section.)

Definition 4.3.1 (Function type). A general function type is introduced as follows:

- **Formation rule**: Given two types $A, B : \mathcal{U}$, one can form the function type $A \rightarrow B : \mathcal{U}$.
- **Introduction rule**: One can either define a function by an explicit definition $f(x) :\equiv \Phi$, where Φ is an expression possibly involving x, or by λ-abstraction $f :\equiv \lambda x.\Phi$.
- **Elimination rule**: If $a : A$ and $\lambda x.\Phi : A \rightarrow B$, then $\lambda x.\Phi(a) : B$.
- **Computation rule**[2]: $\lambda x.\Phi(a) :\equiv \Phi(a)$, i.e. function application is equivalent to the substitution of a for the variable x in the expression Φ. (To be completely correct, one should require the substitution to be *capture-avoiding*, i.e. free variables should remain free and distinct variables should not be assigned the same symbol.)

The **uniqueness principle** for function types should also be included in the definition, i.e. $\lambda x.f(x) \equiv f$. This says that every function is uniquely defined by its image.

An important generalization is obtained when the type of the output of a function is allowed to depend on the type of the input.

[2]In λ-calculus, this is often called β-reduction. (See the next section.)

Definition 4.3.2 (Dependent function types). Given a type $A : \mathcal{U}$ and a type family $B : A \to \mathcal{U}$, one can form the dependent function type

$$\prod_{a:A} B(a) : \mathcal{U}\,. \tag{4.5}$$

When B is a constant family, this type reduces to the ordinary function type $A \to B$. All other defining rules remain (formally) the same as in the nondependent setting.

Remark 4.3.3 (Scope). The Π-symbol scopes over all expressions to the right of the symbol unless delimited (similar to λ-calculus), e.g.

$$\prod_{a:A} B(a) \to C(a) \equiv \prod_{a:A} \big(B(a) \to C(a)\big)\,. \tag{4.6}$$

Example 4.3.4 (Polymorphic functions). An interesting example is obtained when the type A in the above definition is taken to be a universe $\mathcal{U}$ (this is a valid choice since universes are themselves types) together with $B(A) :\equiv A$. In this case, one obtains a function that takes a type as input and then acts on this type (or any other type constructed from it), e.g. the **polymorphic identity function**

$$\mathsf{id} : \prod_{A:\mathcal{U}} A \to A \tag{4.7}$$

defined by

$$\mathsf{id} :\equiv \lambda(A : \mathcal{U}).\lambda(a : A).a\,. \tag{4.8}$$

4.3.2 λ-calculus

@@ COMPLETE (e.g. Curry–Howard or even Curry–Howard–Lambek, typed vs. untyped calculus, ...) @@

4.3.3 Products

As in classic set theory, one of the basic notions is that of a product. This construction is ubiquitous throughout all of mathematics (and computer science). However, as opposed to set theory à la ZFC, products are not explicitly constructed as the set of all pairs of elements of its constituents. On the contrary, in type theory, one can prove that all elements necessarily have to be pairs.

Definition 4.3.5 (Product). First, the binary product of types is defined.

- **Formation rule**: Given any two types $A, B : \mathcal{U}$, one can form the product type $A \times B : \mathcal{U}$.
- **Introduction rule**: Given terms $a : A, b : B$, one can construct the term $(a, b) : A \times B$. This is called the **pairing** of the terms a and b.

- **Elimination and computation rules**: Functions out of a product $A \times B$ are defined through currying, i.e. given a function $A \to B \to C$, one can define a function $A \times B \to C$. Instead of giving an explicit definition every time one wants to construct a new function, a universal point of view is adapted: a single function that turns terms $f : A \to B \to C$ into terms $g : A \times B \to C$ is constructed. To this end, consider the **recursor**

$$\mathsf{rec}_{A\times B} : \prod_{C:\mathcal{U}} (A \to B \to C) \to A \times B \to C \tag{4.9}$$

with the constraint

$$\mathsf{rec}_{A\times B}\big(C, f, (a,b)\big) :\equiv f(a)(b)\,. \tag{4.10}$$

Example 4.3.6 (Projections). Analogous to the projection functions associated to the Cartesian product, one should have functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ that act on constructors as

$$\pi_1(a,b) :\equiv a \qquad \text{and} \qquad \pi_2(a,b) :\equiv b\,. \tag{4.11}$$

Using the recursor, one can define these functions by taking $C = A, f = \lambda a.\lambda b.a$ and $C = B, f = \lambda a.\lambda b.b$, respectively.

Definition 4.3.7 (Nullary product). One can also define a nullary product. In this case, it is called the **unit type 1**.

- **Formation rule**: $\mathbf{1} : \mathcal{U}$.
- **Introduction rule**: There is a unique nullary constructor $* : \mathbf{1}$.
- **Elimination and computation rules**: Since the constructor is a nullary operation, one does not expect to have projection maps and, likewise, one also does not expect function definition to be based on binary currying. Instead, the recursor is defined as follows:

$$\mathsf{rec}_{\mathbf{1}} : \prod_{C:\mathcal{U}} C \to \mathbf{1} \to C\,. \tag{4.12}$$

On the constructor $* : \mathbf{1}$, it is required to act trivially:

$$\mathsf{rec}_{\mathbf{1}}(C, c_0, *) :\equiv c_0\,. \tag{4.13}$$

Definition 4.3.8 (Dependent functions). One can easily generalize the above recursors to **inductors**, to allow for the definition of dependent functions out of product types (these functions are then said to be defined by an **induction principle**). In fact, one only has to change the type judgement of $\mathsf{rec}_{A\times B}$. This is accomplished by replacing $C : \mathcal{U}$ by a type family $C : A \times B \to \mathcal{U}$ and by replacing nondependent function types by dependent function types (the form of the computation rules virtually remain the same):

$$\begin{aligned}\mathsf{ind}_{A\times B} &: \prod_{C:A\times B\to\mathcal{U}}\left(\prod_{a:A,b:B} C(a,b) \to \prod_{x:A\times B} C(x)\right),\\ \mathsf{ind}_{\mathbf{1}} &: \prod_{C:\mathbf{1}\to\mathcal{U}} C(*) \to \prod_{x:\mathbf{1}} C(x)\,.\end{aligned} \tag{4.14}$$

Property 4.3.9 (Uniqueness principle). Using the induction principle, one can prove that every term $x : A \times B$ is necessarily of the form (a, b) for some $a : A, b : B$. Furthermore, one can also prove that $* : \mathbf{1}$ is the unique term in $\mathbf{1}$.

One can also generalize products in such a way that the type of the second factor depends on the type of the first one (in classical set theory, this would correspond to an indexed disjoint union).

Definition 4.3.10 (Dependent pair type). As with function types, the definition is not given as explicit as for nondependent types. Suffice it to say that, given a type $A : \mathcal{U}$ and a type family $B : A \to \mathcal{U}$, one can form the dependent pair type

$$\sum_{a:A} B(a) : \mathcal{U}\,. \tag{4.15}$$

When B is a constant family, the type reduces to the ordinary product type $A \times B$. The recursion and induction functions are defined as in the product case, except for the obvious replacements, such as $A\times B \longrightarrow \sum_{a:A} B(a)$, needed to make everything consistent.

Remark 4.3.11. Dependent pair types are often called **Σ-types** (due to the notation).

Remark 4.3.12 (Scope). Like the Π-symbol, the Σ-symbol scopes over the entire expression to the right unless delimited.

Definition 4.3.13 (Coproduct). Here, a standalone definition is given. The relation with the ordinary product will be mentioned afterwards.

- **Formation rule**: Given two types $A, B : \mathcal{U}$, one can form the coproduct type $A+B : \mathcal{U}$.
- **Introduction rule**: Since in ordinary mathematics (and, in particular, category theory) the coproduct is dual to the product, one expects the projections to be replaced by **injections/inclusions**. In fact, these are taken to be the constructors of coproduct types, i.e. given terms $a : A$ and $b : B$, one can construct the terms $\iota_1(a) : A + B$ and $\iota_2(b) : A + B$.

- **Elimination rules**: Similar to the use of currying for the definition of functions out of a product, functions out of a coproduct are defined in steps. To this intent the recursor and inductor are defined as follows:
$$\begin{aligned}\mathsf{rec}_{A+B} &: \prod_{C:\mathcal{U}}(A \to C) \to (B \to C) \to A + B \to C\,,\\ \mathsf{ind}_{A+B} &: \prod_{C:A+B\to\mathcal{U}}\left(\prod_{a:A} C\big(\iota_1(a)\big)\right) \to \left(\prod_{b:B} C\big(\iota_2(b)\big)\right) \to \prod_{x:A+B} C(x)\,.\end{aligned} \tag{4.16}$$

- **Computation rules**: The recursor acts on the constructors as follows (the inductor virtually has the same action):
$$\begin{aligned}\mathsf{rec}_{A+B}\big(C, f_1, f_2, \iota_1(a)\big) &:\equiv f_1(a)\,,\\ \mathsf{rec}_{A+B}\big(C, f_1, f_2, \iota_2(b)\big) &:\equiv f_2(b)\,.\end{aligned} \tag{4.17}$$

Definition 4.3.14 (Nullary coproduct). As was the case for products, one can also define a nullary version of the coproduct, the **empty type 0**:

- **Formation rule**: $\mathbf{0} : \mathcal{U}$.
- **Introduction rule**: There is no constructor for $\mathbf{0}$.
- **Elimination and computation rules**: Since there is no constructor for $\mathbf{0}$, one can always trivially 'construct' a function out of $\mathbf{0}$:
$$\begin{aligned}\mathsf{rec}_{\mathbf{0}} &: \prod_{C:\mathcal{U}} \mathbf{0} \to C\,,\\ \mathsf{rec}_{\mathbf{0}} &: \prod_{C:\mathbf{0}\to\mathcal{U}} \prod_{x:\mathbf{0}} C(x)\,.\end{aligned} \tag{4.18}$$
This trivial function corresponds to the logical principle *ex falso quodlibet* as introduced in the section on logic above.

Since coproducts in set theory occur as binary disjoint unions, one could expect that there is a way to express coproducts in terms of dependent pair types.

Construction 4.3.15 (Coproducts as Σ-types). First, introduce the type $\mathbf{2} : \mathcal{U}$ (in set theory, this would be the 2-element set). The introduction rule constructs two terms $0, 1 : \mathbf{2}$. The elimination and computation rules say that one can use this type for binary indexing:
$$\mathsf{rec}_{\mathbf{2}} : \prod_{C:\mathcal{U}} C \to C \to \mathbf{2} \to C \tag{4.19}$$
with
$$\begin{aligned}\mathsf{rec}_{\mathbf{2}}(C, c_0, c_1, 0) &:\equiv c_0\,,\\ \mathsf{rec}_{\mathbf{2}}(C, c_0, c_1, 1) &:\equiv c_1\,.\end{aligned} \tag{4.20}$$

Using this type, one can prove that $A + B$ is judgementally equal to $\sum_{x:\mathbf{2}} \mathsf{rec}_{\mathbf{2}}(\mathcal{U}, A, B, x)$. The injections are given by pairing, i.e. $\iota_1(a) \equiv (0, a)$ and $\iota_2(b) \equiv (1, b)$. In a similar way, one can obtain binary products as dependent function types over **2**.

4.3.4 Propositions as types

To conclude this section, an overview of all the concepts introduced above is given from a propositions-as-types perspective. In intuitionistic logic, this is often called the **Brouwer–Heyting–Kolmogorov interpretation** and, more specifically, it should be seen as an incarnation of the Curry–Howard correspondence.

- Types and their terms correspond to propositions and their proofs, respectively. In a proof-relevant context the fact that a type can have multiple terms makes it clear that, although distinct proofs eventually have the same result, the difference in their content can be important as well.
- Function types correspond to implications. A proof of the proposition $A \to B$ boils down to showing that every proof of A gives a proof of B.
- Π-types correspond to universal quantification, i.e. $\prod_{a:A} B(a)$ can be read as $\forall a \in A : B(a)$. Giving a proof of $\prod_{a:A} B(a)$ is the same as giving for every $a : A$ a proof of $B(a)$. This is indeed compatible with the fact that elements of Π-types are dependent functions, i.e. every element $a : A$ gives rise to a (possibly) distinct type/proposition.
- Σ-types correspond to existential quantification, i.e. $\sum_{a:A} B(a)$ can be read as $\exists a \in A : B(a)$. Giving a proof of $\sum_{a:A} B(a)$ is the same as giving a proof for some $(a, B(a))$. This is compatible with the fact that Σ-types can be identified with disjoint unions and hence every element can be associated with a specific constituent type.
- The logical connectives (conjunction and disjunction) correspond to the product and coproduct types.
- The truth values, *true* and *false*, correspond to the unit and empty types, respectively. Furthermore, if the negation of A is defined as the type $\neg A :\equiv A \to \mathbf{0}$, this indeed corresponds to the logical negation by the statements above.

4.3.5 Identity types

One of the most important, but at the same time most subtle, concepts in type theory (especially when moving on to extensions such as homotopy type theory) is the identity type. Since in predicate (and even propositional) logic the equality of two terms is a proposition, one could expect that to every two terms $a, b : A$ there corresponds an associated equality type $a =_A b : \mathcal{U}$. Note that the type of the terms is assumed to be the same since it does not make any sense to compare terms of different types.

Definition 4.3.16 (Equality type[3]). The type corresponding to a propositional equality is defined by the following rules:

- **Formation rule**: Given terms $a, b : A$, one can form the equality type $a =_A b : \mathcal{U}$. When the type A is clear from the context, this is also often written as $a = b : \mathcal{U}$.
- **Introduction rule**: For every term $a : A$, there is a canonical identity element

$$\mathsf{refl}_a : a = a\,. \tag{4.21}$$

  The notation points to the fact that this term can be seen as a proof of the reflexivity of equalities.

- **Elimination and computation rules**: Here, the so-called **path induction principle** for equality types is presented, for the equivalent *based path induction principle* see The Univalent Foundations Program (2013).

  Given a type family

$$C : \prod_{a,b:A} a = b \to \mathcal{U}$$

  and a term

$$I : \prod_{a:A} C(a, a, \mathsf{refl}_a)\,,$$

  there exists a function

$$f : \prod_{a,b:A} \prod_{p:a=b} C(a, b, p) \tag{4.22}$$

  such that

$$f(a, a, \mathsf{refl}_a) :\equiv I(a) \tag{4.23}$$

  for all $a : A$.

Informally this principle says that all terms of the form (a, b, p), with $p : a = b$, are inductively generated by the 'constant' terms (a, a, refl_a). (See the section on homotopy type theory for a more geometric perspective).

Using the notion of identity types one can say when a given type resembles a proposition:

[3] Sometimes called an **identity type**.

Definition 4.3.17 (Mere proposition). A type $A : \mathcal{U}$ for which

$$\mathsf{isProp}(A) :\equiv \prod_{a,b:A} a = b \tag{4.24}$$

is inhabited. This is also called an h**-proposition**.

Given a mere proposition $P : \mathcal{U}$, the related identity type is either uninhabited, if P itself is uninhabited, or has a unique term. Types of this form are said to be **contractible**.

Definition 4.3.18 (Contractible type). A type $A : \mathcal{U}$ for which

$$\mathsf{isContr}(A) :\equiv \sum_{a:A} \prod_{b:A} a = b \tag{4.25}$$

is inhabited.

Definition 4.3.19 (Homotopy level). The homotopy level or h**-level** of a type $A : \mathcal{U}$ is recursively defined as follows:

$$\mathsf{hasHLevel}(0, A) :\equiv \mathsf{isContr}(A) \tag{4.26}$$

$$\mathsf{hasHLevel}(n+1, A) :\equiv \prod_{a,b:A} \mathsf{hasHLevel}(n, a = b)\,. \tag{4.27}$$

A type of homotopy level $n + 2$ is also called a **homotopy** n**-type**.

Example 4.3.20. The lowest h-levels are given by

-2 : Contractible types,

-1 : mere propositions,

0 : sets,

1 : groupoids, ...

Because of the inductive nature of identity types, any homotopy n-type can be truncated to a homotopy k-type for $k < n$. In the case of a (-1)-truncation, the following notion is obtained.

Definition 4.3.21 (Bracket type). Consider a type $A : \mathcal{U}$. The bracket $[A] : \mathcal{U}$ or $[A] : \mathcal{U}$ is inhabited (and uniquely so) if and only if A is inhabited. It is defined by the following rules:

$$\mathsf{isInhab} : A \mapsto [A]\,, \tag{4.28}$$

$$\mathsf{propTrunc} : \prod_{a,b:A} \mathsf{isInhab}(a) = \mathsf{isInhab}(b)\,. \tag{4.29}$$

The term propTrunc exactly says that $[A]$ is a mere proposition.

Definition 4.3.22 (Homotopy fibre). Consider a function $f : A \to B$. For every term $b : B$, the homotopy fibre of f is defined as follows:

$$\mathsf{hfibre}(b, f) :\equiv \sum_{a:A} f(a) = b\,. \tag{4.30}$$

Definition 4.3.23 (Equivalence). A function $f : A \to B$ such that

$$\mathsf{isEquiv}(f) :\equiv \prod_{b:B} \mathsf{isContr}\big(\mathsf{hfibre}(b, f)\big) \tag{4.31}$$

is inhabited.

Slightly different notions exist. A **homotopy equivalence** is a function $f : A \to B$ such that there exists a function $g : B \to A$ and homotopies

$$p : \prod_{a:A} g\big(f(a)\big) = a \qquad \text{and} \qquad q : \prod_{b:B} f\big(g(b)\big) = b. \tag{4.32}$$

It is called an **adjoint equivalence** if it is a homotopy equivalence equipped with higher homotopies representing triangle identities for f and g.

Property 4.3.24. The types $\mathsf{isEquiv}(f)$, $\mathsf{isHomEquiv}(f)$ and $\mathsf{isAdjEquiv}(f)$ are co-inhabited.

4.4 Categorical semantics

4.4.1 Inductive types

Inductive types also admit semantics in category theory. The right concept for these is that of algebras over endofunctors (Definition 2.4.24). The recursion principle and accompanying computation rule of inductive types exactly state for every other F-algebra A there exists a morphism $T \to A$ and that this is a unique algebra morphism, i.e. inductive types are initial algebras over endofunctors. An induction principle assigns to every algebra morphism $B \to T$, where B should be interpreted as the total space $\sum_{x:T} B(t)$, a section $T \to B$. The computation rule then again says that this section is an algebra morphism.

Property 4.4.1. When working in sets, the recursion and induction principles are equivalent.

Remark 4.4.2. When passing from extensional to intensional type theory, one has to replace initial algebras by weakly initial algebras.

4.4.2 Polynomial functors

Definition 4.4.3 (Polynomial). A diagram of the form

$$W \xleftarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z\,. \tag{4.33}$$

In terms of ordinary polynomials, Y gives the index set of the terms, X as a subset of $\mathbb{N}$ gives the degrees of the terms and W gives the coefficients. If g is an identity morphism, one often speaks of **linear functors**.

