

Goal Practice taking limits of infinite **sequences**. Master the various notations used for sequences. Practice a bit with L'Hôpital's rule.

Reference: §11.1

Examples to study first

In each example, **determine whether the given sequence Converges or Diverges**. If it converges, find the **Limit**.

Example $\left\{ \frac{\ln n}{n^3} \right\}_{n=1}^{\infty}$

Solution $\lim_{n \rightarrow \infty} \frac{\ln n}{n^3} = \lim_{x \rightarrow \infty} \frac{\ln x}{x^3} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{3x^2} = \lim_{x \rightarrow \infty} \frac{1}{3x^3} = \boxed{0}$ Converges

Note In the calculation above, the first step changes the n to an x . This is to emphasize that the limit of the **sequence** is the same as the limit of the **function** with the same formula. This is an important conceptual point, and it is also done in the examples below. I am not picky about whether this is shown in your work, however.

Example $\left\{ \frac{e^n}{n^2} \right\}_{n=1}^{\infty}$

Solution $\lim_{n \rightarrow \infty} \frac{e^n}{n^2} = \lim_{x \rightarrow \infty} \frac{e^x}{x^2} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \boxed{\infty}$ Diverges

Example $\left\{ \frac{4 - 9n^3}{5n^3 + 8n^2 - 7n - 6} \right\}_{n=1}^{\infty}$

Solution You have a choice in this problem: you can use L'Hôpital's rule, or you can do the algebra shown below (useful for stacks of polynomials). The algebra is quicker!

$$\lim_{n \rightarrow \infty} \frac{4 - 9n^3}{5n^3 + 8n^2 - 7n - 6} \cdot \frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3}\right)} = \lim_{n \rightarrow \infty} \frac{\overset{0}{\cancel{4}} - 9}{\underset{5}{\cancel{5}} + \overset{0}{\cancel{8}} - \overset{0}{\cancel{7}} - \underset{0}{\cancel{6}}} = \boxed{-\frac{9}{5}} \text{ Converges}$$

Example $\left\{ \left(1 - \sin \left(\frac{6}{n^3} \right) \right)^{n^3} \right\}_{n=1}^{\infty}$

Solution $\lim_{n \rightarrow \infty} \left(1 - \sin \left(\frac{6}{n^3} \right) \right)^{n^3} \stackrel{1^\infty}{=} \lim_{x \rightarrow \infty} \left(1 - \sin \left(\frac{6}{x^3} \right) \right)^{x^3}$

$$= e^{\lim_{x \rightarrow \infty} \ln \left[\left(1 - \sin \left(\frac{6}{x^3} \right) \right)^{x^3} \right]} = e^{\lim_{x \rightarrow \infty} x^3 \ln \left(1 - \sin \left(\frac{6}{x^3} \right) \right)} \stackrel{\infty \cdot 0}{=} \lim_{x \rightarrow \infty} \frac{\ln \left(1 - \sin \left(\frac{6}{x^3} \right) \right)}{\frac{1}{x^3}}$$

$$\stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{\frac{1}{1 - \sin \left(\frac{6}{x^3} \right)} \cdot \left(-\cos \left(\frac{6}{x^3} \right) \right) \cdot \left(-\frac{18}{x^4} \right)^6}{-\frac{3}{x^4}}} = e^{-6} = \boxed{\frac{1}{e^6}} \text{ Converges}$$

Example $\left\{ \frac{(3n-1)!}{(3n+1)!} \right\}_{n=1}^{\infty}$

Solution $\lim_{n \rightarrow \infty} \frac{(3n-1)!}{(3n+1)!} = \lim_{n \rightarrow \infty} \frac{\cancel{(3n-1)!}}{(3n+1)(3n)\cancel{(3n-1)!}} = \lim_{n \rightarrow \infty} \frac{1}{\cancel{(3n+1)}(3n)} \stackrel{0}{=} \boxed{0} \text{ Converges}$

Problems to hand in

List the first five terms of the Sequence. (Start with $n = 1$)

1. $a_n = \frac{(-1)^{n-1}}{5^n}$

2. $a_n = \frac{1}{(n+1)!}$

3. $a_n = \frac{(-1)^n n^2}{n+1}$

Determine whether the given sequence Converges or Diverges. If it converges, find the Limit. Justify your answer.

4. $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$

5. $\left\{ \frac{5n^2 + 3}{2n^2 - 7n} \right\}_{n=1}^{\infty}$

6. $\left\{ \frac{3n^4 - n - 5}{7n^4 + n^2 - 9} \right\}_{n=1}^{\infty}$

7. $\left\{ \frac{\tan^{-1} n}{n} \right\}$

8. $\left\{ \frac{n^2}{e^n} \right\}$

9. $\left\{ n \sin \left(\frac{1}{n} \right) \right\}$

10. $\left\{ \frac{(\ln n)^2}{n} \right\}_{n=1}^{\infty}$

11. $\left\{ \frac{n^{99}}{\ln n} \right\}_{n=2}^{\infty}$

12. $\left\{ \frac{\ln(99)}{n^{99}} \right\}$

13. $\left\{ \left(1 + \frac{1}{n} \right)^n \right\}_{n=1}^{\infty}$

14. $\left\{ \left(1 - \frac{5}{n^6} \right)^{n^6} \right\}_{n=1}^{\infty}$

15. $\left\{ \left(1 - \arcsin \left(\frac{3}{n^2} \right) \right)^{n^2} \right\}$

16. $\{ \ln(2n^2 + 1) - \ln(n^2 + 1) \}$

17. $\left\{ \frac{(n+3)!}{(n+1)!} \right\}_{n=1}^{\infty}$

18. $\left\{ \frac{(2n-1)!}{(2n+1)!} \right\}$

19. $\left\{ \cos^2 \left(\frac{\pi n^6 + 6}{6n^6 + 1} \right) \right\}_{n=1}^{\infty}$

20. $\left\{ \arctan \left(\frac{5n^7 + 1}{5n^7 + 7} \right) \right\}_{n=1}^{\infty}$