Compendium of Mathematics & Physics

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Contents

Chapter 1

Quantum Mechanics

The main reference for this chapter is [49]. In the first two sections the two basic formalisms of Quantum Mechanics are introduced: wave and matrix mechanics. The main reference for the mathematically rigorous treatment of quantum mechanics, in particular in the infinite-dimensional setting, is [105]. The main reference for the generalization to curved backgrounds [113]. Relevant chapters in this compendium are **??**, **??** and **??**.

1.1 Schrödinger picture

Formula 1.1.1 (Time-independent Schrödinger equation).

$$\hat{H}\psi(x) = E\psi(x) \tag{1.1}$$

The operator $\hat{H}$ is called the **Hamiltonian** of the system. The wave function ψ is an element of the vector space $L^2(\mathbb{R}, \mathbb{C}) \otimes \mathcal{H}$ with $\mathcal{H}$ the internal Hilbert space (describing for example the spin or charge of a particle). This is an eigenvalue equation for the energy levels of the system.

Property 1.1.2 (Orthogonality). Let $\{\psi_i\}_{i\in I} \subset L^2(\mathbb{R})$ be a collection of eigenfunctions of the TISE (the internal space is supressed for convenience). These functions can be normalized with respect to the inner product (**??**) such that they obey the following relation:

$$\int_{\mathbb{R}} \overline{\psi_i}(x)\psi_j(x)\,dx = \delta_{ij}. \tag{1.2}$$

The time evolution of a wave function is defined through the following equation:

Formula 1.1.3 (TDSE). The following partial differential equation is called the **(time-dependent) Schrödinger equation**:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t). \tag{1.3}$$

In case $\hat{H}$ is time-independent, the TISE can be obtained from this equation by separation of variables (see below).

Example 1.1.4 (Massive particle in a stationary potential).

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(\frac{\hat{p}^2}{2m} + \hat{V}(x)\right)\psi(x,t) \tag{1.4}$$

Derivation of TISE from TDSE. Starting from the one-dimensional TDSE in position space with a time-independent potential

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left(-\frac{\hbar^2}{2m}\partial_x{}^2 + V(x)\right)\Psi(x,t)\,, \tag{1.5}$$

one can perform a separation of variables and assert a solution of the form $\Psi(x,t) = \psi(x)T(t)$. Inserting this in the previous equation gives

$$i\hbar\psi(x)T'(t) = -\frac{\hbar^2}{2m}T(t)\psi''(x) + V(x)\psi(x)T(t). \tag{1.6}$$

Dividing both sides by $X(x)T(t)$ and rearranging the terms gives

$$i\hbar\frac{T'(t)}{T(t)} = \left(-\frac{\hbar^2}{2m} + V(x)\right)\frac{\psi''(x)}{\psi(x)}. \tag{1.7}$$

Because the left side only depends on t and the right side only depends on x, one can conclude that they are both equal to a constant E. This leads to the following system of differential equations:

$$\begin{cases} i\hbar T'(t) - ET(t) = 0\,, \\ \left(-\frac{\hbar^2}{2m} + V(x)\right)\psi''(x) = E\psi(x). \end{cases} \tag{1.8}$$

The first equation immediately gives a solution for T:

$$T(t) = Ce^{-\frac{i}{\hbar}Et}. \tag{1.9}$$

Rearranging the second solution in the system gives the TISE:

$$\psi''(x) = -\frac{2m}{\hbar^2}\big(E - V(x)\big)\psi(x). \tag{1.10}$$

□

Formula 1.1.5 (General solution). A general solution of the time-dependent Schrödinger equation (for time-independent Hamiltonians) is given by the following formula (cf. Formula **??**):

$$\psi(x,t) = \sum_E c_E\psi_E(x)e^{-\frac{i}{\hbar}Et}, \tag{1.11}$$

where the functions $\psi_E(x)$ are the eigenfunctions of the TISE 1.1.1. The coefficients c_E can be found using the orthogonality relations

$$c_E = \left(\int_{\mathbb{R}} \overline{\psi_E}(x)\psi(x,t_0)\,dx\right)e^{\frac{i}{\hbar}Et_0}. \tag{1.12}$$

?? COMPLETE ??

1.1.1 Hydrogen atom

Consider the hydrigen atom, i.e. a single proton (the nucleus) orbited by a single electron with only the electrostatic Coulomb force acting between them (gravity can safely be neglected):

$$\hat{H} := \frac{\hat{p}_p^2}{2m_p} + \frac{\hat{p}_e^2}{2m_e} - \frac{e^2}{4\pi\varepsilon r^2}. \tag{1.13}$$

It is not hard to see that this is the quantum mechanical version of the Kepler problem (Section **??**). The special property of the Kepler problem was that it contained a "hidden" symmetry that gave rise to the conserved Laplace-Runge-Lenz vector **??**. As is the case for all conserved charges in quantum mechanics, this symmetry induces a degeneracy of the energy eigenvalues. Degeneracy of the magnetic quantum number m follows from rotational symmetry, but the energy levels of the hydrogen atom only depend on the principal quantum number n. The degeneracy of the total angular quantum number l is that due to the "hidden" SO(4)-symmetry. It is often called an "accidental degeneracy" for this reason.

?? COMPLETE ??

1.1.2 Molecular dynamics

Consider the Hamiltonian of two interacting atoms:

$$\hat{H} := \frac{\hat{P}_1^2}{2M_1} + \frac{\hat{P}_2^2}{2M_2} + \frac{q_1 q_2}{4\pi\varepsilon R^2} + \sum_i \frac{\hat{p}_i^2}{2m} - \frac{eq_1}{4\pi\varepsilon r_{i1}^2} - \frac{eq_2}{4\pi\varepsilon r_{i2}^2} + \sum_{i\neq j} \frac{e^2}{4\pi\varepsilon r_{ij}^2}\,, \tag{1.14}$$

where the indices i, j indicate the electrons and capital symbols denote operators associated to the nuclei.

Except for the most simple situations, solving the Schrödinger equation for this Hamiltonian becomes intractable (both analytically and numerically). However, in general one can approximate the situation. The mass of nuclei are much larger than those of the electrons and this influences their motion, they move much slower than the electrons. In essence the nuclei and electrons live on different time scales and this allows to decouple their dynamics:

$$\hat{H}_{\text{nucl}} = \frac{\hat{P}_1^2}{2M_1} + \frac{\hat{P}_2^2}{2M_2} + \frac{Q_1 Q_2}{4\pi\varepsilon R^2} + V_{\text{eff}}(R_1, R_2). \tag{1.15}$$

The electrons generate an effective potential for the nuclei and the Schrödinger equation decouples as follows:

$$\hat{H}_{\text{nucl}}(R)\psi(R) = E\psi(R)\,, \tag{1.16}$$

$$\hat{H}_{\text{el}}(r, R)\phi(r, R) = E_{\text{el}}\phi(r, R). \tag{1.17}$$

This is the so-called **Born-Oppenheimer approximation**. From a more modern physical perspective this approximation can also be seen to be a specific instance of renormalization theory, where the short time-scale (or, equivalently, the high energy-scale) degrees of freedom are integrated out of the theory.

