Reference Point Calculations in MAS

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Reference point calculations for the Metapopulation assessment system are defined for the general case of a total of *P* populations in *D* areas with *G* genders and (*A*+1) age groups. The notation for indexing population, area, gender and age variables is *p*=1…*P* populations, *d*=1...*D* areas, *g*=1…*G* genders, and *a*=0…*A* age groups where age-*A* is a plus group comprised of all fish age-*A* and older. Another convention for notation is that the symbol "" indicates the equilibrium value of a variable used in the calculation of a reference point. We describe iterative algorithms to calculate biological reference points for multiple populations in multiple areas.

Calculation of MSY reference points

One can use the following numerical search algorithm to calculate Maximum Sustainable Yield (MSY) based reference points. To initialize the calculations, we construct a uniform-spaced vector of fishing mortality rates, denoted by $\underline{F}$, with mesh size δ where $\underline{F} = \left(F_1, F_2, ..., F_{Upper}\right)$ and $F_j = \left(j-1\right)\cdot\delta$ and $F_1 = 0$ and F_{Upper} is a maximal value of fishing mortality, say for example, $F_{Upper} = 3.00$ with $\delta = 0.01$.

The algorithm to calculate MSY-based reference points can be categorized into 8 steps.

Step 1. For each population *p*, each area *d* and each gender *g*, calculate the unfished numbers at age in equilibrium ($\underline{\widetilde{N}}_{U,d,g}^{(p)}$) using Algorithm 1 as

(1) $$\underline{\widetilde{N}}_{U,d,g}^{(p)} = \left(\widetilde{N}_{U,d,a=0,g}^{(p)}, \widetilde{N}_{U,d,1,g}^{(p)}, ..., \widetilde{N}_{U,d,A,g}^{(p)}\right)$$

Step 2. For each fishing mortality F_j, each population *p*, each area *d* and each gender *g*, calculate the fished numbers at age in equilibrium ($\underline{\widetilde{N}}^{(p)}_{F,d,g}$) using Algorithm 2 as

(2) $$\underline{\widetilde{N}}^{(p)}_{F_j,d,g} = \left(\widetilde{N}^{(p)}_{F_j,d,a=0,g}, \widetilde{N}^{(p)}_{F_j,d,1,g}, \ldots, \widetilde{N}^{(p)}_{F_j,d,A,g}\right)$$

Step 3. For each fishing mortality F_j, set recruitment strength by population *p*, each area *d* and each gender *g*, as

(3) $$\widetilde{R}^{(p)}_{F_j,d,g} = \widetilde{N}^{(p)}_{F_j,d,a=0,g}$$

Step 4. For each fishing mortality F_j, calculate equilibrium yield per recruit by population, area, and gender as

(4)

$$\widetilde{YPR}\left(F_j\right)^{(p)}_{d,g} = \sum_{a=0}^{A-1}\left\{\frac{\widetilde{W}^{(p)}_{C,d,a,g}\cdot F_j\cdot \widetilde{S}^{(p)}_{d,a,g}}{F_j\cdot \widetilde{S}^{(p)}_{d,a,g}+\widetilde{M}^{(p)}_{d,a,g}}\left(1-\exp\left(-F_j\cdot \widetilde{S}^{(p)}_{d,a,g}-\widetilde{M}^{(p)}_{d,a,g}\right)\right)\cdot\exp\left(-\sum_{k=1}^{a-1}\left(F_j\cdot \widetilde{S}^{(p)}_{d,k,g}+\widetilde{M}^{(p)}_{d,k,g}\right)\right)\right\}$$
$$+\frac{\widetilde{W}^{(p)}_{C,d,A,g}\cdot F_j\cdot \widetilde{S}^{(p)}_{d,A,g}}{F_j\cdot \widetilde{S}^{(p)}_{d,A,g}+\widetilde{M}^{(p)}_{d,A,g}}\exp\left(-\sum_{k=1}^{A-1}\left(F_j\cdot \widetilde{S}^{(p)}_{d,k,g}+\widetilde{M}^{(p)}_{d,k,g}\right)\right)$$

Step 5. For each fishing mortality F_j, calculate equilibrium yield by population, area, and gender as

(5) $$\widetilde{Y}\left(F_j\right)^{(p)}_{d,g} = \widetilde{R}^{(p)}_{F_j,d,g}\cdot \widetilde{YPR}\left(F_j\right)^{(p)}_{d,g}$$

