Compendium of Mathematics & Physics

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Contents

Introduction

Goals

This compendium originated out of the necessity for a compact summary of important theorems and formulas during physics and mathematics classes at university. When the interest in more (and more exotic) subjects grew, this collection lost its compactness and became the chaos it now is. Although there should exist some kind of overall structure, it was not always possible to keep every section self-contained or respect the order of the chapters.

It should definitely not be used as a formal introduction to any subject. It is neither a complete work nor a fact-checked one, so the usefulness and correctness is not guaranteed. However, it can be used as a look-up table for theorems and formulas, and as a guide to the literature. To this end, each chapter begins with a list of useful references. At the same time, only a small number of statements are proven in the text (or appendices). This was done to keep the text as concise as possible (a failed endeavour). However, in some cases the major ideas underlying the proofs are provided.

Structure and conventions

Sections and statements that require more advanced concepts, in particular concepts from later chapters or (higher) category theory, will be labelled by the *clubs* symbol ♣. Some definitions, properties or formulas are given with a proof or an extended explanation whenever I felt like it. These are always contained in a blue frame to make it clear that they are not part of the general compendium. When a section uses notions or results from a different chapter at its core, this will be recalled in a green box at the beginning of the section.

Definitions in the body of the text will be indicated by the use of **bold font**. Notions that have not been defined in this summary but that are relevant or that will be defined further on in the compendium (in which case a reference will be provided) are indicated by *italic text*. Names of authors are also written in *italic*.

Objects from a general category will be denoted by a lower-case letter (depending on the context, upper-case might be used for clarity), functors will be denoted by upper-case letters and the categories themselves will be denoted by symbols in **bold font**. In the later chapters on physics, specific conventions for the different types of vectors will often be adopted. Vectors in Euclidean space will be denoted by a bold font letter with an arrow above, e.g. $\vec{\mathbf{a}}$, whereas vectors in Minkowski space (4-vectors) and differential forms will be written without the arrow, e.g. $\mathbf{a}$. Matrices and tensors will always be represented by capital letters and, dependent on the context, a specific font will be adopted.

Chapter 1

Model theory ♣

General references for this chapter are Hovey (2007); Riehl (2019). For more on monoidal model categories, see Riehl (2013). A good reference for the section on simplicial spaces and, in particular, the theory of Segal spaces is Rezk (2001). For more on Reedy model structures, see Riehl and Verity (2013). A gentle introduction to the theory of homotopy (co)limits can be found in Lambrechts (2013); Riehl (2011). The content and language of **??** will be used throughout this chapter.

Chapter Contents

1.1 Simplicial sets

Definition 1.1.1 (Simplex category). The simplex category Δ has as objects the posets of the form $[n] := \{0 < \cdots < n\}$ and as morphisms the order-preserving maps.

Definition 1.1.2 (Simplicial set). The category **sSet** of simplicial sets is defined as the presheaf category **Psh**(Δ). For all $n \in \mathbb{N}$, the set of n-**simplices in** X is defined as the set $X_n := X([n])$.

Definition 1.1.3 (Simplicial object). By internalizing the notion of a simplicial set, one obtains the definition of a simplicial object, i.e. a simplicial object in a category **C** is a **C**-valued presheaf on Δ.

Remark 1.1.4. Note that the notion of **simplicial category** can mean two distinct things. In general, it will mean a category enriched in **sSet**. However, following the previous definition, it can also mean a simplicial object in the (2-)category **Cat**. It can be shown that all simplicially enriched categories are a specific kind of degenerate simplicial object in **Cat**, where the face and degeneracy maps are identity-on-objects.

Definition 1.1.5 (Standard simplex). For every $n \in \mathbb{N}$, the standard n-simplex $\Delta[n]$ is defined as the Yoneda embedding $\Delta(-, [n])$. One can also define a functor $\Delta_{\text{top}} : \Delta \to$ **Top** that maps $[n]$ to the standard topological n-simplex Δ^n (**??**).

Property 1.1.6. By the Yoneda lemma, there exists a natural bijection between the set of n-simplices of a simplicial set X and the set of maps $\Delta[n] \to X$.

Property 1.1.7 (Face and degeneracy maps). All morphisms in the simplex category Δ are generated by morphisms of the following two types:

- For every n and $i < n$, the unique map $\delta_{n,i} : [n-1] \to [n]$ that misses the ith element.

- For every n and $i \leq n$, the unique map $\sigma_{n,i} : [n+1] \to [n]$ that duplicates the ith element.

Under the action of a presheaf, this gives the **face** and **degeneracy** maps $d_{n,i}$ and $s_{n,i}$. (If the index n is clear, it is often omitted.)

These morphisms satisfy the fundamental **simplicial identities**:

- $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$,

- $d_i \circ s_j = s_{j-1} \circ d_i$ for $i < j$,

- $d_i \circ s_j = \mathbb{1}$ for $i = j$ or $i = j + 1$,

- $d_i \circ s_j = s_j \circ d_{i-1}$ for $i > j + 1$, and

- $s_i \circ s_j = s_{j+1} \circ s_i$ for $i \le j$.

Definition 1.1.8 (**Connected components**). Consider a simplicial set X. Its set of connected components $\pi_0(X)$ is defined as the quotient of X_0 under the relation

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \times X_0 . \tag{1.1}$$

This defines a fucntor $\pi_0 : \mathsf{sSet} \to \mathsf{Set}$. By base change **??**, this also induces a functor on simplicial categories.

1.1.1 Constructions

Construction 1.1.9 (**Nerve and realization**). Consider a general functor $F : \mathsf{S} \to \mathsf{C}$ into a cocomplete category (S will often be a category of geometric shapes such as the simplex category Δ or the *cube category* $\square$). Every such functor induces an adjunction

$$\mathsf{C} \overset{|-|}{\underset{N}{\overset{\longleftarrow}{\underset{\longrightarrow}{\bot}}}} \mathsf{Psh}(\mathsf{S}) . \tag{1.2}$$

The **realization functor** $| - |$ is defined as the left Kan extension $\mathrm{Lan}_y F$. The **nerve functor** $N : \mathsf{C} \to \mathsf{Psh}(\mathsf{S})$ is defined as $Nx := \mathsf{C}(F-, x) = \mathcal{Y}x \circ F$.

This definition can easily be generalized to the enriched setting. Furthermore, if one assumes that C is copowered over $\mathcal{V}$, the realization functor can be expressed as a coend:

$$|X| = \int^{s \in \mathsf{S}} Xs \cdot Fs . \tag{1.3}$$

Example 1.1.10 (**Nerve of a category**). To every small category C, one can associate a simplicial set $N\mathsf{C}$ in the following way. The set $N\mathsf{C}_0$ is given by the set of objects in C and the set $N\mathsf{C}_1$ is given by the set of morphisms in C. Now, for every two composable morphisms f, g, one obtains a canonical commuting triangle by composition. Let $N\mathsf{C}_2$ be the set of all these triangles. The higher simplices are defined analogously. Face maps act by composing morphisms or by dropping the exterior morphisms in a composable string. Degeneracy maps act by inserting an identity morphism.

Equivalently, one can define the **simplicial nerve functor** in the following way. Every poset $[n]$ admits a canonical category structure for which the order-preserving maps give rise to the associated functors. By the above construction, this inclusion $\Delta \hookrightarrow \mathsf{Cat}$ induces a nerve functor

$$N : \mathsf{Cat} \to \mathsf{sSet} : \mathsf{C} \mapsto \mathsf{Cat}(-, \mathsf{C}) . \tag{1.4}$$

This way, one obtains $N\mathbf{C}_k = \mathbf{Cat}([k], \mathbf{C})$. This object is, by definition, equivalent to the collection of all strings of k composable morphisms in $\mathbf{C}$. It can be shown that the simplicial nerve functor is fully faithful.

Example 1.1.11 (Geometric realization). Consider a simplicial set X. From this object, one can construct a topological space as follows. First, take a point for every element in X_0. Then, glue 1-simplices between these points using the face maps. The higher (nondegenerate) simplices are attached analogously.

More abstractly, the geometric realization functor $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ is defined as a (left) Kan extension:

$$|\cdot| := \mathsf{Lan}_y \Delta_{\mathrm{top}}. \tag{1.5}$$

An application of the Yoneda lemma shows that the geometric realization can be expressed as a functor tensor product (**??**):

$$|X| = X \otimes_{\Delta} \Delta_{\mathrm{top}} = \int^{n \in \Delta} \Delta^n \cdot X_n. \tag{1.6}$$

This formula can easily be generalized to the category of simplicial topological spaces (**sSet** is a full subcategory obtained by endowing every set with the discrete topology). In **Top**, the coend can be expressed as the quotient space

$$|X| := \bigsqcup_{n \in \mathbb{N}} X_n \times \Delta^n / \sim, \tag{1.7}$$

where the equivalence relation identifies the points $(x, f_* y)$ and $(f^* x, y)$ for all morphisms $f \in \mathrm{hom}(\Delta)$. The morphisms f^*, f_* are the ones induced by X and $\Delta_{\mathrm{top}} : \Delta \hookrightarrow \mathbf{Top}$. As an immediate example one obtains

$$|\Delta[n]| = \Delta^n, \tag{1.8}$$

which shows that $\Delta[n]$ really deserves to be called the standard n-simplex.

Remark 1.1.12. Note that the composition of the geometric realization and (simplicial) nerve functors $|N \cdot| : \mathbf{Cat} \to \mathbf{Top}$ is also often called the **geometric realization** (of categories).

Example 1.1.13 (Singular set). Given a topological space X, one can define a simplicial set $\mathsf{Sing}(X)$. Its components are defined as the set of morphisms from the standard (topological) n-simplex to X:

$$\mathsf{Sing}(X)_n := \mathbf{Top}(\Delta^n, X). \tag{1.9}$$

This is the object of relevance in the definition of singular (co)homology as given in **??**.

Property 1.1.14 (Classifying space). For a (discrete) group G, one can construct two important objects: the delooping $\mathbf{B}G$ (**??**) and the *classifying space BG* (see **??**). As their

notations imply, there exists a relation between these space. By taking the geometric realization of $\mathbf{B}G$, one obtains BG. In fact, this method can be applied to any monoid A to obtain the so-called (two-sided) **bar construction**.

Construction 1.1.15 (Join). Consider two simplicial sets X, Y. An explicit construction of the join $X \star Y$ is as follows:

$$(X \star Y)_n := X_n \sqcup Y_n \sqcup \bigsqcup_{i+j=n-1} X_i \times Y_j. \tag{1.10}$$

The boundary maps on the first two components are those of X and Y. On the third component it acts by

$$\mathsf{d}_i(x, y) := \begin{cases} (\mathsf{d}_i x, y) & \text{if } i \le j, j \ne 0, \\ (x, \mathsf{d}_{i-j-1} y) & \text{if } i > j, k \ne 0, \end{cases} \tag{1.11}$$

where $x \in X_j$ and $y \in Y_k$. For $j = 0$ and $k = 0$, the projections are used.

1.1.2 Homological algebra

In this section, simplicial sets are related to homological algebra (**??**). A basic introduction is Master (2020).

Construction 1.1.16 (Alternating face map complex). From a simplicial Abelian group A, one can construct a connective chain complex as follows. For every $n \in \mathbb{N}$, define

$$(CA)_n := A_n. \tag{1.12}$$

The boundary maps δ_n are defined as the alternating sum of the face maps:

$$\delta_n := \sum_{i=1}^n (-1)^i dr_i. \tag{1.13}$$

Every group A_{n+1} contains a subgroup $D(A_n)$ generated by the degeneracy maps:

$$D(A_n) := \left\langle \bigcup_{i=1}^n s_i(A_n) \right\rangle. \tag{1.14}$$

If these degenerate simplices are quotiented out, the **normalized complex** is obtained.

This construction can be generalized to any simplicial group:

Construction 1.1.17 (Moore complex). Let G be a simplicial group. For every $n \in \mathbb{N}$, define

$$(NG)_n := \bigcap_{i=1}^n \ker(\mathsf{d}_i^n). \tag{1.15}$$

The differential ∂_n is given by the zeroth face map d_0^n.

Property 1.1.18 (Equivalence). For simplicial Abelian groups, the Moore complex and normalized complex are isomorphic. Moreover, the inclusion of the normalized complex into the alternating face map complex is a quasi-isomorphism.

Theorem 1.1.19 (Dold–Kan correspondence). *The functor that maps simplicial Abelian groups to normalized chain complexes gives an equivalence of categories* $\mathbf{sAb} \cong \mathbf{Ch}^+(\mathbf{Ab})$.

1.2   Localization

Definition 1.2.1 (Category with weak equivalences). A category $\mathbf{C}$ with a subcategory $\mathbf{W}$ such that:

1. $\mathbf{W}$ contains all isomorphisms in $\mathbf{C}$ (in particular, $\mathbf{W}$ is wide).

