0. Read Chapters I and II of *Gödel's Proof* (revised edition), by Nagel and Newman. You can find the full book electronically at a link on the Moodle Page (or get a paperback copy for about $10). You will submit a short reading response on these chapters; I have not determined the exact format of this response but will post it soon.

1. (a) In each of the following cases, write down the formula α_t^u associated to the given formula α, variable u and term t (no explanations necessary):
   i. α is $(\forall x)(=xx')$, u is x, t is $+xx'$
   ii. α is $(\forall x)(=xx')$, u is x', t is $+xx'$
   iii. α is $((\forall x)(>xx') \vee (\forall x')(>xx'))$, u is x, t is $+x'x''$

   (b) In each of the cases in part (a), work out whether t is substitutable in α for u, or not. You should explain your answers clearly using the definition of 'substitutable'.

2. Write down a deduction for each of the following $\mathcal{L}_{NT}$-formulas, using the first-order logical axioms and rules of inference.
   (a) $Sx = Sx$
   (b) $((0 = 0) \vee \neg(0 = 0)) \to (Sx = Sx)$
   (c) $(\forall x)(Sx = Sx)$
   (d) $S0 = S0$

3. Let $\mathcal{L}$ be a first-order language with a unary relation symbol R. Write out an explicit deduction to show that
$$(\forall x)(Rx) \vdash (\forall x')(Rx').$$

4. Let Γ be a set of formulas in a first-order language $\mathcal{L}$. We make the following definition: a formula ϕ in $\mathcal{L}$ is called *decidable by* Γ if either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$.
   (a) Prove that if α is decidable by Γ, then so is $\neg\alpha$. (This doesn't sound like I'm saying anything at all, but there is something subtle to prove. Nonetheless, the proof is not long given what we've already established).
   (b) Prove that if α and β are both decidable by Γ, then so is $\alpha \vee \beta$.
   (c) Suppose that every atomic formula in $\mathcal{L}$ is decidable. Prove that any formula with no quantifiers is decidable.

> **Note** In the following problem, you should assume the Deduction Theorem for our full deductive system. It states: for any set of formulas Γ and any two formulas α and β, *if α is a sentence,* then $\Gamma, \alpha \vdash \beta$ if and only if $\Gamma \vdash \alpha \to \beta$.

5. Let Γ be a set of formulas in a first-order language $\mathcal{L}$. We make the following definitions: Γ is *inconsistent* if there is a formula ϕ such that both $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$. Call Γ *explosive* if it deductively implies *every formula*, i.e. $\Gamma \vdash \phi$ for every formula ϕ in $\mathcal{L}$.
   (a) Prove that Γ is inconsistent if and only if Γ is explosive. Equivalently: if there is even a *single formula* ϕ such that $\Gamma \nvdash \phi$, then Γ is consistent.
   (b) Prove that if ϕ is a sentence, then $\Gamma \vdash \phi$ if and only if $\Gamma \cup \{\neg\phi\}$ is inconsistent.
   (c) Prove that if ϕ is a sentence, then ϕ is undecidable from Γ if and only if both $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg\phi\}$ are consistent.