And calculate equilibrium yield by population and area as

(6) $$\widetilde{Y}\left(F_j\right)^{(p)}_{d} = \sum_{g}\widetilde{Y}\left(F_j\right)^{(p)}_{d,g}$$

And calculate equilibrium yield by population as

(7) $$\widetilde{Y}\left(F_j\right)^{(p)} = \sum_{d}\widetilde{Y}\left(F_j\right)^{(p)}_{d}$$

And calculate equilibrium yield by area as

(8) $$\widetilde{Y}\left(F_j\right)_{d} = \sum_{p}\widetilde{Y}\left(F_j\right)^{(p)}_{d}$$

And calculate the equilibrium yield for all populations and areas as

$$\widetilde{Y}\left(F_j\right) = \sum_p \sum_d \widetilde{Y}\left(F_j\right)_d^{(p)} \tag{9}$$

<u>Step 6.</u> Loop over the set of fishing mortalities $F_j \in \underline{F}$, to find $F_{MSY}^{(p)}$ by population p, such that $\widetilde{Y}\left(F_{MSY}^{(p)}\right) \geq \widetilde{Y}\left(F_j\right)^{(p)}$ for all $F_j \neq F_{MSY}^{(p)}$ and then set *MSY* by population as

$$MSY^{(p)} = \widetilde{Y}\left(F_{MSY}^{(p)}\right) \tag{10}$$

Loop over the set of fishing mortalities $F_j \in \underline{F}$, to find $F_{MSY,d}$ by area, such that $\widetilde{Y}\left(F_{MSY,d}\right) \geq \widetilde{Y}\left(F_j\right)_d$ for all $F_j \neq F_{MSY,d}$ and then set *MSY* by area as

$$MSY_d = \widetilde{Y}\left(F_{MSY,d}\right) \tag{11}$$

Loop over the set of fishing mortalities $F_j \in \underline{F}$, to find F_{MSY} such that $\widetilde{Y}\left(F_{MSY}\right) \geq \widetilde{Y}\left(F_j\right)$ for all $F_j \neq F_{MSY}$ and then set

$$MSY_{Global} = \widetilde{Y}\left(F_{MSY}\right) \tag{12}$$

<u>Step 7.</u> Loop over the set of fishing mortalities $F_j \in \underline{F}$, to calculate the equilibrium female spawning biomass by population and area as

$$\widetilde{SB}_{F_j,d,female}^{(p)} = \sum_a \widetilde{P}_{M,a,female}^{(p)} \cdot \widetilde{W}_{S,a,female}^{(p)} \cdot \widetilde{N}_{F_j,d,a,female}^{(p)} \cdot \exp\left(-\Delta_S \cdot \widetilde{Z}_{F_j,d,a,female}^{(p)}\right) \tag{13}$$

And calculate equilibrium female spawning biomass by population as

$$\widetilde{SB}_{F_j,female}^{(p)} = \sum_d \widetilde{SB}_{F_j,d,female}^{(p)} \tag{14}$$

And calculate equilibrium female spawning biomass by area as

$$\widetilde{SB}_{F_j,d,female} = \sum_p \widetilde{SB}_{F_j,d,female}^{(p)} \tag{15}$$

And calculate equilibrium female spawning biomass for all populations and areas as

$$\widetilde{SB}_{F_j,female} = \sum_p \sum_d \widetilde{SB}_{F_j,d,female}^{(p)} \tag{16}$$

And set the female spawning biomass to produce *MSY* by population as

$$(17)\qquad \widetilde{SB}_{MSY}^{(p)} = \widetilde{SB}_{F_{MSY}^{(p)},female}$$

And set the female spawning biomass to produce *MSY* by area as

$$(18)\qquad \widetilde{SB}_{MSY,d} = \widetilde{SB}_{F_{MSY,d},female}$$

Last set the female spawning biomass to produce *MSY* as

$$(19)\qquad \widetilde{SB}_{MSY_{Global}} = \widetilde{SB}_{F_{MSY},female}$$

Step 8. Loop over the set of fishing mortalities $F_j \in \underline{F}$, to calculate the equilibrium female spawning biomass per recruit by population and area as

$$(20)\qquad \widetilde{SSBR}\left(F_j\right)_d^{(p)} = \frac{\widetilde{SB}_{F_j,d,female}^{(p)}}{\sum_g \widetilde{R}_{F_j,d,g}^{(p)}}$$