1.2 Heisenberg picture

This is the right place to reflect on what the wave function is and how it relates to the state of a system. At every point it gives the probability of observing a particle (or whatever object is being studied). But what if one wants to express this information in terms of momenta instead of positions? The information about the state should not depend on the chosen "representation". To this end, a state vector $|\psi\rangle$ that represents the state of the system as an abstract vector in some Hilbert space is introduced.

Notation 1.2.1 (Dirac notation). This notation is often called the **braket notation**. State vectors $|\psi\rangle$ are called **ket**'s and their duals $\langle\psi|$ are called **bra**'s. The inner product of a state $|\phi\rangle$ and a state $|\psi\rangle$ is denoted by $\langle\phi|\psi\rangle$.

But then, how does one recover the position (configuration) representation $\psi(x)$? This is simply the projection of the state vector $|\psi\rangle$ on the "basis function" $\delta(x)$, i.e. $\psi(x)$ represents an expansion coefficient in terms of a "basis" for the physical Hilbert space. In the same way one can obtain the momentum representation $\psi(p)$ by projecting on the plane waves e^{ipx}.

Remark 1.2.2. It should be noted that neither the "basis states" $\delta(x)$, nor the plane waves e^{ipx} are square-integrable and, hence, they are not elements of the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$. In the next chapter this issue will be resolved through the concept of *rigged Hilbert spaces*.

Formula 1.2.3 (Matrix representation). The following formula gives the matrix representation of an operator $\hat{A}$ with respect to the orthonormal basis $\{|\psi_i\rangle\}_{i\leq n}$ cf. Construction **??**:

$$A_{ij} := \langle\psi_i|\hat{A}|\psi_j\rangle. \tag{1.18}$$

Remark 1.2.4. The basis $\{|\psi_i\rangle\}_{i\leq n}$ need not consist out of eigenfunctions of $\hat{A}$.

?? COMPLETE ??

1.3 Uncertainty principle

Definition 1.3.1 (Compatible observables). Two observables are said to be compatible if they share a complete set of eigenvectors.

Definition 1.3.2 (Expectation value). The expectation value of an operator $\hat{A}$ in a state $|\psi\rangle$ is defined as

$$\langle\hat{A}\rangle_\psi := \langle\psi|\hat{A}|\psi\rangle. \tag{1.19}$$

The subscript ψ is often left implicit. As in ordinary statistics (**??**), the uncertainty or variance is defined as follows:

$$\Delta A := \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2. \tag{1.20}$$

Formula 1.3.3 (Uncertainty relation). Let $\hat{A}, \hat{B}$ be two operator and let $\Delta A, \Delta B$ be the corresponding uncertainties. The (**Robertson**) uncertainty relation reads as follows:

$$\Delta A\Delta B \geq \frac{1}{4}\left|\left\langle\left[\hat{A}, \hat{B}\right]\right\rangle\right|^2. \tag{1.21}$$

1.4 Angular Momentum

1.4.1 Angular momentum operator

Property 1.4.1 (Lie algebra). The angular momentum operators generate a Lie algebra **??**. The Lie bracket is defined by the following commutation relation:

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\varepsilon_{ijk}\hat{J}_k. \tag{1.22}$$

Since rotations correspond to actions of the orthogonal group SO(3) it should not come as a surprise that the above relation is that of the Lie algebra $\mathfrak{so}(3)$ from Example **??**.

Property 1.4.2. The mutual eigenbasis of $\hat{J}^2$ and $\hat{J}_z$ is defined by the following two eigenvalue equations:

$$\hat{J}^2|j,m\rangle = j(j+1)\hbar^2\,|j,m\rangle \tag{1.23}$$

$$\hat{J}_z|j,m\rangle = m\hbar\,|j,m\rangle. \tag{1.24}$$

Definition 1.4.3 (Ladder operators[1]). The raising and lowering operators $\hat{J}_+$ and $\hat{J}_-$ are defined as follows:

$$\hat{J}_+ := \hat{J}_x + i\hat{J}_y \qquad \text{and} \qquad \hat{J}_- := \hat{J}_x - i\hat{J}_y. \tag{1.25}$$

These operators only change the quantum number m_z, not the total angular momentum.

Corollary 1.4.4. From the commutation relations of the angular momentum operators, one can derive the commutation relations of the ladder operators:

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z. \tag{1.26}$$

Formula 1.4.5. The total angular momentum operator $\hat{J}^2$ can now be expressed in terms of $\hat{J}_z$ and the ladder operators using the commutation relation (1.22):

$$\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z. \tag{1.27}$$

Remark 1.4.6 (Casimir operator). From the definition of $\hat{J}^2$ it follows that this operator is a Casimir invariant **??** of $\mathfrak{so}(3)$.

1.4.2 Rotations

Formula 1.4.7. An infinitesimal rotation $\hat{R}(\delta\vec{\varphi})$ is given by the following formula:

$$\hat{R}(\delta\vec{\varphi}) = \mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}} \cdot \delta\vec{\varphi}. \tag{1.28}$$

A finite rotation can be generated by applying this infinitesimal rotation repeatedly:

$$\hat{R}(\vec{\varphi}) = \left(\mathbb{1} - \frac{i}{\hbar}\vec{\boldsymbol{J}} \cdot \frac{\vec{\varphi}}{n}\right)^n = \exp\left(-\frac{i}{\hbar}\vec{\boldsymbol{J}} \cdot \vec{\varphi}\right). \tag{1.29}$$

Formula 1.4.8 (Matrix elements). Applying a rotation over an angle φ about the z-axis to a state $|j, m\rangle$ gives

$$\hat{R}(\varphi\vec{\boldsymbol{e}}_z)|j, m\rangle = \exp\left(-\frac{i}{\hbar}\hat{J}_z\varphi\right)|j, m\rangle = \exp\left(-\frac{i}{\hbar}m\varphi\right)|j, m\rangle. \tag{1.30}$$

Multiplying these states with a bra $\langle j', m'|$ and using the orthonormality of the eigenstates gives the matrix elements of the rotation operator:

$$\hat{R}_{ij}(\varphi\vec{\boldsymbol{e}}_z) = \exp\left(-\frac{i}{\hbar}m\varphi\right)\delta_{jj'}\delta_{mm'}. \tag{1.31}$$

From the expression of the angular momentum operators and the rotation operator it is clear that a general rotation has no effect on the total angular momentum number j. This means that the rotation matrix will be block diagonal with respect to j. This amounts to the following reduction of the representation of the rotation group:

$$\langle j, m'|\hat{R}(\varphi\vec{\boldsymbol{n}})|j, m\rangle = \mathcal{D}^{(j)}_{m,m'}(\hat{R})\,, \tag{1.32}$$

where the functions $\mathcal{D}^{(j)}_{m,m'}(\hat{R})$ are called the **Wigner D-functions**. For every value of j there are $(2j+1)$ values for m. This implies that the matrix $\mathcal{D}^{(j)}(\hat{R})$ is a $(2j+1) \times (2j+1)$-matrix.