Given a polynomial in a locally Cartesian closed category $\mathbf{C}$, the induced polynomial functor is given by the composition

$$\mathbf{C}/W \xrightarrow{f^*} \mathbf{C}/X \xrightarrow{\prod_g} \mathbf{C}/Y \xrightarrow{\sum_h} \mathbf{C}/Z\,, \tag{4.34}$$

where $\prod_g$ and $\sum_h$ are dependent product and sum functors (**??**).

Definition 4.4.4 (*W*-**type**). Consider a type A and a type family $B : A \rightarrow \mathcal{U}$. The W-type $W_{a:A}B(a)$ is obtained as the initial algebra for the polynomial functor

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4.5 Homotopy type theory

4.5.1 Introduction

This section gives a reformulation or extension of the concept introduced before using the language of homotopy theory (and, more generally, algebraic topology). The relevant concepts can be found in **??** and Section 2.5.1. The resulting theory is called homotopy type theory or **HoTT**.

The general idea is to associate types with topological spaces and terms with points in those spaces. The main novelty is given by the identification of (propositional) equalities with paths between points. Since everything happens in a proof-relevant context, two equalities $p, q : a =_A b$ are not necessarily equal themselves and, hence, one can consider equalities between equalities (and so on). In the topological picture this gives rise to homotopies between paths. By going all the way and working out all coherence laws, one obtains the structure of a (weak) ∞-groupoid.[4]

It is also this interpretation that explains the name 'path induction' for the induction principle of equality types. Namely, what this induction principle says is that the free path space ΩA is *inductively generated* by constant loops (ranging over all possible points). This principle, however, sounds quite crazy. How can one build a path between two distinct points from (constant) loops? Here it is important to remind that everything only has to be equal up to homotopy and any path is indeed homotopy-equivalent to a constant loop if one retracts one of the endpoints along the path. It

[4]This characterization is strongly related to the homotopy hypothesis (or theorem when using the right model for ∞-categories).

is thus important that one does not require the homotopies to act rel endpoints (as is often done in classical homotopy theory).

Definition 4.5.1 (Pointed type). A type $A : \mathcal{U}$ together with a distinguished term $a : A$, called the **base point**. Pointed types are often denoted by a pair (A, a). It should be clear that the type of pointed types $\mathcal{U}_\bullet$ is equal to $\sum_{A:\mathcal{U}} A$.

Definition 4.5.2 (Loop space). The loop space $\Omega(A, a)$ of a pointed type (A, a) is the pointed type $(a =_A a, \mathsf{refl}_a)$.

Now, the important aspect of HoTT is that the ∞-groupoid structure of a type can be derived solely from the (path) induction principle of the equality types. Some examples are given.

Property 4.5.3 (Inversion). For every type $A : \mathcal{U}$ and terms $a, b : A$, there exists a function

$$p \mapsto p^{-1} : (a = b) \to (b = a) \tag{4.35}$$

such that $\mathsf{refl}_a^{-1} :\equiv \mathsf{refl}_a$ for all $a : A$.

Property 4.5.4 (Concatenation). For every type $A : \mathcal{U}$ and terms $a, b, c, d : A$, there exists a function

$$p \mapsto q \mapsto p \cdot q : (a = b) \to (b = c) \to (c = d) \tag{4.36}$$

such that $\mathsf{refl}_a \cdot \mathsf{refl}_a :\equiv \mathsf{refl}_a$ for all $a : A$. (Note that the composition does not follow the usual convention of right-to-left. This is why the symbol $\cdot$ and not $\circ$ was used.)

Property 4.5.5. The above operations satisfy the group relations (up to higher equalities):

- $p \cdot \mathsf{refl}_b = p$ and $\mathsf{refl}_a \cdot p = p$ for all $p : a = b$.
- $p \cdot p^{-1} = \mathsf{refl}_a$ and $p^{-1} \cdot p = \mathsf{refl}_b$ for all $p : a = b$.
- $(p^{-1})^{-1} = p$ for all $p : a = b$.
- $p \cdot (q \cdot r) = (p \cdot q) \cdot r$ for all $p : a = b, q : b = c, r : c = d$.

4.5.2 Transport

The relation with homotopy theory and category theory becomes even stronger when looking at function types:

Property 4.5.6. Given a function $f : A \to B$, there exists an **application function**

$$\mathsf{ap}_f : (a =_A b) \to \big(f(a) =_B f(b)\big) \tag{4.37}$$

such that $\mathsf{ap}_f(\mathsf{refl}_a) :\equiv \mathsf{refl}_{f(a)}$ for all $a, b : A$. Furthermore, this function behaves functorially in that it preserves concatenation, inverses and identities (again this should be interpreted in the full weak ∞-sense). From the topological perspective this can be interpreted as if all functions are 'continuous'.

Notation 4.5.7. Because functors in category theory are generally given the same notation when acting on objects or morphisms, the application function ap_f is also often denoted by f.

For dependent functions one can obtain a similar result. However, for this generalization, one needs some kind of 'parallel transport' since for two terms with $a = b$, it does not necessarily hold that $f(a)$ and $f(b)$ have the same type.

Property 4.5.8 (Transport). Given a type family $P : A \to \mathcal{U}$ and an equality $p : a =_A b$, there exists a **transport function**

$$p_* : P(a) \to P(b) \tag{4.38}$$

such that $(\mathsf{refl}_a)_* :\equiv \mathsf{id}(a)$ for all $a : A$. The pushforward notation is used since p_* can be (informally) interpreted as the pushforward of p along P.

From a topological perspective, this transport function allows to regard type families as fibrations (**??**). For every type family $P : A \to \mathcal{U}$, term $\alpha : P(a)$ and equality $p : a = b$, there exists a **lift**

$$\mathsf{lift}(p, \alpha) : (a, \alpha) = \big(b, p_*(\alpha)\big) \tag{4.39}$$

such that

$$\pi_1\big(\mathsf{lift}(p, u)\big) = p\,. \tag{4.40}$$

The equality $\mathsf{lift}(p, u)$ acts between terms of the Σ-type $\sum_{a:A} P(a)$, which can be interpreted as the total space of a **fibration** $\pi_1 : \sum_{a:A} P(a) \to A$. To take this terminology even further, one can could call functions $\sigma : \prod_{a:A} P(a)$ **sections** (of π_1).

Now, as mentioned before, for dependent functions one cannot just compare $f(a)$ and $f(b)$ if $a \not\equiv b$. However, the function $\mathsf{lift}(p, \cdot)$ gives a canonical path from one fibre to the other and every path between these fibres should factor through this canonical path essentially uniquely. Hence, one can define a path between α and β in the total space $\sum_{a:A} P(a)$, lying over $p : a = b$, to be a path $p_*(\alpha) = \beta$ (up to equivalence).

Property 4.5.9. Given a dependent function $f : \prod_{a:A} P(a)$, there exists a function

$$\mathsf{apd}_f : \prod_{p:a=b} p_*\big(f(a)\big) =_{P(b)} f(b)\,. \tag{4.41}$$

Again, with some abuse of notation, this function is also denoted by f

Since an ordinary function is a specific instance of a Π-type, one might expect that the application functions ap_f and apd_f are related in this case. The following property shows that this intuition is not unreasonable.

Property 4.5.10. Consider two types $A, B : \mathcal{U}$ and a function $f : A \to B$. For every equality $p : a =_A b$ and term $\alpha : P(b)$, there exists an equality $\tilde{p} : p_*(\alpha) =_{P(b)} \alpha$. Using this equality one can relate the application functions as follows:

$$\mathsf{apd}_f(p) = \tilde{p}\big(f(a)\big) \cdot \mathsf{ap}_f(p)\,. \tag{4.42}$$

4.5.3 Equivalences

In this paragraph the notions of equivalences and isomorphisms are considered in more detail. As is known from the chapter on category theory, the distinction between the various notions of similarity (or equality) is important yet subtle.

Lead by the intuition from topology a **homotopy** between functions is defined.

Definition 4.5.11 (Homotopy). Consider two sections $f, g : \prod_{a:A} P(a)$. A homotopy between f and g is a term of the type

$$f \sim g :\equiv \prod_{a:A} f(a) = g(a)\,. \tag{4.43}$$

It can be shown that homotopies induce equivalence relations on function types.

It has already been noted that functions can be regarded as functors between ∞-groupoids. Since homotopies act between functions, one might expect that these can be regarded as (weak) natural transformations between the (∞-)functors.

Property 4.5.12. Consider two sections $f, g : \prod_{a:A} P(a)$ and an equality $p : a = b$. If H is a homotopy between f and g, then

$$H(a) \cdot g(p) = f(p) \cdot H(b)\,. \tag{4.44}$$

Using the notion of homotopy one can introduce a first kind of 'equivalence'.

Definition 4.5.13 (Quasi-inverse). Given a function $f : A \to B$, a quasi-inverse of f is a triple (g, α, β), where $g : B \to A$ and

$$\alpha : f \circ g \sim \mathsf{id}_B \qquad\qquad \beta : g \circ f \sim \mathsf{id}_B\,. \tag{4.45}$$

From a homotopy theoretical perspective one would call the pair (f, g) a homotopy equivalence. The corresponding type is given by

$$\mathsf{qInv}(f) :\equiv \sum_{g:B\to A} (f \circ g \sim \mathsf{id}_B) \times (g \circ f \sim \mathsf{id}_A)\,. \tag{4.46}$$

Now, although this type may seem to give the right notion of equivalence, it is better to generalize it since it is in general not very well-behaved. (This is similar to the fact that adjoint equivalences between categories are better behaved than ordinary equivalences.)

In general an equivalence should satisfy three requirements:

1. For every function $f : A \to B$, there exists a function $\mathsf{qInv}(f) \to \mathsf{isEquiv}(f)$.
2. For every function $f : A \to B$, there also exists a function $\mathsf{isEquiv}(f) \to \mathsf{qInv}(f)$.
3. For every two terms $eq_1, eq_2 : \mathsf{isEquiv}(f)$, there exists an equality $eq_1 = eq_2$.

So, inducing an equivalence is logically equivalent to admitting a quasi-inverse and as such finding a quasi-inverse is sufficient to show that a function induces an equivalence.

4.5.4 Equality types: revisited

In the section on (intensional) type theory, equality types were introduced in a general and uniform way. The defining rules did not assume any specific structure on the underlying types. Although this made the technique of path induction widely applicable, it has the downside that one cannot leverage the internal structure of specific types to get more useful characterizations.

First, consider binary products (and by extension Σ-types). Can one express the equality of two elements $x, y : A \times B$ in terms of their projections? The answer is yes: there exists an equivalence

$$(x =_{A\times B} y) \simeq \big(\pi_1(x) =_A \pi_1(y)\big) \times \big(\pi_2(a) =_B \pi_2(y)\big)\,. \tag{4.47}$$

However, one should bear in mind that this is merely an equivalence. A term (resp. proof) of one side gives a term (resp. proof) of the other side, but it is not a judgemental equality (it is not even a propositional one). One could see this as a problem or defect of the theory and to resolve this kind of (apparent) issue the univalence axiom will be introduced at the end of this section. Still, one can leverage this equivalence to give a practical alternative[5] for the defining rules of the equality type in the case of product types:

Remark 4.5.14. The function $\big(\pi_1(a) = \pi_1(b)\big) \times \big(\pi_2(a) = \pi_2(b)\big) \to (a = b)$ associated to the above equivalence can be interpreted as an introduction rule of the equality type for binary products. At the same time one can take the application functions induced by the projections on $A \times B$ as elimination rules for the equality type. The homotopies associated to the equivalence in their turn induce the propositional computation rules and uniqueness principle.

One can also express the transport of properties along an equality $p : x =_{A\times B} y$ in terms of transport in the individual spaces:

Property 4.5.15. Consider two types $A, B : \mathcal{U}$ together with type families $P : A \to \mathcal{U}$ and $Q : B \to \mathcal{U}$. For every term α of the product family $(P\times Q)(x) :\equiv P\big(\pi_1(x)\big)\times Q\big(\pi_2(x)\big)$, the following equality type is inhabited:

$$p_*(\alpha) = \big(p_*(\pi_1(\alpha)), p_*(\pi_2(\alpha))\big)\,. \tag{4.48}$$

Note that all three occurrences of the pushforward p_* denote a different operation or, more precisely, the same operation but applied to different types.

One would intuitively expect that given two functions $f, g : A \to B$ that take the same value at all points, i.e. $f(a) = g(a)$ for all $a : A$, there exists an equality $f =_{A\to B} g$. However, this cannot be proven within the frame work of intensional type theory. This issue should also not come as a shock, since two functions that are defined differently might still take the same value at all points. To resolve this apparent gap in the theory, the following axiom is introduced.

Axiom 4.1 (Function extensionality). Given two functions $f, g : \prod_{a:A} P(a)$, there exists an equivalence $(f = g) \to \prod_{a:A} f(a) = g(a)$ that sends refl_f to $f(\mathsf{refl}_x)$.

Axiom 4.2 (Univalence axiom). Given two types $A, B : \mathcal{U}$, there exists an equivalence $(A =_{\mathcal{U}} B) \to (A \simeq B)$ that takes refl_A to id_A. A universe in which the univalence axiom holds is said to be univalent.

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[5]Note that this is not a judgementally equal alternative. It is merely a convenient interpretation.

4.6 Modal logic

4.6.1 Modalities

The two most important or most well-known modalities are 'neccesity' and 'possibility'. For any proposition p, $\Box p$ means that p is necessarily true and $\Diamond p$ means that p is possibly true. To formalize these modalities a few axioms can be introduced.

Axiom 4.3 (K). $\Box$ preserves implications:

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)\,. \tag{4.49}$$

In intuitionistic logic, a similar axiom has to be introduced for $\Diamond$:

$$\Box(p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q)\,. \tag{4.50}$$

Axiom 4.4 (Necessitation). If p is true, then $\Box p$ is true or, equivalently:

$$\mathtt{true} \rightarrow \Box\mathtt{true}\,. \tag{4.51}$$

Axiom 4.5 (T).

$$\Box p \rightarrow p \tag{4.52}$$

As for Axiom K, a separate axiom has to be introduced for $\Diamond$:

$$p \rightarrow \diamond p\,. \tag{4.53}$$

Axiom 4.6 (S4).

$$\Box p \rightarrow \Box\Box p \tag{4.54}$$

and

$$\Diamond\Diamond p \rightarrow \Diamond p \tag{4.55}$$

The categorical semantics for intuitionistic (propositional) logic is given by the internal language of Cartesian closed categories (Definition 3.1.20). When passing to S4 modal logic, one has to specialize these categories. The three axioms above imply that $\Box$ is a monoidal comonad. The analogous axioms for $\Diamond$ imply that it is a $\Box$-strong monad, i.e. it is a monad equipped with a natural transformation

$$\eta_{x,y} : \Box x \otimes \Diamond y \rightarrow \Diamond(\Box x \otimes y)\,. \tag{4.56}$$

4.6.2 Type theory

Definition 4.6.1 (Modality). Consider an $(\infty, 1)$-topos $\mathbf{H}$. A modality or **modal operator** on $\mathbf{H}$ is an idempotent[6] (co)monad on $\mathbf{H}$ (see also Definition 2.3.27).

Definition 4.6.2 (Modal type). A type $X \in \mathbf{H}$ is said to be modal for a modality $\lozenge$ if the unit $\eta_X : X \to \lozenge X$ or counit $\varepsilon_X : \lozenge X \to X$ is an equivalence.

Property 4.6.3. The modal types of a (co)modality $\lozenge$ on $\mathbf{H}$ are exactly the $\lozenge$-(co)algebras or, equivalently, the objects of the Eilenberg–Moore category $\mathbf{H}^{\lozenge}$ (Definition 2.3.20). This property allows to define modal types for (co)monads that are not idempotent as their (co)algebras. It also means that one can recover the (co)monad objectwise as the composition of the (co)unit and the inclusion.

Definition 4.6.4 (Opposite). Consider a modality $\lozenge : \mathbf{H} \to \mathbf{H}$. Its (left) opposite is another modality $\bigcirc : \mathbf{H} \to \mathbf{H}$ such that $\bigcirc \dashv \lozenge$ together with an adjoint triple

$$\mathbf{H}^{\bigcirc} \cong \mathbf{H}^{\lozenge} \underset{\hookrightarrow}{\overset{\hookrightarrow}{\leftarrow}} \mathbf{H}\,, \tag{4.57}$$

where the inclusions are the inclusions of modal types. Analogously, a right opposite is given by an adjunction $\lozenge \dashv \bigcirc$ together with an adjoint triple

$$\mathbf{H}^{\bigcirc} \cong \mathbf{H}^{\lozenge} \underset{\leftarrow}{\overset{\leftarrow}{\hookrightarrow}} \mathbf{H}\,. \tag{4.58}$$

This, in particular, implies that the modalities $\lozenge$ and $\bigcirc$ are themselves adjoint.

Remark 4.6.5 (Interpretation). Note that the adjoint triples in this definition have a slightly different interpretation due to there only being one inclusion arrow in the second case. If one interprets a modality as projecting out some kind of property of objects, the first opposition embodies the case where two related properties are considered, while the second case involves only one property but projected out in two different ways.

Property 4.6.6 (Extra structure). Since for idempotent (co)monads the forgetful functor $\mathbf{H}^{\lozenge} \hookrightarrow \mathbf{H}$ is fully faithful, the above adjoint triples also induce extra structure on the Eilenberg–Moore category $\mathbf{H}^{\lozenge}$. In the case where $\lozenge$ is a monad, the adjoint triple is also called an **adjoint cylinder** and $\mathbf{H}^{\lozenge}$ has the structure of an essential subtopos. For comonads, $\mathbf{H}^{\lozenge}$ obtains the structure of a bireflective subcategory (Definition 2.2.29).

Notation 4.6.7. One can show that adjoint triples $F \dashv G \dashv H$ are equivalent to adjoint pairs in the 2-category having adjunctions as morphisms, hence to adjunctions of adjunctions. This inspires the following notations:

$$\begin{array}{ccc} F & \dashv & G \\ \perp & & \perp \\ G & \dashv & H \end{array}$$

[6] Sometimes, idempotency is not required.

Definition 4.6.8 (Negative). Consider a comodality $\bigcirc : \mathbf{H} \to \mathbf{H}$. The negative of a modal type X is obtained by removing its 'pure $\bigcirc$-part', i.e. its projection under $\bigcirc$. Categorically this means that one takes the cofibre:

$$\overline{\bigcirc}X := \mathsf{cofib}(\bigcirc X \to X)\,. \tag{4.59}$$

Definition 4.6.9 (Aufhebung). Consider an inclusion of adjoint modalities:

$$\begin{array}{ccc} \Diamond_1 & \dashv & \bigcirc_1 \\ \vee & & \vee \\ \Diamond_2 & \dashv & \bigcirc_2 \end{array} \tag{4.60}$$

Right Aufhebung (of opposites) is given by an inclusion $\Diamond_2 < \bigcirc_1$ of modal types.