2. $\mathbf{W}$ satisfies the '2-out-of-3 property': If any two of $\{f, g, f \circ g\}$ are in $\hom(\mathbf{W})$, so is the third.

Definition 1.2.2 (Weak factorization system). Consider a category $\mathbf{C}$. A pair (L, R) of classes of morphisms in $\mathbf{C}$ is called a weak factorization system (WFS) if it satisfies the following properties:

1. Every morphism in $\mathbf{C}$ factorizes as $g \circ f$ where $f \in L$ and $g \in R$.

2. L consists of exactly those morphisms in $\mathbf{C}$ that have the left lifting property (**??**) with respect to morphisms in R.

3. R consists of exactly those morphisms in $\mathbf{C}$ that have the right lifting property with respect to morphisms in L.

Remark. The original definition by *Quillen* only required that L and R satisfied the lifting properties with respect to each other, not that they were closed under this condition. This was enforced by a further condition that both L and R are closed under retracts in the arrow categories. It can be proven that this is equivalent to the definition above.

Definition 1.2.3 (Homotopical category). A category $\mathbf{C}$ equipped with a subcategory $\mathbf{W}$ such that:

1. $\mathbf{W}$ contains all identity morphisms in $\mathbf{C}$ (in particular, $\mathbf{W}$ is wide).

2. $\mathbf{W}$ satisfies the '2-out-of-6 property': If $f \circ g$ and $g \circ f$ are in $\hom(\mathbf{W})$, so are f, g, h and $f \circ g \circ h$.

It is not hard to see that every homotopical category is, in particular, a category with weak equivalences.

Definition 1.2.4 (Homotopical functor). A functor (between homotopical categories)

that preserves weak equivalences.

Definition 1.2.5 (Gabriel–Zisman localization). Consider a category **C** with a collection of morphisms $M \subset \mathrm{hom}(\mathbf{C})$. The localization of **C** with respect to M is constructed by adding for each morphism $f \in M$ a formal inverse to $\mathrm{hom}(\mathbf{C})$.

More specifically, the localization consists of a category $\mathbf{C}[M^{-1}]$ and a functor $F_M : \mathbf{C} \to \mathbf{C}[M^{-1}]$ inverting M, i.e. mapping all morphisms in M to isomorphisms, that is universal with respect to this property.

When **C** has finite products, the localization $C[x^{-1}]$ at an object $x \in \mathrm{ob}(\mathbf{C})$ is given by the localization at the projections $M_x := \{y \times x \to y \mid y \in \mathrm{ob}(\mathbf{C})\}$.

Definition 1.2.6 (Homotopy category I). When **C** is a category with weak equivalences W, the localization $\mathbf{C}[W^{-1}]$ is called the homotopy category $\mathbf{Ho}(\mathbf{C})$. In this context, the localization functor $\mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ is sometimes denoted by $\gamma_{\mathbf{C}}$.

Remark 1.2.7 (Size issues). When **C** is small, so is its localization. However, even in the case where **C** is locally small, its localization might be large.

Property 1.2.8 (Reflective localization). Consider an adjunction $F \dashv G : \mathbf{C} \to \mathbf{D}$. The right adjoint is fully faithful, i.e. determines a reflective subcategory (**??**), if and only if F realizes **D** as a localization of **C**. The essential image of G consists exactly of the local objects in C.

Definition 1.2.9 (Derived functor). Consider a homotopical functor $F : \mathbf{C} \to \mathbf{D}$ and consider the composition $\gamma := \gamma_{\mathbf{D}} \circ F$ with the localization functor $\gamma_{\mathbf{D}} : \mathbf{D} \to \mathbf{Ho}(\mathbf{D})$. The derived functor $\mathbf{Ho}(F) : \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{D})$ is obtained by applying the universal property of $\mathbf{Ho}(\mathbf{C})$ to γ.

This definition can be rephrased in terms of Kan extensions. Consider a homotopical functor $F : \mathbf{C} \to \mathbf{D}$. The left and right derived functors are defined as the following Kan extensions:

$$
\begin{aligned}
LF &:= \mathrm{Ran}_{\gamma_{\mathbf{C}}}(\gamma_{\mathbf{D}} \circ F), \\
RF &:= \mathrm{Lan}_{\gamma_{\mathbf{C}}}(\gamma_{\mathbf{D}} \circ F).
\end{aligned}
\tag{1.16}
$$

In fact, one can drop the assumption that **D** has weak equivalences (here, F should map weak equivalences to isomorphisms). In this case, one simply has to replace $\gamma_{\mathbf{D}} \circ F$ by F in the above formulas.

Example 1.2.10 (Derived category). Consider an Abelian category **C** together with its category of chain complexes $\mathbf{Ch}(\mathbf{C})$. The derived category $\mathcal{D}(\mathbf{C})$ is defined as the localization of $\mathbf{Ch}(\mathbf{C})$ at the quasi-isomorphisms.

Remark 1.2.11. In this case, it can be shown that one can first restrict to the naive homotopy category $\mathbf{K}(\mathbf{C})$, consisting of chain complexes and chain maps up to chain homotopy, and then localize at the collection of quasi-isomorphisms.

1.3 Model categories

1.3.1 Model structures

Definition 1.3.1 (**Model structure**). Let **C** be a category. A (**Quillen**-)model structure on **C** consists of three classes of morphisms:

- **weak equivalences** W,

- **fibrations** Fib, and

- **cofibrations** Cof,

that satisfy the following two conditions:

1. W turns **C** into a category with weak equivalences (Definition 1.2.1).

2. $(\text{Cof}, \text{Fib} \cap W)$ and $(\text{Cof} \cap W, \text{Fib})$ are weak factorization systems (Definition 1.2.2).

The morphisms in $\text{Fib} \cap W$ and $\text{Cof} \cap W$ are said to be **acyclic** or **trivial**.

Remark. That W contains all isomorphisms follows from the property that any class of morphisms satisfying a lifting property contains all isomorphisms.

Definition 1.3.2 (**Model category**). A bicomplete category equipped with a model structure.[1]

Definition 1.3.3 (**Proper model category**). A model category is said to be left proper (resp. right proper) if weak equivalences are preserved by pushouts along cofibrations (resp. pullbacks along fibrations).

Definition 1.3.4 (**Fibrant object**). An object in a model category for which the terminal morphism is a fibration. Dually, an object in a model category is said to be cofibrant if the initial morphism is a cofibration.

Example 1.3.5 (**Model structure on functor categories**). In some cases, the functor category $[\mathbf{C}, \mathbf{D}]$, for **C** small and **D** a model category, admits two canonical model structures:

- **Injective model structure**: The weak equivalences are the natural transformations that are objectwise weak equivalences and the cofibrations are the natural transformations that are objectwise cofibrations.

- **Projective model structure**: The weak equivalences are the natural transformations that are objectwise weak equivalences and the fibrations are the natural transformations that are objectwise fibrations.

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[1]*Quillen*'s original definition only required the existence of finite limits and finite colimits.

Property 1.3.6 (Resolution). In a model category, the bicompleteness property implies that the initial and terminal object always exist. The weak factorization property then implies that, for every object x, one can find a weakly equivalent fibrant replacement x^f and a weakly equivalent cofibrant replacement x^{cf} by suitably factorizing the initial and terminal morphisms. These replacements are sometimes called **resolutions** or **approximations**.

If the weak factorization system is functorial (**??**), (co)fibrant replacement defines an endofunctor that is weakly equivalent to the identity functor.

Definition 1.3.7 (Quillen adjunction). Let $\mathbf{C}, \mathbf{D}$ be two model categories. An adjunction

$$\mathbf{D} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{C}$$

is called a **Quillen adjunction** if the left adjoint preserves cofibrations and acyclic cofibrations. The model category axioms imply that this is equivalent to requiring that the right adjoint preserves fibrations and acyclic fibrations. The adjoint functors are called (left and right) **Quillen functors**.

If $F \dashv G$ is a Quillen adjunction, such that, for all cofibrant objects x and fibrant objects y, the morphism $Fx \to y$ is a weak equivalence if and only if the adjunct $x \to Gy$ is a weak equivalence, then $F \dashv G$ is called a **Quillen equivalence**.

Quillen equivalences can also be characterized at the homotopical level.

Property 1.3.8. A Quillen adjunction $F \dashv G$ is a Quillen equivalence if and only if the left derived functor LF (or right derived functor RG) is an equivalence.

Property 1.3.9 (Derived adjunction). If $F \dashv G$ is a Quillen adjunction, the derived functors (LF, RG) also form an adjunction.

Property 1.3.10 (Doubly categorical interpretation). The map that sends a model category to its homotopy category and a Quillen functor to its derived functor is a *double pseudofunctor*. Amongst other things, this implies that the composition of derived functors is naturally weakly equivalent to the derived functor of the composition.

Example 1.3.11 (Topological spaces). A first example of model categories is the category **Top** of topological spaces (**??** and **??**). This category can be endowed with a model structure by taking the weak equivalences to be the weak homotopy equivalences (**??**) and by taking the fibrations to be the Serre fibrations (**??**).

Example 1.3.12 (Simplicial sets). As a second example, consider the category **sSet**. This category can be turned into a model category by taking the weak equivalences to be the morphisms that induce weak homotopy equivalences between geometric realizations and by taking the fibrations to be Kan fibrations. With this structure, the

fibrant objects are the Kan complexes and the cofibrations are the monomorphisms, i.e. the levelwise injections. This model structure is generally called **Quillen's model structure** and the resulting model category is denoted by **sSet**$_{\text{Quillen}}$.

Property 1.3.13 (Quillen). The adjoint pair of geometric realization and singular set functors (Example 1.1.11 and Example 1.1.13) gives a Quillen equivalence between the above model categories. This result allows to regard simplicial sets as if they were spaces and vice versa. Consequently, most of homotopy theory can be done in either category.

1.3.2 Monoidal structures

Definition 1.3.14 (Pushout product). Let $(\mathbf{M}, \otimes)$ be a closed symmetric monoidal category and consider two morphisms $f : a \to b$ and $g : x \to y$ in $\mathbf{M}$. By taking suitable tensor products, one can form the span $a \otimes y \leftarrow a \otimes x \to b \otimes x$. If the pushout of this diagram exists, the pushout product $f \square g$ is defined as the unique morphism from this pushout to $b \otimes y$ defined by the obvious diagram.

The dual concept, where one of the arguments is contravariant, is sometimes called a **pullback hom**, **pullback exponential** or **pullback power** depending on which bifunctor is used in the definition. In fact, the requirement that $\mathbf{M}$ carries a monoidal structure can be dropped and any bifunctor $\circledast : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ can be used. This more general construction is sometimes called the **Leibniz construction** and the case where the bifunctor is a tensor bifunctor is then called the **Leibniz tensor**.

If $\mathbf{M}$ is also a model category, it is said to satisfy the **pushout-product axiom** if the pushout product of two cofibrations is again a cofibration, acyclic if any of the input cofibrations is.

Definition 1.3.15 (Quillen bifunctor). A bifunctor satisfying the pushout-product axiom that preserves colimits in both variables is called a **(left) Quillen bifunctor**. It should be noted that the tensor product automatically satisfies this last property in the case of closed monoidal categories.

In fact, the natural setting for defining Quillen bifunctors is that of two-variable adjunctions (**??**). Consider such a triple of bifunctors $(\otimes, \text{Hom}_L, \text{Hom}_R)$.

- The bifunctor $- \otimes - : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is said to be **left Quillen** if it satisfies the pushout-product axiom.

- The bifunctors $\text{Hom}_L : \mathbf{C}^{op} \times \mathbf{E} \to \mathbf{D}$ and $\text{Hom}_R : \mathbf{D}^{op} \times \mathbf{E} \to \mathbf{C}$ are said to be **right Quillen** if the Leibniz product of a cofibration and a fibration is a fibration and if the acyclicity of any of the inputs implies the acyclicity of the result.

It can be shown that, if one of these bifunctors is Quillen, the others are too.

Remark 1.3.16. The fact that for two-variable adjunctions one does not mention preservation of (co)limits follows from the property that left (resp. right) adjoints preserve colimits (resp. limits).

Definition 1.3.17 (Monoidal model category). A model category **M** that carries the structure of a closed symmetric monoidal category $(\mathbf{M}, \otimes, \mathbf{1})$ such that the following compatibility conditions are satisfied:

1. **Pushout-product**: The tensor bifunctor and internal homs define a Quillen two-variable adjunction.

2. **Unit**: For every cofibrant object $x \in \mathrm{ob}(\mathbf{M})$ and every cofibrant replacement $\mathbf{1}^c$ of $\mathbf{1}$, the induced morphism $\mathbf{1}^c \otimes x \to x$ is a weak equivalence.

Definition 1.3.18 (Enriched model category). Let $\mathcal{V}$ be a monoidal model category. A model category **M** is said to be $\mathcal{V}$-enriched if is satisfies the following conditions:

1. **M** is a $\mathcal{V}$-enriched category that is both powered and copowered over $\mathcal{V}$ (**??**).