And calculate equilibrium female spawning biomass per recruit by population as

$$(21)\qquad \widetilde{SSBR}\left(F_j\right)^{(p)} = \frac{\sum_d \widetilde{SB}_{F_j,d,female}^{(p)}}{\sum_d \sum_g \widetilde{R}_{F_j,d,g}^{(p)}}$$

And calculate equilibrium female spawning biomass per recruit by area as

$$(22)\qquad \widetilde{SSBR}\left(F_j\right)_d = \frac{\sum_p \widetilde{SB}_{F_j,d,female}^{(p)}}{\sum_p \sum_g \widetilde{R}_{F_j,d,g}^{(p)}}$$

And calculate equilibrium female spawning biomass per recruit for all populations and areas as

$$(23)\qquad \widetilde{SSBR}\left(F_j\right) = \frac{\sum_p \sum_d \widetilde{SB}_{F_j,d,female}^{(p)}}{\sum_p \sum_d \sum_g \widetilde{R}_{F_j,d,g}^{(p)}}$$

And set the female spawning biomass per recruit to produce *MSY* by population as

(24) $$\widetilde{SBPR}^{(p)}_{MSY} = \widetilde{SBPR}\left(F^{(p)}_{MSY}\right)$$

And set the female spawning biomass per recruit to produce *MSY* by area as

(25) $$\widetilde{SBPR}_{MSY,d} = \widetilde{SBPR}\left(F_{MSY,d}\right)$$

Last set the female spawning biomass per recruit to produce *MSY* as

(26) $$\widetilde{SBPR}_{MSY_{Global}} = \widetilde{SBPR}\left(F_{MSY}\right)$$

<u>Calculation of $F_{X\%}$ reference points</u>

The fishing mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass is $F_{X\%}$. To calculate $F_{X\%}$, one first needs to start with an estimate of the equilibrium spawning biomass as a function of *F*. We can use the calculations of equilibrium spawning biomass by population, area and for all populations and areas based on equations (14), (15) and (16) in the MSY reference point calculations. Without loss of generality, assume we have the equilibrium female spawning biomass by population ($\widetilde{SB}^{(p)}_{F_j,female}$), area ($\widetilde{SB}_{F_j,d,female}$), and for all populations and areas ($\widetilde{SB}_{F_j,female}$) already calculated for the set of fishing mortalities $F_j \in \underline{F}$. Here is the algorithm to calculate the fishing mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass by population ($F^{(p)}_{X\%}$), area ($F_{X\%,d}$) and for all populations and areas ($F_{X\%}$).

<u>Step 1.</u> For each population and $F_j \in \underline{F}$, calculate the ratio $R^{(p)}_j = \dfrac{\widetilde{SB}^{(p)}_{F_j,female}}{\widetilde{SB}^{(p)}_{F=0,female}}$ and the difference $\Delta^{(p)}_j = \left|R^{(p)}_j - X_{\%}\right|$ by population.

<u>Step 2.</u> Next find the index $k \in \{1,2,...,Upper\}$ that produces the smallest difference $\Delta^{(p)}_k \le \Delta^{(p)}_j \; for \; all \; j \in \{1,2,...,Upper\}$ for each population *p*. Then set the fishing mortality mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass by population along with the associated spawning biomass and fishery yield in each population *p* as

(27) $$F^{(p)}_{X\%} = F_k \text{ and } \widetilde{SB}^{(p)}_{X\%} = \widetilde{SB}^{(p)}_{F_k,female} \text{ and } \widetilde{Y}^{(p)}_{X\%} = \widetilde{Y}\left(F_k\right)^{(p)}$$

Step 3. For each area and $F_j \in \underline{F}$, calculate the ratio $R_{j,d} = \dfrac{\widetilde{SB}_{F_j,d,female}}{\widetilde{SB}_{F=0,d,female}}$ and the difference $\Delta_{j,d} = \left| R_{j,d} - X_{\%} \right|$ by area.