Chapter 5

Calculus

Chapter Contents

5.1 General definitions

Definition 5.1.1 (Domain). A connected, open subset of $\mathbb{R}^n$. (Not to be confused with the domain of a function as in Definition 1.5.1.)

Definition 5.1.2 (Factorial).

$$n! := n(n-1)\cdots 1\,, \tag{5.1}$$

where $n \in \mathbb{N}$. The convention is that $0! = 1$. (This, for example, agrees with the combinatorial result that there is a unique way to order zero objects.)

Definition 5.1.3 (Envelope). Consider a set $\mathcal{F}$ of real-valued functions with common domain X. An envelope (function) for $\mathcal{F}$ is any function $F : X \to \mathbb{R}$ such that

$$\forall f \in \mathcal{F}, x \in X : |f(x)| \leq F(x)\,. \tag{5.2}$$

5.2 Continuity

Definition 5.2.1 (Darboux function). A function that has the intermediate value property (**??**).

Theorem 5.2.2 (Darboux). *Every differentiable function defined on a closed interval is Darboux.*

Corollary 5.2.3 (Bolzano). If $f(a) < 0$ and $f(b) > 0$ (or vice versa), there exists at least one point x_0 for which $f(x_0) = 0$.

Theorem 5.2.4 (Weierstrass's extreme value theorem). *Let $I = [a, b]$ be a closed interval and let $f : I \to \mathbb{R}$ be a continuous function. Then f attains a minimum and maximum at least once on I.*

Definition 5.2.5 (Absolute continuity). A function $f : \mathbb{R} \to \mathbb{R}$ is said to be absolutely continuous if for every $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that for every finite collection of disjoint intervals $]x_i, y_i[$ satisfying

$$\sum_i (y_i - x_i) < \delta_\varepsilon\,, \tag{5.3}$$

the function f satisfies

$$\sum_i |f(y_i) - f(x_i)| < \varepsilon\,. \tag{5.4}$$

Property 5.2.6. The different types of continuity form the following hierarchy:

$$\text{Lipschitz-continuous} \subset \text{absolutely continuous} \subset \text{uniformly continuous} \subset \text{continuous}\,.$$

Definition 5.2.7 (Function of bounded variation). A function f is said to be of bounded variation on the interval $[a, b]$ if the following quantity is finite:

$$V_{a,b}(f) := \sup_{P \in \mathcal{P}} \sum_{i=0}^{|P|-1} |f(x_{i+1}) - f(x_i)|\,, \tag{5.5}$$

where the supremum is taken over all partitions of $[a, b]$.

Property 5.2.8. Every function of bounded variation can be decomposed as the difference of two monotonically increasing functions.

Example 5.2.9. Every absolutely continuous function is of bounded variation.

5.3 Convergence

Definition 5.3.1 (Pointwise convergence). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions. The sequence is said to converge pointwise to a limit function f if

$$\forall x \in \mathsf{dom}(f_n) : \lim_{n\to\infty} f_n(x) = f(x)\,. \tag{5.6}$$

Definition 5.3.2 (Uniform convergence). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions. The sequence is said to converge uniformly to a limit function f if

$$\lim_{n\to\infty} \sup_{x\in\mathsf{dom}(f_n)} |f_n(x) - f(x)| = 0\,. \tag{5.7}$$

Definition 5.3.3 (Limit superior). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The limit superior is defined as follows:

$$\limsup_{n\to\infty} x_n := \inf_{n\geq 1} \sup_{k\geq n} x_k\,. \tag{5.8}$$

Definition 5.3.4 (Limit inferior). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The limit superior is defined as follows:

$$\liminf_{n\to\infty} x_n := \sup_{n\geq 1} \inf_{k\geq n} x_k\,. \tag{5.9}$$

Property 5.3.5. A sequence $(x_n)_{n\in\mathbb{N}}$ converges pointwise if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n\,. \tag{5.10}$$

5.4 Series

5.4.1 Convergence tests

Property 5.4.1. A necessary condition for the convergence of a series $\sum_{i=1}^{+\infty} a_i$ is that

$$\lim_{n\to\infty} a_n = 0\,. \tag{5.11}$$

Property 5.4.2 (Absolute/conditional convergence). If $S' = \sum_{i=1}^{+\infty} |a_i|$ converges, so does $S = \sum_{i=1}^{+\infty} a_i$. In this case, S is said to be absolutely convergent. If S converges but S' does not, S is said to be conditionally convergent.

Definition 5.4.3 (Majorizing series). Let $S_a = \sum_{i=1}^{+\infty} a_i$ and $S_b = \sum_{i=1}^{+\infty} b_i$ be two series. The series S_a is said to majorize S_b if for every $k > 0$ the partial sums satisfy $S_{a,k} \geq S_{b,k}$, i.e.

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \tag{5.12}$$

for all $k \in \mathbb{N}$.

Method 5.4.4 (Comparison test). Let S_a, S_b be two series such that S_a majorizes S_b.

- If S_b diverges, then S_a diverges.
- If S_a converges, then S_b converges.
- If S_b converges, nothing can be said about S_a.
- If S_a diverges, nothing can be said about S_b.

Method 5.4.5 (Maclaurin–Cauchy integral test). Let f be a nonnegative, continuous and monotonically decreasing function defined on the interval $[n, +\infty[$ for some $n \in \mathbb{N}$. If $\int_n^{+\infty} f(x)\,dx$ is convergent, so is $\sum_{k=n}^{+\infty} f(k)$. On the other hand, if the integral is divergent, so is the series.

Remark 5.4.6. The function does not have to be nonnegative and decreasing on the complete interval. As long as it does on the interval $[N, +\infty[$ for some $N \geq n$, the statement holds. This can be seen by writing $\sum_{k=n}^{+\infty} f(k) = \sum_{k=n}^{N} f(k) + \sum_{k=N}^{+\infty} f(k)$ and noting that the first term is always finite (and similarly for the integral).

Property 5.4.7. If the integral in the previous theorem converges, the series is bounded in the following way:

$$\int_n^{+\infty} f(x)\,dx \leq \sum_{i=n}^{+\infty} a_i \leq f(n) + \int_n^{+\infty} f(x)\,dx\,. \tag{5.13}$$

Method 5.4.8 (d'Alembert's ratio test). Consider the quantity

$$R := \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|. \tag{5.14}$$

The following cases can be distinguished:

- $R < 1$: the series converges absolutely.
- $R > 1$: the series does not converge.
- $R = 1$: the test is inconclusive.

Method 5.4.9 (Cauchy's root test). Consider the quantity

$$R := \limsup_{n\to\infty} \sqrt[n]{|a_n|}\,. \tag{5.15}$$

The following cases can be distinguished:

- $R < 1$: the series converges absolutely.
- $R > 1$: the series does not converge.
- $R = 1$ and the limit approaches strictly from above: the series diverges.

- $R = 1$: the test is inconclusive.

Definition 5.4.10 (Radius of convergences). The number $1/R$ is called the radius of convergence.

Remark 5.4.11. The root test is stronger than the ratio test. However, if the ratio test can determine the convergence of a series, the radius of convergence of both tests will coincide and, hence, it is a well-defined quantity.

Method 5.4.12 (Gauss's test). If $a_n > 0$ for all $n \in \mathbb{N}$, one can write the ratio of successive terms as follows:

$$\left|\frac{a_n}{a_{n+1}}\right| = 1 + \frac{h}{n} + \frac{B(n)}{n^k}, \tag{5.16}$$

where $k > 1$ and $B(n)$ is a bounded function when $n \longrightarrow \infty$. The series converges if $h > 1$ and diverges otherwise.

Definition 5.4.13 (Asymptotic expansion). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A series $\sum_{i=0}^{+\infty} a_n x^n$ is called an asymptotic expansion of f if there exists an $N \in \mathbb{N}$ such that

$$f(x) - \sum_{i=0}^{n} a_i x^i = O(x^{n+1}) \tag{5.17}$$

for all $x \in \mathbb{R}$ and $n \geq N$.

5.5 Differentiation

Formula 5.5.1 (Derivative). Consider a function $f : \mathbb{R} \to \mathbb{R}$. If it exists, the following limit is called the derivative of f at $x \in \mathbb{R}$:

$$\frac{\mathrm{d}f}{\mathrm{d}x} \equiv f'(x) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{5.18}$$

If the derivative exists at every point of some interval I, then f is said to be differentiable on I. For multivariate functions $f : \mathbb{R}^n \to \mathbb{R}$, one can similarly define the partial derivatives:

$$\frac{\partial f}{\partial x_i} := \frac{f(x + he_i) - f(x)}{h}, \tag{5.19}$$

where e_i is the i^{th} coordinate vector, i.e. the partial derivatives determine the rate of change in the coordinate directions.

Notation 5.5.2. Iterated derivatives are often denoted as follows:

$$f^{(i)}(x) := \frac{\mathrm{d}^i f}{\mathrm{d}x^i}. \tag{5.20}$$

Theorem 5.5.3 (Mean value theorem). *Let f be a continuous function defined on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$. There exists a point $c \in]a, b[$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \,. \tag{5.21}$$

Definition 5.5.4 (Differentiablity class). A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be of class C^n if it is $n \in \mathbb{N}$ times continuously differentiable, i.e. $f^{(i)}$ exists and is continuous for $i = 1, \ldots, n$. Multivariate functions are said to be of class C^n if all of their partial derivatives are of class C^{n-1} or, by recursion, if all mixed partial derivatives up to order n exists and are continuous.

Definition 5.5.5 (Smooth function). A function f is said to be smooth if it is of class C^∞.

Theorem 5.5.6 (Boman). *Consider a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$. If, for every smooth function $g : \mathbb{R} \rightarrow \mathbb{R}^d$, the composition $f \circ g$ is smooth, the function f is also smooth.*

Property 5.5.7 (Taylor expansion). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function. Around every point $x \in \mathbb{R}$, one can express f as the following series:

$$f(y) = f(x) + f'(x)(y - x) + \frac{f''(x)}{2}(y - x)^2 + \cdots = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x)}{n!}(y - x)^n \,. \tag{5.22}$$

For the special case $x = 0$, the name **Maclaurin series** is sometimes used. A similar expression exists for multivariate functions, where derivatives are replaced by partial derivatives.

Definition 5.5.8 (Analytic function). A function f is said to be analytic if it is smooth and if its Taylor series expansion around any point x converges to f in some neighbourhood of x. The set of analytic functions defined on V is denoted by $C^\omega(V)$.

Theorem 5.5.9 (Hadamard lemma). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function defined on an open, star-convex set U. One can expand the function as follows:*

$$f(x) = f(x_0) + \sum_{i=1}^{n} (x^i - x_0^i) g_i(x_0) \,, \tag{5.23}$$

where all functions g_i are also smooth on U.

From this expression, one can also see that the functions g_i, evaluated at 0, give the partial derivatives of f. These functions are sometimes called the **Hadamard quotients**.

Remark 5.5.10. This lemma gives a finite-order approximation of the Taylor expansion of f.

Theorem 5.5.11 (Schwarz[1]). *Consider a function $f \in C^2(\mathbb{R}^n, \mathbb{R})$. The mixed partial derivatives of f coincide for all indices $i, j \leq n$:*

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right). \tag{5.24}$$

Formula 5.5.12 (Derivative of $f(x)^{g(x)}$). Consider a function of the form

$$u(x) = f(x)^{g(x)},$$

with $f, g : \mathbb{R} \to \mathbb{R}$ differentiable. After taking the logarithm and applying the standard rules of differentiation, one can obtain the following expression:

$$\left(f(x)^{g(x)}\right)' = f(x)^{g(x)}\left(g'(x)\ln\left[f(x)\right] + \frac{g(x)}{f(x)}f'(x)\right). \tag{5.25}$$

Definition 5.5.13 (Euler operator). On the space $C^{n>1}(\mathbb{R}^n, \mathbb{R})$, the Euler operator $\mathbb{E}$ is defined as follows:

$$\mathbb{E} := \sum_{i=1}^{n} x_i \frac{\partial}{\partial x^i}. \tag{5.26}$$

Theorem 5.5.14 (Euler). *Let f be a homogeneous function, i.e.*

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^n f(x_1, \ldots, x_n). \tag{5.27}$$

This function satisfies the following equality:

$$\mathbb{E}(f) = nf(x_1, \ldots, x_n). \tag{5.28}$$

5.6 Integration theory

5.6.1 Riemann integral

Definition 5.6.1 (Improper Riemann integral).

$$\int_{-\infty}^{+\infty} f(x)\,dx := \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b f(x)\,dx \tag{5.29}$$

One-sided improper integrals are defined in a similar fashion.

Theorem 5.6.2 (First fundamental theorem of calculus). *Let f be a continuous function defined on an open interval I and consider any number $c \in I$. Then the integral*

$$F(x) = \int_c^x f(x')\,dx' \tag{5.30}$$

is differentiable (and uniformly continuous) and gives an antiderivative of f:

$$\forall x \in I : F'(x) = f(x). \tag{5.31}$$

[1] Also called **Clairaut's theorem**.

Remark 5.6.3. The function F in the previous theorem is called a **primitive (function)** of f. Remark that F is just 'a' primitive function, since adding a constant to F does not change anything because the derivative of a constant is zero (the number $c \in \mathbb{R}$ was arbitrary).

Theorem 5.6.4 (Second fundamental theorem of calculus[2]). *Consider an integrable function $f : [a,b] \to \mathbb{R}$. If $F : [a,b] \to \mathbb{R}$ is an antiderivative of f, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)\,. \tag{5.32}$$

Formula 5.6.5 (Differentiation under the integral sign[3]).

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,y)\,dy = f\big(x,b(x)\big)b'(x) - f\big(x,a(x)\big)a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\,dy \tag{5.33}$$

Definition 5.6.6 (Borel transform). Consider the following function:

$$F(x) := \sum_{n=0}^{+\infty} \frac{a_n}{n!}x^n\,. \tag{5.34}$$

If

$$\int_0^{+\infty} e^{-t}F(xt)\,dt < +\infty \tag{5.35}$$

for all $x \in \mathbb{R}$, then F is called the Borel transform of

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n\,. \tag{5.36}$$

Furthermore, the integral gives a convergent expression for f.

Proof. The function F is defined as follows:

$$F(x) := \sum_{n=0}^{+\infty} \frac{a_n}{n!}x^n\,.$$

[2]Sometimes called the **Newton–Leibniz** theorem.

[3]This is a more general version of the *Leibniz integral rule*.

The Borel transform gives:

$$\begin{aligned}
\int_0^{+\infty} F(xt)e^{-t}\,dt &= \sum_{n=0}^{+\infty}\int_0^{+\infty}\frac{a_n}{n!}x^nt^ne^{-t}\,dt \\
&= \sum_{n=0}^{+\infty}\frac{a_n}{n!}x^n\int_0^{+\infty}t^ne^{-t}\,dt \\
&= \sum_{n=0}^{+\infty}\frac{a_n}{n!}x^n\,\Gamma(n+1) \\
&= \sum_{n=0}^{+\infty}a_nx^n\,,
\end{aligned}$$

where the definition of the Gamma function (Formula 5.6.9) was used on line 3, and the relation (5.6.12) between the factorial function and the Gamma function was used on line 4. □

Theorem 5.6.7 (Watson). *The Borel transform F is unique if the function f is holomorphic (see* **??***) on the domain* $\{z \in \mathbb{C} \mid |\arg(z)| < \frac{\pi}{2} + \varepsilon\}$.

5.6.2 Euler integrals

Formula 5.6.8 (Beta function). The beta function (also known as the **Euler integral of the first kind**) is defined as follows:

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt\,. \tag{5.37}$$

Formula 5.6.9 (Gamma function). The gamma function (also known as the **Euler integral of the second kind**) is defined as follows:

$$\Gamma(x) := \int_0^{+\infty} t^{x-1}e^{-t}\,dt\,. \tag{5.38}$$

Formula 5.6.10. The following formula relates the beta and gamma functions:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,. \tag{5.39}$$

Property 5.6.11 (Recursion). The gamma function satisfies the following recursion relation for all points x in its domain:

$$\Gamma(x+1) = z\Gamma(x)\,. \tag{5.40}$$

Formula 5.6.12 (Factorial). For integers $n \in \mathbb{N}$, the gamma function can be expressed in terms of the factorial (Definition 5.1.2):

$$\Gamma(n) = (n-1)!\,. \tag{5.41}$$

Formula 5.6.13 (Stirling). This formula (originally stated for the factorial of natural numbers) gives an asymptotic expansion of the gamma function:

$$\ln\Gamma(z) \approx z\ln z - z + \frac{1}{2}\ln\left(\frac{2\pi}{z}\right). \tag{5.42}$$

Property 5.6.14 (Euler's reflection formula). When $x \notin \mathbb{Z}$, the following formula holds:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}. \tag{5.43}$$

5.6.3 Gaussian integrals

Formula 5.6.15 (n-dimensional Gaussian integral). An integral of the form

$$I\left(A,\vec{b}\right) := \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\vec{x}\cdot A\vec{x} + \vec{b}\cdot\vec{x}\right) d^n x\,, \tag{5.44}$$

where A is a real symmetric matrix. By performing the transformation $\vec{x} \to A^{-1}\vec{b} - \vec{x}$ and diagonalizing A, one can obtain the following expression:

$$I\left(A,\vec{b}\right) = \sqrt{\frac{(2\pi)^n}{\det(A)}}\exp\left(\frac{1}{2}\vec{b}\cdot A^{-1}\vec{b}\right). \tag{5.45}$$

More generally, one has the following result:

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\vec{x}\cdot A\vec{x}\right) f(\vec{x})\, d^n x = \sqrt{\frac{(2\pi)^n}{\det(A)}}\exp\left(\frac{1}{2}\sum_{i,j=1}^{n} A^{-1}_{ij}\partial_i\partial_j\right) f(\vec{x})\Bigg|_{\vec{x}=0}. \tag{5.46}$$

This result is sometimes called **Wick's lemma**.

Corollary 5.6.16. A functional generalization is given by:

$$\begin{aligned} I(iA, iJ) &= \int \exp\left(-i\int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi(x)A(x,y)\varphi(y)\, d^n x\, d^n y + i\int_{\mathbb{R}^n}\varphi(x)J(x)\, d^n x\right)[d\varphi] \\ &= C\det(A)^{-1/2}\exp\left(\frac{i}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} J(x)A^{-1}(x,y)J(y)\, d^n x\, d^n y\right), \end{aligned} \tag{5.47}$$

where the analytic continuation $I(iA, iJ)$ of Eq. (5.45) was used. One should pay attention to the normalization factor C which is infinite in general.

Method 5.6.17 (Feynman diagrams). The expansion of the exponential in the general expression for Gaussian integrals admits a diagrammatic expression. Let $f(\vec{x})$ be a polynomial function of the coordinates.