[1] Also called the **creation** and **annihilation** operators (especially in quantum field theory).

1.4.3 Spinor representation

Definition 1.4.9 (Pauli matrices).

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.33}$$

From this definition it is clear that the Pauli matrices are Hermitian and unitary. Together with the 2×2 identity matrix, they form a basis for the space of 2×2 Hermitian matrices. For this reason the identity matrix is often denoted by σ_0 (especially in the context of relativistic QM).

Formula 1.4.10. In the spinor representation ($J = \frac{1}{2}$) the Wigner-D matrix reads as follows:

$$\mathcal{D}^{(1/2)}(\varphi\vec{e}_z) = \begin{pmatrix} e^{-i/2\varphi} & 0 \\ 0 & e^{i/2\varphi} \end{pmatrix}. \tag{1.34}$$

1.4.4 Coupling of angular momenta

Due to the tensor product structure of a coupled Hilbert space, the angular momentum operator $\hat{J}_i$ should now be interpreted as $\mathbb{1} \otimes \cdots \otimes \hat{J}_i \otimes \cdots \otimes \mathbb{1}$ (cf. Notation **??**). Because the angular momentum operators $\hat{J}_{l\neq i}$ do not act on the space $\mathcal{H}_i$, one can pull these operators through the tensor product:

$$\hat{J}_i|j_1\rangle \otimes \cdots \otimes |j_n\rangle = |j_1\rangle \otimes \cdots \otimes \hat{J}_i|j_i\rangle \otimes \cdots \otimes |j_n\rangle.$$

The basis used above is called the **uncoupled basis**.

For simplicity the total Hilbert space is from here on assumed to be that of a two-particle system. Let $\hat{J}$ denote the total angular momentum defined as

$$\hat{J} = \hat{J}_1 + \hat{J}_2. \tag{1.35}$$

With this operator one can define a **coupled** state $|J, M\rangle$, where M is the total magnetic quantum number which ranges from $-J$ to J.

Formula 1.4.11 (Clebsch-Gordan coefficients). Because both bases (coupled and uncoupled) span the total Hilbert space $\mathcal{H}$, there exists an invertible transformation between them. The transformation coefficients can be found by using the resolution of the identity:

$$|J, M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2|J, M\rangle. \tag{1.36}$$

These coefficients are called the Clebsch-Gordan coefficients.

Property 1.4.12. By acting with the operator $\hat{J}_z$ on both sides of Equation (1.36) it is possible to prove that the Clebsch-Gordan coefficients are nonzero if and only if $M = m_1 + m_2$.

1.5 Mathematical Formalization

1.5.1 Postulates

Axiom 1.1 (State spaces). The states of a (closed) system are represented by vectors in a (complex) Hilbert space. In the infinite-dimensional setting one often further restricts to separable spaces, i.e. the spaces are required to admit a countable Hilbert basis.

Axiom 1.2 (Observables). A self-adjoint operator. In the finite-dimensional case this is equivalent to an operator that admits a complete set of eigenfunctions.

Axiom 1.3 (Rays). The dynamics of the system do not depend on the global phase or normalization, states are represented by rays in a projective Hilbert space.

1.5.2 Observables

Formula 1.5.1. Let $|\Psi\rangle$ be a state vector representing a given system and let $\{|\psi_i\rangle\}_{i\in I}$ be a complete set of eigenvectors of some observable of the system. The state vector $|\Psi\rangle$ can be expressed as a linear combination of the eigenfunctions:

$$|\Psi\rangle = \sum_i c_i|\psi_i\rangle + \int c_a|\psi_a\rangle\, da\,, \tag{1.37}$$

where the summation ranges over the discrete spectrum and the integral over the continuous spectrum (Section **??**). Note that this expression only makes sense formally, since linear combinations only consist of a finite number of terms, i.e. c_a should be a finite sum of delta functionals.

Formula 1.5.2 (Closure relation). For a complete set of discrete eigenvectors the closure relation (also called the **resolution of the identity**) is given by

$$\sum_n |\psi_n\rangle\langle\psi_n| = \mathbb{1}. \tag{1.38}$$

For a complete set of continuous eigenvectors the following counterpart holds:

$$\int |x\rangle\langle x|\, dx = \mathbb{1}. \tag{1.39}$$

For a mixed set of eigenvectors a similar relation is obtained by summing over the discrete part and integrating over the continuous part. For simplicity the notation of Equation (1.38) will also be used for the continuous part.

Definition 1.5.3 (Canonical commutation relations). Two observables A, B are said to obey a canonical commutation relation (CCR) if they satisfy (up to a constant factor $\hbar$)

$$[A, B] = i. \tag{1.40}$$

The prime examples are the position and momentum operators $\hat{x}, \hat{p}$. Through functional calculus one can also define the exponential operators e^{isA} and e^{itB}. The above relation then induces the so-called **Weyl form** of the CCR:

$$e^{isA}e^{itB} = e^{ist}e^{itB}e^{isA}. \tag{1.41}$$

Theorem 1.5.4 (Stone-von Neumann). *All pairs of irreducible, unitary, one-parameter subgroups satisfying the Weyl form of the CCRs are unitarily equivalent.*

Corollary 1.5.5. The Schrödinger and Heisenberg pictures are unitarily equivalent.

In fact one can generalize the Weyl relation:

Definition 1.5.6 (Weyl system). Let (A, ω) be a symplectic vector space and let $\mathcal{H}$ be a Hilbert space equipped with a continuous map $W : A \to \mathrm{U}(\mathcal{H})$. This data defines a Weyl system if the following equality is satisfied for all $v, v' \in A$:

$$W(v)w(v') = e^{i\omega(v,v')/2}W(v + v'). \tag{1.42}$$

The relation itself is called a **Weyl relation**.

