

Goal Improper integrals of Types *I* and *II*, incorporating various integration techniques and limits via L'Hôpital's rule.

Reference: §7.8

Examples to study first

Example Evaluate $\int_{-\infty}^7 \frac{1}{x^2 - 6x + 25} dx$.

Solution

$$\begin{aligned}
 \int_{-\infty}^7 \frac{1}{x^2 - 6x + 25} dx &= \lim_{t \rightarrow -\infty} \int_t^7 \frac{1}{x^2 - 6x + 25} dx \stackrel{\text{complete square}}{=} \lim_{t \rightarrow -\infty} \int_t^7 \frac{1}{(x-3)^2 + 16} dx \\
 &= \lim_{t \rightarrow -\infty} \int_{t-3}^4 \frac{1}{u^2 + 16} du \quad \begin{array}{l} u = x-3 \\ du = dx \end{array} \\
 &= \lim_{t \rightarrow -\infty} \frac{1}{4} \arctan\left(\frac{u}{4}\right) \Big|_{t-3}^4 = \lim_{t \rightarrow -\infty} \frac{1}{4} \left(\arctan\left(\frac{4}{4}\right) - \arctan\left(\frac{t-3}{4}\right) \right) \\
 &= \lim_{t \rightarrow -\infty} \frac{1}{4} \left(\arctan(1) - \arctan\left(\frac{t-3}{4}\right) \right) = \frac{1}{4} \left(\frac{\pi}{4} - \left(-\frac{\pi}{2}\right) \right) = \frac{1}{4} \left(\frac{3\pi}{4} \right) \\
 &= \boxed{\frac{3\pi}{16}}
 \end{aligned}$$

Example Evaluate $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$.

Solution This is a Type 2 improper integral.

$$\begin{aligned}
 \int_0^1 \frac{\ln x}{\sqrt{x}} dx &= \lim_{s \rightarrow 0^+} \int_s^1 \frac{\ln x}{\sqrt{x}} dx = \lim_{s \rightarrow 0^+} \int_s^1 (\ln x) x^{-\frac{1}{2}} dx \\
 &\stackrel{IBP}{=} \lim_{s \rightarrow 0^+} 2\sqrt{x} \ln x \Big|_s^1 - 2 \int_s^1 \frac{1}{\sqrt{x}} dx \quad \begin{array}{l} u = \ln x \quad dv = x^{-\frac{1}{2}} dx \\ du = \frac{1}{x} dx \quad v = 2\sqrt{x} \end{array} \\
 &= \lim_{s \rightarrow 0^+} 2\sqrt{x} \ln x \Big|_s^1 - 4\sqrt{x} \Big|_s^1 = \lim_{s \rightarrow 0^+} 2\sqrt{1} \ln 1 - 4\sqrt{1} - \left(2\sqrt{s} \ln s^{0 \cdot (-\infty)} - 4\sqrt{s} \right) \\
 &\stackrel{*}{=} \boxed{-4}
 \end{aligned}$$

* L'H Finish: $\lim_{s \rightarrow 0^+} \sqrt{s} \ln s \stackrel{0 \cdot (-\infty)}{=} \lim_{s \rightarrow 0^+} \frac{\ln s}{\frac{1}{\sqrt{s}}} \stackrel{\infty}{=} \lim_{s \rightarrow 0^+} \frac{\frac{1}{s}}{-\frac{1}{2s^{\frac{3}{2}}}} \stackrel{\text{L'H}}{=} \lim_{s \rightarrow 0^+} -2\sqrt{s} = 0$

Example Evaluate $\int_0^6 \frac{8}{x^2 - 4x - 12} dx$.

Solution

$$\int_0^6 \frac{8}{x^2 - 4x - 12} dx = \int_0^6 \frac{8}{(x-6)(x+2)} dx = \lim_{t \rightarrow 6^-} \int_0^t \frac{8}{(x-6)(x+2)} dx$$

$$\stackrel{\text{PFD}}{=} \lim_{t \rightarrow 6^-} \int_0^t \frac{1}{x-6} - \frac{1}{x+2} dx = \lim_{t \rightarrow 6^-} \ln|x-6| - \ln|x+2| \Big|_0^t$$

$$= \lim_{t \rightarrow 6^-} \ln|t-6| - \ln|t+2| - (\ln|-6| - \ln 2)$$

$$= -\infty - \ln 8 - \ln 6 + \ln 2 = \boxed{-\infty} \text{ (the integral diverges)}$$

PFD algebra:

$$\frac{8}{(x-6)(x+2)} = \frac{A}{x-6} + \frac{B}{x+2}$$

Clearing the denominator yields:

$$8 = A(x+2) + B(x-6) = Ax + 2A + Bx - 6B = (A+B)x + (2A-6B)$$

so that $A+B=0$ and $2A-6B=8$.

Solving gives $B=-A$ and $2A+6A=8$, hence $A=1$ and $B=-1$.

Problems to hand in

Compute each of the following Integrals. Simplify when possible.

1. $\int_{-\infty}^0 \frac{1}{3-4x} dx$

2. $\int_1^{\infty} \frac{1}{(2x+1)^3} dx$

3. $\int_2^{\infty} \frac{x}{e^{3x}} dx$

4. $\int_e^{\infty} \frac{\ln x}{x^3} dx$

5. $\int_e^{\infty} \frac{1}{x(\ln x)^2} dx$

6. $\int_e^{\infty} \frac{1}{x \ln x} dx$

7. $\int_{-\infty}^7 \frac{1}{x^2 - 4x + 29} dx$

8. $\int_0^5 \frac{6}{x^2 - 4x - 5} dx$

9. $\int_0^{e^5} \frac{1}{x[25 + (\ln x)^2]} dx$

10. $\int_1^2 \frac{1}{x \ln x} dx$

11. $\int_0^1 x \ln x dx$