Compendium of Mathematics & Physics

Nicolas Dewolf

September 21, 2022

Contents

Chapter 1

Algebraic Geometry

References for this chapter are [8, 9]. For the basics on ring theory, ideals and algebraic dependence, see Sections **??** and **??**. In order to not confuse the letter k, often used for fields, with various indices and dimensions, fields will be denoted by the capital letter K.

1.1 Varieties

From here on K is assumed to be an algebraically closed field. For notational simplicity and to differentiate between K^n as a vector space and as a set, the notion of an affine space is introduced:

Definition 1.1.1 (Affine space). $\mathbb{A}^n$ is defined as the underlying set of the vector space K^n:

$$\mathbb{A}^n := \big\{(a_1, \dots, a_n) \in K^n\big\}. \tag{1.1}$$

Definition 1.1.2 (Algebraic set). Consider a finite set of polynomials in $K[x_1, \dots, x_n]$. It is not hard to show that the zero locus of these polynomials depends only on the ideal spanned by them and, hence, one can define the algebraic set associated to an ideal $I \subseteq K[x_1, \dots, x_N]$ to be

$$V(I) := \big\{(a_1, \dots, a_n) \in \mathbb{A}^n \mid \forall f \in I : f(a_1, \dots, a_k) = 0\big\}. \tag{1.2}$$

A set $S \in \mathbb{A}^n$ is said to be an **(affine)** algebraic set if there exists an ideal I such that $S = V(I)$. An algebraic set $S \in \mathbb{A}^n$ is said to be **irreducible** if it is not the union of two strictly smaller algebraic sets. Irreducible algebraic sets are also called **affine varieties**.

Remark. Some authors (such as [8]) make no distinction between general algebraic sets and affine varieties.

Property 1.1.3. By Hilbert's basis theorem one can express any algebraic set as the zero locus of a finite number of polynomials.

Given an algebraic set S, the set $I(S)$ is defined as the ideal of polynomials that vanish on S. The following theorem gives an important relation between algebraic sets and ideals.

Theorem 1.1.4 (Hilbert's Nullstellensatz). *Let J be an ideal in $K[x_1, \dots, x_n]$ and let $\sqrt{J}$ denote its radical.*

$$I\big(V(J)\big) = \sqrt{J}. \tag{1.3}$$

Similar to the case of the weak Nullstellensatz, one obtains the following result

Corollary 1.1.5. There exists a bijection between the algebraic subsets of $\mathbb{A}^n$ and the radical ideals in $K[x_1, \ldots, x_n]$. The irreducible algebraic sets correspond to the prime ideals (by the *Noetherian decomposition theorem*).

Definition 1.1.6 (Morphism of varieties). Let $V_1 \subset \mathbb{A}^{n_1}, V_2 \subset \mathbb{A}^{n_2}$ be two affine varieties. A morphism $\varphi : V_1 \to V_2$ is a function that can be expressed in the following way:

$$\varphi(x_1, \ldots, x_{n_1}) = \big(f_1(x_1, \ldots, x_{n_1}), \ldots, f_{n_2}(x_1, \ldots, x_{n_1})\big)\,, \tag{1.4}$$

where $f_i \in K[x_1, \ldots, x_{n_1}]$ for all $i \leq n_2$.

A closely related notion is that of rational maps:

Definition 1.1.7 (Rational map). Consider two affine varieties X, Y. A rational map $f : X \to Y$ is an equivalence class of pairs (U, f_U), where U is a nonempty open subset and $f_U : U \to Y$ is a morphism of varieties, under the following relation: $(U, f_U) \sim (V, f_V)$ if and only if $f_U = f_V$ on a nonempty subset of $U \cap V$.

A rational map is said to be **dominant** if for one of its representatives (U, f) the image $f(U)$ is dense. Dominance of rational maps assures that their composition exists and is well-defined.

A rational map $f : X \to Y$ is said to be **birational** if it is dominant and if there exists a rational map $g : Y \to X$ such that $f \circ g = \mathbb{1}_Y$ and $g \circ f = \mathbb{1}_X$.

Definition 1.1.8 (Coordinate ring). Consider the polynomial ring $K[x_1, \ldots, x_n]$ and let V be an algebraic set in $\mathbb{A}^n$. The coordinate ring (or **affine ring**) of V is defined as the following quotient:

$$\Gamma(V) := K[x_1, \ldots, x_n]/I(V). \tag{1.5}$$

The elements of this ring are the K-valued polynomials in the coordinates on V.

If V is irreducible it follows from the Nullstellensatz that $I(V)$ is a prime ideal and, hence, that $\Gamma(V)$ is an integral domain. This property allows to construct the field of fractions **??**. This field is called the **function field** of V and is denoted by $K(V)$. The elements of $K(V)$ are called the **rational functions** on V. It can be shown that the rational functions are exactly the rational maps $V \to \mathbb{A}^1$.

It should be noted that every morphism of varieties induces a K-morphism on the associated affine rings by precomposition. This gives rise to the following property:

Property 1.1.9 (Finitely generated algebras). Γ gives an equivalence between the category of algebraic sets and the category of finitely-generated (reduced) K-algebras. This equivalence passes to an equivalence between the subcategories on affine varieties and integral domains.

Definition 1.1.10 (Dimension). The dimension of an affine variety is given by the **Krull dimension** of its coordinate ring, i.e. the maximum length of a chain of prime ideals in $\Gamma(X)$:

$$\dim(X) := \sup_{n\in\mathbb{N}}(\exists\, \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n). \tag{1.6}$$

By the Nullstellensatz this is equivalent to the maximum length of chains of irreducible algebraic subsets.

The **local dimension** $\dim_p(X)$ of a point $p \in X$ is defined in a similar way, with the start of the chains fixed at $\{p\}$. One then obtains

$$\dim(X) = \max_{p\in X} \dim_p(X). \tag{1.7}$$

1.1.1 Topology

A topology on an affine variety can be constructed in the following way:

Definition 1.1.11 (Zariski topology). A set in $\mathbb{A}^n$ is deemed closed exactly if it is an algebraic set. A basis for this topology is given by the complements of zero loci $B_f = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}$ for $f \in K[x_1, \dots, x_n]$. This topology turns an affine variety into an irreducible space.

On an algebraic subset $V \subset \mathbb{A}^n$ one defines the Zariski topology as the induced topology of the one on $\mathbb{A}^n$. A basis for this induced Zariski topology is given by the sets B_f as above, but where f is now an element of $\Gamma(V)$.

The following property shows that the Zariski topology is very different from the topologies that occur in, for example, analysis:

Property 1.1.12 (Density). Any open subset of an affine variety is dense.

By dualizing one can focus on the coordinate rings and construct varieties as a derived notion. In this approach the main tool is the structure sheaf of a variety. From here on the content of Chapter **??** on sheaf theory will become a prerequisite.

Definition 1.1.13 (Structure sheaf). Consider an affine variety X with its associated coordinate ring $\Gamma(X)$. For any point $x \in X$ one can consider the set of functions $m_x \subset \Gamma(X)$ that vanish at x. This is a prime ideal, so one can construct a localization:

$$\mathcal{O}_x := \Gamma(X)_{m_x} = \{f/g \mid f, g \in \Gamma(X) \wedge g(x) \neq 0\}. \tag{1.8}$$

For every open subset $U \subset X$ one can then define the ring of functions on U as follows:

$$\mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_x. \tag{1.9}$$

$\mathcal{O}_X$ is a sheaf with stalks given by $\mathcal{O}_x$. By Property **??** all stalks $\mathcal{O}_x$ are local rings and, hence, $(X, \mathcal{O}_X)$ is a locally ringed space **??**. The residue field of these local rings is equal to the base field K. The elements of $\mathcal{O}_X(U)$ are called the **regular functions** on U.

This construction can be made more explicit. A map $\varphi : X \to K$ is said to be regular at a point $x \in X$ if there exists an open neighbourhood $U \ni x$ and polynomials $f, g \in \Gamma(X)$ such that $g \neq 0$ and $\varphi = f/g$ on U. As for continuous functions, the map φ is said to be regular on X if it is regular at every point $x \in X$.

Property 1.1.14. Let $f \in \Gamma(X)$ be a function on X and consider the basis set B_f, i.e. the complement of the zero locus of f. The structure sheaf assigns localizations to basis sets:

$$\mathcal{O}_X(B_f) = \Gamma(X)_f\,, \tag{1.10}$$

where $\Gamma(X)_f$ denotes the localization **??** of $\Gamma(X)$ at f. In particular

$$\Gamma(X, \mathcal{O}_X) = \Gamma(X)\,, \tag{1.11}$$

where on the left-hand side Γ denotes the global sections functor **??**. This property explains the notation $\Gamma(X)$ introduced before.

Remark 1.1.15. Both $\mathcal{O}_X(U)$ and $\mathcal{O}_x$ are subrings of the function field $K(X)$.

Alternative Definition 1.1.16 (Affine variety). A topological space X equipped with a sheaf $\mathcal{F}$ of K-valued functions such that X is isomorphic to an irreducible algebraic set Σ and such that $\mathcal{F}$ is isomorphic to the structure sheaf $\mathcal{O}_X$. An open subset of an affine variety is sometimes called a **quasi-affine variety**.

Using the notion of a regular function, the definition of a morphism of affine varieties can be restated:

Alternative Definition 1.1.17 (Morphism). A continuous function between affine varieties $f : X \to Y$ such that precomposition by f preserves regular functions.

Property 1.1.18 (Identity theorem). If two regular maps coincide on a nonempty open subset, they are equal.

Definition 1.1.19 (Generic stalk). For the construction of the stalk of the structure sheaf over a point $x \in X$ one takes a direct limit over all open sets containing x. This way the local ring $\Gamma(X)_{m_x}$ is obtained. Moreover, this is a subring of the field of fractions $K(X)$ of $\Gamma(X)$ (see Definition **??**). Now, using a similar definition, one can recover all of $K(X)$.

Instead of taking a direct limit over the open sets containing a certain point $x \in X$, take a direct limit over all open sets in X:

$$\mathcal{O}_{\tilde{x}} := \varinjlim_{U \subset X} \mathcal{O}_X(U). \tag{1.12}$$

This stalk is called the generic stalk of X and it is isomorphic to $K(X)$.

Definition 1.1.20 (Étale morphism over algebraically closed fields). A morphism $f : X \to Y$, where both X and Y are affine varieties over an algebraically closed field, is said to be formally étale if the induced maps $\widehat{\mathcal{O}}_{f(x)} \to \widehat{\mathcal{O}}_x$ on the completion **??** of local rings are isomorphisms.

1.1.2 Varieties

Definition 1.1.21 (Prevariety). A connected locally ringed space that admits a finite cover by affine spaces.

Remark 1.1.22. It can be shown that every prevariety X is irreducible and, hence, the open sets form a direct system. This way one can define the **generic stalk** of an arbitrary sheaf $\mathcal{F}$, as in the case of affine varieties. For the structure sheaf $\mathcal{O}_X$ this generic stalk is called the **function field** $K(X)$. It coincides with the function field of every open affine subset of X.

Construction 1.1.23 (Gluing). Consider two prevarieties X, Y together with an isomorphism $f : U \cong W$ between open subsets $U \subset X, V \subset Y$. The prevarieties can be glued together along f as follows. First, build the attaching space **??** $U \cup_f Y$ with its canonical topology. Then, define the regular functions on a subset to be those that come from regular functions on (subsets of) X and Y.

Definition 1.1.24 (Variety[1]). A prevariety X for which the diagonal Δ_X is closed in $X \times X$. It should be noted that every affine variety is a variety, but not the other way around.

Remark 1.1.25. The motivation for this definition is Property **??**. In general topology it is well-known that a lot of pathological spaces can be excluded by restricting to Hausdorff spaces, i.e. spaces where distinct points admit disjoint neighbourhoods. Because open subsets of irreducible spaces have nonempty intersections, this property is sadly enough not very useful in the study of varieties. However, the equivalent definition using closedness of the diagonal remains useful if one does not consider the product topology on $X \times X$, but instead uses the "gluing"-topology from Construction 1.1.23 above.

The following two closure properties are very important:

[1] Sometimes also called a **separated prevariety**.

Property 1.1.26. Consider a prevariety morphism $f : X \to Y$, where Y is a variety. The graph of f is closed in $X \times Y$.

Property 1.1.27. Consider two prevariety morphisms $f, g : X \to Y$ where Y is a variety. The set on which f and g coincide is closed in X.

1.1.3 Projective varieties

Definition 1.1.28 (Projective space). Consider the vector space K^n. The projective space $\mathbb{P}_{n-1}(K)$ or $K\mathbb{P}^{n-1}$ is defined as the quotient of K^n under the following equivalence relation:

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_n) \iff \exists \lambda \in K^\times : \forall i \leq n : x_i = \lambda y_i. \tag{1.13}$$

The equivalence class of a vector $(x_1, \dots, x_n)$ is denoted by $[x_1 : \cdots : x_n]$. Because the numbers characterizing an equivalence class are only determined up to a common factor, the coordinates $x_1, \dots, x_n$ are called **homogeneous coordinates**.

Consider the subset

$$K_{\text{hom}}[x_0, \dots, x_n] \subset K[x_0, \dots, x_n]$$

consisting of all homogeneous polynomials. The definition of projective space implies that

$$\forall f \in K_{\text{hom}}[x_0, \dots, x_n] : f(\lambda x_0, \dots, \lambda x_n) = 0 \iff f(x_0, \dots, x_n) = 0$$

and, hence, that the zero loci of homogeneous polynomials are well-defined subsets of the projective space $\mathbb{P}_n(K)$.

Definition 1.1.29 (Projective algebraic set). As in the case of affine algebraic sets one can define two operations. Let I be a homogeneous ideal, i.e. an ideal in $K[x_0, \dots, x_n]$ that is generated by homogeneous polynomials. The projective algebraic set $V_{\mathbb{P}}(I)$ is defined as the zero locus of I:

$$V_{\mathbb{P}}(I) := \big\{x \in \mathbb{P}_n(K) \mid \forall f \in I : f(x) = 0\big\}. \tag{1.14}$$

Given a projective algebraic set $V \in \mathbb{P}_n(K)$, one can define the ideal $I_{\mathbb{P}}(V)$ as follows:

$$I_{\mathbb{P}}(V) := \big(f \in K_{\text{hom}}[x_0, \dots, x_n] \mid \forall x \in V : f(x) = 0\big), \tag{1.15}$$

i.e. the ideal $I_{\mathbb{P}}(V)$ is generated by all homogeneous polynomials that vanish on V. The **Zariski topology** on $\mathbb{P}_n(K)$ is defined such that the closed sets are exactly the projective algebraic sets.

Theorem 1.1.30 (Projective Nullstellensatz). *For all homogeneous ideals I, except $I_0 := (x_1, \dots, x_n)$, one has*

$$I_{\mathbb{P}}\big(V_{\mathbb{P}}(I)\big) = \sqrt{I}. \tag{1.16}$$

Corollary 1.1.31. As before this implies that there exists a bijection between the projective algebraic sets in $\mathbb{P}_n(K)$ and the homogeneous radical ideals (except for I_0) in $K[x_0, \dots, x_n]$.

Definition 1.1.32 (Coordinate ring). As for affine algebraic sets, the coordinate ring of a projective algebraic set V is defined as the quotient

$$\Gamma(V) := K[x_0, \dots, x_n]/I_{\mathbb{P}}(V). \tag{1.17}$$

The construction of regular functions on affine varieties 1.1.13 cannot be extended to projective spaces in a straightforward manner. Consider for example a polynomial $f \in K[x_0, \ldots, x_n]$. This polynomial does not form a well-defined function on a projective algebraic set $V_{\mathbb{P}}(I) \subset \mathbb{P}_n(K)$ even if f is homogeneous, since changing the homogeneous coordinates on $V_{\mathbb{P}}(I)$ changes the value of f (only the zero locus is invariant). However, the ratio of two homogeneous polynomials of the same degree does form a well-defined function on $V_{\mathbb{P}}(I)$.

Since the ideal I is homogeneous, the quotient $R := K[x_0, \ldots, x_n]/I$ is a graded algebra. Denote by $K(X)$ the zeroth order part of the localization of R by the homogeneous elements:

$$K(X) := \{f/g \mid \exists n \in \mathbb{N} : f, g \in R_n\}. \tag{1.18}$$

Now, although an element $f \in R_n$ does not give a well-defined function on X, the property $f(x) \neq 0$ is preserved under rescaling. Hence, one can define a ring $\mathcal{O}_x$ as before:

$$\mathcal{O}_x := \{f/g \in K(X) \mid g(x) \neq 0\}. \tag{1.19}$$

This ring has a maximal ideal $I_x = \{f/g \in K(X) \mid f(x) = 0, g(x) \neq 0\}$ such that all elements in $\mathcal{O}_x$ are invertible and, by Property **??**; $\mathcal{O}_x$ is a local ring. One can then construct a sheaf $\mathcal{O}_X$ using the same procedure as for affine varieties to turn a projective space into a locally ringed space:

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_x. \tag{1.20}$$

Property 1.1.33 (Variety). For every projective variety $X \subset \mathbb{P}_n(K)$ the pair $(X, \mathcal{O}_X)$ is locally isomorphic to an affine variety and as such every projective variety is in particular a variety in the sense of Definition 1.1.24.

Property 1.1.34 ($\mathbb{A}^n$ in $\mathbb{P}_n(K)$). Consider the affine variety $\mathbb{A}^n$. This set admits a bijective mapping onto an open subset of $\mathbb{P}_n(K)$ as follows:

$$\varphi : \mathbb{A}^n \to U_0 : (x_1, \ldots, x_,n) \mapsto [1 : x_1 : \cdots : x_n]. \tag{1.21}$$

It can be shown that this map is a homeomorphism if both spaces are equipped with the Zariski topology.

Property 1.1.35 (Schubert decomposition). The projective space $\mathbb{P}_n(K)$ admits a decomposition of the form

$$\mathbb{P}_n(K) = \bigcup_{i=0}^{n} K^i\,, \tag{1.22}$$

where the union is set-theoretic. However, one can refine this to a statement in topology. The projective space $\mathbb{P}_n(K)$ admits the structure of a CW complex with one k-cell in every dimension (namely $\mathbb{A}^k$). These cells are called **Bruhat** or **Schubert cells**. (The precise distinction is of no relevance here.)

Example 1.1.36 (Finite fields). Consider a finite field $\mathbb{F}_q$. Using the above decomposition one can easily compute the cardinality of $\mathbb{P}_n(\mathbb{F}_q)$:

$$|\mathbb{P}_n(\mathbb{F}_q)| = \sum_{i=0}^{n} |\mathbb{F}_q^i| = \sum_{i=0}^{n} q^i \equiv [n+1]_q\,, \tag{1.23}$$

For example, the **Fano plane** $\mathbb{P}_2(\mathbb{F}_2)$ has cardinality 7.

Construction 1.1.37 (Blow-up). Consider an algebraic set $X \subseteq \mathbb{A}^n$ together with a set of regular functions $\{f_1, \dots, f_k\} \subset \Gamma(X)$. Define the subset $Y := X \backslash V(f_1, \dots, f_k)$. By definition these functions do not all vanish simultaneously on Y and, hence, there exists a well-defined map

$$f : Y \to \mathbb{P}_{k-1}(K) : x \mapsto \big[f_1(x) : \cdots : f_k(x)\big].$$

The graph of this morphism is closed in $Y \times \mathbb{P}_{k-1}(K)$ by Property 1.1.26, but not in $X \times \mathbb{P}_{k-1}(K)$. Its closure in the latter is called the blow-up $\widetilde{X}$ of X at $f_1, \dots, f_k$. The projection map $\pi : \widetilde{X} \to X$ is sometimes also called the blow-up (map). The graph Γ_f is isomorphic to Y and its complement $\pi^{-1}(V(f_1, \dots, f_k))$ in $\widetilde{X}$ is called the **exceptional set** (of the blow-up).

Construction 1.1.38 (Explicit description). Consider an algebraic set $X \subseteq \mathbb{A}^n$ together with its blow-up $\widetilde{X}$ at $\{f_1, \dots, f_k\}$. One can prove that the following inclusion holds:

$$\widetilde{X} \subseteq \big\{(x, y) \in X \times \mathbb{P}_{n-1}(K) \mid \forall i, j \leq n : y_i f_j(x) = y_j f_i(x)\big\}. \tag{1.24}$$

In the case of $X = \mathbb{A}^n$ and $f_i(x) = x_i$ one can even prove that this inclusion is an equality. Since the zero locus of the coordinate functions is $\{0\}$, one finds that the exceptional set of this blow-up is exactly $\mathbb{P}_{n-1}(K)$.

Property 1.1.39. If X is irreducible, there exists a birational morphism $X \to \widetilde{X}$.

1.2 Schemes

Many of the definitions in this section will be dual to the ones in Section **??**.

Definition 1.2.1 (Spectrum). Let R be a commutative ring. The spectrum $\mathrm{Spec}(R)$ is defined as the set of prime ideals of R. This set can be turned into a topological space by equipping it with the **Zariski topology** whose closed subsets are of the form

$$V(I) := \{P \subseteq I \mid I \text{ is an ideal}, P \in \mathrm{Spec}(R)\}. \tag{1.25}$$

A basis for this topology is given by the sets

$$D_f := \{P \not\ni f \mid f \in R, P \in \mathrm{Spec}(R)\}. \tag{1.26}$$

Property 1.2.2. $\mathrm{Spec}(R)$ is a compact T_0-space.

Definition 1.2.3 (Structure sheaf). Given a spectrum $X = \mathrm{Spec}(R)$, one can define a sheaf[2] $\mathcal{O}_X$ by setting $\forall f \in R : \mathcal{O}_X(D_f) := R_f$, where R_f is the localization of R with respect to the monoid of powers of f.

Property 1.2.4. The spectrum $\mathrm{Spec}(R)$ together with its structure sheaf forms a ringed space.

Definition 1.2.5 (Affine scheme). A locally ringed space isomorphic to the spectrum of a commutative ring.

Example 1.2.6 (Spectrum of a field). For any field K, its spectrum $\mathrm{Spec}(K)$ consists of a single point with K as the stalk over it.

Property 1.2.7 (Stalks). Consider an affine scheme $\mathrm{Spec}(R)$ and a point $x \in \mathrm{Spec}(R)$ corresponding to the prime ideal $\mathfrak{p}$. The stalk at x is given by the localization $R_{\mathfrak{p}}$.

[2]In fact this is merely a *B-sheaf* as it is only defined on the basis of the topology. However, every *B-sheaf* can be extended to a sheaf by taking appropriate limits [9].

Definition 1.2.8 (Scheme). A locally ringed space such that for every point there exists an open neighbourhood isomorphic to an affine scheme. A scheme X **over a ring** R means a morphism $X \to \mathrm{Spec}(R)$.

Property 1.2.9 (Integers). $\mathrm{Spec}(\mathbb{Z})$ is the terminal scheme, i.e. for every scheme X there exists only one morphism $X \to \mathrm{Spec}(\mathbb{Z})$. This spectrum itself consists of all the prime numbers, together with a generic point at infinity. The stalks over these elements are given by $\mathbb{Z}_p$ and $\mathbb{Q}$, respectively.

Definition 1.2.10 (Geometric point). The spectrum of an algebraically closed field. A geometric point of a scheme X over a field K is a morphism $\mathrm{Spec}\big(\overline{K}\big) \to X$.

Property 1.2.11 (Affine schemes). There exists an adjunction

$$\mathbf{CRing}^{op} \underset{\mathrm{Spec}}{\overset{\Gamma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Sch}. \tag{1.27}$$

By Property **??** this becomes an (adjoint) equivalence between **AffSch** and $\mathbf{CRing}^{op}$, i.e. a scheme X is affine if and only if the unit $X \to \mathrm{Spec}\left(\Gamma(X, \mathcal{O}_X)\right)$ is an isomorphism or, equivalently, if the map

$$\mathbf{Sch}(Y, X) \to \mathbf{CRing}\big(\Gamma(X, \mathcal{O}_X), \Gamma(Y, \mathcal{O}_Y)\big) \tag{1.28}$$

is bijective for all schemes Y.

Property 1.2.12 (Functor of points). The functor of points of a scheme X is simply the functor represented by X. By the Yoneda lemma, morphisms between such functors, i.e. morphisms between (generalized) points, are equivalent to morphisms between schemes. Moreover, similar to the case of manifolds (this is explained in Chapter 8), restricting representable functors to the subcategory of affine schemes loses no information, i.e. schemes are characterized by morphisms from affine schemes.

Property 1.2.13 (Sober). Every scheme is sober **??**.

Definition 1.2.14 (Integral scheme). A scheme X such that for every open affine subset $\mathrm{Spec}(R) \subseteq X$ the ring R is an integral domain **??**.

Property 1.2.15. By the above property, every integral scheme has a unique generic point.

Definition 1.2.16 (Reduced scheme). A scheme for which all stalks of the structure sheaf are reduced.

Property 1.2.17. A scheme is integral if and only if it is irreducible and reduced.

Definition 1.2.18 (Rational function). In Definition 1.1.8 the field of rational functions, the function field, was constructed as the field of fractions of the ring global sections. For an integral scheme the rings of local sections are again integral domains and, hence, one can again construct these fields. Moreover, in an integral scheme all open affine subsets are dense and, hence, one can define the function field as $K(X) := \mathrm{Frac}(R)$ for any affine open subset $\mathrm{Spec}(R) \subseteq X$. On the other hand, because integral schemes have a unique generic point, the associated prime ideal in any affine neighbourhood is the zero ideal and, consequently, the associated stalk is isomorphic to the function field.

Definition 1.2.19 (Group scheme). A group object in **Sch**.

Example 1.2.20 (Additive group). The additive group $\mathbb{G}_a$ (over some field k) has as underlying object the affine line $\mathrm{Spec}(k[x])$ and as operations:

1. **Addition**: Dual of the ring morphism $k[x] \to k[x_1] \otimes k[x_2] : x \mapsto x_1 + x_2$.
2. **Unit**: Dual of $k[x] \to \mathbb{Z} : x \mapsto 0$.
3. **Inversion**: Dual of $k[x] \to k[x] : x \mapsto -x$.

Equivalently, it is the **Ab**-valued functor that sends a scheme to the Abelian group of global sections of its structure sheaf.

1.2.1 Morphisms

Definition 1.2.21 (Open immersion). An open immersion $f : X \to Y$ of locally ringed spaces is a topological embedding such that the induced map $\mathcal{O}_{f(x)} \to \mathcal{O}_x$ is an isomorphism.

Definition 1.2.22 (Quasicompact). A morphism $f : X \to Y$ of schemes such that Y admits an affine open cover whose preimages are compact[3]. Equivalently, a morphism is quasicompact if the preimage of every compact open subset is compact.

Property 1.2.23. A scheme is compact if and only if it is the union of a finite number of open affine subschemes.

The following definition is analogous to the characterization of Hausdorff spaces **??**:

Definition 1.2.24 (Quasiseparated). A morphism $f : X \to Y$ such that the diagonal morphism $\Delta_f : X \to X \times_f X$ is quasicompact. In the case $Y = \mathrm{Spec}(\mathbb{Z})$, where the diagonal morphism is truly the diagonal map $x \mapsto (x, x)$, X is called a quasiseparated scheme. If one replaces "quasicompact" by "closed", the morphism is said to be **separated**.

Remark 1.2.25. In the early days of scheme theory, what is here called a scheme was known as a "prescheme". Proper "schemes" were what are now called separated schemes. Note that [9] still uses this terminology.

Definition 1.2.26 (Finite type). A scheme morphism $f : X \to Y$ is said to be **locally of finite type** if Y admits an affine open cover $\{\mathrm{Spec}(B_i)\}_{i\in I}$ such that every preimage has an affine open cover $\{\mathrm{Spec}(A_{ij})\}_{j\in J_i}$ where all induced morphisms $B_i \to A_{ij}$ are of finite type. It is said to be of finite type if the index sets J_i are finite for all $i \in I$.

Definition 1.2.27 (Finite presentation). A scheme morphism $f : X \to Y$ is said to be of **finitely presented at a point** $x \in X$ if there exists an affine open set $\mathrm{Spec}(A) \cong U \ni x$ and an affine open set $\mathrm{Spec}(B) \cong V \subset Y$, with $f(U) \subseteq V$, such that the induced ring morphism $B \to A$ is finitely presented **??**. The morphism $f : X \to Y$ is said to be **locally finitely presented** if it is finitely presented at every point $x \in X$ and it is said to be finitely presented if it is locally finitely presented, quasicompact and quasiseparated.

Property 1.2.28. For morphisms with locally Noetherian codomain, morphisms of (locally) finite type and (locally) finite presentation coincide.

Definition 1.2.29 (Flat morphism). A morphism of schemes such that the induced morphism on local rings is flat. If it is also surjective, it is said to be **faithfully flat**.

Definition 1.2.30 (Étale morphism). In Definition 1.1.20 étale morphisms were introduced for affine varieties over algebraically closed fields. However, for more general fields or schemes, this definition does not give the right results. A better approach is to dualize Definition **??**,

[3] In algebraic geometry, compact topological spaces that are not Hausdorff are often called **quasicompact**. However, this convention is not adopted here.

i.e. a scheme morphism $f : X \to Y$ is said to be **formally étale** if for every commutative ring R and nilpotent ideal $I \subset R$ the following lifting exists and is unique:

$$\begin{tikzcd} \mathrm{Spec}(R/I) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

In other words, formally étale morphisms have the unique right lifting property with respect to infinitesimal thickenings. Analogously to the case of rings, the morphism f is said to be **formally smooth** (resp. **formally unramified**) if there is at least (resp. at most) one lift.

Since this definition does allow for some pathological examples, étale morphisms are those formally étale momorphisms that are locally of finite presentation.

Remark. The term ''formally'' should be interpreted in the sense of *formal geometry* (Chapter 8). There, infinitesimal neighbourhoods of points 8.2.17, used for defining the local geometry, are induced by nilpotent ideals. This is further exemplified by Property 1.2.44.

Remark 1.2.31. As a side note the general definition of étale morphism is related to Definition 1.1.20. The completion of the local ring $\mathcal{O}_x$ is defined by the inverse limit over the filtration $\mathcal{O}_x/\mathfrak{m}_x \leftarrow \mathcal{O}_x/\mathfrak{m}_x^2 \leftarrow \mathcal{O}_x/\mathfrak{m}_x^3 \leftarrow \cdots$. Hence, a morphism $\widehat{\mathcal{O}}_x \to \widehat{\mathcal{O}}_y$ between completions is characterized by consecutive lifting problems that are exactly the content of Definition **??**, since the ideals $\mathfrak{m}_x$ are clearly nilpotent in rings of the form $\mathcal{O}_x/\mathfrak{m}_x^k$ for any $k \in \mathbb{N}_0$.

This is generalized to the schemes as follows:

Property 1.2.32. If Y is locally Noetherian and $f : X \to Y$ is locally of finite presentation, étale morphisms can also be characterized in terms of completions. f is étale if and only if the induced morphism on completions is formally étale in the adic topology.

Moreover, if the induced morphisms of residue fields are isomorphisms, f is étale if and only the induced morphisms of completions are isomorphisms (as in the case of affine varieties over algebraically closed fields).

Definition 1.2.33 (Finite étale morphism). A morphism of schemes $f : X \to Y$ for which there exists a covering $\{U_i\}_{i\in I}$ of Y by affine sets $U_i \equiv \mathrm{Spec}(A_i)$ such that $f^{-1}(U_i) \cong \mathrm{Spec}(B_i)$ is a free, separable A_i-algebra B_i for all $i \in I$.

Remark 1.2.34 (Finite étale covering). When a finite étale morphism $f : X \to Y$ exists, X is sometimes called a **finite étale cover(ing)** of Y. The full subcategory of the slice category $\mathbf{Sch}/_Y$ on finite étale coverings is often denoted by $\mathbf{FEt}_X$.

Definition 1.2.35 (Étale fundamental group). Consider a connected scheme X. The (étale) fundamental group $\pi_1(X)$ is the unique (up to isomorphism) profinite group **??** such that $\mathbf{FEt}_X$ is equivalent to the category $\mathbf{Fin}\,\pi_1(X)\mathbf{Set}$ of finite of $\pi_1(X)$-sets.

Remark 1.2.36. Recall Corollary **??** for topological spaces. There, a stricter set of spaces was considered with the benefit that all $\pi_1(X)$-sets could be classified. By passing to profinite groups one can formulate a classification statement in the topological setting that is virtually identical to the above one.

1.2.2 Tangent space

Definition 1.2.37 (Tangent cone). Consider an affine variety $X = V(I)$. The tangent cone to X at the origin is defined as the zero locus of the "initial ideal" of I:

$$C_0X := V\big(\{f^{\mathrm{in}} \mid f \in I\}\big)\,, \tag{1.29}$$

where f^{in} denotes the **initial part** of f, i.e. the sum of the smallest degree monomials in f.

Definition 1.2.38 (Tangent space). Consider an affine variety X and choose a point $x \in X$. By working in a suitable affine chart, one can assume that $x = 0$. This implies that any polynomial $f \in I(X)$ has a vanishing constant term. The tangent space at x is defined as follows:

$$T_xX := V\big(\{f^{[1]} \mid f \in I(X)\}\big)\,, \tag{1.30}$$

where $f^{[1]}$ denotes the linear part of the polynomial f.

Property 1.2.39. For $x = 0$ one obtains that $I(0) = (x_1, \ldots, x_n)/I(X)$. Moreover, there exists a natural isomorphism

$$I(0)/I(0)^2 \cong \mathrm{Hom}_K(T_0X, K). \tag{1.31}$$

The tangent space at 0 can accordingly also be obtained as the dual of $I(0)/I(0)^2$.

It is not so hard to prove that this property can in fact easily be transported to arbitrary points $x \in X$ if one replaces the ideal $I(0)$ by the maximal ideal of the structure sheaf $\mathcal{O}_X$ at x. Therefore, one can give the following general definition:

Definition 1.2.40 (Zariski tangent space). Consider a scheme X with structure sheaf $\mathcal{O}_X$. At every point $x \in X$ the stalk $\mathcal{O}_x$ is a local ring and, hence, one obtain a maximal ideal $\mathfrak{m}_x$. The quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a vector space over the residue field $\mathcal{O}_x/\mathfrak{m}_x$. It is called the Zariski cotangent space at $x \in X$. Its algebraic dual is called the Zariski tangent space at $x \in X$.

Property 1.2.41 (Inverse function theorem). In the smooth setting (Chapter 3 and onwards), the inverse function theorem states that a tuple of functions forms a local coordinate system at a point if their *differentials* generate the *cotangent space* at that point.

Here, the ideal $\mathfrak{m}_x$ consists of the functions vanishing at $x \in X$ and the cotangent space is given by $\mathfrak{m}_x/\mathfrak{m}_x^2$. By Nakayama's lemma **??**, a tuple of elements represent "coordinate functions", i.e. generate $\mathfrak{m}_x$, if their "differential", i.e. their residues in the Zariski cotangent space, generate the cotangent space.

Definition 1.2.42 (Regular scheme). A (Noetherian) scheme is said to be regular or **non-singular** if the local dimension at any point is equal to the dimension of the Zariski tangent space at that point. Equivalently, an affine variety is smooth if at every point the tangent space is isomorphic to the tangent cone.

Alternative Definition 1.2.43 (Tangent cone: Schemes). Assume that X is a Noetherian scheme. The tangent cone C_xX is given by the spectrum of the graded ring $\mathrm{Gr}(\mathcal{O}_x) = \bigoplus_{k\in\mathbb{N}} \mathfrak{m}_x^k/\mathfrak{m}_x^{k+1}$. If X is regular, the tangent cone coincides with the Zariski tangent space.

Recall Definition 1.1.20 of étale morphisms of affine varieties (over algebraically closed fields). The following property relates these to the infinitesimal thickening point-of-view:

Property 1.2.44. A morphism $f : X \to Y$ of varieties induces an isomorphism on completions $\widehat{\mathcal{O}}_{f(x)} \to \widehat{\mathcal{O}}_x$ if and only if it induces an isomorphism on tangent cones.

1.3 Sheaf theory ♣

This section uses the content of Chapter 2.

Property 1.3.1 (Coherent sheaves). A sheaf on a locally Noetherian scheme is coherent **??** if and only if it is quasicoherent and of finite type.

Definition 1.3.2 (Zariski topology). The Zariski topology on the category of schemes is generated by sets $\{f_i : U_i \to X\}_{i\in I}$ of open immersions such that $X = \bigcup_{i\in I} f_i(U_i)$ in the set-theoretic sense. This latter condition is often called **joint surjectivity**. This gives the **big Zariski site** on an a scheme X. The **small Zariski site** of a scheme X is the subsite of $\mathbf{Sch}_{/X}$ on the open immersions.

Remark 1.3.3 (Size issues). There are some size issues with the category of schemes, it being a big category. The result is that in general there exists no sheafification functor, i.e. no general construction of the category of sheaves and, hence, this category can only be turned into a site, not a (Grothendieck) topos.[4] In the case of *topological* or *smooth manifolds*, this problem is often ignored since one has a (small) dense subsite and no issues arise. For schemes, however, the situation is different. For example, the fpqc topology (to be introduced below) does not admit a sheafification functor. Two solutions exists in such a context, either one works within a suitable universe or one only considers a single (pre)sheaf at a time.

Property 1.3.4. The Zariski site is subcanonical.

The following topology is finer than the Zariski topology:[5]

Definition 1.3.5 (Étale cover). An étale cover of a scheme X is a jointly surjective family of étale morphisms that are locally of finite type[6]. The **big étale site** $\mathbf{Sch}_{/X,\text{ét}}$ is the slice category of X equipped with the coverage 2.4.8 of étale covers. The **small étale site** is the subsite on étale morphisms.

Property 1.3.6. A presheaf satisfies descent with respect to the étale topology if and only if it satisfies descent with respect to the following covers:

- Zariski covers, and
- (single) faithfully flat morphisms of affine schemes.

Definition 1.3.7 (Flat topologies). The finest topologies on the category of schemes (or slices thereof) are given by considering flat morphisms. The two most widely used flat topologies are the fppf and fpqc topologies. The **fppf topology** (for "fidèlement plat de presentation finie") is generated by jointly surjective families of flat morphisms that are locally of finite presentation. The **fpqc topology** (for "fidèlement plat et quasi-compacte") is generated by jointly surjective families of flat and quasicompact morphisms.

Remark 1.3.8. With the definition as above the fpqc topology is technically not comparable to the Zariski topology since open immersions need not be quasicompact (unless one restricts to Noetherian schemes). A solution introduced by *Vistoli* and *Kleiman* is to replace quasicompactness by the following seemingly similar definition: There exists an open affine cover $\{U_i\}_{i\in I}$ of X such that each U_i is the image of a compact open subscheme. (If one drops the modifier "locally" in the definition of the fppf topology, a similar problem arises.)

[4] One can actually also work with large sites to obtain topoi as long as they contain a *small topologically generating set* or, even more general, satisfy the *WISC axiom.*

[5] There exists another topology in between the Zariski and étale topologies, the *Nisnevich topology*, but this is of no interest here.

[6] For X locally Noetherian this additional condition is superfluous.

Remark 1.3.9 (Standard covers). An equivalent approach to the definition of flat topologies is by first defining "standard" fppf or fpqc covers on affine schemes. These are similar to the flat covers defined above, but where the families are finite and the codomains affine. The covers of general schemes are then generated by base change.

By interpreting objects in the big topos (over a scheme) as generalized spaces in the sense of Chapter 8, the following definition is obtained:

Definition 1.3.10 (Algebraic space). An object $X \in \mathbf{Sh}(\mathbf{Sch}_{\text{fppf}})$ satisfying

1. The diagonal morphism $X \to X \times X$ is representable 2.4.28.
2. There exists a scheme morphism $Y \to X$ that surjective and étale. This is sometimes called an **atlas**.

Requiring representability of the diagonal can be argued as follows. The fibre product can be seen as a generalized notion of intersection, so representability means that the intersection of any two schemes is again a scheme. Moreover, representability of the diagonal also implies representability of the atlas morphism.

Definition 1.3.11 (Local property). Let P be some property of scheme morphisms, e.g. étale or smoothness. Relative to some topology on **Sch**, the property P is said to be **local on the source** if for every cover $\{X_i \to X\}_{i\in I}$ and every morphism $f : X \to Y$ the following holds:

$$f \text{ satisfies } P \iff \forall i \in I : X_i \to Y \text{ satisfies } P. \tag{1.32}$$

Analogously, the property P is said to be **local on the base** or **local on the target** if for every cover $\{Y_i \to Y\}_{i\in I}$ and every morphism $f : X \to Y$ the following holds:

$$f \text{ satisfies } P \iff \forall i \in I : Y_i \times_Y X \to Y_i \text{ satisfies } P. \tag{1.33}$$

Inspired by locality one can generalize many properties of scheme morphisms to morphisms of algebraic spaces:

Definition 1.3.12 (Property of algebraic spaces). Consider a representable morphism of algebraic spaces $F : X \to Y$. If P is a property of scheme morphisms satisfying:

1. it is preserved under base change, and
2. it is fppf-local on the target,

then f is said to have the property P if for all $T \in \text{ob}(\mathbf{Sch})$ and all $G : h_T \to Y$ the morphism $S_G \to T$ has property P, where $S_G \in \text{ob}(\mathbf{Sch})$ is a representing object of G, i.e. $h_{S_G} \cong h_T \times_Y F$.

Definition 1.3.13 (Étale morphism). A morphism $F : X \to Y$ of algebraic spaces such that for every étale morphism $G : h_S \to X$ the composition $f \circ g$ is also étale.[7]

1.4 Algebraic groups & Galois theory

Definition 1.4.1 (Linear algebraic group). A subgroup of $\text{GL}(n, F)$ defined by a (finite) set of polynomials in the matrix coefficients.

Property 1.4.2. From the definition it is immediately clear that intersections of algebraic groups are again algebraic.

[7] In a similar vein one can define smooth morphisms of algebraic spaces.

Definition 1.4.3 (Absolute Galois group). The Galois group $\mathrm{Gal}(K_S/K)$ of the separable closure **??** of K.

The following property relates Galois theory to *algebraic geometry* (Chapter 1):

Property 1.4.4 (Grothendieck's Galois theory). Consider a field K with separable closure K_S.

$$\mathrm{Gal}(K_S/K) \cong \pi_1\big(\mathrm{Spec}(K)\big)\,, \tag{1.34}$$

where π_1 is the étale fundamental group (the fundamental group at the topological level would not carry a lot of information since $\mathrm{Spec}(K)$ is a one-point space).

?? COMPLETE ??

1.5 Modular forms

This section gives a relation between *number theory* and (algebraic) geometry. The content of Chapter **??** will be used throughout this section.

Definition 1.5.1 (Modular group). In the setting of number theory, the projective special linear group $\mathrm{PSL}(2,\mathbb{Z})$ is often called the modular group. The modular group acts on the complex plane by **Möbius transformations**:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}. \tag{1.35}$$

For this reason the group is sometimes called the **Möbius group**.

Definition 1.5.2 (Modular form). A modular form of weight $k \in \mathbb{R}$ is a holomorphic function on the upper-half plane $f : \mathcal{H} \to \mathbb{C}$ satisfying the following two conditions:

1. **Automorphicity**: For all $g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$ one has $f\big(g(z)\big) = (cz+d)^k f(z)$, and
2. **Bounded growth**: $f(z)$ is bounded for $z \longrightarrow i\infty$.

If the modular form satisfies the stronger condition $f(z) \longrightarrow 0$ when $z \longrightarrow i\infty$, it is said to be **cuspidal** or it is simply called a **cusp form**.

Remark 1.5.3 (Arithmetic group). Modular forms can also be defined for subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ with finite index, the so-called **arithmetic groups**.

Property 1.5.4. The generators of the modular group are given by

$$z \mapsto -\frac{1}{z} \qquad \text{and} \qquad z \mapsto z+1. \tag{1.36}$$

Invariance under the second generator shows that modular forms are, in particular, periodic and, hence, admit a Fourier expansion. Cusp forms are exactly those modular forms with vanishing constant Fourier coefficient.

Chapter 2

Topos theory ♣

The main reference for this chapter is [6, 10]. For an introduction to stacks and descent theory, see [11].

2.1 Elementary topoi

Definition 2.1.1 (Subobject classifier). Consider a finitely complete category (in fact, the existence of a terminal object suffices). A subobject classifier is a mono[1] $\texttt{true} : 1 \hookrightarrow \Omega$ from the terminal object such that for every mono $\phi : x \hookrightarrow y$ there exists a unique morphism $\chi : y \to \Omega$ that fits in the following pullback square:

$$\begin{tikzcd}
x \arrow[r] \arrow[d, hook, "\phi"'] \arrow[dr, phantom, "\text{pb}"] & 1 \arrow[d, hook, "\texttt{true}"] \\
y \arrow[r, "\exists!\chi"'] & \Omega
\end{tikzcd}$$

Figure 2.1: Subobject classifier.

Alternative Definition 2.1.2. Consider a well-powered category $\mathbf{C}$. The assignment of subobjects $\mathrm{Sub}(x)$ to an object $x \in \mathrm{ob}(\mathbf{C})$ defines functor $\mathrm{Sub} : \mathbf{C}^{op} \to \mathbf{Set}$. A subobject classifier Ω is a representation of this functor, i.e. the following isomorphism is natural in x:

$$\mathrm{Sub}(x) \cong \mathbf{C}(x, \Omega). \tag{2.1}$$

Example 2.1.3 (Indicator function). The category **Set** has the 2-element set $\{\texttt{true}, \texttt{false}\}$ as subobject classifier. The morphism $\chi : S \to \Omega$ is the indicator function

$$\chi_S(x) = \begin{cases} \texttt{true} & x \in S \\ \texttt{false} & x \notin S. \end{cases} \tag{2.2}$$

Definition 2.1.4 (Elementary topos). An elementary topos is a finitely complete, Cartesian closed category admitting a subobject classifier. Equivalently, one can define an elementary topos as a finitely complete category that has all power objects exist.

[1] The symbol for this morphism will become clear in Section 2.2.

The power object Px of $x \in \mathrm{ob}(\mathcal{E})$ is related to the subobject classifier Ω by the following relation:

$$Px = \Omega^x. \tag{2.3}$$

Remark 2.1.5 (Finite colimits). The original definition by *Lawvere* also required the existence of finite colimits. However, it can be proven that finite cocompleteness follows from the other axioms.

Theorem 2.1.6 (Fundamental theorem of topos theory). *Let $\mathcal{E}$ be an elementary topos. The slice category $\mathcal{E}_{/x}$ is also a topos for every object $x \in \mathrm{ob}(\mathcal{E})$. The subobject classifier is given by $\pi_2 : \Omega \times x \to x$.*

Property 2.1.7 (Balanced). All monos in a topos are regular. Hence, every mono arises as an equalizer. Since every epic equalizer is necessarily an isomorphism **??**, it follows that every topos is balanced **??**.

Property 2.1.8 (Epi-mono factorization). Every morphism $f : x \to y$ in a topos factorizes uniquely as an epi followed by a mono:

$$x \overset{e}{\twoheadrightarrow} z \overset{m}{\hookrightarrow} y. \tag{2.4}$$

The mono is called the **image** of f.

2.2 Internal logic

In this subsection finitely complete categories that admit a subobject classifier are considered (they do not have to be elementary topoi).

Definition 2.2.1 (Truth value). A global element of the subobject classifier, i.e. a morphism $1 \to \Omega$. The subobject classifier Ω is also sometimes called the **object of truth values**.

Property 2.2.2 (Heyting algebra). For all objects x in an elementary topos, the poset of subobjects $\mathrm{Sub}(x)$ has the structure of a Heyting algebra **??** and, in particular, that of a locale **??**. Hence, every topos canonically gives an external Heyting algebra, namely $\mathrm{Sub}(1)$. Furthermore, every power object is an internal Heyting algebra. This in particular includes the subobject classifier $\Omega = P1$.

?? COMPLETE ??

2.3 Morphisms

Definition 2.3.1 (Base change). Consider a category $\mathbf{C}$ with pullbacks. For every morphism $f : x \to y$ one can define a functor $f^* : \mathbf{C}_{/y} \to \mathbf{C}_{/x}$. This functor acts by pullback along f.

Definition 2.3.2 (Logical morphism). A functor $f : \mathcal{E} \to \mathcal{F}$ between elementary topoi is said to be logical if it preserves finite limits, exponential objects and the subobject classifier.

Property 2.3.3. If a logical morphism has a left adjoint, it also has a right adjoint.

Definition 2.3.4 (Geometric morphism). A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ of elementary topoi consists of an adjunction

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, bend right, "f^*"'] \end{tikzcd}$$

where the left adjoint is left exact. The right adjoint f_* is called the **direct image** part of f and the left adjoint f^* is called the **inverse image** part. If f^* itself has a left adjoint, then f is said to be **essential**.

Definition 2.3.5 (Geometric embedding). A geometric morphism for which the direct image part is fully faithful.

Property 2.3.6 (Characterization of geometric embeddings). Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric embedding and let $W \subset \hom(\mathcal{F})$ be the collection of morphisms that are mapped to isomorphisms under f^*. $\mathcal{E}$ is both equivalent to the full subcategory of $\mathcal{F}$ on W-local objects (Definition **??**) and the localization $\mathcal{F}[W^{-1}]$ at W (Definition **??**).

Property 2.3.7 (Base change). The base change functors on a topos are logical and admit a left adjoint, the postcomposition functor. This implies that these functors can be refined to essential geometric morphisms.

Example 2.3.8 (Topological spaces). Every continuous function $f : X \to Y$ induces a geometric morphism

$$\begin{tikzcd} \mathbf{Sh}(X) \arrow[r, shift right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathbf{Sh}(Y) \arrow[l, shift right, "f^*"'] \end{tikzcd}, \tag{2.5}$$

where the direct image functor f_* is defined as

$$f_*F(U) := F(f^{-1}U) \tag{2.6}$$

for any sheaf $F \in \mathbf{Sh}(X)$ and any open subset $U \in \mathbf{Open}(Y)$. The inverse image functor f^* is defined using the equivalence between sheaves on topological spaces and étalé spaces. Consider a sheaf $E \in \mathbf{Sh}(Y)$ as an étalé space $\pi : E \to Y$. The inverse image of E along a continuous function $f : X \to Y$ is the pullback of π along f.

This example implies that the global elements $* \to X$ of a topological space induce geometric morphisms of the form $\mathbf{Sh}(*) \to \mathbf{Sh}(X)$. By noting that $\mathbf{Sh}(*) = \mathbf{Set}$, one obtains the following generalization:

Definition 2.3.9 (Point). A point of a topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

Notation 2.3.10 (Category of topoi). The category of elementary topoi and geometric morphisms is a 2-category. It is denoted by **Topos**.

However, to obtain the structure of a 2-category, one needs to define an appropriate notion of 2-morphism. Because a geometric morphism consists of an adjunction, one can consider two distinct conventions. Either one can choose the 2-morphisms in **Topos** to be the natural transformations $f^* \Rightarrow g^*$ (with associated transformations $g_* \Rightarrow f_*$) or one can choose them to be the natural transformations $f_* \Rightarrow g_*$ (and associated transformations $g^* \Rightarrow f^*$). This chapter follows [6] and the "inverse image convention" is used, i.e. a 2-morphism $f \Rightarrow g$ consists of natural transformations $f^* \Rightarrow g^*$ and $g_* \Rightarrow f_*$.

2.4 Grothendieck topos

Definition 2.4.1 (Sieve). Let $\mathbf{C}$ be a small category. A sieve S on $\mathbf{C}$ is a fully faithfull discrete fibration $S \hookrightarrow \mathbf{C}$.

A sieve S on an object $x \in \mathbf{C}$ is a sieve in the slice category $\mathbf{C}_{/x}$. This means that S is a subset of $\mathrm{ob}(\mathbf{C}_{/x})$ that is closed under precomposition, i.e. if $y \to x \in S$ and $z \to y \in \hom(\mathbf{C})$, then $z \to y \to x \in S$.

All of this can be summarized by saying that a sieve on an object $x \in \mathrm{ob}(\mathbf{C})$ is a subfunctor of the hom-functor $\mathbf{C}(-, x)$.

Example 2.4.2 (Maximal sieve). Let $\mathbf{C}$ be a category. The maximal sieve on $x \in \mathrm{ob}(\mathbf{C})$ is the collection of all morphisms $\{f \in \hom(\mathbf{C}) \mid \mathrm{cod}(f) = x\}$ or, equivalently, all of $\mathrm{ob}(\mathbf{C}_{/x})$.

Example 2.4.3 (Pullback sieve). Consider a morphism $f : x \to y$. Given a sieve S on y, one can construct the pullback sieve f^*S on x as the sieve of morphisms in S that factor through f:

$$f^*S(x) = \big\{g \,\big|\, f \circ g \in S(y)\big\}. \tag{2.7}$$

Property 2.4.4 (Presheaf topos). Consider the presheaf category $\mathbf{Psh}(\mathbf{C})$ on an arbitrary (small) category $\mathbf{C}$. This category is an elementary topos, where the subobject classifier assigns sieves:

$$\underline{\Omega}(x) := \{S \mid S \text{ is a sieve on } x\}. \tag{2.8}$$

The action on a morphism $f : x \to y$ gives the morphism $\underline{\Omega}(f)$ that sends a sieve S to its pullback sieve f^*S.

The morphism $\texttt{true} : \underline{1} \hookrightarrow \underline{\Omega}$ is defined as the natural transformation assigning to every object its maximal sieve. For every subobject $\underline{Y} \hookrightarrow \underline{X}$ the characteristic morphism $\chi_{\underline{Y}}$ is defined as follows. Consider an object $c \in \mathrm{ob}(\mathbf{C})$. The component $\chi_{\underline{Y}}|_c$ is then given by

$$\chi_{\underline{Y}}|_c(x) := \big\{f \in \mathbf{C}(d, c) \mid \underline{X}(f)(x) \in \underline{Y}(d)\big\}\,, \tag{2.9}$$

for all $x \in \underline{X}(c)$.