Step 4. Next find the index $k \in \{1,2,...,Upper\}$ that produces the smallest difference $\Delta_{k,d} \le \Delta_{j,d}$ *for all* $j \in \{1,2,...,Upper\}$ for each area *d*. Then set the fishing mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass along with the associated spawning biomass and fishery yield in each area *d* as

(28) $$F_{X\%,d} = F_k \text{ and } \widetilde{SB}_{X\%,d} = \widetilde{SB}_{F_k,d,female} \text{ and } \tilde{Y}_{X\%,d} = \tilde{Y}\left(F_k\right)_d$$

Step 5. For each $F_j \in \underline{F}$, calculate the global ratio $R_j = \dfrac{\widetilde{SB}_{F_j,female}}{\widetilde{SB}_{F=0,female}}$ for all populations and area and the global difference $\Delta_j = \left| R_j - X_{\%} \right|$.

Step 6. Next find the index $k \in \{1,2,...,Upper\}$ that produces the smallest difference $\Delta_k \le \Delta_j$ *for all* $j \in \{1,2,...,Upper\}$. Then set the fishing mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass for all populations and areas along with the associated global spawning biomass and fishery yield as

(28) $$F_{X\%} = F_k \text{ and } \widetilde{SB}_{X\%} = \widetilde{SB}_{F_k,female} \text{ and } \tilde{Y}_{X\%} = \tilde{Y}\left(F_k\right)$$

<u>Calculation of $F_{0.1}$, the fishing mortality where the slope of the *YPR* curve is equal to 10% of the slope at *F*=0</u>

<u>Calculate F_{MAX}, the *F* that produces the maximum yield per recruit</u>

Algorithm 1. Calculate Unfished Equilibrium Numbers at Age

In this Appendix we provide details of an algorithm to iteratively calculate unfished equilibrium numbers at age, conditioned on the existence of an equilibrium solution. That is, we need to calculate unfished numbers at age in equilibrium by population, area, and gender $\left(\widetilde{N}_{U,d,a,g}^{(p)}\right)$ to determine the values of unfished female spawning biomasses by population and area to inform the recruitment process models. Here note that the unfished numbers at age by population, area, and gender depend on the population movement and recruitment distribution matrices and are needed to compute the unfished spawning biomasses by population and area, which in turn, are needed to implement the recruitment submodels by population and area for the initial fished equilibrium time period and assessment time horizon. That is, this algorithm will determine the values of the unfished equilibrium female spawning biomasses for the recruitment submodels by population and area, which are derived quantities that depend on the unfished recruitment parameters by population and area.

Here we provide the formulas to calculate the unfished numbers at age for *P* populations, *D* areas, *A+1* ages from *a*=0 to the plus group age *A*, and *G* genders. The inputs for this calculation are:

- The *DxD* matrix of recruitment distribution probabilities from area *k* to area *d* for each population *p* denoted by $\underline{\underline{Q}}^{(p)} = \left(Q_{k\to d}^{(p)}\right)_{DxD}$
- The *Gx1* vector of sex ratio by gender for each population *p* denoted by $\rho_g^{(p)}$
- The *(A+1)x1* vector of natural mortality at age and gender vector for each population *p* denoted by $\underline{M}_{a,g}^{(p)}$
- The *(A+1)x1* vector of mean spawning weight at age and gender vector for each population *p* denoted by $\underline{W}_{S,a,g}^{(p)}$
- The *(A+1)x1* vector of probability of maturity at age and gender vector for each population *p* denoted by $\underline{P}_{M,a,g}^{(p)}$
- The *DxD* matrix of movement probabilities from area *k* to area *d* by age and gender for each population *p* denoted by $\underline{\underline{T}}_{k\to d,a,g}^{(p)} = \left(T_{k\to d,a,g}^{(p)}\right)_{DxD}$
- The fraction of the year prior to spawning offset for each population *p* denoted as $\Delta_S^{(p)}$

Iteration i=1: Calculate the initial unfished numbers at age estimates by population, area, and gender based on unfished recruitment, the recruitment distribution, equilibrium natural mortality and no movement, where $(x)^{[k]}$ denotes the k^{th} iterate of an estimate of a quantity x.

i. In general, population recruitment by area and gender is a function of area-specific recruitment production and the recruitment distribution matrix $\underline{\underline{Q}}^{(p)}$. Set age-0 recruits as a function of unfished recruitment by area $\left(R_{U,d}^{(p)}\right)$ and recruitment distribution by area and gender via