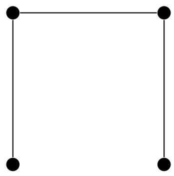
If the number of factors in a monomial is odd, the resulting integral will vanish (since the integral of an odd function over an even domain is zero). For an even number of factors, one gets the following expression:

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\vec{x}\cdot A\vec{x}\right) x^{i_1}\cdots x^{i_k}\, d^n x = \sqrt{\frac{(2\pi)^n}{\det(A)}}\sum_{\sigma\in S_k} A^{-1}_{\sigma(i_1)\sigma(i_2)}\cdots A^{-1}_{\sigma(i_{k-1})\sigma(i_k)}. \tag{5.48}$$

To every coordinate dimension, one can assign a vertex in the plane, i.e. the object x_i can be interpreted as a real-valued function on the set of $k \in \mathbb{N}$ elements. The sum on the right-hand side above can then be expressed as a ‘sum’ over all possible diagrams, where a factor A^{-1}_{ij} is represented by a line connecting the vertices i and j.

Example 5.6.18 (Feynman diagrams). Some simple examples are given:

$$A^{-1}_{13}A^{-1}_{12}A^{-1}_{24} =$$



Higher powers of a given coordinate would then, for example, give rise to diagrams with loops at a given vertex:

$$A^{-1}_{11}A^{-1}_{12}A^{-1}_{22} =$$

Remark 5.6.19 (Normalization). In practice, one often divides all Gaussian integrals by the quantity $I(A, 0)$ to cancel the normalization factor. In the functional setting, this is even imperative since, as mentioned above, the normalization factor diverges for infinite-dimensional spaces.

5.6.4 Generalizations

Definition 5.6.20 (Henstock–Kurzweil integral[4]). Consider the usual definition of the (proper) Riemann integral, where tagged partitions P of $[a, b]$ are chosen and the integral is obtained as the limit of the Riemann sums

$$I = \sum_P f(x_i)(t_i - t_{i-1}) \tag{5.49}$$

as the mesh size of the partitions goes to zero.

Now, to obtain the generalized integral, consider a strictly positive function $\delta : [a, b]\mathbb{R}^{>0}$, the **gauge function**. Given such a gauge, a tagged partition P is said to be **δ-fine** if

$$[t_{i-1}, t_i] \subset [x_i - \delta(x_i), x_i + \delta(x_i)] \tag{5.50}$$

for subintervals in the partition.[5]

If the integral exists, it is given by the number $I \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a gauge $\delta : [a, b] \to \mathbb{R}^{>0}$ such that, if P is δ-fine, then

$$\left| I - \sum_P f(x_i)(t_i - t_{i-1}) \right| < \varepsilon\,. \tag{5.51}$$

[4]Also called the **Perron**, **Lusin**, (**narrow**) **Denjoy** or **gauge** integral.

[5]If the condition $x_i \in [t_{i-1}, t_i]$ in the definition of tagged partitions is dropped, the **McShane integral** is obtained. This can be shown to be equivalent to the *Lebesgue integral* (see **??**).

Remark 5.6.21 (Riemann integral). If the gauge functions are chosen to be constant, the classical (ε, δ)-definition of ordinary Riemann integrals is obtained.

The following statement can be seen as a refinement of **??**. Moreover, it is also sometimes known as the **Borel–Lebesgue theorem**.

Property 5.6.22 (Cousin). For every gauge $\delta : [a, b] \to \mathbb{R}^{>0}$, there exists a δ-fine partition.

Property 5.6.23 (Integrability). If $f : [a, b] \to \mathbb{R}$ is bounded, then the following are equivalent:

- f is Henstock–Kurzweil integrable, and
- f is *Lebesgue integrable* (see **??**).

More generally, a function $f : [a, b] \to \mathbb{R}$ is Henstock–Kurzweil integrable if and only if both f and $|f|$ are *Lebesgue integrable*.

The following property shows that 'improper' Henstock–Kurzweil integrals are only truly improper for unbounded domains.

Property 5.6.24 (Hake).

$$\int_a^b f\,dx = \lim_{c \nearrow b} \int_a^c f\,dx\,, \tag{5.52}$$

whenever either side exists.

One of the most important arguments for using the Henstock–Kurzweil integral is its refinement of the Second Fundamental Theorem of Calculus 5.6.4. Note that the theorem for the Riemann integral required that the derivative was integrable. The gauge integral relaxes this condition.

Theorem 5.6.25 (Second fundamental theorem of calculus). *Let $f : [a, b] \to \mathbb{R}$ be differentiable, then*

$$\int_a^x f'(x')\,dx' = f(x) - f(a) \;\; a.e. \tag{5.53}$$

5.7 Convexity

Definition 5.7.1 (Convex set). A subset of X of a vector space V (**??**) is said to be convex if $x, y \in X$ implies that $\{\lambda x + (1-\lambda)y \mid \lambda \in [0, 1]\} \subset X$, i.e. if all straight lines connecting elements of the set are completely contained in that set. The **convex hull** of a subset X is defined as the smallest convex subset containing X.

Definition 5.7.2 (Extreme point). Consider a convex set X. The extreme points of X are the points $p \in X$ such that, if

$$p = \lambda p_1 + (1-\lambda)p_2 \tag{5.54}$$

for some $p_1, p_2 \in X$ and $\lambda \in [0,1]$, then $p_1 = p_2 = p$.

Definition 5.7.3 (Convex function). Let X be a convex set. A function $f : X \rightarrow \mathbb{R}$ is said to be convex if for all $x, y \in X$ and $\lambda \in [0,1]$:

$$f\big(\lambda x + (1-\lambda)y\big) \leq t\lambda(x) + (1-\lambda)f(y)\,. \tag{5.55}$$

For the definition of a **concave** function, the inequality has to be turned around.

Definition 5.7.4 (Linear map). A function $f : X \rightarrow \mathbb{R}$ is linear if and only if it is both convex and concave.

Theorem 5.7.5 (Karamata's inequality). *Consider an interval $I \subset \mathbb{R}$ and let $f : I \rightarrow \mathbb{R}$ be a convex function. If $(x_1, \ldots, x_n)$ is a tuple that majorizes $(y_1, \ldots, y_n)$, i.e.*

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i \tag{5.56}$$

and

$$x_{(1)} + \cdots + x_{(k)} \geq y_{(1)} + \cdots + y_{(k)} \tag{5.57}$$

for all $k \leq n$, where $x_{(i)}$ denotes the i^{th} largest element of $(x_1, \ldots, x_n)$, then

$$\sum_{i=1}^n f(x_i) \geq \sum_{i=1}^n f(y_i)\,. \tag{5.58}$$

The following inequality can be derived directly from the definition of convexity by induction.

Theorem 5.7.6 (Jensen's inequality). *Let f be a convex function and consider a point $\{a_i\}_{i\leq n}$ in the probability simplex Δ^n (**??**).*

$$f\left(\sum_{i=1}^n a_i x_i\right) \leq \sum_{i=1}^n a_i f(x_i)\,. \tag{5.59}$$

Definition 5.7.7 (Legendre transformation). Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$. In certain cases (especially in physics) it is sometimes useful to replace the argument x by the slope of f at x, i.e. to perform the transformation

$$x \longrightarrow f'(x)\,. \tag{5.60}$$

However, it should be clear that this transformation is not always well-defined and, even if it is, it does not always preserve all the information contained in f.

These conditions are satisfied exactly if f is convex (or concave). In this case, the Legendre transform of f is defined as

$$f^*(x^*) := \sup_x \bigl(x^*x - f(x)\bigr) . \tag{5.61}$$

Now, consider the case where f is differentiable. The above supremum can then be obtained by differentiating the right-hand side and equating it to zero. This results in $x^* = f'(x)$, which is exactly the transformation that was required. By expressing everything in terms of the Legendre tranformed quantity x^*, one can also find the derivative of f^*:

$$\frac{\mathrm{d}f^*}{\mathrm{d}x^*}(x^*) = x(x^*) . \tag{5.62}$$

Property 5.7.8 (Alternative characterization). In fact, up to an additive constant, the condition

$$(f^*)' = (f')^{-1} \tag{5.63}$$

uniquely determines the Legendre transformation.

Remark 5.7.9. These definitions can easily be extended to higher dimensions ($n \geq 2$).

5.8 Trigonometry

Definition 5.8.1 (Trigonometric functions). Consider Fig. 5.1. In a right(-angled) triangle, the trigonometric functions of the angle θ are defined as follows:

$$\sin(\theta) := \frac{y}{x} \qquad \cos\theta := \frac{z}{x} \qquad \tan(\theta) := \frac{y}{z} \tag{5.64}$$

Corollary 5.8.2.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \tag{5.65}$$

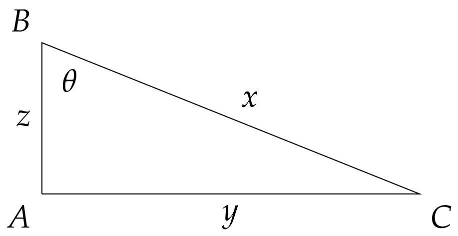


Figure 5.1: Right(-angled) triangle.

Part III

Quantum Theory

Table of Contents

Chapter 6

Quantum Mechanics

The main reference for this chapter is Bransden and Joachain (2000). In the first two sections, the two basic formalisms of quantum mechanics are introduced: wave and matrix mechanics. The main reference for the mathematically rigorous treatment of quantum mechanics, in particular in the infinite-dimensional setting, is Moretti (2016). The main reference for the generalization to curved backgrounds is Schuller (2016). The section on the WKB approximation is based on Bates and Weinstein (1997). Relevant chapters in this compendium are, amongst others, **??**, **??** and **??**.

Chapter Contents

6.1 Introduction

This section will give both an introduction and formal treatment of the objects and notions used in quantum mechanics.

6.1.1 Dirac–von Neumann postulates

Axiom 6.1 (States). The states of a (closed) system are represented by vectors in a (complex) Hilbert space $\mathcal{H}$. In the infinite-dimensional setting, one often further restricts to separable spaces, i.e. the spaces are required to admit a countable Hilbert basis.

Notation 6.1.1 (Dirac notation). State vectors $|\psi\rangle$ are called **ket**'s and their duals $\langle\psi|$ are called **bra**'s. The inner product of a state $|\phi\rangle$ and a state $|\psi\rangle$ is denoted by $\langle\phi\,|\,\psi\rangle$. This notation is often called the **braket notation** (or Dirac notation).

Axiom 6.2 (Observables). Every physical property is represented by a bounded, self-adjoint operator. In the finite-dimensional case, this is equivalent to an operator that admits a complete set of eigenfunctions.

Definition 6.1.2 (Compatible observables). Two observables are said to be compatible if they share a complete set of eigenvectors.

Formula 6.1.3 (Closure relation). For a complete set of eigenvectors, the closure relation (also called the **resolution of the identity**) is given by (see also **??**)

$$\sum_n |\psi_n\rangle\langle\psi_n| + \int_X |x\rangle\langle x|\,dx = \mathbb{1}\,, \tag{6.1}$$

where the sum ranges over the discrete spectrum and the integral over the continuous spectrum. For simplicity, the summation will also be used for the continuous part.

Axiom 6.3 (Born rule). Let $\mathcal{H}$ be the Hilbert space of a physical system and consider an observable $\widehat{O}$. If $|\psi\rangle$ is a state vector and $\widehat{P}_\phi$ is the projection onto an eigenvector $|\phi\rangle$ of $\widehat{O}$, the probability of observing the state $|\phi\rangle$ is given by:

$$\frac{\langle\psi \mid \widehat{P}_\phi \mid \psi\rangle}{\langle\psi \mid \psi\rangle} = \frac{|\langle\psi \mid \phi\rangle|^2}{\langle\psi \mid \psi\rangle}\,. \tag{6.2}$$

Property 6.1.4 (Projectivization). In light of the Born rule, the dynamics of a system does not depend on the global phase or normalization, i.e. states are represented by rays in a projective Hilbert space $\mathcal{H}\mathbb{P}$ (**??**).

Combining Born's rule with **??**, gives the following definition.

Definition 6.1.5 (Expectation value). The expectation value of an observable $\widehat{A}$ in a (normalized) state $|\psi\rangle$ is defined as follows:

$$\langle\widehat{A}\rangle_\psi := \langle\psi|\widehat{A}|\psi\rangle\,. \tag{6.3}$$

The subscript ψ is often left implicit. As in ordinary statistics (**??**), the uncertainty or variance is defined as follows:

$$\Delta A := \langle\widehat{A}^2\rangle - \langle\widehat{A}\rangle^2\,. \tag{6.4}$$

Formula 6.1.6 (Uncertainty relation). Let $\widehat{A}, \hat{B}$ be two observables and let $\Delta A, \Delta B$ be the corresponding uncertainties. The (**Robertson**) uncertainty relation reads as follows:

$$\Delta A \Delta B \geq \frac{1}{4}\left|\left\langle\left[\widehat{A}, \hat{B}\right]\right\rangle\right|^2\,. \tag{6.5}$$

Axiom 6.4 (Projection[1]). Let $\mathcal{H}$ be the Hilbert space of a physical system and consider an observable $\widehat{O}$ with eigenvalues $\{o_i\}_{i\in I}$. After measuring the observable $\widehat{O}$ in the state $|\psi\rangle$, the outcome will be one of the eigenvalues o_i and system will 'collapse' to, i.e. get projected onto, the eigenstate $\widehat{P}_{o_i}|\psi\rangle \equiv |o_i\rangle$.

Axiom 6.5 (Unitary evolution). The evolution of a closed system is unitary, i.e. there exists a unitary operator $\widehat{U}(t, t') \in \mathsf{Aut}(\mathcal{H})$, for all times $t \leq t'$, such that

$$|\psi(t')\rangle = \widehat{U}(t, t')|\psi(t)\rangle\,. \tag{6.6}$$

[1] Also called the **measurement postulate**.

6.2 Schrödinger picture

Since the energy is of paramount importance in physics, the associated eigenvalue equation deserves its own name.

Formula 6.2.1 (Time-independent Schrödinger equation).

$$\widehat{H}|\psi\rangle = E|\psi\rangle \tag{6.7}$$

The operator $\widehat{H}$ is called the **Hamiltonian** of the system. The wave function ψ is an element of the vector space $L^2(\mathbb{R},\mathbb{C})\otimes\mathcal{H}$ with $\mathcal{H}$ the internal Hilbert space (describing, for example, the spin or charge of a particle). This is an eigenvalue equation for the energy levels of the system.

@@ INTRODUCE POSITION/CONFIGURATION REPRESENTATION @@

The time evolution of a wave function was governed by Axiom 6.5. By passing to generators, the following equation is obtained.

Formula 6.2.2 (Time-dependent Schrödinger equation).

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle\,. \tag{6.8}$$

In case $\widehat{H}$ is time independent, the TISE can be obtained from this equation by separation of variables (see below).

Proof (Derivation of TISE from TDSE). Starting from the one-dimensional TDSE in position space with a time-independent Hamiltonian, one can perform a separation of variables and assert a solution of the form $\psi(x,t) = X(x)T(t)$. Inserting this in the previous equation gives

$$i\hbar X(x)T'(t) = \big(\widehat{H}X(x)\big)T(t)\,.$$

Dividing both sides by $X(x)T(t)$ and rearranging the terms gives

$$i\hbar\frac{T'(t)}{T(t)} = \frac{\widehat{H}X(x)}{X(x)}\,.$$

Because the left side only depends on t and the right side only depends on x, one can conclude that they both have to equal a constant $E\in\mathbb{C}$. This leads to the following system of differential equations:

$$\begin{cases} i\hbar T'(t) = ET(t)\,,\\ \widehat{H}X(x) = EX(x)\,. \end{cases}$$

The first equation immediately gives a solution for T:

$$T(t) = C\exp\left(-\frac{iE}{\hbar}t\right). \tag{6.9}$$

The second equation is exactly the TISE (Formula 6.2.1). □

Example 6.2.3 (Massive particle in a stationary potential).

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right)\psi(x,t) \tag{6.10}$$

In this case, the TISE reads as follows:

$$\psi''(x) = -\frac{2m}{\hbar^2}\big(E - V(x)\big)\psi(x). \tag{6.11}$$

Formula 6.2.4 (General solution). A general solution of the TDSE (for time-independent Hamiltonians) is given by the following formula (cf. **??**):

$$\psi(x,t) = \sum_E c_E\psi_E(x)e^{-\frac{i}{\hbar}Et}, \tag{6.12}$$

where the functions $\psi_E(x)$ are the eigenfunctions of the TISE. The coefficients c_E can be found using the orthogonality relations

$$c_E = \left(\int_{\mathbb{R}}\overline{\psi_E}(x)\psi(x,t_0)\,dx\right)e^{\frac{i}{\hbar}Et_0}. \tag{6.13}$$

6.3 Heisenberg–Born–Jordan picture

In the previous section, the central object was the wave function. It was this object that evolved in time and the operators acting on the Hilbert space of physical states were assumed to be fixed. However, it is also possible to transfer this dependence on time to the operators.

Formula 6.3.1 (Time-dependent observables).

$$\widehat{O}_H(t) := e^{\frac{i}{\hbar}\widehat{H}t}\widehat{O}_S(t)e^{-\frac{i}{\hbar}\widehat{H}t} \tag{6.14}$$

The equivalence between the Schödinger and Heisenberg pictures essentially come from the fact that the time-evolving expectation values of operators are given by the following formula:

$$\langle\widehat{O}(t)\rangle = \langle\psi|e^{\frac{i}{\hbar}\widehat{H}t}\widehat{O}(t)e^{-\frac{i}{\hbar}\widehat{H}t}|\psi\rangle. \tag{6.15}$$

The difference between the pictures is simply the choice of whether to include the evolution operator in the states or in the operators.

Using the above transformation, the Schödinger equation (Formula 6.2.2) can also be reexpressed.

Formula 6.3.2 (Time-dependent Schrödinger equation).

$$\frac{\partial \widehat{O}_H}{\partial t}(t) = \frac{i}{\hbar}\left[\widehat{H}_H(t), \widehat{O}_H(t)\right] + \left(\frac{\partial \widehat{O}}{\partial t}(t)\right)_H \tag{6.16}$$

Taking this expression for the Schrödinger equation and taking expectation values (using the linearity of the equation), gives the following (interaction-independent) result.

Theorem 6.3.3 (Ehrenfest). *Let $\widehat{H}$ be the Hamiltonian and consider an observable $\widehat{O}$. The expectation value of this operator evolves as follows:*

$$\frac{\mathrm{d}\langle\widehat{O}\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\langle[\widehat{O}, \widehat{H}]\rangle + \left\langle\frac{\partial \widehat{O}}{\partial t}\right\rangle. \tag{6.17}$$

Remark 6.3.4 (Equivalence). It is important to note that the Schrödinger equation could be replaced by Ehrenfest's theorem. They are entirely equivalent.