2. The copower is a left Quillen bifunctor or, equivalently, the power is a right Quillen bifunctor.

Example 1.3.19 (Simplicial model category). An $\mathbf{sSet}_{\mathrm{Quillen}}$-enriched model category. It can be shown that the full subcategory of a simplicial model category on fibrant-cofibrant objects is enriched over Kan-complexes.

1.3.3   Homotopy category

Definition 1.3.20 (Homotopy). Before being able to construct homotopies in general model categories, thereby generalizing the ideas from **??**, one has to define the counterpart of the unit interval in a model category **M**. To this end, consider an object $x \in \mathrm{ob}(\mathbf{M})$. By definition, one has the unique diagonal and codiagonal morphisms $\Delta : x \to x \times x$ and $\nabla : x \sqcup x \to x$. If these morphisms can be factorized in **M** through weak equivalences

$$x \to \mathsf{Path}(x) \qquad \text{and} \qquad \mathsf{Cyl}(x) \to x\,,$$

a **path** and **cylinder object** are obtained. By choosing the morphism to be a fibration (resp. cofibration), the notion of **good** cylinder (resp. path) object is obtained.

Using these objects, one can define left and right homotopies between parallel morphisms $f, g : x \to y$. A left homotopy between f and g is a morphism $\eta : \mathsf{Cyl}(x) \to y$ such that Diagram 1.1a commutes. Analogously, a right homotopy between f and g is a morphism $\lambda : x \to \mathsf{Path}(y)$ such that Diagram 1.1b commutes.

Example 1.3.21 (Topological spaces). Consider the category **Top**. A cylinder object for a topological space X is given by the product $X \times [0, 1]$. The codiagonal map is
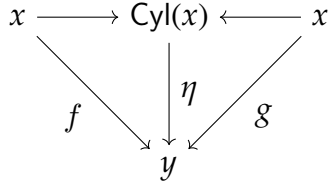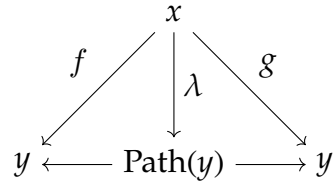
(a) A left homotopy.   (b) A right homotopy.

Figure 1.1: Homotopies in a model category.

factorized as the endpoint inclusion $X \sqcup X \rightrightarrows X \times [0,1]$ followed by the collapse $X \times [0,1] \overset{\pi_X}{\to} X$, where it is not hard to show that the collapse is a homotopy equivalence. A left homotopy with respect to $X \times [0,1]$ is exactly a homotopy in the sense of **??**.

A path object $\mathsf{Path}(X)$ is given by the mapping space $X^{[0,1]}$. The diagonal map is factorized by the basepoint inclusion $X \overset{\iota_0}{\to} X^{[0,1]}$ followed by the endpoint projections $X^{[0,1]} \rightrightarrows X \times X$. That the right homotopies with respect to $X^{[0,1]}$ give the same equivalence classes as the left homotopies is the content of **??**.

Property 1.3.22. If the domain is cofibrant, every left homotopy induces a right homotopy. Dually, if the codomain is fibrant, every right homotopy induces a left homotopy.

Corollary 1.3.23 (Equivalence relation). Whenever the domain $x \in \mathrm{ob}(\mathbf{M})$ is cofibrant and the codomain $y \in \mathrm{ob}(\mathbf{M})$ is fibrant, the relations of being left-homotopic (or, equivalently, right-homotopic) coincide on $\mathbf{M}(x,y)$ and, in particular, define equivalence relations. The equivalence classes are denoted by $[x,y]$.

Property 1.3.24 (Stability under composition). Homotopies are preserved under both precomposition and postcomposition by arbitrary morphisms.

Property 1.3.25 (Weak equivalences). A morphism is a weak equivalence if and only if it is a homotopy inverse.

Definition 1.3.26 (Homotopy equivalence). Two objects $x, y \in \mathrm{ob}(\mathbf{M})$ in a model category are said to be homotopy equivalent if there exists morphisms $f : x \leftrightarrows y : g$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity. The morphisms f, g are then said to be homotopy equivalences.

In Definition 1.2.5, it was shown that one can assign to every category with weak equivalences a homotopy category. When the category has the additional structure of a model category, one can construct an equivalent category.

Alternative Definition 1.3.27 (Homotopy category II). The homotopy category $\mathbf{Ho}(\mathbf{M})$ of a model category $\mathbf{M}$ is the category defined by the following data:

- **Objects**: $\mathrm{ob}(\mathbf{M})$, and

- **Morphisms**: $\mathbf{Ho}(\mathbf{M})(x,y) := [x^{\mathrm{cf}}, y^{\mathrm{cf}}]$.

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In fact, it is easier to restrict to the subcategory $\mathbf{M_{cf}}$ of $\mathbf{Ho(M)}$ on the fibrant-cofibrant objects due to the following property (when restricting to a subcategory, the resulting homotopy category is only equivalent and not isomorphic to the localization).

Property 1.3.28. The homotopy category of a model category is equivalent to those of the full subcategories on (co)fibrant objects:

$$\mathbf{Ho(M)} \cong \mathbf{Ho(M_f)} \cong \mathbf{Ho(M_c)} \cong \mathbf{Ho(M_{cf})}. \tag{1.17}$$

The following theorem gives a stronger version of Property 1.3.25 above.

Theorem 1.3.29 (Whitehead). *A weak equivalence between objects that are both fibrant and cofibrant is a homotopy equivalence.*

Property 1.3.30. A Quillen equivalence of model categories induces an equivalence of homotopy categories.

Property 1.3.31 (Monoidal model categories). The homotopy category of a monoidal model category has a closed monoidal structure defined by the induced derived adjunction. The homotopy category of an enriched model category is enriched, powered and copowered over the homotopy category of its enriching category, where the enriched structure is again given by the induced derived adjunction.

In some cases, it is useful to consider categories that are strictly weaker than model categories but stronger than categories with weak equivalences. The prime example being the full subcategories of a model category on the (co)fibrant objects. These are often easier to handle in the setting of homotopy theory.

Definition 1.3.32 (Category of fibrant objects). A category $\mathbf{C}$ with weak equivalences $\mathbf{W} \hookrightarrow \mathbf{C}$ equipped with another subcategory $\mathbf{F} \hookrightarrow \mathbf{C}$ for which the morphisms are called **fibrations** such that the following conditions are satisfied:

1. $\mathbf{C}$ has finite products.

2. Fibrations and acyclic fibrations are preserved under arbitrary pullbacks.

3. Every object admits a good path object.

4. All objects are fibrant.

Theorem 1.3.33 (Factorization lemma). *In a category of fibrant objects, any morphism can be factorized as the right inverse of an acyclic fibration followed by a fibration.*

The following theorem is important for characterizing functors that preserve weak equivalences.

Theorem 1.3.34 (Ken Brown's lemma). *Let $\mathbf{C}$ be a category of fibrant objects and let $\mathbf{D}$ be a category with weak equivalences. If a functor $F : \mathbf{C} \to \mathbf{D}$ maps acyclic fibrations to weak*

equivalences, it preserves all weak equivalences.

Remark 1.3.35. An analogous theorem exists for *categories of cofibrant objects*.

The above lemma allows to define derived functors for Quillen functors between model categories (the following definition is also better suited for working in the ∞-setting than Definition 1.2.9).

Alternative Definition 1.3.36 (Derived functor). Let $F : \mathbf{C} \to \mathbf{D}$ be a left (resp. right) Quillen functor. The left (resp. right) derived functors are obtained by precomposition with the cofibrant (resp. fibrant) replacement functors.

Property 1.3.37 (Derived functors are absolute). It can be shown that derived functors built using (co)fibrant replacement are given by absolute Kan extensions. So, even though homotopy categories often do not admit all (co)limits, the resulting Kan extensions do exist.

1.3.4   Reedy model structure ♣

Consider a bicomplete category $\mathbf{M}$ (this will often be **sSet**). For any full subcategory inclusion $\mathbf{D} \hookrightarrow \mathbf{C}$, one obtains an induced **truncation** (or **restriction**) functor $\mathrm{tr} : \mathbf{M}^{\mathbf{C}} \to \mathbf{M}^{\mathbf{D}}$. The left and right adjoints of this functor are, respectively, called the **skeleton** and **coskeleton** functors.

Formula 1.3.38. The adjoint functors are defined by Kan extensions and, hence, can be expressed in terms of (co)ends and weighted (co)limits:

$$
\begin{aligned}
\mathrm{sk}(X)x &:= \int^{y \in \mathbf{D}} \mathbf{C}(y, x) \cdot Xy = \mathrm{colim}^{\mathbf{C}(-, x)} X, \\
\mathrm{cosk}(X)x &:= \int_{y \in \mathbf{D}} \big[\mathbf{C}(x, y), Xy \big] = \lim^{\mathbf{C}(x, -)} X.
\end{aligned}
\tag{1.18}
$$

Definition 1.3.39 (Skeletal sets). Let $\mathbf{M} = \mathbf{sSet}$ and consider the inclusion $\Delta_{\leq n} \hookrightarrow \Delta$ of the full subcategory on the objects $\{[0], \dots, [n]\}$. The n-truncation functor tr_n discards all simplices of degree higher than n or, in other words, it 'truncates' a simplicial set at degree n.

The n-skeleton functor sk_n takes a simplicial set X of degree $\leq n$ and freely adds degenerate simplices in degrees $> n$, i.e. it is the smallest simplicial set containing X as a simplicial subset. The n-coskeleton functor cosk_n adds a simplex in degree $> n$ whenever all of its faces are present, i.e. the m-simplices in $\mathrm{cosk}_n X$ are given by the collection of all $(m + 1)$-tuples of $(m - 1)$-simplices that are compatible (along lower simplices).

Property 1.3.40 (Simplicial nerve). The nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ from Example 1.1.10 is a fully faithful functor to the category of 2-coskeletal simplicial sets. This follows from the fact that, in ordinary categories, compositions of morphisms are unique and, hence, all higher-order tuples of composable morphisms are determined by composable pairs. In fact, this is just the characterization of (small) categories as categories

internal to **Set** (under the isomorphism $C_2 \cong C_1 \times_{C_0} C_1$ which will be called the first *Segal condition* in Definition 1.4.19).

Definition 1.3.41 (**Reedy category**). A category **R** equipped with a **degree** function $\mathrm{ob}(\mathbf{R}) \to \alpha$, where α is an ordinal (**??**), and two wide subcategories $\mathbf{R}^{\pm}$ that satisfy the following conditions:

1. Nontrivial morphisms in $\mathbf{R}^+$ (strictly) increase the degree.

2. Nontrivial morphisms in $\mathbf{R}^-$ (strictly) decrease the degree.

3. All morphisms admit a unique factorization as a morphism in $\mathbf{R}^-$ followed by a morphism in $\mathbf{R}^+$. This factorization is sometimes called the (**canonical**) **Reedy factorization**.

Property 1.3.42 (**Minimality**). The Reedy factorization is the (unique) factorization of minimal degree, where the **degree** of a factorization $x \xrightarrow{f} y \xrightarrow{g} z$ is defined as the degree of y.

Property 1.3.43 (**Isomorphisms are trivial**). A morphism in a Reedy category is an isomorphism if and only if it is trivial.

Example 1.3.44. Some common examples of Reedy categories are discrete categories, finite posets, the simplex category $\mathbf{\Delta}$ and opposites of Reedy categories.

For Reedy categories, one can also define n-truncation, n-skeleton and n-coskeleton functors by restricting to the full subcategories on elements of degree $\leq n$.

Definition 1.3.45 (**Matching and latching objects**). Let **R** be a Reedy category and consider a diagram $X \in [\mathbf{R}, \mathbf{C}]$ with **C** small. Consider the skeleton monad and coskeleton comonad (often just called the skeleton and coskeleton functors) $\mathbf{sk}_n := \mathrm{sk}_n \circ \mathrm{tr}_n$ and $\mathbf{cosk}_n := \mathrm{cosk}_n \circ \mathrm{tr}_n$. The latching and matching objects of X at $r \in \mathrm{ob}(\mathbf{R})$ are defined as the restrictions of $\mathbf{sk}_{\deg(r)-1}$ and $\mathbf{cosk}_{\deg(r)-1}$ to the degree-$\deg(r)$ subcategory of **R**, respectively. The counits of these adjunctions give rise to the **latching** and **matching** maps.

One can also define the latching and matching objects through (co)limits. Define the subcategory $\mathbf{R}^+(r)$ as the subcategory of $\mathbf{R}^+/r$ on all objects except the identity. The latching object $L_r X$ can be shown to be isomorphic to the colimit of X over $\mathbf{R}^+(r)$.

Example 1.3.46 (**Simplicial objects**). The above property allows to give a nice interpretation to latching objects in the case of $\mathbf{R} = \mathbf{\Delta}^{\mathrm{op}}$. Using the *Eilenberg–Zilber lemma*, it can be shown that the nth latching object of a simplicial object is given by its collection of degenerate n-simplices.

