1. (Textbook 3.4)
   (Decide irreducibility of five polynomials)

2. (Textbook 3.5)
   (Decide irreducibility of three polynomials)

3. (Textbook 3.12)
   (Some true/false questions about factoring polynomials)

4. Prove, as claimed in class, that if ν is a discrete valuation on a field K, and $a, b \in K$ have $\nu(a) \neq \nu(b)$, then $\nu(a+b) = \min\{\nu(a), \nu(b)\}$.

5. Let ν be a discrete valuation on a field K. Extend ν from K to $K[t]$ with the following definition:
   $$\nu(a_0 + a_1 t + \cdots + a_d t^d) = \min\{\nu(a_0), \nu(a_1), \cdots, \nu(a_d)\}.$$
   Prove that for all $f, g \in K[t]$, $\nu(fg) = \nu(f)\nu(g)$.

   Hint First, prove that $\nu(fg) \geq \nu(f)\nu(g)$. Then, to show that equality holds, it suffices to exhibit a single coefficient with the desired valuation.

6. Let $\alpha \in \mathbb{C}$ be an algebraic number, with minimal polynomial $m \in \mathbb{Q}[t]$. Let $d = \deg_{\mathbb{Q}}(\alpha)$, which we have seen is also equal to ∂m.

   (a) Prove that m is *separable*, as defined in the previous problem set, and therefore has d distinct roots.

   (b) Let $\alpha_1, \cdots, \alpha_d \in \mathbb{C}$ be the distinct roots of m; as mentioned in class, these are called the *conjugates* of α. Prove that all the conjugates α_i have the same minimal polynomial m.

   (c) Denote by $N_{\mathbb{Q}}(\alpha)$ the product of all the conjugates, i.e. $\alpha_1 \alpha_2 \cdots \alpha_d$. Prove that $N_{\mathbb{Q}}(\alpha) \in \mathbb{Q}$.

   (d) Determine a formula for $N_{\mathbb{Q}}(a + b\sqrt{d})$. Here, assume $a, b, d \in \mathbb{Z}$, d is squarefree, and $b \neq 0$.

7. Using the same notation as the previous problem:

   (a) Prove that for each conjugate α_i, there is a unique field homomorphism $\phi_i : \mathbb{Q}(\alpha) \to \mathbb{C}$ such that $\phi_i(\alpha) = \alpha_i$.

   Hint Use the isomorphism $\mathbb{Q}(\alpha) \cong \mathbb{Q}[t]/\langle m \rangle$, and apply the first isomorphism theorem for rings to reduce the problem to classifying certain homomorphisms $\mathbb{Q}[t] \to \mathbb{C}$.

   (b) Prove that in any nonzero field homomorphism $\phi : \mathbb{Q}(\alpha) \to \mathbb{C}$, $\phi(\alpha)$ must be one of the conjugates of α. Conclude that there exist exactly d nonzero homomorphisms $\mathbb{Q}(\alpha) \to \mathbb{C}$.

   Note This is yet possible way to define the degree of an algebraic number, and another point of view on "conjugates:" they are other avatars of α under different choices of embedding of $\mathbb{Q}(\alpha)$ in $\mathbb{C}$.