But, given the abstract state vectors $|\psi\rangle$ from Section 6.1.1, how does one recover the position (configuration) representation $\psi(x)$? This is simply the projection of the state vector $|\psi\rangle$ on the 'basis function' $\delta(x)$, i.e. $\psi(x)$ represents an expansion coefficient in terms of a 'basis' for the physical Hilbert space. In the same way, one can obtain the momentum representation $\psi(p)$ by projecting onto the plane waves e^{ipx}.

Remark 6.3.5. It should be noted that neither the 'basis states' $\delta(x)$, nor the plane waves e^{ipx} are square integrable and, hence, they are not elements of the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$. This issue can be resolved through the concept of *rigged Hilbert spaces*.

@@ COMPLETE @@

6.3.1 Hydrogen atom

Consider the hydrogen atom, i.e. a single proton (the nucleus) orbited by a single electron with only the electrostatic Coulomb force acting between them (gravity can safely be neglected):

$$\widehat{H} := \frac{\widehat{p}_p^2}{2m_p} + \frac{\widehat{p}_e^2}{2m_e} - \frac{e^2}{4\pi\varepsilon r^2}. \tag{6.18}$$

It is not hard to see that this is the quantum mechanical version of the Kepler problem (**??**). The special property of the Kepler problem was that it contained a 'hidden' symmetry that gave rise to the conserved Laplace–Runge–Lenz vector (**??**). As is the case for all conserved charges in quantum mechanics, this symmetry induces a degeneracy of the energy eigenvalues. Degeneracy of the magnetic quantum number $m \in \mathbb{N}$ follows from rotational symmetry, but the energy levels of the hydrogen atom only depend on the principal quantum number $n \in \mathbb{N}$. It is the degeneracy of the total angular quantum number $l \in \mathbb{N}$ that is due to this 'hidden' $SO(4)$-symmetry. It is often called an 'accidental degeneracy' for this reason.

@@ COMPLETE @@

6.3.2 Molecular dynamics

Consider the Hamiltonian of two interacting atoms:

$$\widehat{H} = \frac{\widehat{P}_1^2}{2M_1} + \frac{\widehat{P}_2^2}{2M_2} + \frac{\hat{q}_1\hat{q}_2}{4\pi\varepsilon R^2} + \sum_i \frac{\hat{p}_i^2}{2m} - \frac{e\hat{q}_1}{4\pi\varepsilon r_{i1}^2} - \frac{e\hat{q}_2}{4\pi\varepsilon r_{i2}^2} + \sum_{i\neq j} \frac{e^2}{4\pi\varepsilon r_{ij}^2}, \tag{6.19}$$

where the indices i, j indicate the electrons and uppercase symbols denote operators associated to the nuclei.

Except for the most simple situations, solving the Schrödinger equation for this Hamiltonian becomes intractable (both analytically and numerically). However, in general, one can approximate the situation. The masses of nuclei are much larger than those of the electrons and this influences their motion, they move much slower than the electrons. In essence, the nuclei and electrons live on different time scales and this allows to decouple their dynamics:

$$\widehat{H}_{\text{nucl}} = \frac{\widehat{P}_1^2}{2M_1} + \frac{\widehat{P}_2^2}{2M_2} + \frac{Q_1Q_2}{4\pi\varepsilon R^2} + V_{\text{eff}}(R_1, R_2). \tag{6.20}$$

The electrons generate an effective potential for the nuclei and the Schrödinger equation decouples as follows:

$$\begin{aligned} \widehat{H}_{\text{nucl}}(R)\psi(R) &= E\psi(R)\,, \\ \widehat{H}_{\text{el}}(r,R)\phi(r,R) &= E_{\text{el}}\phi(r,R)\,. \end{aligned} \tag{6.21}$$

This is the so-called **Born–Oppenheimer approximation**. From a more modern physical perspective, this approximation can also be seen to be a specific instance of renormalization theory, where the short time-scale (or, equivalently, the high energy-scale) degrees of freedom are integrated out of the theory.

6.4 Mathematical formalism

6.4.1 Weyl systems

Definition 6.4.1 (Canonical commutation relations). Two observables $\widehat{A}, \widehat{B}$ are said to obey a canonical commutation relation (CCR) if they satisfy (up to a constant factor

$\hbar$)

$$[\widehat{A}, \widehat{B}] = i\,. \tag{6.22}$$

The prime examples are the position and momentum operators $\hat{x}, \hat{p}$. Through functional calculus, one can also define the exponential operators $e^{is\widehat{A}}$ and $e^{it\widehat{B}}$. The above relation then induces the so-called **Weyl form** of the CCR:

$$e^{is\widehat{A}}e^{it\widehat{B}} = e^{ist}e^{it\widehat{B}}e^{is\widehat{A}}\,. \tag{6.23}$$

Theorem 6.4.2 (Stone–von Neumann). *All pairs of irreducible, unitary one-parameter subgroups satisfying the Weyl form of the CCRs are unitarily equivalent.*

Corollary 6.4.3. The Schrödinger and Heisenberg pictures are unitarily equivalent.

In fact, one can generalize the Weyl form of the CCRs.

Definition 6.4.4 (Weyl system). Let (L, ω) be a symplectic vector space and let K be a complex vector space. Consider a map W from L to the space of unitary operators on K. The pair (K, W) is called a Weyl system over (L, ω) if it satisfies

$$W(z)W(z') = e^{i/2\omega(z,z')}W(z + z') \tag{6.24}$$

for all $z, z' \in L$, i.e. W is a projective representation of the Abelian group L and ω is, up to rescaling, the group cocycle inducing it (**??**). The relation itself is called a **Weyl relation**.

Definition 6.4.5 (Heisenberg system). Let W be a Weyl system. The selfadjoint generators $\phi(z)$, which exist by Stone's theorem **??**, of the maps $t \mapsto W(tz)$ are said to form a Heisenberg system. These operators satisfy the following properties:

1. **Positive homogeneity**: $\lambda\phi(z) = \phi(\lambda z)$ for all $\lambda > 0$,
2. **Commutator**: $[\phi(z), \phi(z')] = -i\omega(z, z')$, and
3. **Weak additivity**: $\phi(z + z')$ is the closure (**??**) of $\phi(z) + \phi(z')$.

Remark 6.4.6. It should be noted that the Weyl relations are more fundamental than their infinitesimal counterparts. Only the Weyl relations are well defined on more general spaces and when passing to a relativistic setting.

Recall **??**, where the framework of measure theory and distributions was generalized to the noncommutative context.

Property 6.4.7 (Schrödinger representation). Consider a distribution d on a (real) TVS V. There exists a unique unitary representation U of the additive group V^* on $L^2(V,d)$ such that

$$U(\lambda)f = e^{id(\lambda)}f \tag{6.25}$$

for all bounded tame functions f and such that 1 is cyclic for U in $L^2(V,d)$. Moreover, this representation is continuous with respect to the finest locally convex topology on V (the one generated by all seminorms on V)[2].

@@ EXPLAIN RELEVANCE e.g. Baez, Segal, and Zhou (2014) @@

6.4.2 Dirac–von Neumann postulates: revisited

Section 6.1.1 presented the axioms of quantum mechanics in terms of Hilbert spaces and the operators thereon. However, the incredible insight of *von Neumann* was that one can do away with the Hilbert space. By **??**, the observables of a quantum-mechanical system form a C^*-algebra. Consequently, the idea is to rephrase the axioms in purely C^*-algebraic terms (**??**). By **??**, these two approaches are equivalent.

Axiom 6.6 (Observables). A physical system is characterized by a C^*-algebra, with the observables corresponding to the self-adjoint elements.

Axiom 6.7 (States). A state of a quantum-mechanical system is given by a state of the associated C^*-algebra (**??**).

Axiom 6.8 (Born rule). The expectation value of an observable a in a state ω is given by the evaluation $\omega(a)$.

Remark 6.4.8. Section 7.2 will link this axiom to traces and operator theory through **??** and **??**.

Axiom 6.9 (Projection).

Axiom 6.10 (Unitary evolution).

@@ CORRECT ALL AXIOMS @@

6.4.3 Symmetries

Property 6.4.9 (States). By the postulates of quantum mechanics, states are represented by rays in the projective Hilbert space $\mathcal{H}\mathbb{P}$. The probabilities, given by the Born rule (Axiom 6.3), can be expressed in terms of the *Fubini–Study metric* on $\mathcal{H}\mathbb{P}$ as follows:

$$\mathcal{P}(\psi,\phi) := \cos^2\big(d_{\text{FS}}(\psi,\phi)\big) = \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}, \tag{6.26}$$

where $|\psi\rangle, |\phi\rangle$ are representatives of the states ψ,ϕ in $\mathcal{H}\mathbb{P}$.

[2]This topology is also known as the **algebraic topology**

Definition 6.4.10 (Symmetry). A quantum symmetry (or **quantum automorphism**) is an isometric automorphism of $\mathcal{H}\mathbb{P}$. The group of these symmetries is denoted by $\mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P})$.

The following theorem due to *Wigner* gives a (linear) characterization of quantum symmetries.[3]

Theorem 6.4.11 (Wigner). *Every quantum automorphism of $\mathcal{H}\mathbb{P}$ is induced by a unitary or anti-unitary operator on $\mathcal{H}$.*

This is equivalent to saying that the group morphism

$$\pi : \mathsf{Aut}(\mathcal{H},\mathcal{P}) := \mathsf{U}(\mathcal{H}) \times \mathsf{AU}(\mathcal{H}) \to \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \tag{6.27}$$

is surjective. Together with the kernel $\mathsf{U}(1)$, given by phase shifts, this forms a short exact sequence:

$$1 \longrightarrow \mathsf{U}(1) \longrightarrow \mathsf{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \longrightarrow 1\,. \tag{6.28}$$

In the case of symmetry breaking (e.g. lattice systems), the full symmetry group is reduced to a subgroup $G \subset \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P})$. The group of operators acting on $\mathcal{H}$ is then given by the pullback $\widetilde{G}$ of the diagram

$$\mathsf{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathsf{Aut}_{\mathrm{QM}}(\mathcal{H}\mathbb{P}) \longleftarrow G\,. \tag{6.29}$$

It should also be noted that the kernel of the homomorphism $\widetilde{G} \to G$ is again $\mathsf{U}(1)$. This leads to the property that $\widetilde{G}$ is a $\mathbb{Z}_2$-twisted (hence noncentral) $\mathsf{U}(1)$-extension of G, where the twist is induced by the homomorphism $\phi : \mathsf{Aut}(\mathcal{H},\mathcal{P}) \to \mathbb{Z}_2$ that says whether an operator is implemented unitarily or anti-unitarily.

@@ COMPLETE @@

6.4.4 Symmetric states

Axiom 6.11 (Symmetrization postulate). Let $\mathcal{H}$ be the single-particle Hilbert space. A system of $n \in \mathbb{N}$ identical particles is described by a state $|\Psi\rangle$ belonging to either $S^n\mathcal{H}$ or $\Lambda^n\mathcal{H}$. These **bosonic** and **fermionic** states are, respectively, of the form

$$|\Psi_B\rangle = \sum_{\sigma\in S_n} \big|\psi_{\sigma(1)}\big\rangle \cdots \big|\psi_{\sigma(n)}\big\rangle \tag{6.30}$$

and

$$|\Psi_F\rangle = \sum_{\sigma\in S_n} \mathsf{sgn}(\sigma)\big|\psi_{\sigma(1)}\big\rangle \cdots \big|\psi_{\sigma(n)}\big\rangle\,, \tag{6.31}$$

where the $|\psi_i\rangle$ are single-particle states and S_n is the permutation group on n elements.

[3] It is a particular case of a more general theorem in projective geometry.

Remark 6.4.12. In ordinary quantum mechanics, this is a postulate, but in quantum field theory, this is a consequence of the *spin-statistics theorem*. @@ ADD THIS THEOREM TO [QFT] @@

Definition 6.4.13 (Slater determinant). Let $\{\phi_i(\vec{q})\}_{i\leq n}$ be a set of wave functions, called **spin orbitals**, describing a system of n identical fermions. The totally antisymmetric wave function of the system is given by

$$\psi(\vec{q}_1,\dots,\vec{q}_n) = \frac{1}{\sqrt{n!}}\det\begin{pmatrix}\phi_1(\vec{q}_1) & \cdots & \phi_n(\vec{q}_1)\\ \vdots & \ddots & \vdots \\ \phi_1(\vec{q}_n) & \cdots & \phi_n(\vec{q}_n)\end{pmatrix}. \tag{6.32}$$

A similar function can be defined for bosonic systems using the concept of *permanents*.

6.5 Foundations ♣

6.5.1 Measurement problem

If one looks at the Schrödinger equation (Formula 6.2.2) or Ehrenfest's theorem (Theorem 6.3.3), it is easy to see that time evolution is entirely linear and deterministic. Superpositions are preserved under Hamiltonian flow (a crucial ingredient of quantum mechanics) and, given an initial state, time evolution will always lead to the same final state. However, the Born rule (Axiom 6.3), which governs 'measurements' is very nonlinear and nondeterministic. It is probabilistic and, once a 'measurement' has been performed, the state has 'collapsed' onto an eigenstate of the observable under consideration.

The issue of what constitutes a 'measurement' — Is it a conscious human doing an experiment? Is it a mouse interfering with an experiment? Is it two particles interacting? ...[4] — and why exactly the Born rule holds and what it entails, i.e. how probabilities arise, is known as the measurement problem. On a historical note, it should be noted that, after an initial surge of interest shortly after the 5th Solvay Conference (1927), where quantum mechanics was formally established, the study of the foundations of quantum mechanics (the measurement problem specifically) became an infamous topic due to the pragmatic mentality of nuclear physics during the 20th century.

@@ ADD (dynamical collapse, epistemic) @@

[4] This (perhaps artificial) boundary between classical and quantum is sometimes called the **Heisenberg cut**.

6.5.2 Copenhagen interpretation

The Copenhagen interpretation[5] takes the foundations of quantum mechanics as presented above very literally.

@@ COMPLETE (e.g. collapse) @@

6.5.3 Many-worlds interpretation

This interpretation, originating with *Everett*, posits a different idea, which does away with the need of the explicit Born rule axiom. In this interpretation, there is a kind of 'universal wave function', which governs both the observer and the experiment. A 'measurement' is then simply an entanglement-inducing interaction between these two subsystems.

The main implication of such an interpretation is, however, that the universal wave function branches every time such an interaction occurs. More precisely, assume that 'we', the observers, perform a measurement on some system (for simplicity, assume that the measurement has a binary outcome). The measurement process is then described as follows:

$$|\mathrm{in}\rangle_{\mathrm{obs}}|\mathrm{in}\rangle_{\mathrm{exp}} \longrightarrow \lambda_0|0\rangle_{\mathrm{obs}}|0\rangle_{\mathrm{exp}} + \lambda_1|1\rangle_{\mathrm{obs}}|1\rangle_{\mathrm{exp}}\,. \tag{6.33}$$

Taking this superposition as a physical reality, this means that if we had measured the state 0, a copy of us living on the other branch will have measured 1 (and the other way around).

@@ COMPLETE (e.g. origin of probabilities) @@

6.5.4 Relational quantum mechanics

An important notion in classical physics is that of a *reference frame*, i.e. a choice of axes and scales. Usually, this corresponds to choosing an observer, relative to which one expresses the motion of all other objects. In relativity, the relative treatment of physics was the grand breakthrough by Einstein. However, although this notion had been left aside for a long time in the treatment of quantum mechanics and a specific choice of reference frame was silently assumed, this assumption was not as innocuous as it appears. Superposition and complementarity make a definite choice of absolute reference frame impossible.

To understand the relevance of a relational approach to quantum mechanics, consider the following thought experiment.

[5]This name stems from the fact that its initial proponents were from the group of physicists centered around *Bohr*.

Definition 6.5.1 (Wigner's friend). Consider two observers, Wigner and his friend, performing an experiment as shown diagrammatically in Fig. 6.1. One envisions Wigner standing outside the laboratory, having no way to observe what happens inside the lab, and his friend who performs an experiment inside the lab. The paradox arises from the two ways one can describe the sequence of the friend performing a measurement and Wigner checking up on the results in the classical (Copenhagen) interpretation.

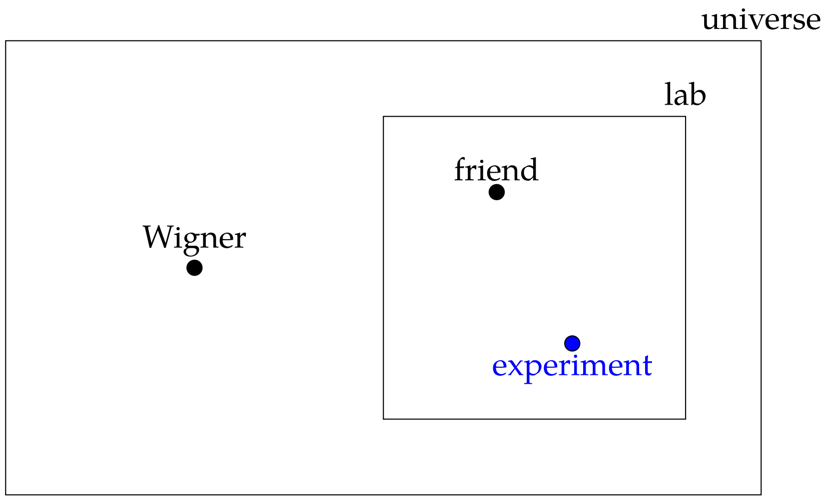


Figure 6.1: Wigner's friend thought experiment.

From the point of view of the friend, at the moment of measurement, the projection/collapse axiom states that the wave function describing friend + experiment 'collapses' to:

$$|\psi\rangle = |\uparrow\rangle_{\text{friend}}|\uparrow\rangle_{\text{exp}}\,. \tag{6.34}$$

However, from the point of view of Wigner, who has not observed the measurement, the state is described by

$$|\psi\rangle' = \alpha|\uparrow\rangle_{\text{friend}}|\uparrow\rangle_{\text{exp}} + \beta|\downarrow\rangle_{\text{friend}}|\downarrow\rangle_{\text{exp}}\,. \tag{6.35}$$

Whereas in the many-world approach one would simply take the branching approach, which is fully unitary and resolves this issue by avoiding collapse, the relational approach takes collapse at face value, but states that observations are relative, i.e. always with respect to some fixed observer (be it a person, a classical object or another quantum-mechanical system). From this point of view, textbook Copenhagen QM is simply quantum mechanics with respect to some god-given observer, and collapse and unitary evolution do not have to be reconciled.

The more general idea is that information and, hence, the values of observables are a relative notion, i.e. variables only attain their values when considered with respect to a certain observer. As such, RQM is an epistemic interpretation of quantum mechanics in that the wave function only captures 'our' information about the system (or universe) and not the 'true' physical state. When applied to the notion of instantaneity (or velocity), this line of thinking will give rise to (special) relativity as in **??** (and **??**).