Definition 1.3.47 (**Boundary**). The boundary of a representable presheaf $\mathbf{R}(-, r)$, denoted by $\partial \mathbf{R}(-, r)$, is defined as its latching object. It can be shown that $\partial \mathbf{R}(-, r)$ consists

of exactly those morphisms that are not in $\mathbf{R}^-$.

Formula 1.3.48. One can show that the latching and matching objects can be obtained through (co)limits weighted by boundaries:

$$
\begin{aligned}
M_r X &\cong \lim{}^{\partial \mathbf{R}(r,-)} X\,, \\
L_r X &\cong \operatorname{colim}{}^{\partial \mathbf{R}(-,r)} X\,.
\end{aligned}
\tag{1.19}
$$

From here on, $\mathbf{M}$ will be assumed to be a model category. In this case, a canonical model structure on the functor category $[\mathbf{R}, \mathbf{M}]$ for Reedy R can be defined.

Definition 1.3.49 (Relative matching and latching objects). Consider the (weighted) colimit bifunctor

$$
\operatorname{colim} : \mathbf{Psh}(\mathbf{R}) \times [\mathbf{R}, \mathbf{M}] \to \mathbf{M}\,.
\tag{1.20}
$$

The Leibniz construction 1.3.14 allows to define the relative latching object of $f : X \to Y$ at $r \in \operatorname{ob}(\mathbf{R})$ as the Leibniz product of the boundary inclusion $\partial \mathbf{R}(-, r) \hookrightarrow \mathbf{R}(-, r)$ and f.

By **??** (**??**) and (1.19), the relative latching map is of the form $Xr \sqcup_{L_r X} L_r Y \to Yr$ and the relative matching map is of the form $Xr \to Yr \times_{M_r Y} M_r X$.

Property 1.3.50 (Reedy model structure). Let $\mathbf{R}$ be a Reedy category and let $\mathbf{M}$ be a model category. The functor category $[\mathbf{R}, \mathbf{M}]$ admits the following model structure:

- **Weak equivalences**: the objectwise weak equivalences.

- **Fibrations**: morphisms for which the relative matching map is a fibration (in $\mathbf{M}$) for all $r \in \operatorname{ob}(\mathbf{R})$

- **Cofibrations**: morphisms for which the relative latching map is a cofibration (in $\mathbf{M}$) for all $r \in \operatorname{ob}(\mathbf{R})$

Property 1.3.51 (Quillen (co)limit functors). Consider a $\mathcal{V}$-enriched model category $\mathbf{M}$ and a Reedy category $\mathbf{R}$. For every Reedy cofibrant functor W in $[\mathbf{R}, \mathcal{V}]$, the weighted limit and colimit functors are right and left Quillen, respectively.

@@ COMPLETE (this section is too abstract and difficult at this point, a more intuitive explanation of matching/latching objects is necessary) @@

1.4 Simplicial spaces

1.4.1 Kan complexes

Definition 1.4.1 (Horn). Consider the standard simplex $\Delta[n]$. For all $n \geq 1$ and $0 \leq k \leq n$, the (n, k)-horn $\Lambda^k[n]$ is defined as the subsimplicial set obtained by removing

the kth face from $\partial\Delta[n]$. When $k = 0$ or $k = n$, the horn is said to be **outer**, otherwise it is said to be **inner**.

Definition 1.4.2 (Kan fibration). A morphism of simplicial sets that has the right lifting property with respect to all horn inclusions $\Lambda^k[n] \hookrightarrow \Delta[n]$.

Definition 1.4.3 (Kan complex). A simplicial set that has all horn fillers or, equivalently, a simplicial set for which the terminal morphism is a Kan fibration. The full subcategory of **sSet** on Kan complexes is denoted by **Kan**.

Property 1.4.4 (Horn filler condition). A simplicial set is the nerve of a (small) category if and only if all of its inner horns admit a unique filler (compositions are unique). If one requires all horns to admit a unique filler, the nerve of a groupoid is obtained.

By relaxing the above requirements, one can generalize the notion of a category (this is due to *Boardman* and *Vogt*).

Definition 1.4.5 (Quasicategory[2]**).** A simplicial set for which all inner horns have (not necessarily unique) fillers. This condition is sometimes called the **Boardman condition**.

At this point, a third instance of a 'homotopy category' can be defined.

Definition 1.4.6 (Homotopy category III). Let X be a quasicategory. The homotopy category **Ho**(X) consists of the following data:

- **Objects**: X_0.

- **Morphisms**: The quotient of X_1 under the relation $f \circ g \sim h$ if there exists a 2-simplex with (suitably oriented) edges f, g and h.

Property 1.4.7 (Fundamental category). If X is a quasicategory, its homotopy category is equivalent to its **fundamental category** $\pi_1(X)$, i.e. the image under the left adjoint of the (simplicial) nerve functor.

The following theorem is a restatement of Property 1.4.4.

Theorem 1.4.8 (Joyal). *A quasicategory is a Kan complex if and only if its homotopy category is a groupoid.*

1.4.2   Simplicial localization

Definition 1.4.9 (Homotopy category IV). Consider a simplicially enriched category **C**. Its homotopy category $\pi_0(\mathbf{C})$ is defined as follows:

- **Objects**: ob(**C**), and

---

[2]Some authors, such as *Joyal*, call these **logoi** (singular: **logos**).

- **Morphisms**: $\pi_0\mathbf{C}(x,y)$.

Two morphism $f, g \in \mathbf{C}(x,y)$ are said to be **homotopic** if they are identified in $\pi_0(\mathbf{C})$. A morphism $f \in \mathbf{C}(x,y)$ is called a **homotopy equivalence** if it admits both a left and a right inverse in $\pi_0(\mathbf{C})$, i.e. if it becomes an isomorphism in the homotopy category.

Definition 1.4.10 (Dwyer–Kan equivalence I). Consider a simplicial functor $F : \mathbf{C} \to \mathbf{D}$ between two simplicially enriched categories. This functor is called a Dwyer–Kan equivalence if:

- F is ∞-**fully faithful**, i.e. the induced map on hom-objects is a weak equivalence.[3]

- The induced map on connected components $\pi_0(F) : \pi_0(\mathbf{C}) \to \pi_0(\mathbf{D})$ is an equivalence (of categories). In fact, this condition can be relaxed to $\pi_0(F)$ being essentially surjective, since together with the previous condition this implies that $\pi_0(F)$ is an equivalence.

Construction 1.4.11 (Hammock localization). The hammock localization (or **simplicial localization**) $L^H\mathbf{C}$ of a category with weak equivalences $(\mathbf{C}, W)$ is the simplicially enriched category constructed as follows:

- **Objects**: $\mathrm{ob}(\mathbf{C})$, and

- **Morphisms**: $L^H\mathbf{C}(x,y)$ is the simplicial set defined as follows:

  First, for every $n \in \mathbb{N}$, one constructs a category with as objects the zigzags of morphisms in $\mathbf{C}$ relating x and y such that all left-pointing morphisms are in W, and as morphisms the endpoint-preserving 'natural transformations' (in the sense that all triangles/squares in the resulting diagram commute). Then, the coproduct is taken over the (simplicial) nerves of the categories for all $n \in \mathbb{N}$. Finally, the quotient is taken by the equivalence relations generated by:

  1. inserting or removing identity morphisms, and

  2. composing morphisms.

The following property relates all constructions of what could be called homotopy categories.

Property 1.4.12. Let $(\mathbf{C}, W)$ be a category with weak equivalences. It can be shown that

$$\mathbf{Ho}(L^H\mathbf{C}) \cong \mathbf{C}[W^{-1}]. \tag{1.21}$$

---

[3]This is a generalization of the ordinary definition for categories. (Simplicial categories will give models for ∞-*categories.*)

This construction gives a more explicit description of the homotopy category. Furthermore, if **M** is a simplicial model category, the categories **M**$_{cf}$ and L^H**M** are Dwyer–Kan equivalent.

Property 1.4.13. Quillen equivalent model categories have Dwyer–Kan equivalent simplicial localizations.

Property 1.4.14. Up to Dwyer–Kan equivalence, every simplicially enriched category can be obtained as the simplicial localization of a category with weak equivalences.

Definition 1.4.15 (Simplicial resolution). Consider an object $x \in \mathrm{ob}(\mathbf{M})$ in a model category **M**. A (co)simplicial resolution of x is a Reedy (co)fibrant (co)simplicial object X together with a weak equivalence $x \simeq X_0$.

Every object in a model category admits a (co)simplicial resolution by taking a (co)fibrant replacement of the constant (co)simplicial object.

Definition 1.4.16 (Bergner model structure). The catgory of simplicially enriched categories admits the following model structure:

- **Weak equivalences**: Dwyer–Kan equivalences,

- **Cofibrant objects**: *simplicial computads*, and

- **Fibrant objects**: **Kan**-enriched categories.

Construction 1.4.17 (Free resolution). Consider a (small) category **C**. From this category, we can construct a simplicial category $\mathfrak{C}\mathbf{C}$ consisting of the following data:

- **Objects**: $\mathrm{ob}(\mathbf{C})$, and

- **Morphisms**: $\mathrm{hom}(\mathfrak{C}\mathbf{C})_n := \mathrm{hom}(F^{n+1}\mathbf{C})$, where F is identity-on-objects and $F\mathbf{C}$ has as morphisms strings of composable morphisms in **C**.

It can be shown that $\mathfrak{C}\mathbf{C}$ is a *simplicial computad* for any simplicially enriched category **C**, $\mathfrak{C}$ acts as the cofibrant replacement functor in the Bergner model structure.

1.4.3 Segal spaces

Example 1.4.18 (Bisimplicial sets). Simplicial objects in **sSet** are, in particular, also **sSet**-enriched and, accordingly, such objects are often called bisimplicial sets or **simplicial spaces**[4]. Since **sSet** is a model category, Example 1.3.5 above says that **ssSet** itself admits a model structure. It can furthermore be shown that the injective model structure on **ssSet** coincides with the Reedy model structure (a rather nontrivial statement).

---

[4]The latter name follows from the fact that topological spaces and simplicial sets are (Quillen-)equivalent.

Definition 1.4.19 (Segal space). Consider a fibrant object X in the injective model structure on **ssSet**. This bisimplicial set is called a Segal space if it satisfies the following weak form of the **Segal condition** for all $n \geq 1$:[5]

$$X_n \simeq \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}} . \tag{1.22}$$

These maps are called **Segal maps** (even when they are not weak equivalences). They are the morphisms induced by **spine** inclusions, i.e. inclusions of the union of edge 1-cells.

Definition 1.4.20 (Segal category). A bisimplicial set X is called a **Segal precategory** if X_0 is discrete. It is called a Segal category if, in addition, all its Segal maps are weak equivalences, i.e. if it is a Segal space with a discrete set of 'objects'.

Definition 1.4.21 (Mapping space). Consider a Segal space X. For every two points $x, y \in X_0$, the mapping space $\mathsf{Map}(x, y)$ is defined as the fibre of $(d_1, d_0) : X_1 \rightrightarrows X_0 \times X_0$ over the point (x, y). The identity at $x \in X_0$ is given by $s_0(x)$.

The following two definitions should be compared to Definitions 1.4.9 and 1.4.10.

Definition 1.4.22 (Homotopy category V). Consider a Segal category X. Its homotopy category $\mathsf{Ho}(X)$ is defined as follows:

- **Objects**: X_0, and

- **Morphisms**: $\pi_0 \mathsf{Map}(x, y)$.

Two points $f, g \in \mathsf{Map}(x, y)$ are said to be **homotopic** if they are identified in $\mathsf{Ho}(X)$. A point $f \in \mathsf{Map}(x, y)$ is called a **homotopy equivalence** if it admits both a left and a right inverse in $\mathsf{Ho}(X)$, i.e. if it becomes an isomorphism. The subspace $X_{\text{hoequiv}} \subset X_1$ consists of the components that contain homotopy equivalences.[6]

Definition 1.4.23 (Dwyer–Kan equivalence II). A map of Segal spaces such that:

1. the induced map on mapping spaces is a weak equivalence, and

2. the induced map between homotopy categories is an equivalence (of categories).

Definition 1.4.24 (Complete Segal space). A Segal space X for which the map $s_0 : X_0 \to X_{\text{hoequiv}}$ is a weak equivalence.

Property 1.4.25 (Dwyer-Kan equivalence). A map of Segal spaces is a Dwyer–Kan equivalence if and only if it is a weak equivalence in the *complete Segal space model struc-*

---

[5]If the Reedy condition is omitted, the limit on the right-hand side has to be replaced by a *homotopy limit*.

[6]It should be noted that if any point in a component is a homotopy equivalence, all points in that component are homotopy equivalences.

ture. A map of complete Segal spaces is a Dwyer–Kan equivalence if and only if it is a levelwise weak equivalence.