The following definition is due to *Giraud* (for the original definition using the notion of a *cover*, see the end of this section):

Definition 2.4.5 (Grothendieck topology). A Grothendieck topology on a category is a map J assigning to every object a collection of sieves satisfying the following conditions:

1. **Identity**[2]: For every object x the maximal sieve M_x is an element of $J(x)$ or, equivalently, all sieves generated by isomorphisms are in $J(x)$.
2. **Base change**: If $S \in J(x)$, then $f^*S \in J(y)$ for every morphism $f : y \to x$.
3. **Locality**: Consider a sieve S on x. If there exists a sieve $R \in J(x)$, such that for every morphism $(f : y \to x) \in R$ the pullback sieve $f^*S \in J(y)$, then $S \in J(x)$.

The sieves in J are called (J-)**covering sieves**. A collection of morphisms with codomain $x \in \mathrm{ob}(\mathbf{C})$ is called a **cover**[3] of x if the sieve generated by these morphisms is a covering sieve on x.

Example 2.4.6 (Topological spaces). These conditions have the following interpretation in the case of topological spaces:

- The collection of all open subsets covers a space U.

[2] The name itself stems from the fact that the maximal sieve is generated from the identity morphism.

[3] Sometimes this term is also used to denote any collection of morphism with common codomain x, i.e. without reference to a covering sieve.

- If $\{U_i\}_{i\in I}$ covers U, then $\{U_i \cap V\}_{i\in I}$ covers $U \cap V$.
- If $\{U_i\}_{i\in I}$ covers U and if for every $i \in I$ the collection $\{U_{ij}\}_{j\in J_i}$ covers U_i, then $\{U_{ij}\}_{i\in I, j\in J_i}$ covers U.

The canonical Grothendieck topology on $\mathbf{Open}(X)$ is given by the sieves $S = \{U_i \hookrightarrow U\}_{i\in I}$, where $\bigcup_{i\in I} U_i = U$. This topology is denoted by $J_{\mathbf{Open}(X)}$.

Definition 2.4.7 (Site). A (small) category equipped with a Grothendieck topology J.

A slightly weaker notion than that of a (Grothendieck) topology is the following:

Definition 2.4.8 (Coverage). Let $\mathbf{C}$ be a category. A coverage on $\mathbf{C}$ is a map that assigns to every object $x \in \mathrm{ob}(\mathbf{C})$ a collection of families $\{f : y \to x\} \subset \hom(\mathbf{C})$, the **covering families** or **covers**, satisfying the following condition. If $\{f : y \to x\}$ is a covering family on x, then for every morphism $g : x' \to x$ there exists a covering family $\{f' : y' \to x'\}$ on x' such that every composite $g \circ f'$ factors through some f.

Definition 2.4.9 (Matching family). Consider a presheaf $F \in \mathbf{Psh}(\mathbf{C})$ together with a sieve S on $x \in \mathrm{ob}(\mathbf{C})$. A matching family for S with respect to F is a natural transformation $\alpha : S \Rightarrow F$ between S, regarded as a subfunctor of $\mathbf{C}(-, x)$, and F.

More explicitly, it is an assignment of an element $x_f \in Fd$ to every morphism $(f : y \to x) \in S$ such that

$$F(g)(x_f) = x_{f\circ g} \tag{2.10}$$

for all morphisms $g : z \to y$. Equivalently, a matching family for S with respect to F is a set of elements $\{x_f\}_{f\in S}$ such that for all covering morphisms $f : y \to x, g : z \to x \in S$ and all morphisms $f' : c \to y, g' : c \to z$ such that $f \circ f' = g \circ g'$ the following equations holds:

$$F(f')(x_f) = F(g')(x_g). \tag{2.11}$$

Given such a matching family, one calls an element $a \in Fx$ an **amalgamation** if it satisfies

$$F(f)(a) = x_f \tag{2.12}$$

for all morphisms $f \in S(y)$. The existence of such an element can also be stated in terms of natural transformations. Consider the obvious inclusion ι_S of S into the the hom-functor $\mathbf{C}(-, x)$. Every morphism with codomain x can be obtained from the identity morphism by precomposition and, hence, a natural transformation $\mathbf{C}(-, x) \Rightarrow F$ is determined by its action on the identity morphisms $\mathbb{1}_x$. The existence of an amalgamation is thus equivalent to the existence of an extension of S along ι_S.

Remark 2.4.10. If the base category has all pullbacks, for example if it is a topos on its own, one can restrict the above commuting diagrams to the pullback diagrams of morphisms in the sieve S.

Definition 2.4.11 (Sheaf). Consider a site $(\mathbf{C}, J)$. A presheaf F on $\mathbf{C}$ is called a J-sheaf if every matching family for every covering sieve in J admits a unique amalgamation[4] or, equivalently, if all sieves admit a unique extension to representable presheaves.

The category $\mathbf{Sh}(\mathbf{C}, J)$ of J-sheaves on the site $(\mathbf{C}, J)$ is the full subcategory of $\mathbf{Psh}(\mathbf{C})$ on the presheaves that satisfy the above condition.

This definition can also be restated in terms of local objects **??**:

[4] If there exists at most one amalgamation, the presheaf is said to be **separated**.

Alternative Definition 2.4.12 (Sheaf). By definition every covering sieve admits a morphism into the Yoneda embedding: $\eta : S \hookrightarrow \mathcal{Y}x$. If the collection of all these morphisms is denoted by $\mathcal{S}$, a presheaf is a sheaf if and only if it is $\mathcal{S}$-local, i.e. if the following morphism is an isomorphism for all $\eta \in \mathcal{S}$:

$$Fx \cong \mathbf{Psh}(\mathcal{Y}x, F) \xrightarrow{\mathbf{Psh}(\eta,F)} \mathbf{Psh}(S, F). \tag{2.13}$$

This is also called the **descent condition** for sheaves. In this context the collection of matching families $\mathrm{Match}(S, F) := \mathbf{Psh}(S, F)$ for a sieve S with respect to a presheaf F is often called the **descent object** of S with respect to F.

Example 2.4.13 (Topological spaces). The usual category of sheaves $\mathbf{Sh}(X)$ on a topological space X is obtained as the category of sheaves on the site $(\mathbf{Open}(X), J_{\mathbf{Open}(X)})$. Since the morphisms in the covering sieves are exactly the inclusion maps $U_i \hookrightarrow U$, the pullback of two such morphisms is given by the intersection $U_i \cap U_j$. Hence, the condition for a matching family, as formulated in 2.4.9 above, gives the second part of Definition **??**. The uniqueness of an amalgamation is equivalent to the first part of that definition.

For topological spaces, sheaves are easily represented visually. A matching family assigns to every set U_i of an open cover $\mathcal{U} \equiv \{U_i\}_{i\in I}$ of U an element $x_i \in FU_i$, such that the restrictions coincide on double overlaps, as in step (1) in the figure below.

Cover | Matching family | Unique amalgamation

$$\begin{tikzcd}[column sep=small]
 & U_i \cap U_j \arrow[dl] \arrow[dr] & & & & x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j} & & & & x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j} & \\
U_i \arrow[dr] & & U_j \arrow[dl] \arrow[r, "(1)"] & {} & x_i \arrow[ur] & & x_j \arrow[ul] \arrow[r, "(2)"] & {} & x_i = x|_{U_i} \arrow[ur] \arrow[dr] & & x_j = x|_{U_j} \arrow[ul] \arrow[dl] \\
 & U & & & & & & & & x &
\end{tikzcd}$$

The descent condition then states that for every such matching family, there exists a unique element x on U, such that the elements of the matching family are restrictions of x as in step (2) of the figure above.

The classical example would be the assignment of the set of continuous functions to open subsets of a topological space. When two functions, defined on two open sets, coincide on the intersection, there exists a unique continuous function defined on the union, such that it restricts to the given functions.

Above, in Definition 2.4.8, the notion of a coverage was introduced. It should be clear that every coverage generates a sieve. Furthermore, although coverages are weaker and easier to handle, they are in fact equivalent for the purpose of sheaf theory:

Property 2.4.14. Consider a covering family C and let S_C be the sieve it generates. A presheaf is a sheaf for C if and only if it is a sheaf for S_C.

Example 2.4.15 (Canonical topology). The canonical topology on a category is the largest (finest) Grothendieck topology for which all representable presheaves are sheaves. A subcanonical topology is then defined as a subtopology of the canonical one, i.e. any Grothendieck topology for which all representable presheaves are sheaves.

Example 2.4.16 (Minimal and maximal topologies). The minimal Grothendieck topology on a category is the one for which only the maximal sieves are covering sieves. In this topology all presheaves are sheaves. The maximal Grothendieck topology is the one for which all sieves are covering sieves. In this topology only the terminal element of the associated presheaf category is a sheaf.

Definition 2.4.17 (Grothendieck topos). A category equivalent to the category of sheaves on a (small) site. This site is often called the **site of definition** for the given topos.

Property 2.4.18. Every Grothendieck topos is an elementary topos.

Property 2.4.19. For every Grothendieck topos there exists a site of definition for which the Grothendieck topology is (sub)canonical.

Construction 2.4.20 (Sheafification). Given a presheaf $\mathcal{F}$, one can construct a sheaf $\overline{\mathcal{F}}$ along the same lines of Construction **??**.

Definition 2.4.21 (Global sections functor). Every Grothendieck topos $\mathcal{E}$ admits a geometric morphism to **Set**, where the right adjoint assigns to an object x its set of global elements:

$$\Gamma : \mathcal{E} \to \mathbf{Set} : x \mapsto \mathcal{E}(1, x). \tag{2.14}$$

When $\mathcal{E}$ is the sheaf topos over a topological space, this is exactly the global sections functor **??**. The left adjoint assigns to every set S the copower $S \cdot 1 \equiv \bigsqcup_{s \in S} 1$. When $\mathcal{E}$ is a sheaf topos, this adjoint is exactly the constant sheaf functor. It is sometimes denoted by LConst.

A different approach for defining sheaf topoi is through an embedding of sheaves into presheaves.

Definition 2.4.22 (Local isomorphism). A system of local isomorphisms in $\mathbf{Psh}(\mathbf{C})$ is a class of morphisms in $\mathbf{Psh}(\mathbf{C})$ forming a system of weak equivalences **??** closed under pullbacks along morphisms out of representable presheaves.

Property 2.4.23 (Local isomorphisms and Grothendieck topologies). A system of local isos induces a *system of local epis* in the following way. $f : X \to Y$ is a local epi if $\mathrm{im}(f) \to Y$ is a local iso. A Grothendieck topology is defined by declaring a presheaf $F \in \mathbf{Psh}(\mathbf{C})$ to be a covering sieve at $X \in \mathrm{ob}(\mathbf{C})$ if $F \hookrightarrow \mathcal{Y}X$ is a local epi.

Alternative Definition 2.4.24 (Sheaf topos). A category $\mathbf{Sh}(\mathbf{C})$ equipped with a geometric embedding into $\mathbf{Psh}(\mathbf{C})$.

Proof of equivalence. By Property 2.3.6 such a category is equivalent to the full subcategory on S-local presheaves for some system of local isomorphisms S and, therefore, also to a sheaf topos in the sense of Grothendieck by the property above.

Remark 2.4.25 (Descent condition). This is essentially a restatement of the descent condition 2.4.12. Covering sieves, regarded as subfunctors, are in particular local isomorphisms. Stability of sieves under pullback together with the co-Yoneda lemma **??**, which says that every presheaf is a colimit of representables, generates the full collection of local isomorphisms.

By Property **??**, the construction of Grothendieck topoi as localizations of presheaf categories is equivalent to a definition in terms of reflections (this is due to *Street*):

Corollary 2.4.26. Given a small category $\mathbf{C}$, there exists a bijection between the Grothendieck topologies on $\mathbf{C}$ and the equivalence classes of left exact reflective subcategories of $\mathbf{Psh}(\mathbf{C})$.

Definition 2.4.27 (Representable morphism). A natural transformation of presheaves $F \to G$ is said to be representable if for every representable presheaf h_X and every morphism $h_X \to G$ the pullback $h_X \times_G F$ is representable.

Property 2.4.28 (Diagonals). The diagonal morphism $\Delta_F : F \to F \times F$, for F a presheaf on a category with pullbacks, is representable if and only if any of the following two equivalent properties holds:

1. For every two representable presheaves h_X, h_Y and natural transformations $h_X \to F, h_Y \to F$ the pullback $h_X \times_F h_Y$ is representable.
2. Every natural transformation $h_X \to F$ from a representable presheaf is representable.

2.4.1 Topological sheaves

See Chapter ?? for the application of sheaves to topology.

Property 2.4.29 (Presheaf topos). Consider the presheaf category

$$\mathbf{Psh}(X) := \mathbf{Psh}(\mathbf{Open}(X))$$

over a topological space (X, τ). Unpacking Property 2.4.4 shows that this category is an elementary topos where the subobject classifier is given by

$$\Omega(U) = \{V \in \tau \mid V \subseteq U\}. \tag{2.15}$$

Construction 2.4.30 (Sheaves and étalé bundles). Let X be a topological space. The functor

$$I : \mathbf{Open}(X) \to \mathbf{Top}_{/X} : U \mapsto (U \hookrightarrow X)$$

induces the following adjunction:

$$\begin{tikzcd} \mathbf{Top}_{/X} \arrow[r, shift right, "\Gamma"'] \arrow[r, phantom, "\perp"] & \mathbf{Psh}(X) \arrow[l, shift right, "E"'] \end{tikzcd}. \tag{2.16}$$

The slice category on the left-hand side is the category of (topological) bundles (Chapter 4) over X. Both directions of the adjunction have a clear interpretation. The right adjoint assigns to every bundle its sheaf of local sections and the left adjoint assigns to every presheaf its bundle of germs.

By restricting to the subcategories on which this adjunction becomes an adjoint equivalence, one obtains the **étalé space** and **sheaf categories** respectively:

$$\mathbf{Et}(X) \cong \mathbf{Sh}(X). \tag{2.17}$$

The category on the right-hand side is the category of sheaves on a topological space X. The category on the left is the full subcategory on local homeomorphisms, i.e. the étalé spaces ??.

Property 2.4.31 (Associated sheaf). The inclusion functor $\mathbf{Sh}(X) \hookrightarrow \mathbf{Psh}(X)$ admits a left adjoint, the **sheafification functor**, that assigns to every presheaf its associated sheaf. This functor is given by the composition $\Gamma \circ E$, which is just Construction ??.

The fact that the counit of the adjunction 2.4.30 restricts to an isomorphism on the full subcategory $\mathbf{Sh}(X)$ is equivalent to the fact that the sheafification of a sheaf Γ is again Γ.

Definition 2.4.32 (Petit and gros topoi[5]). Consider a topological space X together with its category of opens $\mathbf{Open}(X)$. The petit topos over X is defined as the sheaf topos $\mathbf{Sh}(X) \equiv \mathbf{Sh}(\mathbf{Open}(X))$. It represents X as a "generalized space". (By Construction 2.4.30 the objects in a petit topos are the étalé spaces over the given base space.) Topoi equivalent to such petit topoi are sometimes said to be **spatial**. However, one can also build a topos whose objects are themselves generalized spaces. To this end, choose a site S of "probes" and call the sheaf topos $\mathbf{Sh}(S)$ a gros topos. See Section 8.2 for more information.

Property 2.4.33 (Localic reflection). Mapping a topological space to its sheaf of continuous sections defines a functor $\mathbf{Sh} : \mathbf{Top} \to \mathbf{Topos}$ by Example 2.3.8. When restricted to the full subcategory of sober spaces ??, this functor becomes fully faithful.

Generalizing to locales even gives a reflective inclusion ??.

This property states that no information is lost when regarding (sober) topological spaces as sheaf topoi. This also explains the name "petit topos".

[5] For those that do not master French, petit and gros mean small and big, respctively.

Definition 2.4.34 (Localic topos). Multiple equivalent definitions exist:

1. A topos equivalent to a sheaf topos over a locale **??** equipped with the topology of jointly surjective morphisms.
2. A topos generated under colimits of subobjects of the terminal object 1.
3. A topos $\mathcal{E}$ for which the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$ is localic, i.e. every object in $\mathcal{E}$ is a subquotient of an object in the inverse image Γ^*

Given a geometric morphism to some base topos $\mathcal{S}$, one can define $\mathcal{S}$-localic topoi by generalizing the third point.

The following property shows that the locale in the first definition has a specific meaning:

Property 2.4.35. By Property 2.2.2, for every topos the poset $\mathrm{Sub}(1)$ is a locale. Every localic topos $\mathcal{E}$ satisfies $\mathcal{E} \cong \mathrm{Sh}\big(\mathrm{Sub}(1)\big)$, where $\mathrm{Sub}(1)$ is equipped with the topology of jointly surjective morphisms.

The equivalence between localic topoi and locales carries over to the notion of $\mathcal{S}$-localic topoi:

Property 2.4.36. The (2-)category of localic topoi over a base topos $\mathcal{S}$ is equivalent to the (2-)category $\mathbf{Loc}(\mathcal{S})$ of locales internal to $\mathcal{S}$.

Property 2.4.37. Given a locale X, the category $\mathbf{Loc}\big(\mathbf{Sh}(X)\big)$ is equivalent to the slice category $\mathbf{Loc}_{/X}$.

2.4.2 Lawvere-Tierney topology

Definition 2.4.38 (Lawvere-Tierney topology). As noted in Section 2.2 on the internal logic of elementary topoi, the subobject classifier Ω has the structure of an internal Heyting algebra and, in particular, that of a meet-semilattice, where the meet is given by the pullback of morphisms. This internal poset, viewed as an internal category, admits the construction of a closure operator $j : \Omega \to \Omega$ (Definition **??**) satisfying the following condition:

$$j \circ \wedge = \wedge \circ (j \times j). \tag{2.18}$$

This condition states (in a nontrivial way) that j is (internally) order-preserving.

Remark 2.4.39. The condition satisfied by the unit morphism in the definition of a closure operator can also be reformulated in this context as follows:

$$j \circ \mathtt{true} = \mathtt{true}. \tag{2.19}$$

The Lawvere-Tierney operator also induces a "closure operator" on all posets $\mathrm{Sub}(x)$ in the topos. Given an object x and a subobject $u \in \mathrm{Sub}(x)$, one defines the closure $j_*(u) \in \mathrm{Sub}(x)$ as the subobject classified by the characteristic morphism $j \circ \chi_u : x \to \Omega$.

Definition 2.4.40 (Dense object). Given a Lawvere-Tierney topology $j : \Omega \to \Omega$, a subobject $u \in \mathrm{Sub}(x)$ is said to be dense (in x) if it satisfies $j_*(u) = x$.

Alternative Definition 2.4.41 (Sheaf). Given a Lawvere-Tierney topology $j : \Omega \to \Omega$ on a topos $\mathcal{E}$, one calls an object $s \in \mathrm{ob}(\mathcal{E})$ a j-sheaf if for all dense morphisms $u \hookrightarrow x$ the induced map

$$\mathcal{E}(x, s) \to \mathcal{E}(u, s)$$

is a bijection.

Property 2.4.42. For the presheaf topos on a small category $\mathbf{C}$, the Grothendieck topologies on $\mathbf{C}$ are equivalent to Lawvere-Tierney topologies on $\mathbf{Psh}(\mathbf{C})$.

Sketch of proof. Since a Grothendieck topology assigns to every object a collection of sieves, Property 2.4.4 implies that $J(x) \subseteq \Omega_{\mathbf{Psh}}(x)$ for all $x \in \mathrm{ob}(\mathbf{C})$. By the base change condition of Grothendieck topologies, this relation is natural in x and, hence, J is a subobject of $\Omega_{\mathbf{Psh}}$. One thus finds a characteristic morphism $j : \Omega_{\mathbf{Psh}} \to \Omega_{\mathbf{Psh}}$ that can be proven (by the other conditions of Grothendieck topologies) to define a Lawvere-Tierney topology on $\mathbf{Psh}(\mathbf{C})$. Conversely, a Lawvere-Tierney topology is a morphism $j : \Omega \to \Omega$ and, hence, determines a unique subobject of $\Omega_{\mathbf{Psh}}$, i.e. a unique collection of sieves for every object $x \in \mathrm{ob}(\mathbf{C})$. From the conditions of Lawvere-Tierney topologies one can then prove that this collection satisfies the conditions of a Grothendieck topology.

Remark 2.4.43. It follows that Lawvere-Tierney topologies generalize Grothendieck topologies from presheaf topoi to arbitrary elementary topoi.

2.5 Stacks

2.5.1 2-sheafs

An important subject, especially in the context of gauge theories in physics, is that of groupoid-valued (pre)sheaves. To this end, sites are generalized to higher category theory.

Definition 2.5.1 (2-presheaf). Consider a 2-category $\mathbf{C}$. A 2-presheaf on $\mathbf{C}$ is a pseudofunctor $F : \mathbf{C}^{op} \to \mathbf{Cat}$. When $\mathbf{C}$ is the categorification of a 1-category, i.e. when it has discrete hom-categories, 2-presheaves are better known as **prestacks**.

Definition 2.5.2 (2-coverage). Virtually the same as an ordinary coverage 2.4.8, but factorization is only required to exist up to an isomorphism. A 2-category equipped with a 2-coverage is called a **2-site**.

As for 1-sites, every coverage generates a unique sieve. It is the full subcategory on those morphisms that factor through a covering map in the given coverage (again up to isomorphism).

As in the case of ordinary categories (Definition 2.4.12), one can define 2-sheaves through a descent condition:

Definition 2.5.3 (2-sheaf). A 2-presheaf $F : \mathbf{C}^{op} \to \mathbf{Cat}$ on a 2-site $(\mathbf{C}, J)$ is said to be a 2-sheaf with respect to J if for all sieves $S \in J$ the following functor is an equivalence:

$$Fc \cong \mathbf{Psh}_2(\mathcal{Y}c, F) \to \mathbf{Psh}_2(S, F), \tag{2.20}$$

where the fist equivalence is just the 2-Yoneda lemma.

Remark 2.5.4. It should be noted that 2-(pre)sheaves can also be defined on ordinary (1-)sites. Sieves, regarded as subfunctors of the Yoneda embedding, take values in $\mathbf{Set}$. By composing these with the embedding $\mathbf{Set} \hookrightarrow \mathbf{Cat}$ of sets as (discrete) categories, one obtains 2-subfunctors of the 2-Yoneda embedding. Often 2-sheaves over 1-sites are called **stacks** (although this terminology is also used for general 2-sites).

Definition 2.5.5 (Prestack of groupoids). Consider a category $\mathbf{C}$. A prestack of groupoids on $\mathbf{C}$ is a $\mathbf{Grpd}$-valued prestack on $\mathbf{C}$.

The category of (groupoid-valued) prestacks becomes $\mathbf{Grpd}$-enriched if one takes the hom-category between two prestacks F, G to consist of the following data:

- **Objects**: The natural transformations $\alpha : F \Rightarrow G$ (note that the components are themselves functors).
- **Morphisms**: The "strict modifications" in the sense that they map objects in $\mathbf{C}$ to natural transformations satisfying the whiskering condition (see also Definition **??**)

$$\mathbb{1}_{Ff} \cdot \mathfrak{m}_b = \mathfrak{m}_a \cdot \mathbb{1}_{Gf}. \tag{2.21}$$

For ordinary sites and presheaves, descent was defined in terms of matching families. Since presheaves are now taking values in a 2-category, the matching families are a bit more complex. However, this structure is already familiar from differential geometry and algebraic topology, where it is known under the name of the *Čech nerve*:

Definition 2.5.6 (Čech groupoid). Consider a site $(\mathbf{C}, J)$. To every covering family $\mathcal{U} \equiv \{f_i : x_i \to x\}_{i\in I}$ one can assign an internal groupoid in presheaves $C(\mathcal{U})$ consisting of the following data:

- **Objects**: $\bigsqcup_i \mathcal{Y}x_i$
- **Morphisms**: $\bigsqcup_{i,j} \mathcal{Y}x_i \times_{\mathcal{Y}x} \mathcal{Y}x_j$

This is equivalent to the ($\mathbf{Grpd}$-valued) presheaf that assigns to every object $y \in \mathrm{ob}(\mathbf{C})$ the groupoid consisting of the following data:

- **Objects**: The pairs $(i, g_i : y \to x_i)$ where $x_i \in \mathcal{U}$.
- **Morphisms**: A unique arrow $(i, g_i) \to (j, g_j)$ if and only if $f_i \circ g_i = f_j \circ g_j$.

Comparing the definition of morphisms in the Čech groupoid to the condition for matching families in Definition 2.4.9, shows that one could presume that the Čech groupoid is related to the matching families. This intuition is indeed correct:

Property 2.5.7 (Matching families). Any ordinary presheaf F can be considered to be $\mathbf{Grpd}$-valued by postcomposing with the embedding $\mathbf{Set} \hookrightarrow \mathbf{Grpd}$. For any covering family $\mathcal{U}$, there exists an isomorphism

$$[\mathbf{C}^{op}, \mathbf{Grpd}](C(\mathcal{U}), F) \cong \mathbf{Psh}_2(\mathcal{U}, F). \tag{2.22}$$

Because the Čech groupoid (co)represents a descent object, it is sometimes called a **codescent object**.

It is exactly this (co)descent property of the Čech groupoid that will be used in Chapter 8 to define (higher) smooth groupoids.

People with some experience in algebraic topology will also notice that the Čech groupoid only contains the first degrees of the Čech complex. The full Čech complex can be obtained from the following construction:

Definition 2.5.8 (Čech nerve). Consider a morphism $f : y \to x$ in a category $\mathbf{C}$. The Čech nerve $C_\bullet(f)$ is the simplicial object **??** that is defined as the $(k+1)$-fold pullback of f along itself in degree k. For a covering family $\mathcal{U} \equiv \{f_i : x_i \to x\}$, the Čech nerve is defined as $C_\bullet(\mathcal{U}) := C_\bullet(\bigsqcup_i x_i \to x)$.

For ∞-sheaves the full Čech nerve will be used. However, for 2-sheaves and, in particular, stacks, only its 3-coskeleton is necessary. The extra information will encode the *cocycle condition* (4.1) known for example from the study of fibre bundles.

2.5.2 Stacks on a 1-site

For the definition of stacks, one needs the notions of fibred categories or, equivalently, pseudofunctors as defined in Section **??**. The definitions are recalled here:

Consider a functor $\Pi : \mathbf{A} \to \mathbf{B}$. A morphism f in $\mathbf{A}$ is said to be Π-Cartesian if for every morphism φ in $\mathbf{A}$ and factorization of $\Pi\varphi$ through Πf in $\mathbf{B}$, there exists a unique factorization of φ through f. f is called the inverse image of Πf.

A fibred category consists of a functor $\Pi : \mathbf{A} \to \mathbf{B}$ such that for each morphism in $(f : c \to d)\mathbf{B}$ with $d \in \mathrm{im}(\Pi)$ and each lift $y \in \mathbf{A}_d$ there exists at least one inverse image in $(\widetilde{f} : x \to y) \in \mathbf{A}$ of f. By the Grothendieck construction every fibred category gives rise to a pseudofunctor $F : \mathbf{B}^{op} \to \mathbf{Cat}$ by sending objects to their fibres under Π and sending morphisms f to their pullback functors f^*.

Definition 2.5.9 (Descent datum). Consider a category $\mathbf{C}$ with a covering family $\mathcal{U} \equiv \{f_i : x_i \to x\}$ and a pseudofunctor $F : \mathbf{C}^{op} \to \mathbf{Cat}$. The projections associated to the pullback $x_i \cap x_j := x_i \times_x x_j$ will be denoted by π_i and π_j (and analogously for iterated pullbacks). A descent datum for $\mathcal{U}$ with respect to F is a pair of families $(\{g_i\}, \{f_{ij}\})_{i,j\in I}$, where $\{g_i\}$ is a matching family for $\mathcal{U}$ with respect to F and every f_{ij} is an isomorphism $\pi_i^* x_i \cong \pi_j^* x_j$. This data is required to satisify the following **cocycle condition**:

$$\pi_{ik}^* f_{ik} = \pi_{ij}^* f_{ij} \circ \pi_{jk}^* f_{jk}. \tag{2.23}$$

Morphisms $(\{g_i\}, \{f_{ij}\}) \to (\{g_i'\}, \{f_{ij}'\})$ between descent data are families of morphisms $\{\phi_i : g_i \to g_i'\}$ that satisfy

$$\pi_i^* \phi_i \circ f_{ij} = f_{ij}' \circ \pi_j^* \phi_j. \tag{2.24}$$

The category of descent data for $\mathcal{U}$ with respect to F will be denoted by $\mathrm{Descent}(\mathcal{U}, F)$.

Construction 2.5.10. Consider an object ξ in Fx. From this object one can construct a descent datum as follows. The objects g_i are the pullbacks $f_i^*\xi$ and the isomorphisms $f_{ij} : \pi_i^* f_i^* \xi \cong \pi_j^* f_j^* \xi$ are obtained from the fact that both these objects are (Cartesian) pullbacks of the same morphisms. Arrows in Fx induce morphisms of descent data by (Cartesian) pullbacks along the covering maps. This construction defines a functor $Fx \to \mathrm{Descent}(\mathcal{U}, F)$. It can be shown that this construction is independent of a choice of cleavage up to equivalence.

Definition 2.5.11 (Stack). Consider a fibred category F over a site $(\mathbf{C}, J)$.

- F is called a **separated prestack** if for each covering family $\mathcal{U}$ on $x \in \mathrm{ob}(\mathbf{C})$, the functor $Fx \to \mathrm{Descent}(\mathcal{U}, F)$ is fully faithful.
- F is called a **stack** if for each covering family $\mathcal{U}$ on $x \in \mathrm{ob}(\mathbf{C})$ the functor $Fx \to \mathrm{Descent}(\mathcal{U}, F)$ is an equivalence.

This is a generalization of the descent condition 2.4.12. This can be seen by observing that $\mathrm{Descent}(\mathcal{U}, F) \cong \mathbf{Psh}_2(S(\mathcal{U}), F)$, where $S(\mathcal{U})$ is the sieve generated by $\mathcal{U}$ regarded as a fibred category. When F is fibred over groupoids, it is called a **stack of groupoids**. This forms the category $\mathbf{Sh}_{(2,1)}(\mathbf{C})$ of $(2,1)$-sheaves. In fact, it is this subcategory that is usually meant when considering stacks.

A more conceptual (although completely equivalent) generalization from (1-)sheaves to 2-sheaves can be obtained by starting from Property 2.5.7. There it was shown that matching families for (1-)presheaves can be obtained as natural transformations from the Čech groupoid.

Property 2.5.12 (Descent data and Čech nerve). Let $C(\mathcal{U})$ denote the 3-coskeleton of the Čech nerve $C_\bullet(\mathcal{U})$. Pseudonatural transformations $C(\mathcal{U}) \Rightarrow F$ can be shown to be equivalent to tuples $(c, \{c_i\}, \{c_{ij}\}, \{c_{ijk}\})$, where $c_i \in Fx_i$, that fit into cubes lying in the image of $C_2(\mathcal{U})$ in which all edges consist of Cartesian morphisms. Arrows between such cubes are given by arrows between the vertices that make the "obvious" diagrams commute.

By comparing these cubes to the previous definition of descent data, one obtains the following equivalence:

$$\text{Descent}(\mathcal{U}, F) \cong [\mathbf{C}^{op}, \mathbf{Cat}](C(\mathcal{U}), F). \tag{2.25}$$

?? FINISH THIS ??

Remark 2.5.13 (1-sheaves). Although most of the above seems very abstract and complex compared to ordinary sheaves, it is not quite so. In fact, when restricting to pseudofunctors of the form $\mathbf{C}^{op} \rightarrow \mathbf{Set}$, where the embedding $\mathbf{Set} \hookrightarrow \mathbf{Cat}$ sends sets to discrete categories, one obtains ordinary sheaves as a subcategory of stacks. For example, by the equivalence between pseudofunctors and Grothendieck fibrations, it is known that the Cartesian pullbacks f^* are in fact just the images of morphism f under the pseudofunctor F. This way the condition $\pi_1^* c_i \cong \pi_2^* c_j$ can be rewritten as $Ff_i'(c_i) = Ff_j'(c_j)$, which is nothing but the matching family condition (2.11).

2.6 Higher topos theory

In this section the notion of topos is generalized from ordinary category theory to higher category theory. In particular, ∞-sheaves will be defined. This will require a suitable foundation for ∞-category theory. To this end the language of (simplicial) model categories as introduced in Chapter **??** will be used.

Definition 2.6.1 (∞-groupoid). Objects of the full simplicial subcategory of $\mathbf{sSet}_{Quillen}$ on Kan complexes. From Property **??**, it is immediately clear how this generalizes the definition of ordinary groupoids. For groupoids one needs unique horn fillers (composition in ordinary categories is unique), while for ∞-groupoids this is allowed to be unique up to higher coherence.

Definition 2.6.2 ($(\infty,1)$-category). An $\infty\mathbf{Grpd}$-enriched category or, equivalently, a simplicially enriched category for which all hom-objects are Kan complexes. The functor category between $(\infty,1)$-categories is defined through the (simplicial) nerve and realization functors **??**:

$$[\mathbf{C}, \mathbf{D}] := |\mathbf{sSet}(N\mathbf{C}, N\mathbf{D})|. \tag{2.26}$$

Property 2.6.3 (Čech model structure). For any small category $\mathbf{C}$, the ∞-category of $\infty\mathbf{Grpd}$-valued ∞-sheaves can be represented by the category $[\mathbf{C}^{op}, \mathbf{sSet}]$ of simplicial presheaves on $\mathbf{C}$ by a theorem of *Lurie* **??**, i.e. there exists an ∞-equivalence between $\mathbf{Sh}_{(\infty,1)}(\mathbf{C})$ and the full subcategory on fibrant-cofibrant objects of the (left Bousfield) localization of $[\mathbf{C}^{op}, \mathbf{sSet}]$ at the Čech nerve projections. The resulting model structure is called the **Čech model structure**.

A presheaf X is fibrant in this model structure if the map

$$\text{Hom}(M, X) \rightarrow \text{Hom}(\mathcal{C}(\mathcal{U}), X) \tag{2.27}$$

is a weak equivalence for all open covers $\mathcal{U}$, i.e. exactly if X satisfies the descent condition and, hence, is an ∞-stack.

The most straightforward definition of an ∞-sheaf generalizes Definition 2.4.12:

Definition 2.6.4 (∞-sheaf). Consider an ∞-site $(\mathbf{C}, J)$ and let S denote the collection of monomorphisms in $\mathbf{Psh}_\infty(\mathbf{C})$ induced by the covering sieves. An ∞-presheaf on $\mathbf{C}$ is called a J-sheaf if it is S-local. A presheaf with values in an ∞-category $\mathbf{D}$ is called a sheaf if the representable presheaf $\mathbf{D}(x, F-)$ is a J-sheaf for all $x \in \mathrm{ob}(\mathbf{D})$.

In terms of the Čech nerve $\mathcal{C}$, the descent condition can be written as follows:

$$Fx \simeq \mathbf{Psh}_\infty(\mathcal{C}(\mathcal{U}), F) \tag{2.28}$$

for all covers $\mathcal{U}$ of x, where $\simeq$ denotes a weak equivalence.

Definition 2.6.5 (∞-stack). An $(\infty, 1)$-sheaf taking values in $\infty\mathbf{Grpd}$.

Property 2.4.21 can be generalized as follows:

Property 2.6.6. For every ∞-topos $\mathbf{H}$ there exists a geometric morphism $(\mathrm{Disc} \dashv \Gamma) : \mathbf{H} \leftrightarrows \infty\mathbf{Grpd}$. Any morphism into a discrete object $\mathrm{Disc}(X)$ is constant.

The left adjoint is sometimes called the **discrete object functor**. This terminology stems from the case of the forgetful functor $\Gamma : \mathbf{Top} \to \mathbf{Set}$, where the (fully faithful) left adjoint equips a set with the discrete topology.

Example 2.6.7 (Sheaves on manifolds). One of the archetypal examples of ∞-topoi is the topos of sheaves over smooth manifolds. By the Yoneda embedding one can regard a manifold as a sheaf and the global sections functor maps this representable sheaf to the manifold itself: $\Gamma(M) = M$. For a Lie group one can construct the classifying stack $\mathbf{B}G$. The global sections functor maps this stack to the delooping groupoid BG.

Definition 2.6.8 (Mapping stack). Consider two ∞-stacks $X, Y \in \mathbf{Sh}_{(\infty,1)}(\mathbf{C})$. The mapping stack is defined as follows:

$$[X, Y](U) := \mathbf{Sh}_{(\infty,1)}(\mathbf{C})(X \times U, Y), \tag{2.29}$$

where on the right-hand side, U denotes the representable ∞-stack.

?? FINISH (PERHAPS MOVE infinity-CATEGORY STUFF TO CHAPTER ON MODEL THEORY) ??

2.7 Cohomology

In this section, cohomology will be generalized to the ∞-categorical setting.

First, take a topological space X and an ∞-groupoid G. Geometric realization **??** gives an equivalence $\infty\mathbf{Grpd} \cong \mathbf{Top}$ and, therefore, one can define the intrinsic cohomology of X with coefficients in G as follows:

$$H(X; G) := \pi_0 \mathbf{Top}(X, |G|). \tag{2.30}$$

X can also be identified with its petit (∞-)topos $\mathbf{Sh}_{(\infty,1)}(X)$, in which X sits as the terminal object. From this point of view the intrinsic cohomology of X with coefficients in G is

$$\overline{H}(X; G) := \pi_0 \mathbf{Sh}_{(\infty,1)}(X)(X, \mathrm{LConst}\, G) \cong \pi_0 \circ \Gamma \circ \mathrm{LConst}(G). \tag{2.31}$$

This is the **cohomology with constant coefficients** of X with coefficients in G. If X is paracompact, the two cohomologies coincide: $H(X; G) \cong \overline{H}(X; G)$.

Now, it is time to pass to general cohomology:

Definition 2.7.1 (Intrinsic cohomology). Consider a $(\infty, 1)$-category $\mathbf{H}$. For every two objects $X, A \in \mathbf{H}$, the hom-space $\mathbf{H}(X, A)$ is an ∞-groupoid. Define the following notions:

- The objects in $\mathbf{H}(X, A)$ are called **cocycles**.
- The morphism in $\mathbf{H}(X, A)$ are called **coboundaries**.
- The set of connected components

$$H(X; A) := \pi_0\mathbf{H}(X, A) = \mathrm{Hom}_{\mathbf{Ho(H)}}(X, A)\,, \tag{2.32}$$

  where $\mathbf{Ho}(\mathbf{H})$ is the homotopy category **??** of $\mathbf{H}$, is called the intrinsic cohomology of X with coefficients in A.

If the object A admits an n-delooping $\mathbf{B}^n A$, the n^{th} cohomology group of X is defined as

$$H^n(X; A) := H(X; \mathbf{B}^n A). \tag{2.33}$$

Example 2.7.2 (Singular cohomology). Consider a topological space X. For every group G one can define the first delooping **??**, so one can also define the zeroth and first cohomology groups $H^{0,1}(X; G)$. Only when G is Abelian do higher deloopings exists (in fact, if G is Abelian all higher deloopings exist), and so in this case higher cohomology groups $H^{\geq 2}(X; G)$ can be defined. It can be shown that these coincide with the singular cohomology groups of X.

Example 2.7.3 (Group cohomology). Consider a (discrete) group G together with its delooping groupoid $\mathbf{B}G$. The group cohomology **??** of a group with coefficients in an Abelian group A is given by the intrinsic cohomology of $\infty\mathbf{Grpd}$ of the delooping groupoids:

$$H(G; A) \cong \pi_0\infty\mathbf{Grpd}(\mathbf{B}G, \mathbf{B}A). \tag{2.34}$$

By replacing the topos $\mathbf{H}$ by a slice topos $\mathbf{H}_{/X}$ one obtains twisted cohomology:

Definition 2.7.4 (Twisted cohomology). Consider a $(\infty, 1)$-topos $\mathbf{H}$ with some object $X \in \mathrm{ob}(\mathbf{H})$. The mapping space $\mathbf{H}(X, A)$, the cocycles of X with coefficients in A, is easily seen to be isomorphic to the mapping space $\mathbf{H}_{/X}(X, X \times A)$, where the second argument is equipped with the canonical projection morphism. Morphisms in this space are just sections of the trivial A-∞-bundle over X. General twisted cohomology can then be defined as the space of sections of an arbitrary A-∞-bundle over X.

By passing to classifying morphisms of bundles one obtains the twist $\chi : X \to \mathbf{BAut}(A)$ and the universal bundle $\rho_A : A/\!\!/\mathbf{Aut}(A) \to \mathbf{BAut}(A)$. χ-twisted cohomology is then given by (hte connected components of) the following mapping space:

$$\mathbf{H}_{/\mathbf{BAut}(A)}(\chi, \rho_A). \tag{2.35}$$

2.8 Cohesion

In this section the terminology "(Grothendieck) topos **over** a topos $\mathcal{S}$" will mean a topos equipped with a geometric morphism to $\mathcal{S}$.

Definition 2.8.1 (Local topos). Consider a topos $\mathcal{E}$ over a base topos $\mathcal{S}$. $\mathcal{E}$ is said to be ($\mathcal{S}$-)local if the geometric morphism $(f^* \dashv f_*) : \mathcal{E} \leftrightarrows \mathcal{S}$ admits a right adjoint $f^!$ such that one of the following equivalent statements holds:

- $f^!$ is fully faithful.
- f^* is fully faithful.

- $f^!$ is an $\mathcal{S}$-indexed functor **??**.
- $f^!$ is Cartesian closed **??**.

If one takes $\mathcal{S} = \mathbf{Set}$, the conditions are automatically satisfied since all functors are **Set**-indexed.

The right adjoint is sometimes called the **codiscrete object functor** coDisc (in fact, this terminology is applied more generally when $\mathcal{E}$ is just any category). If this functor exists, $\mathcal{E}$ is said to have **codiscrete objects**.

Property 2.8.2. A topos is local if and only if 1 is tiny **??**.

Definition 2.8.3 (Locally connected topos). An object in a category is said to be **connected** if its representable functor preserves finite coproducts. A topos is said to be **locally connected** if all objects can be written as coproducts of connected objects. This defines a functor

$$\Pi_0 : \mathcal{E} \to \mathbf{Set} : \bigsqcup_{i\in I} X_i \mapsto I \tag{2.36}$$

left adjoint to the discrete object functor (which is itself left adjoint to the global section functor). This functor is suitably called the **connected components functor**.

A topos is locally connected if and only if its global section geometric morphism is essential. More generally, a topos over some base topos $\mathcal{S}$ is said to be **locally connected** if its associated geometric morphism is essential and the left adjoint is $\mathcal{S}$-indexed. In the case of $(\infty, 1)$-topoi, the image of the functor Π_0 is called the **fundamental ∞-groupoid**.

Definition 2.8.4 (Connected topos). A topos over a base topos is said to be **connected** if the inverse image part of the associated geometric morphism is fully faithful. For sheaf topoi over a topological space X this is exactly the requirement that X is connected.

For locally connected topoi this amounts to the property that the left adjoint in its adjoint triple preserves the terminal object. Furthermore, a locally connected topos is said to be **strongly connected** if the left adjoint in its adjoint triple preserves finite products (in particular turning it into a connected topos).

Property 2.8.5. Every local topos is connected.

Definition 2.8.6 (Cohesive topos). A local, strongly connected topos. This implies the existence of an adjoint quadruple $(\Pi_0, \mathrm{Disc}, \Gamma, \mathrm{coDisc})$ where both Disc and coDisc are fully faithful.

Property 2.8.7 (Cohesive modalities). The adjoint quadruple on a cohesive topos induces an adjoint triple of modalities **??**, i.e. idempotent (co)monads (see Section **??** for a formal introduction in the context of type theory):

$$(\int \dashv \flat \dashv \sharp) := (\mathrm{Disc}\circ\Pi_0 \dashv \mathrm{Disc}\circ\Gamma \dashv \mathrm{coDisc}\circ\Gamma). \tag{2.37}$$

These are respectively called the **shape**, **flat** and **sharp** modalities. The modal types of the flat and sharp modalities are called the **discrete** and **codiscrete objects**, respectively.

?? COMPLETE (e.g. work by Schreiber) ??

Chapter 3

Manifolds

References for this chapter (and Part **??** in general) are [12--17].

3.1 Charts

Definition 3.1.1 (Chart). Consider a topological space M and consider an open subset of $U \subseteq M$ such that there exists a homeomorphism $\varphi : U \to O$ where O is an open subset of $\mathbb{R}^n$. The pair (U, φ) is called a chart on M.

Definition 3.1.2 (Transition map). Let (U_1, φ_1) and (U_2, φ_2) be two charts on M. The mapping $\varphi_1^{-1} \circ \varphi_2$, defined on the intersection $U_1 \cap U_2$, is called the transition map between the charts.

If $\varphi_1^{-1} \circ \varphi_2$ is continuous, the charts are said to be C^0-compatible. However, because the composition of any two continuous functions is also continuous, every two charts on a topological space are automatically C^0-compatible.

Definition 3.1.3 (Atlas). Let M be a topological space and let $\{(U_i, \varphi_i)\}_i$ be a collection of pairwise compatible charts covering M. This collection of charts is called an atlas on M. From the above remark on C^0-compatibility it follows that every atlas is a C^0-atlas. By requiring the transition functions to satisfy additional conditions, other types of atlases can be defined.

Definition 3.1.4 (Maximal atlas). Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two atlases on the same topological space. If $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$ is again an atlas, the atlases are said to be **equivalent** or **compatible**. A maximal union of compatible atlases is called a maximal atlas.

Definition 3.1.5 (Manifold). A topological space equipped with a maximal C^0-atlas is called a **topological** manifold. An alternative definition (often used in topology) is that of a locally Euclidean Hausdorff space. The topology is generated by the collection of charts.

Remark. In the literature second-countability is often added to the definition of a topological manifold. This ensures that the space has (among others) the property of paracompactness **??** and, hence, lends itself to the construction of partitions of unity (which are for example necessary for the introduction of integration theory as in Section 5.8).

(For an alternative definition of manifolds in the context of *smooth spaces* see Section 8.2.)

If all transition maps are C^k-diffeomorphisms, the manifold is called a C^k-manifold. The limiting case, a C^∞-manifold, is also called a **smooth manifold**. If the transition maps are not only smooth, but even analytic **??**, the manifold is called an **analytic** or C^ω-manifold. A topological manifold equipped with a maximal atlas for which the transition maps are piecewise-linear is called a **PL manifold**.

Definition 3.1.6 (Structure sheaf ♣). Let M be a C^k-manifold. The structure sheaf $\mathcal{O}_M$ is defined as the sheaf **??** that assigns to every open set $U \subseteq M$ the set of C^k-functions $f : U \rightarrow \mathbb{R}$.

Generally, one can define for all $j \leq k$ the sheaf $\mathcal{O}_M^j$ as the sheaf that assigns to every open set $U \subseteq M$ the set of C^j-functions $f : U \rightarrow \mathbb{R}$.

From the "sheafy" point of view one can equivalently define a smooth manifold as a locally ringed space **??** that is locally isomorphic to $\mathbb{R}^n$ equipped with its standard space of differentiable functions. (This is an extension of the algebro-geometric constructions from Sections 1.1 and 1.2.)

Property 3.1.7. Two C^k-manifolds are isomorphic if and only if their associated structure sheaves are isomorphic. Moreover, if the manifolds are second-countable and paracompact, they are isomorphic if their function algebras are isomorphic as rings. The manifolds can, up to isomorphism, be completely reconstructed from this algebraic data. (This can be seen as an analogue of the Gel'fand-Naimark theorem **??**. However, no compactness is required here.)

Theorem 3.1.8 (Whitney). *Every C^k-atlas on a paracompact space contains a C^∞-atlas. Furthermore, two C^k-atlases are equal if and only if they contain the same C^∞-atlas. It follows that every differentiable manifold is automatically smooth.*

Theorem 3.1.9 (Radó-Moise). *In dimensions $1, 2$ and 3 there exists for every topological manifold a unique smooth structure.*

Theorem 3.1.10. *For dimensions higher than 4 there exist only finitely many distinct smooth structures on compact manifolds. In fact, for PL manifolds the number of smooth structures is fixed for each dimension (except for 4).*

Remark 3.1.11. In dimension 4 there only exist partial results. For noncompact manifolds there uncountably many distinct smooth structures exist, while for compact manifolds no complete characterization has been found.

Definition 3.1.12 (Smooth function). Let $f : M \rightarrow N$ be a function between two smooth manifolds. It is said to be smooth if there exist charts (U, φ) and (V, ψ) for M and N with $f(U) \subseteq V$ such that the function

$$f_{\varphi\psi} = \psi \circ f \circ \varphi^{-1} \tag{3.1}$$

is smooth on $\mathbb{R}^n$. This function is called a **local representation** of f.

Definition 3.1.13 (Diffeomorphism). A homeomorphism f such that both f and f^{-1} are smooth.

Notation 3.1.14. The set of all C^∞-functions on a manifold M, defined on a neighbourhood of $p \in M$, is denoted by $C_p^\infty(M)$. This set forms a commutative ring when equipped with the usual sum and product (composition) of functions.

Remark 3.1.15. Depending on the choice of chart one can define other types of functions in the same way, e.g. C^k-functions or piecewise linear functions.

Definition 3.1.16 (Differentiably good cover). A good cover **??** for which the intersections are diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

If a manifold admits a finite (differentiably) good cover, it is said to be of **finite type**.

Property 3.1.17. Every paracompact smooth manifold admits a (differentiably) good cover. Furthermore, if the manifold is compact, it admits a finite good cover.

3.2 Tangent vectors

Definition 3.2.1 (Tangent vector). Let M be a smooth manifold and consider a point $p \in M$. A tangent vector to M at p is a differential operator on the germs of smooth functions at p, i.e. a map $v_p : C^\infty_p(M) \to \mathbb{R}$ satisfying the properties

1. **Linearity**: $v_p(\lambda f + g) = \lambda v_p(f) + v_p(g)$, and
2. **Leibniz property**: $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$

for all $f, g \in C^\infty_p(M)$ and $\lambda \in \mathbb{R}$. Maps with these properties are also called **derivations**[1].

Property 3.2.2 (Constant functions). Constant functions $c : p \mapsto c$ lie in the kernel of all tangent vectors:

$$v_p(c) = 0. \tag{3.2}$$

Definition 3.2.3 (Tangent space). The set of all tangent vectors at a point $p \in M$ admits the structure of a vector space T_pM. A canonical choice of basis vectors is given by

$$\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty_p(M) \to \mathbb{R} : f \mapsto \frac{\partial}{\partial x^i}\big(f \circ \varphi^{-1}\big)(\varphi(p))\,, \tag{3.3}$$

where (U, φ) is a coordinate chart such that $p \in U$ with local coordinates $(x^1, \ldots, x^n)$. The above basis vector are also often denoted by ∂_i.

Due to the explicit dependence of the tangent vectors on the point $p \in M$, it is clear that for curved manifolds the tangent spaces belonging to different points are not the same. However, they are related through the following property:

Property 3.2.4. For a smooth connected manifold, the tangent spaces satisfy

$$\dim(T_pM) = \dim(M) \tag{3.4}$$

for all $p \in M$. Theorem **??** then implies that the tangent spaces over two distinct points $p, q \in M$ are isomorphic. (A way to relate distinct tangent spaces will be presented in Sections 5.7 and 6.4.)

Alternative Definition 3.2.5 (Tangent space). Let (U, φ) be a chart around the point $p \in M$. Two smooth curves γ_1, γ_2 through $p \in M$ are said to be tangent at p if their local representations are tangent at 0:

$$\frac{\mathrm{d}(\varphi \circ \gamma_1)}{\mathrm{d}t}(0) = \frac{\mathrm{d}(\varphi \circ \gamma_2)}{\mathrm{d}t}(0). \tag{3.5}$$

This defines an equivalence relation[2] on the set of smooth curves through p. The tangent space at p is then defined as the set of equivalence classes of tangent curves through p. These equivalence classes can be explicitly constructed as follows. The tangent vector to the curve $c(t)$ through p is defined by the following formula:

$$v_p(f) := \left.\frac{\mathrm{d}(f \circ c)}{\mathrm{d}t}\right|_{t=0}. \tag{3.6}$$

[1] More generally, every operation that satisfies the Leibniz property is called a derivation.

[2] The relation is well-defined because the transition functions (and their Jacobian matrices) are invertible and thus nonsingular.

Applying the chain rule gives

$$v_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))\frac{\mathrm{d}x^i}{\mathrm{d}t}(0)\,, \tag{3.7}$$

where $x^i := (\varphi \circ c)^i$. The first factor depends only on the point p, while the second factor is equal for all tangent curves through p. It is thus clear that curves satisfying equation (3.5) define the same tangent vector.

Proof of equivalence. Let (U, φ) be a chart around the point $p \in M$. Using the first definition of a tangent vector 3.2.3, i.e.

$$\left.\frac{\partial}{\partial q^i}\right|_p : C_p^\infty(M, \mathbb{R}) \to \mathbb{R} : f \mapsto \frac{\partial}{\partial q^i}\big(f \circ \varphi^{-1}\big)(\varphi(p))\,,$$

one can rewrite Equation (3.7)

$$v_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial q^i}(\varphi(p))\frac{\mathrm{d}q^i}{\mathrm{d}t}(0)$$

as follows:

$$v_p(f) = \left.\frac{\partial f}{\partial q^i}\right|_p \frac{\mathrm{d}q^i}{\mathrm{d}t}(0).$$

Because the partial derivatives as defined in 3.2.3 form a basis for the tangent space (by construction), one can see that this equation is in fact an expansion of the tangent vector v_p in terms of that basis. It follows that vectors tangent to curves[a] are also tangent vectors according to the first definition.