@@ COMPLETE @@

For example, consider three observers: Alice, Bob and Charlie. Assume that each observer has a spin-$\frac{1}{2}$ particle and that, relative to Alice, the joint state is given by

$$|\psi\rangle^A_{ABC} = |\uparrow\rangle^A_A\Big(|\uparrow\rangle^A_B + |\downarrow\rangle^A_B\Big)|\downarrow\rangle^A_C\,. \tag{6.36}$$

Note that this state is separable. Now, what would the state be relative to Bob? If one supposes that changes of reference frame are *coherent* (to be formalized below), the joint state will be

$$|\psi\rangle^B_{ABC} = |\uparrow\rangle^B_B\Big(|\uparrow\rangle^B_A|\downarrow\rangle^B_C + |\downarrow\rangle^B_A|\uparrow\rangle^B_C\Big)\,. \tag{6.37}$$

A mere change of reference frame, an operation that would classically leave the physics invariant, has transformed a product state into an entangled state.

Axiom 6.12 (Relational physics). Given $n \in \mathbb{N}$ systems[6], any state is described relative to one of these systems. Given a choice of 'observing system', let it be system i, the state of system i is given by a fiducial state $|0\rangle^i_i$.

Axiom 6.13 (Coherent change). Consider a change of reference frame $0 \longrightarrow i$ such that

$$\begin{cases} |\psi\rangle^0 \longrightarrow |\psi\rangle^i \\ |\phi\rangle^0 \longrightarrow |\phi\rangle^i\,. \end{cases} \tag{6.38}$$

Then

$$\alpha|\psi\rangle^0 + \beta|\phi\rangle^0 \longrightarrow \alpha|\psi\rangle^i + \beta|\phi\rangle^i \tag{6.39}$$

for all $\alpha, \beta \in \mathbb{C}$.

Abstractly, a (classical) reference frame is defined as follows in the spirit of **??** and **??**.

Definition 6.5.2 (Reference frame). Let X be an object of interest. Whereas a coordinate chart on X, modeled on an object Y, is given by a morphism $Y \rightarrow X$, a **coordinate system** on X is given by an isomorphism $Y \cong X$, i.e. a global coordinate chart. A reference frame is coordinate system for which Y corresponds the a physical system.

[6] An abstraction of the notion of observer.

Let the system of interest X admit a group action that is both free and transitive, turning it into a G-torsor (**??**). At the level of sets, one has $X \cong G$ and a choice of origin, i.e. a specific choice of isomorphism, corresponds to a choice of reference frame (the identity element corresponding to the fiducial state above). A change of reference frames $s^0 \longrightarrow s^i$, frome system 0 to system i, is given by the right regular action of the relative coordinate of i on all relative coordinates:

$$\phi^{0\to i}(e, g_1^0, \dots, g_n^0) \mapsto (g_0^i, g_1^0 g_0^i, \dots, e, \dots, g_n^0 g_0^i)\,, \tag{6.40}$$

where the relation $g_i^0 = (g_0^i)^{-1}$ was used. It should be noted that this boils down to a *passive transformation*. When passing to the quantization of these systems, one should assume that G is locally compact and comes equipped with the canonical Haar measure (**??**). In this case, a quantization is given by the space of square-integrable functions $L^2(G)$, where basis states are labeled by group elements.

@@ VERIFY THIS STATEMENT @@

The change-of-reference-frame operator is given as follows:

$$\widehat{U}^{0\to i} := \mathsf{SWAP}_{0,i} \circ \int_G \mathbb{1}_{L^2(G)} \otimes \widehat{U}_R(g_i^0)^{\otimes i-2} \otimes |g_0^i\rangle\langle g_i^0| \otimes \widehat{U}_R(g_i^0)^{\otimes n-i-2}\, dg_i^0\,, \tag{6.41}$$

where

$$\widehat{U}_R(g) : |x\rangle \mapsto |xg^{-1}\rangle \tag{6.42}$$

is the unitary implementation of the right regular action and dg denotes integration with respect to the Haar measure on G. It can be shown that $\widehat{U}^{0\to i}$ is unitary, its inverse being given by $\widehat{U}^{i\to 0}$ and composition is transitive. It can be shown that this procedure can be extended to any one-particle Hilbert space $\mathcal{H}$ as long as the inclusion $G \to \mathcal{H}$ is injective and maps G to an orthonormal basis of (a subset of) $\mathcal{H}$.

6.6 Angular Momentum

6.6.1 Angular momentum operator

Property 6.6.1 (Lie algebra). The angular momentum operators generate a Lie algebra (**??**). The Lie bracket is defined by the following commutation relation:

$$[\hat{J}_i, \hat{J}_j] = i\hbar\varepsilon_{ijk}\hat{J}_k\,. \tag{6.43}$$

Since rotations correspond to actions of the orthogonal group SO(3), it should not come as a surprise that the above relation is exactly the defining relation of the Lie algebra $\mathfrak{so}(3)$ from **??**.

Property 6.6.2. The mutual eigenbasis of $\hat{J}^2$ and $\hat{J}_z$ is defined by the following two eigenvalue equations:

$$\hat{J}^2|j,m\rangle = j(j+1)\hbar^2\,|j,m\rangle\,, \tag{6.44}$$

$$\hat{J}_z|j,m\rangle = m\hbar\,|j,m\rangle\,. \tag{6.45}$$

Definition 6.6.3 (Ladder operators[7]). The raising and lowering operators $\hat{J}_+$ and $\hat{J}_-$ are defined as follows:

$$\hat{J}_+ := \hat{J}_x + i\hat{J}_y \qquad \text{and} \qquad \hat{J}_- := \hat{J}_x - i\hat{J}_y\,. \tag{6.46}$$

These operators only change the quantum number $m_z \in \mathbb{N}$, not the total angular momentum.

Corollary 6.6.4. From the commutation relations of the angular momentum operators, one can derive the commutation relations of the ladder operators:

$$\left[\hat{J}_+,\hat{J}_-\right] = 2\hbar\hat{J}_z\,. \tag{6.47}$$

Formula 6.6.5. The total angular momentum operator $\hat{J}^2$ can now be expressed in terms of $\hat{J}_z$ and the ladder operators using the commutation relation (6.43):

$$\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z\,. \tag{6.48}$$

Remark 6.6.6 (Casimir operator). From the definition of $\hat{J}^2$, it follows that this operator is a Casimir invariant (**??**) of $\mathfrak{so}(3)$.

6.6.2 Rotations

Formula 6.6.7. An infinitesimal rotation $\hat{R}(\delta\vec{\boldsymbol{\varphi}})$ is given by the following formula:

$$\hat{R}(\delta\vec{\boldsymbol{\varphi}}) = \mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\delta\vec{\boldsymbol{\varphi}}\,. \tag{6.49}$$

A finite rotation can be generated by applying this infinitesimal rotation repeatedly:

$$\hat{R}(\vec{\boldsymbol{\varphi}}) = \left(\mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\frac{\vec{\boldsymbol{\varphi}}}{n}\right)^n = \exp\left(-\frac{i}{\hbar}\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{\varphi}}\right). \tag{6.50}$$

Formula 6.6.8 (Matrix elements). Applying a rotation over an angle φ about the z-axis to a state $|j,m\rangle$ gives

$$\hat{R}(\varphi\vec{e}_z)|j,m\rangle = \exp\left(-\frac{i}{\hbar}\hat{J}_z\varphi\right)|j,m\rangle = \exp\left(-\frac{i}{\hbar}m\varphi\right)|j,m\rangle\,. \tag{6.51}$$

[7] Also called the **creation** and **annihilation** operators (especially in quantum field theory).

Multiplying these states with a bra $\langle j', m'|$ and using the orthonormality of the eigenstates, gives the matrix elements of the rotation operator:

$$\widehat{R}_{ij}(\varphi\vec{e}_z) = \exp\left(-\frac{i}{\hbar}m\varphi\right)\delta_{jj'}\delta_{mm'}\,. \tag{6.52}$$

From the expression of the angular momentum operators and the rotation operator, it is clear that a general rotation has no effect on the total angular momentum number $j \in \mathbb{N}$. This means that the rotation matrix will be block diagonal with respect to j. This amounts to the following reduction of the representation of the rotation group:

$$\langle j, m'|\widehat{R}(\varphi\vec{n})|j, m\rangle = \mathcal{D}^{(j)}_{m,m'}(\widehat{R})\,, \tag{6.53}$$

where the functions $\mathcal{D}^{(j)}_{m,m'}(\widehat{R})$ are called the **Wigner D-functions**. For every value of j, there are $(2j+1)$ values for m. This implies that the matrix $\mathcal{D}^{(j)}(\widehat{R})$ is a $(2j+1)\times(2j+1)$-matrix.

6.6.3 Spinor representation

Definition 6.6.9 (Pauli matrices).

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.54}$$

From this definition, it is clear that the Pauli matrices are Hermitian and unitary. Together with the 2×2 identity matrix, they form a basis for the space of 2×2 Hermitian matrices. For this reason, the identity matrix is often denoted by σ_0 (especially in the context of relativistic QM).

Formula 6.6.10. In the spinor representation ($J = \frac{1}{2}$), the Wigner-D matrix reads as follows:

$$\mathcal{D}^{(1/2)}(\varphi\vec{e}_z) = \begin{pmatrix} e^{-i/2\varphi} & 0 \\ 0 & e^{i/2\varphi} \end{pmatrix}. \tag{6.55}$$

6.6.4 Coupling of angular momenta

Due to the tensor product structure of a coupled Hilbert space, the angular momentum operator $\hat{J}_i$ should now be interpreted as $\mathbb{1}\otimes\cdots\otimes\hat{J}_i\otimes\cdots\otimes\mathbb{1}$ (cf. **??**). Because the angular momentum operators $\hat{J}_{k\neq i}$ do not act on the space $\mathcal{H}_i$, one can pull these operators through the tensor product:

$$\hat{J}_i|j_1\rangle\otimes\cdots\otimes|j_n\rangle = |j_1\rangle\otimes\cdots\otimes\hat{J}_i|j_i\rangle\otimes\cdots\otimes|j_n\rangle\,. \tag{6.56}$$

The basis used above is called the **uncoupled basis**.

For simplicity, the total Hilbert space is, from here on, assumed to be that of a two-particle system. Let $\hat{J}$ denote the total angular momentum:

$$\hat{J} = \hat{J}_1 + \hat{J}_2 . \tag{6.57}$$

With this operator, one can define a **coupled** state $|J, M\rangle$, where M is the total magnetic quantum number which ranges from $-J$ to J.

Formula 6.6.11 (Clebsch–Gordan coefficients). Because both bases (coupled and uncoupled) span the total Hilbert space $\mathcal{H}$, there exists an invertible transformation between them. The transformation coefficients can be found by using the resolution of the identity:

$$|J, M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2 \,|\, J, M\rangle . \tag{6.58}$$

These coefficients are called the Clebsch–Gordan coefficients.

Property 6.6.12. By acting with the operator $\hat{J}_z$ on both sides of Formula 6.6.11, it is possible to prove that the Clebsch–Gordan coefficients are nonzero if and only if $M = m_1 + m_2$.

6.7 Approximation methods

6.7.1 WKB approximation

The Wentzel–Kramers–Brillouin (WKB) approximation[8] starts from the ansatz

$$\psi(\vec{q}) := \exp\big(iS(\vec{q})/\hbar\big) , \tag{6.59}$$

with $S : \mathbb{R}^n \to \mathbb{R}$ a phase function that is to be determined. Inserting this in the TDSE (in configuration representation) gives:

$$\left[\frac{\|\vec{\nabla}S(\vec{q})\|^2}{2m} + \big(V(\vec{q}) - E\big) - \frac{i\hbar\Delta S(\vec{q})}{2m}\right] \exp\big(iS(\vec{q})/\hbar\big) = 0 . \tag{6.60}$$

To first order, i.e. for slowly varying potentials, the last term can be ignored. In this case, the phase function satisfies the Hamilton–Jacobi equation (**??**):

$$H\big(\vec{q}, S'(\vec{q})\big) = \frac{\|\vec{\nabla}S(\vec{q})\|^2}{2m} + \big(V(x) - E\big) = 0 . \tag{6.61}$$

In physics, the Hamilton–Jacobi equation without time derivative is often called the **eikonal equation**[9]. This leads to the following result.

[8]This approach to solving second-order ODEs was essentially introduced a century earlier by *Green* and *Liouville*.

[9]This name stems from optics.

Property 6.7.1. A function $S : \mathbb{R}^n \to \mathbb{R}$ is a phase function for a first-order solution to the Schrödinger equation if its differential lies in a level set of the classical Hamiltonian $H : T^*\mathbb{R}^n \to \mathbb{R}$. These solutions are said to be **admissible**.

To obtain higher-order approximations, the solution has to be generalized beyond a pure phase function:

$$\psi(\vec{q}) = a(\vec{q}) \exp\big(iS(\vec{q})/\hbar\big) . \tag{6.62}$$

Assuming S is admissible, the factor $a : \mathbb{R}^n \to \mathbb{R}$ satisfies the **homogeneous transport equation**:

$$a\Delta S + 2\vec{\nabla} a \cdot \vec{\nabla} S = 0 . \tag{6.63}$$

If a satisfies this equation, ψ is called a **semiclassical state**. Note that this equation is equivalent to $a^2\vec{\nabla}S$ being divergence free or, equivalently:

$$\mathcal{L}_{\pi_* X^H}(a^2 \,\mathrm{Vol}) = 0 . \tag{6.64}$$

Since Lie derivatives pull back under diffeomorphisms (**??**) and the image $\mathrm{im}(\mathrm{d}S)$ gives a trivial subbundle of $T^*\mathbb{R}^n$, this is also equivalent to

$$\mathcal{L}_{X^H}(a^2\pi^* \,\mathrm{Vol}) = 0 . \tag{6.65}$$

This quadratic behaviour in a leads to the idea that the correct object for representing quantum states is a half-density (see also **??**). This leads to the following statement:

> A second-order solution to the Schrödinger equation is given by a pair (S, a), where S is an admissible phase function and $a \in \Omega^{1/2}(\mathrm{im}(\mathrm{d}S))$ is a half-form that is invariant under the (classical) Hamiltonian flow.

The generalization to curved spaces, i.e. replacing $\mathbb{R}^{2n}$ by a symplectic manifold M, will be covered in Section 6.8.2.

6.8 Curved backgrounds ♣

Using the tools of distribution theory and differential geometry (**?? ??**, **??** and onwards), one can introduce quantum mechanics on curved backgrounds (in the sense of 'space', not 'spacetime').

6.8.1 Extending quantum mechanics

Remark 6.8.1 (Rigged Hilbert spaces). A first important remark to be made is that the classical definition of the wave function as an element of $L^2(\mathbb{R}^d, \mathbb{C})$ is not sufficient, even in flat Cartesian space. A complete description requires the introduction of so-called *Gel'fand triples* or *rigged Hilbert spaces*, where the space of square-integrable functions is replaced by the Schwartz space (**??**) of rapidly decreasing functions. The linear functionals on this space are then given by the tempered distributions.

When working on curved spaces or even in non-Cartesian coordinates on flat space, one can encounter problems with the definition of the self-adjoint operators $\widehat{q}^i$ and $\widehat{p}_i$. The naive definition $\widehat{q}^i = q^i, \widehat{p}_i = -i\partial_i$ gives rise to extra terms that break the canonical commutation relations and the selfadjointness of the operators (e.g. the angular position operator $\widehat{\varphi}$ on the circle together with its conjugate $\widehat{L}$) when calculating inner products.

An elegant solution to this problem is obtained by giving up the definition of the wave function as a well-defined function $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$. Assume that the physical space has the structure of a Riemannian manifold (M, g) and that the 'naive' wave functions take values in a vector space V. Then, construct a vector bundle E with typical fibre V over M. By **??**, an invariant description of the 'true' wave function is a map $\Psi : F(E) \rightarrow V$ or, locally, the pullback $\psi := \varphi^*\Psi$ for some local section $\varphi : U \subseteq M \rightarrow F(E)$. The Levi-Civita connection on M also induces a covariant derivative ∇ on E that can be used to define differential operators.

Now, a general inner product can be introduced:

$$\langle \psi, \phi \rangle := \int_M \overline{\psi(x)}\phi(x)\,\mathrm{Vol}_M \,. \tag{6.66}$$

Because the factor $\sqrt{\det(g)}$ transforms in the inverse manner of the measure dx, the integrand is invariant under coordinate transforms (something that is generally required of physical laws). Using this new inner product, one can for example check the selfadjointness of the momentum operator $\widehat{P}_i := -i\nabla_i$:

$$\begin{aligned}
\langle\psi, \widehat{P}_i\phi\rangle &= \int_M \overline{\psi(x)}(-i\nabla_i)\phi(x)\sqrt{\det(g)}\,dx \\
&\overset{??}{=} \int_M \overline{\psi(x)}(-i\partial_i - i\omega_i)\phi(x)\sqrt{\det(g)}\,dx \\
&= \int_M \overline{(-i\partial_i\psi)(x)}\phi(x)\sqrt{\det(g)}\,dx + i\int_M \overline{\psi(x)}\phi(x)\left(\partial_i\sqrt{\det(g)}\right)dx \\
&\qquad - i\int_M \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx \\
&= \langle\widehat{P}_i\psi, \phi\rangle - i\int_M \overline{\psi(x)}\overline{\omega_i}\phi(x)\sqrt{\det(g)}\,dx \\
&\qquad + i\int_M \overline{\psi(x)}\phi(x)\left(\partial_i\sqrt{\det(g)}\right)dx \\
&\qquad - i\int_M \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx\,.
\end{aligned}$$

Selfadjointness then requires that

$$\sqrt{\det(g)}(\omega_i + \overline{\omega_i}) = \partial_i\sqrt{\det(g)} \tag{6.67}$$

or

$$2\mathrm{Re}(\omega_i) = \partial_i \ln\left(\sqrt{\det(g)}\right). \tag{6.68}$$

@@ COMPLETE (rewrite in global terms) @@

6.8.2 WKB approximation

Property 6.7.1 is generalized quite trivially after replacing $\mathbb{R}^n$ by a configuration manifold Q. A further step is provided by also generalizing **??**.

Property 6.8.2. A Lagrangian submanifold $\iota : L \hookrightarrow T^*Q$ will be called an admissible phase function for a first-order solution to the Schrödinger equation if it satisfies the classical Hamilton–Jacobi equation, i.e. lies in a level set of the classical Hamiltonian $H : T^*Q \to \mathbb{R}$, for a regular value.

To obtain a second-order solution, one also needs prefactor for the semiclassical states. The homogeneous transport equation (6.63) is generalized as follows:

$$a\Delta S + 2\mathcal{L}_{\nabla S}a = 0\,, \tag{6.69}$$

where Δ is the Laplace–Beltrami operator on Q. As before, a general second-order solution, assuming S is admissible, is given by a half-form $a \in \Omega^{1/2}(L)$ satisfying

$$\mathcal{L}_Y a = 0\,, \tag{6.70}$$

where Y is the (nonsingular) vector field on L induced by X^H. This then gives a second-order solution on Q by pulling back along the inverse $(\pi \circ \iota)^{-1}$, which is a diffeomorphism since L is projectable. Moreover, if L is exact (**??**), then S is induced by a primitive of the induced Liouville form $\iota^*\alpha$. If both the exactness and projectability conditions are dropped, the notion of a **geometric solution** are obtained.