1.4.4   Coherent nerve

The nerve and realization functors (Construction 1.1.9) can also be modified to incorporate the higher homotopical data present in a simplicially enriched category.

First, define a cosimplicial simplicially enriched category $C : \Delta \to$ **sSetCat** that assigns to every finite ordinal $[n]$, the category consisting of the following data:

- **Objects**: $[n] \equiv \{0, 1, \ldots, n\}$, and

- **Morphisms**: $C[n](i, j) := N(P_{ij})$, where N is the ordinary nerve functor on **Cat** and P_{ij} is the poset consisting of all subsets of $\{i, \ldots, j\}$ that contain both i and j.

The (homotopy) **coherent nerve functor** (or **simplicial nerve functor**[7]) is the nerve functor induced by this cosimplicial object. It not only knowns about all possible morphisms, it also knows about all possible ways how one can obtain this morphism.

Remark 1.4.26. The cosimplicial object $C : \Delta \to$ **sSetCat** is in fact just the free resolution functor $\mathfrak{C}$ in the Bergner model structure (Construction 1.4.17) restricted to the subcategory Δ.

Property 1.4.27. The homotopy coherent nerve functor N_{Δ} is uniquely determined by the following relation:

$$\mathbf{sSet}(\Delta[n], N_{\Delta}\mathbf{C}) = \mathbf{sSetCat}(C[n], \mathbf{C}). \tag{1.23}$$

Property 1.4.28 (Simplicial realization). The left adjoint of the homotopy coherent nerve functor satisfies the following relation for all finite ordinals $[n]$:

$$|\Delta[n]| \cong C[n]. \tag{1.24}$$

Property 1.4.29 (Quasicategories). If **C** is **Kan**-enriched, its coherent nerve is a quasicategory. By Example 1.3.19, this allows to associate a quasicategory to any simplicial model category by passing to the coherent nerve of its fibrant-cofibrant subcategory.

At this point, it is interesting to reconsider the simplicial nerve construction (Example 1.1.10). Although the functor $N :$ **Cat** $\to$ **sSet** is fully faithful, there is a problem in that weak equivalences of simplicial sets do not necessarily correspond to (weak) equivalences of categories. The issue comes from the definition of the canonical model structure (Example 1.3.12) on **sSet**. Here, the weak equivalences are those morphisms

---

[7]This terminology was also used for the ordinary nerve functor taking values in **sSet**. In general, it should be clear from the context which one is meant.

that become weak equivalences after geometric realization. During this step, information about the direction of morphisms is lost and it becomes impossible to distinguish categories from groupoids. Only when all morphisms are assumed to be invertible, i.e. the category is assumed to be a groupoid, does one recover the converse statement.

A first solution is to use a different model structure with less weak equivalences.

Definition 1.4.30 (Joyal model structure). sSet admits a model structure defined by the following data:

- **Weak equivalences**: *weak categorical equivalences*, i.e. maps $f : X \to Y$ such that the induced simplicial functor $C(f) : C(X) \to C(Y)$ is a Dwyer–Kan equivalence.

- **Cofibrations**: monomorphisms.

- **Fibrant objects**: quasicategories (Definition 1.4.5).

When characterizing those simplicial sets that are nerves of categories, groupoids and categories are distinguished by whether all horns admit a (unique) filler or only inner horns admit a (unique) filler. However, when the fibrant objects ar<e Kan complexes, those simplicial sets that admit all horn fillers, the homotopical structure only knows about nerves that come from a groupoid. By passing to the larger class of quasicategories, this problem is solved.

Instead of changing the model structure on **sSet** to overcome the issues with taking nerves of categories or groupoids, one can also change the construction of the nerve functor. An alternative approach was introduced by *Rezk*.

Definition 1.4.31 (Classifying diagram). Consider a (small) category **C** together with the functor category $\mathbf{C}^{[n]}$. The classifying diagram of **C** is the bisimplicial set $\widetilde{N}\mathbf{C}$ defined levelwise as follows:

$$\widetilde{N}\mathbf{C}_k := N\big(\mathsf{Core}(\mathbf{C}^{[k]})\big), \tag{1.25}$$

where N is the ordinary nerve functor and Core denotes the core of a category (**??**).

The reason for why this construction is better for distinguishing categories stems from the fact that information about isomorphisms is already captured at degree 0, while information about noninvertible morphisms is only captured from degree 1 onwards.

Property 1.4.32. If **C** is small, then $\widetilde{N}\mathbf{C}$ is a complete Segal space.

@@ COMPLETE @@

1.5 Cofibrantly generated categories

1.5.1 Transfinite constructions

Before proceeding, some notions from ordinary category need to be specialized to the context of regular cardinals (**??**). In this section, the symbol κ will always denote such a regular cardinal.

Definition 1.5.1 (κ-filtered category). A category in which every diagram with less than κ arrows admits a cocone.

Definition 1.5.2 (κ-directed limit). Consider a poset I such that every subposet of cardinality less than κ has a lower bound (upper bound for directed colimits). Such a set is said to be κ-(co)directed. A limit of a diagram over this poset is called a κ-(co)directed (co)limit.

The following definition is a categorification of the previous one.

Definition 1.5.3 (κ-filtered limit). Consider a diagram $D : \mathsf{I} \to \mathsf{C}$. The limit (resp. colimit) of D is said to be κ-cofiltered (resp. κ-filtered) if I is a κ-cofiltered (resp. κ-filtered) category.

It should be noted that an analogue of **??** also holds in the κ-context, i.e. a category has all κ-directed limits if and only if it has all κ-filtered limits (and analogously for colimits).

Definition 1.5.4 (Small object). An object for which there exists a regular cardinal κ such that its covariant hom-functor preserves all κ-filtered colimits. These objects are also said to be κ**-compact** or κ**-presentable** (cf. **??**).

Definition 1.5.5 (Accessible category). A locally small category C for which there exists a regular cardinal κ such that C has all κ-filtered colimits and such that it is generated under κ-filtered colimits by a small subcategory D of κ-small objects, i.e. $\mathsf{C} \cong \mathsf{Ind}_\kappa(\mathsf{D})$.

Alternative Definition 1.5.6. A category equivalent to the category of (κ-)flat functors (**??**) on a small category or, by **??**, the category of (κ)ind-objects on a small category.

Definition 1.5.7 (Locally presentable category). A cocomplete, accessible category. It can be shown that such categories are also complete and, hence, bicomplete.

Property 1.5.8. Every locally presentable category can be obtained as a full reflective subcategory of a presheaf category under an **accessible embedding**, i.e. a finitary embedding of accessible categories.

Property 1.5.9 (Gabriel–Ulmer duality). First, define the following two 2-categories. **LFP** consists of the following data:

- **Objects**: locally finitely presentable categories,

- **1-Morphisms**: finitary, right adjoint functors (**??**), and

- **2-Morphisms**: natural transformations.

Lex consists of the following data:

- **Objects**: small, finitely complete categories,

- **1-Morphisms**: left exact functors, and

- **2-Morphisms**: natural transformations.

There exists an equivalence **Lex**$^{op} \cong$ **LPF** that sends a small, finitely complete category to its category of left exact copresheaves.

Theorem 1.5.10 (Adjoint functor theorem). *Consider a functor $F : \mathbf{C} \to \mathbf{D}$ between locally presentable categories. F admits a right adjoint if and only if it preserves all colimits. F admits a left adjoint if and only if it is accessible and if it preserves all limits.*

For ordinary categories, the axioms guarantee the existence of a (unique) composition of any finite number of (composable) morphisms. However, in some cases it is useful, or even necessary, to talk about the 'composite' of an infinite number of morphisms.

Definition 1.5.11 (Transfinite composition). Consider a category $\mathbf{C}$ with a collection of morphisms $I \subseteq \hom(\mathbf{C})$ and let α be an infinite ordinal. An α-indexed **transfinite sequence** of morphisms in I is a diagram of the form $D : \alpha \to \mathbf{C}$ such that:

1. Successor morphisms in α are mapped to elements of I.

2. D is 'continuous' in the sense that for every limit ordinal $\beta < \alpha$: $D\beta \cong \operatorname*{colim}_{\gamma < \beta} D\gamma$.

$D\lambda$ denotes the restriction of D to the (full) subdiagram $\{\gamma \mid \gamma < \lambda\}$ of α. The transfinite composition of this sequence is the induced morphism $D_0 \to D\alpha \equiv \operatorname{colim} D$.

Definition 1.5.12 (Cell complex). Consider a cocomplete category $\mathbf{C}$ with a collection of morphisms $I \subseteq \hom(\mathbf{C})$. A **relative** I-cell complex is a transfinite composition of pushouts (of coproducts[8]) of morphisms in I. An I-cell complex is an object such that the unique morphism from the initial object is a relative I-cell complex.

Notation 1.5.13 (Relative cell complexes). The set of all relative I-cell complexes is often denoted by $\operatorname{cell}(I)$.

---

[8]It can be shown that closure under coproducts follows automatically.

1.5.2 Cofibrant generation

Definition 1.5.14 (**Saturated class**)**.** A collection of morphisms $I \subseteq \hom(\mathbf{C})$ in a category is said to be (**weakly**) **saturated** if it is closed under:

1. pushouts,

2. retracts (in the arrow category of $\mathbf{C}$), and

3. transfinite compositions.

Definition 1.5.15 (**Cofibrantly generated WFS**)**.** A weak factorization system (R, L) on a category $\mathbf{C}$ (Definition 1.2.2) is said to be cofibrantly generated by a collection of morphisms $I \subseteq \hom(\mathbf{C})$ if

$$R = \mathsf{rlp}(I). \tag{1.26}$$

The following is a famous result by *Quillen*.

Theorem 1.5.16 (**Small object argument**)**.** *Let $\mathbf{C}$ be a locally presentable category with a set of morphisms $I \subseteq \hom(\mathbf{C})$. The WFS given by $\mathsf{llp}(I) = \mathsf{cell}(I)$ is cofibrantly generated.*

Remark 1.5.17. This theorem can be generalized to cocomplete categories where the domains of morphisms in I are small relative[9] to $\mathsf{cell}(I)$. Sets of morphisms with this property are said to **admit a small object argument**.

Definition 1.5.18 (**Cofibrantly generated model category**)**.** A model category $\mathbf{C}$ is said to be cofibrantly generated by two sets of morphisms $I, J \subseteq \hom(\mathbf{C})$ if it satisfies the following conditions:

1. *I and J both admit the small object argument.*

2. The fibrations are given by $\mathsf{rlp}(J)$.

3. The acyclic fibrations are given by $\mathsf{rlp}(I)$.

It can be shown that the last two conditions are equivalent to the following ones:

2*. The cofibrations are the retracts (in the arrow category) of $\mathsf{cell}(I)$.

3*. The acyclic cofibrations are the retracts (in the arrow category) of $\mathsf{cell}(J)$.

The morphisms in I and J are called the **generating cofibrations** and **generating acyclic cofibrations**, respectively.

Sometimes, it is desirable to replace the model structure on a category $\mathbf{C}$ by one that has more weak equivalences (denote the new one by $\mathbf{C}_0$). If the cofibrations remain

---

[9]*Small relative to a set of morphisms* is defined just as ordinary smallness, but with general κ-filtered colimits replaced by those that start from morphisms in the given set.

the same, this new structure has some nice properties:

- The fibrations are a subclass of the original ones.

- The acyclic fibrations remain the same.

- The identity functors $\mathbb{1}_{C_0} : C_0 \leftrightarrows C : \mathbb{1}_C$ form a Quillen adjunction.

- Every object in C is weakly equivalent (in C_0) to one in C_0.

This procedure can be made explicit for a specific class of model categories. Let C be a left proper, cofibrantly generated, simplicial model category and consider a collection $S \subset \text{hom}(C)$ of cofibrations with cofibrant domain. First, the notion of 'S-local objects' is introduced.

Definition 1.5.19 (Local object). A fibrant object $x \in \text{ob}(C)$ is said to be S-local if, for all morphisms in S, the image under the Yoneda embedding of x is an acyclic Kan fibration. Analogously, a morphism is called an S-local weak equivalence if, for all S-local fibrant objects, its image under their Yoneda embeddings is an acyclic Kan fibration.

Property 1.5.20. Every weak equivalence is an S-local weak equivalence: $W \subset W_S$.

Construction 1.5.21 (Left Bousfield localization). Given a model category C with the same assumptions as before, the (left) Bousfield localization $L_S C$ is defined as the same category but with the following model structure:

- **Cofibrations**: $\text{cof}(C)$, and

- **Acyclic cofibrations**: cofibrations that are S-local weak equivalences.

If it exists, the Bousfield localization $L_S C$ of C at S is the universal left Quillen functor $\gamma : C \to L_S C$ such that its left derived functor sends the image of S to isomorphisms in $\text{Ho}(C)$. The identity functor gives a Quillen adjunction between C and $L_S C$ and the induced adjunction on homotopy categories defines a reflective localization.