To prove the other direction one has to show that the partial derivative operators can be constructed as vectors tangent to curves. A tangent vector can be expressed, according to the first construction, in the following way:

$$v_p = v^i \left.\frac{\partial}{\partial q^i}\right|_p\,,$$

where the definition $v = (v^1, \dots, v^n)$ was used. One can then construct the curve $\gamma : t \mapsto \varphi^{-1}(q_0 + vt)$. It is obvious that the tangent vector v_p is tangent to the curve γ. From this it follows that there exists an isomorphism between the tangent vectors from the first definition and the equivalence classes of vectors tangent to curves from the second definition. □

Although the previous equivalence implies that the tangent space construction using germs of curves gives a vector space, one could also check the vector space axioms directly. First, one should prove that the sum of vectors tangent to the curves γ and δ is again a vector tangent to some curve $\chi : \mathbb{R} \to M$. To this end, define the curve

$$\chi(t) \equiv \varphi^{-1} \circ \Big(\varphi \circ \gamma(t) + \varphi \circ \delta(t) - \varphi(p)\Big),$$

where φ is again the coordinate map in some chart (U, φ) around $p \in M$. Using Equation

(3.7) one can find

$$\begin{aligned}
v_{p,\chi}(f) &= \frac{\partial(f\circ\varphi^{-1})}{\partial q^i}(\varphi(p))\frac{\mathrm{d}(\varphi^i\circ\chi)}{\mathrm{d}t}(0)\\
&= \frac{\partial(f\circ\varphi^{-1})}{\partial q^i}(\varphi(p))\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi^i\circ\gamma+\varphi^i\circ\delta-\varphi^i(p)\big)\\
&= \frac{\partial(f\circ\varphi^{-1})}{\partial q^i}(\varphi(p))\left(\frac{\mathrm{d}(\varphi^i\circ\gamma)}{\mathrm{d}t}+\frac{\mathrm{d}(\varphi^i\circ\delta)}{\mathrm{d}t}\right)\\
&= v_{p,\gamma}(f)+v_{p,\delta}(f).
\end{aligned}$$

The constant term $-\varphi(p)$ in the definition of χ is necessary to make sure that $\chi(0)=\gamma(0)=\delta(0)=p$. The axiom of scalar multiplication by a number $\lambda\in K$ can be proven similarly by defining the curve

$$\chi(t)=\varphi^{-1}\circ\Big[\lambda\Big(\varphi\circ\gamma(t)\Big)\Big].$$

□

[a] More precisely, representatives of equivalence classes of vectors tangent to curves.

3.3 Submanifolds

3.3.1 Immersions and submersions

In this section the tangent map induced by a smooth function $f:M\to N$ is denoted by $T_pf:T_pM\to T_{f(p)}N$. A formal definition is given in Equation (5.1). For now this will be the map that is locally represented by the Jacobian of f.

Definition 3.3.1 (Immersion). A differentiable function $f:M\to N$ between smooth manifoldsfor which the derivative is everywhere injective or, equivalently, such that its derivative has maximal rank everywhere:

$$\mathrm{rk}(T_pf)=\dim(M)\qquad\forall p\in M. \tag{3.8}$$

Definition 3.3.2 (Critical point). A point $p\in\mathrm{dom}(f)$ is said to be critical if the rank of the Jacobian T_pf is not maximal. The image of a critical point is called a **critical value**.

At a critical point $p\in M$ the Hessian of f gives a well-defined quadratic form. A critical point is said to be **nondegenerate** if the Hessian is nonsingular there.

Property 3.3.3 (Criticality). A point $p\in\mathrm{dom}(f)$ is critical if and only if there exists a chart (U,φ) containing p for which $\partial_if(p)=0$.

Theorem 3.3.4 (Sard). *Consider a differentiable function $\psi:M\to N$, where $\dim(M)=m$ and $\dim(N)=n$ and let $k_0=\max\{1,m-n+1\}$. If ψ is of class C^k, with $k\geq k_0$, the set of critical values of ψ has Lebesgue measure 0.*

Definition 3.3.5 (Regular point). A regular point of f is a point $p\in M$ such that T_pf is surjective.

Definition 3.3.6 (Regular value). Let $f:M\to N$ be a differentiable function between smooth manifolds. A point $y\in N$ is called a **regular value** if every point in the preimage $f^{-1}(y)$ is a regular point or, equivalently, if it is not a critical value.

Corollary 3.3.7. It follows from Property 3.3.3 that a point $p\in\mathrm{dom}(f)$ is regular if and only if $\partial_if(p)\neq0$ for all charts (U,φ) containing p.

Definition 3.3.8 (Submersion). A differentiable function $f : M \to N$ between smooth manifolds such that all $p \in M$ are regular or, equivalently, such that

$$\mathrm{rk}(T_pf) = \dim(N) \qquad \forall p \in M. \tag{3.9}$$

3.3.2 Submanifolds

Definition 3.3.9 (Embedding). A differentiable function between smooth manifolds that is both an immersion and an embedding in the topological sense **??**. This implies that the submanifold topology coincides with the subspace topology **??**.

Definition 3.3.10 (Embedded submanifold). Let M be a manifold. A smooth manifold N is called an embedded or **regular submanifold** (of M) if there exists an embedding $f : M \hookrightarrow N$.

Definition 3.3.11 (Slice). Consider two positive integers $m < n$. The space $\mathbb{R}^m$ can be canonically identified with a subspace of $\mathbb{R}^n$ as follows:

$$\mathbb{R}^m \cong \mathbb{R}^m \times \{0, \dots, 0\} \hookrightarrow \mathbb{R}^m \times \mathbb{R}^{n-m} \cong \mathbb{R}^n. \tag{3.10}$$

Subspaces obtained in this way, i.e. by setting a number of coordinates equal to 0 (or any other constant), are called slices.

Alternative Definition 3.3.12 (Embedded submanifold). A subset N of M for which there exists a positive integer k and such that for every point $p \in N$ there exists a chart (U, φ) that satisfies

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^k \times \underbrace{\{0, \dots, 0\}}_{\dim(M)-k}). \tag{3.11}$$

The set $U \cap N$ is called a **slice** of (U, φ) in analogy with the previous definition of a (standard) slice.

Definition 3.3.13 (Immersed submanifold). Let M, N be smooth manifolds. N is said to be an immersed submanifold of M if there exists an immersion $i : N \hookrightarrow M$. Locally every immersed submanifold looks like a regular submanifold. Globally, however, the topology does not have to coincide with the subspace topology.

Theorem 3.3.14 (Submersion theorem[3]). *Consider a smooth map $f : M_1 \to M_2$ between smooth manifolds and let $y \in M_2$ be a regular value. Then $N = f^{-1}(y)$ is a submanifold of M_1 with codimension* $\dim(M_2)$*.*

Definition 3.3.15 (Closed embedded manifold). Let N be an immersed submanifold of M. If the inclusion map $i : N \hookrightarrow M$ is closed (or, equivalently, proper), N is in fact an embedded submanifold. It is called a closed embedded manifold.

Example 3.3.16 (Stiefel manifold). Let V be an inner product space **??** over a field K. The set of orthonormal k-frames can be embedded in $K^{n\times k}$. It is a compact embedded submanifold, called the Stiefel manifold of k-frames over V.

Definition 3.3.17 (Transversal intersection). Consider a smooth manifold M. Two submanifolds $X, Y \subset M$ are said to be transversal (or to intersect transversally) if at each intersection point p the following relation holds:

$$T_pX + T_pY = T_pM. \tag{3.12}$$

If the dimensions of X and Y are complementary (in M), the sum becomes a direct sum. If two submanifolds do not intersect at all, they are vacuously transversal (independent of their dimension).

[3]Also called the **regular value theorem**.

Property 3.3.18 (Codimension). The codimension of transversal intersections is equal to the sum of the codimensions of the intersecting submanifolds. It follows that if the submanifolds have complementary dimensions, the intersection consists of isolated points.

Definition 3.3.19 (Intersection number). By the above property two closed submanifolds $X, Y \subset M$ with complementary dimension that intersect transversally, have a finite number of intersection points. Given an orientation on M, the oriented sum of intersection points is called the intersection number $I(X, Y)$.

To extend this definition to nontransversal intersections, one can observe that the definition is homotopy invariant: given a homotopy $H : X \times [0, 1] \to Z$, if $H(X, 0) \pitchfork Y$ and $H(X, 1) \pitchfork Y$, then

$$I(H(X, 0), Y) = I(H(X, 1), Y). \tag{3.13}$$

So to define the intersection number of nontransversally intersecting submanifolds, one simply chooses a transverse (homotopical) deformation. By the invariance property, the result does not depend on the choice of deformation.

Property 3.3.20 (Euler characteristic). Consider a closed manifold M. The Euler characteristic **??** is given by

$$\chi(M) = I(\Delta_M, \Delta_M), \tag{3.14}$$

where $\Delta_M \in M \times M$ is the diagonal of M.

3.4 Manifolds with boundary

Definition 3.4.1 (Manifold with boundary). Let $\mathbb{H}^n$ denote the upper half space:

$$\mathbb{H}^n := \mathbb{R}^{n-1} \times \mathbb{R}^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}. \tag{3.15}$$

An n-dimensional manifold with boundary is defined as a topological space M equipped with a maximal atlas consisting of (regular) charts (U, φ) such that U is diffeomorphic to $\mathbb{R}^n$ (these points are called **interior points**) and **boundary charts** (V, ϕ) such that V is diffeomorphic to $\mathbb{H}^n$ (these points are called **boundary points**).

Remark 3.4.2 (Boundary). The boundary ∂M, consisting of all boundary points of M as defined in the above definition, should not be confused with the topological boundary of M. In general these are different sets. Similarly, the interior $\mathrm{Int}(M) = M \backslash \partial M$, in the sense of manifolds, should not be confused with the topological interior.

Property 3.4.3. Let M be an n-dimensional manifold with boundary and let (U, φ) be a chart for $p \in \partial M$.

$$\varphi(p) \in \partial \mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\} \tag{3.16}$$

Definition 3.4.4 (Manifold with corners). Analogous to the definition of a manifold with boundaries one can define a manifold with corners using **corner charts** of the form

$$\varphi : U \to \mathbb{R}^k \times (\mathbb{R}^+)^l.$$

In contrast to the case of manifolds with boundary one does need to add an extra requirement when working with higher order corners. For every two charts (U, φ) and (V, ψ) the transition function should preserve the corners:

$$\varphi \circ \psi^{-1}(V \cap \{0\} \times \mathbb{R}^k) \subset \{0\} \times \mathbb{R}^k.$$

Remark 3.4.5. In the topological setting every manifold with corners (even higher order ones) is homeomorphic to a manifold with boundary. However, when working with smooth structures this result fails. There exists no such diffeomorphism and accordingly one has to make a distinction between the type of corners.

3.4.1 Cobordisms ♣

Definition 3.4.6 (Cobordism). Two manifolds X, Y are said to be **(co)bordant** if there exists a manifold with boundary M such that $\partial M = X \sqcup Y$. The manifold M is called a (co)bordism between X and Y.

Remark. In the category of oriented manifolds one can also define a cobordism, but there the manifolds X, Y should respect the orientation of ∂M.

Definition 3.4.7 (Cobordism group). Under the operation of disjoint union the closed n-dimensional manifolds, modulo cobordisms, form a commutative group Ω_n. Under Cartesian products these match together to form a commutative graded ring $\Omega = \bigoplus_{n=0}^{\infty} \Omega_n$.

?? COMPLETE ??

3.5 Morse theory

3.5.1 Morse functions

Definition 3.5.1 (Morse function). Let M be a smooth manifold. A smooth function is called a Morse function if it has no degenerate critical points 3.3.2.

Property 3.5.2 (Density). The set of Morse functions is open and dense in the C^2-topology (see Section **??** on jet spaces).

Definition 3.5.3 (Palais-Smale condition). A smooth function $f \in C^1(M)$ is said to satisfy the Palais-Smale condition if every sequence $(x_n)_{n\in\mathbb{N}} \subset M$ with

1. $|f(x_n)|$ bounded for all $n \in \mathbb{N}$, and
2. $\|Df(x_n)\| \longrightarrow 0$

contains a convergent subsequence. It is clear that every smooth function on a compact manifold or every proper function satisfies this condition.

Corollary 3.5.4. If $f \in C^1(M)$ is Morse and satisfies the Palais-Smale condition, it has only finitely many critical points in every bounded subset or in any set where f is bounded.

Corollary 3.5.5. Let γ be a flow line on M such that $f(\gamma)$ is bounded. The flow line is complete and its limits are critical points of f. Moreover, the convergence at $t \longrightarrow \pm\infty$ is exponential.

Definition 3.5.6 (Morse index). Consider a Morse function $f \in C^\infty(M)$. The number of negative eigenvalues at a critical point $p \in M$ is called the (Morse) index of f at p. This is often denoted by $\lambda_p(f)$.

To any Morse function one can associate a series called the **Morse counting-series**:

$$M_t(f) := \sum_{p\in\mathrm{crit}(f)} t^{\lambda_p(f)}. \tag{3.17}$$

If M is compact, the nondegeneracy condition implies that the above sum only has a finite number of terms.

Property 3.5.7 (Morse lemma). Consider a Morse function $f : M \to \mathbb{R}$ and let $p \in M$ be a nondegenerate critical point of f. There exists a chart $(U, x_1, \dots, x_n)$ around p such that $x_i(p) = 0$ and

$$f|_U(x) = f(p) - x_1^2 - \cdots + x_k^2 + \cdots , \tag{3.18}$$

where k is the Morse index of f.

Corollary 3.5.8. The critical points of a Morse function are isolated.

Remark 3.5.9 (Morse-Palais lemma). The Morse lemma can be generalized to open subsets of Banach spaces (and thus to infinite-dimensional manifolds).

Definition 3.5.10 (Self-indexing function). A Morse function is said to be self-indexing if at every critical points its value is equal to its index.

?? COMPLETE ??

3.5.2 Morse-Bott functions

By the Morse lemma, the critical points of a Morse function are isolated. When this condition is relaxed, a more general class of functions is obtained (it is assumed that M comes equipped with a covariant derivative):

Definition 3.5.11 (Morse-Bott function). A smooth function $f : M \to \mathbb{R}$ for which the critical set $\mathrm{Crit}(f)$ is a submanifold of M and at every point $p \in \mathrm{Crit}(f)$ the tangent space is the kernel of the Hessian of f, i.e. its Hessian is nondegenerate in the normal directions at every critical point.

3.5.3 Morse homology

Definition 3.5.12 (Gradient-like vector field). Consider a Morse function $f \in C^\infty(M)$. A vector field X is said to be gradient-like with respect to f if is satisfies the following conditions:

1. For all $p \notin \mathrm{Crit}(f) : X|_p(f) > 0$.
2. For all $p \in \mathrm{Crit}(f)$ there exists a Morse chart containing p such that

$$X = -2 \sum_{i=1}^{\lambda_p(f)} x^i \partial_i + 2 \sum_{i=\lambda_p(f)+1}^{\dim(M)} x^i \partial_i. \tag{3.19}$$

Its flow lines have the same orientation away from critical points and it coincides with the gradient at critical points. Furthermore, such vector fields always exist.

Property 3.5.13. Let $f \in C^\infty(M)$ be a Morse function on a compact manifold and consider a gradient-like vector field X (with respect to f). For every $p \in M$ the limits of the flow line of $-X$, passing through p, are critical points of f.

Definition 3.5.14 (Stable and unstable manifold). Let $f \in C^\infty(M)$ be a Morse function and consider a gradient-like vector field X (with respect to f). For every critical point p of f, one defines the stable and unstable manifold of X as follows:

$$W_p^\pm(X) := \big\{x \in M \mid \lim_{t\to\pm\infty} \Phi_t(x) = p\big\}, \tag{3.20}$$

where Φ_t denotes the flow of $-X$. These sets carry the structure of a smooth manifold, locally diffeomorphic to $\mathbb{R}^{\dim(M)-\lambda_p(f)}$ and $\mathbb{R}^{\lambda_p(f)}$, respectively.

Definition 3.5.15 (Morse-Smale pair). Let $f \in C^\infty(M)$ be a Morse function and consider a gradient-like vector field X (with respect to f). If for all critical points $p, q \in \mathrm{Crit}(f)$ one has that

$$W_p^+(X) \pitchfork W_q^-(X), \tag{3.21}$$

the pair (f, X) is called a Morse-Smale pair.

Property 3.5.16. If M is compact, there exists a self-indexing Morse-Smale pair.

Property 3.5.17. For every Morse function on a compact manifold there exists a generic metric such that $(f, \nabla f)$ is Morse-Smale.

From here on it will be assumed that given a Morse function $f \in C^\infty(M)$, the pair $(f, \nabla f)$ is Morse-Smale. By $\mathcal{M}(p, q)$ one denotes the set of integral curves of $-\nabla f$ that start at p and end at q, i.e. the integral curves γ that satisfy $\gamma([0, 1]) \subset W_p^-(\nabla f) \cap W_q^+(\nabla f)$. By the structure of the stable and unstable manifolds, this solution space has dimension $\lambda_p(f) - \lambda_q(f)$. Integral curves can be arbitrarily reparametrized. To obtain a well-defined moduli space $\overline{\mathcal{M}}(p, q)$, this $\mathbb{R}$-action is quotiented out (it is free and proper, so the resulting space is again a smooth manifold).

Definition 3.5.18 (Morse homology). The chain groups are defined as follows

$$CM_k(M, f) := \bigoplus_{\substack{p \in \mathrm{Crit}(f) \\ \lambda_p(f) = k}} \mathbb{Z}\langle p \rangle. \tag{3.22}$$

For critical points $p, q \in \mathrm{Crit}(f)$ such that $\lambda_p(f) = \lambda_q(f) + 1$, the moduli space is a discrete, compact set. This allows to define the boundary operator as follows:

$$\partial p := \sum_{\substack{q \in \mathrm{Crit}(f) \\ \lambda_q(f) = \lambda_p(f) - 1}} \left|\overline{\mathcal{M}}(p, q)\right| \langle q \rangle. \tag{3.23}$$

One can show that $\partial^2 = 0$ and Morse homology is defined as the homology of this complex:

$$HM_\bullet(M, f) := \frac{\ker(\partial)}{\mathrm{im}(\partial)}. \tag{3.24}$$

3.6 Surgery theory ♣

Definition 3.6.1 (Dehn twist). Consider an orientable surface M together with a simple closed curve c. A tubular neighbourhood[4] T of c is homeomorphic to an annulus and hence allows a parametrization $(e^{i\alpha}, t)$ where $\alpha \in [0, 2\pi[$ and $t \in [0, 1]$. A Dehn twist about c is an automorphism that is given by $(e^{i\alpha}, t) \mapsto (e^{i(\alpha + 2\pi t)}, t)$ on T and restricts to the identity outside of it.

?? COMPLETE ??

[4]See Definition 5.2.10 for a formal definition.

Chapter 4

Fibre Bundles

This chapter is formulated in sufficient generality so as to encompass both the topological and smooth setting (or any other setting one might find useful). To this end the generic terms "space", "group" and "morphism" are used. The reader should choose in which category he wants to work, e.g. topological space, topological group and continuous map in the case of **Top**.

4.1 Bundles

Definition 4.1.1 (Bundle). A triple (E, B, π) where E and B are spaces and π is a morphism. Sometimes the map π is also required to be surjective. However, under this additional restriction one cannot make the association $\mathbf{Bundle}(B) \cong \mathbf{C}/B$ of categories anymore.

An explicit example in the category **Diff** is the following:

Example 4.1.2 (Fibred manifold). A surjective submersion 3.3.8

$$\pi : E \to B,$$

where E is called the **total space**, B the **base space** and π the **projection**. For every point $p \in B$, the set $\pi^{-1}(p)$ is called the **fibre over** p.

The most important example of a bundle is a fibre bundle. Before being able to give the definition, an important concept needs to be introduced:

Definition 4.1.3 (Cocycle). Let B be a space and G a group. A G-valued cocycle on B with respect to an open cover $\{U_i\}_{i\in I}$ is a family of morphisms $g_{ij} : U_i \cap U_j \to G$ that satisfy the following **Čech cocycle condition**:

$$g_{ij} = g_{ik} \circ g_{kj}. \tag{4.1}$$

Two cocycles (U_i, g_{ij}) and (V_i, h_{ij}) are said to be equivalent if there exist morphisms $\lambda_{i,j} : U_i \cap V_j \to G$ such that

$$\lambda_{i,r} g_{ij} \lambda_{j,s}^{-1} = h_{rs} \tag{4.2}$$

whenever this is well-defined. The resulting quotient set is denoted by $\check{H}^1(B; G)$.[1]

Property 4.1.4 (Normalization). Let $\{g_{ij}\}_{i,j\in I}$ be a cocycle on B. It satisfies the following properties for all $x \in B$:

[1]The notation stems from the fact that this is the first Čech cohomology group with values in G (Section **??**).

- $g_{ij}(x) = (g_{ji}(x))^{-1}$, and
- $g_{ii}(x) = e$.

Definition 4.1.5 (Fibre bundle). A tuple (E, B, π, F, G) where E, B and F are spaces and G is a group, called the **structure group**, such that there exists a surjective morphism

$$\pi : E \to B$$

and an open cover $\{U_i\}_{i\in I}$ of B together with a family of isomorphisms $\{\varphi_i : \pi^{-1}(U_i) \to U_i \times F\}_{i\in I}$ that make the following diagram commute for all $i \in I$:

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr, "\pi"'] & & U_i \times F \arrow[dl, "\mathrm{pr}_1"] \\
& U_i &
\end{tikzcd}$$

As for general bundles, one calls E and B the **total space** and **base space**, respectively. The space F is called the **(typical) fibre**. The pair (U_i, φ_i) is sometimes called a **bundle chart** and the set $\{(U_i, \varphi_i)\}_{i\in I}$ is often called a **local trivialization**[2]. The cover $\{U_i\}_{i\in I}$ itself is called a **trivializing cover** of the bundle.

The **transition maps** $\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ can be identified with a cocycle $g_{ji} : U_i \cap U_j \to G$ as follows. The transition maps restrict to the identity on B and, hence, act only on the fibres:

$$\varphi_j \circ \varphi_i^{-1}(b, x) = (b, g_{ji}(b) \cdot x). \tag{4.3}$$

The compatibility conditions satisfied by the functions g_{ji}, obtained by considering triple intersections, are exactly the cocycle conditions (4.1). Moreover, it can be shown that this action of G on every fibre is faithful **??**.

Remark 4.1.6. One should pay attention to the fact that the bundle charts are not coordinate charts in the sense of manifolds 3.1.1 because the image of φ_i is not an open subset of $\mathbb{R}^n$. However, they serve the same purpose as they are used to locally describe the total space P.

Notation 4.1.7. A fibre bundle (E, B, π, F, G) is often denoted by $F \hookrightarrow E \xrightarrow{\pi} B$ or even $\pi : E \to B$ if the fibre is not important. A drawback of such notations is that the structure group of the bundle is not shown.

Definition 4.1.8 (Numerable fibre bundle). A fibre bundle that admits a local trivialization over a numerable open cover **??**.

Definition 4.1.9 (Compatible bundle charts). A bundle chart (V, ψ) is said to be **admissable** or compatible with a trivializing cover $\{(U_i, \varphi_i)\}_{i\in I}$ if, whenever $V \cap U_i \neq \emptyset$, there exists a map $h_i : V \cap U_i \to G$ such that

$$\psi \circ \varphi_i^{-1}(b, x) = (b, h_i(b)x) \tag{4.4}$$

for all $b \in V \cap U_i$ and $x \in F$. Two trivializing covers are said to be equivalent if all bundle charts are mutually compatible. As in the case of manifolds, this gives rise to the notion of a **G-atlas**. A **G-bundle** is then defined as a fibre bundle equipped with an equivalence class of G-atlases.

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f_B"'] & B_2 \end{tikzcd}$$

Figure 4.1: Bundle map between fibre bundles.

Definition 4.1.10 (Bundle map). A bundle map between two fibre bundles $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ is a pair (f_E, f_B) of morphisms that make Diagram 4.1 commute. The map f_E is said to **cover** f_B. If such a couple exists, the base map f_B is uniquely determined by f_E and, therefore, a bundle map is often just denoted by $f_E : E_1 \to E_2$.

Definition 4.1.11 (Equivalent fibre bundles). Two fibre bundles $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ with the same typical fibre and structure group are said to be equivalent if there exist trivializations $\{(U_i, \varphi_i)\}_{i\in I}$ and $\{(U_i, \varphi_i')\}_{i\in I}$ such that the associated cocycles are equivalent (note that the cover $\{U_i\}_{i\in I}$ is the same for both trivializations). An explicit form of the functions λ is given by

$$\lambda_i := \varphi_i' \circ \varphi_i^{-1}. \tag{4.5}$$

Property 4.1.12 (Isomorphism). Two fibre bundles over the same base space are equivalent if and only if they are isomorphic.

Definition 4.1.13 (Trivial bundle). A fibre bundle (E, B, π, F) is said to be trivial if there exists an equivalence $E \cong B \times F$.

4.2 Constructions

Construction 4.2.1 (Fibre bundle construction theorem[3]). Let B and F be spaces and let G be a group equipped with a faithful (left) action on F. Suppose that a cover $\{U_i\}_{i\in I}$ of B and a collection of morphisms $\{g_{ji} : U_i \cap U_j \to G\}$ that satisfy the cocycle condition 4.1.3 are given. A fibre bundle over B can be constructed as follows:

1. Construct for every set U_i the Cartesian product $U_i \times F$.
2. Construct the disjoint union $T := \bigsqcup_{i\in I} U_i \times F$ and equip it with the disjoint union topology **??**.
3. From this disjoint union construct a quotient space, equipped with the quotient space topology **??**, induced by the following equivalence relations for all $i, j \in I$:

   $$(b, f) \sim (b, g_{ji}(b) \cdot f), \tag{4.6}$$

   where $b \in U_i \cap U_j$ and $f \in F$ (note that the disjoint union indices are suppresed in this notation).

The fibre bundle E is equal to this quotient space, where the projection π is the quotient space projection $\pi : E \to B : [(b, f)] \mapsto b$, where $[A]$ denotes the equivalence class of A in E. Local trivializations are given by the maps $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$ that satisfy

$$\varphi_i^{-1} : (b, f) \mapsto [(b, f)]. \tag{4.7}$$

[2] This terminology follows from the fact that the bundle is locally isomorphic to a (trivial) product space: $E \cong U \times F$.

[3] Sometimes also called the **clutching theorem**, see below for an explanation.

Remark 4.2.2 (Topology). Although the resulting fibre bundle is by construction bijective (as a set) to the Cartesian product $B \times F$ or the disjoint union $\bigsqcup_{b\in B} F$, this does not hold on the level of topologies. It is not equipped with the disjoint union topology.

Property 4.2.3 (Homotopy invariance). Homotopic transition functions give rise to equivalent bundles.

Remark 4.2.4 (Clutching). The above construction is often called the clutching construction, especially when constructing vector bundles over a (hyper)sphere S^n. In that case, the covering consists of two hemispheres that intersect on the equator S^{n-1} and the function g_{21} is called the **clutching function**.

Definition 4.2.5 (Pullback bundle). Let $\pi : E \to B$ be a fibre bundle and let $f : B' \to B$ be a morphism of spaces. The pullback **??** of the bundle projection and f gives the total space of the pullback bundle f^*E:

$$f^*E := \big\{(b', e) \in B' \times E \,\big|\, f(b') = \pi(e)\big\}. \tag{4.8}$$

The topology on f^*E is induced by the subspace topology of the product $B' \times E$. The projection onto the second factor gives a map of total spaces $f^*E \to E$.

Definition 4.2.6 (Fibre product). Let (E_1, B, π_1) and (E_2, B, π_2) be two fibre bundles over the same base space B. Their fibre product is defined as follows:

$$E_1 \times_B E_2 := \big\{(p, q) \in E_1 \times E_2 \,\big|\, \pi_1(p) = \pi_2(q)\big\}. \tag{4.9}$$

It is the pullback of one bundle along the projection of another bundle.

4.3 Sections

Definition 4.3.1 (Section). A (**global**) section of a fibre bundle $\pi : E \to B$ is a morphism $s : B \to E$ such that $\pi \circ s = \mathbb{1}_B$, i.e. it is a section of π in the sense of Definition **??**. For any open subset $U \subset B$, a **local** section is defined as a morphism $s_U : U \to E$ such that $\pi \circ s_U = \mathbb{1}_U$.

Notation 4.3.2. The set of all global sections of a bundle E is denoted by $\Gamma(E)$. The set of local sections over U is sometimes denoted by $\Gamma(U, E)$. With this latter notation one also has $\Gamma(E) \equiv \Gamma(B, E)$.

Property 4.3.3 (Pullback of sections). Consider a fibre bundle $\pi : E \to B$ together with a morphism of spaces $f : B' \to B$. The sections of E pullback to the pullback bundle f^*E by defining $f^*s := s \circ f$.

Chapter 5

Vector Bundles

The main reference for this chapter is [18].

5.1 Tangent bundle

The tangent space, as introduced in Section 3.2, can also be introduced in a more abstract and general way. Because it is the most important example of a vector bundle, tangent bundles are introduced first.

Construction 5.1.1 (Tangent bundle). Let M be an n-dimensional manifold with atlas $\{(U_i, \varphi_i)\}_{i\leq n}$. Construct for every open set U an associated set $TU := U \times \mathbb{R}^n$ and construct for every smooth function f an associated smooth function on TU, called the **differential** or **derivative** of f, by

$$Tf : U \times \mathbb{R}^n \to f(U) \times \mathbb{R}^n : (p, v) \mapsto \big(f(p), Df(p)v\big)\,, \tag{5.1}$$

where $Df(p) : \mathbb{R}^n \to \mathbb{R}^n$ is the Jacobian of f at p.

By applying this definition to the transition functions ψ_{ji} one obtains a new set of functions

$$T\psi_{ji} : TU_i \to TU_j$$

given by

$$T\psi_{ji}(\varphi_i(p), v) := \big(\varphi_j(p), D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(p))v\big). \tag{5.2}$$

Because the transition functions are diffeomorphisms, the associated Jacobians are invertible. This implies that the maps $T\psi_{ji}$ are represented by elements of $\mathrm{GL}(\mathbb{R}^n)$. The tangent bundle is then obtained by applying the fibre bundle construction theorem 4.2.1 to the triple $(M, \mathbb{R}^n, \mathrm{GL}(\mathbb{R}^n))$ together with the cover $\{U_i\}_{i\leq n}$ and the cocycle $\{T\psi_{ji}\}_{i,j\leq n}$.

Definition 5.1.2 (Natural chart). The charts in the atlas of this bundle are sometimes called natural charts or **adapted charts** because the first n coordinates are equal to the coordinates on the base manifold.

Definition 5.1.3 (Tangent space). Consider a point $p \in M$. The definition of the tangent space in the bundle setting is given by the fibre

$$T_pM := \pi^{-1}_{TM}(p). \tag{5.3}$$

If one uses the natural charts identify T_pM with the set $\varphi_i(p) \times \mathbb{R}^n$, it can be seen that T_pM is isomorphic to $\mathbb{R}^n$ (as a vector space).

Property 5.1.4 (Smooth structure). An atlas on TM is given by the charts (TU_i, θ) with

$$\theta : TU_i \to \mathbb{R}^{2n} : (p, X) \mapsto (\varphi_i \circ \pi(p), X^1, \dots, X^n), \tag{5.4}$$

where (U_i, φ_i) is a chart around $p \in M$ such that a tangent vector X is trivialized as $X^i\partial_i \in T_pM$.

Property 5.1.5 (Dimension). Let M be an n-dimensional manifold. Using the charts on M and the adapted charts on TM, one can see that TM is locally isomorphic to $\mathbb{R}^{2n}$. This implies that

$$\dim(TM) = 2\dim(M). \tag{5.5}$$

Remark 5.1.6 (Physics). Now, it should be clear that the statement "*a vector is something that transforms like a vector*", which one often hears in introductory physics courses, comes from the fact that

$$\text{a vector } v \in T_pM \text{ is tangent to } \varphi_i(p) \text{ in a chart } (U_i, \varphi_i)$$

if and only if

$$D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(p))v \text{ is tangent to } \varphi_j(p) \text{ in a chart } (U_j, \varphi_j).$$

Definition 5.1.7 (Differential). The map T defined in Equation (5.1) can be generalized to arbitrary smooth manifolds as the map $Tf : TM \to TN$, which is locally represented by the Jacobian. Furthermore, let $p \in U \subseteq M$ and let $V = f(U)$. By looking at the restriction of Tf to T_pM, one can see that it maps T_pU to $T_{f(p)}V$ linearly.

Property 5.1.8. The map $Tf : TM \to TN$ has the following properties:

- T preserves identities: $T\mathbb{1}_M = \mathbb{1}_{TM}$.
- Let f, g be two smooth functions on smooth manifolds, then $T(f \circ g) = Tf \circ Tg$.

This turns the map T into an endofunctor **??** on the category of smooth manifolds. One can view T as a "functorial derivative".

Definition 5.1.9 (Rank). Let $f : M \to N$ be a differentiable function between smooth manifolds. Using the fact that Tf is fibrewise linear, the rank of f at $p \in M$ is defined as the rank of the differential $Tf : T_pM \to T_{f(p)}N$ in the sense of Definition **??**.

Theorem 5.1.10 (Inverse function theorem). *A smooth function $f : M \to N$ between smooth manifolds is a local diffeomorphism at $p \in M$ if and only if its differential $Tf : T_pM \to T_{f(p)}N$ is an isomorphism at p.*

Definition 5.1.11 (Parallelizable manifold). A manifold with a trivial tangent bundle.

5.2 Vector bundles

Instead of restricting the typical fibre to be a Euclidean space with the same dimension as the base manifold, one can generalize the construction of the tangent bundle in the following way:

Construction 5.2.1 (Vector bundle). Consider a (topological) manifold with atlas $\{(U_i, \varphi_i)\}_{i\in I}$ together with a cocycle $\{g_{ji} : U_i \cap U_j \to G\}_{i,j\in I}$ with values in a group G and a representation $\rho : G \to \mathrm{GL}(V)$ on a (finite-dimensional) vector space V. This data can be used to construct a fibre bundle through Construction 4.2.1. The dimension of the typical fibre V is called the **rank** of the vector bundle.

Remark 5.2.2. As was also the case for tangent bundles, the choice of charts on E is not random. To preserve the linear structure of fibres, the use of the natural charts is imperative.

Example 5.2.3 (Line bundle). A vector bundle with a one-dimensional fibre. A common example in quantum mechanics is the $\mathbb{C}$-line bundle over some smooth manifold whose sections correspond to the wave functions of a given system. (See Section **??** on geometric quantization for more information.)

Property 5.2.4 (Vector bundles over a sphere). The clutching theorem 4.2.4 and the homotopy invariance imply that vector bundles over the sphere are determined by homotopy classes of functions $S^{n-1} \to \mathrm{GL}(p, K)$, i.e. they are classified by the homotopy group $\pi_{n-1}(\mathrm{GL}(p, K))$.

5.2.1 Sections

Definition 5.2.5 (Frame). A frame of a vector bundle E is a tuple $(s_1, \ldots, s_n)$ of sections such that $(s_1(b), \ldots, s_n(b))$ is a basis for the fibre $\pi^{-1}(b)$ for all $b \in B$. By passing to local sections, one obtains a local frame.

Property 5.2.6 (Trivial bundles). A vector bundle is trivial if and only if it admits a global frame.

Theorem 5.2.7 (Serre & Swan). *The set of smooth sections of a smooth vector bundle over a smooth manifold M is a finitely-generated projective $C^\infty(M)$-module. More generally, the set of sections of a vector bundle over a compact Hausdorff manifold B is a finitely-generated projective $C(B)$-module.*

Definition 5.2.8 (Zero section). The section s_0 of a vector bundle $E \to B$ that assigns to every point $b \in B$ the zero vector of the associated fibre E_b. For every vector bundle $\pi : E \to B$ one can embed the base manifold B in the bundle E through the zero section $s_0 : B \to E$. The complement of the image of this section is often denoted by E_0.

5.2.2 Tubular neighbourhoods

Definition 5.2.9 (Normal bundle). Consider a smooth manifold M with a submanifold S and consider for every point $p \in S$ the tangent spaces T_pS and T_pM. Since T_pS is a subspace of T_pM, one can construct the quotient space $N_pS := T_pM/T_pS$. The normal bundle of S in M is defined as the vector bundle with fibres N_pS.

Definition 5.2.10 (Tubular neighbourhood). Consider a smooth manifold M with an embedded submanifold S. A tubular neighbourhood of S in M is a vector bundle $\pi : E \to S$ such that (an open neighbourhood of the zero section of) E is diffeomorphic to an open neighbourhood of S in M.

Theorem 5.2.11 (Tubular neighbourhood theorem). *Every embedded submanifold admits a tubular neighbourhood, namely its normal bundle. Furthermore, all tubular neighbourhoods are diffeomorphic.*

Corollary 5.2.12 (Submanifolds and NDR pairs). Consider a smooth manifold M with a submanifold S. The pair (M, S) is an NDR pair **??**. In particular, consider a smooth fibre bundle $\pi : E \to B$. If π admits a global section, one can embed B in E as a submanifold and, hence, the pair (E, B) is an NDR pair.

5.2.3 Sums and product

Definition 5.2.13 (Whitney sum). Consider two vector bundles E, E' with typical fibres W, W' over the same base space. One can construct a new vector bundle $E \oplus E'$ by taking the typical fibre to be the direct sum $W \oplus W'$, i.e. the fibre over p is given by $W_p \oplus W'_p$. This operation is called the Whitney sum or **direct sum** of vector bundles.

The existence property for complements **??** from linear algebra can be generalized in the following way:

Property 5.2.14. Let B be a paracompact Hausdorff manifold and let E be a vector bundle over B. Every vector subbundle F of E admits an orthogonal complement $F^\perp$.

Property 5.2.15. Let B be a compact Hausdorff manifold. Every vector bundle E over B admits a complementary vector bundle E^c such that $E \oplus E^c \cong B \times \mathbb{R}^n$ for some $n \in \mathbb{N}$.

Definition 5.2.16 (Stable isomorphism). Two vector bundles E, E' over a base space B are said to be stably isomorphic if there exist integers $m, n \in \mathbb{N}$ such that

$$E \oplus (B \times \mathbb{R}^m) \cong E' \oplus (B \times \mathbb{R}^n). \tag{5.6}$$

Construction 5.2.17 (Tensor product). The tensor product $E \otimes E' \to B$ of two vector bundles $\pi : E \to B$ and $\pi' : E' \to B$ is given by the fibrewise tensor product.

Construction 5.2.18 (Exterior tensor product). The exterior tensor product $E \boxtimes E' \to B \times B'$ of two vector bundles $\pi : E \to B$ and $\pi' : E' \to B'$ is given by the tensor product of fibres over the Cartesian product $B \times B'$.

5.2.4 Associated vector bundles

Construction 5.2.19 (Associated vector bundle). Consider a representation

$$\rho : \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^l)$$

together with the tangent bundle cocycle $\{t_{ji} := D(\psi_{ji}) \circ \varphi_i\}_{i,j \leq n}$. The composite

$$\rho \circ t_{ji} : U_i \cap U_j \xrightarrow{t_{ji}} \mathrm{GL}(\mathbb{R}^n) \xrightarrow{\rho} \mathrm{GL}(\mathbb{R}^l)$$

is again a cocycle and can be used to define a new vector bundle on M through the fibre bundle construction theorem. The vector bundle $\rho(TM)$ is called the associated (vector) bundle of the tangent bundle induced by ρ.

Example 5.2.20 (Contravariant vectors). By noting that the k^{th} tensor power $\otimes^k$ induces a representation given by the tensor product of representations, one can construct the bundle of order-k (contravariant) tensors $\otimes^k(TM)$.

Example 5.2.21 (Cotangent bundle). Another useful construction is given by the contragredient representation $(\rho^T)^{-1} = (\rho^{-1})^T$. The vector bundle constructed this way, where the cocycle is given by $(t_{ji}^T)^{-1}$, is called the cotangent bundle on M and is denoted by T^*M. Elements of the fibres are called **covariant vectors** or **covectors**.

Notation 5.2.22. A combination of the cocycle t_{ji} and its dual $(t_{ji}^T)^{-1}$ can also be used to define the bundle of (k, l)-tensors on M. This bundle is denoted by $T^{(k,l)}M$.

Definition 5.2.23 (Twisted bundle). Given a vector bundle $\pi : E \to B$ and a line bundle $\psi : L \to B$, one calls the tensor product $E \otimes L$ the "L-twisted" version of E.

For every vector bundle one can define a canonical line bundle:

Construction 5.2.24 (Determinant line bundle). Consider a rank-n vector bundle $\pi : E \to B$. The determinant map induces an associated line bundle $\det(\pi) : \bigwedge^n E \to B$, where the transition functions on the fibres are given by the determinant of the transition functions of E. Bundles twisted by a determinant line bundle $\det(E)$ are called **densitized bundles**.

Example 5.2.25 (Canonical bundle). Consider a smooth manifold M. The canonical (line) bundle of M is given by $\det(T^*M)$, the determinant line bundle of the cotangent bundle of M.

5.3 Vector fields

From here on the theory will be specialized to the smooth setting, i.e. all manifolds and all morphisms will be assumed to be smooth (unless stated otherwise).

Definition 5.3.1 (Vector field). A section $s \in \Gamma(TM)$ of the tangent bundle. By the Serre-Swan theorem 5.2.7 the set of vector fields forms a $C^\infty(M)$-module.

Notation 5.3.2. The set of all vector fields on a manifold M is often denoted by $\mathfrak{X}(M)$.

Definition 5.3.3 (Index). Consider a vector field X on an n-dimensional manifold M and let $p \in M$ be an isolated zero of X. Because p is isolated, one can find a small $(n-1)$-sphere around p that does not contain any other zeroes of X. The index $\mathrm{ind}_X(p)$ of X at p is defined as the degree **??** of the function

$$f : S^{n-1} \to S^{n-1} : m \mapsto \frac{X(m)}{\|X(m)\|}.$$

Property 5.3.4 (Winding number). The winding number of a vector field X along a curve γ on which X does not vanish is equal to the sum of indices of zeroes of X lying inside γ.

Theorem 5.3.5 (Poincaré-Hopf). *Let M be a compact manifold and consider a vector field X having only isolated zeroes. The Euler characteristic* **??** *is given by*

$$\chi(M) = \sum_{p \in X^{-1}(0)} \mathrm{ind}_X(p). \tag{5.7}$$

Remark 5.3.6. This is essentially a restatement of Property 3.3.20, where one submanifold is given by the zero section and the other is given by the graph of X.

An immediate consequence of the Poincaré-Hopf theorem is the following well-known result:

Theorem 5.3.7 (Hairy ball theorem). *There exists no nowhere-vanishing vector field on an even-dimensional sphere S^{2n}.*

Definition 5.3.8 (Pullback). Consider a diffeomorphism $\varphi : M \to N$ and let X be a vector field on N. The pullback of X along φ is defined as follows:

$$(\varphi^* X)_p := T\varphi^{-1}(X_{\varphi(p)}). \tag{5.8}$$

Definition 5.3.9 (Pushforward). Let X be a vector field on M and let $\varphi : M \to N$ be a diffeomorphism. The pushforward of X along φ is defined as follows:

$$(\varphi_* X)_{\varphi(p)} := T\varphi(X_p). \tag{5.9}$$

This can be rewritten using the pullback as follows:

$$\varphi_* X = (\varphi^{-1})^* X. \tag{5.10}$$

Equivalently, one can define a vector field on N as

$$(\varphi_* X)_q(f) := X_{\varphi^{-1}(q)}(f \circ \varphi). \tag{5.11}$$

Example 5.3.10 (Projective space). Consider the manifold $\mathbb{CP}^n$ for some $n \in \mathbb{N}$. Local coordinate patches are given by charts

$$\mathbb{C}^n \to \pi_i : U_i : (x_1, \ldots, x_n) \mapsto [x_1 : \ldots : x_i : 1 : x_{i+1} : \ldots : x_n], \tag{5.12}$$

where $U_i := \{[x] \in \mathbb{CP}^n \mid x \in \mathbb{C}^{n+1}, x_i \neq 0\}$. To determine the vector fields induced by the basis vector fields ∂_i, however, it is more useful to start from the quotient map $\pi : \mathbb{C}^{n+1}/\{0\} \to \mathbb{CP}^n$. This map acts on the subset $\{x \in \mathbb{C}^{n+1} \mid x_i \neq 0\}$ as

$$(x_1, \dots, x_{n+1}) \mapsto \left[\frac{x_1}{x_i} : \dots : \frac{x_{i-1}}{x_i} : 1 : \frac{x_{i+1}}{x_i} : \dots : \frac{x_{n+1}}{x_i}\right] \tag{5.13}$$

to give the chart U_i. Locally on this subset, pushing the $n+1$ basis vector fields ∂_i forward along π gives

$$\begin{cases} \pi_*\partial_j = \dfrac{1}{x_i}\widetilde{\partial}_j & j \neq i \\ \pi_*\partial_i = -\displaystyle\sum_{j\neq i} \frac{x_j}{x_i^2}\widetilde{\partial}_j & i. \end{cases} \tag{5.14}$$

The kernel of this projection is given by the Euler vector field

$$\mathbb{E} := x^i\partial_i \tag{5.15}$$

from Definition **??**

5.3.1 Integral curves

Definition 5.3.11 (Integral curve). Let $X \in \mathfrak{X}(M)$ and let $\gamma : \,]a, b[\, \to M$ be a curve on M. γ is called an integral curve of X if

$$\gamma'(t) = X(\gamma(t)) \tag{5.16}$$

for all $t \in \,]a, b[$, where $\gamma'(t) := T\gamma(t, 1)$.

This equation can be viewed as a system of ordinary differential equations. Using the Picard-Lindelöf existence theorem **??**, together with the initial value condition $\gamma(0) = p$, one can find a unique maximal curve satisfying the above equation. This solution, denoted by γ_p, is called the **integral curve of X through p**.

Definition 5.3.12 (Flow). Let $X \in \mathfrak{X}(M)$ and consider its integral curve γ_p through a point $p \in M$. The function σ_t defined by

$$\sigma_t(p) := \gamma_p(t)\,, \tag{5.17}$$

is called the flow of X at time t. The **flow domain** is defined as the set

$$D(X) := \big\{(t, p) \in \mathbb{R} \times M \,\big|\, p \in M, t \in \,]a_p, b_p[\,\big\}\,, \tag{5.18}$$

where $]a_p, b_p[$ is the maximal interval on which γ_p is defined.

Property 5.3.13. Suppose that $D(X) = \mathbb{R} \times M$. The flow σ_t has the following properties for all $s, t \in \mathbb{R}$:

- σ_t is smooth,
- $\sigma_0 = \mathbb{1}_M$,
- $\sigma_{s+t} = \sigma_s \circ \sigma_t$, and as a consequence
- $\sigma_{-t} = (\sigma_t)^{-1}$.

These properties say that σ_t is a smooth, bijective group action of the additive group of real numbers on M. This implies that σ_t is a (smooth) flow in the general mathematical sense.

Definition 5.3.14 (Complete vector field). A vector field on a manifold M for which the flow domain for every flow is all of $\mathbb{R} \times M$.

Property 5.3.15. If the manifold M is compact, every vector field $X \in \mathfrak{X}(M)$ is complete.

Property 5.3.16 (Winding number). The winding number of a vector field along a closed integral curve is 1.

5.3.2 Lie derivative

Formula 5.3.17 (Lie derivative for smooth functions). Consider $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. The Lie derivative of f with respect to X at $p \in M$ is defined as

$$\mathcal{L}_X f(p) := \lim_{t\rightarrow 0} \frac{f(\gamma_p(t)) - f(p)}{t}. \tag{5.19}$$

The definition of the Lie derivative closely resembles the definition of the ordinary derivative on Euclidean space. This is not a coincidence:

Formula 5.3.18. Working out the definition of the Lie derivative and rewriting it as an operator equality gives

$$\mathcal{L}_X = \sum_k X^k \frac{\partial}{\partial x^k}. \tag{5.20}$$

It is clear that this is just the vector field X expressed in the basis 3.2.3. This way, one also recovers the behaviour of a tangent vector as a derivation. For smooth functions $f : M \rightarrow \mathbb{R}$ this gives

$$\mathcal{L}_X f(p) = X_p(f). \tag{5.21}$$

Explanation. In this derivation Landau's little-o notation is used:

$$\lim_{t\rightarrow 0} \frac{o(t)}{t} = 0. \tag{5.22}$$

Now, assume that X is a smooth vector field and f is a smooth function. Because the Lie derivative is a local operation one can work in a local chart such that γ is, again locally, equivalent to a curve[a] $\beta_p : U \rightarrow \mathbb{R}^n$ and such that one can expand $\beta_p(t)$ around $p \in U$:

$$\begin{aligned}
\mathcal{L}_X f(p) &= \lim_{t\rightarrow 0}\left[\frac{f(\beta_p(0) + t\beta_p'(0) + o(t)) - f(p)}{t}\right] \\
&= \lim_{t\rightarrow 0}\left[\frac{f(p + tX(p) + o(t)) - f(p)}{t}\right] \\
&= \lim_{t\rightarrow 0}\left[\frac{f(p) + tDf(p)\cdot X(p) + o(t) - f(p)}{t}\right] \\
&= \sum_k \frac{\partial f}{\partial x^k}(p) X_k(p) + \lim_{t\rightarrow 0}\frac{o(t)}{t} \\
&= \sum_k \frac{\partial f}{\partial x^k}(p) X_k(p),
\end{aligned} \tag{5.23}$$

where the defining condition 5.3.11 for integral curves was used on the second line. If this equation is rewritten as an operator equality, one obtains

$$\mathcal{L}_X = \sum_k X_k \frac{\partial}{\partial x^k}. \tag{5.24}$$

[a] The vector field $X(p) = (p, Y(p))$, where Y is a smooth vector field on $\mathbb{R}^n$, can also be identified with Y itself. This is implicitly done in the derivation by using the notation X for both vector fields.

Formula 5.3.19 (Lie derivative for vector fields). Let $X, Y \in \mathfrak{X}(M)$.

$$\mathcal{L}_X Y(p) := \left.\frac{\mathrm{d}}{\mathrm{d}t}(\sigma_t^* X)(\gamma_p(t))\right|_{t=0}. \tag{5.25}$$

Explanation. For vector fields one cannot just take the difference at two different points because the tangent spaces generally do not coincide. This can be resolved by using the flow 5.3.12:

$$\mathcal{L}_X Y = \lim_{t\to 0} \frac{(T\sigma_t)^{-1} X(\gamma_p(t)) - X(p)}{t}, \tag{5.26}$$

where $T\sigma_t$ is the differential 5.1.7 of the flow, which satisfies $(T\sigma)^{-1} = T\sigma_{-t}$. To see that this definition makes sense, one has to show that $(T\sigma_t)^{-1}[X(\gamma_p(t))] \in T_pM$. This goes as follows:

$$\begin{aligned}
(T\sigma_t)^{-1} X(\gamma_p(t))(f) &= T\sigma_{-t} X(\gamma_p(t))(f) \\
&= X(\sigma_{-t} \circ \gamma_p(t))(f \circ \sigma_{-t}) \\
&= X(\sigma_{-t} \circ \sigma_t(p))(f \circ \sigma_{-t}) \\
&= X(p)(f \circ \sigma_{-t}) \\
&\in T_pM
\end{aligned}$$

for all $f \in C^k(M, \mathbb{R})$. On the third line the definition of the flow 5.3.12 was used. One can also rewrite the second term in the numerator of (5.26) using the flow:

$$X(p) = X(\sigma_0(p)) = T\sigma_0(X).$$

Using the definition of the pushforward of vector fields (5.9), the Lie derivative can be rewritten as follows:

$$\begin{aligned}
\mathcal{L}_X Y &= \lim_{t\to 0} \frac{\sigma_{-t*} X(\gamma_p(t)) - \sigma_{0*} X(\gamma_p(0))}{t} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}(\sigma_{-t*} X)(\gamma_p(t))\right|_{t=0}.
\end{aligned}$$

Finally, by using the relation between pushforward and pullback (5.10) this becomes

$$\mathcal{L}_X Y = \left.\frac{\mathrm{d}}{\mathrm{d}t}(\sigma_t^* X)(\gamma_p(t))\right|_{t=0}. \tag{5.27}$$

Property 5.3.20. Let $X, Y \in \mathfrak{X}(M)$ be C^k-vector fields. The Lie derivative has the following properties:

- $\mathcal{L}_X Y$ is a vector field.

- **Lie bracket**: The Lie derivative of vector fields coincides with the commutator:

$$\mathcal{L}_X Y = [X, Y]. \tag{5.28}$$

The fact that this is indeed a derivation on $C^{k-1}(M, \mathbb{R})$ follows from Schwarz's theorem **??**. This result shows that the Lie derivative on vector fields turns the space $\mathfrak{X}(M)$ into a (real) Lie algebra.

- The previous item also implies that the Lie derivative is antisymmetric:

$$\mathcal{L}_X Y = -\mathcal{L}_Y X. \tag{5.29}$$

Definition 5.3.21 (Holonomic basis). Consider a smooth manifold M and an open subset $U \subseteq M$. A local frame $\{e_i\}_{i\leq \dim(M)}$ for TU is said to be holonomic if all the Lie derivatives vanish on U:

$$\mathcal{L}_{e_i} e_j = 0. \tag{5.30}$$

Equivalently, a basis is holonomic if the associated structure coefficients of the Lie algebra $\mathfrak{X}(M)$ vanish on U.

Property 5.3.22. For every holonomic basis there exists a coordinate system on M such that the basis coincides with the coordinate-induced basis.

5.4 Differential k-forms

Definition 5.4.1 (Differential form). A differential k-form is a map

$$\omega : T^k M \rightarrow \mathbb{R} \tag{5.31}$$

such that the restriction of ω to each fibre of the bundle $T^k M$ is multilinear and antisymmetric. The space of all differential k-forms on a manifold M is denoted by $\Omega^k(M)$. Just like $\mathfrak{X}(M)$, it forms a $C^\infty(M)$-module.