(27) $$\left(N_{U,d,a=0,g}^{(p)}\right)^{[1]} = \rho_g^{(p)} \sum_k R_{U,k}^{(p)} \cdot Q_{k \to d}^{(p)}$$

ii. Set age-a survivors by area and gender for ages a=1 to A-1 via

(28) $$\left(N_{U,d,a,g}^{(p)}\right)^{[1]} = \left(N_{U,d,a-1,g}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{a-1,g}^{(p)}\right)$$

iii. Set age-A group of survivors by area and gender via

(29) $$\left(N_{U,d,A,g}^{(p)}\right)^{[1]} = \frac{\left(N_{U,d,A-1,g}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{A-1,g}^{(p)}\right)}{1-\exp\left(-M_{A,g}^{(p)}\right)}$$

iv. Set unfished spawning biomass by population, area and gender via

(30) $$\left(SB_{U,d,g}^{(p)}\right)^{[1]} = \sum_a P_{M,a,g}^{(p)} \cdot W_{S,a,g}^{(p)} \cdot \left(N_{U,d,a,g}^{(p)}\right)^{[1]} \cdot \exp\left(-\Delta_S^{(p)} \cdot M_{a,g}^{(p)}\right)$$

Iteration i=2: Calculate the next iterate of unfished numbers at age estimates by population, area, and gender based on unfished recruitment, the recruitment distribution, equilibrium survival, movement probabilities by area, age, and gender, and the previous iterate. Unfished recruitment production by area is a function of area-specific spawning biomasses which need to be iteratively calculated to account for the movement probabilities that redistribute fish.

i. Set age-0 recruits as a function of unfished recruitment and recruitment distribution by area and gender (Note that this calculation of unfished recruitment by population, area and gender does not change between iterations and can be done once, but is listed here to show the iterative process and emphasize the dependence on the unfished recruitment parameters by population and area, $R_{U,d}^{(p)}$) via

(31) $$\left(N_{U,d,a=0,g}^{(p)}\right)^{[2]} = \left(N_{U,d,a=0,g}^{(p)}\right)^{[1]}$$

ii. Set age-a survivors for ages a=1 to A-1 by population, area, and gender that did not emigrate plus age-a surviving immigrants from other areas via

$$\left(N_{U,d,a,g}^{(p)}\right)^{[2]} = \sum_k \left(N_{U,k,a-1,g}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{a-1,g}^{(p)}\right) \cdot T_{k\to d,a-1,g}^{(p)} \tag{32}$$

iii. Set age-(A-1) survivors that did not emigrate plus age-(A-1) immigrants from other areas plus age-A group survivors that did not emigrate plus age-A group immigrants from other areas via

$$\begin{aligned}\left(N_{U,d,A,g}^{(p)}\right)^{[2]} &= \sum_k \left(N_{U,k,A-1,g}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{A-1,g}^{(p)}\right) \cdot T_{k\to d,A-1,g}^{(p)} \\ &\quad + \sum_k \left(N_{U,k,A,g}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{A,g}^{(p)}\right) \cdot T_{k\to d,A,g}^{(p)}\end{aligned} \tag{33}$$

v. Set unfished spawning biomass by population, area and gender via

$$\left(SB_{U,d,g}^{(p)}\right)^{[2]} = \sum_a P_{M,a,g}^{(p)} \cdot W_{S,a,g}^{(p)} \cdot \left(N_{U,d,a,g}^{(p)}\right)^{[2]} \cdot \exp\left(-\Delta_S^{(p)} \cdot M_{a,g}^{(p)}\right) \tag{34}$$

Iteration i=j+1: Calculate the (j+1)$^{\text{st}}$ iterate of equilibrium fished numbers at age estimates by population, area, and gender based on unfished recruitment, the recruitment distribution, equilibrium survival, movement probabilities by area, age, and gender, and the j^{th} iterate.

i. Set age-0 recruits by population, area, and gender via

$$\left(N_{U,d,a=0,g}^{(p)}\right)^{[j+1]} = \left(N_{U,d,a=0,g}^{(p)}\right)^{[j]} \tag{35}$$

ii. Set age-a survivors for ages a=1 to A-1 by population, area, and gender that did not emigrate plus age-a surviving immigrants from other areas via

$$\left(N_{U,d,a,g}^{(p)}\right)^{[j+1]} = \sum_k \left(N_{U,k,a-1,g}^{(p)}\right)^{[j]} \cdot \exp\left(-M_{a-1,g}^{(p)}\right) \cdot T_{k\to d,a-1,g}^{(p)} \tag{36}$$

iii. Set age-(A-1) survivors that did not emigrate plus age-(A-1) immigrants from other areas plus age-A group survivors that did not emigrate plus age-A group immigrants from other areas via

