Solutions to Practice Final A

1. Compute the following derivatives.

a. $\dfrac{d}{dx}[e^x \sin(5x)]$

b. $\dfrac{dy}{dx}$, where $e^{2x} - e^y + xy = e^2$.

c. $G'(\ln(3))$, where $G(x) = h(e^x)$ and it is known that $h(3) = 2$ and $h'(3) = -5$.

d. $g''(x)$, where $g(x) = \sqrt{x}\ln x + \ln(\sqrt{x})$.

Solutions. (a): $\dfrac{d}{dx}[e^x \sin(5x)] = e^x \sin(5x) + 5e^x \cos(5x) = \boxed{(\sin 5x + 5\cos 5x)e^x}$

(b): Starting from $e^{2x} - e^y + xy = e^2$, then $2e^{2x} - y'e^y + y + xy' = 0$, giving $y' \cdot (-e^y + x) = -2e^{2x} - y$, and so $\boxed{y' = \dfrac{2e^{2x} + y}{e^y - x}}$

(c): Since $G(x) = h(e^x)$, we have $G'(x) = h'(e^x) \cdot e^x$, and therefore $G'(\ln(3)) = h'(e^{\ln 3}) \cdot e^{\ln 3} = h'(3) \cdot 3 = -5 \cdot 3 = \boxed{-15}$

(d): Write $g(x) = \sqrt{x}\ln x + \ln(\sqrt{x}) = x^{1/2}\ln x + \dfrac{1}{2}\ln x$.

So $g'(x) = \dfrac{1}{2}x^{-1/2}\ln x + x^{1/2} \cdot x^{-1} + \dfrac{1}{2}x^{-1} = \dfrac{1}{2}x^{-1/2}\ln x + x^{-1/2} + \dfrac{1}{2}x^{-1}$, and therefore

$g''(x) = -\dfrac{1}{4}x^{-3/2}\ln x + \dfrac{1}{2}x^{-1/2} \cdot x^{-1} - \dfrac{1}{2}x^{-3/2} - \dfrac{1}{2}x^{-2} = \boxed{-\dfrac{1}{4}x^{-3/2}\ln x - \dfrac{1}{2}x^{-2}}$

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2. Calculate the following limits.

a. $\displaystyle\lim_{x\to -1} \frac{x^2 + 4x + 3}{2x^2 + 3x + 1}$

b. $\displaystyle\lim_{x\to 9} \frac{3 - \sqrt{x}}{x - 9}$

c. $\displaystyle\lim_{x\to 1} \frac{2f(x) - 4x}{f(2x) - 5}$, where $f(x) = x^2 + 1$

d. $\displaystyle\lim_{x\to 2^-} \frac{|x - 2|}{x^2 - 4}$

Solutions. (a): $\displaystyle\lim_{x\to -1} \frac{x^2 + 4x + 3}{2x^2 + 3x + 1} = \lim_{x\to -1} \frac{(x+1)(x+3)}{(x+1)(2x+1)} = \lim_{x\to -1} \frac{x+3}{2x+1} = \frac{2}{-1} = \boxed{-2}$

(b): $\displaystyle\lim_{x\to 9} \frac{3 - \sqrt{x}}{x - 9} = \lim_{x\to 9} \frac{(3 - \sqrt{x})(3 + \sqrt{x})}{(x - 9)(3 + \sqrt{x})} = \lim_{x\to 9} \frac{9 - x}{(x - 9)(3 + \sqrt{x})} = \lim_{x\to 9} \frac{-1}{3 + \sqrt{x}}$

$\displaystyle = \frac{-1}{3 + \sqrt{9}} = \boxed{-\frac{1}{6}}$

(c): $\displaystyle\lim_{x\to 1} \frac{2f(x) - 4x}{f(2x) - 5} = \lim_{x\to 1} \frac{2x^2 + 2 - 4x}{4x^2 + 1 - 5} = \lim_{x\to 1} \frac{2(x^2 - 2x + 1)}{4(x^2 - 1)}$