Differential forms can also be constructed as sections of an associated vector bundle:

Alternative Definition 5.4.2. Consider the representation

$$\rho_k : \mathrm{GL}(\mathbb{R}^{m*}) \rightarrow \mathrm{GL}(\Lambda^k \mathbb{R}^{m*}) : A \mapsto A \wedge \cdots \wedge A.$$

This representation induces an associated vector bundle $\rho_k(\pi_{T^*M})$ of the cotangent bundle on M. A differential k-form is given by a section of $\rho_k(\pi_{T^*M})$:

$$\Omega^k(M) := \Gamma(\rho_k(\pi_{T^*M})). \tag{5.32}$$

Construction 5.4.3 (Exterior algebra). One can construct a Grassmann algebra **??** by equipping the graded vector space

$$\Omega^\bullet(M) := \bigoplus_{k\geq 0} \Omega^k(M) \tag{5.33}$$

with the wedge product of differential forms that is induced by the wedge product on $\Lambda^k \mathbb{R}^m$. This graded algebra is associative, graded-commutative and unital with the constant function $1 \in C^\infty(M)$ as the identity element.

Definition 5.4.4 (Pullback). Let $f : M \to N$ be a smooth function between manifolds and let ω be a differential k-form on N. The pullback of ω by f is defined as

$$f^*\omega := \omega \circ f_*. \tag{5.34}$$

This defines a map $f^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$.

Definition 5.4.5 (Pushforward). Let $f : M \to N$ be a diffeomorphism between manifolds and let ω be a differential k-form on M. The pushforward of ω by f is defined as

$$f_*\omega := \omega \circ (f^{-1})_*. \tag{5.35}$$

Remark. Note that the pushforward of differential k-forms is only defined for diffeomorphisms, in contrast to pullbacks which only require smooth functions. This also explains why differential forms are the most valuable elements in differential geometry. (Vector fields cannot even be pulled back by general smooth maps.)

Formula 5.4.6 (Dual basis). Consider the coordinate basis from Definition 3.2.3 for the tangent space T_pM. From this set one can construct a natural dual basis for the cotangent space T_p^*M using the natural pairing:

$$\left\langle \frac{\partial}{\partial x^i}, \mathrm{d}x^j \right\rangle = \delta_i^j. \tag{5.36}$$

It should be noted that $\mathrm{d}x^i$ is not just a notation. In the next section it will be shown that these basis vectors can be obtained by applying the *exterior derivative* to the coordinate functions x^i.

5.4.1 Exterior derivative

Definition 5.4.7 (Exterior derivative). The exterior derivative d_k is a morphism constructed on the graded algebra of differential k-forms:

$$\mathrm{d}_k : \Omega^k(M) \to \Omega^{k+1}(M). \tag{5.37}$$

For $k = 0$ it is defined by

$$\mathrm{d}f := \sum_{i=1}^n \frac{\partial f}{\partial x_i}\mathrm{d}x_i \tag{5.38}$$

The object $\mathrm{d}f \in \Omega^1(M)$ is often called the **differential** of f. This formula can be generalized to higher degree forms as follows:

$$\mathrm{d}(f\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) := \mathrm{d}f \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}. \tag{5.39}$$

Remark 5.4.8. Equation (5.38) should be compared with the (informal) formula for the differential of a function that is often used in physics. The main difference is that here the quantities dx^i are not infinitesimal quantities but vectors of unit norm.

Property 5.4.9. The exterior derivatives satisfy the following properties for all $k \geq 0$:

- **Nilpotency**: For all $\omega \in \Omega^k(M)$:

$$\mathrm{d}_k \circ \mathrm{d}_{k+1} = 0. \tag{5.40}$$

- **Linearity**: d_k is an $\mathbb{R}$-linear map.

These two items say that $(\Omega^\bullet(M), \mathrm{d})$ is not just a graded algebra, but in fact a dg-algebra **??**.

- **Graded Leibniz rule**: (hence d is a graded derivation):

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^j \omega_1 \wedge \mathrm{d}\omega_2, \tag{5.41}$$

  where $\omega_1 \in \Omega^j(M)$ and $\omega_2 \in \Omega^k(M)$.

- **Naturality**: If $f \in C^\infty(M)$, then $f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$.

Example 5.4.10. Let $f \in C^\infty(M, \mathbb{R})$ and let γ be a curve on M. From Definition 5.4.6 of the basis $\{\mathrm{d}x_k\}_{k\leq n}$ one obtains the following result:

$$\langle \mathrm{d}f(x), \gamma'(t)\rangle = \sum_k \frac{\partial f}{\partial x_k}(x)\gamma_k'(t) = (f \circ \gamma')(t). \tag{5.42}$$

Example 5.4.11. An explicit formula for the exterior derivative of a k-form Φ is

$$\begin{aligned}\mathrm{d}\Phi(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\Phi(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \Phi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),\end{aligned} \tag{5.43}$$

where $\hat{X}$ indicates that this argument is omitted.

5.4.2 Lie derivative

Formula 5.4.12 (Lie derivative of smooth functions). Using the definition of the exterior derivative of smooth functions (5.38) and the definition of the dual basis 5.4.6, one can rewrite the Lie derivative 5.3.17 as

$$Xf(p) = \mathrm{d}f_p(X(p)). \tag{5.44}$$

Formula 5.4.13 (Lie derivative of differential forms).

$$\mathcal{L}_X\omega(p) := \lim_{t\to 0} \frac{\sigma_t^*\omega - \omega}{t}(p) \tag{5.45}$$

Property 5.4.14. The Lie derivative has the following Leibniz-type property with respect to differential forms:

$$\mathcal{L}_X(\omega(Y)) = (\mathcal{L}_X\omega)(Y) + \omega(\mathcal{L}_X Y), \tag{5.46}$$

where X, Y are two vector fields and ω is a one-form.

5.4.3 Interior product

Definition 5.4.15 (Interior product). Aside from the exterior derivative one can also define another operation on the algebra of differential forms:

$$\iota_X : (\iota_X\omega)(v_1, \ldots, v_{k-1}) \mapsto \omega(X, v_1, \ldots, v_{k-1}). \tag{5.47}$$

This antiderivation of degree -1 is called the **interior product** or **interior derivative**. It can be seen as a generalization of the contraction map **??**.

Notation 5.4.16. In certain situations the above notation might become cumbersome. For this reason the notation $X \lrcorner\, \omega$ is frequently used.

Formula 5.4.17 (Cartan's magic formula[1]). Let X be a vector field and let ω be a differential k-form. The Lie derivative of ω along X is given by the following formula:

$$\mathcal{L}_X\omega = \iota_X(\mathrm{d}\omega) + \mathrm{d}(\iota_X\omega). \tag{5.48}$$

[1] Sometimes called **Cartan's (infinitesimal) homotopy formula**.

5.4.4 Lie derivative of tensor fields

Formula 5.4.18 (Lie derivative of tensor fields). By comparing the definitions of the Lie derivatives of vector fields 5.3.19 and differential forms 5.4.13, one can see that both definitions are identical upon replacing X by ω. This leads to the following definition of the Lie derivative of a general tensor field $\mathcal{T} \in \Gamma(T^{(k,l)}M)$:

$$\mathcal{L}_X\mathcal{T}(p) := \frac{\mathrm{d}}{\mathrm{d}t}\sigma_t^*\mathcal{T}(\gamma_p(t))\bigg|_{t=0}. \tag{5.49}$$

Alternative Definition 5.4.19 (Lie derivative of tensor fields). The Lie derivative of tensor fields can also be defined as the unique differential operator satisfying the following axioms:

1. $\mathcal{L}_X$ coincides with X on $C^\infty(M)$.
2. $\mathcal{L}_X$ satisfies the Leibniz rule with respect to tensor products.
3. $\mathcal{L}_X$ satisfies the Leibniz rule with respect to the contraction of forms and vector fields.
4. $\mathcal{L}_X$ commutes with the exterior derivative.

Property 5.4.20 (Derivations). Every derivation D of the tensor algebra can be decomposed as

$$D = \mathcal{L}_X + S \tag{5.50}$$

for some vector field X and some endomorphism S.

5.4.5 Vector-valued differential forms

Definition 5.4.21 (Vector-valued form). Consider a vector space V and let $E \to M$ be a vector bundle with typical fibre V. A vector-valued differential form on M can be defined in two ways. A vector-valued k-form can be defined as a map $\omega : \Gamma(T^kM) \to V$ or, more generally, as a section of the following associated bundle:

$$\Omega^k(M;E) := \Gamma(E \otimes \Lambda^kT^*M). \tag{5.51}$$

The latter construction is also often called a **vector bundle-valued differential form**.

Construction 5.4.22 (Wedge product). Let $\omega \in \Omega^p(M;E_1)$ and $\nu \in \Omega^q(M;E_2)$. The wedge product of these differential forms is defined as follows:

$$\omega\wedge\nu(v_1,\ldots,v_{p+q}) := \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}}\mathrm{sgn}(\sigma)\omega(v_{\sigma(1)},\ldots,v_{\sigma(p)})\otimes\nu(v_{\sigma(p+1)},\ldots,v_{\sigma(q)}). \tag{5.52}$$

This is a direct generalization of the formula for the wedge product of ordinary differential forms, where the scalar product (product in the algebra $\mathbb{R}$) is replaced by the tensor product (product in the tensor algebra). It should be noted that the result of this operation is not a section of any of the original bundles E_1 or E_2, but rather of the tensor product bundle $E_1 \otimes E_2$.

Remark 5.4.23 (Differential vs. pushforward). At this point the reason why the pushforward 5.9 is also sometimes called the differential (as in Definition 5.1.1) can be given.

It can be shown that for any vector bundle E and any manifold M, the space of sections of the homomorphism bundle $\mathrm{Hom}(TM,E)$ is isomorphic to the space of E-valued differential forms $\Omega^1(M;E)$. Now, consider a smooth function $f : M \to N$. Its pushforward is a map $f_* : TM \to TN$. Locally, the corresponding differential form is given by

$$\mathrm{d}f := \mathrm{d}f^i \otimes \partial_i, \tag{5.53}$$

where $(f^1, \ldots, f^{\dim(N)})$ is a local expression for f and $\{\partial_1, \ldots, \partial_{\dim(N)}\}$ is a local frame for TN. It is straightforward to show that acting with this differential form on a vector field in $\mathfrak{X}(M)$ gives the same result as acting with the pushforward f_*.

Construction 5.4.24 (Exterior derivative). The definition of an exterior derivative on E-valued differential forms is more involved than in the case of ordinary forms. The naive thing to do would be defining a derivative through the Leibniz formula. However, without further structure on E there is no natural way of differentiating sections of E.

If E is *flat*, i.e. if its transition functions are locally constant, one can choose a frame of sections $e^i : U \to E|_U$ induced by the trivializing maps $E|_U \to U \times \mathbb{R}^n$. Locally, one can then express any E-valued differential form as $\omega|_U = \sum_i \omega_i \otimes e^i$, where the ω_i are ordinary differential forms. After defining $\mathrm{d}e^i := 0$, one can again construct an exterior derivative through the Leibniz formula.

Remark 5.4.25. It should be noted that the definition of d depends on the choice of trivialization since the sections e^i depend on this choice.

Definition 5.4.26 (Lie algebra-valued form). A vector-valued differential form where the vector space V is equipped with a Lie algebra structure.

Formula 5.4.27 (Wedge product). Let $\omega \in \Omega^p(M; \mathfrak{g})$ and $\nu \in \Omega^q(M; \mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra. The wedge product of these differential forms is defined as follows:

$$[\omega \wedge \nu](v_1, \ldots, v_{p+q}) := \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma) \big[\omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), \nu(v_{\sigma(p+1)}, \ldots, v_{\sigma(q)})\big], \tag{5.54}$$

where $[\cdot, \cdot]$ denotes the Lie bracket on $\mathfrak{g}$.

Formula 5.4.28. Let $\{e_a\}_{a \leq \dim(\mathfrak{g})}$ be a basis for the Lie algebra $\mathfrak{g}$. One can write any Lie algebra-valued differential forms as $\phi = \phi^a \otimes e_a$ and $\psi = \psi^b \otimes eb$, where ϕ^a and ψ^b are ordinary differential forms. The above formula for the wedge product can now be rewritten more elegantly as

$$[\phi \wedge \psi] = (\phi^a \wedge \psi^b) \otimes [e_a, e_b], \tag{5.55}$$

where $\wedge$ is the wedge product on $\Omega^\bullet(M)$.

Corollary 5.4.29 (Graded algebra). Using the above formula it is easy to verify a number of properties similar to the ones of ordinary differential forms. As an example the analogue of the graded-commutativity property on $\Omega^\bullet(M)$ is given:

$$[\phi \wedge \psi] = (-1)^{pq+1}[\psi \wedge \phi], \tag{5.56}$$

where $\phi \in \Omega^p(M; \mathfrak{g})$ and $\psi \in \Omega^q(M; \mathfrak{g})$. Here, the extra factor -1 comes from the antisymmetry of the Lie bracket.

Analogously, one can prove that the Lie algebra-valued wedge product satisfies a graded Jacobi-type identity:

$$(-1)^{pr}[\phi \wedge [\psi \wedge \theta]] + (-1)^{pq}[\psi \wedge [\theta \wedge \phi]] + (-1)^{qr}[\theta \wedge [\phi \wedge \psi]] = 0, \tag{5.57}$$

where $\theta \in \Omega^r(M; \mathfrak{g})$.

5.5 Frobenius's theorem

Definition 5.5.1 (Distribution). A section of the Grassmann k-plane bundle 5.6.1.

Definition 5.5.2 (Integrable distribution). Let M be a manifold and consider a distribution of k-planes $W \in \Gamma(\mathrm{Gr}(k, TM))$. A submanifold $N \subseteq M$ is said to integrate W (or to be **integral**) with initial condition $p_0 \in M$ if $p_0 \in N$ and if $\forall p \in N : W(p) = T_pN$. W is said to be integrable if there exists such a submanifold N.

Property 5.5.3. If a distribution on M is integrable, M can be written as the (disjoint) union of maximal connected, integrable manifolds. These submanifolds are also called the **leaves** of the distribution (the decomposition in leaves defines a *foliation*).

Definition 5.5.4 (Frobenius's integrability condition). A distribution W on a manifold M is said to satisfy the Frobenius integrability condition on an open set $U \subseteq M$ if for every two vector fields X, Y defined on U, such that $X(p) \in W(p)$ and $Y(p) \in W(p)$ for all $p \in U$, the Lie bracket $[X, Y](p)$ is also an element of $W(p)$ for all $p \in U$.

Theorem 5.5.5 (Frobenius's integrability theorem). *Let W be a distribution over a manifold M. W is integrable if and only if it satisfies the Frobenius integrability condition.*

This theorem also admits a formulation in terms of differential forms.

Property 5.5.6 (Differential formulation). Consider a rank-r distribution D. The annihilator $I(D)$ is the ideal containing all differential forms satisfying

$$\omega(X_1, \dots, X_k) = 0 \tag{5.58}$$

whenever all $X_i \in D$. The Frobenius theorem says that the following conditions are equivalent:

- D is integrable.
- $I(D)$ is a differential ideal.

Property 5.5.7 (Pfaffian system). The fact that D is a distribution implies that locally there exists a set of $\dim(M) - r$ linearly independent, annihilating one-forms that generate $I(D)$. These one-forms are said to form a Pfaffian system. Furthermore, every ideal that is locally generated by linearly independent one-forms defines a smooth distribution.

Given a Pfaffian system $\{\theta^\alpha\}_{\alpha \leq \dim(M)-r}$, a third equivalent condition for (**complete**) integrability is that

$$\mathrm{d}\theta^1 \wedge \theta^2 \wedge \cdots = 0. \tag{5.59}$$

Property 5.5.8 (Integral manifold). An integral manifold of a rank-r Pfaffian system $\{\theta^\alpha\}_{\alpha \leq \dim(M)-r}$ is a smooth function $f : N \to M$ such that

$$f^*\theta^\alpha = 0 \tag{5.60}$$

for all $1 \leq \alpha \leq \dim(M) - r$. Often f will be a submanifold inclusion.

If the system is integrable, there exist $\dim(M) - r$ independent functions y^α such that $\{\mathrm{d}y^\alpha\}_{\alpha \leq \dim(M)-r}$ generates the same differential ideal as the Pfaffian system. The integral submanifolds of the distribution are given by the systems

$$\begin{cases} y^1 = c^1, \\ y^2 = c^2, \\ \quad\vdots \end{cases} \tag{5.61}$$

for constants c^α.

Definition 5.5.9 (Characteristic system). Consider a system P of exterior differential equations $\{\omega^\alpha = 0\}$ where the degree of all forms is nonzero. As for a smooth distribution, one can locally find a set of one-forms that generate the same ideal. This is again called the **Pfaffian system**. The characteristic system of P is defined as the Pfaffian system of the differential closure of P.

If P also contained functions, i.e. degree-zero forms, the characteristic system is that of $\{\omega^\alpha, \mathrm{d}f^\beta\}$ extended by the functions f^β themselves.

Definition 5.5.10 (Integral invariant). Consider a Pfaffian system $P \equiv \{\theta^\alpha\}$. A differential form ω is called an invariant of P if

$$\mathcal{L}_X\omega = 0 \tag{5.62}$$

whenever $\theta(X) = 0$ for al $\theta \in P$. These forms are also called absolute integral invariants because the integral of ω over every chain is invariant under the transformations generated by these X. If $\mathrm{d}\varphi$ is an (absolute integral) invariant of P, then φ is called a relative integral invariant.

5.6 Grassmann bundle

Looking at Property **??** and noting that $\mathrm{GL}(n,\mathbb{R})$ is a Lie group, it is clear that one can endow the Grassmannian $\mathrm{Gr}(k,\mathbb{R}^n)$ from Definition **??** with a differentiable structure, thereby turning it into a smooth manifold. This allows the construction of a new bundle, but because the Grassmannian is not a vector space, the resulting bundle will be a general fibre bundle and not a vector bundle.

Construction 5.6.1 (Grassmann bundle). Define a new set of transition functions

$$\psi_{ji} : (\varphi_i(p), V) \mapsto \big(\varphi_j(p), t_{ji}(p)\cdot V\big)\,, \tag{5.63}$$

where $\{t_{ji}\}_{i,j\leq n}$ is the tangent bundle cocycle. These transition functions can be used to create a new fibre bundle with typical fibre $\mathrm{Gr}(k,\mathbb{R}^n)$. The fibre over a point $p \in M$ is the Grassmannian $\mathrm{Gr}(k, T_pM)$ associated to the tangent space over p.

By replacing the tangent bundle TM by an arbitrary vector bundle E (and accordingly replacing the cocycle t with the cocycle of E) one can define the Grassmann bundle of a general vector bundle.

Notation 5.6.2. The Grassmann k-plane bundle of a vector bundle E is denoted by $\mathrm{Gr}(k,E)$.

Definition 5.6.3 (Tautological bundle). Consider the Grassmannian $\mathrm{Gr}(n,V)$ of an $(n+k)$-dimensional vector space V. The total space of the tautological k-bundle $\gamma_{n,k}$ is defined as the set of points (W,w) where $W \in \mathrm{Gr}(n,V)$ and $w \in W$. Local trivializations are constructed as follows:

$$\varphi_Z : \pi^{-1}(U) \to \mathrm{Gr}(n,V)\times Z : (W,w) \mapsto (W, \mathrm{pr}_Z(w))\,, \tag{5.64}$$

where pr_Z is the orthogonal projection onto the subspace $Z \in \mathrm{Gr}(n,V)$. This bundle inherits a natural vector bundle structure from V.

Definition 5.6.4 (Twisting sheaf). Consider the tautological line bundle J over a projective space $\mathbb{CP}^n \cong \mathrm{Gr}(1,\mathbb{C}^n)$. The dual line bundle $\mathrm{Hom}(J,\mathbb{C})$ is often denoted by $\mathcal{O}_{\mathbb{CP}^n}(1)$ or $\mathcal{O}(1)$ and is sometimes called **Serre's twisting sheaf**. Tensor powers of this bundle are accordingly denoted by $\mathcal{O}_{\mathbb{CP}^n}(k)$. To also allow for factors of J one can extend the notation to negative indices: $\mathcal{O}_{\mathbb{CP}^n}(-k)$. The tautological bundle is then denoted by $\mathcal{O}_{\mathbb{CP}^n}(-1)$.

Property 5.6.5 (Euler sequence). The twisting sheaf fits in a short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^n} \longrightarrow \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \longrightarrow T\mathbb{CP}^n \longrightarrow 0. \tag{5.65}$$

By dualizing one can obtain an exact sequence in terms of the de Rham complex.

Proof. Note that vector fields on $\mathbb{CP}^n$ are, locally, obtained by pairing (coordinate-induced) basis vectors on $\mathbb{C}^{n+1}$ with linear functions on $\mathbb{C}^{n+1}$, i.e. with sections of $\mathcal{O}_{\mathbb{CP}^n}(1)$. This leads to the central term $\mathcal{O}_{\mathbb{CP}^n}(1) \otimes \mathbb{C}^{n+1} \cong \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}$. When projecting these vector fields down to $T\mathbb{CP}^n$, the kernel is generated by the Euler vector field (Example 5.3.10). Not only scalar multiples are sent to 0, all multiples by functions that descend to well-defined functions on $\mathbb{CP}^n$ will vanish. Hence, the kernel is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{CP}^n}$.

A more invariant proof is given as follows. For every one-dimensional space $L \in \mathbb{CP}^n$, the tangent space $T_L\mathbb{CP}^n$ is isomorphic to $\mathrm{Hom}(L, \mathbb{C}^{n+1}/L)$ since, by linearity, a variation of the space is determined by the image of a single point and moving this point along the space L itself does not change anything. By basic linear algebra one has $\mathrm{Hom}(L, \mathbb{C}^{n+1}/L) \cong L^* \otimes \mathbb{C}^{n+1}/L$ or $\mathbb{C}^{n+1}/L \cong \mathrm{Hom}(L, \mathbb{C}^{n+1}/L) \otimes L$. This gives the exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{C}^{n+1} \longrightarrow T_L\mathbb{CP}^n \otimes L \longrightarrow 0.$$

After globalizing one obtains

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{CP}^n}^{\oplus(n+1)} \longrightarrow T\mathbb{CP}^n \otimes \mathcal{O}_{\mathbb{CP}^n}(-1) \longrightarrow 0.$$

After tensoring with the twisting sheaf, one obtains the Euler sequence. □

5.7 Linear connections

5.7.1 Koszul connections

Definition 5.7.1 (Koszul connection). Let $\pi : E \to M$ be a vector bundle over a manifold M. A Koszul connection (or **linear connection**) on E is a (smooth) linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the Leibniz property

$$\nabla(f\sigma) = f\nabla\sigma + \mathrm{d}f \otimes \sigma \tag{5.66}$$

for all $f \in C^\infty(M)$. When evaluated on a vector field X, ∇_X is often called a **covariant derivative**.

Property 5.7.2. Because $\nabla\sigma$ takes a vector field as input, which is a $C^\infty(M)$-linear operation, the connection satisfies the following linearity property:

$$\nabla_{fX+Y}\sigma = f\nabla_X\sigma + \nabla_Y\sigma. \tag{5.67}$$

Formula 5.7.3. Let E, E' be two vector bundles over the same manifold M. Koszul connections on E and E' induce a connection on the tensor product bundle $E \otimes E'$ as follows:

$$\nabla(X \otimes Y) := \nabla X \otimes Y + X \otimes \nabla Y \tag{5.68}$$

for $X \in \Gamma(E), Y \in \Gamma(E')$.

Example 5.7.4 (Affine connection). Let M be a manifold. An affine connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a Koszul connection on the tangent bundle.

Property 5.7.5 (Local behaviour). Consider a vector $v \in T_pM$. If two vector fields $X, Y \in \Gamma(TM)$ coincide on some neighbourhood U of p, then $\nabla_v X = \nabla_v Y$ at p. Furthermore, given a curve $c : [0,1] \to M$ and two vector fields $X, Y \in \Gamma(TM)$ such that $X \circ c = Y \circ c$, one finds that $\nabla_{\dot{c}} X = \nabla_{\dot{c}} Y$. This implies that an affine connection only depends on the local behaviour of the given section.

Remark 5.7.6. The above property shows the major difference between the Lie derivative and the covariant derivative when acting on sections of the tangent bundle σ. Lie derivatives depend on the local behaviour of both X and σ. The covariant derivative on the other hand only depends on the value of X at $p \in M$ and on the local behaviour of σ.

Definition 5.7.7 (Parallel tensor fields). A tensor field T is said to be parallel with respect to a connection ∇ if it satisfies $\nabla T = 0$. It is said to be parallel with respect to a vector field X if $\nabla_X T = 0$.

Example 5.7.8. Important examples in the case of the Levi-Civita connection on a Riemannian manifold are the volume form Vol and the metric g (see Chapter **??**).

Property 5.7.9 (Affinity). Consider two affine connections $\nabla, \overline{\nabla}$ on a smooth manifold M. The operator $\nabla - \overline{\nabla}$ is an endomorphism of E, i.e. $\nabla - \overline{\nabla} \in \Omega^1(M; \mathrm{End}(E))$. It follows that the set of affine connections forms an affine space (hence the name).

Definition 5.7.10 (Connection coefficients). Let E be a smooth rank-k vector bundle. Consider a Koszul connection ∇, a (local) frame $\{e_i\}_{1\leq i\leq k}$ and a (local) coframe $\{f^i\}_{1\leq i\leq k}$ on E. For every vector field e_i one can (locally) write

$$\nabla e_i = \Gamma^k{}_{ji} e_k \otimes f^j. \tag{5.69}$$

The quantities $\Gamma^k{}_{ji}$ are called the connection coefficients or **Christoffel symbols** of ∇. For a general vector field $\sigma = \sigma^i e_i$ one then obtains (if $\{e_i, f^i\}_{1\leq i\leq k}$ are coordinate-induced):

$$\begin{aligned}\nabla\sigma &= (\nabla\sigma^i) \otimes e_i + \sigma^i(\nabla e_i)\\ &= (\partial_j\sigma^k) e_k \otimes f^j + \sigma^i(\Gamma^k{}_{ji} e_k \otimes f^j)\\ &= (\partial_j\sigma^k + \Gamma^k{}_{ji}\sigma^i) e_k \otimes f^j.\end{aligned} \tag{5.70}$$

Definition 5.7.11 (Curvature). Let ∇ be a Koszul connection on a vector bundle $E \to M$. The associated curvature 2-form $F_\nabla \in \Omega^2(M; \mathrm{End}(E))$ is defined as follows:

$$F_\nabla(X, Y)\sigma := [\nabla_X, \nabla_Y]\sigma - \nabla_{[X,Y]}\sigma. \tag{5.71}$$

Definition 5.7.12 (Torsion). Let ∇ be an affine connection on smooth manifold M. The associated torsion 2-form $T \in \Omega^2(M; TM)$ is defined as follows:

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]. \tag{5.72}$$

Property 5.7.13 (Bianchi identities). Let ∇ be an affine connection on smooth manifold M. The associated curvature and torsion forms are related as follows:

$$R(X, Y)Z - T(T(X, Y), Z) + \text{cyclic} = \nabla_X T(Y, Z) \tag{5.73}$$

$$\nabla_X R(Y, Z) + R(T(X, Y), Z) + \text{cyclic} = 0, \tag{5.74}$$

where cyclic indicates that all terms obtained by cyclic permutations of the vector fields have to be included.

5.7.2 Induced connections

Formula 5.7.14 (Connection on tensors). Applying the Leibniz property of a Koszul connection to tensor contractions gives the following form of the induced connection on the (k,l)-tensor bundle:

$$\begin{aligned}\nabla_Y T(\omega^1,\ldots,\omega^k,X_1,\ldots,X_l) := Y\Big(T(\omega^1,\ldots,\omega^k,X_1,\ldots,X_l)\Big)&\\ -\sum_{i=1}^{k} T(\omega^1,\ldots,\nabla_Y\omega^i,\ldots,\omega^k,X_1,\ldots,X_l)&\\ -\sum_{i=1}^{l} T(\omega^1,\ldots,\omega^l,X_1,\ldots,\nabla_Y X_i,\ldots,X_l),&\end{aligned}\tag{5.75}$$

where $Y, X_1,\ldots,X_l \in \mathfrak{X}(M)$ and $\omega^1,\ldots,\omega^k \in \Omega^1(M)$.

Corollary 5.7.15 (Iterated derivatives). By noting that the covariant derivative of a vector field is a vector-valued differential form, one can use the previous formula to compute the covariant derivative of the covariant derivative:

$$(\nabla_X\nabla)_Y Z = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z).\tag{5.76}$$

Parentheses were added to make it clear that the outer covariant derivatives act on the result of the inner derivatives. If these parentheses would not have been added, these terms could have been confused with the second covariant derivative (whose definition also follows from a Leibniz-type argument):

$$\nabla^2_{X,Y} S := \nabla_X(\nabla_Y S) - \nabla_{\nabla_X Y} S.\tag{5.77}$$

As an example of the second covariant derivative the definition of the Hessian on arbitrary smooth manifolds is given:

Definition 5.7.16 (Hessian). Consider a manifold with connection ∇. The Hessian of a function $f \in C^\infty(M)$ is defined as the iterated covariant derivative:

$$\mathrm{Hess}(f) := \nabla^2 f,\tag{5.78}$$

where one should note that by the above definition the first covariant derivative also acts on the second one, i.e

$$\nabla^2 f(X,Y) = \nabla_X(\nabla_Y f) - \nabla_{\nabla_X Y} f.\tag{5.79}$$

For a scalar function one knows that $\nabla f = \mathrm{d}f$ and for covector fields one knows that (in local coordinates)

$$\nabla_i\sigma_j = \partial_i\sigma_j - \Gamma^k_{ij}\sigma_k,$$

where Γ^k_{ij} are the connection coefficients. Combining these facts one obtains the following local expression for the Hessian of f:

$$\mathrm{Hess}(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k}\right) dx^i \otimes dx^j.\tag{5.80}$$

Definition 5.7.17 (Pullback connection). Let $E \to N$ be a vector bundle with Koszul connection ∇ and let $f : M \to N$ be a smooth function. On the pullback bundle 4.2.5 there exists a unique Koszul connection ∇' satisfying

$$\nabla'(f^*\chi) = f^*(\nabla\chi)\tag{5.81}$$

for any section χ of E.

Definition 5.7.18 (Invariant connection). Let G be a Lie group acting on a vector bundle $E \to M$. A Koszul connection ∇ on E is said to be invariant with respect to the G-action if it satisfies:

$$g^*\nabla = \nabla \tag{5.82}$$

for all $g \in G$.

5.8 Integration Theory

For the theory of measure spaces and Lebesgue integration, see Chapter **??**.

5.8.1 Orientation and densities

One can define an orientation of manifolds by generalizing the situation for vector spaces **??**:

Definition 5.8.1 (Orientable manifold). First, the definition of the volume element needs to be slightly modified. A **volume form** on M is a nowhere-vanishing top-dimensional differential form $\mathrm{Vol} \in \Omega^{\dim(M)}(M)$. The definition of an orientation is now virtually the same as for vector spaces.

An **oriented atlas** is given by all charts of M for which the pullback of the Euclidean volume form is a positive multiple of Vol. This also implies that the transition functions have a positive Jacobian determinant. The existence of such a volume form turns a differentiable manifold into an **orientable manifold**.

Alternatively, an (orientable) manifold with volume form Vol is said to be **positively oriented** if it comes equipped with a smooth choice of bases $\{v_1, \dots, v_n\}$ for T_pM such that

$$\mathrm{Vol}_p(v_1, \dots, v_n) > 0. \tag{5.83}$$

Example 5.8.2. Let $M = \mathbb{R}^n$. The canonical Euclidean volume form is given by the determinant map

$$\det : (u_1, \dots, u_n) \mapsto \det(u_1, \dots, u_n), \tag{5.84}$$

where the u_n's are expressed in the canonical basis $(e_1, \dots, e_n)$. The terminology of "volume forms" is justified by noting that the determinant map gives the signed volume of the n-dimensional parallelotope spanned by the vectors $\{u_1, \dots, u_n\}$.

Property 5.8.3. Let ω_1, ω_2 be two volume forms on M. Because the space of top-degree forms is one-dimensional, there exists a smooth function f such that

$$\omega_1 = f\omega_2.$$

Furthermore, the sign of this function is constant on every connected component of M.

One can also rephrase orientability of manifolds in terms of bundles:

Definition 5.8.4 (Orientation bundle). Consider a manifold M. The transition function A of TM is given by the Jacobian of the transitions functions on M. The associated line bundle with transition function $\operatorname{sgn}\det(A)$ is called the orientation bundle $o(M)$.

In general one can define the orientation bundle $o(E)$ for any vector bundle E, where one replaces the Jacobian in the above construction by the transition maps of E. From this it is clear that the orientation bundle $o(M)$ is the same as $o(TM)$.

Alternative Definition 5.8.5 (Orientable manifold). A manifold is orientable if its orientation bundle is trivial.

Remark 5.8.6. By definition of the orientation bundle, the transition functions are those that have a positive determinant. This gives the equivalence with Definition 5.8.1. In the next chapter on principal bundles yet another (equivalent) definition of orientability in terms of G-structures will be given (see Example 6.5.5).

Further below, integration theory will be generalized from orientable manifolds to non-orientable manifolds. To achieve this goal the notion of differential forms needs to be generalized. A good introduction for this is [19].

Definition 5.8.7 (Pseudoscalars). Let G be a Lie group and consider a group morphism $\phi : G \to \mathrm{O}(p, q)$ for some $p, q \in \mathbb{N}$. The pseudoscalar representation of G, induced by ϕ, is defined as the one-dimensional representation given by

$$\mathbf{1}_{\text{sgn}} : g \mapsto \det(\phi(g)). \tag{5.85}$$

The notation $\mathbf{1}_{\text{sgn}}$ refers to the fact that this is a generalization of the *alternating* (or *sign*) *representation* of the permutation groups S_n. Any Riemannian manifold admits a canonical pseudoscalar bundle Ψ associated to its (orthogonal) frame bundle.

Sections of a vector bundle with transition functions defined by $\mathbf{1}_{\text{sgn}}$ are generally called **pseudoscalar fields**. When using the pseudoscalar representation of the transition functions of the tangent bundle TM to construct an associated bundle, one obtains the pseudoscalar bundle Ψ_M. A vector bundle twisted by the pseudoscalar bundle Ψ often receives the prefix ''pseudo'', e.g. the Ψ-twisted k-form bundle is called the bundle of k-**pseudoforms**.

Definition 5.8.8 (Tensor density). Consider a vector bundle $E \to M$ defined by transition functions A. The associated bundle of (tensor) s-densities is obtained by using the representation

$$\rho : A \mapsto \det(A)^{-s}. \tag{5.86}$$

The number s is called the **weight** of the density. For $E \equiv TM$ one obtains the (tensor) s-densities on M, which in the case of $s = 1$ are equivalent to top-dimensional forms on M. When twisting a vector bundle by an s-density bundle, the prefix ''s-weighted'' is often added.

Example 5.8.9 (Pseudovectors). The representation

$$\rho : A \mapsto \operatorname{sgn}\det(A)A \tag{5.87}$$

gives rise to a bundle similar to the tangent bundle, where the sign of the cocycles t_{ji} now has an influence on the fibres. Sections of such bundles are called **pseudovector fields**. This construction is equivalent to twisting the tangent bundle by the pseudoscalar bundle Ψ (hence its name).

Remark 5.8.10 (Honest densities). One should pay incredible attention to the definition of a **density** (i.e. without the prefix ''tensor''). A density is defined as an n-pseudoform, i.e. a section of the **density bundle** $|\Omega|(M) := \Omega^n(M) \otimes o(M)$. Here, the transition function is $|\det(A)|$, where A is the transition function of T^*M. These are the objects one can integrate over any manifold, even the non-orientable ones. They are essentially maps $\Gamma(\det(T^*M)) \to C^\infty(M)$. A naive way to construct a density on a manifold M is by choosing a volume form $\mathrm{Vol}(M)$ and taking the absolute value $|\mathrm{Vol}(M)|$.

One can also define **honest** s**-densities** $|\Omega|^s(M)$ by combining Definition 5.8.8 with the orientation bundle to obtain transition maps $|\det(A)|^s$, where A is again the cotangent transition function. This is also the only possible way to generalize the (tensor) s-densities to real s.

Property 5.8.11 (Orientability). A smooth manifold is orientable if and only if its canonical line bundle 5.2.25 is trivial. Furthermore, for orientable manifolds there exists an isomorphism $\Gamma(\det(T^*M)) \cong \Gamma(|\Omega|(M))$.

5.8.2 Orientation in homology

In this section a characterization of orientability in terms of the homology of a manifold is given. Smoothness is not required here. See Sections **??** and **??** for an introduction to homology.

Begin with the canonical example $\mathbb{R}^n$. Intuitively one would expect an orientation on Euclidean space to be a property that is preserved under rotations and reversed by reflections. On the sphere these operations have degree 1 and -1 respectively, so the perfect choice for an orientation would be the generator of $H_n(S^n) \cong \mathbb{Z}$. Luckily, there exists an isomorphism $H_n(S^n) \cong H_n(\mathbb{R}^n, \mathbb{R}^n\backslash\{*\})$. So, for every point $x \in \mathbb{R}^n$ one can define a local orientation as a choice of generator of the local homology group $H_n(\mathbb{R}^n, \mathbb{R}^n\backslash\{x\})$.

For a given manifold M one then defines a global orientation (if it exists) as a choice of local orientation for every point $p \in M$ such that every two points admitting a common covering chart have consistent local orientations.

Property 5.8.12 (Orientability). If a closed connected manifold is ($\mathbb{Z}$-)orientable, there exists an isomorphism

$$H_n(M) \cong H_n(M, M\backslash\{p\}) \tag{5.88}$$

for all points $p \in M$. A choice of class in $H_n(M)$ that maps to a generator of $H_n(M, M\backslash\{p\})$ for all $p \in M$ is called a **fundamental class** or **orientation class**.

In the case where M is not connected, the fundamental class equals the direct sum of the generators of the connected components (following the idea of the additivity axiom **??**).

The above definition and property can be generalized to arbitrary unital rings R:

Definition 5.8.13 (R-orientability). A manifold is M-orientable if a consistent choice of local R-orientation exists or, equivalently, if $H_n(M; R) \cong R$.

Property 5.8.14 (Non-orientable manifolds). If M is not R-orientable, the map

$$H_n(M; R) \to H_n(M, M\backslash\{p\}; R)$$

is still injective with image $\{r \in R \mid 2r = 0\}$. In particular, every closed manifold is $\mathbb{Z}_2$-orientable.

Property 5.8.15 (Orientability implies R-orientability). By the *universal coefficient theorem* it follows that a $\mathbb{Z}$-orientable manifold is also R-orientable for all unital rings R. Conversely, a manifold is $\mathbb{Z}$-orientable if it is R-orientable for all unital rings R.

5.8.3 Integration of top-dimensional forms

Definition 5.8.16 (Measure zero). A subset $U \subset M$ of an orientable manifold is said to be of measure zero (or **null**) if it is the countable union of inverse images (with respect to the chart maps on M) of null sets in $\mathbb{R}^n$.

Definition 5.8.17 (Integrable form). A differential form for which its components with respect to any basis of $\Omega^k(M)$ are Lebesgue integrable on $\mathbb{R}^n$.

Formula 5.8.18 (Integration with compact support). Consider a top-dimensional form $\omega \in \Omega^{\dim(M)}$ on M with compact support on a coordinate patch $U \subset M$.

$$\int_M \omega = \int_U \omega := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \omega_{12\ldots n}(x)\, dx^1\, dx^2 \cdots dx^n. \tag{5.89}$$

This integral is well-defined because under an orientation-preserving change of coordinates the component $\omega_{1\ldots n}$ transforms as $\omega'_{1\ldots n} = \det(J)\omega_{1\ldots n}$, where J is the Jacobian of the coordinate transformation. Inserting this in the integral and replacing dx_i by dx'_i then gives the well-known change-of-variables formula from (Lebesgue) integration theory.

If one requires the manifold M to be paracompact, such that every open cover $\{U_i \subseteq M\}_{i\in I}$ admits a subordinate partition of unity $\{\phi_i\}_{i\in I}$, one can define the integral of a general compactly supported form $\omega \in \Omega^n(M)$ as follows:

$$\int_M \omega := \sum_{i\in I} \int_{U_i} \rho_i \omega. \tag{5.90}$$

Remark 5.8.19. Although integration was only defined for compactly supported forms, the general formula can also be applied to general forms. It is well-defined whenever the forms $\rho_i\omega$ are integrable and the sum in the definition converges.

Property 5.8.20 (Compact manifolds). Let M be a smooth compact manifold. Because every form on M is automatically compactly supported, all forms are integrable on M.

Property 5.8.21 (Invariance under pullbacks). Consider an orientation-preserving diffeomorphism $f : M \to N$.

$$\int_M f^*\omega = \int_N \omega \tag{5.91}$$

Notation 5.8.22. Because the integral of differential forms is linear in the integrand and additive over disjoint unions, it can be interpreted as a linear pairing. This motivates the following notation:

$$\langle M, \omega\rangle := \int_M \omega. \tag{5.92}$$

5.8.4 Stokes's theorem

Theorem 5.8.23 (Stokes's theorem). *Let M be an orientable manifold with boundary ∂M and let ω be a differential k-form on M.*

$$\int_{\partial M} \omega = \int_M \mathrm{d}\omega. \tag{5.93}$$

Corollary 5.8.24. The Kelvin-Stokes theorem **??**, the divergence theorem **??** and Green's identity **??** are immediate results of this (generalized) Stokes's theorem.

Definition 5.8.25 (Calibration). A degree-p calibration on a smooth manifold M with volume form Vol is a differential form $\omega \in \Omega^p(M)$ satisfying the following conditions:

1. **Closedness**: $\mathrm{d}\omega = 0$.
2. **Volume**: Over any dimension-p submanifold, the integral of ω is smaller than its volume, with at least one submanifold saturating the inequality.

A submanifold is said to be **calibrated** if the restriction of the calibration to this submanifold coincides with the induced volume form.

Property 5.8.26. Calibrated submanifolds minimize the volume within their homology class.

5.8.5 Distributions ♣

For more information on the theory of distributions on Euclidean space, see Chapter **??**.

There are two ways to introduce distributions on general manifolds. Either one uses the locally Euclidean character, defines distributions on charts and glues them together using some compatibility data (see for example [12]) or one defines them as the dual of the space of smooth functions (with compact support) as in the Euclidean case. In this section the second approach is followed.

The base manifold M will be required to be paracompact and second-countable. Moreover, it is assumed that a Riemannian metric g is given (see Chapter **??**). This data allows to turn the space of smooth sections of any tensor bundle over M into a Fréchet space **??** using a generalization of the seminorms (**??**), where the (partial) derivatives ∂_i are replaced by covariant derivatives ∇_i. The norm will now also be the one induced (fibrewise) by g. In a similar way one can for every compact subset $K \subset M$ define the space $\mathcal{D}(K, \otimes^p)$ of smooth p-tensor fields with support in K. By taking the direct limit (with its associated topology) one obtains the space of smooth compactly supported p-tensor fields $\mathcal{D}(M, \otimes^p)$.

Definition 5.8.27 (Tensor distribution). The space of tensor distributions of order p is defined as the continuous dual of $\mathcal{D}(M, \otimes^p)$.

Much of the theory of distributions on Euclidean space can be generalized to smooth manifolds without too much trouble (for example one again obtains a dense inclusion $\mathcal{D} \hookrightarrow \mathcal{D}'$). An interesting generalization is the definition of the covariant derivative:

Definition 5.8.28 (Covariant derivative). Let (M, g) be a Riemannian manifold with associated Levi-Civita connection ∇. The covariant derivative of a tensor distribution T is defined using duality as follows (as in the case of Euclidean space this can be interpreted as an extension of the integration-by-parts formula):

$$\langle \nabla T, \sigma \rangle := -\langle T, g \cdot \nabla \sigma \rangle, \tag{5.94}$$

where $g \cdot \nabla\sigma$ denotes the **internal contraction** (generalizing the divergence of a vector field) which, in local coordinates, is given by

$$(g \cdot \nabla\sigma)^{i_1 \dots i_p} = \nabla_j \sigma^{j i_1 \dots i_p}. \tag{5.95}$$

Definition **??** can easily be generalized to smooth manifolds:

Definition 5.8.29 (Wave front set). The wave front set of a distribution $\phi \in D'(M)$ is defined as follows:

$$\mathrm{WF}(\phi) := \{(x, v) \in T^*M_0 \mid v \in \Sigma_x(\phi)\}, \tag{5.96}$$

where the singular fibre $\Sigma_x(\phi)$ is defined as in the Euclidean case (since by localization one can restrict to a chart containing x and work in local coordinates).

?? COMPLETE? ??

5.9 Cohomology

5.9.1 de Rham complex

Definition 5.9.1 (Exact form). If $\omega \in \Omega^k(M)$ can be written as $\omega = \mathrm{d}\chi$ for some $\chi \in \Omega^{k-1}(M)$, it is said to be exact.

Definition 5.9.2 (Closed form). Let $\omega \in \Omega^k(M)$. If $\mathrm{d}\omega = 0$, it is said to be closed.

Definition 5.9.3 (de Rham complex). The sequence

$$0 \longrightarrow \Omega^0(M) \longrightarrow \Omega^1(M) \longrightarrow \cdots \longrightarrow \Omega^{\dim(M)}(M) \longrightarrow 0 \tag{5.97}$$

together with the sequence of exterior derivatives d_k forms a cochain complex by the nilpotency of the exterior derivative. This complex is called the de Rham complex $\Omega^\bullet_{\mathrm{dR}}(M)$. It encodes the information that every exact form is closed. The converse, however, is not true in general (see Theorem 5.9.8 below for more information).

Definition 5.9.4 (de Rham cohomology). Following Definition **??** the k^{th} de Rham cohomology group on M is defined as the k^{th} (co)homology group of the de Rham complex:

$$H^k_{\mathrm{dR}}(M) := \frac{\ker(\mathrm{d}_k)}{\mathrm{im}(\mathrm{d}_{k-1})}. \tag{5.98}$$

This quotient space is a vector space. Two elements of the same equivalence class in $H^k_{\mathrm{dR}}(M)$ are said to be **cohomologous**.

Definition 5.9.5 (Integral form). A closed k-form ω that lies in the image of the inclusion $H^k_{\mathrm{dR}}(M,\mathbb{Z}) \hookrightarrow H^k_{\mathrm{dR}}(M,\mathbb{R})$. Equivalently, a closed k-form is integral if integrating it over any k-cycle with integral coefficients gives an integer.

Formula 5.9.6 (Cup product). Let $[\nu] \in H^k_{\mathrm{dR}}$ and $[\omega] \in H^l_{\mathrm{dR}}$. The cup product on de Rham cohomology is given by

$$[\nu] \smile [\omega] := [\nu \wedge \omega]. \tag{5.99}$$

The following theorem allows to write $H^\bullet(M)$ for the de Rham cohomology on M:

Theorem 5.9.7 (de Rham). *The de Rham cohomology over a smooth manifold is isomorphic to its singular cohomology (Section* **??***).*

Theorem 5.9.8 (Poincaré lemma[2]). *For every point $p \in M$ there exists a neighbourhood on which the de Rham cohomology is trivial:*

$$\forall p \in M : \exists U \subseteq M : H^k(U) = 0. \tag{5.100}$$

This implies that every closed form is locally exact, i.e. if $\mathrm{d}\omega = 0$ at the point $p \in M$, there exist a neighbourhood $U \subseteq M$ of p and a differential form λ such that

$$\omega = \mathrm{d}\lambda \tag{5.101}$$

at all points $p' \in U$. More generally, this lemma says that the following isomorphism exists for every smooth manifold M:

$$H^\bullet(M \times \mathbb{R}^n) \cong H^\bullet(M). \tag{5.102}$$

In fact, this can even be further generalized due to the homotopy axiom of de Rham cohomology:

$$H^\bullet(E) \cong H^\bullet(M) \tag{5.103}$$

for every vector bundle E over M.

[2]The original theorem states that on a contractible space **??** every closed form is exact.

Definition 5.9.9 (Relative cohomology). Consider the submanifold inclusion $\iota : S \hookrightarrow M$. The relative de Rham complex is defined as follows:

$$\Omega^n(M,S) := \Omega^n(M) \oplus \Omega^{n-1}(S), \tag{5.104}$$

where the coboundary operator d is defined by

$$\mathrm{d}(\omega,\lambda) := (\mathrm{d}\omega, \iota^*\omega - \mathrm{d}\lambda). \tag{5.105}$$

The relative de Rham cohomology $H^\bullet(M,S)$ is defined as the cohomology of this complex. Classes are represented by closed forms on M that restrict to exact forms on S.

This definition can in fact be generalized to any smooth map $f : M \to N$ by replacing ι^* in the coboundary by f^*. This cohomology ring is denoted by $H^\bullet(f)$. For all smooth functions f, the following long exact sequence is exists:

$$\cdots \longrightarrow H^k(f) \longrightarrow H^k(M) \longrightarrow H^k(N) \longrightarrow H^{k+1}(f) \longrightarrow \cdots . \tag{5.106}$$

Definition 5.9.10 (Twisted de Rham complex). Consider the usual (graded) de Rham ring $\Omega^\bullet(M)$ on a smooth manifold M. For every degree-3 class $\alpha \in H^3(M)$, one can define the twisted de Rham differential

$$\mathrm{d}_\alpha := \mathrm{d} + \alpha \wedge . \tag{5.107}$$

Nilpotency follows from that of d and the degree of α. The cohomology of this complex is called the (α-)twisted de Rham cohomology of M.

The de Rham theorem above can be generalized to the setting of equivariant cohomology **??**:

Property 5.9.11 (Equivariant de Rham theorem ♣). Consider a smooth manifold M with a smooth G-action and let $W(\mathfrak{g})$ be the Weil algebra **??** of G. Construct the dgca $\Omega^\bullet(M) \otimes W(\mathfrak{g})$ with differential $\mathrm{d}_{\mathrm{dR}} + \mathrm{d}_W$. The infinitesimal G-action gives a map $\mathfrak{g} \to \mathfrak{X}(M)$, so Cartan calculus can be extended to all of $\Omega^\bullet(M) \otimes W(\mathfrak{g})$. The **basic differential forms** are defined as the kernel of the Cartan operators $\iota_\xi, \mathcal{L}_\xi$ for all $\xi \in \mathfrak{g}$. This subcomplex, with the induced differential $\mathrm{d}_{\mathrm{dR}} + \mathrm{d}_W$, is called the **Weil model** of equivariant de Rham cohomology.

If G is compact and connected, the cohomology of the Weil model is isomorphic to the G-equivariant cohomology $H^\bullet_G(M)$ from Definition **??**.

Property 5.9.12 (Cohomological models). The intersection of the basic subcomplex with $\Omega^\bullet(M)$ can be identified with the complex of tensorial (basic) differential forms (see Definition 6.3.14), i.e. the pullback $\pi^*\Omega^\bullet(M) \subset \Omega^\bullet(P)$. Moreover, the intersection with the Weil algebra gives a model for the classifying space BG (the Weil algebra itself gives a model for the total spacee EG).

5.9.2 Integration

At this point a little side note can be given about why the de Rham cohomology groups 5.9.4 really constitute a cohomology theory. Some concepts from homology are needed that can be found in Section **??**.

Let M be a compact manifold and let $\{\lambda_i : \Delta^k \to M\}$ be the set of singular k-simplexes on M. Suppose that one wants to integrate a form over a singular k-chain $C = \sum_{i=0}^k a_i\lambda_i$ on M. Through integration one can pair the k-form ω and the singular chain C as if they are dual objects (hence p-forms are also called p-**cochains**) to produce a real number:

$$\langle\cdot, C\rangle : \Omega^k(M) \to \mathbb{R} : \omega \mapsto \int_C \omega = \sum_{i=0}^k a_i \int_{\Delta_k} \lambda_i^*\omega, \tag{5.108}$$

where λ_i^* pulls ω back to Δ^k, which is a subset of $\mathbb{R}^k$ as required. Stokes's theorem 5.8.23 then says that

$$\int_C \mathrm{d}\omega = \int_{\partial C} \omega. \tag{5.109}$$

Using the pairing $\langle\cdot,\cdot\rangle$ this can be rewritten more explicitly as

$$\langle \mathrm{d}\omega, C\rangle = \langle \omega, \partial C\rangle. \tag{5.110}$$

The operators d and ∂ can thus be interpreted as formal adjoints. After confirming (again using Stokes' theorem) that all chains C and cochains ω belonging to the same equivalence classes $[C] \in H_k(M;\mathbb{R})$ and $[\omega] \in H^k(M;\mathbb{R})$ give rise to the same number $\langle\omega, C\rangle$, one can see that the singular homology groups and the de Rham cohomology groups on M are well-defined dual groups. The name *co*homology is thus well-chosen for 5.9.4.

Remark 5.9.13 (L^2-cohomology). Given a *Riemannian metric* (Chapter **??**), one can define a notion of square-integrable differential forms. The cohomology theory of these forms is called L^2-cohomology.

5.9.3 Cohomology with compact support

Because integration is involved in all statements in this section, it will be assumed that all manifolds and bundles are orientable (unless states otherwise).