$$\begin{aligned}\left(N_{U,d,A,g}^{(p)}\right)^{[j+1]} &= \sum_k \left(N_{U,k,A-1,g}^{(p)}\right)^{[j]} \cdot \exp\left(-M_{A-1,g}^{(p)}\right) \cdot T_{k\to d,A-1,g}^{(p)} \\ &\quad + \sum_k \left(N_{U,k,A,g}^{(p)}\right)^{[j]} \cdot \exp\left(-M_{A,g}^{(p)}\right) \cdot T_{k\to d,A,g}^{(p)}\end{aligned} \tag{37}$$

vi. Set unfished spawning biomass by population, area and gender via

$$\left(SB_{U,d,g}^{(p)}\right)^{[j+1]} = \sum_{a} P_{M,a,g}^{(p)} \cdot W_{S,a,g}^{(p)} \cdot \left(N_{U,d,a,g}^{(p)}\right)^{[j+1]} \cdot \exp\left(-\Delta_S^{(p)} \cdot M_{a,g}^{(p)}\right) \tag{38}$$

Continue the iterations until the convergence criteria below is achieved or the maximum number of iterations has been reached.

<u>Convergence Criterion</u>

Calculate the distance between successive sets of unfished equilibrium spawning biomass estimates by population, area and gender, denoted by $\Delta^{[j]}$, by applying the L_1, or least absolute deviations norm, to the set of estimates as

$$\Delta^{[j]} = \sum_{p}\sum_{d}\sum_{g}\left|\left(SB_{U,d,g}^{(p)}\right)^{[j+1]} - \left(SB_{U,d,g}^{(p)}\right)^{[j]}\right| \tag{39}$$

Stop the iterations when the set of unfished spawning biomass estimates have converged. That is, stop when $\Delta^{[j]} < \varepsilon$ for a small positive constant $\varepsilon > 0$.

If the iterations converge, then the outputs are the vectors of unfished numbers at age by population, area, and gender $\underline{\widetilde{N}}_{U,d,g}^{(p)} = \left(\widetilde{N}_{U,d,a,g}^{(p)}\right)_{(A+1)x1}$ in equilibrium along with the unfished spawning biomasses by population, area, and gender $\left(\widetilde{SB}_{U,d,g}^{(p)}\right)$ in equilibrium.

Algorithm 2. Calculate Equilibrium Fished Numbers at Age

Similarly, one needs to calculate fished numbers at age in equilibrium prior to the start of the assessment time horizon by population, area, and gender $\left(\widetilde{N}^{(p)}_{F,d,a,g}\right)$ as a function of population recruitment distribution by area, movement probabilities, and the equilibrium total mortality at age. The equilibrium fished numbers at age by population, area, and gender depend on the population movement and recruitment distribution matrices and are needed to compute the fished equilibrium spawning biomasses by population and area, which in turn, are needed to calculate the equilibrium numbers at by population, area, and gender for the initial fished equilibrium time period to the start of the assessment time horizon.

In this Appendix, we provide the formulas to calculate the unfished numbers at age for *P* populations, *D* areas, *A+1* ages from *a*=0 to the plus group age *A*, and *G* genders. Similar to the unfished equilibrium calculation, the inputs for fished equilibrium calculation are:

- The *DxD* matrix of recruitment distribution probabilities from area *k* to area *d* for each population *p* denoted by $\underline{\underline{Q}}^{(p)} = \left(Q^{(p)}_{k\to d}\right)_{DxD}$
- The *Gx1* vector of sex ratio by gender for each population *p* denoted by $\rho^{(p)}_g$
- The *(A+1)x1* vector of natural mortality at age and gender vector for each population *p* denoted by $\underline{M}^{(p)}_{a,g}$
- The *(A+1)x1* vector of mean spawning weight at age and gender vector for each population *p* denoted by $\underline{W}^{(p)}_{S,a,g}$
- The *(A+1)x1* vector of probability of maturity at age and gender vector for each population *p* denoted by $\underline{P}^{(p)}_{M,a,g}$
- The *DxD* matrix of movement probabilities from area *k* to area *d* by age and gender for each population *p* denoted by $\underline{\underline{T}}^{(p)}_{k\to d,a,g} = \left(T^{(p)}_{k\to d,a,g}\right)_{DxD}$
- The fraction of the year prior to spawning offset for each population *p* denoted as $\Delta^{(p)}_S$
- The vector of unfished numbers at age by population, area, and gender in equilibrium denoted by $\underline{\widetilde{N}}^{(p)}_{U,d,g} = \left(\widetilde{N}^{(p)}_{U,d,a,g}\right)_{(A+1)x1}$
- The unfished spawning biomasses by population, area, and gender in equilibrium denoted by $\left(\widetilde{SB}^{(p)}_{U,d,g}\right)$