For every vector $v \in A$, the map $t \mapsto W(tv)$ is a continuous unitary one-parameter subgroup, so by Stone's theorem **??** one obtains a self-adjoint generator $\phi(v)$. The map $v \mapsto \phi(v)$ is called the associated **Heisenberg system**.

Remark 1.5.7. It should be noted that the Weyl relations are more fundamental than their infinitesimal counterpart. Only the Weyl relations are well-defined on more general spaces and when passing to a relativistic setting.

1.5.3 Symmetries

Definition 1.5.8 (State space). By the postulates of quantum mechanics, the states in a quantum theory are represented by rays in the projective Hilbert space $\mathbb{P}\mathcal{H}$. Probabilities are defined through the *Fubini-Study metric* on $\mathbb{P}\mathcal{H}$ as follows:

$$\mathcal{P}(\psi,\phi) := \cos^2\left[d_{\mathrm{FS}}(\psi,\phi)\right] = \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}\,, \tag{1.43}$$

where $|\psi\rangle, |\phi\rangle$ are representatives of the states ψ,ϕ in $\mathbb{P}\mathcal{H}$.

Definition 1.5.9 (Symmetry). A quantum symmetry (or **quantum automorphism**) is an isometric automorphism of $P\mathcal{H}$. The group of these symmetries is denoted by $\mathrm{Aut}_{\mathrm{QM}}(\mathbb{P}\mathcal{H})$.

The following theorem due to *Wigner* gives a (linear) characterization of quantum symmetries:[2]

Theorem 1.5.10 (Wigner). *Every quantum automorphism of $\mathbb{P}\mathcal{H}$ is induced by a unitary or anti-unitary operator on $\mathcal{H}$.*

This is equivalent to saying that the group morphism

$$\pi : \mathrm{Aut}(\mathcal{H},\mathcal{P}) := \mathrm{U}(\mathcal{H}) \times \mathrm{AU}(\mathcal{H}) \to \mathrm{Aut}_{\mathrm{QM}}(\mathbb{P}\mathcal{H})$$

is surjective. Together with the kernel $\mathrm{U}(1)$, given by phase shifts, this forms a short exact sequence:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathrm{Aut}_{\mathrm{QM}}(\mathbb{P}\mathcal{H}) \longrightarrow 1. \tag{1.44}$$

In the case of symmetry breaking (e.g. lattice systems), the full symmetry group is reduced to a subgroup $G \subset \mathrm{Aut}_{\mathrm{QM}}(\mathbb{P}\mathcal{H})$. The group of operators acting on $\mathcal{H}$ is then given by the pullback $\widetilde{G}$ of the diagram

$$\mathrm{Aut}(\mathcal{H},\mathcal{P}) \longrightarrow \mathrm{Aut}_{\mathrm{QM}}(\mathbb{P}\mathcal{H}) \longleftarrow G.$$

It should also be noted that the kernel of the homomorphism $\widetilde{G} \to G$ is again $\mathrm{U}(1)$. This leads to the property that $\widetilde{G}$ is a $\mathbb{Z}_2$-twisted (hence noncentral) $\mathrm{U}(1)$-extension of G (where the twist is induced by the homomorphism $\phi : \mathrm{Aut}(\mathcal{H},\mathcal{P}) \to \mathbb{Z}_2$ that says whether an operator is unitary or anti-unitary).

?? COMPLETE ??

1.5.4 Symmetric states

Axiom 1.4 (Symmetrization postulate). Let $\mathcal{H}$ be the single-particle Hilbert space. A system of n identical particles is described by a state $|\Psi\rangle$ belonging to either $S^n\mathcal{H}$ or $\Lambda^n\mathcal{H}$, i.e. **bosonic** and **fermionic** states are of the form

$$|\Psi_B\rangle = \sum_{\sigma\in S_n} |\psi_{\sigma(1)}\rangle\cdots|\psi_{\sigma(n)}\rangle \tag{1.45}$$

and

$$|\Psi_F\rangle = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)|\psi_{\sigma(1)}\rangle\cdots|\psi_{\sigma(n)}\rangle, \tag{1.46}$$

respectively, where the $|\psi_i\rangle$ are single-particle states and S_n is the permutation group on n elements.

[2] It is a particular case of a more general theorem in projective geometry.

Remark 1.5.11. In ordinary quantum mechanics this is a postulate, but in quantum field theory this is a consequence of the *spin-statistics theorem.*

Definition 1.5.12 (Slater determinant). Let $\{\phi_i(\vec{\boldsymbol{q}})\}_{i\leq n}$ be a set of wave functions, called **spin orbitals**, describing a system of n identical fermions. The totally antisymmetric wave function of the system is given by

$$\psi(\vec{\boldsymbol{q}}_1,\ldots,\vec{\boldsymbol{q}}_n) = \frac{1}{\sqrt{n!}}\det\begin{pmatrix}\phi_1(\vec{\boldsymbol{q}}_1) & \cdots & \phi_n(\vec{\boldsymbol{q}}_1)\\ \vdots & \ddots & \vdots\\ \phi_1(\vec{\boldsymbol{q}}_n) & \cdots & \phi_n(\vec{\boldsymbol{q}}_n)\end{pmatrix} \tag{1.47}$$

A similar function can be defined for bosonic systems using the concept of *permanents.*

1.6 Curved backgrounds ♣

Using the tools of distribution theory and differential geometry (Chapters **??**, **??** and onwards), one can introduce quantum mechanics on curved backgrounds (in the sense of "space", not "spacetime").

Remark 1.6.1 (Rigged Hilbert spaces). A first important remark to be made is that the classical definition of the wave function as an element of $L^2(\mathbb{R}^d,\mathbb{C})$ is not sufficient, even in flat Cartesian space. A complete description requires the introduction of so-called *Gelfand triples* or *rigged Hilbert spaces*, where the space of square-integrable functions is replaced by the Schwartz space **??** of rapidly decreasing functions. The linear functionals on this space are then given by the tempered distributions.

When working on curved spaces or even in non-Cartesian coordinates on flat space, one can encounter problems with the definition of the self-adjoint operators $\hat{q}^i$ and $\hat{p}_i$. The naive definition $\hat{q}^i = q^i, \hat{p}_i = -i\partial_i$ gives rise to extra terms that break the canonical commutation relations and the self-adjointness of the operators (e.g. the angular position operator $\hat{\varphi}$ on the circle together with its conjugate $\hat{L}$) when calculating inner products.