The following definition characterizes cohomology with compact support directly through its relation to compact sets:

Definition 5.9.14 (Cohomology with compact support). Consider a manifold M (not necessarily orientable). The cohomology with compact support $H_c^\bullet(M)$ is defined as the following direct limit:

$$H_c^\bullet(M) := \varinjlim_{\text{compact } K} H^\bullet(M, M\backslash K). \tag{5.111}$$

Property 5.9.15 (Relation to reduced cohomology). For a topological space X, the inclusion $U \hookrightarrow X$ for any open U induces a long exact sequence in compactly supported cohomology. Performing excision by $V := X\backslash U$ in the above definition of compact cohomology gives $H^\bullet(X, X\backslash K \cup V) \cong H^\bullet(U, U\backslash K)$ and thus $H_c^\bullet(X, V) \cong H_c^\bullet(U)$.

If X is chosen to be the one-point compactification $\widehat{M}$ of M and if $U = M$, the aforementioned long exact sequence implies that

$$H_c^\bullet(M) \cong H^\bullet(\widehat{M}, *) \cong \widetilde{H}^\bullet(\widehat{M}), \tag{5.112}$$

where the fact that both $*$ and $\widehat{M}$ are compact (this is true for Hausdorff spaces) is used.

Theorem 5.9.16 (Poincaré duality). *Let M be an m-dimensional manifold. The pairing $\int : H^k(M) \otimes H_c^{m-k}(M) \to \mathbb{R}$ induces an isomorphism on cohomology:*

$$H^k(M) \cong \left(H_c^{m-k}(M)\right)^*. \tag{5.113}$$

If M is of finite type, the converse also holds:

$$H_c^k(M) \cong \left(H^{m-k}(M)\right)^*. \tag{5.114}$$

Corollary 5.9.17 (Poincaré lemma for compact cohomology). Let M be a (not necessarily orientable) manifold of finite type. For every rank-n vector bundle E over M the following isomorphism exists:

$$H_c^\bullet(E) \cong H_c^{\bullet-n}(M). \tag{5.115}$$

Definition 5.9.18 (Poincaré dual). Let M be an m-dimensional manifold and let $i : S \to M$ be a closed k-dimensional submanifold. The Poincaré dual of S in M is the unique cohomology class $[\eta_S] \in H^{m-k}(M)$ such that

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S \tag{5.116}$$

for all compactly supported $\omega \in H_c^k(M)$. If S is compact in M, two Poincaré duals exist:

- **Closed dual**: The Poincaré dual obtained by using the fact that S is compact and hence closed in M.
- **Compact dual**: Because S is compact, all forms $\omega \in H^k(M)$ (not only the compactly supported ones) can be integrated over S and, assuming M is of finite type, Poincaré duality implies that there exists a unique cohomology class with compact support η'_S such that

$$\int_S i^*\omega = \int_M \omega \wedge \eta'_S \tag{5.117}$$

  for all $\omega \in H^k(M)$.

Remark 5.9.19. Because the compact Poincaré dual induces a pairing on all closed forms ω, which include the compactly supported ones, the compact dual is equal to the closed Poincaré dual as a differential form. However, as elements in cohomology these can be quite different.

Property 5.9.20 (Localization principle). The support of the compact Poincaré dual of a compact submanifold S may be shrunk to any neighbourhood of S. More generally, the support of the (closed) Poincaré dual of a closed submanifold S can be shrunk to any tubular neighbourhood of S.

Formula 5.9.21 (Transversal intersections). The Poincaré dual of a transversal intersection is equal to the wedge product of the individual Poincaré duals:

$$\eta_{S\pitchfork T} = \eta_S \wedge \eta_T. \tag{5.118}$$

Definition 5.9.22 (Compact vertical cohomology). Let $\pi : E \to M$ be a smooth vector bundle over M. A differential form $\omega \in \Omega^\bullet(E)$ is said to be an element of $\Omega_{cv}^\bullet(E)$ if $\text{supp}(\omega) \cap \pi^{-1}(K)$ is compact for every compact subset $K \subset M$. The cohomology of this complex is called the **de Rham cohomology with compact support in the vertical direction**.

Remark. The above definition implies that $\omega \in \Omega_{cv}^\bullet(E)$ is compactly supported on each fibre $\pi^{-1}(p), p \in M$. This observation explains the name of the cohomology theory.

Definition 5.9.23 (Fibre integration). Differential forms with vertically compact support on a rank-n vector bundle $\pi : E \to M$ can be divided into two classes:

- **Type 1**: those locally of the form $f(x,u)\pi^*\phi \wedge (\mathrm{d}u_{i_1} \wedge \cdots \wedge \mathrm{d}u_{i_k})$, where ϕ is a form on the base manifold M, f has compact support and $k < n$.
- **Type 2**: those locally of the form $f(x,u)\pi^*\phi \wedge (\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n)$ where ϕ is a form on the base manifold M and f has compact support.

The fibre integration map $\pi_* : \Omega^\bullet_{cv}(E) \rightarrow \Omega^{\bullet-n}(M)$ is defined as follows. If ω is of type 1, then $\pi_*\omega := 0$. If ω is of type 2, then

$$\pi_*\omega := \left(\int\!\!\cdots\!\!\int f(x,u)\,du_1\cdots du_n\right)\phi. \tag{5.119}$$

Formula 5.9.24 (Projection formula). Consider a rank-n vector bundle $\pi : E \rightarrow M$. For every pair of forms $\phi \in \Omega^\bullet(M)$ and $\omega \in \Omega^\bullet_{cv}(E)$, the following formula holds:

$$\pi_*(\pi^*\phi \wedge \omega) = \phi \wedge \pi_*\omega. \tag{5.120}$$

Furthermore, if $\phi \in \Omega^k_c(M)$ and $\omega \in \Omega^{m+n-k}_{cv}(E)$,

$$\int_E \pi^*\phi \wedge \omega = \int_M \phi \wedge \pi_*\omega. \tag{5.121}$$

Theorem 5.9.25 (Thom isomorphism). *For every rank-n vector bundle $\pi : E \rightarrow M$, where M is (not necessarily orientable and) of finite type, fibre integration gives the following isomorphisms:*

$$\pi_* : H^\bullet_{cv}(E) \cong H^{\bullet-n}(M) : \mathcal{T}. \tag{5.122}$$

Corollary 5.9.26 (Poincaré lemma for vertically compact cohomology).

$$H^\bullet_{cv}(M \times \mathbb{R}^n) \cong H^{\bullet-n}(M). \tag{5.123}$$

Formula 5.9.27 (Thom isomorphism). Denote the **Thom class** of M by $\Phi := \mathcal{T}(1) \in H^0(M)$. Because $\mathcal{T}$ and π_* are mutual inverses and, hence, $\pi_*\Phi = 1$, the projection formula implies that

$$\mathcal{T}(\omega) = \pi^*\omega \wedge \Phi. \tag{5.124}$$

Property 5.9.28 (Orientation class). The Thom class Φ restricts to a generator of the cohomology of the typical fibre:

$$H^n_c(V) \cong \widetilde{H}^n(S^n) \cong H^n(S^n).$$

For compact orientable manifolds such a generator gives rise to a generator of the homology group H_n, i.e. it gives rise to an orientation class 5.8.12.

Property 5.9.29 (Poincaré dual). The Poincaré dual of a closed submanifold is equal to the Thom class of its normal bundle.

The construction of the Thom isomorphism involves some technicalities. For example, throughout the literature, the Thom isomorphism is stated in various forms using compactly supported cohomology, relative cohomology or reduced cohomology.

Definition 5.9.30 (Thom space). Let $E \rightarrow M$ be a vector bundle. For every fibre in E one can construct its one-point compactification **??** and by gluing these together the **sphere bundle** $\mathrm{Sph}(E)$ is obtained. The quotient space $\mathrm{Sph}(E)/B$, where all the adjoined points are identified, is called the Thom space $\mathrm{Th}(E)$.

By equipping E with a metric (see Chapter **??**), one can give an alternative definition. Let V be the typical fibre of E. A new bundle, the unit sphere bundle $S(E)$, where the typical fibre is the unit sphere $S(V) := \{v \in V \mid \|v\| = 1\}$, can now be constructed. (It should be noted that this new bundle is not a vector bundle since the unit sphere is not a vector space.) A similar construction leads to the unit disk bundle $D(E)$, where the typical fibre is the unit disk $D(V) := \{v \in V \mid \|v\| \leq 1\}$. The Thom space $\mathrm{Th}(E)$ can be shown to be isomorphic to the quotient space $D(E)/S(E)$.

Property 5.9.31. If the base manifold M is compact, the Thom space is obtained as the one-point compactification of the total space E.

Property 5.9.32 (Different forms of Thom isomorphism). Let $E \to M$ be a vector bundle and denote the complement of the zero section by E_0 as in 5.2.8. Homotopy invariance implies that

$$H^\bullet(D(E), S(E)) \cong H^\bullet(E, E_0). \tag{5.125}$$

Then, using Result 5.2.12 together with the dual of Property **??**, one can show that the reduced cohomology of the Thom space $\mathrm{Th}(E)$ is isomorphic to the relative cohomology of the pair (E, E_0):

$$\widetilde{H}^\bullet(\mathrm{Th}(E)) \cong H^\bullet(E, E_0). \tag{5.126}$$

To relate this to vertically compact cohomology, Property 5.9.15 can be adapted. Compact support gave rise to the (reduced) cohomology of the compactified space. By analogy, vertically compact support corresponds to compactifications of the fibres, which is exactly how the Thom space is constructed.

The above arguments finally lead to the following triangle of isomorphisms:

$$\begin{tikzcd} & \widetilde{H}^\bullet(\mathrm{Th}(E)) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H^\bullet(E, E_0) \arrow[rr, "\cong"'] & & H^\bullet_{cv}(E) \end{tikzcd} \tag{5.127}$$

Definition 5.9.33 (Thom spectrum ♣). Let $E \to M$ be a vector bundle. The Thom spectrum of E is defined as the suspension spectrum of its Thom space:

$$(\Sigma^\infty \mathrm{Th}(E))_n \cong \mathrm{Th}(\mathbb{R}^n \oplus E), \tag{5.128}$$

where $\mathrm{Th}(\mathbb{R}^n \oplus E) \cong \Sigma^n \mathrm{Th}(E)$ was used.

Now, consider the sequence $(\xi_n)_{n\in\mathbb{N}}$ of *universal vector bundles*. For every ξ_n, define the n^{th} component space as follows:

$$MO_n := \mathrm{Th}(\xi_n). \tag{5.129}$$

The Whitney sum $\xi_n \oplus \mathbb{R}$ can be obtained as a pullback of ξ_{n+1}. This map induces a morphism $\Sigma MO_n \to MO_{n+1}$, which gives the n^{th} structure map of "the" Thom spectrum MO.[3]

Definition 5.9.34 (Euler class). Consider a vector bundle $E \to M$ together with its Thom class Φ. The Euler class $e(E)$ is defined as the pullback $s_0^*\Phi$ of the Thom class along the zero section of E.

Property 5.9.35. If the orientation of E is reversed, the Euler class changes sign.

The following property distinguishes the Euler class among all characteristic classes of E:

Property 5.9.36 (Normalization). If the vector bundle admits a nowhere-vanishing section, its Euler class vanishes.

[3]Note that the Thom spectrum as defined here is not an Ω-spectrum **??**, it is merely a sequential spectrum (prespectrum).

5.9.4 Čech-de Rham complex

Theorem 5.9.37 (Mayer-Vietoris sequence). *Consider a smooth manifold M with an open covering $U \cup V$. The cohomology of U, V is related to that of M by the following short exact sequence:*

$$0 \longrightarrow H^\bullet(M) \overset{\iota_U \oplus \iota_V}{\longrightarrow} H^\bullet(U) \oplus H^\bullet(V) \overset{\pi_2 - \pi_1}{\longrightarrow} H^\bullet(U \cap V) \longrightarrow 0. \tag{5.130}$$

Definition 5.9.38 (Čech-de Rham complex). The Čech complex **??** associated to the constant sheaf $\flat\mathbb{R}$, i.e. the sheaf of locally constant functions.

The Mayer-Vietoris sequence can be generalized to a statement for the Čech-de Rham complex:

Property 5.9.39 (Mayer-Vietoris sequence). The horizontal complex

$$0 \longrightarrow \Omega^\bullet(M) \longrightarrow \prod_{i_0} \Omega^\bullet(U_{i_0}) \longrightarrow \prod_{i_0, i_1} \Omega^\bullet(U_{i_0 i_1}) \longrightarrow \cdots \tag{5.131}$$

is acyclic, i.e. the δ-cohomology of the Čech-de Rham complex vanishes.

An important corollary is that one can compute the (de Rham) cohomology of M using the above double complex:

Property 5.9.40. The restriction map $\Omega^\bullet(M) \to C^\bullet(\mathcal{U}; \Omega^\bullet)$ induces an isomorphism in cohomology.

One can also augment the Čech-de Rham complex in the other direction by the kernel of the de Rham differential in degree 1. These are the locally constant functions on the intersections $U_{i_0 \ldots i_p}$. The cohomology of this augmenting sequence $C^\bullet(\mathcal{U}, \flat\mathbb{R})$ is called the **Čech cohomology** of M. By the same reason as for why the Mayer-Vietoris sequence implied the above theorem, the following theorem is obtained:

Theorem 5.9.41 (Čech = de Rham). *For a smooth manifold M, admitting a good cover $\mathcal{U}$, the Čech cohomology of $\mathcal{U}$ is isomorphic to the de Rham cohomology of M:*

$$H^\bullet(M) \cong \check{H}^\bullet(\mathcal{U}; \flat\mathbb{R}). \tag{5.132}$$

By noting that good covers are *cofinal* in the set of open covers, one can pass to the full Čech cohomology:

$$H^\bullet(M) \cong \check{H}^\bullet(M; \flat\mathbb{R}). \tag{5.133}$$

Corollary 5.9.42. All compact manifolds admit a finite good cover and hence have finite-dimensional de Rham cohomology.

Property 5.9.43 (Exponential sequence). Consider the following exact sequence of topological groups:

$$0 \longrightarrow \mathbb{Z} \overset{2\pi}{\longrightarrow} \mathbb{C} \overset{\exp}{\longrightarrow} \mathrm{U}(1) \longrightarrow 0. \tag{5.134}$$

Let $(M, \mathcal{O}_M)$ be a complex manifold (or a smooth manifold and restrict the above sequence to the real numbers). The exact sequence induces an exact sequence of structure sheaves:

$$0 \longrightarrow \flat\mathbb{Z} \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M^\times \longrightarrow 0. \tag{5.135}$$

This in turn induces a long exact sequence in cohomology and by Example **??** (if M is paracompact) the connecting homomorphism leads to an isomorphism

$$H^{\bullet+1}(M; \mathbb{Z}) \cong \check{H}^\bullet(M; \mathrm{U}(1)). \tag{5.136}$$

Note that the cohomology on the right-hand side is not singular cohomology. Singular U(1)-valued cohomology could also be related to integral cohomology through the *universal coefficient theorem*, but extra terms involving Ext functors would appear.

5.9.5 Non-orientable manifolds

This section gives a differential-geometric incarnation of Section **??** on local coefficients.

Definition 5.9.44 (Twisted cohomology). Let $E \to M$ be a flat vector bundle over M. By Construction 5.4.24 the algebra $\Omega^\bullet(M) \otimes E$ can be given the structure of a differential graded algebra **??** and, hence, gives rise to a cohomology theory $H^\bullet(M; E)$. This is called the E-twisted de Rham cohomology of M.

Remark 5.9.45. According to the remark following Construction 5.4.24 attention should be payed to which trivialization was used in the construction of $H^\bullet(M; E)$. However, it can be shown that two trivializations give rise to the same E-twisted cohomology if they admit a common refinement for which the induced sections differ by a locally constant matrix $a \in \mathrm{GL}(n, \mathbb{R})$.

If one takes $E = o(M)$ to be the orientation line bundle over M, the (honest) densities of Remark 5.8.10 are obtained. The cohomology of this complex is simply called the **twisted de Rham cohomology**.

Property 5.9.46 (Isomorphism). The twisted cohomologies defined by two trivializations induced from atlases on M are isomorphic.[4]

Property 5.9.47 (Trivial twisting). If M is orientable, its twisted cohomology is isomorphic to its ordinary (de Rham) cohomology. More generally, the E-twisted de Rham cohomology is isomorphic to the ordinary de Rham cohomology whenever E is trivial.

Poincaré duality 5.9.16 can be generalized almost verbatim to the twisted case:

Theorem 5.9.48 (Poincaré duality). *Integration of densities induces the following isomorphism:*

$$H^k(M) \cong \left(H_c^{m-k}(M; o(M))\right)^*. \tag{5.137}$$

If M is of finite type, the converse also holds:

$$H_c^k(M) \cong \left(H^{m-k}(M; o(M))\right)^*. \tag{5.138}$$

The Thom isomorphism also holds for non-orientable bundles:

Theorem 5.9.49 (Thom isomorphism). *Let $E \to M$ be a rank-n vector bundle. Fibre integration gives the following isomorphism:*

$$H_{cv}^{\bullet+n}(E) \cong H^\bullet(M; o(E)). \tag{5.139}$$

5.9.6 Generalized cohomology ♣

In this section some statements from singular/de Rham cohomology on vector bundles are generalized to statements about generalized (Eilenberg-Steenrod) cohomology theories (Section **??**). In the remainder of this section $E^\bullet$ will denote a multiplicative generalized cohomology theory.[5]

Definition 5.9.50 (Orientation). Let $\pi : E \to M$ be a rank-n vector bundle. An E-orientation or E-**Thom class** is a cohomology class $u \in \widetilde{E}^n(\mathrm{Th}(V))$ that restricts to a generator on every fibre of V.

[4] Although one almost always works with a natural trivialization, i.e. the open subsets of M are obtained from charts on M, this is technically not necessary. For more "exotic" cases, the isomorphisms not always exist.

[5] "Multiplicative" indicates that there exists a cup product such that every group $E^\bullet(M)$ becomes a graded ring.

5.10 Differential operators

In this section the study of PDEs is generalized to vector bundles. The case of PDEs on $\mathbb{R}^n$ was treated in Chapter **??**.

Definition 5.10.1 (Differential operator). A (linear) differential operators between two vector bundels E, F over the same base manifold M is a linear map $D : \Gamma(E) \to \Gamma(F)$ that can locally be expressed as a system of partial differential equations. The principal symbols on different charts glue together to give a globally defined morphism

$$\sigma_D : T^*M \to \mathrm{Hom}(\pi^*E, \pi^*F), \tag{5.140}$$

where $\pi : T^*M \to M$ is the cotangent bundle projection.

By extension of the ordinary theory of PDEs one says that the differential operator is **elliptic** (hyperbolic, ...) if its associated PDE is elliptic (hyperbolic, ...). E.g. a differential operator is hyperbolic if the the above morphism is invertible for all cotangent vectors.

Definition 5.10.2 (Normally hyperbolic operators). Consider a (pseudo)Riemannian vector bundle E (see Chapter **??**). A linear differential operator on E is said to be normally hyperbolic if its principal symbol is proportional to the given metric.

5.10.1 Elliptic complexes

Definition 5.10.3 (Elliptic complex). Consider a collection of vector bundles $\{\pi_n : E_n \to M\}_{n\in\mathbb{N}}$ together with a sequence of differential operators $\big(D_n : C^\infty(E_n) \to C^\infty(E_{n+1})\big)_{n\in\mathbb{N}}$. This system is called an elliptic complex if it is a cochain complex, i.e. $D_{n+1} \circ D_n = 0$, and if the induced sequence $\big(\sigma_p(\xi)(D_n)\big)_{n\in\mathbb{N}}$ is exact for all $x \in M$ and $\xi \neq 0$.

?? COMPLETE ??

Chapter 6

Principal Bundles

The main reference for this chapter is [20]. The theory of principal bundles uses the language of (Lie) group theory quite heavily. For all things related to group theory the reader is referred to Sections **??** and **??**. For more information on Lie groups and their associated Lie algebras the reader is referred to Chapter **??**.

6.1 Principal bundles

Definition 6.1.1 (Principal bundle). A fibre bundle $\pi : P \to M$ equipped with a right action $\rho : P \times G \to P$ that satisfies two properties:

1. **Free action**: ρ is free. This implies that the orbits are isomorphic to the structure group.
2. **Fibrewise transitivity**: The action preserves fibres, i.e. $y \cdot g \in F_b$ for all $y \in F_b, g \in G$. In turn this implies that the fibres over M are exactly the orbits of ρ.

Together these properties imply that the typical fibre F and structure group G can be identified. The right action of G on P will often be denoted by R_g (unless this would give conflicts with the same notation for the action of G on itself).

Remark 6.1.2 (G-torsor). Although the fibres are homeomorphic to G, they do not carry a group structure due to the lack of a distinct identity element. This turns them into G-torsors **??**. However, it is possible to locally (i.e. in a neighbourhood of a point $p \in M$) endow the fibres with a group structure by choosing an element of every fibre to be the identity element.

Property 6.1.3. A corollary of the definition is that the bundle $\pi : P \to M$ is isomorphic to the bundle $\xi : P \to P/G$, where P/G denotes the orbit space of P with respect to the G-action (which can be proven to be proper) and ξ is the quotient projection.

In fact this property can be used to give an alternative characterization of smooth principal bundles:

Property 6.1.4 (Quotient manifold theorem). Consider a smooth manifold P equipped with a free and proper (right) action of a Lie group G. The following statements hold:

- The orbit space P/G is a smooth manifold.
- The projection $P \to P/G$ is a submersion.
- P is principal G-bundle over P/G.

Property 6.1.5 (Dimension). The dimension of P is given by

$$\dim(P) = \dim(B) + \dim(G). \tag{6.1}$$

Property 6.1.6. Every local trivialization φ_i is G-equivariant:

$$\varphi_i(z \cdot g) = \varphi_i(z) \cdot g. \tag{6.2}$$

Definition 6.1.7 (Principal bundle map). A bundle map $F : P_1 \to P_2$ between principal G-bundles is a pair of morphisms (f_B, f_P) such that:

1. (f_B, f_P) is an ordinary bundle map 4.1.10.
2. f_P is G-equivariant.

The map f_P is said to **cover** f_B.

The following property proves that the equivariance condition on principal bundle maps is in fact a very strong condition:

Property 6.1.8. Every principal bundle map covering the identity is an isomorphism.

Definition 6.1.9 (Vertical automorphism). Consider a principal G-bundle $\pi : P \to B$. An automorphism f of this bundle is said to be vertical if it covers the identity, i.e. $\pi \circ f = \pi$. It is the subgroup $\mathrm{Aut}_V(P) \subset \mathrm{Aut}(P)$ of vertical automorphisms that is known as the **group of gauge transformations** or **gauge group**[1] in physics.

Remark. It should be clear that the above definition can easily be generalized to arbitrary fibre bundles.

6.1.1 Associated bundles

Construction 6.1.10 (Associated principal bundle). For every fibre bundle one can construct an associated principal G-bundle by replacing the fibre F by G itself using the fibre bundle construction theorem 4.2.1, where the left action of G is given by left multiplication in G.

Property 6.1.11. A fibre bundle ξ is trivial if and only if its associated principal bundle is trivial. More generally, two fibre bundles are isomorphic if and only if their associated principal bundles are isomorphic.

Example 6.1.12 (Frame bundle). Let V be an n-dimensional vector space and denote the set of frames **??** of V by FV. It follows from the fact that every basis transformation is given by the action of an element of the general linear group that FV is isomorphic to $\mathrm{GL}(V) \cong \mathrm{GL}(\mathbb{R}^n)$.

Given an n-dimensional vector bundle E, one can construct an associated principal bundle by replacing every fibre $\pi^{-1}(b)$ by $F(\pi^{-1}(b)) \cong \mathrm{GL}(\mathbb{R}^n)$. The right action on this bundle by $g \in \mathrm{GL}(\mathbb{R}^n)$ is given by the basis transformation $\widetilde{e}_j = g^i_j e_i$. This bundle is denoted by FE or FM in the case of the tangent bundle $E = TM$.

Construction 6.1.13 (Associated bundle to a principal bundle). Consider a principal G-bundle $\pi : P \to B$ and let F be a space equipped with a left G-action $\rhd$. One can construct an associated bundle $P_F \equiv P \times_\rhd F$ in the following way:

1. Define an equivalence relation $\sim_G$ on the product space $P \times F$ by

$$(p, f) \sim_G (p', f') \iff \exists g \in G : (p', f') = (p \cdot g, g^{-1} \rhd f). \tag{6.3}$$

[1] This should not be confused with the structure group G, which is also sometimes called the gauge group in physics.

2. Define the total space of the associated bundle as the following quotient space:

$$P_F := (P \times F)/\sim_G . \tag{6.4}$$

3. Define the projection $\pi_F : P_F \to B$ as follows:

$$\pi_F : [p, f] \mapsto \pi(p), \tag{6.5}$$

where $[p, f]$ is the equivalence class of $(p, f) \in P \times F$ in the quotient space P_F.

Example 6.1.14 (Tangent bundle). Starting from the frame bundle FM over a manifold M, one can reconstruct (up to a bundle isomorphism) the tangent bundle TM in the following way. Consider the left G-action $\rhd$ of a matrix group given by

$$\rhd: G \times \mathbb{R}^n \to \mathbb{R}^n : (g \rhd f)^i \mapsto g^i{}_j f^j. \tag{6.6}$$

The tangent bundle is isomorphic to the associated bundle $FM \times_\rhd \mathbb{R}^n$, where the bundle map is defined as $[e, v] \mapsto v^i e_i \in TM$.

Example 6.1.15 (Adjoint bundle). Consider a principal G-bundle P. G acts on itself by conjugation, i.e. the adjoint action:

$$\mathrm{Ad} : G \times G \to G : (g, h) \mapsto g^{-1}hg. \tag{6.7}$$

This action induces an associated bundle $\mathrm{Ad}(P) := P \times_G G$, suitable named the adjoint bundle of P.

Property 6.1.16 (Vertical automorphisms). There exists an isomorphism between the vertical automorphism group $\mathrm{Aut}_V(P)$ and the group of sections of the adjoint bundle $\mathrm{Ad}(P)$.

Construction 6.1.17 (Associated bundle map). Given a principal bundle map (f_P, f_B) between two principal bundles one can construct an associated bundle map between any two of their associated bundles with the same typical fibre in the following way:

- The total space map $\widetilde{f}_P : P \times_G F \to P \times_{G'} F$ is given by

$$\widetilde{f}_P([p, f]) := [f_P(p), f]. \tag{6.8}$$

- The base space map is simply given by f_B itself:

$$\widetilde{f}_B(b) = f_B(b). \tag{6.9}$$

6.1.2 Sections

Although every vector bundle has at least one global section, namely the zero section, a general principal bundle does not necessarily have a global section. This is made clear by the following property:

Property 6.1.18 (Trivial bundles). A principal G-bundle P is trivial if and only if there exists a global section of P. Furthermore, there exists a bijection between the set of global sections $\Gamma(P)$ and the set of trivializations $\mathrm{Triv}(P)$.

Corollary 6.1.19. Every local section $\sigma : U \to P$ induces a local trivialization φ by

$$\varphi^{-1} : (m, g) \mapsto \sigma(m) \cdot g. \tag{6.10}$$

The converse is also true: Consider a local trivialization $\psi^{-1} : U \times G \to \pi^{-1}(U)$. A local section can be obtained by taking $\sigma(u) := \psi^{-1}(u, e)$.

Property 5.2.6 can now be reformulated as follows:

Property 6.1.20 (Trivial vector bundles). A vector bundle is trivial if and only if its associated frame bundle admits a global section. This can easily be interpreted as follows. If one can for every fibre choose a basis in a smooth way, one can also express the restriction of any vector field to a fibre in terms of this basis in a smooth way.

Property 6.1.21 (Higgs fields). Let (P, B, π, G) be a principal bundle and let P_F be an associated bundle. There exists a bijection between the sections of P_F and the G-equivariant maps $\phi : P \to F$, i.e. maps satisfying $\phi(p \cdot g) = g^{-1} \cdot \phi(p)$.

An explicit correspondence is given by

$$\sigma_\phi : B \to P_F : b \mapsto [p, \phi(p)], \tag{6.11}$$

where p is any point in $\pi^{-1}(\{b\})$. This is well-defined due to Equation (6.3). In the other direction one finds

$$\phi_\sigma : P \to F : p \mapsto j_p^{-1} \circ \sigma(\pi(p)), \tag{6.12}$$

where $j_p : F \to P_F : f \mapsto [p, f]$. Either of these maps is sometimes called a **Higgs field** in the physics literature.

6.2 Universal bundle

Definition 6.2.1 (Universal bundle). Consider a topological group G. A universal bundle of G is any principal bundle of the form

$$G \hookrightarrow EG \to BG$$

where EG is weakly contractible. The space BG is called the **classifying space** of G.

Property 6.2.2. A principal G-bundle $EG \to BG$ is universal if and only if EG is weakly contractible.

Definition 6.2.3 (n-universal bundle). A principal bundle with an $(n-1)$-connected total space.

Property 6.2.4 (Delooping). For every topological group one can prove that the loop space of BG is (weakly) homotopy equivalent to G, i.e. $\Omega BG \cong G$. As such it also deserves the name of delooping.

Property 6.2.5 (Groups). Let G be a group (regarded as a discrete toological space). Because the fundamental group of a topological group is Abelian by Property **??**, the classifying space BG is a group if and only if G is Abelian.

This also has an abstract nonsense generalization. The classifying space functor $B :$ **TopGrp** $\to$ **Top** is product-preserving and, hence, it maps group objects to group objects. So, Abelian groups are mapped to topological groups and, even better, to Abelian groups. An important consequence is that all Abelian topological groups are in particular infinite loop spaces.

Property 6.2.6 (Classification). The collection of principal G-bundles over a paracompact Hausdorff space X is in bijection with $[X, BG]$, the set of homotopy classes of continuous functions $f : X \to BG$. This bijection is given by the pullback-construction $f \mapsto f^*EG$.

Due to the homotopical nature of this classification one can also replace G by any homotopy equivalent space. For Lie groups the natural choice is a *maximal compact subgroup* since these are deformation retracts and hence homotopy equivalent.

Corollary 6.2.7 (Vector bundles). Since every vector bundle is uniquely related to its frame bundle, there exists a bijection between principal GL-bundles and vector bundles. This implies that rank-k vector bundles are classified by the homotopy mapping space $[X, B\mathrm{GL}(k)]$. Because $\mathrm{O}(k)$ is the maximal compact subgroup of $\mathrm{GL}(k)$, one also obtains the result that any real vector bundle over a paracompact space admits a *Riemannian structure* (see Chapter **??**).

Property 5.2.4 now follows from Eckmann-Hilton duality **??** together with the above delooping property.

Remark 6.2.8. There also exists a slightly different notion of universal bundles and their associated classifying property. When one requires the total space of the universal bundle to be contractible instead of weakly contractible, the mapping space $[X, BG]$ only classifies numerable principal bundles 4.1.8, but now over arbitrary base spaces X.

An explicit construction of the numerable universal bundle for any topological group G was given by *Milnor*:

Construction 6.2.9 (Milnor ♣). First, consider the infinite join E_∞ equipped with the strong topology. This space is constructed as the direct limit of finite joins **??**:

$$E_n = \underbrace{G \circ \cdots \circ G}_{n \text{ times}},$$

where E_n is embedded in E_{n+1} using the identity element, i.e. every element of E_∞ corresponds to an element of some finite join. Then, construct the quotient of E_n (resp. E_∞) by the canonical right action of G on E_n (resp. E_∞). The bundle $p_n : E_n \to B_n$ (resp. $p : E_\infty \to B_\infty$) is an n-universal bundle (resp. ∞-universal bundle). It follows from the above property that $p : E_\infty \to B_\infty$ is a universal bundle for G.

Construction 6.2.10 (Category theory ♣). Let G be a topological group and consider the delooping (groupoid) $\mathbf{B}G$ from Definition **??**. This groupoid can also be obtained as the *action groupoid* associated to the trivial action of G on $\{*\}$. The regular action of G on itself also induces an action groupoid $\mathbf{E}G := G//G$. The map $G \to \{*\}$ in turn induces a map of groupoids $\mathbf{E}G \to \mathbf{B}G$ which under geometric realisation gives us a universal bundle map.

6.3 Connections

6.3.1 Vertical vectors

Because smooth fibre bundles are also smooth manifolds, one can define the traditional notions such as the tangent bundle. Due to the composite nature of these geometric objects, one can decompose the tangent bundle in horizontal and vertical (sub)bundles:

Definition 6.3.1 (Vertical vector). Let $\pi : E \to M$ be a smooth fibre bundle. The subbundle $\mathrm{Vert}(TE) := \ker(\pi_*)$ of TE is called the vertical (sub)bundle over E. The sections of this bundle are called vertical vector fields.

For principal G-bundles an alternative definition exists:

Alternative Definition 6.3.2. Consider a smooth principal G-bundle (P, M, π, G). First, construct a map ι_p for every element $p \in P$:

$$\iota_p : G \to P : g \mapsto p \cdot g. \tag{6.13}$$

Then, define a tangent vector $v \in T_pP$ to be vertical if it lies in the image of $\iota_{p,*}$, i.e. $\mathrm{Vert}(T_pP) := \mathrm{im}(\iota_{p,*})$. This definition is equivalent to the previous one because of the short exact sequence

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\iota_{p,*}} T_pP \xrightarrow{\pi_*} T_pM \longrightarrow 0. \tag{6.14}$$

Property 6.3.3 (Dimension of vertical bundle). It follows from the second definition that the vertical vectors of a principal G-bundle are nothing but the pushforward of the Lie algebra $\mathfrak{g}$ under the right action of G on P. Furthermore, the exactness of the sequence implies that $\iota_{p,*} : \mathfrak{g} \to \mathrm{Vert}(T_pP)$ is an isomorphism of vector spaces. In particular, it implies that

$$\dim(\mathrm{Vert}(T_pP)) = \dim(\mathfrak{g}) = \dim(G). \tag{6.15}$$

Definition 6.3.4 (Fundamental vector field). Let P be a principal G-bundle and consider $A \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra corresponding to G. The vertical vector field $A^\# : P \to TP$, given by

$$A^\#(p) := \iota_{p,*}(A) \in \mathrm{Vert}(T_pP)\,, \tag{6.16}$$

is called the fundamental vector field associated to A. The action of the vector field $A^\#$ is given by

$$A^\#_p(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(p \cdot \exp(tA))\bigg|_{t=0}\,, \tag{6.17}$$

where $f \in C^\infty(P)$.

Property 6.3.5. The map $(\cdot)^\# : \mathfrak{g} \to \Gamma(TP)$ is a Lie algebra morphism:

$$[A, B]^\# = [A^\#, B^\#]\,, \tag{6.18}$$

where the Lie bracket on the left is the one in $\mathfrak{g}$ and the Lie bracket on the right is the one in $\mathfrak{X}(M)$ given by (5.28).

Property 6.3.6. The vertical bundle satisfies the following equivariance condition:

$$R_{g,*}(\mathrm{Vert}(T_pP)) = \mathrm{Vert}(T_{p\cdot g}P). \tag{6.19}$$

By differentiating the equality

$$R_g \circ \iota_p = \iota_{p\cdot g} \circ \mathrm{ad}_{g^{-1}}$$

and using Example **??** and Definition 6.3.4, one can obtain the following algebraic reformulation:

$$R_{g,*}\Big(A^\#(p)\Big) = \big(\mathrm{Ad}_{g^{-1}}A\big)^\#(p \cdot g). \tag{6.20}$$

6.3.2 Ehresmann connections

The exact sequence (6.14) does not split canonically. However, one can make a choice of splitting:

Definition 6.3.7 (Ehresmann connection). Consider a smooth fibre bundle E. An (Ehresmann) connection on E is the selection of a subspace $\mathrm{Hor}(T_eE) \leq T_eE$ for every $e \in E$ such that:

1. The horizontal and vertical bundles are complementary: $\mathrm{Vert}(T_eE) \oplus \mathrm{Hor}(T_eE) = T_eE$.
2. The choice of subspace depends smoothly on $e \in E$ in the sense of distributions 5.5.1.

The vectors in $\mathrm{Hor}(T_eE)$ are said to be **horizontal** (with respect to the chosen connection).

Definition 6.3.8 (Horizontal bundle). The horizontal (sub)bundle $\mathrm{Hor}(TE)$ is defined as $\bigsqcup_{e\in E} \mathrm{Hor}(T_eE)$ with the bundle structure induced from TE.

Definition 6.3.9 (Principal connection). A principal connection on a smooth principal G-bundle P is a G-equivariant Ehresmann connection, i.e. an Ehresmann connection for which the horizontal subspaces satisfy the following G-equivariance condition:

$$R_{g,*}(\mathrm{Hor}(T_pP)) = \mathrm{Hor}(T_{p\cdot g}P). \tag{6.21}$$

Remark 6.3.10. Note that this condition was automatically satisfied for vertical bundles as in Equation (6.19).

Property 6.3.11 (Dimension). Properties 6.1.5 and 6.3.3, together with the direct sum decomposition of TP, imply the following relation for all $p \in P$:

$$\dim(\mathrm{Hor}(T_pP)) = \dim(M). \tag{6.22}$$

All dimensional relations between the data of a principal bundle (P, M, π, G) are now summarized:

$$\begin{aligned} \dim(P) &= \dim(M) + \dim(G) \\ \dim(M) &= \dim(\mathrm{Hor}(T_pP)) \\ \dim(G) &= \dim(\mathrm{Vert}(T_pP)) \end{aligned} \tag{6.23}$$

for all $p \in P$.

Definition 6.3.12 (Dual connection). First, define the dual of the horizontal bundle:

$$\mathrm{Hor}(T_p^*P) := \big\{h \in T_p^*P \mid h(v) = 0, v \in \mathrm{Vert}(T_pP)\big\}. \tag{6.24}$$

It is the space of one-forms that vanish on the vertical subspace. A dual connection can then be defined as the selection of a vertical covector bundle $\mathrm{Vert}(T_p^*P)$ satisfying the conditions of Definitions 6.3.7 and 6.3.9, where Vert and Hor should now be interchanged. Note that here the horizontal bundle is canonically defined.

Definition 6.3.13 (Horizontal and vertical forms). Let $\theta \in \Omega^k(P)$ be a differential k-form.

- θ is said to be horizontal if
$$\theta(v_1, \ldots, v_k) = 0 \tag{6.25}$$
whenever at least one of the v_i is in $\mathrm{Vert}(T_pP)$.
- θ is said to be vertical if
$$\theta(v_1, \ldots, v_k) = 0 \tag{6.26}$$
whenever at least one of the v_i is in $\mathrm{Hor}(T_pP)$.

For functions $f \in \Omega^0(P)$ it is vacuously true that they are both vertical and horizontal.

Definition 6.3.14 (Tensorial form). Consider a differential form θ on a principal G-bundle P with values in a vector space V equipped with a representation $\rho : G \to V$. This form is said to be tensorial or **basic of type** (V, ρ) if it is horizontal and if it satisfies the equivariance condition

$$R_g^*\theta = \rho(g^{-1})\theta. \tag{6.27}$$

An equivalent construction exists. Let $E := P \times_\rho V$ be the associated vector bundle of (V, ρ). Tensorial k-forms of type (V, ρ) are naturally isomorphic to E-valued k-forms. The isomorphism is given fibrewise by

$$\phi \mapsto \overline{\phi} := f^{-1}(\pi^*\phi)\,, \tag{6.28}$$

where $f : V \to E_{\pi(u)} \cong (\pi^*E)_u : v \mapsto [u, v]$.

6.3.3 Connection forms

Definition 6.3.15 (Connection one-form). Let P be a principal bundle G-bundle. A connection one-form, associated to a given principal connection, is a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(P;\mathfrak{g})$ that satisfies the following two conditions:

1. **Cancellation of fundamental vector fields**:
$$\omega(A^{\#}) = A \tag{6.29}$$
for all $A \in \mathfrak{g}$.
2. **Equivariance**:
$$\omega \circ R_{g,*} = \mathrm{Ad}_{g^{-1}} \circ \omega \tag{6.30}$$
for all $g \in G$.

The horizontal subspaces are recovered as the kernel of the connection one-form: $\mathrm{Hor}(T_pP) = \ker \omega|_p$.

Property 6.3.16 (Connection form from principal connection). Consider a principal G-bundle P. Given a principal connection on P, the associated connection one-form is given by the following map:
$$\omega := (\iota_{p,*})^{-1} \circ \mathrm{pr}_{\mathrm{Vert}}\,, \tag{6.31}$$
where $\mathrm{pr}_{\mathrm{Vert}}$ is the canonical projection $TP \to \mathrm{Vert}(TP)$.

Property 6.3.17 (Pullback connection). Consider two principal G-bundles P_1, P_2. Let ω be a connection one-form on P_1 and let $F : P_1 \to P_2$ be a bundle map. The pullback $F^*\omega$ defines a principal connection on P_2 called the pullback connection.

6.3.4 Maurer-Cartan form

Definition 6.3.18 (Maurer-Cartan form). For every $g \in G$ the tangent space T_gG is isomorphic to $T_eG \equiv \mathfrak{g}$. A canonical isomorphism $T_gG \to \mathfrak{g}$ is given by the Maurer-Cartan form
$$\Omega := L_{g^{-1},*}. \tag{6.32}$$

Construction 6.3.19. Consider the one-point manifold $M = \{*\}$. When constructing a principal G-bundle over M, one can see that the total space $P = \{*\}\times G$ can be identified with the structure group G. From the relations in Property 6.3.11 it follows that the horizontal spaces are null-spaces (this trivially defines a smooth distribution and a connection in the sense of Ehresmann 6.3.7) and that the vertical spaces are equal to the tangent spaces, i.e. $\mathrm{Vert}(T_gG) = T_gG$, where the identification $P \cong G$ (as manifolds) is used.

The simplest way to define a connection form ω on this bundle would be the trivial projection $\mathbb{1}_{TP} : TP \to TP = \mathrm{Vert}(TP)$. However, the image of this map would be T_gG and not $\mathfrak{g}$ as required. This can be solved by using the Maurer-Cartan form:
$$\omega(v) := \Omega(v). \tag{6.33}$$

Property 6.3.20. The Maurer-Cartan form is the unique principal connection on the bundle $G \hookrightarrow G \to \{*\}$.

Definition 6.3.21 (Darboux derivative). Consider a smooth function $f : M \to G$ between a manifold and a Lie group. The Darboux derivative of f is defined as follows:

$$\omega_f := f^*\Omega. \tag{6.34}$$

The function f is called an **integral** or **primitive** of ω_f.

Property 6.3.22. Let M be a connected manifold. If two functions $f, f' : M \to G$ have the same Darboux derivative, there exists an element $g \in G$ such that $f(p) = g \cdot f'(p)$ for all $p \in M$.

Theorem 6.3.23 (Fundamental theorem of calculus). *Consider a smooth manifold M and a Lie group G with Lie algebra $\mathfrak{g}$. If $\omega : TM \to \mathfrak{g}$ satisfies the Maurer-Cartan equation*

$$\mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] = 0\,, \tag{6.35}$$

then (locally) there exists a smooth function $f : M \to G$ such that $\omega = f^*\Omega$.

6.3.5 Local representations

Definition 6.3.24 (Yang-Mills field). Consider a principal G-bundle $P \to M$ and an open subset $U \subseteq M$. Given a principal connection ω on P and a local section $\sigma : U \to P$, the Yang-Mills field $\omega^U \in \Omega^1(U; \mathfrak{g})$ is defined as follows:

$$\omega^U := \sigma^*\omega. \tag{6.36}$$

Definition 6.3.25 (Local representation). Consider a principal bundle $P \to M$ and let (U, φ) be a bundle chart on P. The local representation of a principal connection ω on P with respect to the chart (U, φ) is defined as $(\varphi^{-1})^*\omega$.

Formula 6.3.26. Consider a principal connection ω on a principal G-bundle $P \to M$. Because of Property 6.1.19 every local section $\sigma : U \to P$ induces both a Yang-Mills field ω^U and a local representation of ω. These two forms are related by the following equation:

$$\sigma^*\omega|_{(m,g)}(v, A) = \mathrm{Ad}_{g^{-1}}\big(\omega^U_m(v)\big) + \Omega_g(A)\,, \tag{6.37}$$

where $v \in T_mU$, $A \in \mathfrak{g}$ and Ω is the Maurer-Cartan form on G.

Formula 6.3.27 (Compatibility condition). Consider a principal bundle (P, M, π, G) and two open subsets U, V of M. Given two local sections $\sigma_U : U \to P$, $\sigma_V : V \to P$ and a principal connection ω on P, one can define two Yang-Mills field ω^U and ω^V.

On the intersection $U \cap V \subset M$ there exists a (unique) gauge transformation $\xi : U \cap V \to G$ such that $\sigma_V(m) = \sigma_U(m) \cdot \xi(m)$. Using this gauge transformation one can relate ω^U and ω^V as follows:

$$\omega^V = \mathrm{Ad}_{\xi^{-1}}\omega^U + \xi^*\Omega\,, \tag{6.38}$$

where Ω is the Maurer-Cartan form on G. This formula holds more generally to (locally) relate the connection one-forms ω and $\xi^*\omega$ for any gauge transformation $\xi \in \mathrm{Aut}_V(P)$.

Example 6.3.28 (General linear group). Let $G = GL(n, \mathbb{R})$. The second term in Equation (6.38) can be written as follows:[2]

$$(\xi^*\Omega)^i{}_j = (\xi(m)^{-1})^i{}_k \frac{\partial}{\partial x^\mu}\xi(p)^k{}_j \mathrm{d}x^\mu \tag{6.39}$$

at every point $m \in M$. Formally, this can be written coordinate-independently as

$$\xi^*\Omega = \xi^{-1}\mathrm{d}\xi. \tag{6.40}$$

[2] A derivation can be found in Lecture 22 of [17].

Example 6.3.29 (Christoffel symbols). Let $\Gamma^i{}_{j\mu}, \overline{\Gamma}^k{}_{l\nu}$ be the Yang-Mills fields corresponding to a connection on the frame bundle of some manifold M, where the sections are induced by a choice of coordinates (x^i and y^i, respectively). In this case, the expansion coefficients of the Yang-Mills field are called the **Christoffel symbols** (compare this to Definition 5.7.10). Using Equations (6.38) and (6.40) this becomes:

$$\overline{\Gamma}^i{}_{j\mu} = \frac{\partial y^\nu}{\partial x^\mu}\left(\frac{\partial x^i}{\partial y^k}\Gamma^k{}_{l\nu}\frac{\partial y^l}{\partial x^j} + \frac{\partial x^i}{\partial y^k}\frac{\partial^2 y^k}{\partial x^j \partial x^\nu}\right). \tag{6.41}$$

6.3.6 Parallel transport

Definition 6.3.30 (Horizontal lift). Consider a principal bundle (P, M, π, G) and a curve $\gamma : [0,1] \to M$. Given an Ehresmann connection Hor, for every point $p_0 \in \pi^{-1}(\gamma(0))$ there exists a unique curve $\widetilde{\gamma}_{p_0} : [0,1] \to P$ satisfying the following conditions:

1. $\widetilde{\gamma}_{p_0}(0) = p_0$,
2. $\pi \circ \widetilde{\gamma}_{p_0} = \gamma$, and
3. $\widetilde{\gamma}'_{p_0}(t) \in \mathrm{Hor}(T_{\widetilde{\gamma}_{p_0}(t)}P)$ for all $t \in [0,1]$.

The curve $\widetilde{\gamma}_{p_0}$ is called the **horizontal lift** of γ starting at p_0. When it is clear from the context what the basepoint p_0 is, the subscript is often omitted and one writes $\widetilde{\gamma}$ instead of $\widetilde{\gamma}_{p_0}$.

Remark 6.3.31 (Horizontal curve). Curves satisfying the last condition in the above property are said to be horizontal.

Method 6.3.32. Consider a principal bundle (P, M, π, G). Let γ be a curve in M and let ω be a principal connection one-form on P. For general structure groups G, the horizontal lift can be found as follows. Let δ be a curve in P that projects onto γ, i.e. $\pi \circ \delta = \gamma$, such that $\widetilde{\gamma}_{p_0}(t) = \delta(t) \cdot g(t)$ for some curve g in G. The latter curve g can be found as the unique solution of the following first-order ODE:

$$\mathrm{Ad}_{g(t)^{-1}}\omega_{\delta(t)}(X_{\delta,\delta(t)}) + \Omega_{g(t)}(Y_{g,g(t)}) = 0\,, \tag{6.42}$$

where X_δ, Y_g are tangent vectors to the curves δ and g respectively and where Ω is the Maurer-Cartan form on G. The solution is uniquely determined through the initial value condition $\delta(0) \cdot g(0) = p_0$.

Remark 6.3.33. When given a local section $\sigma : U \to P$, one can rewrite the above ODE in a more explicit form. First, remark that the section induces a curve $\delta = \sigma \circ \gamma$. Taking the derivative then yields $X_\delta = \sigma_*(X_\gamma)$. Using this one can rewrite the ODE as

$$\mathrm{Ad}_{g(t)^{-1}}\omega_{\delta(t)}(\sigma_* X_{\gamma,\gamma(t)}) + \Omega_{g(t)}(Y_{g,g(t)}) = 0. \tag{6.43}$$

After using the equality $f^*\omega = \omega \circ f_*$ and introducing the Yang-Mills field $A = \sigma^*\omega$, this becomes

$$\mathrm{Ad}_{g(t)^{-1}}A(X_{\gamma,\gamma(t)}) + \Omega_{g(t)}(Y_{g,g(t)}) = 0. \tag{6.44}$$

Example 6.3.34. For matrix Lie groups this ODE can be reformulated as follows. Given the trivial section $s : U \to U \times G : x \mapsto (x, e)$, where U is an open subset of M, the horizontal lift of γ can locally be parametrized as

$$\widetilde{\gamma}(t) = \underbrace{(s \circ \gamma)(t)}_{\delta(t)} \cdot g(t) = (\gamma(t), g(t))\,,$$

where g is a curve in G. To determine $\widetilde{\gamma}$ it is thus sufficient to find g. The ODE (6.42) then becomes

$$g'(t) = -\omega(\gamma(t), e, \gamma'(t), 0)g(t). \tag{6.45}$$

Using the trivial section s one can further rewrite this formula. First, consider the action of the Yang-Mills field $s^*\omega$ on the derivative $\gamma_* = (\gamma(t), \gamma'(t))$. Using the fact that it is linear in the second argument, it can be rewritten as

$$s^*\omega(\gamma(t), \gamma'(t)) = A(\gamma(t))\gamma'(t)\,,$$

where $A : M \to \mathrm{Hom}(\mathbb{R}^{\dim(M)}, \mathfrak{g})$ gives a linear map for each point $\gamma(t) \in M$. The action can also be rewritten using the relation $f^*\omega = \omega \circ f_*$ as

$$s^*\omega(\gamma(t), \gamma'(t)) = \omega\big(s_*(\gamma(t), \gamma'(t))\big) = \omega(\gamma(t), e, \gamma'(t), 0).$$

Combining these relations with the ODE (6.45) gives

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + A(\gamma(t))\gamma'(t)\right)g(t) = 0, \tag{6.46}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ is the matrix given by element-wise multiplication of the derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ and the identity matrix I.

The ODE (6.42) can now be solved. Direct integration and iteration gives

$$g(t) = \left[\mathbb{1} - \int_0^t A(\gamma'(t_1))\,dt_1 + \int_0^t\int_0^{t_1} A(\gamma'(t_1))A(\gamma'(t_2))\,dt_2\,dt_1 - \cdots\right]g(0)\,, \tag{6.47}$$

where A is the Yang-Mills field associated to the local section σ. This can be rewritten using the standard *Dyson trick* (see Formula **??**):

$$g(t) = \left[\mathbb{1} - \int_0^t (\gamma'(t_1))\,dt_1 + \frac{1}{2!}\int_0^t\int_0^t \mathcal{T}\Big(A(\gamma'(t_1))A(\gamma'(t_2))\Big)\,dt_2\,dt_1 - \cdots\right]g(0). \tag{6.48}$$

By noting that this formula is equal to the path-ordered exponential series one finds

$$g(t) = \mathcal{T}\exp\left(-\int_0^t A(\gamma'(t'))\,dt'\right)g(0). \tag{6.49}$$

Definition 6.3.35 (Parallel transport). The parallel transport map along the curve γ is defined as follows:

$$\mathrm{Par}_t^\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(t)) : p_0 \mapsto \widetilde{\gamma}_{p_0}(t). \tag{6.50}$$

This map is G-equivariant and it restricts to an isomorphism on the fibres. The group element given by the path-ordered exponential in Equation (6.49) is generally called the **holonomy** along the curve γ.

Using the above constructions that assign Lie group elements to paths, one can give an alternative definition of principal connections:

Alternative Definition 6.3.36 (Principal connection ♣). Let M be a smooth manifold and consider its path groupoid[3] $\mathcal{P}_1(M)$ which has the points of M as objects and homotopy classes of smooth paths in M as morphisms. Let (P, M, π, G) be a principal G-bundle over M

[3]See Definition 8.3.20 for a rigorous exposition.

and denote the delooping **??** of G by $\mathbf{B}G$. The assignment of holonomies to smooth paths locally defines a functor

$$\mathrm{hol}_i : \mathcal{P}_1(U_i) \to \mathbf{B}G \tag{6.51}$$

for every chart $U_i \subseteq M$. Globally, these can be glued together using the transition cocycles g_{ij} (in there incarnation as natural isomorphisms) to obtain a functor

$$\mathrm{hol} : \mathcal{P}_1(M) \to \mathbf{Trans}_1(P) \subset G\mathbf{Torsor}\,, \tag{6.52}$$

where $\mathbf{Trans}_1(P)$ is the full subcategory of the category of G-torsors on the fibres of P (Remark 6.1.2).

It can be shown that any functor of this type gives rise to a principal connection on P and, conversely, every principal connection gives rise to a holonomy functor through the parallel transport constructions as given above. ?? COMPLETE ??