Remark 1.5.22. The notions of local object/morphisms can be slightly generalized (they give equivalent localizations). An S-**local morphism** $f : a \to b$ is a morphism such that for all objects $x \in \text{ob}(M)$ for which the map

$$C(g^c, x^f) : C(d^c, x^f) \to C(c^c, x^f) \tag{1.27}$$

is a weak equivalence for all $g \in S$ (x is said to be S-**local**), the map

$$C(f^c, x^f) : C(b^c, x^f) \to C(a^c, x^f) \tag{1.28}$$

is also a weak equivalence. This is the homotopical version of **??**.

Definition 1.5.23 (Combinatorial model category). A locally presentable, cofibrantly generated model category.

The following theorem is an analogue of Property 1.4.14 for combinatorial model categories.

Theorem 1.5.24 (**Dugger**). *Every combinatorial model category is Quillen equivalent to a (left) Bousfield localization of a category of simplicial presheaves on a small category (with the global projective model structure[10]).*

@@ FINISH @@

1.6 Homotopy (co)limits

Consider a category $\mathbf{C}$ with weak equivalences, together with diagrams $D, D' : \mathbf{I} \to \mathbf{C}$. Assume that there exists a weak equivalence between D and D', i.e. a natural transformation that consists of componentwise weak equivalences. Clearly, this induces a morphism between (co)limits, but it would be nice if the construction of (co)limits would also preserve the homotopical structure, i.e. weakly equivalent diagrams should have weakly equivalent (co)limits.

The main purpose of this section is to introduce a modification of the ordinary (co)limit functors that takes into account the homotopical structure of the underlying categories. A first step is the modification of (co)products.

Definition 1.6.1 (**Homotopy (co)products**). In ordinary categories, the universal property of a product (and, dually, for a coproduct) characterizes it as an object with an isomorphism

$$\mathbf{C}(-, c) \cong \prod_{i \in I} \mathbf{C}(-, c_i) \tag{1.29}$$

in that every collection of morphisms $\{f_i : x \to c_i\}_{i \in I}$ can be factorized as follows

$$
\begin{array}{ccc}
x & & \\
\exists! \Big\downarrow & \searrow^{f_i} & \\
c & \xrightarrow[\pi_i]{} & c_i \, .
\end{array}
$$

To obtain a homotopical version, the commutativity of this diagram is relaxed up to a homotopy/path in $\mathbf{C}(x, c_i)$. This leads to the definition of a homotopy product as an object $c \in \mathrm{ob}(\mathbf{C})$ together with a natural weak (homotopy) equivalence

$$\psi_x : \mathbf{C}(x, c) \simeq \prod_{i \in I} \mathbf{C}(x, c_i) \, . \tag{1.30}$$

---

[10]Example 1.3.5

The homotopy (co)products are unusual in the sense that they can be obtained as (co)limits in the homotopy category **Ho**(**C**). More generally, one can define homotopy (co)limits in any homotopical category by passing to derived functors.

Definition 1.6.2 (Homotopy (co)limits). Let **A** be a homotopical category and let **B** be a small category. The homotopy limit and colimit functors are defined as the derived functors of the limit and colimit functors (co)lim : [**B**, **A**] → **A**.

Remark 1.6.3. The reason for why homotopy (co)products can be obtained as ordinary (co)limits is related to the fact that their indexing categories are discrete. In this case, the adjoint of the derived functor is equivalent to the diagonal functor on the homotopy category.

Property 1.6.4 (Model categories). Consider the specific case where **C** is a model category. If [**D**, **C**] admits an injective (resp. projective) model structure, the homotopy limit (resp. colimit) always exists and can be obtained through fibrant (resp. cofibrant) replacement as in Definition 1.3.36.

Example 1.6.5 (Reedy categories). Consider the general case of diagrams $D : \mathbf{R} \to \mathbf{C}$ with **R** Reedy. First, note that the constant functor $\Delta : \mathbf{C} \to [\mathbf{R}, \mathbf{C}]$ maps weak equivalences to (pointwise) weak equivalences.[11] If the Reedy structure is such that the constant functor preserves cofibrations, this functor is left-Quillen and Ken Brown's lemma 1.3.34 implies that its right Quillen adjoint, the limit functor, preserves weak equivalences. In this case, one can define the **homotopy limit** holim D as the functor $\lim(D \circ Q_f)$, where Q_f is the fibrant replacement-functor (which, in this case, acts pointwise). A dual construction gives rise to **homotopy colimits**.

1.6.1   Simplicially enriched diagrams

In the setting where diagrams are enriched over **sSet**, one can define homotopy (co)limits in a more sophisticated way.

Definition 1.6.6 (Homotopy colimit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$ with **C** copowered over **sSet**. The homotopy colimit of D is defined as the following tensor product (**??**):

$$\mathrm{hocolim}\, D := N(-/\mathbf{I}) \otimes_{\mathbf{I}} D = \int^{i \in \mathbf{I}} N(i/\mathbf{I}) \cdot Di\,, \tag{1.31}$$

where N is the nerve functor (Example 1.1.10).

A similar definition for homotopy limits can be used when the category is powered over **sSet**.

Definition 1.6.7 (Homotopy limit). Consider a diagram $D : \mathbf{I} \to \mathbf{C}$ with **C** powered

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[11]This is also true when **R** is not Reedy.

over **sSet**. The homotopy limit of D is defined as the following hom-like object:

$$\operatorname{holim} D := \int_{i \in \mathsf{I}} [N(\mathsf{I}/i), Di]. \tag{1.32}$$

This is exactly the characterization of the homotopy limit as the **sSet**-natural transformations between $N(\mathsf{I}/-)$ and D.

Remark 1.6.8 (Bousfield–Kan map). The expressions from the above formulas are also known as the **Bousfield–Kan formulas**. It should be noted that the above definitions are not strictly equivalent to the ones from the previous section. To be precise, the Bousfield–Kan formulas are only weakly equivalent to the general definitions if the objects in **C** are replaced by their (co)fibrant replacements, i.e. if in the above expressions D is postcomposed by a (co)fibrant replacement functor.

@@ COMPLETE (CHECK THESE STATEMENTS) @@

By Definition 1.3.27, one can construct morphisms in a homotopy category as morphisms from a cofibrant replacement to a fibrant replacement. This allows to define diagrams-up-to-homotopy (in two settings).

Definition 1.6.9 (Homotopy coherent diagram). A morphism in the homotopy category of the Bergner model category. When considering diagrams taking values in a **Kan**-enriched category, one can use the free resolution functor to characterize them as functors $D : \mathfrak{C}\mathsf{I} \to \mathbf{C}$. Since the simplicial realization functor extends the free resolution functor $\mathfrak{C}$ to simplicial sets, one can also define homotopy coherent diagrams on simplicial sets.

Natural transformations between such diagrams are given by homotopy coherent diagrams $\mathfrak{C}(\mathsf{I} \times \Delta[1]) \to \mathbf{C}$ in analogy with ordinary homotopies. However, these do no compose uniquely in the sense that one does not obtain a well-defined diagram on $\mathsf{I} \times \Delta[2]$. Therefore, one does not obtain a category of homotopy coherent diagrams. For simplicial sets I, it can be shown that the natural structure is that of a quasicategory:

$$\mathbf{CohDgrm}(\mathsf{I}, \mathbf{C}) \cong (N\mathbf{C})^{\mathsf{I}}, \tag{1.33}$$

where N is the simplicial nerve functor.

@@ COMPLETE @@

Chapter 2

Electromagnetism

Chapter Contents

2.1   Electricity

Definition 2.1.1 (Coulomb). The SI unit of charge is the coulomb (symbol: C). One coulomb equals the amount of charge that passes through a conductor when the electric current equals one ampère (1A).

Property 2.1.2 (Conservation of charge). Charge is a conserved quantity.

Formula 2.1.3 (Coulomb's law). The force between two point charges q_1, q_2 is given by

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2},\tag{2.1}$$

where r is the distance between the charges and ε_0 is the permittivity of the vacuum ($\varepsilon_0 \approx 8.854\mathrm{e}{-}12\ \frac{\mathrm{F}}{\mathrm{m}}$).

Definition 2.1.4 (Electric potential). The electric field gives the force experienced by a (positive) unit charge. The electric potential is then defined as the work done by this charge when moving through the field:

$$V(\vec{\mathbf{a}}, \vec{\mathbf{b}}) := \int_{\vec{\mathbf{a}}}^{\vec{\mathbf{b}}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}.\tag{2.2}$$

By fixing an endpoint (some fixed reference location[1]), the electric field and electric potential are related in the same way as an ordinary potential and its associated (conservative) force (cf. **??**):

$$E = -\nabla V.\tag{2.3}$$

Definition 2.1.5 (Capacitance). The capacitance is a (geometrical) property that reflects the amount of charge an object can store:

$$C := \frac{Q}{V}.\tag{2.4}$$

2.1.1   Conductors and resistors

Definition 2.1.6 (Current). The amount of charge that passes through a given point in one second:

$$I := \frac{\mathrm{d}Q}{\mathrm{d}t}.\tag{2.5}$$

Formula 2.1.7 (Power). Given that the electric potential determines the work done by a positive charge, the electric power of a current I is given by

$$P = VI.\tag{2.6}$$

Property 2.1.8 (Electric field of conductor). The electric field inside a conductor vanishes at equilibrium: all free charges are located at the surface. Moreover, the electric field at the surface points along the normal vector.

Definition 2.1.9 (Drift velocity). The average velocity of independent charge carriers is the drift velocity $\vec{\mathbf{v}}_d$. It is important to note that v_d is not equal to the propagation speed of the electric signal, which is the result of particle collisions. In fact, it is several orders of magnitude smaller!

---

[1]As usual, the potential is only determined up to some constant.

Definition 2.1.10 (Mobility).

$$\mu := \frac{v_d}{E}, \tag{2.7}$$

where E is the external electric field.

Definition 2.1.11 (Conductivity).

$$\sigma := nq\mu, \tag{2.8}$$

where $n \in \mathbb{N}$ is the number of charge carriers.

Definition 2.1.12 (Resistivity).
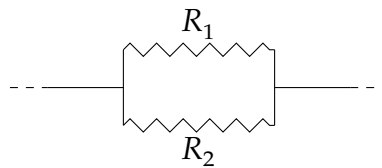
$$\rho := \frac{1}{\sigma} \tag{2.9}$$

Formula 2.1.13 (Pouillet's law). The resistance of a wire is given by the following formula:

$$R = \frac{\rho l}{A}, \tag{2.10}$$

where

- ρ is the resistivity of the material,

- l is the length of the resistor, and

- A is the cross-sectional area of the resistor.

Property 2.1.14 (Resistor circuits). Consider an electrical circuit of the form

The potential difference over both resistors is equal:

$$R_1 I_1 = R_2 I_2. \tag{2.11}$$

Theorem 2.1.15 (Kirchhoff's first law). *The oriented (algebraic)[2] sum of currents meeting at any point is zero.*

Theorem 2.1.16 (Kirchhoff's second law). *The oriented sum of potential differences in a closed circuit is zero.*

---

[2]Currents out of a point are counted with a minus sign.

2.1.2 Ohm's law

Formula 2.1.17 (Free current density). The current density generated by free charges is given by

$$\vec{J} = nq\vec{v}_d.$$

(2.12)

Formula 2.1.18 (Ohm's law).

$$\vec{J} = \sigma \cdot \vec{E},$$

(2.13)

where σ is the conductivity tensor. (Compare this to Definition 2.1.10 and Definition 2.1.11 for 1D systems.)

Formula 2.1.19 (Ohm's law in wires). The following formula can be obtained by combining Definition 2.1.12 and Formulas 2.1.13 and 2.1.18, and by assuming that the conductivity tensor is a scalar (this follows from the isotropic behaviour of most resistors):

$$V = RI.$$

(2.14)

2.1.3 Electric dipoles

Formula 2.1.20 (Electric dipole).

$$\vec{p} := q\vec{l},$$

(2.15)

where

- q is the positive charge, and

- $\vec{l}$ is the vector pointing from the negative charge to the positive charge.

Assume that the positive charge is located at the origin and that the vector $\vec{l}$ points in the direction of the x-axis, i.e. $\vec{l} = l\hat{e}_x$ for some $l > 0$ (the negative charge is, hence, located to the left of the origin). The electric potential of the dipole is given by:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{q}{r + l\cos(\theta)}\right) = \frac{q}{4\pi\varepsilon_0}\frac{l\cos(\theta)}{r(r + l\cos(\theta))},$$

(2.16)

where θ is the angle between $\hat{e}_x$ and $\vec{r} + \vec{l}$ (see Fig. 2.1). In the far-field regime, where $l\cos(\theta) \ll r$, the potential is given by

$$V(\vec{r}) \approx \frac{q}{4\pi\varepsilon_0}\frac{l\cos(\theta)}{r^2},$$

(2.17)

i.e. it decays quadratically.