An elegant solution to this problem is obtained by giving up the definition of the wave function as a well-defined function $\psi:\mathbb{R}^d\rightarrow\mathbb{C}$. Assume that the physical space has the structure of a Riemannian manifold (M,g) and that the "naive" wave functions take values in a vector space V. Then, construct a vector bundle E with typical fibre V over M. By Property **??** an invariant description of the "true" wave function is a map $\Psi: F(E)\rightarrow V$ or, locally, the pullback $\psi := \varphi^*\Psi$ for some local section $\varphi: U\subseteq M\rightarrow F(E)$. The Levi-Civita connection on M also induces a covariant derivative ∇ on E that can be used to define differential operators.

Now, a general inner product can be introduced:

$$\langle\psi,\phi\rangle := \int \overline{\psi(x)}\phi(x)\,dx. \tag{1.48}$$

Because the factor $\sqrt{\det(g)}$ transforms in the inverse manner of the measure dx, the integrand is invariant under coordinate transforms (something that is generally required of physical laws). Using this new inner product one can for example check the self-adjointness of the momentum operator $\hat{P}_i := -i\nabla_i$:

$$
\begin{aligned}
\langle\psi, \hat{P}_i\phi\rangle &= \int \overline{\psi(x)}(-i\nabla_i)\phi(x)\sqrt{\det(g)}\,dx \\
&\overset{(??)}{=} \int \overline{\psi(x)}(-i\partial_i - i\omega_i)\phi(x)\sqrt{\det(g)}\,dx \\
&= \int \overline{(-i\partial_i\psi)(x)}\phi(x)\sqrt{\det(g)}\,dx + i\int \overline{\psi(x)}\phi(x)\Big(\partial_i\sqrt{\det(g)}\Big)\,dx \\
&\qquad - i\int \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx \\
&= \langle\hat{P}_i\psi, \phi\rangle - i\int \overline{\psi(x)}\overline{\omega_i}\phi(x)\sqrt{\det(g)}\,dx \\
&\qquad + i\int \overline{\psi(x)}\phi(x)\Big(\partial_i\sqrt{\det(g)}\Big)\,dx \\
&\qquad - i\int \overline{\psi(x)}\omega_i\phi(x)\sqrt{\det(g)}\,dx.
\end{aligned}
$$

Self-adjointness then requires that

$$\sqrt{\det(g)}(\omega_i + \overline{\omega_i}) = \partial_i\sqrt{\det(g)} \tag{1.49}$$

or

$$2\mathrm{Re}(\omega_i) = \partial_i \ln\Big(\sqrt{\det(g)}\Big). \tag{1.50}$$

?? COMPLETE ??

1.7 Topos theory ♣

Definition 1.7.1 (Bohr topos). Consider a C^*-algebra A of bounded observables on a Hilbert space $\mathcal{H}$. Denote by $\mathrm{Pos}(A)$ the poset **??** of commutative C^*-subalgebras. This set can be equipped with the **Alexandroff topology**, i.e. the topology for which the open sets are the upward closes subsets. The topological space $(\mathrm{Pos}(A), \tau_{\mathrm{Alex}})$ is called the Bohr site of A.

The sheaf topos over the Bohr site is called the Bohr topos. It can be turned into a ringed topos, where the internal ring object (even an internal C^*-algebra) is given by the tautological functor

$$A : \mathrm{Pos}(A) \to \mathbf{Set}(A) : C \mapsto C. \tag{1.51}$$

Chapter 2

Dirac Equation

References for this chapter are [39]. (Note that the authors use the mostly-pluses signature there.) For the mathematical background of Clifford algebras and Spin groups, see Chapter **??** and, in particular, Section **??**. For the extension to (pseudo-)Riemannian manifolds, see Section **??**.

2.1 Dirac matrices

Definition 2.1.1 (Dirac matrices). The Dirac (or **gamma**) matrices are defined by the following equality:

$$\{\gamma^\mu, \gamma^\nu\}_+ = 2\eta^{\mu\nu}\mathbb{1}, \tag{2.1}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. This has the form of Equation (**??**), i.e. the Dirac matrices form the generating set of a Clifford algebra, called the **Dirac algebra**.

There exist multiple distinct representations of the Clifford generators in signature $(1,3)$. The first one is called the **Dirac representation**. Here, the timelike Dirac matrix γ^0 is defined as

$$\gamma^0 := \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \tag{2.2}$$

The spacelike Dirac matrices γ^k $(k = 1,2,3)$ are defined using the Pauli matrices 1.4.9 σ^k:

$$\gamma^k := \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}. \tag{2.3}$$

The **Weyl** or **chiral** representation[1] is defined by replacing the timelike matrix γ^0 by

$$\gamma^0 := \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{2.4}$$

In signature $(3,1)$ one obtains the Weyl representation by defining $\sigma^\mu := (\mathbb{1}, \sigma_i)$ and $\overline{\sigma}^\mu := \sigma_\mu$:

$$\gamma^\mu := \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix}. \tag{2.5}$$

Remark 2.1.2. In the remainder of this compendium the Weyl representation will be used.

[1]This representation is widely used in quantum field theory and supergravity.

Notation 2.1.3 (Feynman slash notation). Let $\mathbf{a} \equiv a^\mu \mathbf{e}_\mu \in M^4$ be a general 4-vector. The Feynman slash is defined as follows:

$$\not{a} := a^\mu \gamma_\mu. \tag{2.6}$$

In fact this is just the embedding of Minkowski space in its Clifford algebra:

$$/ : M^4 \to C\ell(M^4, \eta) : a^\mu \mathbf{e}_\mu \mapsto a^\mu \gamma_\mu. \tag{2.7}$$

2.2 Spinors

2.2.1 Dirac equation

Formula 2.2.1 (Dirac equation). In covariant form the Dirac equation reads as

$$(i\hbar \not{\partial} - mc)\psi = 0, \tag{2.8}$$

where m denotes the mass and c denotes the speed of light.

Definition 2.2.2 (Dirac adjoint).

$$\overline{\psi} := i\psi^\dagger \gamma^0 \tag{2.9}$$

When working in the Dirac representation the factor i should be dropped.

Definition 2.2.3 (Majorana adjoint). In the context of SUSY it is often convenient to work with a different adjoint spinor. Let $\mathcal{C} := i\gamma^3\gamma^1$ denote the charge conjugation operator. The Majorana adjoint is defined by

$$\overline{\psi} := \psi^t C. \tag{2.10}$$

Formula 2.2.4 (Parity). The parity operator is defined as follows:

$$\hat{P}(\psi) = \gamma^0 \psi. \tag{2.11}$$

2.2.2 Chiral spinors

In even dimensions one can define an additional matrix that satisfies Equation (2.1):

Definition 2.2.5 (Chiral matrix). Assume that the dimension, given by $d = m + n$, is even. The chiral (helicity) matrix can be defined as follows:[2]

$$\gamma_{d+1} := \gamma_1 \gamma_2 \cdots \gamma_d. \tag{2.12}$$

In odd dimensions ($d = 2m + 1$) a generating set for the Clifford algebra can be obtained by taking the generating set from one dimension lower and adjoining the element $k\gamma_*$ where $k^2 = (-1)^{n+d/2}$. This gives two inequivalent representations of the Clifford algebra (depending on the sign). From here on the following redefinition will be used:

$$\gamma_{d+1} \longrightarrow k\gamma_{d+1}. \tag{2.13}$$

This has the benefit that $\gamma_{d+1}^2 = \mathbb{1}$.