6.3.7 Holonomy group

Definition 6.3.37 (Holonomy group). Consider a principal bundle (P, M, π, G) and choose a point $m \in M$. Let $\Omega^{ps}_m M \subset \Omega_m M$ denote the subset of the based loop space consisting of piecewise smooth loops with basepoint $m \in M$. The holonomy group $\mathrm{Hol}_p(\omega)$ based at $p \in \pi^{-1}(m)$ with respect to the connection form ω is given by

$$\mathrm{Hol}_p(\omega) := \{g \in G \mid p \sim p \cdot g\}\,, \tag{6.53}$$

where two points $p, q \in P$ are identified if there exists a loop $\gamma \in \Omega^{ps}_m M$ such that the horizontal lift $\widetilde{\gamma}$ connects p and q.

Definition 6.3.38 (Reduced holonomy group). The subgroup of the holonomy group induced by contractible loops.

Definition 6.3.39 (Holonomy bundle). Let M be a path-connected manifold and consider a principal bundle P over M with principal connection ω. One can equip P with an equivalence relation $\sim$ such that $p \sim q$ if and only if there exists a horizontal curve connecting p and q. For every point $p \in P$ one can then construct the following set:

$$H(p) := \{q \in P \mid p \sim q\}. \tag{6.54}$$

Path-connectedness of the base manifold implies that $H(p)$ and $H(q)$ are isomorphic for all $p, q \in P$. Using this fact one can show that $\sqcup_p H(p)$ is in fact a principal bundle itself. Its structure group is $\mathrm{Hol}_p(\omega)$ for any $p \in P$.

6.4 Covariant derivatives

6.4.1 Koszul connections

Definition 6.4.1 (Horizontal lifts on associated bundles). Let $P_F := P \times_G F$ be an associated bundle of a principal bundle (P, M, π, G) and let γ be a curve in M with horizontal lift $\widetilde{\gamma}_p$ in P. The horizontal lift of γ to P_F through the point $[p, f] \in P_F$ is defined as follows:

$$\widetilde{\gamma}^{P_F}_{[p,f]}(t) := [\widetilde{\gamma}_p(t), f]. \tag{6.55}$$

Although the element f seems to stay constant along the horizontal lift, it in fact changes according to Equation (6.3).

Definition 6.4.2 (Parallel transport). Similar to the case of principal bundles P, the parallel transport map on an associated bundle P_F is defined as

$$\mathrm{Par}_t^\gamma : \pi_F^{-1}(\gamma(0)) \to \pi_F^{-1}(\gamma(t)) : [p, f] \mapsto \widetilde{\gamma}_{[p,f]}^{P_F}(t). \tag{6.56}$$

Example 6.4.3 (Vector bundles). Consider a principal bundle (P, M, π, G) and suppose that the Lie group G acts on a vector space V through a representation $\rho : G \to \mathrm{GL}(V)$. One can construct an associated vector bundle $\pi_1 : P \times_{\mathrm{GL}(V)} V \to M$. Moreover, by working over a chart (U, φ) one can locally write P and P_V as product bundles. Parallel transport on this vector bundle is then defined as follows. Let γ be a curve in M such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Furthermore, let the horizontal lift $\widetilde{\gamma}(t) = (\gamma(t), g(t))$ satisfy $\widetilde{\gamma}(0) = (x_0, h)$ as initial condition. Parallel transport of the point $(x_0, v_0) \in U \times V$ along γ is given by the following map:

$$\mathrm{Par}_t^\gamma : \pi_1^{-1}(x_0) \to \pi_1^{-1}(\gamma(t)) : (x_0, v_0) \mapsto \big(\gamma(t), \rho\big(g(t)h^{-1}\big)v_0\big). \tag{6.57}$$

It should be noted that this map is independent of the initial element $h \in G$ despite the presence of the factor h^{-1}. Moreover, Par_t^γ is an isomorphism of vector spaces and can thus be used to identify distant fibres (as long as they lie in the same path-component).

Remark 6.4.4. For every vector bundle one can construct the frame bundle and use the parallel transport map on this bundle to define parallel transport of vectors. Therefore, the previous construction is applicable to any vector bundle.

Definition 6.4.5 (Covariant derivative). Consider a vector bundle $\pi : E \to M$ with typical fibre V and its associated principal $\mathrm{GL}(V)$-bundle with principal connection ω. Let $\sigma : M \to E$ be a section of the vector bundle and let X be a vector field on M. The covariant derivative of σ with respect to X is defined as follows:

$$\nabla_X\sigma|_{x_0} := \lim_{t\to 0} \frac{(\mathrm{Par}_t^\gamma)^{-1}\sigma(\gamma(t)) - \sigma(x_0)}{t}\,, \tag{6.58}$$

where γ is any curve satisfying $\gamma(0) = x_0$ and $\gamma'(0) = X(x_0)$. Let $\widetilde{\gamma}$ and X^H be the horizontal lifts of γ and X, respectively. An equivalent expression is the following one:

$$\nabla_X\sigma = \pi_*(\sigma_*X - X^H \circ \sigma). \tag{6.59}$$

One can also rephrase the above definition in terms of the horizontal vector field associated to the lift $\widetilde{\gamma}$ (akin to Definition 5.3.17). By Property 6.1.21 every section σ of an associated bundle corresponds to a G-equivariant map $\phi(\sigma) : P \to V$. In terms of this map one obtains

$$\phi(\nabla_X\sigma) = X^H(\phi(\sigma))\,, \tag{6.60}$$

where X^H acts componentwise on V.

Property 6.4.6. The map

$$\Gamma(TM) \times \Gamma(E) \to \Gamma(E) : (X, \sigma) \mapsto \nabla_X\sigma \tag{6.61}$$

defines a Koszul connection 5.7.1. It follows that every principal connection on a principal bundle induces a Koszul connection on all of its associated vector bundles.

6.4.2 Exterior covariant derivative

Definition 6.4.7 (Exterior covariant derivative). Let P be a principal bundle equipped with a principal connection ω and let $\theta \in \Omega^k(P)$ be a differential k-form. The exterior covariant derivative $\mathrm{D}\theta$ is defined as follows:

$$\mathrm{D}\theta(v_0, \ldots, v_k) := \mathrm{d}\theta(v_0^H, \ldots, v_k^H)\,, \tag{6.62}$$

where d is the exterior derivative 5.4.7 and v_i^H is the projection of v_i on the horizontal subspace $\mathrm{Hor}(T_pP)$. From this definition it follows that the exterior covariant derivative $\mathrm{D}\theta$ is a horizontal form 6.3.13.

Remark 6.4.8. The exterior covariant derivative can also be defined for general vector-valued k-forms. This can be done by defining it component-wise with respect to a given basis. Afterwards one can prove that the choice of basis plays no role.

For tensorial forms of type (V, ρ) this is given by the following expression:

$$\mathrm{D}\theta = \mathrm{d}\theta + \omega \mathbin{\overline{\wedge}} \theta\,, \tag{6.63}$$

where $\overline{\wedge}$ denotes the combination of the wedge product and the action ρ.

Property 6.4.9 (Tensorial). If Φ is an equivariant form, then $\mathrm{D}\Phi$ is a tensorial form.

The compatibility condition for connection one-forms (6.38) can be restated in terms of the covariant derivative:

Property 6.4.10 (Gauge transformation). Consider a principal bundle (P, M, π, G) and a connection one-form ω. For every gauge transformation $\xi \in \mathrm{Aut}_V(P)$ one (locally) has the following expression:

$$\xi^*\omega = \omega + \xi^{-1}\mathrm{D}\xi\,, \tag{6.64}$$

where D is the exterior covariant derivative associated to ω.

Formula 6.4.11. Using the Koszul connection on the tangent bundle TP one can rewrite the action of the exterior covariant derivative as follows:

$$\begin{aligned}\mathrm{D}\theta(v_0, \ldots, v_k) = \sum_i^k (-1)^i \nabla_{v_i}\theta(v_0, \ldots, \hat{v}_i, \ldots, v_k) \\ + \sum_{i<j}^k (-1)^{i+j}\theta([v_i, v_j], v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k)\,,\end{aligned} \tag{6.65}$$

where, as usual, $\hat{v}_i$ indicates that this vector is omitted. This formula should remind the reader of the analogous formula for the ordinary exterior derivative (5.44). As an example the formula for a one-form Φ is given:

$$\mathrm{D}\Phi(X, Y) = \nabla_X(\Phi(Y)) - \nabla_Y(\Phi(X)) - \Phi([X, Y]). \tag{6.66}$$

Because of Property 6.1.21 one can use the following construction to find an explicit expression for the covariant derivative on an associated vector bundle:

Construction 6.4.12 (Covariant derivative). Let (P, M, π, G) be a principal bundle and let $P_V := P \times_G V$ be an associated vector bundle. Given a section $\sigma : M \to P_V$, one can construct a G-equivariant map $\phi : P \to V$ using Equation (6.12). The exterior covariant derivative of ϕ is given by Equation (6.63):

$$\mathrm{D}\phi(X) = \mathrm{d}\phi(X) + \omega \triangleright \phi(X)\,, \tag{6.67}$$

where $X \in T_pP$. Now, given an additional (local) section $\varphi : U \subseteq M \to P$, one can pull back this derivative to the base manifold M. This gives

$$(\varphi^*\mathrm{D}\phi)(Y) = \mathrm{d}(\varphi^*\phi)(Y) + \varphi^*\omega \triangleright \varphi^*\phi(Y)\,, \tag{6.68}$$

where $Y = \pi_*X \in T_mM$. After introducing the notations $S := \varphi^*\phi$ and $\nabla_Y S := (\varphi^*\mathrm{D}\phi)(Y)$ and remembering the definition of the Yang-Mills field 6.3.24, this becomes

$$\nabla_Y S = \mathrm{d}S(Y) + \omega^U(Y) \triangleright S. \tag{6.69}$$

Example 6.4.13. Let $G = \mathrm{GL}(n, \mathbb{R})$. In local coordinates Equation (6.69) can be rewritten as follows:

$$(\nabla_Y S)^i = \frac{\partial S^i}{\partial x^k}Y^k + \Gamma^i{}_{jk}S^jY^k. \tag{6.70}$$

This is exactly the formula known from classical differential geometry and relativity.

6.4.3 Curvature

Definition 6.4.14 (Curvature). Let ω be a principal connection one-form. The curvature Ω of ω is defined as the exterior covariant derivative $\mathrm{D}\omega$.

Property 6.4.9 implies the following important statement:

Property 6.4.15 (Tensorial). In contrast to a connection one-form, the associated curvature is a tensorial $\mathfrak{g}$-valued two-form or, equivalently, an $\mathrm{End}(P)$-valued two-form.

Definition 6.4.16 (Flat connection). A principal connection is said to be flat if its curvature vanishes everywhere. A bundle is said to be flat if it admits a flat connection.

Property 6.4.17 (Local systems). If the connection ∇ on a vector bundle is flat, the flat sections constitute a (linear) local system **??**. Moreover, this sheaf characterizes the bundle and connection up to isomorphism, i.e. there exists an equivalence of categories of flat vector bundles and linear local systems.

Formula 6.4.18 (Curvature on associated bundles). The above definition of the curvature, together with Equation (6.63) or, equivalently, Construction 6.4.12, implies that one can express the action of the curvature on sections of associated bundles as follows:

$$\mathrm{D}^2\phi = \Omega \triangleright \phi, \tag{6.71}$$

where $\phi \in \Omega^\bullet(M; E)$. This curvature form Ω coincides with the one from Definition 5.7.11.

Example 6.4.19. Let ω_G be the Maurer-Cartan form on a Lie group G. Because the only horizontal vector field on the bundle $G \hookrightarrow G \to \{*\}$ is the zero vector, the curvature of ω_G is 0. It follows that the Maurer-Cartan form is a flat connection.

Property 6.4.20 (Second Bianchi identity). Let ω be a principal connection one-form with curvature Ω. The curvature is covariantly constant:

$$\mathrm{D}\Omega = 0. \tag{6.72}$$

Remark 6.4.21. One should pay attention to the fact that this result does not generalize to arbitrary differential forms. Only the exterior derivative satisfies the coboundary condition $d^2 \equiv 0$, the exterior covariant derivative does not.

Formula 6.4.22 (Cartan structure equation). Let ω be a principal connection one-form and let Ω be its curvature form. The curvature can be expressed in terms of the connection as follows:

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega]. \tag{6.73}$$

The Maurer-Cartan equation in the (geometric) fundamental theorem of calculus 6.3.23 exactly states the vanishing of the algebraic curvature associated to a general $\mathfrak{g}$-valued one-form.

The following property is an immediate consequence of Frobenius's integrability theorem 5.5.5 and the fact that a connection vanishes on the horizontal subbundle:

Property 6.4.23 (Integrability). Let ω be a principal connection one-form. The associated horizontal distribution

$$p \mapsto \mathrm{Hor}(T_pP)$$

is integrable if and only if the connection ω is flat. In contrast, the vertical distribution is always integrable.

Similar to Definition 6.3.24 one can also define the Yang-Mills field strength:

Definition 6.4.24 (Field strength). Let $\pi : P \to M$ be a principal bundle equipped with a principal connection one-form ω and associated curvature Ω. Given a local section $\sigma : U \subseteq M \to P$, one defines the (Yang-Mills) field strength F as the pullback $\sigma^*\Omega$.

Theorem 6.4.25 (Ambrose-Singer). *The Lie algebra of the holonomy group* $\mathrm{Hol}_p(\omega)$ *is spanned by the elements of the form* $\Omega_q(X, Y)$*, where* q *ranges over the holonomy bundle* $H(p)$ *and* X, Y *are horizontal.*

6.4.4 Torsion

Definition 6.4.26 (Solder form). Let (P, M, π, G) be a principal bundle and let V be a $\dim(M)$-dimensional vector space equipped with a representation[4] $\rho : G \to \mathrm{GL}(V)$ such that $TM \cong P \times_G V$ as associated bundles. A solder(ing) form θ on P is a tensorial one-form 6.3.14 of type (V, ρ).

Definition 6.4.27 (Torsion). Let (P, M, π, G) be a principal bundle equipped with a principal connection ω and a solder form θ. The torsion Θ is defined as the exterior covariant derivative $D\theta$. This is the content of the **Cartan structure equation**:

$$\Theta = \mathrm{d}\theta + \omega \mathbin{\bar\wedge} \theta\,, \tag{6.74}$$

where the wedge product is defined analogously to the wedge products 5.4.22 and 5.4.27 using the induced representation of $\mathfrak{g}$ on V:

$$\omega \mathbin{\bar\wedge} \theta(v, w) := \omega(v) \triangleright \theta(w) - \omega(w) \triangleright \theta(v). \tag{6.75}$$

Property 6.4.28 (First Bianchi identity). Let ω be a principal connection one-form, Ω its associated curvature, θ a solder form and Θ its associated torsion.

$$\mathrm{D}\Theta = \Omega \mathbin{\bar\wedge} \theta \tag{6.76}$$

6.5 Reduction of the structure group

Definition 6.5.1 (Reduction). Consider a principal bundle $G \hookrightarrow P \to M$ and let H be a subgroup of G. If the transition functions of P can be chosen to take values in H, it is said that the structure group G can be reduced to H.

More generally, a principal bundle $H \hookrightarrow \widetilde{P} \to M$ with structure group H is called an H-reduction of P if there exists a bundle isomorphism $\widetilde{P} \times_H G \to P$. This allows for morphisms besides inclusions, such as covering maps $\lambda : H \to G$. (See for example the definition of spinor bundles in Section **??**.) As such the name "reduction" is not the best choice of terminology. For covering maps the term **lift(ing)** is sometimes used.

Definition 6.5.2 (G-structure). Consider a manifold M. A G-structure on M is the reduction of the structure group $\mathrm{GL}(n)$ of the frame bundle FM to the group $\iota : G \to \mathrm{GL}(n)$.

Definition 6.5.3 (Integrability). A G-structure P on M is said to be integrable if for every point $p \in M$ there exists a chart $U \ni p$ such that the associated holonomic frame $\{\partial_i\}_{i \leq \dim(M)}$ induces a local section of P.

Property 6.5.4. Consider a smooth manifold M equipped with a G-structure. If this structure is integrable, it admits a torsion-free connection.

[4] In general this will be $V = \mathbb{R}^{\dim(M)}$ and $G = \mathrm{GL}(n, \mathbb{R})$.

Example 6.5.5 (Orientable manifold). An n-dimensional manifold is orientable if and only if the structure group can be reduced to $\mathrm{GL}^+(n)$, the group of invertible matrices with positive determinant. Furthermore, this structure is always integrable if it exists.

Example 6.5.6 (Riemannian manifold). An $\mathrm{O}(n)$-structure turns M into a *Riemannian manifold* **??**. Because the cotangent bundle T^*M transforms under the contragredient representation, which coincides with the regular representation in the case of $\mathrm{O}(n)$, of the transition maps of the tangent bundle TM, these two bundles are equivalent. The isomorphism is given by the musical isomorphism(s) **??**. Riemannian structures are always integrable.

The following property gives a classification of bundle reductions:

Property 6.5.7 (Equivariant morphisms). Consider a principal G-bundle P and let F be a set that admits a transitive action $\varphi : G \to \mathrm{Aut}(F)$. For every $f \in F$ and every equivariant morphism $\psi : P \to F$ there exists a reduction of G to the isotropy subgroup G_f defined by

$$P_f := \{p \in P \mid \psi(p) = f\}. \tag{6.77}$$

One can generalize this definition to arbitrary Lie group actions by restricting to the equivariant morphisms that take value in a single orbit.[5]

Consider a subgroup inclusion $\iota : H \hookrightarrow G$. If H is closed, the action of G on G/H is transitive and one can specialize the above construction to the coset space G/H. It follows that reductions are classified by equivariant maps into the coset space G/H or, according to Property 6.1.21, by the (global) sections of the associated coset bundle $P \times_G G/H$.

Corollary 6.5.8. If G is connected, every principal G-bundle is reducible to a maximal compact subgroup of G.

Definition 6.5.9 (Reducible connection). Consider a principal G-bundle P equipped with a connection one-form ω. If a bundle map F induces an H-reduction of P, then the connection ω is said to be reducible (and to be compactible with the given reduction) if $F^*\omega$ takes values in $\mathfrak{h}$.

Property 6.5.10. Consider a principal bundle P together with a reduction P_f induced by an equivariant morphism $\psi : P \to F$ with $f \in F$. A principal connection on P is reducible to P_f if and only if ψ is parallel with respect to this connection, i.e. $\mathrm{D}\psi = 0$.

The following two properties characterize bundle reductions in terms of holonomy bundles:

Property 6.5.11 (Holonomy bundles and reductions). The holonomy bundle $H(p)$ is a reduction of P for every $p \in P$. Furthermore, any connection ω is reducible to $H(p)$ and it can be proven that this reduction is minimal, i.e. there exists no further reduction.

Corollary 6.5.12. A principal bundle (and any associated connection) is irreducible to a subgroup of the structure group[6] if and only if it is equivalent to its holonomy bundle.

The following property is less known in the literature:

Property 6.5.13 (Flat connections). A bundle is flat if and only if its structure group G can be lifted to the discrete group G^δ, i.e. the same group but with the discrete topology. An equivalent condition is that the structure group can be lifted to the fundamental group of the base space $\pi_1(M)$ (this latter condition is related to the fact that for flat connections parallel transport is path-independent and, hence, is fully characterized by the loops in M).

Note that once such a lift is chosen or, equivalently, if the structure group of the bundle is discrete, a unique flat connection exists.

[5] Since transitive actions have a unique orbit, this is a well-defined generalization.

[6] Lifts as in the case of Spin-structures do not fall under the holonomy classification.

Remark 6.5.14. The above condition can also be applied to define flatness for topological bundles where the notion of connections does not make sense.

6.6 Characteristic classes

Definition 6.6.1 (Characteristic class). Let M be a manifold. A characteristic class is a map from isomorphism classes of vector bundles or principal bundles $E \to M$ to cohomology classes $c(E) \in H^\bullet(M; R)$ such that if there exists a morphism $f : N \to M$, then $c(f^*E) = f^*c(E) \in H^\bullet(N; R)$. The coefficient ring R is often assumed to be the base field ($\mathbb{R}$ or $\mathbb{C}$), but this is not always the case (e.g. the *Stiefel-Whitney classes* from Section **??**).

Using the classification property 6.2.6, one can give a concise construction of characteristic classes in the case of principal bundles:

Construction 6.6.2. Consider a principal bundle (P, M, π, G) with classifying map $\varphi \in [M, BG]$. For every $c \in H^\bullet(BG)$ one defines a characteristic class $c(P) \in H^\bullet(M)$ as the pullback of c under φ.

As the definition implies, both vector bundles and principal bundles admit a theory of characteristic classes. However, in the literature most authors always focus on either one of them and, hence, it is not always easy to see which theorems can be translated and how to do this whenever possible. The relation between the two theories is given by the associated bundle construction 6.1.10 (see [21] for more information). The characteristic classes of a vector bundle are defined as the ones of its frame bundle. Because of this duality one can freely switch between the language of vector bundles and principal bundles, depending on where the results will be applied.

Because the statement of the *splitting principle* is quite different when given in the language of principal bundles or that of vector bundles, it will be stated for both cases. First an additional construction is needed:

Definition 6.6.3 (Flag bundle). Let $\pi : E \to M$ be a vector bundle. Using the definition of the flag manifold **??** one can construct for every fibre E_p a space $\mathrm{Fl}(E_p)$ that has the complete flags of E_p as points (expressed as a sequence of one-dimensional subspaces). Using the bundle construction theorem, one can then obtain the flag bundle $\pi_{\mathrm{Fl}} : \mathrm{Fl}(E) \to M$ that has the flag manifolds as fibres.

Theorem 6.6.4 (Splitting principle). *Consider a vector bundle $\pi : E \to M$. Its flag bundle has the following properties:*

- *The pullback bundle $\pi^*_{\mathrm{Fl}}E$ can be decomposed as a Whitney sum of line bundles.*
- *The induced morphism on cohomology $\pi^*_{\mathrm{Fl}} : H^\bullet(M) \to H^\bullet(\mathrm{Fl}(E))$ is injective.*

For the following form of the splitting principle, see [22, 23].

Theorem 6.6.5 (Splitting principle). *Consider a principal bundle (P, M, π, G) where the structure group G is compact. Every compact Lie group contains a maximal torus $T \cong \mathbb{T}^n$, where $\mathbb{T}$ is the standard 1-torus $S^1 \cong \mathrm{U}(1)$. The inclusion $\iota : T \hookrightarrow G$ induces a G-bundle $B\iota : BT \to BG$ with fibre G/T and total space EG. The pullback of $B\iota$ along the classifying map $p \in [M, BG]$ of P defines another G-bundle $\rho : p^*B\iota \to M$ (also with fibre G/T). This fibre bundle has the following properties:*

- *ρ^*p admits a reduction of the structure group to T.*

- *The induced morphism on cohomology $\rho^* : H^\bullet(M) \to H^\bullet(\rho^*P)$ is injective.*

Because $B\mathbb{T}^n \cong (B\mathbb{T})^n$, one can use the fibration $B\iota$ to pull back any class $c \in H^\bullet(BG)$ to a tuple of classes in $H^\bullet(B\mathrm{U}(1))$. Therefore, every characteristic class of ρ^*P is a tuple of characteristic classes of circle bundles. The injectivity of ρ^* implies that every characteristic class of P can be characterized by such a tuple.

6.6.1 Chern-Weil theory

The characteristic classes of a vector bundle can be constructed from the connection and curvature forms on the vector bundle. The resulting expressions are polynomial in the curvature forms.

Definition 6.6.6 (Chern-Weil morphism). Let $\pi : E \to M$ be a vector bundle with structure group G and denote the connection one-form and curvature two-form by ω and Ω respectively. There exists a morphism of algebras

$$K[\mathfrak{g}]^G \to \Omega^\bullet(E) : P \mapsto P(\Omega)\,, \tag{6.78}$$

where K is the base field, satisfying:

- $P(\Omega)$ is closed.
- $P(\Omega)$ pulls back uniquely to a (closed) form $\overline{P}(\Omega) := \pi^*P(\Omega)$ on M.
- $\overline{P}(\Omega)$ does not depend on the chosen connection, i.e. for two connection one-forms ω, ω', the difference $\overline{P}(\Omega) - \overline{P}(\Omega')$ is exact.

In the remainder of this section this approach will be followed to find explicit descriptions of characteristic classes of vector bundles and principal bundles.

6.6.2 Complex bundles

In this section only complex bundles are considered. This allows for the choice of $\mathfrak{u}(n)$-valued connection one-forms. See Chapter 7 for more information.

Definition 6.6.7 (Chern class). Consider a rank-n vector bundle $\pi : E \to M$ with curvature two-form Ω. Using Chern-Weil theory one defines the Chern classes $c_k(E)$ as follows:

$$\det\left(\mathbb{1} + \frac{it}{2\pi}\Omega\right) =: \sum_{k=1}^{n} c_k(E)t^k. \tag{6.79}$$

The k^{th} Chern class is a cohomology class in $H^{2k}(M)$.

Definition 6.6.8 (Chern polynomial). Let $c_k(E)$ denote the k^{th} Chern class of E. The Chern polynomial is defined as follows:

$$c_t(E) := \sum_{k=1}^{\infty} c_k(E)t^i. \tag{6.80}$$

The **total Chern class** is defined by taking $t = 1$.

Definition 6.6.9 (Chern character). Consider a rank-n vector bundle $\pi : E \to M$ with curvature two-form Ω. Using Chern-Weil theory one defines the Chern character as follows:

$$\mathrm{ch}(E) := \mathrm{tr}\left(\exp\left(\frac{i\Omega}{2\pi}\right)\right). \tag{6.81}$$

If $c_i := c_i(E)$ denotes the i^{th} Chern class of E, the Chern character can also be expressed as

$$\mathrm{ch}(E) = \sum_{k=0}^{n} \frac{c_1^k + \cdots + c_n^k}{k!}. \tag{6.82}$$

The term with prefactor $1/k!$ is a homogeneous polynomial of degree k. One sometimes calls this term the k^{th} Chern character. Using Chern-Weil theory, this form is proportional to $\mathrm{tr}(\Omega^k)$.

Formula 6.6.10 (Whitney product formula[7]). The following equality holds for all bundles E_1, E_2:

$$c_t(E_1 \oplus E_2) = c_t(E_1)c_t(E_2). \tag{6.83}$$

Corollary 6.6.11 (Chern root). The product formula and the splitting principle imply that the Chern polynomial of any rank-n vector bundle can be decomposed as follows:

$$c_t(E) = \prod_{i=1}^{n}(1 + x_i t), \tag{6.84}$$

where in the case of decomposable vector bundles $E \equiv \oplus_{i=1}^n L_i$ the x_i are the first Chern classes $c_1(L_i)$. The factors x_i are called the **Chern roots**.

By working out the above formula one can see that the coefficient in degree k, i.e. the k^{th} Chern class, is given by the k^{th} elementary symmetric polynomial:

$$c_k(E) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}. \tag{6.85}$$

Definition 6.6.12 (Canonical class). Consider a smooth manifold M. The first Chern class of the canonical bundle $\bigwedge^n T^*M$ is called the canonical class of M.

Definition 6.6.13 (Theta characteristic). Consider a smooth manifold M together with its canonical class K_M. The theta characteristic, if it exists, is a characteristic class Θ such that $\Theta \cup \Theta = K_M$, where $\cup$ is the cup-product in cohomology 5.9.6.

After finding the Chern roots of a vector bundle E, one can use them to define various other classes:

Construction 6.6.14 (Genus). Let $f \in K[[t]]$ be a formal power series with constant term 1. For any $k \in \mathbb{N}$ one can easily see that $f(x_1) \cdots f(x_k)$ is a symmetric power series (also with constant term 1). For every such f define the f-genus by the formula[8]

$$G_f(E) := \det f\left(\frac{it}{2\pi}\Omega\right). \tag{6.86}$$

The coefficients of this power series define characteristic classes of E.

Example 6.6.15 (Chern class). The total Chern class is recovered as the genus of $f = 1 + x$.

The following genus is very important, especially in the context of the *Atiyah-Singer index theorem* (see further below):

[7] This formula is also called the **Whitney sum formula**.

[8] In the case that E splits as a sum for line bundles, one simply obtains the product $f(x_1) \cdots f(x_k)$.

Example 6.6.16 (Todd class). Consider the function

$$Q(x) := \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}B_i}{(2i)!}x^{2i}\,, \tag{6.87}$$

where B_i is the i^{th} *Bernoulli number*. Let $\pi : E \to M$ be a rank-n vector bundle. If x_i are the Chern roots of E, the Todd class is defined as

$$\mathrm{td}(E) := \prod_{i=1}^{n} Q(x_i). \tag{6.88}$$

The characteristic function of the Todd genus is the unique power series with constant term 1 that has the property that for all $n \in \mathbb{N}$ the n^{th} degree term in $f(x)^{n+1}$ has coefficient 1.

Another genus that is used in the context of the index theorems is the following one:

Example 6.6.17 ($\hat{A}$-genus[9]). The $\hat{A}$-genus is defined through the following function:

$$Q(x) := \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 - \frac{x}{24} + \frac{7x^2}{5760} - \cdots\,. \tag{6.89}$$

6.6.3 Real bundles

In the case of real vector bundles, which will be assumed to come equipped with a fibre metric as to allow for $\mathfrak{o}(n)$-valued connection one-forms, one can also define a set of characteristic classes.

Definition 6.6.18 (Pontryagin class). Consider a vector bundle $\pi : E \to M$. The Pontryagin classes of E are defined as follows:

$$p_k(E) := (-1)^k c_{2k}(E^{\mathbb{C}}) \in H^{4k}(M)\,, \tag{6.90}$$

where $E^{\mathbb{C}}$ is the complexification of E. If E has the structure of a complex vector bundle, one can use the relation $E^{\mathbb{C}} \cong E \oplus \overline{E}$ to express the Pontryagin classes purely in terms of the Chern classes of E, e.g.

$$p_1(E) = c_1^2(E) - 2c_2(E). \tag{6.91}$$

When the vector bundles in question are orientable, the structure group can further be reduced to $\mathrm{SO}(n)$. If the rank is even, one can define the following characteristic class:

Definition 6.6.19 (Euler class). Let $\pi : E \to M$ be an orientable vector bundle of rank $2k$. The Euler class of E is defined as follows:

$$e(E) := p_k(E) \cup p_k(E). \tag{6.92}$$

Property 6.6.20. Using the fact that one can write the total Pontryagin class using Chern-Weil theory as

$$p(E) = \det\left(1 - \frac{1}{2\pi}\Omega\right) \tag{6.93}$$

and that the determinant is the square of the *Pfaffian*, one can equivalently define the Euler class as follows:

$$e(E) := \mathrm{Pf}\left(-\frac{1}{2\pi}\Omega\right). \tag{6.94}$$

[9] This is pronounced as *A-roof genus*.

6.6.4 Cohomology of Lie groups

Using the language of characteristic classes one can find a concise description of the (continuous) group cohomology of Lie groups. First of all there is the isomorphism between continuous group cohomology and cohomology of classifying spaces:

$$H^\bullet(BG;\mathbb{Z}) \cong H^\bullet_c(G;\mathbb{Z}). \tag{6.95}$$

?? COMPLETE ??

6.6.5 Chern-Simons forms

By Chern-Weil theory, the image of invariant polynomials under the Chern-Weil morphism is closed. This not only allows to interpret them as cohomology classes as done above, but it also implies that (locally) one can find a trivialization:

$$\langle \Omega_A, \cdots \rangle_n = \mathrm{dCS}^n(A), \tag{6.96}$$

where $\langle \cdots \rangle_n$ denotes an invariant polynomial of degree $2n$. Such a form is called a Chern-Simons form or **secondary characteristic form**.

More generally, consider the *concordance* $P \times [0,1]$ for some principal bundle P with itself together with a connection $\hat{A}$. This connection defines a path between two connections A, A' on P. The relative Chern-Simons form is defined as

$$\mathrm{CS}^n(A,A') := \int_{[0,1]} \langle \Omega_{\hat{A}}, \cdots \rangle_n. \tag{6.97}$$

The differential of this form gives the difference of characteristic forms:

$$\mathrm{dCS}^n(A,A') = \langle \Omega_{A_1}, \cdots \rangle_n - \langle \Omega_{A_0}, \cdots \rangle_n. \tag{6.98}$$

However, note that the Chern-Simons form is only defined up to an exact form.

Example 6.6.21 (Killing form). Consider the Killing transgression form, which for $\mathfrak{su}(n)$ is induced by the trace functional. The related Chern-Simons form is given by

$$\langle \mathrm{d}A, A\rangle + \frac{2}{3}\langle A, [A \wedge A]\rangle. \tag{6.99}$$

This form is the exterior derivative of the second Chern character, which for $\mathrm{SU}(n)$-bundles is equivalent to the second Chern class. A similar expression can be obtained for the Chern-Simons form associated to all other Chern characters.

6.7 Differential cohomology ♣

In the foregoing sections a multitude of objects were introduced that are related to principal fibre bundles. For example, connections and their associated curvature forms could be used to construct differential quantities, while characteristic classes contained data about the topology of the bundle. However, even in the simple case of U(1)-bundles, neither the (first) Chern class, nor the curvature form are are able to uniquely characterize the bundle.

6.7.1 Differential characters

In this section all (co)chains, (co)cycles and (co)boundaries are assumed to be smooth. By doing this no generality is lost since every continuous chain is homotopic to a smooth one.

Definition 6.7.1 (Differential character). Consider a positive integer $k \geq 1$ and let M be a manifold. A (**Cheeger-Simons**) differential character of **degree** k is a group homomorphism $\chi : Z_{k-1}(M) \rightarrow \mathrm{U}(1)$ that is given by integration on boundaries:[10]

$$\chi(\partial\gamma) = \exp\left(2\pi i \int_\gamma \omega(\chi)\right) \tag{6.100}$$

for some $\omega(\chi) \in \Omega^k(M)$. The group of differential characters of degree k is denoted by $\hat{H}^k(M;\mathbb{Z})$. For $k = 0$ the convention $\hat{H}^0(M;\mathbb{Z}) := H^0(M;\mathbb{Z})$ is used.

Property 6.7.2 (Thin invariance). Differential characters vanish on boundaries of thin chains, i.e. for chains $\gamma \in C_k(M)$ such that $\int_\gamma \omega = 0$ for all $\omega \in \Omega^k(M)$ one has $\chi(\partial\gamma) = 1$.

Property 6.7.3 (Curvature). Every differential character is represented by a unique, closed and integral k-form. The map $\mathrm{curv} : \hat{H}^k(M;\mathbb{Z}) \rightarrow \Omega^k_{\mathrm{int}}(M) : \chi \mapsto \omega(\chi)$ is called the curvature map. If $\mathrm{curv}(\chi) = 0$, the character χ is said to be **flat**.

Property 6.7.4 (Characteristic class). Every differential character gives rise to a characteristic class as follows. The group of cocycles is free and the quotient map $\mathbb{R} \rightarrow \mathrm{U}(1)$ is onto, so every differential character lifts to a group homomorphism $\widetilde{\chi} : Z_{k-1}(M) \rightarrow \mathbb{R}$ such that $\chi(z) = \exp(2\pi i\widetilde{\chi}(z))$. The map

$$\mathrm{ch}(\chi) : C_k(M) \rightarrow \mathbb{Z} : \gamma \mapsto \int_\gamma \mathrm{curv}(\chi) - \widetilde{\chi}(\partial\gamma) \tag{6.101}$$

induces a well-defined map $\mathrm{ch} : \hat{H}^k(M;\mathbb{Z}) \rightarrow H^k(M;\mathbb{Z})$. If $\mathrm{ch}(\chi) = 0$, the character χ is said to be **topologically trivial**. The characteristic class associated to a differential character is sometimes called the **Dixmier-Douady** class (see e.g. [24]).

Example 6.7.5 (Circle bundles). Consider a $\mathrm{U}(1)$-bundle $\pi : P \rightarrow M$ with connection ω. Holonomy around closed curve γ gives a parallel transport map

$$P \rightarrow P : p \mapsto p \cdot g(p,\gamma) \tag{6.102}$$

for a smooth function $g : \Omega_p P \rightarrow \mathrm{U}(1)$. In fact, g only depends on the homology of γ and the projection $\pi(p)$, so one obtains a map $g \in \hat{H}^2(M;\mathbb{Z})$ with curvature $\frac{-1}{2\pi i}\Omega$ and characteristic class $c_1(P)$. The converse also holds, every different character of degree 2 determines a principal $\mathrm{U}(1)$-bundle with connection (up to connection-preserving isomorphism). This leads to the following equivalence:

$$\begin{aligned}\hat{H}^2(M;\mathbb{Z}) \cong \{&\text{isomorphism classes of } (P,\nabla) \mid P \text{ a circle bundle and} \\ &\nabla \text{ a principal connection}\}.\end{aligned} \tag{6.103}$$

The curvature and characteristic class maps fit in some exact sequences:

Property 6.7.6 (Curvature exact sequence). The first sequence is induced by the curvature map. A vanishing curvature form says that the character vanishes identically on boundaries. This is exactly the property satisfied by cohomology classes:

$$0 \longrightarrow H^{k-1}(M;\mathrm{U}(1)) \longrightarrow \hat{H}^k(M;\mathbb{Z}) \xrightarrow{\mathrm{curv}} \Omega^k_{\mathrm{int}}(M) \longrightarrow 0. \tag{6.104}$$

[10]Some authors omit the exponential function by working modulo $\mathbb{Z}$. This just replaces the multiplicative $\mathrm{U}(1)$-group by the isomorphic additive $\mathbb{R}/\mathbb{Z}$-group.

The first cohomology group classifies flat circle bundles by Property 6.5.13, so this sequence says that, by extending the above example to higher n-bundles (this can be formalized cf. *bundle gerbes*), two circle n-bundles with the same curvature differ by a flat circle $(n-1)$-bundle.

Property 6.7.7 (Characteristic class exact sequence).

$$0 \longrightarrow \Omega^{k-1}(M)/\Omega^{k-1}_{\text{int}}(M) \longrightarrow \hat{H}^k(M;\mathbb{Z}) \xrightarrow{\text{ch}} H^k(M;\mathbb{Z}) \longrightarrow 0 \tag{6.105}$$

The first map is induced by the holonomy functional

$$\iota : \Omega^{k-1}(M) \to \hat{H}^k(M;\mathbb{Z}) : \omega \to \exp\left(2\pi i \int_{-} \omega\right), \tag{6.106}$$

which has the closed integral forms as kernel. This exact sequence says that two connections on the same principal U(1)-bundle differ by a global connection form (up to an integral form).

6.7.2 Combining singular and de Rham cohomology

There is an alternative to the Cheeger-Simons approach. Let C^n and Z^n again denote the smooth cochain and cocycle groups.

Definition 6.7.8 (Differential cocycle). A tuple $(c,h,\omega) \in C^n(M;\mathbb{Z}) \times C^{n-1}(M) \times \Omega^n(M)$ such that

$$\delta c = 0 \tag{6.107}$$

$$\mathrm{d}\omega = 0 \tag{6.108}$$

$$\delta h = \omega - c. \tag{6.109}$$

A differential cocycle thus consists of a singular cocycle (topological information) and a de Rham cocycle (differential information), that are equal up to a (singular) coboundary.

The cochain complex $C^n(M;\mathbb{Z}) \times C^{n-1}(M) \times \Omega^n(M)$ with differential

$$\mathrm{d} : (c,h,\omega) \mapsto (\delta c, \omega - c - \delta h, \mathrm{d}\omega) \tag{6.110}$$

defines a cohomology theory $\hat{H}(n)^\bullet(M)$.

Property 6.7.9 (Relation to differential characters). Differential characters and differential cocycles are related as follows:

$$\hat{H}^k(M) \cong \hat{H}(k)^k(M). \tag{6.111}$$

Given a differential cocycle (c,h,ω), the curvature and characteristic class of the associated differential character are ω and c, respectively. The function $e^{2\pi i h}$ is called the **monodromy** of the cocycle. It can be checked that Equation (6.101) is exactly the third relation in the definition of cocycles above. The mod $\mathbb{Z}$-reduction of h gives the differential character associated to the cocycle.

Example 6.7.10. The first ordinary differential cohomology group $\hat{H}^1(M;\mathbb{Z})$ is isomorphic to the group of smooth functions $C^\infty(M;\mathrm{U}(1))$.

6.7.3 Deligne cohomology

The following theorem states that the differential characters are essentially the unique objects with these properties and that they define a generalized cohomology theory:

Theorem 6.7.11 (Simons-Sullivan). *There is an essentially unique functor*

$$\hat{H}^\bullet(-;\mathbb{Z}) : \mathbf{Diff} \to \mathbf{Ab}^{\mathbb{Z}}$$

such that there exist four natural transformations

1. **Flat class**: $j : H^{\bullet-1}(-;\mathrm{U}(1)) \to \hat{H}^\bullet(-;\mathbb{Z})$,
2. **Topological trivialization**: $\iota : \Omega^{\bullet-1}/\Omega^{\bullet-1}_{\mathrm{int}} \to \hat{H}^\bullet(-;\mathbb{Z})$,
3. **Characteristic class**: $\mathrm{ch} : \hat{H}^\bullet(-;\mathbb{Z}) \to H^\bullet(-;\mathbb{Z})$, *and*
4. **Curvature**: $\mathrm{curv} : \hat{H}^\bullet(-;\mathbb{Z}) \to \Omega^\bullet_{\mathrm{int}}$

that fit in the following commutative diagram, where the diagonal sequences are exact:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{\bullet-1}(-;\mathrm{U}(1)) \arrow[rr, "\text{Bockstein}"] \arrow[dr, "j"] & & H^\bullet(-;\mathbb{Z}) \arrow[ur] \arrow[dr, "\text{de Rham}"] & \\
H^{\bullet-1}_{\mathrm{dR}} \arrow[ur] \arrow[dr, two heads] & & \hat{H}^\bullet(-;\mathbb{Z}) \arrow[ur, "\mathrm{ch}"] \arrow[dr, "\mathrm{curv}"] & & H^\bullet_{\mathrm{dR}} \\
& \Omega^{\bullet-1}/\,\Omega^{\bullet-1}_{\mathrm{int}} \arrow[ur, "\iota"] \arrow[rr, "\mathrm{d}"'] & & \Omega^\bullet_{\mathrm{int}} \arrow[ur, hook] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

Functors satisfying the above properties are said to define **ordinary differential cohomology theories**.

Another approach to differential cohomology is given by the Deligne complex.

Definition 6.7.12 (Deligne complex). Let $\mathbf{B}^k\mathrm{U}(1)_{\mathrm{conn}}$ denote the cochain complex

$$\mathcal{O}^\times_M \xrightarrow{\mathrm{d}\log} \Omega^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^k. \tag{6.112}$$

(Smooth) Deligne cohomology is defined as follows:

$$H^{k+1}_D(M;\mathbb{Z}) := \check{H}^0(M;\mathbf{B}^k\mathrm{U}(1)_{\mathrm{conn}}), \tag{6.113}$$

where $\check{H}^\bullet$ denotes Čech cohomology **??** and the cochain complex $\mathbf{B}^k\mathrm{U}(1)_{\mathrm{conn}}$ is turned into a cochain complex by inverting the degrees.

Property 6.7.13 (Deligne-Beilinson product). Consider the Deligne complex for two integers $k,l\in\mathbb{N}$. There exists a cup product

$$\cup : \mathbf{B}^k\mathrm{U}(1)_{\mathrm{conn}} \otimes \mathbf{B}^l\mathrm{U}(1)_{\mathrm{conn}} \to \mathbf{B}^{k+l+1}\mathrm{U}(1)_{\mathrm{conn}} : x\otimes y \mapsto x\cup y := \begin{cases} x\wedge \mathrm{d}y & \deg(y)=l \\ 0 & \text{otherwise.}\end{cases} \tag{6.114}$$

Example 6.7.14 (Circle bundles). A (Čech-)Deligne cocycle in degree 2 consists of data (A_i, g_{ij}) such that

$$\begin{tikzcd}
A_i \arrow[r, hook] & A_i - A_j = \mathrm{d}\log g_{ij} = g_{ij}^{-1}\mathrm{d}g_{ij} \\
& g_{ij} \arrow[u, "\mathrm{d}\log"] \arrow[r, hook] & g_{jk}g_{ki}^{-1}g_{ij} = 1,
\end{tikzcd}$$

where the inclusion arrows denote the restriction to intersections $U_{ij} := U_i \cap U_j$. Property 6.3.27 and the subsequent example, specialized in the case of U(1)-bundles, show that the above data are exactly the components of a princpal circle bundle with connection.

Remark 6.7.15. As was the case for differential characters, higher Deligne cohomology classes classify higher U(1)-bundles with connection. The main benefit of this approach is that one gets an ‘‘explicit’’ description of the local data. See [25] for a good introduction.

Remark 6.7.16 (Trivial bundles and twisted bundles). From Čech-Deligne cohomology, one knows that a trivial k-bundle α is defined by a $(k-1)$-cochain β such that

$$(\delta\beta)_{i_0\ldots i_k} = \alpha_{i_0\ldots i_k}\,, \tag{6.115}$$

i.e. a trivial k-bundle is equivalent to a twisted $(k-1)$-bundle.

?? COMPLETE ??

6.8 Cartan connections

In the first part of this section a short overview of Klein's **Erlangen program** that unifies (and generalizes) Euclidean and non-Euclidean geometries will be given. In the second part of this section Cartan's generalization in terms of bundles is explained. A reference for this section is [26].

6.8.1 Klein geometry

Definition 6.8.1 (Klein geometry). Consider a Lie group G together with a closed subgroup H. If it is connected, the orbit space G/H is called a Klein geometry and G is called the **principal group**. If the principal group is also connected, the Klein geometry is said to be **geometrically oriented**.

If the associated Lie algebras are denoted by $\mathfrak{g}, \mathfrak{h}$ respectively, the pair $(\mathfrak{g}, \mathfrak{h})$ is called a **Klein pair**. In fact, any pair $(\mathfrak{g}, \mathfrak{h} \leq \mathfrak{g})$ can be called a Klein pair.

Property 6.8.2. It is clear that every Klein geometry gives a homogeneous space and, hence, a principal bundle of rank $\dim(G) - \dim(H)$.

Example 6.8.3 (Euclidean space). Consider the Euclidean group $\mathrm{Euc}(n) := \mathbb{R}^n \rtimes \mathrm{O}(n)$, i.e. the symmetry group of the Euclidean space $\mathbb{R}^n$. This group clearly acts transitively and the subgroup $\mathrm{O}(n)$ can be seen to leave the origin fixed. This implies that $\mathbb{R}^n$ is a homogenous space and even a Klein geometry of the form $\mathrm{Euc}(n)/\mathrm{O}(n)$.

Definition 6.8.4 (Effective Klein pair). The action of G on G/H is not necessarily effective, i.e. the kernel

$$\ker(\rho) = \{x \in G \mid \forall g \in G : g^{-1}xg \in H\}\,, \tag{6.116}$$

is not necessarily trivial. If it is, the Klein geometry is said to be effective. In terms of the associated Klein pair this means that $\mathfrak{h}$ contains no nontrivial ideals of $\mathfrak{g}$. A Klein geometry is said to be locally effective if the kernel is discrete.

Definition 6.8.5 (Reductive Klein pair). A Klein pair $(\mathfrak{g}, \mathfrak{h})$ for which $\mathfrak{g}$ admits a decomposition of the form

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\,, \tag{6.117}$$

where $\mathfrak{m}$ is an $\mathfrak{h}$-module.

Definition 6.8.6 (Model geometry). A model geometry consists of the following data:

1. an effective Klein pair $(\mathfrak{g}, \mathfrak{h})$,
2. a Lie group H such that $\mathrm{Lie}(H) = \mathfrak{h}$, and
3. a representation $\mathrm{Ad} : H \to \mathrm{Aut}(\mathfrak{g})$ that restricts to the adjoint representation $\mathrm{Ad}_H : H \to \mathrm{Aut}(\mathfrak{h})$.

Definition 6.8.7 (Local Klein geometry). A local Klein geometry consists of the following data:

1. a Lie group G,
2. a closed subgroup $H \subset G$, and
3. a subgroup $\Gamma \subset G$ acting by covering transformations on G/H such that the left coset space $\Gamma\backslash G/H$ is connected.

6.8.2 Cartan geometry

The definition of a Klein geometry can be rephrased in the language of bundle theory. First, an alternative characterization of Lie groups in terms of the Maurer-Cartan connection is given:

Alternative Definition 6.8.8 (Lie group). Let G be a smooth manifold and let $\mathfrak{g}$ be a Lie algebra. G is a Lie group, with Lie algebra $\mathfrak{g}$, if it comes equipped with a $\mathfrak{g}$-valued one-form ω satisfying the following conditions:

1. **Maurer-Cartan equation**: $\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0$,
2. **Soldering**: ω restricts to an isomorphism on every fibre, and
3. **Completeness**: ω is complete, i.e. every vector field that maps constantly to $\mathfrak{g}$ is complete.

?? FIX/COMPLETE THIS PROPERTY ??

In a similar way Klein geometries can be characterized as follows:

Property 6.8.9. The bundle $\pi : G \to G/H$ of a Klein geometry G/H admits a one-form $\omega : TG \to \mathfrak{g}$ that satisfies the following conditions:

- ω satisfies the Maurer-Cartan equation.
- ω restricts to an isomorphism on every fibre.
- ω is complete.
- ω is H-equivariant: $R_h^*\omega = \mathrm{Ad}(h^{-1})\omega$.
- ω cancels $\mathfrak{h}$-fundamental vector fields: $\omega(A^\#) = A$ for all $A \in \mathfrak{h}$.

The last two conditions show that ω defines a principal connection one-form, while the first condition states that this connection is flat. In fact this one-form is exactly the Maurer-Cartan form on G, where the last conditions are obtained by restricting to the subgroup $H \subset G$.

By dropping the flatness and completeness conditions, one obtains the notion of Cartan geometries:

Definition 6.8.10 (Cartan geometry). Consider a principal H-bundle $\pi : P \to M$ and a Lie algebra $\mathfrak{g}$ such that $\mathfrak{h} \leq \mathfrak{g}$ (in general it is assumed that these form a model geometry). A Cartan geometry is characterized by a one-form $\omega : TP \to \mathfrak{g}$ satisfying the following conditions:

1. ω restricts to an isomorphism on every fibre.
2. ω is H-equivariant.
3. ω cancels $\mathfrak{h}$-fundamental vector fields.

The one-form ω is called the Cartan connection.

Definition 6.8.11 (Curvature). By analogy with the Maurer-Cartan condition and the Cartan structure equation 6.4.22, the curvature of a Cartan connection is defined as follows:

$$\Omega := \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega]. \tag{6.118}$$

By restricting to reductive model spaces an important decomposition of the Cartan connection is obtained:

Property 6.8.12. Consider a Cartan geometry $\pi : P \to M$ with a reductive model space $(\mathfrak{g}, \mathfrak{h})$ such that the Cartan connection can be decomposed as $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{m}}$. This decomposition has the following important properties:

- The form $\omega_{\mathfrak{h}}$ defines a principal connection on the Cartan geometry P.
- The form $\omega_{\mathfrak{m}}$ defines a solder form on M.
- The decomposition of the associated curvature form Ω gives the curvature and torsion of the induced principal connection and solder forms respectively.

Furthermore, the Cartan geometry $\pi : P \to M$ gives a reduction of the frame bundle FM induced by the solder form $\omega_{\mathfrak{m}}$.

?? COMPLETE ??

6.9 $\mathcal{D}$-geometry ♣

Definition 6.9.1 (Sheaf of differential operators). Let X be a smooth manifold (or variety, see Chapter 1) and denote its structure sheaf by $\mathcal{O}_X$. $\mathcal{D}_X$ denotes the sheaf of $\mathcal{O}_X$-algebras of vector fields (as derivations) on X:

$$\mathcal{D}_X(U) := \{v \in \mathrm{End}(\mathcal{O}_X(U)) \mid v(fg) = v(f)g + fv(g)\}. \tag{6.119}$$

Definition 6.9.2 ($\mathcal{D}_X$-module). An $\mathcal{O}_X$-module equipped with an action of $\mathcal{D}_X$. This is equivalent to a linear map

$$\nabla : \mathcal{D}_X \to \mathrm{End}(\mathcal{O}_X) \tag{6.120}$$

satisfying the following properties:

1. **$\mathcal{O}_X$-linearity**: $\nabla_{fv}\sigma = f\nabla_v\sigma$,
2. **Leibniz rule**: $\nabla_v(f\sigma) = v(f)\sigma + f\nabla_v\sigma$, and
3. **Flatness**: $\nabla_{[v,w]}\sigma = [\nabla_v, \nabla_w]\sigma$.

If X is locally free **??**, i.e. corresponds to a locally trivial bundle, one obtains the algebraic reformulation of a vector bundle with flat connection.

Chapter 7

Complex Geometry

7.1 Complex structures

Definition 7.1.1 (Almost complex structure). Let M be a smooth manifold. An almost complex structure on M is a (complexified) smooth $(1,1)$-tensor field $J : TM \to TM$ such that $J|_p : T_pM \to T_pM$ satisfies $J|_p^2 = -1$ for all $p \in M$. Such a structure allows to treat the tangent spaces as complex vector spaces:

$$(a + ib)X := aX + bJX. \tag{7.1}$$

A general vector bundle equipped with such a tensor field is called a **complex vector bundle**. The underlying (real) vector bundle of a complex vector bundle E is often denoted by $E_{\mathbb{R}}$.

This definition implies the following property:

Property 7.1.2. An almost complex manifold is even-dimensional and orientable.

An almost complex structure induces a decomposition of the tangent bundle in so-called holomorphic and antiholomorphic components:

$$TM^{\mathbb{C}} = TM^+ \oplus TM^-,$$

where both bundles have the same dimension (and are isomorphic as real vector bundles to TM). When the coordinates on M are denoted by $\{x^k\}_{k\leq 2n}$, bases for these two subbundles are given by

$$\left\{\frac{\partial}{\partial z^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^{2k-1}} - i\frac{\partial}{\partial x^{2k}}\right)\right\}_{k\leq n}$$

and

$$\left\{\frac{\partial}{\partial \bar{z}^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^{2k-1}} + i\frac{\partial}{\partial x^{2k}}\right)\right\}_{k\leq n},$$

respectively.

