積分ノート

Masanari Kimura

1　関数の微分形を含む積分

1.1

$$
\begin{aligned}
\int \frac{1}{x(\log x)^2} dx &= \int \frac{(\log x)'}{(\log x)^2} dx \\
&= -\frac{1}{\log x} + C
\end{aligned}
$$

1.2

$$
\begin{aligned}
\int \frac{\log(\log x)}{x \log x} dx &= \int \frac{\log t}{t} dt \\
&= \int \log t (\log t)' dt \\
&= \frac{1}{2}(\log t)^2 + C \\
&= \frac{1}{2}(\log \log x)^2 + C
\end{aligned}
$$

1.3

$$
\begin{aligned}
\int_0^1 (x + 2x^3)\sqrt{1 + 2x^2} dx &= \int_0^1 x(1 + 2x^2)\sqrt{1 + 2x^2} dx \\
&= \int_0^1 x(1 + 2x^2)^{\frac{3}{2}} dx \\
&= \frac{1}{4} \int_0^1 (1 + 2x^2)'(1 + 2x^2)^{\frac{3}{2}} dx \\
&= \frac{1}{4} \Big[\frac{2}{5}(1 + 2x^2)^{\frac{5}{2}} \Big]_0^1 \\
&= \frac{1}{10}(3^{\frac{5}{2}} - 1)
\end{aligned}
$$

2　点対称性の利用

2.1

$$
\int_0^\pi \frac{x \sin x}{8 + \sin^2 x} dx = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{8 + \sin^2 x} dx
$$

$$
\begin{aligned}
&= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{9-\cos^2 x}dx \\
&= \frac{\pi}{2}\int_{-1}^{1} \frac{1}{9-t^2}dt \\
&= \frac{\pi}{2}\int_{-1}^{1} \frac{1}{(3-t)(3+t)}dt \\
&= \frac{\pi}{2}\int_{-1}^{1} \frac{1}{6}\Big(\frac{1}{3-t}+\frac{1}{3+t}\Big)dt \\
&= \frac{\pi}{12}\Big[-\log|3-t|+\log|3+t|\Big]_{-1}^{1} \\
&= \frac{\pi}{12}\Big[-\log 2+2\log 2+2\log 2-\log 2\Big] \\
&= \frac{\pi}{6}\log 2
\end{aligned}
$$

最初の変形で $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$ を使った.

3　同形出現

3.1

$$
\begin{aligned}
\int e^x \sin x dx &= e^x \sin x - \int e^x \cos x dx \\
&= e^x \sin x - \Big(e^x \cos x + \int e^x \sin x dx\Big) \\
&= e^x \sin x - e^x \cos x - \int e^x \sin x dx \\
2\int e^x \sin x dx &= e^x(\sin x - \cos x) \\
\int e^x \sin x dx &= \frac{1}{2}e^x(\sin x - \cos x)
\end{aligned}
$$

3.2

$$
\begin{aligned}
\int e^x \cos x dx &= e^x \cos x + \int e^x \sin x dx \\
&= e^x \cos x + e^x \sin x - \int e^x \cos x dx \\
2\int e^x \cos x dx &= e^x(\sin x + \cos x) \\
\int e^x \cos x dx &= \frac{1}{2}e^x(\sin x + \cos x)
\end{aligned}
$$