Formula 2.1.21 (Energy). If an electric dipole is placed in an external electric field, its potential energy is given by

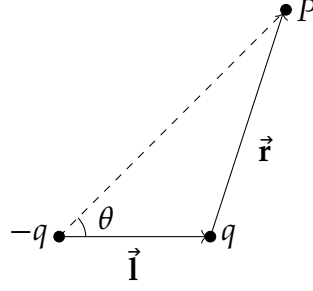$$V = -\vec{p} \cdot \vec{E}.$$

(2.18)

Figure 2.1: Electric dipole configuration.

Formula 2.1.22 (Torque). If an electric dipole is placed in an external electric field, the torque on this system is given by

$$\vec{\tau} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}. \tag{2.19}$$

Formula 2.1.23 (Multipole expansion). Consider a general charge distribution $\rho : \mathbb{R}^3 \to \mathbb{R}$ with support in a bounded region Γ and assume that $\|\vec{\mathbf{r}}\| \gg \mathrm{diam}(\Gamma)$. If $\vec{\mathbf{r}}$ is oriented along the z-axis, one can write the (squared) distance to an infinitesimal volume element as

$$R^2 = r^2 + (r')^2 - 2rr' \cos(\theta) \tag{2.20}$$

or

$$\frac{1}{R} = \frac{1}{r} \frac{1}{\sqrt{1 - \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos(\theta)}}, \tag{2.21}$$

where $\vec{\mathbf{r}}'$ is the position of the volume element and θ is the polar angle of this position vector. In the general integral

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{\mathbf{r}}')}{\|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\|} \, dV', \tag{2.22}$$

one can now expand the denominator in powers of $1/r$,

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{K_0}{r} + \frac{K_1}{r^2} + \frac{K_2}{r^3} + \cdots \right). \tag{2.23}$$

Since the second factor in Eq. (2.21) is the generating function of Legendre polynomials, one obtains:

- $K_0 := \int \rho(\vec{\mathbf{r}}) \, dV$

- $K_1 := \int r \cos(\theta) \rho(\vec{\mathbf{r}}) \, dV$

- $K_2 := \displaystyle\int \frac{3\cos^2(\theta) - 1}{2} r^2 \rho(\vec{\mathbf{r}}) \, dV$

- $\cdots$

- $K_n := \displaystyle\int P_n\big(\cos(\theta)\big) r^n \rho(\vec{\mathbf{r}}) \, dV$,

with P_n the nth *Legendre polynomial*. The first term gives an approximation where all charge is concentrated at a single point (this is called the **monopole approximation**). At very large distances, where the charge distribution appears as a single point, this gives a good approximation.

When the position vector $\vec{\mathbf{r}}$ is not assumed to be oriented along the z-axis, the Legendre polynomials have to be replaced by spherical harmonics.

2.2   Magnetism

Definition 2.2.1 (Tesla). The SI unit of magnetic field strength/magnetic induction is the tesla (symbol: T). It is defined such that a charge of one coulomb ($1\mathsf{C}$) moving at one meter per second experiences a force of one newton ($1\mathsf{N}$) when the magnetic induction equals one tesla: $\mathsf{T} = \frac{\mathsf{N}}{\mathsf{A}}$.

Definition 2.2.2 (Magnetic field). The **magnetizing field** $\vec{\mathbf{H}}$ is the field generated by all external sources. When applying an external (magnetic) field, some materials will try to oppose this external influence. Similar to polarization in the case of electricity, one can define the **magnetization** (in linear media):

$$\vec{\mathbf{M}} := \chi \vec{\mathbf{H}}, \tag{2.24}$$

where χ is the **magnetic susceptibility**.

@@ ADD polarization @@

The **magnetic induction** $\vec{\mathbf{B}}$ is the field generated by both the external sources and the internal magnetization. This is the total magnetic field and, hence, the only field that can be measured. In vacuum, the following relation between the magnetic induction, the magnetizing field and the magnetization exists:

$$\vec{\mathbf{B}} = \mu_0\left(\vec{\mathbf{H}} + \vec{\mathbf{M}}\right). \tag{2.25}$$

By combining the previous two formulas, one obtains (again only in linear media)

$$\vec{\mathbf{B}} = \mu_0(1 + \chi)\vec{\mathbf{H}}. \tag{2.26}$$

The proportionality constant in this formula is called the **magnetic permeability**:

$$\mu := \mu_0(1 + \chi), \tag{2.27}$$

where μ_0 is the magnetic permeability of the vacuum. The factor $1 + \chi$ is called the **relative permeability** and it is often denoted by μ_r.

Remark 2.2.3 (Tensorial formulation). In anisotropic materials, one has to use a tensorial formulation:

$$B_i = \sum_j \mu_{ij} H_j \, , \tag{2.28}$$

$$M_i = \sum_j \chi_{ij} H_j \, . \tag{2.29}$$

In this case, both μ and χ are $(1,1)$-tensors, i.e. linear maps.

@@ CHECK (linear/tensorial) @@

2.2.1   Lorentz force

The relation between the force acting on a charged particle and the strength of the electric and magnetic fields is given by the following law.

Formula 2.2.4 (Lorentz force).

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \tag{2.30}$$

The charge of a particle acts as the constant of proportionality, just as the mass in the case of gravity.

Formula 2.2.5 (Lorentz force density).

$$\vec{\mathbf{f}} = \rho\vec{\mathbf{E}} + \vec{\mathbf{J}} \times \vec{\mathbf{B}} \tag{2.31}$$

2.2.2   Energy and momentum

Definition 2.2.6 (Poynting vector).

$$\vec{\mathbf{S}} := \vec{\mathbf{E}} \times \vec{\mathbf{H}} \tag{2.32}$$

Formula 2.2.7 (Energy density).

$$W = \frac{1}{2}\left(\vec{\mathbf{E}} \cdot \vec{\mathbf{D}} + \vec{\mathbf{B}} \cdot \vec{\mathbf{H}}\right) \tag{2.33}$$

2.2.3   Electric charges in a magnetic field

Solving the Lorentz force law in a uniform magnetic field gives the following solution.

Formula 2.2.8 (Gyroradius). Consider a charge q moving with velocity $\vec{\mathbf{v}}$ in a magnetic field B such that $\vec{\mathbf{v}} \perp \vec{\mathbf{B}}$. The charge will follow a circular path with radius

$$r = \frac{mv_\perp}{|q|B} \, . \tag{2.34}$$

Formula 2.2.9 (Gyrofrequency[3]). The circular velocity is given by

$$\omega = \frac{|q|B}{m}.\tag{2.35}$$

2.3 Maxwell equations

Formula 2.3.1 (Gauss's law for electricity).

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon}\tag{2.36}$$

Formula 2.3.2 (Gauss's law for magnetism).

$$\nabla \cdot \vec{\mathbf{B}} = 0\tag{2.37}$$

If magnetic monopoles would exist, the right-hand side would contain a (magnetic) charge density as in the case of electricity.

Formula 2.3.3 (Faraday's law).

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}\tag{2.38}$$

Formula 2.3.4 (Maxwell's law[4]).

$$\nabla \times \vec{\mathbf{B}} = \varepsilon\mu\frac{\partial \vec{\mathbf{E}}}{\partial t} + \mu\vec{\mathbf{J}}\tag{2.39}$$

Property 2.3.5 (Electric-magnetic duality). Note that in vacuum, where $\rho = 0$ and $\vec{\mathbf{J}} = 0$, there exist a duality between the electric and magnetic fields:

$$\vec{\mathbf{E}} \longrightarrow \vec{\mathbf{B}} \qquad \vec{\mathbf{B}} \longrightarrow -\vec{\mathbf{E}}.\tag{2.40}$$

(To obtain this duality with matter, one would need to include magentic charges, but magnetic monopoles have not been observed.)

In vacuum, one can consider the unified field $\vec{\mathcal{E}} := \vec{\mathbf{E}} + i\vec{\mathbf{B}}$ and duality then boils down to

$$\vec{\mathcal{E}} \longrightarrow i\vec{\mathcal{E}}.\tag{2.41}$$

@@ EXPAND ON EM-duality @@

Property 2.3.6 (Continuity equation). By applying **??**, Maxwell's equations also readily give rise to the continuity equation (**??**):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\mathbf{J}} = 0.\tag{2.42}$$

---

[3]Also called the **Larmor** or **cyclotron frequency**.
[4]Also called the **Maxwell–Ampère law**.

2.4 Potentials

2.4.1 Decomposition in potentials

The Helmholtz decomposition **??** together with Gauss's law 2.3.2 imply the following general form for $\vec{\mathbf{B}}$:

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}, \tag{2.43}$$

where $\vec{\mathbf{A}}$ is called the **magnetic potential**.

Combining Eq. (2.43) with Faraday's law 2.3.3 (and doing some rewriting), gives the following general form for $\vec{\mathbf{E}}$:

$$\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t}, \tag{2.44}$$

where V is called the **electrostatic potential**.

Property 2.4.1. Substituting the previous two expressions into Formula 2.3.1 and Formula 2.3.4, gives the following two (coupled) conditions for the electromagnetic potentials:

$$\Delta \vec{\mathbf{A}} - \varepsilon\mu \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = \nabla\left(\nabla \cdot \vec{\mathbf{A}} + \varepsilon\mu \frac{\partial V}{\partial t} \right) - \mu \vec{\mathbf{J}}, \tag{2.45}$$

$$\Delta V - \varepsilon\mu \frac{\partial^2 V}{\partial t^2} = -\frac{\partial}{\partial t}\left(\nabla \cdot \vec{\mathbf{A}} + \varepsilon\mu \frac{\partial V}{\partial t} \right) - \frac{\rho}{\varepsilon}. \tag{2.46}$$

2.4.2 Gauge theory

Looking at Eq. (2.43), it is clear that a transformation

$$\vec{\mathbf{A}} \longrightarrow \vec{\mathbf{A}} + \nabla\chi \tag{2.47}$$

has no effect on $\vec{\mathbf{B}}$ due to **??**. To compensate for this transformation in Eq. (2.44), one also has to perform a transformation

$$V \longrightarrow V - \frac{\partial \chi}{\partial t}. \tag{2.48}$$

These transformations are called **gauge transformations** (compare to the previous chapter).

Definition 2.4.2 (Gauge fixing conditions). Conditions that fix a certain gauge (or class of gauge transformations). These select one of many physically equivalent configurations. Mathematically this corresponds to picking a representative for any orbit of the gauge transformations. (See **??** for more information.)

Remark 2.4.3. It often happens that a gauge condition does not completely fix the gauge, i.e. the condition does not pick a unique representative but only fixes a subset of the gauge orbit that still has some remaining symmetry or freedom.

Example 2.4.4 (Lorenz gauge). A first example of a gauge fixing condition is the Lorenz gauge[5]:

$$\nabla \cdot \vec{\mathbf{A}} + \varepsilon\mu\frac{\partial V}{\partial t} = 0. \tag{2.49}$$

When using this gauge fixing condition, Eqs. (2.45) and (2.46) uncouple. This allows to rewrite them as

$$\Box\vec{\mathbf{A}} = -\mu\vec{\mathbf{J}}, \tag{2.50}$$

$$\Box V = -\frac{\rho}{\varepsilon}, \tag{2.51}$$

which, in vacuum, are simply wave equations!

To see which gauge functions χ satisfy the Lorenz condition, perform a transformation as explained above:

$$\vec{\mathbf{A}}' = \vec{\mathbf{A}} + \nabla\chi \qquad \text{and} \qquad V' = V - \frac{\partial\chi}{\partial t}. \tag{2.52}$$

Substituting these transformations in Eq. (2.49) and using the fact that both sets of potentials $(\vec{\mathbf{A}}, V)$ and $(\vec{\mathbf{A}}', V')$ satisfy the Lorenz gauge (2.49), gives the following condition for the gauge function χ:

$$\Box\chi = 0. \tag{2.53}$$

Example 2.4.5 (Coulomb gauge).

$$\nabla \cdot \vec{\mathbf{A}} = 0 \tag{2.54}$$

2.4.3 Differential-geometric perspective

Using the tools introduced from **??** (e.g. differential forms), one can rewrite all of the above formulas in a more elegant form. This will also allow to generalize them to higher dimensions and to more general settings. See, for example, Sontz (2015) for a complete derivation and interpretation.

