

Written problems:

1. Let f, g, h be positive-valued functions, and assume that $h(x) \geq 1$ for all x . Prove that if $f(x) = \mathcal{O}(h(x))$ and $g(x) = \mathcal{O}(1)$, then $f(x) + g(x) = \mathcal{O}(h(x))$ and $f(x)g(x) = \mathcal{O}(h(x))$.
2. For any positive integer n , denote by $B(n)$ the number of bits of n . In other words, $B(n)$ is the unique integer such that

$$2^{B(n)-1} \leq n < 2^{B(n)}.$$

- (a) Let $f(n)$ be a function from positive integers to positive real numbers. Prove that $f(n) = \mathcal{O}(n)$ if and only if $f(n) = \mathcal{O}(2^{B(n)})$.
 - (b) Prove that $f(n) = \mathcal{O}(\sqrt{n})$ if and only if $f(n) = \mathcal{O}(\sqrt{2^{B(n)}})$.
 - (c) Let d be a positive integer. Prove that $f(n) = \mathcal{O}((\log n)^d)$ if and only if $f(n) = \mathcal{O}(B(n)^d)$.
3. This problem considers a slightly more general form of babystep-giantstep, in which the babystep list and giantstep list need not have the same length. Let p be a prime number, and let g, h be two units modulo p . Suppose that two positive integers M, N are chosen, and we construct two lists as follows.
 - The babystep list consists of $g^i \pmod p$ for $i = 0, 1, \dots, M-1$.
 - The giantstep list consists of $hg^{-Mj} \pmod p$ for $j = 0, 1, \dots, N-1$.
 - (a) Prove that there is a collisions between these two lists if and only if there exists a solution x to the discrete logarithm problem $g^x \equiv h \pmod p$ with $0 \leq x < MN$.
 - (b) Under what circumstances will there be *multiple* collisions between the two lists?
 - (c) Suppose that M, N are chosen such that $MN \geq \text{ord}_p(g)$, and further suppose that the lists do not collide. Prove that the discrete logarithm problem $g^x \equiv h \pmod p$ has no solution.
 4. Solve each system of congruences. Your answer should take the form of a single congruence of the form $x \equiv c \pmod m$ describing all solutions to the system.

- (a) $x \equiv 1 \pmod 3$
 $x \equiv 2 \pmod 5$

- (b) $x \equiv 6 \pmod{11}$
 $x \equiv 2 \pmod{10}$

- (c) $x \equiv 2 \pmod 3$
 $x \equiv 1 \pmod{10}$
 $x \equiv 3 \pmod 7$

- (d) $x \equiv 6 \pmod 8$
 $x \equiv 3 \pmod 9$
 $x \equiv 0 \pmod{17}$

Programming problems:

1. Implement the Babystep-Giantstep algorithm, efficiently enough to solve the discrete logarithm problem for primes up to 40 bits in length. That is, write a function `bsgs(g,h,p)` that finds an integer x such that $g^x \equiv h \pmod{p}$. You may assume that p is prime, $1 \leq g, h < p$, and that a solution x exists.

The test bank on Gradescope will be identical to the test bank for `disclog` on Problem Set 2, but now your code must solve all 40 cases for full credit.

(It is fine to implement any algorithm you can devise to solve the discrete logarithm problem, but BSGS is probably the easiest to implement based on what we've discussed in class.)

Note If implemented a certain way, your solution to this problem can be almost exactly identical to your solution to the following problem, and indeed can consist of a one-line call to the function you write for the following problem.

2. Write function `bsgsBoundedOrder(g,h,p,N)` to solve the discrete logarithm function $g^x \equiv h \pmod{p}$ under the assumption that the order of g is at most N . You may assume that p is a prime number, but unlike the previous problem it will be quite large (256 bits). The integer N will be various sizes, up to 40 bits. Any correct solution x will be marked correct, even if it is not the smallest possible solution.

Note The time limit is set to 1 second on Gradescope as usual, which may be slightly tight on the largest cases. Let me know if you are running into the time limit so that I can suggest improvements. I may extend the time limit if it seems that generally efficient implementations are not quite meeting it.

3. Suppose that you are given two integers m_1, m_2 with $\gcd(m_1, m_2) = 1$, and two integers a_1, a_2 . Write a function `crt(a1,m1,a2,m2)` that efficiently determines the unique integer x such that

$$\begin{aligned} x &\equiv a_1 \pmod{m_1}, \\ x &\equiv a_2 \pmod{m_2}, \end{aligned}$$

and $0 \leq x < m_1 m_2$. The fact that x exists and is unique comes from the Chinese Remainder Theorem.

4. (This will be an ingredient in the Pohlig-Hellman algorithm, to be discussed soon) Write a function `ppFactor(N)` which accepts an integer $N \geq 2$, and returns a list of the prime powers (all powers of different primes) factoring N , in any order. For example, if $N = 12$ the function should return either `[4,3]` or `[3,4]`. The integer N may be quite large (up to 1024 bits), but you may assume that all of the prime-power factors are 16 bits or smaller.

Suggested approach: There are many ways to do this, and certainly many more efficient than what I'm about to describe, but here is one relatively quick-to-implement approach. Write a `for` loop to iterate through all numbers p from 2 to 2^{16} . For each number, check whether it divides N . If so, divide N by p repeatedly until it is no longer divisible by p (and replace N by the new value), then add the appropriate power of p to the list you will eventually return. As long as you shrink N as you go, you will never find that $p \mid N$ unless p is in fact prime, since any smaller factor would have already been found to divide N .