In $d = 3 + 1$ one generally takes the following representation for γ_5:[3]

$$\gamma_{d+1} := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{2.14}$$

[2]Some authors add a constant to this definition.

[3]Such a block diagonal form can always be chosen by working in a *helicity-adapted basis*.

Definition 2.2.6 (Chiral projection). The chiral projections of a spinor ψ are defined as follows:

$$\psi_L := \frac{1+\gamma_{d+1}}{2}\psi \tag{2.15}$$

and

$$\psi_R := \frac{1-\gamma_{d+1}}{2}\psi. \tag{2.16}$$

Every spinor can then be written as a sum of its chiral parts:

$$\psi = \psi_L + \psi_R. \tag{2.17}$$

2.2.3 Dirac algebra in $d = 4$

For a lot of calculations, especially in quantum electrodynamics, one needs the properties of the gamma matrices. The most relevant relations in $d = 3 + 1$ are listed below:

Formula 2.2.7 (Trace algebra).

$$\mathrm{tr}(\gamma^\mu) = \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0 \tag{2.18}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu} \tag{2.19}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda) = 4(\eta^{\mu\nu}\eta^{\kappa\lambda} - \eta^{\mu\kappa}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\kappa}) \tag{2.20}$$

$$\mathrm{tr}(\gamma^5) = \mathrm{tr}(\gamma^\mu\gamma^5) = \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5) = \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5) = 0 \tag{2.21}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda\gamma^5) = -4i\varepsilon^{\mu\nu\kappa\lambda} \tag{2.22}$$

$$\mathrm{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_k}) = \mathrm{tr}(\gamma^{\mu_k}\cdots\gamma^{\mu_1}) \tag{2.23}$$

Formula 2.2.8 (Contraction identities).

$$\gamma^\mu\gamma_\mu = 4 \tag{2.24}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{2.25}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho} \tag{2.26}$$

$$\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda\gamma_\mu = -2\gamma^\lambda\gamma^\kappa\gamma^\nu \tag{2.27}$$

2.2.4 Fierz identities

Using a spinor $u \in S$ and a cospinor $\overline{v} \in S^*$ one can build a bilinear form $\overline{v}u$. However, for two spinors u, ω and two cospinors $\overline{v}, \overline{\rho}$ one can interpret the expression $(\overline{v}u)(\overline{\rho}\omega)$ either as a quadrilinear form on $u \otimes \overline{v} \otimes \omega \otimes \overline{\rho}$ or as a quadrilinear form on $\omega \otimes \overline{v} \otimes u \otimes \overline{\rho}$. Because $C\ell_{3,1}(\mathbb{C})$ is isomorphic to the endomorphism ring on S, there must exist coefficients a^{ij} where $i, j = 1, \ldots, 2^D$ such that

$$(\overline{v}u)(\overline{\rho}\omega) = \sum_{i,j=1}^{2^d} a^{ij}(\overline{v}\gamma_i\omega)(\overline{\rho}\gamma_j u). \tag{2.28}$$

By using the trace orthogonality relations one can find that

$$\alpha^{ij} = \begin{cases} 0 & i \neq j \\ \frac{1}{2^{\lfloor d/2 \rfloor}} & i = j. \end{cases} \tag{2.29}$$

The above equality can then also be rewritten as follows:

$$\delta^a_b\delta^c_d = \frac{1}{2^{\lfloor d/2 \rfloor}}\sum_{i=1}^{2^d}(\gamma_i)^a_d(\gamma_i)^c_b. \tag{2.30}$$

This expression (and the techniques used to find it) allow one to rearrange almost all multilinear expressions involving spinors and cospinors.

Chapter 3

Quantum Information Theory

3.1 Entanglement

3.1.1 Schmidt decomposition

Construction 3.1.1 (Schmidt decomposition). Consider a bipartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. For any such state there exist orthonormal sets $\{|e_i\rangle, |f_j\rangle\}_{i,j\leq\kappa}$ such that

$$|\psi\rangle = \sum_{i=1}^{\kappa} \lambda_i |e_i\rangle \otimes |f_i\rangle, \tag{3.1}$$

where the coefficients λ_i are nonnegative real numbers. All objects in this expression can be obtained from a singular value decomposition of the coefficient matrix $\mathbf{C}$ of $|\psi\rangle$ in some bases of $\mathcal{H}_1$ and $\mathcal{H}_2$. The number κ is called the **Schmidt rank** of $|\psi\rangle$.

Definition 3.1.2 (Entangled states). Consider a state $|\psi\rangle$ and consider its Schmidt decomposition. If the Schmidt rank is 1, i.e. the state can be written as $|\psi\rangle = |v\rangle \otimes |w\rangle$, the state is said to be **separable**. Otherwise the state is said to be entangled.

3.1.2 Bell states

Definition 3.1.3 (Bell state). A (binary) Bell state (also called a **cat state** or **Einstein-Podolsky-Rosen pair**) is defined as the following entangled state:

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big). \tag{3.2}$$

In fact this state can be extended to a full maximally entangled basis for the 2-qubit Hilbert space:

$$\begin{aligned} |\Phi^-\rangle &:= \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) \\ |\Psi^+\rangle &:= \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big) \\ |\Psi^-\rangle &:= \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big). \end{aligned} \tag{3.3}$$

[1] Sometimes called **superdense coding**.

Method 3.1.4 (Dense coding[1]). Consider the Bell state $|\Phi^+\rangle$. By acting with one of the (unitary) spin-flip operators X, Y, Z one can obtain any of the other three Bell states:

$$\begin{aligned} X|\Phi^+\rangle &= |\Phi^-\rangle \\ Y|\Phi^+\rangle &= |\Psi^+\rangle \\ Z|\Phi^+\rangle &= |\Psi^-\rangle. \end{aligned} \tag{3.4}$$

In a typical Alice-and-Bob-style experiment one can ask the question if this observation allows to achieve a better-than-classical communication channel. If Alice performs a spin flip on her qubit, although the resulting state has instantly "changed" (cf. *spooky action at a distance*), Bob still cannot uniquely determine what this state is (since the resulting state is still maximally entangled). However, if Alice sends her qubit to Bob, the latter can perform a measurement on the composite system to find out what the state is and in this way determine which operation Alice performed ($\mathbb{1}, X, Y, Z$). Alice has thus effectively sent 2 classical bits of information through 1 qubit. Note that due to the fact that Alice still has to send her qubit through classical means, no faster-than-light communication is achieved.