Definition 2.4.6 (Field strength). Define the electric and magnetic field forms as follows:

$$\mathbf{E} = E_1 \mathrm{d}x^1 + E_2 \mathrm{d}x^2 + E_3 \mathrm{d}x^3 \tag{2.55}$$

---

[5]Named after *L. Lorenz*, not to be confused with *H. Lorentz*.

and

$$\mathbf{B} = B_1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_2 \mathrm{d}x^3 \wedge \mathrm{d}x^1 + B_3 \mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{2.56}$$

Using these differential forms, one can write the field strength as follows (cf. Eq. (2.41)):

$$\mathbf{F} = \mathbf{B} - \mathrm{d}t \wedge \mathbf{E}. \tag{2.57}$$

Formula 2.4.7 (Maxwell's equations). Define the electric 4-current as

$$\mathbf{J} = \rho \mathrm{d}t - J_1 \mathrm{d}x^1 - J_2 \mathrm{d}x^2 - J_3 \mathrm{d}x^3. \tag{2.58}$$

Maxwell's equations (Section 2.3) can now be rewritten as follows:

$$\mathrm{d}\mathbf{F} = 0, \tag{2.59}$$

$$*\mathrm{d}*\mathbf{F} = \mathbf{J}, \tag{2.60}$$

where $*$ is the Hodge operator **??**. (Note that the second line is essentially the codifferential from **??**.) Moreover, electric-magnetic duality corresponds to

$$\mathbf{F} \longrightarrow *\mathbf{F} \tag{2.61}$$

in this notation.

Definition 2.4.8 (Potential). The Bianchi identity (2.59) and Poincaré's lemma **??** imply that there exists a differential one-form $\mathbf{A}$ such that

$$\mathbf{F} = \mathrm{d}\mathbf{A}. \tag{2.62}$$

This one-form is called the potential or **gauge field**. It can be related to the ordinary scalar potential V and vector potential $\vec{\mathbf{A}}$ as follows:

$$\mathbf{A} = -V\mathrm{d}t + A_1 \mathrm{d}x^1 + A_2 \mathrm{d}x^2 + A_3 \mathrm{d}x^3. \tag{2.63}$$

Property 2.4.9 (Gauge transformation). Because $\mathrm{d}^2 \equiv 0$, the above equation is invariant under a transformation $\mathbf{A} \longrightarrow \mathbf{A} + \mathrm{d}f$ for any $f \in C^\infty(\mathbb{R}^3)$. When written out in local coordinates, this gives exactly the gauge transformations from Eq. (2.52).

@@ COMPLETE (perhaps move to chapter on gauge theory) @@

List of Symbols

The following abbreviations and symbols are used throughout the compendium.

Abbreviations

AIC	Akaike information criterion
ARMA	autoregressive moving-average model
BCH	Baker–Campbell–Hausdorff
BPS	Bogomol'nyi–Prasad–Sommerfield
BPST	Belavin–Polyakov–Schwarz–Tyupkin
BRST	Becchi–Rouet–Stora–Tyutin
CCR	canonical commutation relation
CDF	cumulative distribution function
CFT	conformal field theory
CIS	completely integrable system
CP	completely positive
CPTP	completely positive, trace-preserving
CR	Cauchy–Riemann
dga	differential graded algebra
dgca	differential graded-commutative algebra
EMM	equivalent martingale measure
EPR	Einstein–Podolsky–Rosen
ESM	equivalent separating measure
ETCS	Elementary Theory of the Category of Sets
FIP	finite intersection property
FWHM	full width at half maximum
GA	geometric algebra
GHZ	Greenberger–Horne–Zeilinger

GNS   Gel′fand–Naimark–Segal

HJE   Hamilton–Jacobi equation

HoTT   Homotopy Type Theory

KKT   Karush–Kuhn–Tucker

LIVF   left-invariant vector field

MCG   mapping class group

MPO   matrix-product operator

MPS   matrix-product state

MTC   modular tensor category

NDR   neighbourhood deformation retract

OPE   operator product expansion

OTC   over the counter

OZI   Okubo–Zweig–Iizuka

PAC   probably approximately correct

PDF   probability density function

PID   principal ideal domain

PL   piecewise-linear

PMF   probability mass function

POVM   positive operator-valued measure

PRP   predictable representation property

PVM   projection-valued measure

RKHS   reproducing kernel Hilbert space

SVM   support-vector machine

TDSE   time-dependent Schrödinger equation

TISE   time-independent Schrödinger equation

TQFT   topological quantum field theory

TVS   topological vector space

UFD   unique factorization domain

VC   Vapnik–Chervonenkis

VIF   variance inflation factor

VOA   vertex operator algebra

WKB   Wentzel–Kramers–Brillouin

ZFC   Zermelo–Frenkel set theory with the axiom of choice

Operations

Ad_g adjoint representation of a Lie group G

ad_X adjoint representation of a Lie algebra $\mathfrak{g}$

arg   argument of a complex number

$\square$ d'Alembert operator

$\deg(f)$ degree of a polynomial f

e identity element of a group

$\Gamma(E)$ set of global sections of a fibre bundle E

$\text{Im}, \mathfrak{J}$ imaginary part of a complex number

$\text{Ind}_f(z)$ index of a point $z \in \mathbb{C}$ with respect to a function f

$\hookrightarrow$ injective function

$\cong$ is isomorphic to

$A \multimap B$ linear implication

$N \triangleleft G$ N is a normal subgroup of G

Par_t^γ parallel transport map along a curve γ

$\text{Re}, \mathfrak{R}$ real part of a complex number

Res   residue of a complex function

$\twoheadrightarrow$ surjective function

$\{\cdot, \cdot\}$ Poisson bracket

$X \pitchfork Y$ transversally intersecting manifolds X, Y

∂X boundary of a topological space X

$\overline{X}$ closure of a topological space X

$X^\circ, \mathring{X}$ interior of a topological space X

$\sphericalangle(\cdot, \cdot)$ angle between two vectors

$X \times Y$ cartesian product of two sets X, Y

$X + Y$ sum of two vector spaces X, Y

$X \oplus Y$ direct sum of two vector spaces X, Y

$V \otimes W$ tensor product of two vector spaces V, W

$\mathbb{1}_X$ identity morphism on an object X

$\approx$ is approximately equal to

$\hookrightarrow$	is included in
$\cong$	is isomorphic to
$\mapsto$	mapsto

Objects

Ab	category of Abelian groups
$\mathrm{Aut}(X)$	automorphism group of an object X
$\mathcal{B}_0(V, W)$	space of compact bounded operators between two Banach spaces V, W
$\mathcal{B}_1(\mathcal{H})$	space of trace-class operators on a Hilbert space
$\mathcal{B}(V, W)$	space of bounded linear maps between two vector spaces V, W
CartSp	category of Euclidean spaces and 'suitable' morphisms (e.g. linear maps, smooth maps, ...)
$C(X, Y)$	set of continuous functions between two topological spaces X, Y
S'	centralizer of a subset (of a ring)
$C_\bullet$	chain complex
Ch(A)	category of chain complexes with objects in an additive category **A**
C$^\infty$, **SmoothSet**	category of smooth sets
$C_p^\infty(M)$	ring of smooth functions $f : M \to \mathbb{R}$ on a neighbourhood of $p \in M$
$C\ell(A, Q)$	Clifford algebra over an algebra A induced by a quadratic form Q
$C^\omega(V)$	set of all analytic functions defined on a set V
$\mathrm{Conf}(M)$	conformal group of a (pseudo-)Riemannian manifold M
C$^\infty$**Ring**, **C**$^\infty$**Alg**	category of smooth algebras
$S_k(\Gamma)$	space of cusp forms of weight $k \in \mathbb{R}$
Δ_X	diagonal of a set X
Diff	category of smooth manifolds
DiffSp	category of diffeological spaces and smooth maps
$\mathcal{D}_M$	sheaf of differential operators
D^n	standard n-disk
$\mathrm{dom}(f)$	domain of a function f
$\mathrm{End}(X)$	endomorphism monoid of an object X
$\mathcal{E}\mathrm{nd}$	endomorphism operad
FormalCartSp$_{\mathbf{diff}}$	category of infinitesimally thickened Euclidean spaces

Frac(I)	field of fractions of an integral domain I
$\mathfrak{F}(V)$	space of Fredholm operators on a Banach space V
$\mathbb{G}_a$	additive group (scheme)
GL(V)	general linear group: group of automorphisms of a vector space V
GL($n, \mathfrak{K}$)	general linear group: group of invertible $n \times n$-matrices over a field $\mathfrak{K}$
Grp	category of groups and group homomorphisms
Grpd	category of groupoids
Hol$_p(\omega)$	holonomy group at a point p with respect to a principal connection ω
Hom$_\mathbf{C}(V, W)$, $\mathbf{C}(V, W)$	collection of morphisms between two objects V, W in a category $\mathbf{C}$
hTop	homotopy category
$I(S)$	vanishing ideal on an algebraic set S
$I(x)$	rational fractions over an integral domain I
im(f)	image of a function f
$K^0(X)$	K-theory over a (compact Hausdorff) space X
Kan	category of Kan complexes
$K(A)$	Grothendieck completion of a monoid A
$\mathcal{K}_n(A, v)$	Krylov subspace of dimension n generated by a matrix A and a vector v
L^1	space of integrable functions
Law	category of Lawvere theories
$\mathfrak{Lie}$	category of Lie algebras
Lie	category of Lie groups
$\mathfrak{X}^L$	space of left-invariant vector fields on a Lie group
llp(I)	set of morphisms having the left lifting property with respect to I
LX	free loop space on a topological space X
Man^p	category of C^p-manifolds
Meas	• category of measurable spaces and measurable functions, or
	• category of measure spaces and measure-preserving functions
M^4	four-dimensional Minkowski space
$M_k(\Gamma)$	space of modular forms of weight $k \in \mathbb{R}$
$\mathbb{F}^X$	natural filtration of a stochastic process $(X_t)_{t \in T}$
$N\mathbf{C}$	simplicial nerve of a small category $\mathbf{C}$

$O(n, \mathfrak{K})$	group of $n \times n$ orthogonal matrices over a field $\mathfrak{K}$
Open(X)	category of open subsets of a topological space X
$P(X), 2^X$	power set of a set X
$\mathrm{Pin}(V)$	pin group of the Clifford algebra $C\ell(V, Q)$
Psh$(\mathbf{C}), \widehat{\mathbf{C}}$	category of presheaves on a (small) category $\mathbf{C}$
$R((x))$	ring of (formal) Laurent series in x with coefficients in R
$\mathrm{rlp}(I)$	set of morphisms having the right lifting property with respect to I
$R[[x]]$	ring of (formal) power series in x with coefficients in R
S^n	standard n-sphere
$S^n(V)$	space of symmetric rank n tensors over a vector space V
Sh(X)	category of sheaves on a topological space X
Sh$(\mathbf{C}, J)$	category of J-sheaves on a site $(\mathbf{C}, J)$
Δ	simplex category
$\mathrm{sing\ supp}(\phi)$	singular support of a distribution ϕ
$\mathrm{SL}_n(\mathfrak{K})$	special linear group: group of all $n \times n$-matrices with unit determinant over a field $\mathfrak{K}$
$W^{m,p}(U)$	Sobolov space in L^p of order m
Span$(\mathbf{C})$	span category over a category $\mathbf{C}$
$\mathrm{Spec}(R)$	spectrum of a commutative ring R
sSet$_{\mathrm{Quillen}}$	Quillen's model structure on simplicial sets
$\mathrm{supp}(f)$	support of a function f
$\mathrm{Syl}_p(G)$	set of Sylow p-subgroups of a finite group G
$\mathrm{Sym}(X)$	symmetric group of a set X
S_n	symmetric group of degree n
$\mathrm{Sym}(X)$	symmetric group on a set X
$\mathrm{Sp}(n, \mathfrak{K})$	group of matrices preserving a canonical symplectic form over a field $\mathfrak{K}$
$\mathrm{Sp}(n)$	compact symplectic group
$\mathbb{T}^n$	standard n-torus (n-fold Cartesian product of S^1)
$T_{\leq t}$	set of all elements smaller than (or equal to) $t \in T$ for a partial order T
$\mathrm{TL}_n(\delta)$	Temperley–Lieb algebra with $n - 1$ generators and parameter δ
Top	category of topological spaces and continuous functions
Topos	(2-)category of (elementary) topoi and geometric morphisms

$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\mathsf{U}(n, \mathfrak{K})$	group of $n \times n$ unitary matrices over a field $\mathfrak{K}$
$V(I)$	algebraic set corresponding to an ideal I
$\mathbf{Vect}(X)$	category of vector bundles over a manifold X
$\mathbf{Vect}_{\mathfrak{K}}$	category of vector spaces and linear maps over a field $\mathfrak{K}$
Y^X	set of functions between two sets X, Y
$\mathbb{Z}_p$	group of p-adic integers
$\varnothing$	empty set
$\pi_n(X, x_0)$	n^{th} homotopy space over X with basepoint x_0
$[a, b]$	closed interval
$]a, b[$	open interval
$\Lambda^n(V)$	space of antisymmetric rank-n tensors over a vector space V
ΩX	(based) loop space on a topological space X
$\Omega^k(M)$	$C^\infty(M)$-module of differential k-forms on a manifold M
$\rho(A)$	resolvent set of a bounded linear operator A
$\mathfrak{X}(M)$	$C^\infty(M)$-module of vector fields on a manifold M

Units

C	Coulomb
T	Tesla

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