To pass to this more general situation, some more structure is needed. If L is not exact, the Liouville form does not admit a global primitive. However, L does admit a (good) cover $\{U_k\}_{k\in I}$ such that on every patch, a second-order solution can be found, and then the problem becomes how to glue these together. The gluing condition is the following integrality condition:

$$\phi_k(x) - \phi_l(x) \in 2\pi\hbar\mathbb{Z}\,, \tag{6.71}$$

where ϕ_k is the phase function on U_k, for all $x \in U_k \cap U_l$. Note that this condition can only be satisfied for all $\hbar \in \mathbb{R}^+$ if $[\alpha] = 0$. However, this is exactly the condition that should be relaxed. Luckily, $\hbar$ should be a fixed value.

Definition 6.8.3 (Quantizable Lagrangian). A projectable Lagrangian submanifold $L \subset T^*M$ is said to be quantizable if there exists an $\hbar \in \mathbb{R}^+$ such that the restriction of the Liouville class to L is $\hbar$-integral, i.e. the integrality condition (6.71) is satisfied. All values $\hbar$ for which the integrality condition is satisfied, are said to be **admissible**.

Remark 6.8.4. Note that the admissible values for $\hbar$ will form a decreasing sequence of the form

$$\hbar_0, \frac{\hbar_0}{2}, \dots\,, \tag{6.72}$$

where $\hbar_0$ is the greatest admissible value.

For the weakening of the projectability condition, see Bates and Weinstein (1997). However, even without weakening that condition, there is still a remaining issue to the quantization of classical solutions. This will involve Maslov indices (**??**) and Morse theory (**??**).

6.9 Topos theory ♣

Definition 6.9.1 (Bohr topos). Consider a C^*-algebra A of bounded observables on a Hilbert space $\mathcal{H}$. Denote by $\mathsf{ComSub}(A)$ the poset (Definition 1.7.1) of commutative C^*-subalgebras. This set can be equipped with the **Alexandrov topology**[10], i.e. the topology for which the open sets are the upward closed subsets. The topological space $(\mathsf{ComSub}(A), \tau_{\text{Alex}})$ is called the Bohr site of A.

[10] There exist an equivalences $\mathbf{Pre} \cong \mathbf{AlexTop}$ and $\mathbf{Pos} \cong \mathbf{AlexTop}_{T_0}$.

The sheaf topos over the Bohr site is called the Bohr topos $\mathbf{Bohr}(A)$. It can be turned into a ringed topos, where the ring object (which is even an internal commutative C^*-algebra) is given by the tautological functor

$$\underline{A} : \mathsf{ComSub}(A) \to \mathbf{Set} : C \mapsto C\,. \tag{6.73}$$

Property 6.9.2. A morphism in $\mathbf{C^*Alg}$ is commutativity reflecting if and only if the induced morphism on posets admits a right adjoint. Moreover, there exists a bijection between the following two classes of morphisms:

- Geometric morphisms $f : \mathbf{Bohr}(B) \to \mathbf{Bohr}(A)$ admitting a right adjoint together with epimorphisms of internal algebras $\underline{A} \to f^*\underline{B}$.
- Commutativity-reflecting functions $f : A \to B$ that restrict to algebra morphisms on all commutative subalgebras.

Definition 6.9.3 (Spectral presheaf). The presheaf on a Bohr site assigning to every commutative subalgebra its Gel′fand spectrum.

Theorem 6.9.4 (Kochen–Specker). *If $A = \mathcal{B}(\mathcal{H})$ with $\mathrm{dim}(\mathcal{H}) > 2$, the spectral presheaf has no global element.*

Property 6.9.5 (Gleason's theorem). There exists a natural bijection between the quantum states of a C^*-algebra A and the classical states of $\underline{A}$ internal to $\mathbf{Bohr}(A)$.

Definition 6.9.6 (Bohrification). Consider a C^*-algebra A together with its Bohr topos $\mathbf{Bohr}(A)$. To its internal C^*-algebra $\underline{A}$, one can assign an internal locale $\underline{\Sigma}_A$ by (internal) Gel′fand duality (**??**). Under the equivalence **??**, one then obtains a locale Σ_A. The functor

$$\Sigma : \mathbf{C^*Alg} \to \mathbf{Loc} : A \mapsto \Sigma_A \tag{6.74}$$

is called Bohrification. This locale can be constructed as the disjoint union

$$\Sigma_A = \bigsqcup_{C \in \mathsf{ComSub}(A)} \Phi_C\,, \tag{6.75}$$

the étale locale corresponding to the spectral presheaf. Its open sets are given by those subsets whose restrictions to commutative subalgebras are open such that these restrictions are compatible with subalgebra inclusions.

This topological bundle $\Sigma_A \to \mathsf{Alex}(\mathsf{ComSub}(A))$ also admits a topos-theoretic incarnation. There exists a (canonical) morphism of ringed topoi

$$\pi : \mathbf{Bohr}(A) \to \big(\mathbf{Sh}\big(\mathsf{Alex}(\mathsf{ComSub}(A))\big), \underline{\mathbb{R}}\big)\,, \tag{6.76}$$

whose underlying geometric morphism is simply the identity.

Example 6.9.7 (Gel'fand spectrum). If A is a commutative C^*-algebra, its Bohrification is not isomorphic to its ordinary Gel'fand spectrum Φ_A. However, after replacing the topology on $\mathbf{Bohr}(A)$ by the *double negation topology* and repeating the above construction, one obtains

$$\Phi_A \cong \Sigma_A^{\neg\neg}. \tag{6.77}$$

Property 6.9.8 (States). A positive and normalized section of the morphism $\pi : \mathbf{Bohr}(A) \rightarrow \big(\mathbf{Sh}\big(\mathsf{Alex}(\mathsf{ComSub}(A))\big), \underline{\mathbb{R}}\big)$ in the category of $\underline{\mathbb{R}}$*-module topoi*.

By Property 6.9.2 above, the following relation is obtained.

Property 6.9.9 (Observables). Morphisms $\mathbf{Bohr}(A) \rightarrow \mathbf{Bohr}(C(\mathbb{R})_0)$ admitting a right adjoint together with an epimorphism $C_0(\mathbb{R}) \rightarrow A$ correspond to observables on A.

Chapter 7

Quantum Information Theory

The section on (quantum) reference frames is based on De La Hamette and Galley (2020).

Chapter Contents

7.1 Entanglement

7.1.1 Introduction

Construction 7.1.1 (Schmidt decomposition). Consider a bipartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. For any such state, there exist orthonormal sets $\{|e_i\rangle, |f_j\rangle\}_{i,j\leq\kappa}$ such that

$$|\psi\rangle = \sum_{i=1}^{\kappa} \lambda_i |e_i\rangle \otimes |f_i\rangle\,, \tag{7.1}$$

where the coefficients λ_i are nonnegative real numbers. All objects in this expression can be obtained from a singular value decomposition of the coefficient matrix $\mathbf{C}$ of $|\psi\rangle$ in some bases of $\mathcal{H}_1$ and $\mathcal{H}_2$. The number $\kappa \in \mathbb{N}$ is called the **Schmidt rank** of $|\psi\rangle$.

Definition 7.1.2 (Entangled states). Consider a state $|\psi\rangle$ and consider its Schmidt decomposition. If the Schmidt rank is 1, i.e. the state can be written as $|\psi\rangle = |v\rangle \otimes |w\rangle$, the state is said to be **separable**. Otherwise, the state is said to be entangled.

The following theorem follow from the linearity of quantum mechanics.

Theorem 7.1.3 (No-cloning). *There is no unitary operator $\widehat{U}$ on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that*

$$\widehat{U}|\psi\rangle_1|\phi\rangle_2 = e^{i\alpha(\psi,\phi)}|\psi\rangle_1|\psi\rangle_2 \tag{7.2}$$

for all (normalized) $|\psi\rangle_1 \in \mathcal{H}_1$ and $|\phi\rangle_1 \in \mathcal{H}_2$.

Theorem 7.1.4 (No-deleting). *Consider a tripartite system $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ such that $\mathcal{H}_1 \cong \mathcal{H}_2$. If $\widehat{U}$ is a unitary operator on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ such that*

$$\widehat{U}|\psi\rangle_1|\psi\rangle_2|\phi\rangle_3 = |\psi\rangle_1|0\rangle_2\big|\phi_\psi\big\rangle_3 \tag{7.3}$$

for all $|\psi\rangle_1 \in \mathcal{H}_1$, where the final ancilla state $\big|\phi_\psi\big\rangle_3$ might depend on the initial state $|\psi\rangle_1$, then $\widehat{U}$ is simply a swap, i.e. $|\psi\rangle_1 \mapsto \big|\phi_\psi\big\rangle_3$ is an isometric embedding.

7.1.2 Bell states

Definition 7.1.5 (Bell state). A (binary) Bell state (also called a **cat state** or **Einstein–Podolsky–Rosen pair**) is defined as the following entangled state:

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)\,. \tag{7.4}$$

In fact, this state can be extended to a full maximally entangled basis for the 2-qubit Hilbert space:

$$\begin{aligned} |\Phi^-\rangle &:= \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big)\,,\\ |\Psi^+\rangle &:= \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)\,,\\ |\Psi^-\rangle &:= \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)\,. \end{aligned} \tag{7.5}$$

Method 7.1.6 (Dense coding[1]). Consider the Bell state $|\Phi^+\rangle$. By acting with one of the (unitary) spin-flip operators $\widehat{X}, \widehat{Y}, \widehat{Z}$, one can obtain any of the other three Bell states:

$$\begin{aligned} \widehat{X}|\Phi^+\rangle &= |\Phi^-\rangle\,,\\ \widehat{Y}|\Phi^+\rangle &= |\Psi^+\rangle\,,\\ \widehat{Z}|\Phi^+\rangle &= |\Psi^-\rangle\,. \end{aligned} \tag{7.6}$$

[1] Sometimes called **superdense coding**.

In a typical Alice-and-Bob-style experiment, one can ask whether this observation allows to achieve a better-than-classical communication channel. If Alice performs a spin flip on her qubit, although the resulting state has instantly 'changed' (cf. *spooky action at a distance*), Bob still cannot uniquely determine what this state is (since the resulting state is still maximally entangled). However, if Alice sends her qubit to Bob, the latter can perform a measurement on the composite system to find out what the state is and in this way determine which operation Alice performed ($\mathbb{1}, \widehat{X}, \widehat{Y}, \widehat{Z}$). Alice has thus effectively sent 2 classical bits of information through 1 qubit. Note that due to the fact that Alice still has to send her qubit through classical means, no faster-than-light communication is achieved.

Definition 7.1.7 (GHZ state). The Greenberger–Horne–Zeilinger state is defined as the multiparticle qudit ($d, N > 2$) version of the Bell state and is, therefore, also referenced to as a cat state:

$$|\mathsf{GHZ}\rangle = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} |i\rangle^{\otimes N} . \tag{7.7}$$

7.1.3 SRE states

@@ ADD @@

7.2 Density operators

Definition 7.2.1 (Density operator). Consider a (finite-dimensional) Hilbert space $\mathcal{H}$. A density operator on $\mathcal{H}$ is a linear operator $\rho \in \mathsf{End}(\mathcal{H})$ satisfying the following properties:

1. **Positivity**: $\langle v \,|\, \rho v\rangle \geq 0$ for all $v \in \mathcal{H}$,
2. **Hermiticity**: $\rho^\dagger = \rho$, and
3. **Unit trace**: $\mathrm{tr}(\rho) = 1$.

More concisely, density operators are the representing objects of normal states (**??**) on $\mathcal{B}(\mathcal{H})$.

Example 7.2.2 (Classical probability). A diagonal density matrix corresponds to a discrete probability distribution.

Definition 7.2.3 (Pure state). A state is said to be pure if it is described by an outer product of a state vector or, equivalently, by an idempotent density matrix:

$$\rho = |\psi\rangle\langle\psi| . \tag{7.8}$$

A density matrix that is not of this form gives rise to a **mixed state**.

Definition 7.2.4 (Reduced density operator). Let $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be the state of a bipartite system. The reduced density operator ρ_A of A is defined as follows:

$$\rho_A := \mathrm{tr}_B|\Psi\rangle\langle\Psi|\,. \tag{7.9}$$

Definition 7.2.5 (Purification). Let ρ_A be the density operator of a system A. A purification of ρ_A is a pure state $|\Psi\rangle$ of some composite system $A \otimes B$ such that

$$\rho_A = \mathrm{tr}_B|\Psi\rangle\langle\Psi|\,. \tag{7.10}$$

Property 7.2.6. Any two purifications of the same density operator ρ_A are related by a transformation $\mathbb{1}_A \otimes \widehat{V}$ with $\widehat{V}$ an isometry.

7.3 Channels

The following definition generalizes the content of **??** to the setting of partial information. When generalizing the projections in a PVM (spectral measure), one obtains a POVM.

Definition 7.3.1 (Positive operator-valued measure). First, let $\mathcal{H}$ be a finite-dimensional Hilbert space. A POVM on $\mathcal{H}$ consists of a finite set of positive (semi)definite operators $\{P_i\}_{i\leq n}$ such that

$$\sum_{i=1}^n P_i = \mathbb{1}_{\mathcal{H}}\,. \tag{7.11}$$

The probability to obtain state i, given a general state $\hat{\rho}$, is given by $\mathrm{tr}(\hat{\rho}P_i)$. Note that the operators are not necessarily orthogonal projectors, so n can be greater than $\dim(\mathcal{H})$.

Now, consider a measurable space (X, Σ) and a (possibly infinite-dimensional) Hilbert space $\mathcal{H}$. A POVM on X consists of a function $P : \Sigma \to \mathcal{B}(\mathcal{H})$ satisfying the following conditions:

1. P_E is positive and self-adjoint for all $E \in \Sigma$,
2. $P_X = \mathbb{1}_{\mathcal{H}}$, and
3. for all disjoint $(E_n)_{n\in\mathbb{N}} \subset \Sigma$:

$$\sum_{n\in\mathbb{N}} P_{E_n} = P_{\cup_{n\in\mathbb{N}} E_n}\,. \tag{7.12}$$

The following theorem can be derived from Stinespring's theorem **??**.

Theorem 7.3.2 (Naimark dilation theorem). *Every POVM P on $\mathcal{H}$ can be realized as a PVM Π on a, possibly larger, Hilbert space $\mathcal{K}$, i.e. there exists a bounded operator $V : \mathcal{K} \to \mathcal{H}$ such that*

$$P(\cdot) = V\Pi(\cdot)V^\dagger\,. \tag{7.13}$$

In the finite-dimensional setting, V can be chosen to be an isometry.

Recall the content of **??**.

Definition 7.3.3 (Completely positive trace-preserving). Consider a map $\Phi : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ between bounded operators on two (finite-dimensional) Hilbert spaces. This map preserves density matrices if it positive (**??**) and if it is trace-preserving (**??**). Furthermore, to ensure that an operation applied to a subsystem does not interfere with the positivity of the complete system, they are also required to be completely positive (**??**).

Completely positive, trace-preserving (CPTP) maps are often called **quantum channels** or **superoperators**.

Theorem 7.3.4 (Choi–Jamiołkowski). *The following map between quantum channels $\Phi : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ and density operators $\rho \in \mathsf{End}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is an isomorphism:*

$$\Phi \mapsto (\mathbb{1}_{\mathcal{H}_1} \otimes \Phi)|\mathsf{GHZ}\rangle\langle\mathsf{GHZ}|, \tag{7.14}$$

where the GHZ state was introduced in Definition 7.1.7.

7.4 Quantum logic

7.4.1 Birkhoff-von Neumann logic

Consider classical propositional logic. This is governed by the Boolean property, i.e. the set of all propositions admits the structure of a complete Boolean algebra (Property 4.1.12). Now, the question becomes how to treat propositions in quantum mechanics (as needed in, for example, quantum computing). In the sense of *von Neumann*, the propositions should be characterized by $\{0, 1\}$-valued observables or, equivalently, by projection operators. As such, the natural lattice to consider logic is that of closed subspaces of the state space $\mathcal{H}$ (which is a Hilbert space). Birkhoff–von Neumann logic is the study of such lattices.

Note that, in contrast to classical logic, the lattices of closed subspaces are not Boolean. The lattices are merely complete, orthomodular lattices (Definition 1.7.42).[2] Now, although these lattices are themselves very interesting, there relevance for quantum logic are heavily discussed for several reasons (e.g. lack of distributivity, lack of a clear implication operator, lack of an extension to predicate logic). In the next section it will be explained how this issue can be avoided by embedding Birkhoff–von Neumann logic into the more general framework of linear logic.

[2]It should be noted that complete, orthomodular lattices are, in general, very different from those originating from Hilbert spaces (cf. *Piron's theorem*).

7.4.2 Linear logic

A basic component of standard logic are the *structural inference rules*. These are inference rules that do not involve any logical operations. The following two inference rules for first-order logic control context extension (for an introduction to sequent calculus, see Section 4.1.3):

- **Contraction**:

$$\frac{\Gamma, p_1 : P, p_2 : P \vdash t_{p_1,p_2} : T}{\Gamma, p : P \vdash t_{p,p} : T\,.} \tag{7.15}$$

  This rule states that, in a valid judgement, premises might be used more than once.

- **Weakening**:

$$\frac{\Gamma \vdash P : \mathsf{Type} \qquad \Gamma \vdash t : T}{\Gamma, P \vdash t : T\,.} \tag{7.16}$$

  This rule states that, any premise can be added to (the premises of) a valid judgement.

In terms of categorical semantics, these two rules correspond (in the independent setting) to the diagonal and projection morphisms in Cartesian categories (Example 3.1.2).

Now, when considering quantum mechanics, two important results are the no-cloning and no-deleting theorem (Theorem 7.1.3 and Theorem 7.1.4). These correspond to the fact that the categories **FinVect** and **Hilb** are monoidal, but not Cartesian monoidal, i.e. the tensor product does not admit diagonal and projection morphisms. The natural type of logic in this setting is than a *substructural* one where the contraction and weakening rules are not valid.

In linear logic, the following propositions exist:

1. **Variables**: Every propositional variable is a proposition.
2. **Negation**: If P is a proposition, so is $P^\perp$.
3. **Connectives**: If P, Q are propositions, then
   - **Additive conjunction**: $P\&Q$ is a proposition. (Read: P with Q.)
   - **Additive disjunction**: $P \oplus Q$ is a proposition. (Read: P plus Q.)
   - **Multiplicative conjunction**: $P \otimes Q$ is a proposition. (Read: P times Q.)
   - **Multiplicative disjunction**: $P \parr Q$ is a proposition. (Read: P par Q.)