Iteration i=1: Calculate the initial equilibrium fished numbers at age estimates by population, area, and gender based on unfished recruitment, the recruitment distribution, equilibrium total mortality and no movement, where $\underline{(x)^{[j]}}$ denotes the j^{th} iterate of an estimate of a quantity x.

i. In general, population recruitment by area and gender is a function of area-specific recruitment production and the recruitment distribution matrix $\underline{\underline{Q}}^{(p)}$. Set the initial age-0 fished recruits as a function of unfished recruitment by and recruitment distribution by area and gender via

(40) $$\left(N_{F,d,a=0,g}^{(p)}\right)^{[1]} \equiv \left(R_{F,d,g}^{(p)}\right)^{[1]} = \sum_k Q_{k\to d}^{(p)} \cdot \widetilde{N}_{U,k,a=0,g}^{(p)}$$

ii. Set initial age-a survivors by area and gender for ages a=1 to A-1 from the initial fished recruits and equilibrium total mortality by area and gender via

(41) $$\left(N_{F,d,a,g}^{(p)}\right)^{[1]} = \left(N_{F,d,a-1,g}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{a-1,g}^{(p)}\right)$$

iii. Set initial Age-A group of survivors from the initial fished recruits and equilibrium total mortality by area and gender via

(42) $$\left(N_{F,d,A,g}^{(p)}\right)^{[1]} = \frac{\left(N_{F,d,A-1,g}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{A-1,g}^{(p)}\right)}{1-\exp\left(-M_{A,g}^{(p)}\right)}$$

iv. Set equilibrium fished spawning biomass by population, area and gender via

(43) $$\left(SB_{F,d,g}^{(p)}\right)^{[1]} = \sum_a P_{M,a,g}^{(p)} \cdot W_{S,a,g}^{(p)} \cdot \left(N_{F,d,a,g}^{(p)}\right)^{[1]} \cdot \exp\left(-\Delta_S^{(p)} \cdot M_{a,g}^{(p)}\right)$$

Iteration i=2: Calculate the next iterate of the equilibrium fished numbers at age estimates by population, area, and gender based on the recruitment submodel, the recruitment distribution, equilibrium total mortality, movement probabilities by area, age, and gender, and the previous iterate. Equilibrium fished recruitment production by area is a function of area-specific spawning biomasses which need to be iteratively calculated to account for the movement probabilities that redistribute fish.

i. Set age-0 recruits as a function of the area-specific stock-recruitment submodel $f_d^{(p)}$ and recruitment distribution by area and gender. This step initiates the recruitment dynamics via

(44) $$\left(R_{F,d}^{(p)}\right)^{[2]} = f_d^{(p)}\left(\left(SB_{F,d,g=female}^{(p)}\right)^{[1]} \mid \underline{\theta}_d^{(p)},\ SB_{U,d,g=female}^{(p)}\right)$$ and

(45) $$\left(N_{F,d,a=0,g}^{(p)}\right)^{[2]} = \rho_g^{(p)} \sum_k Q_{k\to d}^{(p)} \cdot \left(R_{F,k}^{(p)}\right)^{[2]}$$

ii. Set age-*a* survivors for ages *a*=1 to *A*-1 by population, area, and gender as survivors that did not emigrate plus age-*a* surviving immigrants from other areas This step turns on the movement dynamics via

(46) $$\left(N_{F,d,a,g}^{(p)}\right)^{[2]} = \sum_k \left(N_{F,k,a-1,g}^{(p)}\right)^{[1]} \cdot \exp\left(-Z_{F,k,a-1,g}^{(p)}\right) \cdot T_{k\to d,a-1,g}^{(p)}$$

iii. Set age-(*A*-1) survivors that did not emigrate plus age-(*A*-1) immigrants from other areas plus age-*A* group survivors that did not emigrate plus age-*A* group immigrants from other areas via