Remark 7.1.3. The reason that the almost complex structure is defined on the complexified tangent bundle has to do with the fact that J is only diagonalizable on a complex vector space (because it squares to a negative value).

Example 7.1.4 (Complex vector spaces). Consider a complex vector space V. By looking at Property **??** and using the canonical isomorphism $V \cong T_vV$ for vector spaces, one can see that the automorphism $v \mapsto iv$ induced by the imaginary unit gives rise to an almost complex structure on V.

Property 7.1.5 (Reduction of structure group). A $2m$-dimensional manifold M admits an almost complex structure if and only if the structure group of the tangent bundle TM can be reduced from $\mathrm{GL}(\mathbb{R}^{2n})$ to $\mathrm{GL}(\mathbb{C}^n)$.

Moreover, the set of of almost complex structures on $\mathbb{C}^n$ is given by the homogeneous space $\mathrm{GL}(\mathbb{R}^{2n})/\mathrm{GL}(\mathbb{C}^n)$. By globalizing this one obtains that the set of almost complex structures on a complex vector bundle (E, J_0) is given by $\mathrm{Aut}(E_\mathbb{R})/\mathrm{Aut}(E, J_0)$.

Definition 7.1.6 (Complex dimension). The integer n in this property is called the complex dimension of M. It is denoted by $\dim_\mathbb{C}(M)$.

Definition 7.1.7 (Complex manifold). A topological space M for which there exists an open cover $\{U_i\}_i$ such that for every U_i there exists a homeomorphism $\varphi_i : U_i \to \mathbb{C}^n$ onto some open subset of $\mathbb{C}^n$. The transition functions $\varphi_{ji} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are also required to be holomorphic.

Property 7.1.8. An almost complex manifold is complex if and only if the $\mathrm{GL}(\mathbb{C}^n)$-structure is integrable.

The integrability condition can be rephrased algebraically as follows:

Theorem 7.1.9 (Newlander-Nirenberg). *An almost complex manifold is complex if and only if the* ***Nijenhuis tensor*** N_J *vanishes:*

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0 \tag{7.2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. *Locally this can be written as*

$$J^\nu_\rho \partial_\nu J^\mu_\sigma - J^\nu_\sigma \partial_\nu J^\mu_\rho - J^\mu_\nu \partial_\rho J^\nu_\sigma + J^\mu_\nu \partial_\sigma J^\mu_\rho = 0. \tag{7.3}$$

Theorem 7.1.10 (Oka coherence theorem). *The structure sheaf, the sheaf of holomorphic functions, of a complex manifold is coherent* **??**.

Property 7.1.11 (Characteristic classes). Let E be a complex vector bundle and denote its real underlying bundle by $E_\mathbb{R}$.

$$c_1(E) \bmod 2 = w_2(E_\mathbb{R}) \tag{7.4}$$

Definition 7.1.12 (Stein manifold). A complex manifold M satisfying the following two conditions where $\mathcal{O}(M)$ denotes the ring of holomorphic functions on M:

1. **Holomorphic convexity**: For every compact subset $K \subset M$ the **holomorphic convex hull**

   $$\overline{K} := \left\{z \in M \,\middle|\, \forall f \in \mathcal{O}(M) : |f(z)| \leq \sup_{z' \in K} |f(z')|\right\} \tag{7.5}$$

   is compact.
2. **Holomorphic separability**: For every two distinct points $z, z' \in M$, there exist a function $f \in \mathcal{O}(M)$ such that $f(z) \neq f(z')$.

Alternative Definition 7.1.13. A Stein manifold of (complex) dimension $n \in \mathbb{N}$ is a complex manifold admitting a proper holomorphic immersion into $\mathbb{C}^n$.

Property 7.1.14. Stein manifold are noncompact.

The following type of cover resembles that of Definition 3.1.16 and Property 3.1.17:

Property 7.1.15 (Stein cover). Every complex manifold admits a cover by Stein manifolds such that any finite intersection of covering sets is again Stein.

Property 7.1.16 (Oka-Grauert principle). The classification of holomorphic and topological vector bundles over a Stein manifold coincide or, more generally, all cohomological problems on Stein manifolds only have topological obstructions.

Remark 7.1.17 (Oka manifolds). This principle has a more general homotopical/model-theoretic (Chapter **??**) formulation. There exists a class of complex manifolds, the Oka manifolds, such that the inclusion

$$\mathrm{Map}_{\mathrm{hol}}(A,S) \hookrightarrow \mathrm{Map}(A,S) \tag{7.6}$$

of holomorphic functions into continuous functions is a weak homotopy equivalence for all Oka manifolds A and Stein manifolds S.

Theorem 7.1.18 (Cartan's theorem A). *Every coherent sheaf* **??** *on a Stein manifold is spanned by its global sections.*

Theorem 7.1.19 (Cartan's theorem B). *The sheaf cohomology of any coherent sheaf over a Stein manifold vanishes in positive degree:*

$$H^{\geq 1}(M;\mathcal{F}) = 0 \tag{7.7}$$

for all $\mathcal{F} \in \mathrm{Coh}(M)$.

Definition 7.1.20 (Metalinear structure). Consider the determinant morphism

$$\det : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}^\times.$$

The metalinear group can be considered as the domain of the holomorphic square root of det:

$$\mathrm{ML}(n,\mathbb{C}) := \left\{(A,z) \in \mathrm{GL}(n,\mathbb{C}) \times \mathbb{C}^\times \,\middle|\, \det(A) = z^2\right\}. \tag{7.8}$$

An equivalent definition, which will be used in the remainder of the text, makes use of the special linear group:

$$\mathrm{ML}(n,\mathbb{C}) = \frac{\mathrm{SL}(n,\mathbb{C}) \times \mathbb{C}}{2\mathbb{Z}}, \tag{7.9}$$

where $\mathbb{Z}$ acts on the product group as $k : (A,z) \mapsto (e^{-2\pi ik/n}A, z + 2\pi ik/n)$. This group is the double cover of $\mathrm{GL}(n,\mathbb{C})$.

Similar to the definition of spinor and metaplectic structures (Definitions **??** and **??**), one can also define metalinear structures on a manifold. The metalinear frame bundle is a lift of the (complex) frame bundle along the canonical morphism $\mathrm{ML}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$ such that it "commutes" with the bundle map $F_{\mathrm{ML}}M \to FM$.

Property 7.1.21 (Existence). A smooth manifold M admits a metalinear structure if and only if its first Stiefel-Whitney class $w_1 \in H^1(M;\mathbb{Z}_2)$ squares to 0. In particular, every orientable manifold admits a metalinear structure. The set of nonequivalent metalinear structures is parametrized by $H^1(M;\mathbb{Z}_2)$.

Remark 7.1.22. The above definitions can be restricted to real manifolds and real metalinear structures.

Definition 7.1.23 (Half-form). Consider a smooth manifold M equipped with a metalinear frame bundle $F_{\mathrm{ML}}M$. The bundle of half-forms $\Omega^{1/2}(M)$ is defined as the associated $\mathbb{C}$-line bundle constructed from the action $(g,\lambda) \mapsto z\lambda$, where $g \equiv (A,z) \in \mathrm{ML}(n,\mathbb{C})$.

This can be seen as a (holomorphic) square root of the determinant line bundle. Consider the bundle of 1-densities $|\Omega^1|(M)$ from Definition 5.8.10. There exists a map $\Omega^{1/2}(M) \otimes \Omega^{1/2}(M) \rightarrow |\Omega|^1(M)$ defined by sending the pair (μ,ν) to the (tensor) product $\mu\overline{\nu}$ along the covering map $F_{\mathrm{ML}}M \rightarrow FM$. If one does not use the conjugation, a section of the ordinary n-form bundle $\Omega^n(M)$ is obtained.

Property 7.1.24 (Metaplectic structure). Let (M,ω) be a symplectic manifold and consider a Lagrangian subbundle $L \subset TM$. The tangent bundle TM admits a metaplectic structure if and only if L admits a metalinear structure.

7.2 Complex differential forms

Property 7.2.1. On a complex manifold there exist coordinates $\{z^\mu\}_{\mu\leq n}$ such that the almost complex structure J can be written as (when extended to $TM^{\mathbb{C}}$)

$$J = i\partial_\mu \otimes \mathrm{d}z^\mu - i\partial_{\overline{\mu}} \otimes \mathrm{d}\overline{z}^\mu. \tag{7.10}$$

This coordinate expression can be used to find a coordinate transformation from the real coordinates $\{x^\mu\}_{\mu\leq 2n}$ to the complex coordinates $\{z^\mu, \overline{z}^\mu\}_{\mu\leq n}$.

Remark 7.2.2. Note that on the complexified tangent bundle their exist two kind of imaginary units: i and J. The differential forms dz and $\mathrm{d}\overline{z}$ are both linear with respect to the scalar i, but only $\mathrm{d}z$ is linear with respect to J, i.e. $\mathrm{d}z \circ J = \mathrm{d}z$ and $\mathrm{d}\overline{z} \circ J = -\mathrm{d}\overline{z}$.

Using the basis forms $\mathrm{d}z^\mu, \mathrm{d}\overline{z}^\mu$ one can also define complex Grassmann algebras $\Omega^{p,q}(M)$, analoguous to $\Omega^k(X)$ for smooth manifolds:

$$\Omega^{1,0}(M) := \mathrm{span}_{C^\infty(M,\mathbb{C})}\{\mathrm{d}z^\mu\} \tag{7.11}$$

$$\Omega^{0,1}(M) := \mathrm{span}_{C^\infty(M,\mathbb{C})}\{\mathrm{d}\overline{z}^\mu\} \tag{7.12}$$

$$\Omega^{p,q}(M) := \left(\bigwedge_{i=1}^{p} \Omega^{1,0}(M)\right) \wedge \left(\bigwedge_{j=1}^{q} \Omega^{0,1}(M)\right). \tag{7.13}$$

Property 7.2.3. The spaces $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$ are stable, i.e. they transform tensorially, under holomorphic coordinate transformations. On the space

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

of forms of total degree k one can then define the canonical projection maps $\pi^{p,q} : \Omega^k \rightarrow \Omega^{p,q}$.

Definition 7.2.4 (Dolbeault operator). Consider a general $(p+q)$-form $\omega \in \Omega^{p,q}(M)$. The de Rham differential maps this form to a $(p+q+1)$-form. This form is in general an element of $\sum_{r+s=p+q+1} \Omega^{r,s}(M)$. Using the projection maps $\pi^{p,q}$ one can define two additional differential operators:

$$\partial := \pi^{p+1,q} \circ \mathrm{d}\,, \tag{7.14}$$

$$\overline{\partial} := \pi^{p,q+1} \circ \mathrm{d}. \tag{7.15}$$

The latter is called the Dolbeault operator. A form is said to be **holomorphic** if it satisfies $\overline{\partial}\omega = 0$, in analogy with the classical Cauchy-Riemann condition (**??**).

Property 7.2.5. By explicitly writing out the action of the de Rham differential d on a general (p,q)-form one obtains the following decomposition:

$$\mathrm{d} = \partial + \overline{\partial}. \tag{7.16}$$

Note that for an almost complex manifold this relation in general does not hold. An almost complex manifold is integrable if and only if this expression holds. By using the coboundary property of d one also obtains

$$\partial^2 = \overline{\partial}^2 = 0\,, \tag{7.17}$$

$$\partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{7.18}$$

Remark 7.2.6 (Integrability). It can be shown that J is integrable, i.e. the almost complex structure is complex, if and only if the induced Dolbeault operator $\overline{\partial}$ squares to zero.

More generally, a complex vector bundle E is said to be **holomorphic** if it admits a trivialization by holomorphic transition functions or, equivalently, if it admits a Dolbeault operator $\overline{\partial}$: $\Omega^{\bullet,\bullet}(M;E) \to \Omega^{\bullet,\bullet+1}(M;E)$ that squares to zero. Note that, in contrast to the case of TM, a holomorphic vector bundle only admits the natural definition of a $\overline{\partial}$-operator. To have a ∂-operator, one should consider antiholomorphic vector bundles.

Theorem 7.2.7 (Koszul-Malgrange). *Let $E \to M$ be a holomorphic vector bundle. There exists a unique connection ∇ on E such that $\nabla^{0,1} = \overline{\partial}$.*

Formula 7.2.8. Analogous to the definition of the de Rham codifferential (**??**), one can define the adjoints of the Dolbeault operators:

$$\partial^\dagger := - * \partial * \tag{7.19}$$

$$\overline{\partial}^\dagger := - * \overline{\partial} *\,, \tag{7.20}$$

where the fact that the real dimension of a complex manifold is even is used: $(-1)^{n(k+1)+1} = -1$.

Corollary 7.2.9. Using these definitions one can write the Hodge Laplacian **??** as:

$$\Delta = 2(\partial\partial^\dagger + \partial^\dagger\partial) = 2(\overline{\partial}\overline{\partial}^\dagger + \overline{\partial}^\dagger\overline{\partial}). \tag{7.21}$$

7.3 Kähler manifolds

In analogy with the definition of Riemannian manifolds **??** one can also define metrics for complex vector bundles:

Definition 7.3.1 (Hermitian manifold). A complex vector bundle equipped with a Hermitian bundle metric. A connection that is compatible with this metric is called a Hermitian connection.

Definition 7.3.2 (Kähler manifold). Consider a smooth manifold M equipped with a Riemannian structure g, a symplectic structure ω and an almost complex structure J. This manifold is called a Kähler manifold if the structures satisfy any of the following equivalent sets of compatibility conditions:

1. The almost complex structure J is integrable[1], and

[1] If not, the manifold is said to be almost Kähler.

2. The symplectic form is **compatible** with the almost complex structure:

$$\omega(v,w) = \omega(Jv,Jw) \tag{7.22}$$

and

$$\omega(v,Jv) > 0; \tag{7.23}$$

or

1. M is Hermitian with metric $h(v,w) := g(v,w) + ig(v,Jw)$, and
2. The two-form $\omega(v,w) := g(v,Jw)$ is closed and, hence, symplectic[2];

or

1. M is Hermitian with metric $h(v,w) := g(v,w) + ig(v,Jw)$, and
2. J is parallel with respect to the Levi-Civita connection on (M,g):

$$\nabla_X J = 0. \tag{7.24}$$

Remark 7.3.3. The property that says that J acts isometrically can be interchanged for the statement that J acts as a symplectomorphism. These two statements are equivalent for a Kähler manifold.

Property 7.3.4 (Tame structures). The compatibility conditions between a symplectic form ω and an almost complex structure J can be weakened to only be $\omega(v,Jv) > 0$. In this case J is said to be ω**-tame**.

The set of all almost complex structures that are tamed by a given symplectic form (on a finite-rank vector bundle) is nonempty and contractible. This also holds for the stronger compatibility condition. The converse also holds, i.e. given an almost complex structure, the set of all symplectic forms that tame it (or that are compatible with it) is nonempty and convex (hence also contractible).

Remark 7.3.5. Note that every ω-tame almost complex structure J also induces a Riemannian metric after symmetrization. When the tame structure is also compatible, this symmetrized metric coincides with $\omega(\cdot, J\cdot)$.

Definition 7.3.6 (Kähler form). When any of the equivalent sets of conditions in Definition 7.3.2 is satisified, the central object is the Kähler form or **fundamental form**:

$$\omega(v,w) := g(v,Jw). \tag{7.25}$$

Because it is closed, it determines a cohomology class $[\omega] \in H^2_{\text{dR}}(M;\mathbb{R})$. This class is called the **Kähler class** of M.

Formula 7.3.7. The metric $g \equiv g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ can be rewritten as

$$g = g_{\mu\overline{\nu}}\big(\mathrm{d}z^\mu \otimes \mathrm{d}\overline{z}^\nu + \mathrm{d}\overline{z}^\nu \otimes \mathrm{d}z^\mu\big). \tag{7.26}$$

The Kähler form can then be written as

$$\omega = ig_{\mu\overline{\nu}}\mathrm{d}z^\mu \wedge \mathrm{d}\overline{z}^\nu. \tag{7.27}$$

[2]The nondegeneracy condition is automatically satisfied because of the nondegeneracy of the metric.

Definition 7.3.8 (Kähler potential). Using the $\partial\overline{\partial}$-lemma 7.5.4 one can locally write the Kähler form as

$$\omega = i\partial\overline{\partial}K(z,\overline{z})\,, \tag{7.28}$$

where the function $K : M \to \mathbb{R}$ is called the **Kähler potential**. Expression (7.27) implies that one can locally rewrite the metric as

$$g_{\mu\overline{\nu}} = \partial_\mu\partial_{\overline{\nu}}K(z,\overline{z}). \tag{7.29}$$

Property 7.3.9. The Christoffel symbols associated to the Levi-Civita connection on (M,g) admit a simple expression when M is Kähler: only the ${\Gamma_{\mu\nu}}^\lambda$ and ${\Gamma_{\overline{\mu\nu}}}^{\overline{\lambda}}$ components do not vanish. They are given by

$${\Gamma_{\mu\nu}}^\lambda = g^{\lambda\overline{\rho}}\partial_\mu g_{\nu\overline{\rho}}\,, \tag{7.30}$$

$${\Gamma_{\overline{\mu\nu}}}^{\overline{\lambda}} = g^{\overline{\lambda}\rho}\partial_{\overline{\mu}} g_{\overline{\nu}\rho}. \tag{7.31}$$

Accordingly, the only nonvanishing component of the Riemann curvature tensor is

$$R_{\overline{\mu}\nu\overline{\lambda}\rho} = g_{\overline{\lambda}\kappa}\partial_{\overline{\mu}}{\Gamma_{\nu\rho}}^\kappa. \tag{7.32}$$

Definition 7.3.10 (Kähler transformation). From Definition 7.3.8 one can conclude that the Kähler potential is not unambiguously defined. The following transformation leaves the Kähler form invariant:

$$K'(z,\overline{z}) = K(z,\overline{z}) + f(z) + \overline{f(\overline{z})}. \tag{7.33}$$

On overlapping coordinate charts the transformation between Kähler potentials is exactly of this form.

Definition 7.3.11 (Calabi-Yau manifold). A Kähler manifold with trivial canonical bundle $\Omega^{n,0}(M)$. Equivalently, a $2n$-dimensional Riemannian manifold with special holonomy group in $\mathrm{SU}(n)$.[3] This implies, in particular, that the manifold is Ricci flat.

Property 7.3.12 (Calabi-Yau conjecture). For a compact Calabi-Yau manifold, the first Chern class vanishes.

Definition 7.3.13 (Hypercomplex manifold). A smooth manifold equipped with three distinct complex structures $I, J, K : TM^{\mathbb{C}} \to TM^{\mathbb{C}}$ that satisfy the quaternion algebra relation:

$$I \circ J = K. \tag{7.34}$$

Definition 7.3.14 (Hyperkähler manifold). A hypercomplex manifold that admits a (Riemannian) metric that is Kähler with respect to all complex structures.

7.3.1 Killing vectors

Definition 7.3.15 (Holomorphic Killing vector). Consider the set of Killing vector fields associated to the metric g. Within this set of vector fields one can consider those k_A that satisfy

$$\mathcal{L}_{k_A}J = 0. \tag{7.35}$$

or, equivalently by the Kähler condition,

$$\mathcal{L}_{k_A}\omega = 0. \tag{7.36}$$

These are called holomorphic Killing vector fields because their components are locally holomorphic. This can easily be shown by writing the Killing condition in terms of covariant derivatives and by using Equation (7.10).

[3] The original definition by *Yau* was that of a compact, Ricci-flat Kähler manifold with vanishing first Chern class.

Definition 7.3.16 (Moment map). Let k be a holomorphic Killing vector field. From $\mathrm{d}\omega = 0$ one can, using Cartan's magic formula 5.4.17 and the above condition, derive that $\iota_k\omega$ is closed. Poincaré's lemma further implies that there exists a real function $\mathcal{P}(z, \overline{z})$ such that

$$\iota_k\omega = \mathrm{d}\mathcal{P}. \tag{7.37}$$

Using Equation (7.27) one can then find the following expression for the Killing vector fields:

$$k^\mu = -ig^{\mu\overline{\nu}}\partial_{\overline{\nu}}\mathcal{P}. \tag{7.38}$$

7.3.2 Dirac operators

Property 7.3.17. Let M be a compact Kähler (or Hermitian) manifold. M admits a Spin-structure if and only if there exists a square root of its (holomorphic) canonical line bundle by 7.1.11.

Property 7.3.18 (Spin$^{\mathbb{C}}$-structures). By Properties **??** and 7.1.11, every almost complex manifold admits a Spin$^{\mathbb{C}}$-structure. If there exists a line bundle L, such that $c_1(L) = w_2(M)$ mod 2, then M admits a Spin$^{\mathbb{C}}$-structure with L as its determinant line bundle.

Definition 7.3.19 (Kähler-Dirac operator). Consider a Riemannian manifold (M, g). The Kähler-Dirac operator is a square root of the Hodge Laplacian **??**:

$$\mathrm{D} := \mathrm{d} + \delta. \tag{7.39}$$

Definition 7.3.20 (Dolbeault-Dirac operator). Let M be a Kähler manifold equipped with a Spin-structure. By the first property above, this implies that the canonical line bundle admits a square root $\sqrt{\Omega^{n,0}}$. It can be shown that the spin bundle on M satisfies

$$S \cong \Omega^{0,n}(M) \otimes \sqrt{\Omega^{n,0}} \tag{7.40}$$

and that the Dirac operator can be identified with $\overline{\partial} + \overline{\partial}^*$, where $\overline{\partial}$ is the Dolbeault operator 7.2.4. For the action on the θ-characteristic, choose a connection ∇:

$$\overline{\partial}(\omega \otimes \psi) := (\overline{\partial}\omega) \otimes \psi + \sum_{i=1}^{2n} \pi^{0,k+1}(\mathrm{d}x^i \wedge \omega) \otimes \nabla_i\psi \tag{7.41}$$

$$\overline{\partial}^*(\omega \otimes \psi) := (\overline{\partial}^*\omega) \otimes \psi - \sum_{i=1}^{2n} (\partial_i \lrcorner\, \omega) \otimes \nabla_i\psi\,, \tag{7.42}$$

where $\omega \in \Omega^{0,k}(M)$.

7.4 Complex curves

7.4.1 Riemann surfaces and orbifolds

Definition 7.4.1 (Riemann surface). A complex manifold of (complex) dimension one. It is often assumed to be connected.

Property 7.4.2 (Genus classification). Compact Riemann surfaces are, topologically, characterized by their genus.

Example 7.4.3 (Riemann sphere). The sphere S^2 admits the structure of a Riemann surface: $\mathbb{CP}^1$. The automorphism group of $\mathbb{CP}^1$ is the modular group $\mathrm{PSL}(2, \mathbb{C})$ from Definition 1.5.1. It can be proven that, given any two triples of points, there exists a unique Möbius transformation mapping them onto each other.

Theorem 7.4.4 (Uniformization theorem). *Every simply-connected Riemann surface is biholomorphically equivalent to either the Riemann sphere, the complex plane or the upper half plane.*

Corollary 7.4.5. Because the universal cover of a Riemann surface is again a Riemann surface, the uniformization theorem implies that every Riemann surface can be obtained as the quotient of $\mathbb{CP}^1, \mathbb{C}$ or $\mathcal{H}$ by a freely-acting discrete group.

Moreover, $\mathbb{CP}^1$ only covers itself and $\mathbb{C}$ only admits $\mathbb{Z}$ or a discrete lattice as freely-acting discrete automorphism groups (leading to $\mathbb{C}$, $\mathbb{R} \times S^1 \cong \mathbb{C}\backslash\{0\}$ and $\mathbb{T}^2$ as quotients). All other Riemann surfaces are obtained from the halfplane $\mathcal{H}$.

Definition 7.4.6 (Stable Riemann surface). A Riemann surface Σ of genus g with n marked points is said to be stable if its punctured Euler characteristic

$$\chi(\Sigma\backslash\{z_1, \ldots, z_n\}) = 2 - 2g - n \tag{7.43}$$

is negative.

Property 7.4.7 (Automorphism group). For a stable Riemann surface (Σ, J), every element of $\mathrm{Aut}_n(\Sigma, J)$ that is not the identity is also not homotopic to the identity. In particular, (Σ, J) has a finite automorphism group. For nonstable Riemann surfaces, the automorphism group is always a smooth Lie group.

Construction 7.4.8 (Moduli space). Denote by $\mathcal{M}_{g,n}$ the set of isomorphism classes of Riemann surfaces of genus g with n marked points and let Σ be an oriented, closed surface with n marked points. $\mathcal{M}_n(\Sigma)$ denotes the moduli space of almost complex structures up to automorphisms that preserves the order of the marked points:

$$\mathcal{M}_n(\Sigma) := \mathcal{J}(\Sigma)/\mathrm{Diff}_{+,n}(\Sigma). \tag{7.44}$$

In general, the action of diffeomorphisms is not free or proper, since $\mathrm{Aut}_n(\Sigma, J)$ fixes J, so the quotient is not necessarily a smooth manifold. By restricting the action to the identity-component and the surfaces to stable surfaces, one obtains the **Teichmüller space** $\mathcal{T}(\Sigma, n)$. $\mathcal{M}_n(\Sigma)$ can then be obtained by further quotienting out the action of the mapping class group **??**.

Even when restricting to stable surfaces, the moduli space does not have the structure of a smooth manifold due to the existence of the marked points. To accurately describe the geometrical structure one needs to generalize the notion of a manifold (Chapter 3):

Definition 7.4.9 (Orbifold). Let X be a topological space. An orbifold chart on X is a tuple $(U, G, \varphi : U \to V/G)$ such that $U \subset M$ is open, $V \subset \mathbb{R}^n$ is open and connected, G is a finite group and φ is a homeomorphism.

A **subchart** $(U', G', \varphi' : U' \to V'/G')$ of $(U, G, \varphi : U \to V/G)$ is a triple such that there exist inclusions $U' \subset U$, $V' \subset V$ and a homomorphism $G' \to G$ that all commute in the obvious way with the additional property that the stabilizer of every point in U' is preserved. This property implies that the stabilizer of any point X can be uniquely defined as the stabilizer of any of its preimages.

Two orbifold charts are said to be **compatible** if their intersection is contained in a subchart of both charts. An **orbifold atlas** is defined as a cover of X by compatible orbifold charts.

Definition 7.4.10 (Morphism of orbifolds). A general definition of orbifold morphisms is quite technical. Here a specific situation is considered, that where the fibres over the orbifold are manifolds themselves.

A morphism of orbifolds "with manifold fibres" is a continuous function $f : X \to Y$ with for every point $y \in Y$ a choice of orbifold chart $\varphi_y : U_y \to V_y/G$ (containing y), a smooth intertwiner $F : V_x \to V_y$, and an isomorphism of V_x/G with a suborbifold of X such that the following equation is satisfied:

$$\varphi_y^{-1} \circ F = f \circ \varphi_x^{-1}. \tag{7.45}$$

Most notions of differential geometry carry over to the orbifold setting quite naturally. For example, a differential form on a chart $(U, G, \varphi : U \to V/G)$ is defined as a G-invariant differential form on V and integration is defined by averaging over the preimage of a chain:

$$\int_C \omega := \frac{1}{|G|} \int_{\varphi^{-1}(C)} \omega_U \,, \tag{7.46}$$

where ω_U is the orbifold representative of ω for the chart $(U, G, \varphi : U \to V/G)$ containing C. A vector bundle on an orbifold consists of an ordinary vector bundle with a fibrewise lift of the G-action.

Definition 7.4.11 (Euler characteristic). Let X be a G-orbifold. The Euler characteristic of X is defined as the average of the Euler characteristics of its fixed-point spaces:

$$\chi(X) := \frac{1}{|G|} \sum_{g \in G} \chi(X^g). \tag{7.47}$$

Now that orbifolds have been introduced, the structure of $\mathcal{M}_{g,n}$ can be analyzed. When $2 - 2g - n < 0$, the moduli space has the structure of a complex orbifold of (complex) dimension $3g - 3 + n$ and the stabilizer at a point is equal to the automorphism group of a representative of that equivalence class. To endow the moduli space with such a structure, one can pull back a "universal curve".

Definition 7.4.12 (Family of Riemann surfaces). A family of Riemann surfaces of genus g with n marked points is a function $p : C \to U$ that admits n disjoint sections $s_i : U \to C$ and for which the fibre over every point is a Riemann surface. The intersections of the fibres with the sections are exactly the marked points.

?? CHECK THIS DEFINITION ??

Given two such families p, p' and a subset $V \subset U'$, the restriction $p'|_V$ is called a **pullback** of p if there exists a function $\varphi : V \to U$ such that $C'|_V \cong \varphi^* C$.

Property 7.4.13 (Universal curve over $\mathcal{M}_{g,n}$). Let C be Riemann surface of genus g with n marked points (with negative Euler characteristic). Denote its (finite) automorphism group by G. There exist

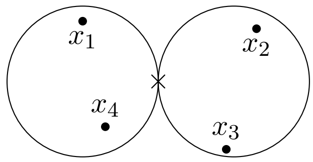
- an open, bounded, connected set $U \subset \mathbb{C}^{3g-3+n}$,
- a family $p : \mathcal{C} \to U$ of Riemann surfaces of genus g with n marked points, and
- a G-action on $\mathcal{C}$ descending to a G-action on U

such that the following conditions are satisfied:

1. The fiber $C_0 := p^{-1}(0)$ is (isomorphic to) C.
2. The G-action preserves C_0 and acts as its automorphism group.
3. For any family $p' : \mathcal{C}' \to U'$ of Riemann surfaces of genus g with n marked points such that $p'^{-1}(w) \cong C$ for some $w \in U'$, there exists an open subset $b \in V \subset U'$ and a map $\varphi : V \to U$, unique up to the G-action, such that $p'|_V = \varphi^* p$.

When one takes all sets U/G corresponding to the Riemann surfaces of genus g with n marked points, one gets an orbifold covering. The corresponding orbifold is the moduli space $\mathcal{M}_{g,n}$. The sets $\mathcal{C}$ give an orbifold covering of an orbifold $\mathcal{C}_{g,n}$. Moreover, there exists an orbifold morphism $\pi : \mathcal{C}_{g,n} \rightarrow \mathcal{M}_{g,n}$, called the universal curve over $\mathcal{M}_{g,n}$. The fibres of the universal curve are exactly the Riemann surfaces, each only appearing once.

Although a proper geometric structure has been placed on the moduli space, a further problem remains. In general, this space is not compact, which might make it difficult to handle for some purposes. Consider for example the case $g = 0, n = 4$. By Example 7.4.3 one can uniquely map three marked points to for example the points 0, 1 and ∞. The fourth point can, however, still be mapped freely to any point $c \in \mathbb{CP}^1\backslash\{0, 1, \infty\}$. (This number is the **modulus** that gives rise to the term "moduli space".) For $c \rightarrow 0$, the marked points x_1 and x_4 would coincide. However, after a coordinate transformation $x \longrightarrow x/c$, the marked points x_2 and x_3 would coincide. To remove this ambiguity one should replace $\mathbb{CP}^1$ by two copies of $\mathbb{CP}^1$ that intersect (transversally) at a single point:



From the point of view of x_2 and x_3 (the original coordinate chart), the marked points x_1 and x_4 have collapsed, while from the point of view of the latter (after the transformation $x \longrightarrow x/c$) the former have collapsed. This "bubbling" is what will generally happen at singular points. Curves that have this kind of singularity are called "stable curves":

Definition 7.4.14 (Stable curve). A compact, complex algebraic curve with n marked points that satisfies the following conditions:

1. It has only simple **nodal singularities** (i.e. singularities of the form $xy = 0$).
2. The marked points are distinct and do not coincide with the nodes.
3. The curve has a finite number of automorphisms.

One can **smoothen** a stable curve by replacing the neighbourhood of every node that consists of two disks by a cylinder. A stable curve can also be **normalized** by replacing these intersecting disks by disjoint disks.

Remark 7.4.15. The last condition above is equivalent to requiring that every connected component of the normalization has negative Euler characteristic, i.e. has itself a finite number of automorphisms.

Property 7.4.16 (Deligne-Mumford compactification). There exists a morphism of compact, complex orbifolds $\overline{\pi} : \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ such that:

- $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is a suborbifold with preimage $\mathcal{C}_{g,n}$ under $\overline{\pi}$,
- the fibres of $\overline{\pi}$ are stable curves of genus g with n marked points,
- each stable curve is isomorphic to a unique fibre, and
- the stabilizer of a point in $\overline{\mathcal{M}}_{g,n}$ is isomorphic to the automorphism group of the corresponding stable curve.

7.4.2 Pseudoholomorphic maps

Definition 7.4.17 (Cauchy-Riemann operator). Consider a complex vector bundle (E, J) over an almost complex manifold (M, j). A Cauchy-Riemann operator on E is a complex-linear map

$$\mathrm{D} : \Gamma(E) \to \Gamma(\overline{\mathrm{Hom}}(TM, E)) \tag{7.48}$$

satisfying the Leibniz rule

$$\mathrm{D}(f\sigma) = (\overline{\partial} f)\sigma + f(\mathrm{D}\sigma)\,, \tag{7.49}$$

where $\overline{\mathrm{Hom}}$ denotes the bundle of antilinear morphisms, i.e. those vector bundle morphisms that anticommute with the almost complex structures.

This strongly resembles the definition of a Koszul connection 5.7.1. The following property shows that this is no coincidence:

Property 7.4.18. For every Cauchy-Riemann operator D on a Hermitian vector bundle (E, J) over an almost complex manifold (M, j) there exists a Hermitian connection ∇ such that

$$\mathrm{D}\sigma = \nabla\sigma + J \circ \nabla\sigma \circ j \tag{7.50}$$

for all sections $\sigma \in \Gamma(E)$.

Definition 7.4.19 (Pseudoholomorphic function). Consider two almost complex manifolds (M, J) and (N, j). A function $f : (M, J) \to (N, j)$ is said to be pseudoholomorphic if it satisfies the **Cauchy-Riemann equations**

$$f_* + j \circ f_* \circ J = 0 \tag{7.51}$$

or, equivalently,

$$j \circ f_* = f_* \circ J\,, \tag{7.52}$$

i.e. the differential is complex-linear. This can rephrased in terms of a Cauchy-Riemann-like operator

$$\overline{\partial}_J : C^\infty(M, N) \to \Gamma(\overline{\mathrm{Hom}}(TM, TM)) : f \mapsto f_* + j \circ f_* \circ J\,, \tag{7.53}$$

The kernel of this operator consists of exactly the pseudoholomorphic functions. Note that the image of f under the above nonlinear Cauchy-Riemann operator is actually an element of $\Gamma(\overline{\mathrm{Hom}}(TM, f^*TN))$, since the tangent space at a point $f \in C^\infty(M, N)$ is modelled on $\Gamma(f^*TN)$.[4] By Remark 5.4.23 this operator can also be written in terms of the differential $\mathrm{d}f$.

Remark 7.4.20 (Pseudoholomorphic curves). In practice one often restricts to pseudoholomorphic curves, i.e. pseudoholomorphic functions where the domain is a Riemann surface. On one hand in two dimensions one does not lose generality by only considering complex manifolds, since the integrability condition $\overline{\partial}^2 = 0$ is always satisfied. On the other hand, it is better to restrict to two-dimensional manifolds in the domain, because in general there are no nonconstant pseudoholomorphic functions (even locally) when the domain has a higher dimension.

[4] To make this statement precise one needs to endow $C^\infty(M, N)$ with the right topology, since this manifold is infinite-dimensional.

Property 7.4.21 (Regularity). If a function of class C^1 satisfies the Cauchy-Riemann equations, it is automatically smooth.

Definition 7.4.22 (Pseudoholomorphic polygon). For every polygon in $\mathbb{C}$, i.e. a complex disk with a finite number of marked points, the Riemann mapping theorem gives a biholomorphism to the interior of the disk D. The *Schwarz-Christoffel* formula gives an expression for this map, which can even be extended to the n-punctured disk D_n. A pseudoholomorphic n-gon in an almost complex manifold M is a pseudoholomorphic map $f : D_n \to M$ such that $\lim_{w\to w_i} f(w) = q_i$, where q_i is a vertex of the polygon and w_i is a puncture of D_n.

Property 7.4.23 (Symplectic submanifolds). Let (M, ω) be a symplectic manifold and consider an ω-tame almost complex structure J. Every (complex) line in a tangent space is also a symplectic subspace. Globally, this means that every J-holomorphic curve corresponds to a symplectic submanifold.

Definition 7.4.24 (Energy). Consider a symplectic manifold (M, ω) equipped with an ω-tame almost complex structure J. By symmetrizing the form $\omega(\cdot, J\cdot)$ one obtains a (Riemannian) metric g. The energy of a J-holomorphic curve $f : (\Sigma, J_0) \to (M, J)$ is defined as follows:

$$\mathrm{E}(f) := \int_\Sigma f^*\omega = \frac{1}{2}\int_\Sigma \|\mathrm{d}f\|^2\mathrm{Vol}_\Sigma. \tag{7.54}$$

This quantity is always nonnegative and is zero if and only if f is locally constant. (Note that for the second expression one needs the induced metric on both Σ and M.)

Construction 7.4.25 (Moduli space). Choose two integers $g, n \geq 1$ and a homology class $A \in H_2(M)$. The moduli space $\mathcal{M}^A_{g,n}$ is defined as the set of equivalence classes of tuples $(\Sigma, J_0, f, z_1, \ldots, z_n)$, where (Σ, J_0) is a Riemann surface of genus g, $f : (\Sigma, J_0) \to (M, J)$ is a pseudoholomorphic curve such that $f_*[\Sigma] = A$ and $\{z_i\}_{i\leq n}$ are marked points of Σ. Two such tuples are deemed equivalent if there exists an automorphism (i.e. a biholomorphic or conformal diffeomorphism) that preserves the order of the marked points.

Example 7.4.26. When $M = \{*\}$, the moduli space reduces to $\mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus g with n marked points.

7.5 Cohomology

7.5.1 Dolbeault cohomology

Theorem 7.5.1 (Hodge decomposition). *Let M be a compact Kähler manifold.*

$$H^k_{dR}(M) \cong \bigoplus_{p+q=k} H^{p,q}(M) \tag{7.55}$$

for all $k \in \mathbb{N}$

By analogy with the Poincaré lemma for smooth manifolds one can prove the following theorems:

Theorem 7.5.2 (∂-lemma). *Let $\alpha \in \Omega^{p,q}(M)$. If $\partial\alpha = 0$, locally there exists a complex form $\beta \in \Omega^{p-1,q}$ such that $\alpha = \partial\beta$.*

Theorem 7.5.3 ($\overline{\partial}$-lemma). *Let $\alpha \in \Omega^{p,q}(M)$. If $\overline{\partial}\alpha = 0$, locally there exists a complex form $\beta \in \Omega^{p,q-1}$ such that $\alpha = \overline{\partial}\beta$.*

Theorem 7.5.4 ($\partial\overline{\partial}$-lemma). *Let $\alpha \in \Omega^{p,q}(M)$. If $\mathrm{d}\alpha = 0$, locally there exists a complex form $\beta \in \Omega^{p-1,q-1}$ such that $\alpha = \partial\overline{\partial}\beta$.*

7.5.2 Lagrangian Floer homology ♣

In this section a (co)homology theory is constructed from the intersection theory of Lagrangian submanifolds (see Chapter **??** for an introduction).

Recall the pseudoholomorphic polygons from Section 7.4.2. In the study of symplectic manifolds, one often has a set of boundary conditions $\{L_i, q_i\}_{i\leq n}$, where the L_i are Lagrangian submanifolds and the $q_i \in L_i \cap L_{i+1 \bmod n}$ are intersection points. A pseudoholomorphic n-gon satisfies these boundary conditions if the edges of the domain are mapped to the submanifolds L_i and the marked points to the intersection points q_i.

Property 7.5.5 (Moduli space). To fully appreciate Floer theory, one needs to study the moduli space of pseudoholomorphic polygons. One of the most important parts being compactness and the *Gromov compactification*. For certain one-parameter families of polygons, the limit is a degenerate configuration that is not part of the moduli space itself. Consider for example the situation in Figure 7.1. The Lagrangian submanifolds are indicated by black lines (infinite straight lines are Lagrangian submanifolds of the complex plane). The interior of the polygon is sketched by a red line and the marked points are indicated by red dots.

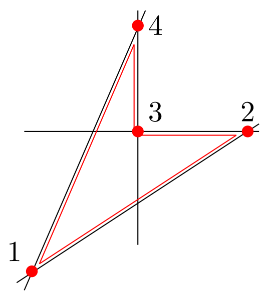


Figure 7.1: Pseudoholomorphic 4-gon.

Now, one can make a one-parameter family of 4-gons that satisfies the same boundary conditions by, instead of going directly from vertex 3 to vertex 4, first going a bit down and then going up again, i.e. making a ‘‘slit’’. This still satisfies all the properties to apply the Riemann mapping theorem, but gives a different polygon map. At a certain point a new vertex is created on the lower Lagrangian, a so-called **nodal** vertex, and the 4-gon is broken up into two 3-gons. From the domain point of view what happens is that two boundary punctures have collided and a new disk (with two punctures) has been attached. Adding such configurations to the moduli space to account for these limit operations gives rise to a *Deligne-Mumford compactification*. Because finding all possible ways to arrange $n+1$ points on different attached disks (without altering their order) is equivalent to finding all possible parenthesations of $n+1$ symbols, one finds that the compactified moduli space of boundary-punctured disks $\overline{\mathcal{M}}_{n+1}$ is isomorphic to the n^{th} Stasheff polytope K_n.

In the case of bigons (for simplicity), three types of degeneracies can occur:

- **Strip breaking**: Here, energy concentrates at one of the marked points. There exists a sequence of bigons $(f_n)_{n\in\mathbb{N}}$, here viewed as homotopies, and a diverging sequence of numbers $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n(s-a_n, t)$ is not the constant strip. This corresponds to the situation in Figure 7.1.
- **Disk bubbling**: Here, energy concentrates at a point on the boundary. There exists a sequence of bigons that can be rescaled such that the limit is a pseudoholomorphic disk entirely contained in one of the boundary Lagrangians.
- **Sphere bubbling**: Here energy concentrates at a point in the interior of a bigon. There

exists a sequence of bigons that can be rescaled such that the limit is a pseudoholomorphic sphere entirely contained in the interior of M.

The latter two issues can be avoided by imposing topological restrictions such as by restricting to *montone Lagrangian submanifolds* or by requiring that $[\omega]\cdot\pi_2(M,L)=0=[\omega]\cdot\pi_2(M,L')=0$.

Definition 7.5.6 (Floer complex). Let M be a symplectic manifold equipped. For every two transversally intersecting Lagrangian submanifolds, the chain group is defined as the free vector space generated by the intersection points: $CF(L,L'):=\Lambda^{L\cap L'}$, where, to avoid technical difficulties, one should take the field Λ to be a *Novikov field.*

The differential is defined by counting pseudoholomorphic bigons:

$$\partial\langle p\rangle := \sum_{q\in L\cap L'\setminus\{p\}} n(p,q)\langle q\rangle\,, \tag{7.56}$$

where $n(p,q)$ denotes the number of (isomorphism classes of[5]) pseudoholomorphic bigons with boundary conditions $\{L,L',p,q\}$ of finite symplectic energy.

To be more precise, denote by $\mathcal{M}(p,q;[f])$ the moduli space of pseudoholomorphic bigons of finite symplectic energy in a fixed homotopy class $[f]\in\pi_2(M,L\cap L')$. The Maslov index of such a class can be defined using the spectral flow approach:

$$\mu([f]) := \mathrm{ind}(\mathrm{D}_{\overline{\partial}_J})\,, \tag{7.57}$$

where $\mathrm{D}_{\overline{\partial}_j}$ denotes the linearization of the Cauchy-Riemann operator associated to J (it can be shown that this operator is Fredholm and, hence, admits a well-defined index). When all bigons are regular, i.e. when the linearized Cauchy-Riemann operator is surjective everywhere, the moduli space has dimension $\mu([f])-1$. Moreover, the *Gromov compactness theorem* (see the property above) states that this manifold is compact.

Putting a well-defined grading on $CF(L,L')$ is a bit more subtle. Let $\mathrm{LGr}(TM)$ denote the Lagrangian Grassmann bundle over M. The Maslov index of a bigon should only depend on the difference in degrees of its marked points and not on its homotopy class. This is implemented as follows (only a $\mathbb{Z}$-grading is considered here).

Definition 7.5.7 (Maslov covering). A **Maslov covering** of M is a $\mathbb{Z}$-covering $\mathcal{L}\rightarrow\mathrm{LGr}(TM)$ such that the fibre over every Lagrangian is given by the $\mathbb{Z}$-covering of $\mathrm{LGr}(\mathbb{R}^{\dim(M)})$. This corresponds to patching together the universal covers of the Lagrangian Grassmannians at every point of M.

A Maslov covering exists if and only if the first Chern class is 2-torsion:

$$2c_1(M)=0. \tag{7.58}$$

(This can be restated in cohomological conditions involving the Chern class [27]: $c_1(M)$ should be 2-torsion and $\mu\in H^1$ should vanish for both L and L'. This allows to lift a Lagrangian submanifold to a graded Lagrangian submanifold, a section of the universal cover of the Lagrangian Grassmann bundle.)

$$\partial\langle p\rangle = \sum_{\substack{q\in L\cap L'\\ \mathrm{ind}([f])=1}} \big|\mathcal{M}(p,q;[f])\big|\,T^{\omega([f])}\langle q\rangle\,, \tag{7.59}$$

where T is the generator of the *Novikov field* Λ and $\omega([f])$ denotes the symplectic energy of $[f]$.

[5]The domain, i.e. the disk with two boundary punctures, has as automorphism group $\mathbb{R}$. It scales the distance between the punctures.

Property 7.5.8 (Hamiltonian isotopy). Consider two Lagrangian submanifolds L, L'. If there exists a Hamiltonian isotopy $L' \to L''$, then

$$HF(L, L') \cong HF(L, L''). \tag{7.60}$$

If there exists a Hamiltonian isotopy $L \to L'$, then

$$HF(L, L') \cong H^\bullet(L)\,, \tag{7.61}$$

where the left-hand side denotes singular cohomology.

Definition 7.5.9 (Symplectic action functional). For every two Lagrangian submanifolds $L, L' \subset M$, one can consider the space of smooth paths connecting L and L':

$$\mathcal{P}(L, L') := \{\gamma \in C^1([0,1], M) \mid \gamma(0) \in L, \gamma(1) \in L'\}. \tag{7.62}$$

The symplectic action functional is defined as follows:

$$A : \widetilde{\mathcal{P}}(L, L') \to \mathbb{R} : [\gamma, h] \mapsto \int_{[0,1]\times[0,1]} h^*\omega\,, \tag{7.63}$$

where ω is the symplectic form on M and $\widetilde{\mathcal{P}}$ denotes the universal cover (of a connected component) of $\mathcal{P}$, i.e. the set of equivalence classes of pairs (γ, h) where h is a homotopy between γ and a fixed point in the connected component.[6] If the symplectic form is trivialisable on h, Stokes's theorem implies that this integral is (up to a constant depending on the choice of base point) equal to

$$\int_{[0,1]} \gamma^*\theta\,, \tag{7.64}$$

where θ is the symplectic potential.

One can also calculate the differential of the action functional or, by introducing a (family of) compatible almost complex structure(s), the gradient of the action functional:

$$\mathrm{grad}A([\gamma, h]) = J_t\frac{\partial\gamma}{\partial t}. \tag{7.65}$$

It follows that the critical points of the functional correspond to constant paths, i.e. intersection points of the Lagrangian submanifolds. Moreover, the flow equation of this gradient is exactly the Cauchy-Riemann equation of J-holomorphic curves with the given Lagrangian boundary conditions. In (finite-dimensional) Morse homology (Section 3.5) one counts flow lines of the Hamiltonian flow, so it follows that one can interpret Lagrangian Floer homology as the infinite-dimensional analogue of Morse homology for the symplectic action functional.

Definition 7.5.10 (Fukaya category). Consider a symplectic manifold (M, ω). The associated Fukaya category consists of the following data:

1. **Objects**: Lagrangian submanifolds of M.
2. **Morphisms**: When $L \pitchfork L'$, $\mathrm{Hom}(L, L') := CF(L, L')$, the (Lagrangian) Floer chain group.

This definition can be generalized to obtain an A_∞-category. Whenever the Lagrangian submanifolds intersect transversally, a multiplication map

$$\mu : \mathrm{Hom}(L', L'') \otimes \mathrm{Hom}(L, L') \to \mathrm{Hom}(L, L'') \tag{7.66}$$

can be defined. Consider intersection points $q_1 \in L \cap L', q_2 \in L' \cap L''$ and $q \in L \cap L''$. The coefficient of q in $\mu(q_2, q_1)$ is obtained by counting pseudo-holomorphic 3-gons with boundary on the Lagrangians and marked points q_1, q_2 and q.

[6] To be correct one should use the so-called *Novikov covering*.

Remark 7.5.11 (Bubbles). Aside from breaking and splitting of holomorphic polygons, another situation where degenerate polygons arise is the so-called bubbling phenomenon, where a holomorphic sphere (0-gon) pops up. Unless the symplectic manifold and the boundary Lagrangian are exact, these give rise to a "vacuum constant" in the sense that the A_∞-structure is modified to a curved A_∞-structure with $m_0 \neq 0$.

?? COMPLETE ??

Chapter 8

Higher-dimensional Geometry ♣

In this chapter certain constructions and theorems introduced in the previous chapters are generalized to the setting of higher categories and supergeometries. As such it can been as an analogue to Chapter **??** for (differential) geometry.

The main references are [28, 29]. The section on spectral geometry is based on the thesis [30], while that on smooth spaces is inspired by [31]. For an introduction to (higher) category theory, see Chapter **??**. Section **??** gives a different approach to the higher-dimensional analogues of Lie algebras.

?? CITE BAEZ, SCHREIBER, BARTELS, ... ??

8.1 Infinite-dimensional geometry

In many situations, when consider function spaces, the objects under consideration do not form a finite-dimensional manifold. However, with some care, one can drop this size condition. In Chapter **??** it was shown how one can extend calculus from $\mathbb{R}^n$ to infinite-dimensional vector spaces.

The first approach uses a locally convex TVS **??** as local model space:

Definition 8.1.1 (E-manifold). Let E be a locally convex TVS. A Hausdorff space is called an E-manifold if there exists an atlas of charts (U, φ), where $\varphi : U \to \varphi(U) \subset E$ is a homeomorphism and the transition maps are Gateaux-smooth.

Definition 8.1.2 (Kinematic tangent bundle). Let M be an E-manifold with a smooth atlas $\{(U_i, \varphi)\}_{i\in I}$. The kinematic tangent bundle of M is defined as the quotient of

$$\bigsqcup_{i\in I} U_i \times E \tag{8.1}$$

by the equivalence relations $(x, w) \sim \big(x, \mathrm{d}\psi_{ji}(\varphi_i(x); v)\big)$.

Remark 8.1.3. For infinite-dimensional E, this tangent bundle is not isomorphic to the definition in terms of derivations. The above construction is "kinematical" because the pair (x, v) represent a vector tangent to a curve at the point $x \in M$.

Since infinite-dimensional vector spaces are in general not reflexive, simply defining the cotangent bundle to be the fibrewise dual of the kinematic tangent bundle would lead to even more size issues. *Kriegl* and *Michor* have shown that one can cook up to 12 sensible definitions of a cotangent bundle (this also includes "operational" definitions using derivations). However, only one of these definitions is well-behaved with respect to Lie derivatives, exterior derivatives and pullbacks. Luckily this is also the most widely used definition in the finite-dimensional setting:

Definition 8.1.4 (Kinematical cotangent bundle). Let M be an E-manifold. Consider the set of bounded, alternating linear maps $E^{\times k} \to \mathbb{R}$. This lifts to a vector bundle $L^k_{\text{alt}}(TM, M \times \mathbb{R})$.

8.2 Smooth spaces

In this section some generalizations of spaces that are better behaved when considering their properties as a whole are introduced. Before moving to the smooth setting, a bit of history will be given, starting from the ordinary topological setting.

The first problem in the study of the global properties of spaces arose in algebraic topology. When consider mapping spaces it is sometimes useful to use the currying operation

$$C(X \times Y, Z) \to C\big(X, C(Y, Z)\big).$$

However, in general, this is not a homeomorphism, i.e. currying does not define an adjunction and, therefore, **Top** is not Cartesian closed **??**. This problem was treated by *Steenrod* and others, and the solution was simply to restrict to a smaller class of better behaved spaces: the compactly generated Hausdorff spaces.[1]

Whilst studying varieties in algebraic geometry people experienced similar problems. For this reason *Grothendieck* invented schemes (see Chapter 1 and Section 1.2 in particular). The main takeaway of this approach was that it is often better to work with a well-behaved category containing some ‘‘nasty’’ objects, than to work with a ‘‘nasty’’ category containing only nice objects.

The category **Diff** of finite-dimensional smooth manifolds suffers the same problems, namely the space of smooth functions $C^\infty(X, Y)$ is in general some kind of infinite-dimensional manifold and, hence, cannot be defined internally. It becomes even worse if one studies the mapping spaces between those. *Kriegl* and *Michor* have introduced a framework in which one can work safely, but the main problem with their solution is that not all spaces of interest are included. Certain other operations such as quotients and (co)limits are also not guaranteed to exist within that category.

8.2.1 Concrete sites

Definition 8.2.1 (Diffeological space). Let X be a set. A diffeology $\mathcal{D}$ on X is defined as a collection of functions $f : U \subseteq \mathbb{R}^n \to X$, called **plots**, satisfying the following conditions (where U, V and W are open sets):

1. $\mathcal{D}$ contains all functions $f : \mathbb{R}^0 \to X$ and all constant functions.
2. If $\{U_i\}_{i\in I}$ is an open cover of U and if $f|_{U_i} \in \mathcal{D}$ for all $i \in I$, then $f \in \mathcal{D}$.
3. If $f \in \mathcal{D}$ and $g : W \subseteq \mathbb{R}^m \to \text{dom}(f)$ is smooth, then $f \circ g \in \mathcal{D}$.