Definition 3.1.5 (GHZ[2] state). The GHZ state is defined as the multiparticle qudit ($d, N > 2$) version of the Bell state above and is, hence, also referenced to as a cat state:

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle^{\otimes N}. \tag{3.5}$$

3.2 Density operators

Definition 3.2.1 (Density operator). Consider a (finite-dimensional) Hilbert space $\mathcal{H}$. A density operator on $\mathcal{H}$ is a linear operator $\rho \in \mathrm{End}(\mathcal{H})$ satisfying the following properties:

1. **Positivity**: $\langle v|\rho v\rangle \geq 0$ for all $v \in \mathcal{H}$.
2. **Hermiticity**: $\rho^\dagger = \rho$.
3. **Unit trace**: $\mathrm{tr}(\rho) = 1$.

More concisely, density operators are the representing objects of normal states **??** on $\mathcal{B}(\mathcal{H})$.

Example 3.2.2 (Classical probability). A diagonal density matrix corresponds to a (classical) discrete probability distribution.

Definition 3.2.3 (Pure state). A state is said to be pure if it is described by an outer product of a state vector or, equivalently, by an idempotent density matrix. A density matrix that is not of this form gives rise to a **mixed state**.

Definition 3.2.4 (Reduced density operator). Let $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be the state of a bipartite system. The reduced density operator $\hat{\rho}_A$ of A is defined as follows:

$$\hat{\rho}_A := \mathrm{tr}_B\, |\Psi\rangle\langle\Psi|. \tag{3.6}$$

Definition 3.2.5 (Purification). Let $\hat{\rho}_A$ be the density operator of a system A. A purification of $\hat{\rho}_A$ is a pure state $|\Psi\rangle$ of some composite system AB such that

$$\hat{\rho}_A = \mathrm{tr}_B\, |\Psi\rangle\langle\Psi|. \tag{3.7}$$

Property 3.2.6. Any two purifications of the same density operator $\hat{\rho}_A$ are related by a transformation $\mathbb{1}_A \otimes \hat{V}$ with $\hat{V}$ an isometry.

[2] *Greenberger-Horne-Zeilinger*

3.3 Operations

The following definition generalizes the content of Section **??** to a setting of partial information:

Definition 3.3.1 (Positive operator-valued measure). First, let $\mathcal{H}$ be a finite-dimensional Hilbert space. A POVM on $\mathcal{H}$ consists of a finite set of positive (semi)definite operators $\{P_i\}_{i\leq n}$ such that

$$\sum_{i=1}^{n} P_i = \mathbb{1}_{\mathcal{H}}. \tag{3.8}$$

The probability to obtain state i, given a general state $\hat{\rho}$, is given by $\mathrm{tr}(\hat{\rho}P_i)$. Note that the operators are not necessarily orthogonal projectors, so n can be greater than $\dim(\mathcal{H})$.

Now, consider a measurable space (X, Σ) and a (possibly infinite-dimensional) Hilbert space $\mathcal{H}$. A POVM on X consists of a function $P : \Sigma \to \mathcal{B}(\mathcal{H})$ satisfying the following conditions:

1. P_E is positive and self-adjoint for all $E \in \Sigma$,
2. $P_X = \mathbb{1}_{\mathcal{H}}$, and
3. for all disjoint $(E_n)_{n\in\mathbb{N}} \subset \Sigma$:

$$\sum_{n\in\mathbb{N}} P_{E_n} = P_{\bigcup_{n\in\mathbb{N}} E_n}. \tag{3.9}$$

The following theorem can be derived from the Stinespring theorem **??**:

Theorem 3.3.2 (Naimark dilation theorem). *Every POVM P on $\mathcal{H}$ can be realized as a PVM Π on a possibly larger Hilbert space $\mathcal{K}$, i.e. there exists a bounded operator $V : \mathcal{K} \to \mathcal{H}$ such that*

$$P(\cdot) = V\Pi(\cdot)V^{\dagger}. \tag{3.10}$$

In the finite-dimensional setting V can be chosen to be an isometry.

Definition 3.3.3 (Completely positive trace-preserving). Consider a map $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ between (trace-class) operators on two (finite-dimensional) Hilbert spaces. This map preserves density matrices if it positive **??** and if it is trace-preserving **??**. Furthermore, to ensure that an operation applied to a subsystem does not interfere with the positivity of the complete system, they are also required to be completely positive **??**.

Completely positive, trace-preserving (CPTP) maps are often called **quantum channels**.

The following property can be derived from the Stinespring theorem **??**:

Property 3.3.4 (Kraus decomposition). Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces of dimensions m and n, respectively. A linear map $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ is completely positive if and only if there exist bounded operators $\{A_i\}_{i\leq mn}$ such that

$$\Phi(B) = \sum_{i=1}^{mn} A_i^{\dagger} B A_i. \tag{3.11}$$

Furthermore, it is trace-preserving if and only if

$$\sum_{i=1}^{mn} A_i^{\dagger} A_i = \mathbb{1}. \tag{3.12}$$

A decomposition of the above form is also often called an **operator-sum decomposition**.

List of Symbols

The following symbols are used throughout the summary:

Abbreviations

AIC	Akaike information criterion
ARMA	autoregressive moving-average model
BCH	Baker-Campbell-Hausdorff
CCR	canonical commutation relation
CDF	cumulative distribution function
CFT	conformal field theory
CIS	completely integrable system
CP	completely positive
CPTP	completely positive, trace-preserving
CR	Cauchy-Riemann
DGA	differential graded algebra
DGCA	differential graded-commutative algebra
EPR	Einstein-Podolsky-Rosen
ETCS	Elementary Theory of the Category of Sets
FWHM	full width at half maximum
GA	geometric algebra
GHZ	Greenberger-Horne-Zeilinger
GNS	Gel'fand-Naimark-Segal
HoTT	Homotopy Type Theory
KKT	Karush-Kuhn-Tucker
LIVF	left-invariant vector field
MPO	matrix product operator
MPS	matrix product state
MTC	modular tensor category
NDR	neighbourhood deformation retract
OPE	operator product expansion
OZI	Okubo-Zweig-Iizuka
PAC	probably approximately correct
PL manifold	piecewise-linear manifold
PVM	projection-valued measure