(47) $$\begin{aligned}\left(N_{F,d,A,g}^{(p)}\right)^{[2]} &= \sum_k \left(N_{F,k,A-1,g}^{(p)}\right)^{[1]} \cdot \exp\left(-Z_{F,k,A-1,g}^{(p)}\right) \cdot T_{k\to d,A-1,g}^{(p)} \\ &+ \sum_k \left(N_{F,k,A,g}^{(p)}\right)^{[1]} \cdot \exp\left(-Z_{F,k,A,g}^{(p)}\right) \cdot T_{k\to d,A,g}^{(p)}\end{aligned}$$

iv. Set equilibrium fished spawning biomass by population, area and gender via

(48) $$\left(SB_{F,d,g}^{(p)}\right)^{[2]} = \sum_a P_{M,a,g}^{(p)} \cdot W_{S,a,g}^{(p)} \cdot \left(N_{F,d,a,g}^{(p)}\right)^{[2]} \cdot \exp\left(-\Delta_S \cdot Z_{F,d,a,g}^{(p)}\right)$$

<u>Iteration i=j+1: Calculate the (*kj*1)[st] iterate of equilibrium fished numbers at age estimates by population, area, and gender based on the recruitment submodels and recruitment distribution, equilibrium total mortality, movement probabilities by area, age, and gender, and the *j*[th] iterate.</u>

i. Set age-0 recruits as a function of the recruitment submodel $f_d^{(p)}$ and recruitment distribution by area and gender via

(49) $$\left(R_{F,d}^{(p)}\right)^{[j+1]} = f_d^{(p)}\left(\left(SB_{F,d,g=female}^{(p)}\right)^{[j]} \mid \underline{\theta}_d^{(p)}\,,\ SB_{F,d,g=female}^{(p)}\right) \text{ and}$$

(50) $$\left(N_{F,d,a=0,g}^{(p)}\right)^{[j+1]} = \rho_g^{(p)} \sum_k Q_{k\to d}^{(p)} \cdot \left(R_{F,k}^{(p)}\right)^{[j]}$$

ii. Set age-*a* survivors for ages *a*=1 to *A*-1 by population, area, and gender that did not emigrate plus age-*a* surviving immigrants from other areas via

(51) $$\left(N_{F,d,a,g}^{(p)}\right)^{[j+1]} = \sum_k \left(N_{F,k,a-1,g}^{(p)}\right)^{[j]} \cdot \exp\left(-Z_{F,k,a-1,g}^{(p)}\right) \cdot T_{k\to d,a-1,g}^{(p)}$$

iii. Set age-(A-1) survivors that did not emigrate plus age-(A-1) immigrants from other areas plus age-A group survivors that did not emigrate plus age-A group immigrants from other areas via

(52)
$$\begin{aligned}\left(N_{F,d,A,g}^{(p)}\right)^{[j+1]} = & \sum_{k}\left(N_{F,k,A-1,g}^{(p)}\right)^{[j]} \cdot \exp\left(-Z_{F,k,A-1,g}^{(p)}\right) \cdot T_{k\to d,A-1,g}^{(p)} \\ & + \sum_{k}\left(N_{F,k,A,g}^{(p)}\right)^{[j]} \cdot \exp\left(-Z_{F,k,A,g}^{(p)}\right) \cdot T_{k\to d,A,g}^{(p)}\end{aligned}$$

Set unfished spawning biomass by population, area and gender via

(53)
$$\left(SB_{F,d,g}^{(p)}\right)^{[j+1]} = \sum_{a} P_{M,a,g}^{(p)} \cdot W_{S,a,g}^{(p)} \cdot \left(N_{F,d,a,g}^{(p)}\right)^{[j+1]} \cdot \exp\left(-\Delta_S \cdot Z_{F,d,a,g}^{(p)}\right)$$

Continue the iterations until convergence is achieved or the maximum number of iterations has been reached.

If the algorithm converges, then one has determined the fished numbers at age by population, area, and gender in equilibrium $\left(\widetilde{N}_{F,d,a,g}^{(p)}\right)$ along with the fished spawning biomass by population, area, and gender in equilibrium $\left(\widetilde{SB}_{F,d,g}^{(p)}\right)$. This population-specific information is used to set the initial conditions at the start (first year) of the initialization time period, prior to the stock assessment time horizon. These initial conditions, along with initial numbers-at-age deviation parameters, determine the population dynamics for the initialization time period.