The set X can be turned into a topological space by equipping it with the **$\mathcal{D}$-topology**, the final topology with respect to $\mathcal{D}$.

Remark 8.2.2. Note that in contrast to ordinary manifolds, the plots in a diffeology can have domains of different dimensions.

[1] This is in general not a problem since all interesting spaces, such as CW complexes, belong to this class.

Definition 8.2.3 (Smooth map). Let $(X, \mathcal{D})$ and $(Y, \mathcal{D}')$ be diffeological spaces. A map $g : X \to Y$ is said to be smooth if for every $f \in \mathcal{D}$ the composite $g \circ f \in \mathcal{D}'$. The diffeological spaces together with their differentiable morphisms form a category **DiffSp**.

Definition 8.2.4 (Chen space). If the open sets in the definition of a diffeological space are replaced by convex sets, the notion of smooth spaces due to *Chen* are obtained.

Alternative Definition 8.2.5 (Manifold). A diffeological space is called an n-manifold if it is locally diffeomorphic to a Euclidean space. A map between manifolds is smooth in the diffeological sense if and only if it smooth in the sense of Definition 3.1.12.

There exist two trivial smooth structures:

Example 8.2.6 (Discrete structure). The smooth structure defined by taking the plots to be the constant functions.

Example 8.2.7 (Indiscrete structure). The smooth structure obtained by taking all functions to be plots.

Definition 8.2.8 (Smooth set). By omitting the reference to an underlying set in the definition of smooth spaces above, a more general definition can be obtained. This way the category **SmoothSet** is obtained as the sheaf category on the site of Cartesian spaces $\mathbf{Sh}(\mathbf{CartSp}_{\mathbf{diff}})$. The topology on this site is generated by the coverage of differentiably good covers 3.1.16 (in fact, this topology coincides with the usual one consisting of open covers). Diffeological spaces can be recovered by passing to the full subcategory on *concrete sheafs*. The category of smooth spaces/sets is often denoted by $\mathbf{C}^\infty$.

?? ADD INFORMATION ON CONCRETE SHEAFS ??

Example 8.2.9 (Differential forms). Consider the k^{th} de Rham functor Ω^k on the category **Diff**. This functor assigns to every smooth manifold its space of differential k-forms (Section 5.4). Locally defined forms can be glued together if they agree on intersections and, hence, they satisfy the sheaf condition. This shows that Ω^k is a smooth space, albeit one that is far from an ordinary smooth manifold.

One can go even further. Consider the subfunctor Ω^2_{cl} that assigns closed two-forms to a smooth manifold. This also defines a smooth space and, hence, one can consider the slice category $\mathbf{C}^\infty/\Omega^2_{\text{cl}}$. It is not hard to show that the category **SpMfd** of symplectic manifolds admits an embedding into this slice category.

Property 8.2.10. There exists an adjunction

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift right, "diff"'] \arrow[r, phantom, "\perp"] & \mathbf{C}^\infty \arrow[l, shift right, "top"'] \end{tikzcd}. \tag{8.2}$$

The functor *diff* endows a topological space X with the smooth structure for which every continuous map $U \to X$ is a plot. The adjoint functor *top* sends a smooth space to the topological space equipped with the finest topology for which all plots become continuous maps.

Definition 8.2.11 (Smooth algebra). For any smooth manifold M the algebra of smooth functions can be obtained as a hom-object:

$$C^\infty(M) \equiv C^\infty(M, \mathbb{R}) = \mathbf{Diff}(M, \mathbb{R}).$$

Since hom-functors are (finite) product-preserving, one can see that the multiplication $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is induced by the multiplication on $\mathbb{R}$:

$$C^\infty(M, \mathbb{R} \times \mathbb{R}) \cong C^\infty(M) \times C^\infty(M).$$

Furthermore, the hom-functor is covariant in the second argument and, hence, a copresheaf on the category $\mathbf{CartSp_{diff}}$ of Euclidean (Cartesian) spaces and smooth morphisms is obtained. Generalizing this situation, smooth algebras are defined as finite product-preserving copresheaves on $\mathbf{CartSp_{diff}}$. This (functor) category is denoted by $\mathbf{C^\infty Alg}$.

Definition 8.2.12 (Underlying algebra). Given a smooth algebra $R \in \mathbf{C^\infty Alg}$, its underlying algebra $U(R)$ is defined as the set $R(\mathbb{R})$ equipped with the canonically induced ring operations.

Definition 8.2.13 (Finitely generated smooth algebra). Since ordinary R-algebras are finitely generated if and only if they are of the form $R[x_1, \dots, x_k]/I$ for some integer $k \in \mathbb{N}$ and some ideal I, a smooth algebra is said to be finitely generated if it is of the form $C^\infty(\mathbb{R}^n)/I$ for some $n \in \mathbb{N}$ and some ideal I in the underlying algebra.

Definition 8.2.14 (Smooth locus). Let $\mathbf{C^\infty Alg^{fin}}$ denote the category of finitely generated smooth algebras. The category of **smooth loci** is defined as $(\mathbf{C^\infty Alg^{fin}})^{op}$. The smooth locus corresponding to a smooth algebra R is often denoted by ℓR.

8.2.2 Supergeometry

In this section the definition of smooth spaces (and sets) is generalized to the odd (fermionic) sector, i.e. "super smooth sets" will be defined.

Definition 8.2.15 (Superscheme). The category of affine superschemes is defined as the opposite of the category of supercommutative superalgebras, i.e. the commutative monoids internal to **sVect**. More generally, one can define affine schemes internal to any symmetric monoidal category.

Property 8.2.16. The category **CAlg** is a full subcategory of **sCAlg**. Moreover, this inclusion is part of an adjoint triple, where the left and right adjoints are given by projection onto the even part and quotienting by the odd part.

Definition 8.2.17 (Infinitesimally thickened space). First, consider a point $\mathbb{R}^0$. Its infinitesimal thickening should be a space such that every function that vanishes at the origin is actually nilpotent (this is essentially a version of the Kock-Lawvere axiom **??**). The straightforward definition is the following one:

$$\mathbb{D} := \mathrm{Spec}(A)\,, \tag{8.3}$$

where $A := \mathbb{R} \oplus V$ for V a finite-dimensional nilpotent ideal. A Euclidean space can be infinitesimally thickened by taking the product with $\mathbb{D}$ (or at the algebraic level by taking the tensor product with A). A morphism of such spaces is defined by an R-algebra homomorphism between their assciated algebras. These form the category $\mathbf{FormalCartSp_{diff}}$.

Example 8.2.18 (First-order neighbourhood). By taking $A = \mathbb{R}[\varepsilon]/\varepsilon^2$ one exactly obtains the first-order infinitesimal neighbourhood of Definition **??**. The morphism dual to the mapping implied by the Kock-Lawvere axiom **??** gives an inclusion map $\mathbb{D}^1 \hookrightarrow \mathbb{R}^1$. (This example can easily be generalized to k^{th}-order neighbourhoods.)

Property 8.2.19 (Morphisms). First, consider the morphisms from a Euclidean space into an infinitesimal neighbourhood $\mathbb{D}^k$. Since such morphisms are dual to algebra homomorphisms, one should look at morphisms of the form $\mathbb{R}[\varepsilon]/\varepsilon^{k+1} \to C^\infty(\mathbb{R}^n)$. However, being an algebra homomorphism implies that $f(1) = 1$ and that nilpotents are mapped to nilpotents. The algebra of smooth functions on a Euclidean space does not contain nilpotents and, hence, there exists a unique function into an infinitesimal neighbourhood (the one that factorizes through the one-point set).

For morphisms out of (first-order) infinitesimal neighbourhoods one obtains the property known from synthetic geometry that morphisms of the form $\mathbb{R}^n \times \mathbb{D}^1 \to \mathbb{R}^n$ are in bijection with vector fields on $\mathbb{R}^n$.

Definition 8.2.20 (Formal smooth set). A sheaf on the site of infinitesimally thickened Euclidean spaces (covers are of the form $\{U_i \times \mathrm{Spec}(A) \mid U_i \hookrightarrow \mathbb{R}^n\}$). The category of formal smooth sets or, equivalently, the sheaf topos on $\mathbf{FormalCartSp_{diff}}$ is also called the **Cahiers topos**. The sets in the image of a formal smooth set X are called the sets of **plots** of X and can be interpreted as sets of functions into X (in analogy with the definition of smooth spaces).

The following definition is dual to **??**:

Definition 8.2.21 (Reduction). Given an infinitesimally thickened space $\mathbb{R}^n \times \mathbb{D}$, its reduction $\mathfrak{R}$ is defined to be $\mathbb{R}^n$. Every reduction induces a canonical morphism $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \mathbb{D}$. Plots can be reduced by precomposing with a reduction morphism.

The **infinitesimal neighbourhood** (to arbitrary order) of a formal smooth subset $Y \hookrightarrow X$ is defined by taking its plots to be those plots of X for which the reductions factorize through plots of Y.

Definition 8.2.22 (Shape modality). The **(infinitesimal) shape** or **de Rham shape** $\mathfrak{J}X$ of a formal smooth set X is defined as the formal smooth set obtained by reducing the plots of X:

$$\mathfrak{J}X(U) := X(\mathfrak{R}(U)). \tag{8.4}$$

For its incarnation as a modal operator, see Section **??**. This modality is sometimes also denoted by $\int$.

In analogy to Definitions **??** and 1.2.30 one can also define local diffeomorphisms/étale morphisms between formal smooth sets:

Definition 8.2.23 (Local diffeomorphism[2]). A morphism of formal smooth sets $f : X \to Y$ such that the thickened plots of X can be identified with those of Y whose reduction comes from a Euclidean plot of X. More elegantly (or abstractly) this means that the naturality square of the shape modality (interpreted as a monad) forms a pullback square:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{pb}"] & \mathfrak{J}X \arrow[d, "\mathfrak{J}f"] \\
Y \arrow[r, "\eta_Y"'] & \mathfrak{J}Y
\end{tikzcd}$$

[2]Also called a **(formally) étale morphism**.

This can be related to the definition in commutative algebra (and dually, that in algebraic geometry) as follows. The shape modality is right adjoint to the reduction modality. Sending a ring (extension) to its representable presheaf and using the Yoneda lemma gives the diagram

$$\begin{tikzcd} \mathbf{CRing}(A,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & \mathbf{CRing}(A,B/I) \arrow[d] \\ \mathbf{CRing}(R,B) \arrow[r] & \mathbf{CRing}(R,B/I) \end{tikzcd}$$

This square being a pullback exactly corresponds to the lifting condition in Definition **??**.

Alternative Definition 8.2.24 (Smooth manifold). A diffeological space (in its incarnation as a smooth formal set) equipped with a family of local diffeomorphisms from Euclidean spaces (also regarded as formal smooth sets) such that every point of the space lies in the image of at least one such morphism and such that the final topology induced by the plots of the smooth set is paracompact Hausdorff.

Although this section started with the promise that spaces would be generalized to the fermionic setting, only bosonic spaces were constructed up to this point. However, everything introduced in this section was formulated in such a way that supergeometry can be included through a minor modification:

Definition 8.2.25 (Superpoint). A space of the form $\mathrm{Spec}(A)$ for $A := \mathbb{R}\oplus V$ with V a finite-dimensional superalgebra **??** that forms a nilpotent ideal of A. When A is taken to be the Grassmann algebra **??** on n generators, the odd point $\mathbb{R}^{0|n}$ is obtained. The **super Euclidean space** $\mathbb{R}^{m|n}$ is obtained as the product of an ordinary Euclidean space $\mathbb{R}^m$ and the superpoint $\mathbb{R}^{0|n}$, i.e. its algebra of smooth functions is $C^\infty(\mathbb{R}^m\times\Pi\mathbb{R}^m)$.

Definition 8.2.26 (Super smooth set). A sheaf on the category of super Euclidean spaces $\mathbf{SuperCartSp}_{\mathsf{diff}}$.

8.2.3 Graded manifolds

In this section some of the notions from Part **??** will be generalized to supermanifolds and general graded manifolds. The general notation (x^i) will be used for the collection of both even and odd coordinates.

Example 8.2.27 (Supermanifold). A super smooth set in the form of a locally ringed space $(M,\mathcal{A})$ that is locally isomorphic to a super Euclidean space, i.e. $\mathcal{A}$ is locally given by $C^\infty(M)\otimes\Lambda^\bullet\mathbb{R}^n$ for some $n\in\mathbb{N}$. More generally, a **graded manifold** is a locally ringed space that is locally isomorphic to $\big(\mathbb{R}^m, C^\infty(\mathbb{R}^m)\otimes\mathrm{Sym}(V^*)\big)$ for a graded vector space V. (A supermanifold can be recovered by taking $V=\Pi\mathbb{R}^n$.)

Theorem 8.2.28 (Batchelor). *Let $(M,\mathcal{A})$ be an $\mathbb{N}$-graded manifold. There exists a vector bundle $E\to M$ such that $\mathcal{A}$ is isomorphic to the structure sheaf $\Gamma(\Lambda^\bullet E)$, i.e. $\mathcal{A}$ is locally given by $\mathrm{Sym}(\Lambda^\bullet E^*)$. If $(M,\mathcal{A})$ is a supermanifold, there exists a vector bundle $E\to N$ such that $\mathcal{A}$ is locally given by $\Lambda^\bullet E^*$.*

Definition 8.2.29 (Vector field). A graded vector field of degree k is a degree-k derivation on $C^\infty(M)$. The integer k is called the **degree**.

Definition 8.2.30 (Cohomological vector field). A graded vector field X of degree 1 that satisfies $[X,X]=0$. Every degree-1 graded vector field satisfies

$$[X,X]=2X\circ X\,, \tag{8.5}$$

which implies that every cohomological vector field defines a coboundary operator on $C^\infty(M)$. A graded manifold equipped with a cohomological vector field is called a **differential-graded manifold** (dg-manifold).

Example 8.2.31 (de Rham differential). Consider the de Rham complex $\Omega^\bullet(M)$ with differential d. This differential corresponds to a cohomological vector field Q on ΠTM, locally defined by

$$Q := \sum_{i=1}^n \mathrm{d}x^i\partial_i. \tag{8.6}$$

Note that the differentials $\mathrm{d}x^i$ are here regarded as coordinate functions on ΠTM. The **degree** of a homogeneous element of $\Omega^\bullet(M)$ is defined as the difference of its graded degree and its form degree. The de Rham complex itself then corresponds to the algebra $C^\infty(\Pi TM)$.

Definition 8.2.32 (Poisson manifold). Consider a degree-k symplectic form ω. This form induces a Poisson structure on the algebra $C^\infty(M)$ as follows:

$$\{f,g\} := (\partial_i^R f)\omega^{ij}(\partial_j^L g). \tag{8.7}$$

It is not hard to check that this operation is graded-commutative. As in Section **??**, a Hamiltonian vector field can be defined for any smooth function $H \in C^\infty(M)$:

$$\omega(X_H,\cdot) = -\mathrm{d}H(\cdot). \tag{8.8}$$

Property 8.2.33 (Euler vector field). Consider the graded vector field

$$E := \sum_{i=1}^n \deg(x^i)x^i\partial_i. \tag{8.9}$$

The Lie derivative $\mathcal{L}_E$, defined through the Cartan formula

$$\mathcal{L}_E := \iota_E\mathrm{d} + (-1)^{\deg(E)}\mathrm{d}\iota_E\,, \tag{8.10}$$

acts on homogeneous forms by multiplication by their degree.

Property 8.2.34. Every closed differential form of degree $k \neq 0$ is exact. More generally, the de Rham cohomology of a graded manifold is isomorphic to the de Rham cohomology of its body. This for example implies that a degree-l symplectic vector field X is Hamiltonian with respect to a degree-k symplectic form if $k + l \neq 0$.

Corollary 8.2.35 (dg-symplectic manifold). Consider a Hamiltonian cohomological vector field X. There exists a Hamiltonian function H such that

$$Xf = \{H,f\} \tag{8.11}$$

for all $f \in C^\infty(M)$. If the symplectic form has degree k, the function H can be chosen to be of degree $k+1$ and, accordingly, $\{H,H\}$ will be of degree $k+2$. Now, the identity $[X,X] = 0$ also implies that $\{H,H\}$ is a constant and, since all constants are of degree 0, it follows that

$$\{H,H\} = 0 \tag{8.12}$$

whenever $k \neq -2$. This equation is often called the **classical master equation**. A graded manifold equipped with both a symplectic form and a symplectic cohomological vector field is called a **differential-graded symplectic manifold**.

If ω is of degree 1, it was shown by *Schwarz* that (M,ω) is symplectomorphic to ΠT^*M such that the Poisson bracket is mapped to the Schouten-Nijenhuis bracket and the Hamiltonian is mapped to a Poisson bivector field exactly if it satisfies the master equation.

8.2.4 Gauge theory

Recall the notions of Chapter 2, in particular the notions of stacks and higher topoi. The $(\infty, 1)$-category of smooth ∞-stacks can be described in terms of (left Bousfield) localization of a suitable presheaf category by Lurie's theorem **??**.

The first possibility is the category of ∞-presheaves on **Diff** with the localization at open covers. While, the second possibility is the dense subsite $\mathbf{CartSp}_{\mathrm{diff}}$ with localization at good open covers. Both will result in a Čech model structure 2.6.3. However, the exact properties will differ.

Example 8.2.36 (Classifying stacks). Consider the example of a Lie group G and its classifying stack $\mathbf{B}G$. In the first model structure, the mapping space $\mathbf{H}(M, \mathbf{B}G)$, for M a smooth manifold, is just presented[3] by $\mathrm{Hom}(M, \mathbf{B}G)$, since M is cofibrant as a representable presheaf and $\mathbf{B}G$ is fibrant by gluing over covers. So mapping spaces $\mathbf{H}(M, \mathbf{B}G)$ are just given by groupoids of G-bundles over M.

On the subsite $\mathbf{CartSp}_{\mathrm{diff}}$, the presheafs represented by manifolds are not cofibrant anymore. However, Čech nerves of open covers give a cofibrant replacement. On the other hand, over Cartesian spaces the stacks are trivial and can be presented as action groupoids $*/\!\!/G$ (the ordinary deloopings). A fibrant replacement is given by the presheaf

$$U \mapsto N_\Delta(*/\!\!/C^\infty(U, G)). \tag{8.13}$$

This presheaf is also equivalent to the groupoid of G-bundles (over U). The derived mapping space in this situation is given by (normalized) G-valued Čech cocycles.

8.3 Higher geometry

In this section some notions about groups, Lie groups and groupoids (Sections **??**, **??** and **??**) are extended the setting of higher category theory.

8.3.1 Groups

Definition 8.3.1 (Lie groupoid[4]). A groupoid internal to **Diff**.

Note that Definition **??** requires the existence of pullbacks. In the category **Diff** this is equivalent to assuming that the source and target morphisms are (surjective) submersions.

Remark 8.3.2. In the Ehresmannian approach one gives the manifold of composable morphisms $D_1 \times_{D_0} D_1$ as part of the data. Hence, no further assumptions have to be made about the source and target morphisms.

Definition 8.3.3 (Lie algebroid). A vector bundle $\pi : E \to M$ together with a vector bundle morphism $\rho : E \to TM$, called the **anchor map**, and a Lie bracket on $\Gamma(E)$ such that the following Leibniz-type property is satisfied:

$$[X, fY] = f[X, Y] + \rho(X)(f)Y. \tag{8.14}$$

This property also implies that ρ preserves the Lie bracket:

$$\rho([X, Y]) = [\rho(X), \rho(Y)]. \tag{8.15}$$

[3]This means the homotopy-invariant hom-object in the underlying presheaf category, where the domain is replaced by a cofibrant object and the codomain by a fibrant object.

[4]In a similar way one could define *topological groupoids, étalé groupoids*, ...

In local coordinates x^i and for a local basis of sections s_α, the bracket and anchor can be expressed in terms of structure functions:

$$\rho(s_\alpha) = R^i_\alpha \partial_i \,, \tag{8.16}$$

$$[s_\alpha, s_\beta] = C^\gamma_{\alpha\beta}(x) s_\gamma. \tag{8.17}$$

The Lie algebroid properties then imply the following conditions on these structure functions:

$$R^j_\alpha \frac{\partial R^i_\beta}{\partial x^j} - R^j_\beta \frac{\partial R^i_\alpha}{\partial x^j} = R^i_\gamma C^\gamma_{\alpha\beta} \tag{8.18}$$

and

$$R^i_\alpha \frac{\partial C^\kappa_{\beta\gamma}}{\partial x^i} + C^\kappa_{\alpha\mu} C^\mu_{\beta\gamma} + (\alpha \leftrightarrow \beta \leftrightarrow \gamma) = 0. \tag{8.19}$$

Example 8.3.4 (Tangent Lie algebroid). The tangent bundle over a smooth manifold is a Lie algebroid with $\rho \equiv \mathbb{1}_{TM}$.

Consider the **pair groupoid M × M**, i.e. the groupoid with the following data:

- **Objects**: M, and
- **Morphisms**: $M \times M$, i.e. between every two points there exists a unique morphism.

Both the fundamental groupoid $\mathbf{\Pi_1}(M)$ (Definition **??**) and the pair groupoid $\mathbf{M} \times \mathbf{M}$ integrate the tangent Lie algebroid.

One can generalize the dual construction of L_∞-algebras **??** even further:

Definition 8.3.5 (L_∞-algebroid). Consider the construction of the Chevalley-Eilenberg algebra for a L_∞-algebra. By replacing the base field by a smooth algebra $C^\infty(M)$ for some smooth manifold M and the (graded) vector space V by a module of sections $\Gamma(E)$ of a (graded) vector bundle $E \to M$, one obtains the notion of a L_∞-algebroid.

Property 8.3.6. L_∞-algebras can be recovered by considering the special case $M = \{*\}$.

Example 8.3.7 (de Rham complex). Consider the tangent algebroid of a smooth manifold M. The associated Chevalley-Eilenberg complex is equivalent to the de Rham complex $\Omega^\bullet(M)$.

Definition 8.3.8 (Weak 2-group). Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a monoidal category. This category is called a weak 2-group, **categorical group** or **gr-category** if it satisfies the following conditions:

1. All morphisms are invertible.
2. Every object is weakly invertible with respect to the monoidal structure.

By Property **??** one can equivalently define a weak 2-group as a 2-category with a single object, weakly invertible 1-morphisms and invertible 2-morphisms.

Definition 8.3.9 (2-groupoid). A 2-groupoid is a 2-category in which all 1-morphisms are invertible and every 2-morphisms has a "vertical" inverse. (The "horizontal" inverse can be constructed from the other ones.)

Definition 8.3.10 (Strict 2-group). A (strict) 2-group is defined as a (strict) 2-groupoid with only one object. From this it follows that the set of 1-morphisms forms a group and so does the set of 2-morphisms under horizontal composition. However, the 2-morphisms do not form a group under vertical composition because the sources/targets may not match.

This definition is equivalent to the following internal version. A (strict) 2-group is a group object in **Cat** or an internal category in **Grp**. If **Grp** is replaced by **Lie**, the notion of a (strict) Lie 2-group is obtained.

Definition 8.3.11 (∞-groupoid). A ∞-category in which all morphisms are invertible. This is equivalent to a $(\infty, 0)$-category in the language of (n, r)-categories.

Property 8.3.12 (Lie crossed modules). The 2-category of (strict) 2-groups is biequivalent to the 2-category of (Lie) crossed modules **??**. Given a 2-group $\mathcal{G}$, a crossed module is obtained as follows:

- $G := \mathrm{ob}(\mathcal{G})$,
- $H := \{h \in \hom(\mathcal{G}) \mid \mathfrak{s}(f) = e\}$,
- $t(h) := \mathfrak{t}(h)$, and
- $\alpha(g)h := \mathbb{1}_g h \mathbb{1}_g^{-1}$,

where $\mathfrak{s}, \mathfrak{t}$ are the source and target morphisms in $\mathcal{G}$.

To every Lie crossed module one can also assign a **differential crossed module**. This consists of the following data:

- two Lie algebras $\mathfrak{g}, \mathfrak{h}$,
- a Lie algebra morphism $\partial : \mathfrak{h} \to \mathfrak{g}$, and
- a Lie algebra morphism $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$.

The equivariance and Peiffer conditions induce similar conditions for the above data:

- $\partial(\rho(h)g) = [h, \partial g]$, and
- $\rho(\partial h)(h') = [h, h']$,

where $g \in \mathfrak{g}$ and $h, h' \in \mathfrak{h}$. The biequivalence of crossed modules and strict 2-groups induces a biequivalence of differential crossed modules and strict Lie 2-algebras.

Example 8.3.13 (Automorphism 2-group). Given a Lie group H, one can construct a crossed module with $G := \mathrm{Aut}(H)$, t assigning inner automorphisms (conjugations) and α the obvious map. The associated 2-group $\mathrm{AUT}(H)$ gives a 2-group of symmetries of H, i.e. it is the automorphism 2-group of H in the 2-category **Lie**.

Definition 8.3.14 (Exponentiable group). A smooth group for which every smooth function $f : [0, 1] \to \mathfrak{g}$ corresponds to a smooth function $g : [0, 1] \to G$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = f(t)g(t) \tag{8.20}$$

with $g(0) = e$. A smooth 2-group is said to be exponentiable if both of its component groups are exponentiable. Since all Lie groups are exponentiable, all Lie 2-groups are also exponentiable

Remark 8.3.15 (Lie's third theorem). In ordinary Lie theory Lie's third theorem states that every (finite-dimensional) Lie algebra can be obtained as the infinitesimal version of a Lie group. However, this does not carry over to the 2-group setting. Consider for example the Lie 2-algebras $\mathfrak{g}_\lambda$ constructed in Example **??**. As shown in [32] only $\mathfrak{g}_0$ gives rise to a Lie 2-group (or even a topological 2-group).

8.3.2 Spaces

To overcome the problem encountered in Definition 8.3.1 above, one should pass from **Diff** to $\mathbf{C^\infty}$. It can be shown that this category admits all pullbacks, quotients, path spaces, etc.

Definition 8.3.16 (Smooth 2-space). A category internal to $\mathbf{C}^\infty$.

In the remainder of this chapter all spaces will be assumed to be smooth in this general sense. The notions of 2-groups as introduced in the previous section are easily generalized to this more general setting.

Definition 8.3.17 (2-group action). Consider a smooth 2-group $\mathcal{G}$ and a smooth 2-space E. A strict action of $\mathcal{G}$ on E is a smooth homomorphism $\mathcal{G} \to \mathrm{AUT}(E)$, i.e. a smooth map preserving products and inverses.

Definition 8.3.18 (Thin homotopy). Let M be a smooth manifold. A smooth homotopy $H : [0,1]^2 \to M$ is said to be thin if

$$H(s,t) = F(s) \tag{8.21}$$

for some smooth F near $t = 0, 1$ and if it pulls back every two-form to 0:

$$\forall \omega \in \Omega^2(M) : H^*\omega = 0. \tag{8.22}$$

Definition 8.3.19 (Lazy path). Let M be a smooth manifold. A path $f : [0,1] \to M$ is said to be lazy (or to have **sitting instants**) if it is locally constant on some neighbourhoods of 0 and 1.

Definition 8.3.20 (Path groupoid). Let M be a smooth space. The path groupoid $\mathcal{P}_1(M)$ is the smooth groupoid conssting of the following data:

- **Objects**: M, and
- **Morphisms**: thin homotopy classes of lazy paths with fixed endpoints on M.

The laziness combined with the first condition of thin homotopies implies that the morphisms of this groupoid are (locally) constant near the full boundary of their domain.

In fact, by suitably generalizing the smoothness properties of the homotopies and paths, one can extend this definition to surfaces, volumes and so on. This results in the n-path n-groupoid $\mathcal{P}_n(M)$.

Remark 8.3.21. The restriction to lazy paths is required to ensure the smoothness of composite paths. The quotient by thin homotopies is required to ensure the validity of the associativity and invertibility properties.

The following definition generalized Definition 1.3.10:

Definition 8.3.22 (Algebraic stack). A stack $X \in \mathbf{Sh}_{(2,1)}(\mathbf{Sch}_{\text{fppf}})$ on the big fppf-site such that

1. The diagonal $\Delta_X : X \to X \times X$ is representable by an algebraic space 1.3.10.
2. There exists a scheme $S \in \mathbf{Sch}$ and a morphism $h_S \to X$ that is surjective and smooth.

If the covering morphism is étale and not just smooth, the notion of **Deligne-Mumford stacks** is obtained. To distinguish these cases, algebraic stacks are sometimes called **Artin stack**.

?? COMPLETE ??

8.4 2-Bundles

A first step is the generalization of the categorical definition of a general bundle 4.1.1, i.e. as an object of a slice category:

Definition 8.4.1 (Smooth 2-bundle). A triple (E, B, π) where both E and B are smooth 2-spaces and π is a smooth map.

Definition 8.4.2 (Locally trivial 2-bundle). A locally trivial 2-bundle with typical fibre F over a smooth 2-space B is defined as a 2-bundle (E, B, π) with an open cover $\{U_i\}_{i\in I}$ of B such that for every $i \in I$ there exists an equivalence $\varphi_i : E|_{U_i} \cong U_i \times F$ that makes the diagram below commute:

$$\begin{tikzcd}
E|_{U_i} \arrow[rr, "\varphi_i"] \arrow[dr, "\pi"'] & & U_i \times F \arrow[dl, "\mathrm{pr}_1"] \\
& U_i &
\end{tikzcd}$$

It should be noted that the existence of such a cover is not a trivial matter. The general definition becomes quite involved when allowing for arbitrary smooth 2-spaces B. For convenience it will always be assumed that B is an ordinary smooth space regarded as a 2-space with only trivial morphisms.

As was the case in Definition 4.1.5, one can also characterize locally trivial 2-bundles by their transition data. Since the trivilizations φ_i are equivalences, they admit an inverse (up to an invertible 2-map) and one can thus construct transition maps $\varphi_i\varphi_j^{-1} = U_{ij} \times F \cong U_{ij} \times F$ as usual. By the commutative diagram above, these transition maps only act on the fibre F. Because $\varphi_i\varphi_j^{-1}$ is itself an (auto)equivalence, the action on F is given by a functor $g_{ij} : U_{ij} \to \mathrm{AUT}(F)$, where the 2-space $\mathrm{AUT}(F)$ is the *coherent 2-group*[5] of autoequivalences of F together with invertible 2-maps between them.

The interesting (and important) part is how the cocycle conditions (4.1) and 4.1.4 for the maps g_{ij} are modified. Since the equivalences g_{ij} are only invertible up to 2-maps, one cannot expect these conditions to hold as equations. Instead, two higher transition maps (i.e. natural isomoprhisms) $h_{ijk} : g_{ij} \circ g_{jk} \Rightarrow g_{ik}$ and $k_i : g_{ii} \Rightarrow \mathrm{id}_F$ are obtained. These higher data should in turn satisfy the necessary conditions coming from associativity and unitality constraints (similar to the coherence conditions from Section **??**).

Definition 8.4.3 ($\mathcal{G}$-bundle). A locally trivial 2-bundle with typical fibre F is said to have the 2-group $\mathcal{G}$ as its structure (2-)group if the transition data factor through an action $\mathcal{G} \to \mathrm{AUT}(F)$. If $F = \mathcal{G}$, the 2-bundle is called a **principal $\mathcal{G}$-2-bundle**.

Remark 8.4.4 (Gerbes). If the transition maps k_i are chosen to be trivial and $\mathcal{G}$ is chosen to be respectively the trivial Lie 2-group associated to an Abelian Lie group G or the automorphism 2-group of a Lie group H, one obtains Abelian and non-Abelian *gerbes*. In fact, it can be shown that the 2-category of principal 2-bundles is equivalent to the 2-category of gerbes for every Lie 2-group of the aforementioned type.

By categorifying Definition 6.3.36 of principal connections, one can define connections for principal n-bundles:

[5]Instead of the strict invertibility of maps in the definition of 2-groups above, one should allow for invertibility up to 2-isomorphisms that themselves satisfy certain coherence conditions.

Definition 8.4.5 (n-connection). Let M be a smooth space and let G be a Lie n-groupoid. Given a locally trivial principal n-bundle P over M, an n-connection with n-holonomy is defined by the following data:

- for every coordinate chart $U_i \subset M$ a local holonomy n-functor

$$\mathrm{hol}_i : \mathcal{P}_n(U_i) \to G\,, \tag{8.23}$$

- for every double intersection U_{ij} a 1-transfor (i.e. an n-natural transformation)

$$g_{ij} : \mathrm{hol}_i \Rightarrow \mathrm{hol}_j\,, \tag{8.24}$$

- for every triple intersection U_{ijk} a 2-transfor

$$f_{ijk} : g_{ij} \circ g_{jk} \Rrightarrow g_{ik}\,, \tag{8.25}$$

- and so on ...

This is equivalently given by a global n-functor

$$\mathrm{hol} : \mathcal{P}_n(M) \to \mathbf{Trans}_n(P). \tag{8.26}$$

?? ADD GERBES (e.g. BRYLINSKI) ??

8.5 Space and quantity

In this section the general notions of spaces and observables are again considered. From the start everyhting will be formulated in an enriched setting where $\mathcal{V}$ is a cosmos **??**. The categories $\mathbf{C}$ of interest will be assumed to be small.

In the previous sections spaces modelled on a base space X, or more generally, on a category of spaces $\mathbf{S}$ were modelled as (concrete) sheaves on a suitable site. Here this notion is relaxed as much as possible:

Definition 8.5.1 (Space). A (generalized) space modelled on a category $\mathbf{C}$ is a presheaf on $\mathbf{C}$.

As before, the object $X(C)$ can be interpreted as the collection of "probes" from C to X. The Yoneda lemma assures that ordinary test spaces in $\mathbf{C}$ can be viewed as spaces modelled on $\mathbf{C}$ and that their probes are indeed the ordinary maps in $\mathbf{C}$.

In a similar vein one can define observables as maps out of a space:

Definition 8.5.2 (Quantity). A (generalized[6]) quantity on a category $\mathbf{C}$ is a copresheaf on $\mathbf{C}$.

Property 8.5.3 (Isbell duality). Given a space X one can look at the quantities that live on it (in ordinary geometry this would have been its algebra of functions). This defines a functor:

$$\mathcal{O} : \mathbf{Psh}(\mathbf{C}) \to \mathbf{coPsh}^{op}(\mathbf{C}) : X \mapsto \mathrm{Hom}_{\mathbf{Psh}(\mathbf{C})}(X, \mathcal{Y}-). \tag{8.27}$$

Similarly, given a quantity Q one can ask on which space it behaves as the algebra of functions. This also defines as functor:

$$\mathrm{Spec} : \mathbf{coPsh}^{op}(\mathbf{C}) \to \mathbf{Psh}(\mathbf{C}) : Q \mapsto \mathrm{Hom}_{\mathbf{coPsh}(\mathbf{C})}(\mathcal{Y}^{op}-, Q)\,, \tag{8.28}$$

[6]It is generalized because it is "measures" a category instead of a single object.

where $\mathcal{Y}^{op}$ denotes the co-Yoneda embedding $\mathbf{C} \to [\mathbf{C}, \mathcal{V}]^{op} : c \mapsto \mathbf{C}(c, -)$.

The incredible result is now that $(\mathcal{O} \dashv \mathrm{Spec})$ is an adjunction, called the **Isbell adjunction**. Objects that are preserved (up to isomorphism) under the associated (co)monad are said to be **Isbell self-dual**.

Example 8.5.4 (Cartesian spaces). When working over the site **CartSp** (with its usual topology) and restricting to coherent sheaves and product-preserving presheaves, the Isbell adjunction maps spaces to smooth algebras.

List of Symbols

The following symbols are used throughout the summary:

Abbreviations

FIP	finite intersection property

Operations

$\deg(f)$	degree of the polynomial f
e	identity element of a group
∂X	boundary of a topological space X
$\overline{X}$	closure of a topological space X
$X^\circ, \mathring{X}$	interior of a topological space X
$X \times Y$	cartesian product of the sets X and Y
$\mathbb{1}_X$	identity morphism on the object X
$\approx$	is approximately equal to
$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto

Objects

$\mathbf{Ab}$	category of Abelian groups
$\mathrm{Aut}(X)$	automorphism group of an object X
$\mathbf{C}^\infty$	category of smooth spaces
$C(X,Y)$	set of continuous functions between two topological spaces X and Y
$\mathbf{C^\infty Ring}, \mathbf{C^\infty Alg}$	category of smooth algebras
$\mathbf{DiffSp}$	category of diffeological spaces and smooth maps
D^n	standard n-disk
$\mathrm{End}(X)$	endomorphism monoid of a an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad
$\mathbf{FormalCartSp_{diff}}$	category of infinitesimally thickened Euclidean spaces
$\mathbb{G}_a$	additive group (scheme)
$\mathbf{Grp}$	category of groups and group homomorphisms
$\mathbf{Grpd}$	category of groupoids
$\mathrm{Hom}_{\mathbf{C}}(V,W)$, $\mathbf{C}(V,W)$	set of homomorphisms from an object V to an object W in a category $\mathbf{C}$
$\mathbf{Law}$	category of Lawvere theories

$\mathbf{Open}(X)$	category of open subsets of a topological space X
$\mathbf{Psh}(\mathbf{C}), \widehat{\mathbf{C}}$	category of presheaves on a (small) category $\mathbf{C}$
$\mathbf{Sh}(\mathbf{C}, J)$	category of J-sheaves on a site $(\mathbf{C}, J)$
S^n	standard n-sphere
$\mathbf{Span}(\mathbf{C})$	span category over $\mathbf{C}$
$\mathrm{Spec}(R)$	spectrum of a commutative ring R
$\mathrm{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
S_n	symmetric group of degree n
$\mathrm{Sym}(X)$	symmetric group on the set X
T^n	standard n-torus (the n-fold Cartesian product of S^1)
$\mathbf{Top}$	category of topological spaces
$\mathbf{Topos}$	the 2-category of (elementary) topoi and geometric morphisms
$\emptyset$	empty set
$[a, b]$	closed interval
$]a, b[$	open interval

Bibliography

[1] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5. Springer Science & Business Media, 2013.

[2] Fosco Loregian. Coend calculus. arXiv:1501.02503.

[3] Gregory M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64. CUP Archive, 1982.

[4] John C. Baez and Peter May. *Towards Higher Categories*, volume 152 of *IMA Volumes in Mathematics and its Applications*. Springer, 2009.

[5] Tom Leinster. Basic bicategories. 1998. arXiv:math/9810017.

[6] Peter T. Johnstone. *Topos Theory*. Dover Publications, 2014.

[7] Sjoerd E. Crans. Localizations of transfors. 1998.

[8] Andreas Gathmann. Algebraic geometry. `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf`.

[9] David Mumford. *The Red Book of Varieties and Schemes: Includes the Michigan Lectures (1974) on Curves and Their Jacobians*, volume 1358. Springer Science & Business Media, 1999.

[10] Olivia Caramello. Lectures on topos theory at the university of Insubria. `https://www.oliviacaramello.com/Teaching/Teaching.htm`.

[11] Angelo Vistoli. Notes on Grothendieck topologies, fibered categories and descent theory. *arXiv:math/0412512*, 2004.

[12] Yvonne Choquet-Bruhat, Cecile DeWitt-Morette, and Margaret Dillard-Bleick. *Analysis, Manifolds and Physics, Part 1: Basics*. North-Holland.

[13] Yvonne Choquet-Bruhat and Cecile DeWitt-Morette. *Analysis, Manifolds and Physics, Part 2*. North-Holland.

[14] Gerd Rudolph and Matthias Schmidt. *Differential Geometry and Mathematical Physics: Part II. Fibre Bundles, Topology and Gauge Fields*. Springer, 2017.

[15] Ivan Kolar, Peter W. Michor, and Jan Slovak. *Normal Operations in Differential Geometry*. Springer.

[16] Charles Nash and Siddharta Sen. *Topology and Geometry for Physicists*. Dover Publications.

[17] Frederic Schuller. Lectures on the geometric anatomy of theoretical physics. `https://www.youtube.com/channel/UC6SaWe7xeOp31Vo8cQG1oXw`.

[18] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*. Graduate Texts in Mathematics. Springer New York, 1995.

[19] Chris Tiee. Contravariance, covariance, densities, and all that: An informal discussion on tensor calculus. `https://ccom.ucsd.edu/~ctiee/notes/tensors.pdf`, 2006.

[20] Stephen B. Sontz. *Principal Bundles: The Classical Case*. Springer.

[21] Derek Sorensen. An introduction to characteristic classes. `http://derekhsorensen.com/docs/sorensen-characteristic-classes.pdf`, 2017.

[22] Peter May. A note on the splitting principle. *Topology and Its Applications*, 153(4):605--609, 2005.

[23] Arun Debray. Characteristic classes. `https://web.ma.utexas.edu/users/a.debray/lecture_notes/u17_characteristic_classes.pdf`.

[24] Jean-Luc Brylinski. *Loop Spaces, Characteristic Classes and Geometric Quantization*. Birkhauser.

[25] Nigel Hitchin. Lectures on special Lagrangian submanifolds. `https://arxiv.org/abs/math/9907034v1`, 1999.

[26] Richard W. Sharpe. *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, volume 166. Springer Science & Business Media, 2000.

[27] Nima Moshayedi. 4-manifold topology, donaldson-witten theory, floer homology and higher gauge theory methods in the BV-BFV formalism. 2021.

[28] John C. Baez and Urs Schreiber. Higher gauge theory. 2005. arXiv:math/0511710.

[29] Urs Schreiber. *From Loop Space Mechanics to Nonabelian Strings*. PhD thesis, 2005.

[30] Richard Sanders. Commutative spectral triples & the spectral reconstruction theorem.

[31] John Baez and Alexander Hoffnung. Convenient categories of smooth spaces. *Transactions of the American Mathematical Society*, 363(11):5789--5825, 2011.

[32] John C. Baez and Aaron D. Lauda. Higher-dimensional algebra V: 2-groups. 2003. arXiv:math/0307200.

[33] Thomas Augustin, Frank PA Coolen, Gert De Cooman, and Matthias CM Troffaes. *Introduction to imprecise probabilities*. John Wiley & Sons, 2014.

[34] Tetsuji Miwa, Michio Jimbo, Michio Jimbo, and E Date. *Solitons: Differential Equations, Symmetries and Infinite-dimensional Algebras*, volume 135. Cambridge University Press, 2000.

[35] Vladimir I. Arnol'd. *Mathematical Methods of Classical Mechanics*, volume 60. Springer Science & Business Media, 2013.

[36] Edwin J. Beggs and Shahn Majid. *Quantum Riemannian Geometry*. Springer, 2020.

[37] Marc Henneaux and Claudio Teitelboim. *Quantization of Gauge Systems*. Princeton university press, 1992.

[38] Mark Hovey. *Model Categories*. Number 63. American Mathematical Soc., 2007.

[39] Vladimir Vovk, Alex Gammerman, and Glenn Shafer. *Algorithmic Learning in a Random World*. Springer Science & Business Media, 2005.

[40] Mukund Rangamani and Tadashi Takayanagi. *Holographic Entanglement Entropy.* Springer, 2017.

[41] Shun-ichi Amari. *Information Geometry and Its Applications.* Springer Publishing Company, Incorporated, 2016.

[42] Charles W. Misner, Kip S. Thorne, and John A. Wheeler. *Gravitation.* Princeton University Press, 2017.

[43] Carlo Rovelli and Francesca Vidotto. *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory.* Cambridge University Press, 2014.

[44] John C. Baez, Irving E. Segal, and Zhengfang Zhou. *Introduction to Algebraic and Constructive Quantum Field Theory.* Princeton University Press, 2014.

[45] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[46] Bruce Blackadar. *Operator Algebras: Theory of C$*$-Algebras and von Neumann Algebras.* Springer, 2013.

[47] Marek Capinski and Peter E. Kopp. *Measure, Integral and Probability.* Springer Science & Business Media, 2013.

[48] Georgiev Svetlin. *Theory of Distributions.* Springer, 2015.

[49] Martin Schottenloher. *A Mathematical Introduction to Conformal Field Theory*, volume 759. 2008.

[50] Dusa McDuff and Deitmar Salamon. *Introduction to Symplectic Topology.* Oxford Graduate Texts in Mathematics. Oxford University Press, 2017.

[51] Mikhail. M. Kapranov and Vladimir A. Voevodsky. *2-categories and Zamolodchikov Tetrahedra Equations*, volume 56 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1994.

[52] Geoffrey Compère. *Advanced Lectures on General Relativity*, volume 952. Springer, 2019.

[53] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor Categories*, volume 205. American Mathematical Soc., 2016.

[54] Charles A. Weibel. *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[55] Peter J. Hilton and Urs Stammbach. *A Course in Homological Algebra.* Springer.

[56] Antoine Van Proeyen and Daniel Freedman. *Supergravity.* Cambridge University Press.

[57] William S. Massey. *A Basic Course in Algebraic Topology.* Springer.

[58] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory.* Westview Press.

[59] Nadir Jeevanjee. *An Introduction to Tensors and Group Theory for Physicists.* Birkhauser.

[60] Herbet Goldstein, John L. Safko, and Charles P. Poole. *Classical Mechanics.* Pearson.

[61] Franco Cardin. *Elementary Symplectic Topology and Mechanics.* Springer.

[62] Walter Greiner and Joachim Reinhardt. *Field Quantization.* Springer.

[63] Walter Greiner. *Quantum Mechanics.* Springer.

[64] B. H. Bransden and Charles J. Joachain. *Quantum Mechanics.* Prentice Hall.

[65] Heydar Radjavi and Peter Rosenthal. *Invariant Subspaces.* Dover Publications.

[66] Max Karoubi. *K-Theory: An Introduction.* Springer.

[67] Damien Calaque and Thomas Strobl. *Mathematical Aspects of Quantum Field Theories.* Springer, 2015.

[68] Stephen B. Sontz. *Principal Bundles: The Quantum Case.* Springer.

[69] William Fulton and Joe Harris. *Representation Theory: A First Course.* Springer.

[70] Peter Petersen. *Riemannian Geometry.* Springer.

[71] Ian M. Anderson. *The Variational Bicomplex.*

[72] Joel Robbin and Dietmar Salamon. The maslov index for paths. *Topology*, 32(4):827--844, 1993.

[73] Edward Witten. Global anomalies in string theory. In *Symposium on Anomalies, Geometry, Topology*, 6 1985.

[74] F.A. Berezin and M.S. Marinov.

[75] Paul A. M. Dirac. Generalized Hamiltonian dynamics. *Canadian Journal of Mathematics*, 2:129–148, 1950.

[76] Emily Riehl and Dominic Verity. The theory and practice of Reedy categories. *Theory and Applications of Categories*, 29, 2013.

[77] Emily Riehl. Homotopical categories: From model categories to $(\infty, 1)$-categories. 2019. arXiv:1904.00886.

[78] Floris Takens. A global version of the inverse problem of the calculus of variations. *Journal of Differential Geometry*, 14(4):543--562, 1979.

[79] John C. Baez and Alissa S. Crans. Higher-dimensional algebra VI: Lie 2-algebras. 2003. arXiv:math/0307263.

[80] Edward Witten. Supersymmetry and Morse theory. *J. Diff. Geom*, 17(4):661--692, 1982.

[81] Jade Master. Why is homology so powerful? 2020. arXiv:2001.00314.

[82] Marcus Berg, Cécile DeWitt-Morette, Shangjr Gwo, and Eric Kramer. The Pin groups in physics: C, P and T. *Reviews in Mathematical Physics*, 13(08):953--1034, 2001.

[83] Arindam Banerjee, Srujana Merugu, Inderjit S Dhillon, and Joydeep Ghosh. Clustering with Bregman divergences. *Journal of Machine Learning Research*, 6(Oct):1705--1749, 2005.

[84] Jean-Daniel Boissonnat, Frank Nielsen, and Richard Nock. Bregman Voronoi diagrams. *Discrete & Computational Geometry*, 44(2):281--307, 2010.

[85] Richard Palais. The symmetries of solitons. *Bulletin of the American Mathematical Society*, 34(4):339--403, 1997.

[86] Michael F. Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175--186, 1988.

[87] Jens Eisert, Christoph Simon, and Martin B Plenio. On the quantification of entanglement in infinite-dimensional quantum systems. *Journal of Physics A: Mathematical and General*, 35(17):3911--3923, 2002.

[88] Benoît Tuybens. Entanglement entropy of gauge theories. 2017.

[89] Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338--353, 1965.

[90] John C. Baez, Alexander E. Hoffnung, and Christopher Rogers. Categorified symplectic geometry and the classical string. *Communications in Mathematical Physics*, 293:701--725, 2010.

[91] Charles Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3):973--1007, 2001.

[92] Glenn Shafer and Vladimir Vovk. A tutorial on conformal prediction. *J. Mach. Learn. Res.*, 9:371-–421, 2008.

[93] Victor Chernozhukov, Kaspar Wüthrich, and Zhu Yinchu. Exact and robust conformal inference methods for predictive machine learning with dependent data. In *Proceedings of the 31st Conference On Learning Theory*, volume 75 of *Proceedings of Machine Learning Research*, pages 732--749. PMLR, 2018.

[94] Irina Markina. Group of diffeomorphisms of the unit circle as a principal $U(1)$-bundle.

[95] Alexander E. Hoffnung. Spans in 2-categories: A monoidal tricategory. 2011. arXiv:1112.0560.

[96] Eugenia Cheng and Nick Gurski. The periodic table of n-categories for low dimensions ii: Degenerate tricategories. 2007. arXiv:0706.2307.

[97] Mehmet B. Şahinoğlu, Dominic J. Williamson, Nick Bultinck, Michael Mariën, Jutho Haegeman, Norbert Schuch, and Frank Verstraete. Characterizing topological order with matrix product operators. 2014. arXiv:1409.2150.

[98] Dominic J. Williamson, Nick Bultinck, Michael Mariën, Mehmet B. Şahinoğlu, Jutho Haegeman, and Frank Verstraete. Matrix product operators for symmetry-protected topological phases: Gauging and edge theories. *Phys. Rev. B*, 94, 2016.

[99] Guifré Vidal. Efficient classical simulation of slightly entangled quantum computations. *Phys. Rev. Lett.*, 91, 2003.

[100] Aaron D. Lauda and Hendryk Pfeiffer. Open–closed strings: Two-dimensional extended TQFTs and Frobenius algebras. *Topology and its Applications*, 155(7):623--666, 2008.

[101] Domenico Fiorenza. An introduction to the Batalin-Vilkovisky formalism. 2004. arXiv:math/0402057v2.

[102] Stefan Cordes, Gregory Moore, and Sanjaye Ramgoolam. Lectures on 2d Yang-Mills theory, equivariant cohomology and topological field theories. arXiv:hep-th/9411210v2.

[103] Donald C. Ferguson. A theorem of Looman-Menchoff. `http://digitool.library.mcgill.ca/thesisfile111406.pdf`.

[104] Holger Lyre. Berry phase and quantum structure. arXiv:1408.6867.

[105] Florin Belgun. Gauge theory. `http://www.math.uni-hamburg.de/home/belgun/Gauge4.pdf`.

[106] Vladimir Itskov, Peter J. Olver, and Francis Valiquette. Lie completion of pseudogroups. *Transformation Groups*, 16:161--173, 2011.

[107] Richard Borcherds. Lie groups. `https://math.berkeley.edu/~reb/courses/261/`.

[108] Andrei Losev. From Berezin integral to Batalin-Vilkovisky formalism: A mathematical physicist's point of view. 2007.

[109] Edward Witten. Coadjoint orbits of the Virasoro group. *Comm. Math. Phys.*, 114(1):1--53, 1988.

[110] Sidney R. Coleman and Jeffrey E. Mandula. All possible symmetries of the S-matrix. *Phys. Rev.*, 159, 1967.

[111] Emily Riehl. Monoidal algebraic model structures. *Journal of Pure and Applied Algebra*, 217(6):1069--1104, 2013.

[112] Valter Moretti. Mathematical foundations of quantum mechanics: An advanced short course. *International Journal of Geometric Methods in Modern Physics*, 13, 2016.

[113] Antonio Michele Miti. Homotopy comomentum maps in multisymplectic geometry, 2021.

[114] Niclas Sandgren and Petre Stoica. On moving average parameter estimation. Technical Report 2006-022, Department of Information Technology, Uppsala University, 2006.

[115] John E. Roberts. Spontaneously broken gauge symmetries and superselection rules. 1974.

[116] Jean Gallier. Clifford algebras, Clifford groups, and a generalization of the quaternions, 2008. arXiv:0805.0311.

[117] Bozhidar Z. Iliev. Normal frames for general connections on differentiable fibre bundles. arXiv:math/0405004.

[118] Piotr Stachura. Short and biased introduction to groupoids. arXiv:1311.3866.

[119] Nima Amini. Infinite-dimensional Lie algebras. `https://people.kth.se/~namini/PartIIIEssay.pdf`.

[120] Peter Selinger. Lecture notes on lambda calculus.

[121] Jonathan R. Shewchuk. An introduction to the conjugate gradient method without the agonizing pain. Technical report, 1994.

[122] Pascal Lambrechts.

[123] Emily Riehl. Homotopy (limits and) colimits. `http://www.math.jhu.edu/~eriehl/hocolimits.pdf`.

[124] Will J. Merry. Algebraic topology. `https://www.merry.io/algebraic-topology`.

[125] Stacks project. `https://stacks.math.columbia.edu/`.

[126] The nlab. `https://ncatlab.org/nlab`.

[127] Wikipedia. `https://www.wikipedia.org/`.

[128] Joost Nuiten. Cohomological quantization of local prequantum boundary field theory. Master's thesis, 2013.

Index