RKHS	reproducing kernel Hilbert space
SVM	support-vector machine
TQFT	topological quantum field theory
VIF	variance inflation factor
ZFC	Zermelo-Frenkel set theory with the axiom of choice
TVS	topological vector space
Operations	
Ad_g	adjoint representation of a Lie group G
ad_X	adjoint representation of a Lie algebra $\mathfrak{g}$
arg	argument of a complex number
$\square$	d'Alembert operator
$\deg(f)$	degree of the polynomial f
e	identity element of a group
$\Gamma(E)$	set of global sections of a fibre bundle E
Im	imaginary part of a complex number
$\mathrm{Ind}_f(z)$	index of a point $z \in \mathbb{C}$ with respect to a function f
$\hookrightarrow$	injective function
$\cong$	is isomorphic to
Par_t^γ	parallel transport map with respect to the curve γ
Re	real part of a complex number
Res	residue of a complex function
$\twoheadrightarrow$	surjective function
$\{\cdot,\cdot\}$	Poisson bracket
∂X	boundary of a topological space X
$\overline{X}$	closure of a topological space X
$X^\circ, \mathring{X}$	interior of a topological space X
$\sphericalangle(\cdot,\cdot)$	angle between two vectors
$X \times Y$	cartesian product of the sets X and Y
$X + Y$	sum of the vector spaces X and Y
$X \oplus Y$	direct sum of the vector spaces X and Y
$V \otimes W$	tensor product of the vector spaces V and W
$\mathbb{1}_X$	identity morphism on the object X
$\approx$	is approximately equal to
$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto
Collections	
Ab	category of Abelian groups
$\mathrm{Aut}(X)$	automorphism group of an object X
$\mathcal{B}_0(V,W)$	space of compact bounded operators between the Banach spaces V and W

$\mathcal{B}(V, W)$	space of bounded linear maps from the space V to the space W
CartSp	the category of Euclidean spaces and "suitable" homomorphisms (e.g. linear maps, smooth maps, ...)
$C_\bullet$	chain complex
$\mathbf{Ch}(\mathbf{A})$	category of chain complexes with objects in the additive category $\mathbf{A}$
$\mathbf{C}^\infty$	category of smooth spaces
$C_p^\infty(M)$	ring of smooth functions $f : M \to \mathbb{R}$ on a neighbourhood of $p \in M$
$C^\omega(V)$	the set of all analytic functions defined on the set V
$\mathrm{Conf}(M)$	conformal group of (pseudo-)Riemannian manifold M
$C(X, Y)$	set of continuous functions between two topological spaces X and Y
$\mathbf{C^\infty Ring}, \mathbf{C^\infty Alg}$	category of smooth algebras
Diff	category of smooth manifolds
DiffSp	category of diffeological spaces and smooth maps
D^n	standard n-disk
$\mathrm{dom}(f)$	domain of a function f
$\mathrm{End}(X)$	endomorphism monoid of a an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad
$\mathbf{FormalCartSp_{diff}}$	category of infinitesimally thickened Euclidean spaces
$\mathrm{GL}(V)$	general linear group, the group of automorphisms of a vector space V
$\mathrm{GL}(n, K)$	general linear group: the group of all invertible $n \times n$-matrices over the field K
Grp	category of groups and group homomorphisms
Grpd	category of groupoids
$\mathrm{Hol}_p(\omega)$	holonomy group at the point p with respect to the principal connection ω
$\mathrm{Hom}_{\mathbf{C}}(V, W)$	set of homomorphisms from an object V to an object W in a category $\mathbf{C}$
hTop	homotopy category
$\mathrm{im}(f)$	image of a function f
$K^0(X)$	K-theory over a (compact Hausdorff) space X
Kan	category of Kan complexes
$\mathcal{K}_n(A, v)$	Krylov subspace of dimension n generated by the matrix A and the vector v
L^1	space of integrable functions
Law	category of Lawvere theories
Lie	category of Lie groups
$\mathfrak{Lie}$	category of Lie algebras
$\mathfrak{X}^L$	space of left-invariant vector fields on a Lie group
LX	free loop space on X
$\mathbf{Man}^p$	category of C^p-manifolds
Meas	category of measure spaces and measure-preserving functions
$N\mathbf{C}$	the simplicial nerve of a small category $\mathbf{C}$
$\mathbf{Open}(X)$	category of open subsets of a topological space X
$\mathrm{O}(n, K)$	group of $n \times n$ orthogonal matrices over a field K
$P(S), 2^S$	power set of S

$\mathrm{Pin}(V)$	pin group of the Clifford algebra $C\ell(V,Q)$
$\mathbf{Psh}(\mathbf{C}), \widehat{\mathbf{C}}$	category of presheaves on a (small) category $\mathbf{C}$
$\mathbf{Sh}(X)$	category of sheaves on a topological space X
$\mathbf{Sh}(\mathbf{C},J)$	category of J-sheaves on a site $(\mathbf{C},J)$
$\boldsymbol{\Delta}$	simplex category
$\mathrm{SL}_n(K)$	special linear group: group of all invertible n-dimensional matrices with unit determinant over the field K
S^n	standard n-sphere
$S^n(V)$	space of symmetric rank n tensors over a vector space V
$W^{m,p}(U)$	the Sobolov space in L^p of order m
$\mathbf{Span}(\mathbf{C})$	span category over $\mathbf{C}$
$\mathrm{Spec}(R)$	spectrum of a commutative ring R
$\mathrm{supp}(f)$	support of a function f
$\mathrm{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
S_n	symmetric group of degree n
$\mathrm{Sym}(X)$	symmetric group on the set X
$\mathrm{Sp}(n,K)$	group of matrices preserving a canonical symplectic form over the field K
$\mathrm{Sp}(n)$	compact symplectic group
$\mathrm{TL}_n(\delta)$	Temperley-Lieb algebra with $n-1$ generators and parameter δ.
T^n	standard n-torus (the n-fold Cartesian product of S^1)
Top	category of topological spaces
Topos	the 2-category of (elementary) topoi and geometric morphisms
$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\mathrm{U}(n,K)$	group of $n\times n$ unitary matrices over a field K
$\mathbf{Vect}(X)$	category of vector bundles over a manifold X
$\mathbf{Vect}_K$	category of vector spaces and linear maps over a field K
Y^X	set of functions from a set X to a set Y
$\emptyset$	empty set
$\pi_n(X,x_0)$	n^{th} homotopy space over X with basepoint x_0
$[a,b]$	closed interval
$]a,b[$	open interval
$\Lambda^n(V)$	space of antisymmetric rank n tensors over a vector space V
ΩX	(based) loop space on X
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on the manifold M
$\rho(A)$	resolvent set of a bounded linear operator A
$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on the manifold M
Units	
C	coulomb
T	tesla

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