4. **Constants**:
   - **Additive truth**: $\top$,
   - **Additive falsity**: $\mathbf{0}$,
   - **Multiplicative truth**: $\mathbf{1}$,
   - **Multiplicative falsity**: $\perp$.
5. **Exponential connectives**: If P is a proposition, then
   - **Exponential conjunction**: $!P$ is a proposition. (Read: of course P.)
   - **Exponential disjunction**: $?P$ is a proposition. (Read: why not P.)

Given a context, the following inference rules are valid:[3]

1. **Identity***: If P is a propositional variable, then $P \vdash P$.
2. **Exchange***: Sequents are remain valid under permutations.
3. **Restricted weakening***: If P is a proposition, then

$$\frac{\Gamma \vdash \Theta}{\Gamma, !P \vdash \Theta} \tag{7.17}$$

   and, dually,

$$\frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, ?P}. \tag{7.18}$$

4. **Restricted contraction***: If P is a proposition, then

$$\frac{!P, !P \vdash \Theta}{!P \vdash \Theta} \tag{7.19}$$

   and, dually,

$$\frac{\Gamma \vdash ?P, ?P}{\Gamma \vdash ?P}. \tag{7.20}$$

5. **Negation**: If P is a proposition, then

$$\frac{\Gamma \vdash \Theta, P}{\Gamma, P^\perp \vdash \Theta} \tag{7.21}$$

[3]The form of these rules heavily depends on the exchange rule (the second item). Care must be taken if this rule is weakened.

and, conversely,

$$\frac{\Gamma, P \vdash \Theta}{\Gamma \vdash \Theta, P^{\perp}}. \tag{7.22}$$

Note that these rules allow to write any sequent in right form, i.e. $\vdash \Gamma^{\perp}, P$.

6. **Additive conjunction**: If P, Q are propositions, then

$$\frac{P \vdash \Theta \qquad Q \vdash \Theta}{P \& Q \vdash \Theta} \tag{7.23}$$

and, conversely,

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \& Q}. \tag{7.24}$$

7. **Additive disjunction**: If P, Q are propositions, then

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \oplus Q} \tag{7.25}$$

and, conversely,

$$\frac{P \vdash \Theta \qquad Q \vdash \Theta}{P \oplus Q \vdash \Theta}. \tag{7.26}$$

8. **Multiplicative conjunction**: If P, Q are propositions, then

$$\frac{P, Q \vdash \Theta}{P \otimes Q \vdash \Theta} \tag{7.27}$$

and, conversely,

$$\frac{\Gamma \vdash P \qquad \Lambda \vdash Q}{\Gamma, \Lambda \vdash P \otimes Q}. \tag{7.28}$$

9. **Multiplicative disjunction**: If P, Q are propositions, then

$$\frac{\Gamma \vdash P, Q}{\Gamma \vdash P \parr Q} \tag{7.29}$$

and, conversely,

$$\frac{P \vdash \Delta \qquad Q \vdash \Theta}{P \parr Q \vdash \Theta}. \tag{7.30}$$

10. **Truth and falsity**:

$$
\begin{array}{cc}
\Gamma \vdash \top & \mathbf{0} \vdash \Theta \\[1ex]
\dfrac{\Gamma \vdash \Theta}{\Gamma, \mathbf{1} \vdash \Theta} & \vdash \mathbf{1} \\[2ex]
\dfrac{\vdash \Theta}{\vdash \Theta, \bot} & \bot \vdash\ .
\end{array}
\tag{7.31}
$$

11. **Exponential conjunction**: If P is a proposition, then

$$\frac{P \vdash \Theta}{!P \vdash \Theta} \tag{7.32}$$

and, conversely, whenever Γ consists solely of !-propositions and Θ consists solely of ?-propositions,

$$\frac{\Gamma \vdash P}{\Gamma \vdash !P}. \tag{7.33}$$

12. **Exponential disjunction**: If P is a proposition, then

$$\frac{\Gamma \vdash P}{\Gamma \vdash ?P} \tag{7.34}$$

and, conversely, whenever Γ consists solely of !-propositions and Θ consists solely of ?-propositions,

$$\frac{P \vdash \Theta}{?P \vdash \Theta}. \tag{7.35}$$

The inference rules with an asterisk are the structural rules. Note that a *cut-elimination theorem* holds and, hence, the identity and cut rules for general propositions can be derived from the rules above.

Linear implication is characterized as follows:

$$P \vdash Q \iff \vdash P^{\perp} \mathbin{⅋} Q \iff \vdash P \multimap Q\,. \tag{7.36}$$

Remark 7.4.1 (Resource theory). Before passing to the properties that follow from the basic rules and the categorical semantics of linear logic (eventually passing to linear type theory), it is useful to rephrase the connectives and their inference rules in terms of 'resources'.

In this interpretation, an implication $A \implies B$ would mean that the resources A can be used to obtain the resources B. However, in ordinary logic, if A and $A \implies B$ hold, then one can derive that B holds, but A also still holds, something that does not work with resources. If you can use resources A to construct B, the resources A are (usually[4]) used up.[5] One, hence, needs a more subtle and nuanced framework to capture these notions: linear logic. The implication that will be used, where resources are spent, is denoted by $A \multimap B$ for clarity. The two conjunctives, $\otimes$ and $\&$, mean that two resources are available concurrently and separately, respectively. So, if $A \multimap B$ holds, then, since the contraction rule is not valid, one does not have $A \multimap B \otimes B$. However, one does have $A \otimes A \multimap B \otimes B$.

The connectives in linear logic satisfy (or generalize) many of the properties of ordinary logic.

Property 7.4.2 (Distributivity).

$$\begin{aligned} P \otimes (Q \oplus R) &= (P \otimes Q) \oplus (P \otimes R) \\ P \parr (Q \& R) &= (P \parr Q) \& (P \parr R) \\ P \otimes \mathbf{0} &= \mathbf{0} \\ P \parr \top &= \top \end{aligned} \tag{7.37}$$

The exponential connectives can be used to turn additive connectives into multiplicative ones (and the other way around) as with the ordinary exponential function in calculus.

Property 7.4.3.

$$\begin{aligned} !(P \& Q) &= !P \otimes !Q \\ ?(P \oplus Q) &= !P \parr !Q \\ !\top &= \mathbf{1} \\ ?\mathbf{0} &= \bot \end{aligned} \tag{7.38}$$

Moreover, due to the apparent similarity with the operators in (S4) modal logic (Section 4.6), the exponential connectives are sometimes also called **modalities**.

Linear negation can also be defined alternatively.

Property 7.4.4 (Negation). Linear negation admits the following recursive definition:

- $P^{\perp\perp} = P$,
- $(P \& Q)^{\perp} = P^{\perp} \oplus Q^{\perp}$,
- $(P \otimes Q)^{\perp} = P^{\perp} \parr Q^{\perp}$,

[4]This is not the case with catalysts

[5]One could also give this a causal flavour (Girard, 1995).

- $\top^\perp = \mathbf{0}$,
- $\mathbf{1}^\perp = \perp$, and
- $(!P)^\perp = ?P^\perp$.

Property 7.4.5 (Categorical semantics). Whereas standard Boolean logic is the internal logic of Cartesian closed categories — where conjunction, disjunction and implication correspond, respectively to products, coproducts and internal homs — linear logic is the internal logic of (a subclass of) $*$-autonomous categories (Definition 3.4.8).

The multiplicative conjunction $\otimes$ corresponds to the tensor product, hence the notation. Similar to ordinary logic, the linear implication $\multimap$ corresponds to taking internal homs. The important part, now, is that negation comes as a separate entity, in this case given by taking duals: $x^\perp \equiv x^*$. The multiplicative disjunction $⅋$ is then constructed through Property 7.4.4 (which corresponds to de Morgan duality as in Property 3.4.9).

For the additive connectives, one needs the existence of finite products. The $*$-autonomy then also implies the existence of finite coproducts (again through de Morgan duality).

For the exponential connectives, some more structure is needed. As with modal logic, the structure is given by the existence of a suitable (co)monad.

List of Symbols

The following abbreviations and symbols are used throughout the compendium.

Abbreviations

AIC	Akaike information criterion
ARMA	autoregressive moving-average model
BCH	Baker–Campbell–Hausdorff
BPS	Bogomol'nyi–Prasad–Sommerfield
BPST	Belavin–Polyakov–Schwarz–Tyupkin
BRST	Becchi–Rouet–Stora–Tyutin
CCR	canonical commutation relation
CDF	cumulative distribution function
CFT	conformal field theory
CIS	completely integrable system
CP	completely positive
CPTP	completely positive, trace-preserving
CR	Cauchy–Riemann
dga	differential graded algebra
dgca	differential graded-commutative algebra
EMM	equivalent martingale measure
EPR	Einstein–Podolsky–Rosen
ESM	equivalent separating measure
ETCS	Elementary Theory of the Category of Sets
FIP	finite intersection property
FWHM	full width at half maximum
GA	geometric algebra
GHZ	Greenberger–Horne–Zeilinger

GNS	Gel′fand–Naimark–Segal
HJE	Hamilton–Jacobi equation
HoTT	Homotopy Type Theory
KKT	Karush–Kuhn–Tucker
LIVF	left-invariant vector field
MCG	mapping class group
MPO	matrix-product operator
MPS	matrix-product state
MTC	modular tensor category
NDR	neighbourhood deformation retract
OPE	operator product expansion
OTC	over the counter
OZI	Okubo–Zweig–Iizuka
PAC	probably approximately correct
PDF	probability density function
PID	principal ideal domain
PL	piecewise-linear
PMF	probability mass function
POVM	positive operator-valued measure
PRP	predictable representation property
PVM	projection-valued measure
RKHS	reproducing kernel Hilbert space
SVM	support-vector machine
TDSE	time-dependent Schrödinger equation
TISE	time-independent Schrödinger equation
TQFT	topological quantum field theory
TVS	topological vector space
UFD	unique factorization domain
VC	Vapnik–Chervonenkis
VIF	variance inflation factor
VOA	vertex operator algebra
WKB	Wentzel–Kramers–Brillouin

ZFC	Zermelo–Frenkel set theory with the axiom of choice

Operations

Ad_g	adjoint representation of a Lie group G
ad_X	adjoint representation of a Lie algebra $\mathfrak{g}$
arg	argument of a complex number
$\square$	d'Alembert operator
$\mathsf{deg}(f)$	degree of a polynomial f
e	identity element of a group
$\Gamma(E)$	set of global sections of a fibre bundle E
Im, $\mathfrak{I}$	imaginary part of a complex number
$\mathsf{Ind}_f(z)$	index of a point $z \in \mathbb{C}$ with respect to a function f
$\hookrightarrow$	injective function
$\cong$	is isomorphic to
$A \multimap B$	linear implication
$N \lhd G$	N is a normal subgroup of G
Par_t^γ	parallel transport map along a curve γ
Re, $\mathfrak{R}$	real part of a complex number
Res	residue of a complex function
$\twoheadrightarrow$	surjective function
$\{\cdot,\cdot\}$	Poisson bracket
$X \pitchfork Y$	transversally intersecting manifolds X, Y
∂X	boundary of a topological space X
$\overline{X}$	closure of a topological space X
$X^\circ, \mathring{X}$	interior of a topological space X
$\sphericalangle(\cdot,\cdot)$	angle between two vectors
$X \times Y$	cartesian product of two sets X, Y
$X + Y$	sum of two vector spaces X, Y
$X \oplus Y$	direct sum of two vector spaces X, Y
$V \otimes W$	tensor product of two vector spaces V, W
$\mathbb{1}_X$	identity morphism on an object X
$\approx$	is approximately equal to

$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto

Objects

Ab	category of Abelian groups
$\mathsf{Aut}(X)$	automorphism group of an object X
$\mathcal{B}_0(V,W)$	space of compact bounded operators between two Banach spaces V,W
$\mathcal{B}_1(\mathcal{H})$	space of trace-class operators on a Hilbert space
$\mathcal{B}(V,W)$	space of bounded linear maps between two vector spaces V,W
CartSp	category of Euclidean spaces and 'suitable' morphisms (e.g. linear maps, smooth maps, ...)
$C(X,Y)$	set of continuous functions between two topological spaces X,Y
S'	centralizer of a subset (of a ring)
$C_\bullet$	chain complex
$\mathbf{Ch}(\mathbf{A})$	category of chain complexes with objects in an additive category $\mathbf{A}$
$\mathbf{C}^\infty$, **SmoothSet**	category of smooth sets
$C^\infty_p(M)$	ring of smooth functions $f: M \to \mathbb{R}$ on a neighbourhood of $p \in M$
$C\ell(A,Q)$	Clifford algebra over an algebra A induced by a quadratic form Q
$C^\omega(V)$	set of all analytic functions defined on a set V
$\mathsf{Conf}(M)$	conformal group of a (pseudo-)Riemannian manifold M
$\mathbf{C}^\infty\mathbf{Ring}$, $\mathbf{C}^\infty\mathbf{Alg}$	category of smooth algebras
$S_k(\Gamma)$	space of cusp forms of weight $k \in \mathbb{R}$
Δ_X	diagonal of a set X
Diff	category of smooth manifolds
DiffSp	category of diffeological spaces and smooth maps
$\mathcal{D}_M$	sheaf of differential operators
D^n	standard n-disk
$\mathsf{dom}(f)$	domain of a function f
$\mathsf{End}(X)$	endomorphism monoid of an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad
$\mathbf{FormalCartSp}_{\mathbf{diff}}$	category of infinitesimally thickened Euclidean spaces

$\mathsf{Frac}(I)$	field of fractions of an integral domain I
$\mathfrak{F}(V)$	space of Fredholm operators on a Banach space V
$\mathbb{G}_a$	additive group (scheme)
$\mathsf{GL}(V)$	general linear group: group of automorphisms of a vector space V
$\mathsf{GL}(n, \mathfrak{K})$	general linear group: group of invertible $n \times n$-matrices over a field $\mathfrak{K}$
Grp	category of groups and group homomorphisms
Grpd	category of groupoids
$\mathsf{Hol}_p(\omega)$	holonomy group at a point p with respect to a principal connection ω
$\mathsf{Hom}_{\mathbf{C}}(V, W)$, $\mathbf{C}(V, W)$	collection of morphisms between two objects V, W in a category $\mathbf{C}$
hTop	homotopy category
$I(S)$	vanishing ideal on an algebraic set S
$I(x)$	rational fractions over an integral domain I
$\mathsf{im}(f)$	image of a function f
$K^0(X)$	K-theory over a (compact Hausdorff) space X
Kan	category of Kan complexes
$K(A)$	Grothendieck completion of a monoid A
$\mathcal{K}_n(A, v)$	Krylov subspace of dimension n generated by a matrix A and a vector v
L^1	space of integrable functions
Law	category of Lawvere theories
Lie	category of Lie groups
$\mathfrak{Lie}$	category of Lie algebras
$\mathfrak{X}^L$	space of left-invariant vector fields on a Lie group
$\mathsf{llp}(I)$	set of morphisms having the left lifting property with respect to I
LX	free loop space on a topological space X
$\mathbf{Man}^p$	category of C^p-manifolds
Meas	• category of measurable spaces and measurable functions, or • category of measure spaces and measure-preserving functions
M^4	four-dimensional Minkowski space
$M_k(\Gamma)$	space of modular forms of weight $k \in \mathbb{R}$
$\mathbb{F}^X$	natural filtration of a stochastic process $(X_t)_{t \in T}$
$N\mathbf{C}$	simplicial nerve of a small category $\mathbf{C}$

$\mathsf{O}(n, \mathfrak{K})$	group of $n \times n$ orthogonal matrices over a field $\mathfrak{K}$
$\mathbf{Open}(X)$	category of open subsets of a topological space X
$P(X), 2^X$	power set of a set X
$\mathsf{Pin}(V)$	pin group of the Clifford algebra $C\ell(V, Q)$
$\mathbf{Psh}(\mathbf{C}), \widehat{\mathbf{C}}$	category of presheaves on a (small) category $\mathbf{C}$
$R((x))$	ring of (formal) Laurent series in x with coefficients in R
$\mathsf{rlp}(I)$	set of morphisms having the right lifting property with respect to I
$R[[x]]$	ring of (formal) power series in x with coefficients in R
S^n	standard n-sphere
$S^n(V)$	space of symmetric rank n tensors over a vector space V
$\mathbf{Sh}(X)$	category of sheaves on a topological space X
$\mathbf{Sh}(\mathbf{C}, J)$	category of J-sheaves on a site $(\mathbf{C}, J)$
$\boldsymbol{\Delta}$	simplex category
$\mathsf{sing\ supp}(\phi)$	singular support of a distribution ϕ
$\mathsf{SL}_n(\mathfrak{K})$	special linear group: group of all $n \times n$-matrices with unit determinant over a field $\mathfrak{K}$
$W^{m,p}(U)$	Sobolov space in L^p of order m
$\mathbf{Span}(\mathbf{C})$	span category over a category $\mathbf{C}$
$\mathsf{Spec}(R)$	spectrum of a commutative ring R
$\mathbf{sSet}_{\text{Quillen}}$	Quillen's model structure on simplicial sets
$\mathsf{supp}(f)$	support of a function f
$\mathsf{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
$\mathsf{Sym}(X)$	symmetric group of a set X
S_n	symmetric group of degree n
$\mathsf{Sym}(X)$	symmetric group on a set X
$\mathsf{Sp}(n, \mathfrak{K})$	group of matrices preserving a canonical symplectic form over a field $\mathfrak{K}$
$\mathsf{Sp}(n)$	compact symplectic group
$\mathbb{T}^n$	standard n-torus (n-fold Cartesian product of S^1)
$T_{\leq t}$	set of all elements smaller than (or equal to) $t \in T$ for a partial order T
$\mathsf{TL}_n(\delta)$	Temperley–Lieb algebra with $n - 1$ generators and parameter δ
$\mathbf{Top}$	category of topological spaces and continuous functions
$\mathbf{Topos}$	(2-)category of (elementary) topoi and geometric morphisms

$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\mathsf{U}(n, \mathfrak{K})$	group of $n \times n$ unitary matrices over a field $\mathfrak{K}$
$V(I)$	algebraic set corresponding to an ideal I
$\mathbf{Vect}(X)$	category of vector bundles over a manifold X
$\mathbf{Vect}_{\mathfrak{K}}$	category of vector spaces and linear maps over a field $\mathfrak{K}$
Y^X	set of functions between two sets X, Y
$\mathbb{Z}_p$	group of p-adic integers
$\emptyset$	empty set
$\pi_n(X, x_0)$	n^{th} homotopy space over X with basepoint x_0
$[a, b]$	closed interval
$]a, b[$	open interval
$\Lambda^n(V)$	space of antisymmetric rank-n tensors over a vector space V
ΩX	(based) loop space on a topological space X
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on a manifold M
$\rho(A)$	resolvent set of a bounded linear operator A
$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on a manifold M

Units

C	Coulomb
T	Tesla

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