1. In general, we call a field extension *Galois* if it is both normal and separable. Prove that if L/K is a finite extension, then the following conditions are equivalent.

   - L/K is Galois.
   - $\Gamma(L/K)$ has exactly $[L : K]$ elements.
   - K is the fixed field of $\Gamma(L/K)$ (that is, if $\alpha \in L$ satsifes $\phi(\alpha) = \alpha$ for all $\alpha \in \Gamma(L, K)$, then in fact $\alpha \in K$).

   > **Note** These equivalences are fairly standard results, the proofs of which you may be able to find somewhere in our text or other algebra texts. You should feel free to consult all such sources to study the proofs. After you have done so, try to write as convincing (to you) and concise a proof as you can based on the tools we've developed in this course.

2. Let L/K be a finite Galois extension, and M an intermediate extension.

   (a) Prove that for all $\phi \in \Gamma(M/K)$, there exists $\psi \in \Gamma(L/K)$ such that $\phi(\alpha) = \psi(\alpha)$ for all $\alpha \in M$.

   (b) Prove that if M/K is a normal extension, then $\psi(M) = M$ for all $\psi \in \Gamma(L/K)$. Conclude that there is a surjective group homomorphism $\Gamma(L/K) \to \Gamma(M/K)$. What is its kernel?

3. (This is essentially Textbook 12.2, except that you should not assume the extension lies in $\mathbb{C}$). Let L/K be a finite Galois extension, and M, N be two intermediate subfields with $N \supseteq M$.

   (a) Prove that N/M is a normal extension if and only if N^* is a normal subgroup of M^*.

   (b) Prove that if N/M is normal, then the quotient group M^*/N^* is isomorphic is $\Gamma(N/M)$.

4. Let L/K be a finite Galois extension. For any intermediate field M and any $\phi \in \Gamma(L/K)$, denote by $\phi(M)$ the field $\{\phi(\alpha) : \alpha \in M\}$. The fields $\{\phi(M) : \phi \in \Gamma(L/K)\}$ are called the *conjugate fields* of M. Denote by $\mathrm{Stab}(M) \subseteq \Gamma(L/K)$ the subgroup of automorphisms ϕ such that $\phi(M) = M$.

   (a) Prove that restriction from L to M gives a *surjective* group homomorphism $\mathrm{Stab}(M) \to \Gamma(M/K)$, and state its kernel.

   (b) Prove that the number of conjugate fields of M is equal to $[L : K]/|\mathrm{Stab}(M)|$. (Hint: you can do this with the *orbit-stabilizer theorem,* if you know what that is already, or if you look it up. You can also do it with bare hands; feel free to ask me for a hint.)

   (c) Deduce that the number of conjugate fields of M is equal to $[M : K]/|\Gamma(M/K)|$. (Combining with the first problem, this shows that M/K is normal if and only if M has no conjugate fields besides itself.)

   {IOU: I may add another problem or two to this set later, but I am posting these problems for now so that you can start thinking about them.}