$\displaystyle = \lim_{x\to 1} \frac{2(x-1)(x-1)}{4(x-1)(x+1)} = \lim_{x\to 1} \frac{2(x-1)}{4(x+1)} = \frac{0}{8} = \boxed{0}$

(d): $\displaystyle\lim_{x\to 2^-} \frac{|x - 2|}{x^2 - 4} = \lim_{x\to 2^-} \frac{-(x-2)}{(x-2)(x+2)} = \lim_{x\to 2^-} \frac{-1}{x+2} = \boxed{-\frac{1}{4}}$

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3. Compute the following integrals.

a. $\displaystyle\int \frac{(x+2)^2}{x}\,dx$

b. $\displaystyle\int_{\pi/12}^{\pi/6} \sec^2(2x)\,dx$

c. $\displaystyle\int \frac{5}{3x+2}\,dx$

d. $\displaystyle\int_1^e \frac{1}{x}\cos\left(\frac{\pi}{4}\ln x\right)\,dx$

e. $\displaystyle\int_0^3 |x-1|\,dx$ (*Hint*: cut the interval into two pieces and do each piece separately.)

Solutions. (a): $\displaystyle\int \frac{(x+2)^2}{x}\,dx = \int \frac{x^2+4x+4}{x}\,dx = \int x+4+\frac{4}{x}\,dx = \boxed{\frac{x^2}{2}+4x+4\ln|x|+C}$

(b): $\displaystyle\int_{\pi/12}^{\pi/6} \sec^2(2x)\,dx \quad [u=2x, du=2\,dx]$

$\displaystyle = \frac{1}{2}\int_{\pi/6}^{\pi/3}\sec^2 u\,du = \frac{1}{2}\tan u\Big|_{\pi/6}^{\pi/3} = \frac{1}{2}\Big(\tan\frac{\pi}{3}-\tan\frac{\pi}{6}\Big) = \frac{1}{2}(\sqrt{3}-\frac{1}{\sqrt{3}}) = \frac{1}{2\sqrt{3}}(3-1) = \boxed{\frac{1}{\sqrt{3}}}$

(c): $\displaystyle\int \frac{5}{3x+2}\,dx \quad [u=3x+2, du=3\,dx] \quad = \frac{5}{3}\int\frac{du}{u} = \frac{5}{3}\ln|u|+C = \boxed{\frac{5}{3}\ln|3x+2|+C}$

(d): $\displaystyle\int_1^e \frac{1}{x}\cos\left(\frac{\pi}{4}\ln x\right)dx \quad [u=\frac{\pi}{4}\ln x, du=\frac{\pi}{4x}\,dx] \quad = \frac{4}{\pi}\int_0^{\pi/4}\cos u\,du$

$\displaystyle = \frac{4}{\pi}\sin u\Big|_0^{\pi/4} = \frac{4}{\pi}\Big(\frac{\sqrt{2}}{2}-0\Big) = \boxed{\frac{2\sqrt{2}}{\pi}}$

(e): $\displaystyle\int_0^3 |x-1|\,dx = \int_0^1 1-x\,dx + \int_1^2 x-1\,dx = \Big[x-\frac{x^2}{2}\Big]_0^1 + \Big[\frac{x^2}{2}-x\Big]_1^3$

$\displaystyle = 1-\frac{1}{2}-(0-0)+(\frac{9}{2}-3)-(\frac{1}{2}-1) = \boxed{\frac{5}{2}}$

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4. Let $f(x) = \dfrac{4}{x+3}$. Calculate $f'(1)$ using the **limit definition of the derivative**.

Solution. $\displaystyle f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\frac{4}{4+h}-1}{h} = \lim_{h\to 0}\frac{4-(4+h)}{h(4+h)}$

$\displaystyle = \lim_{h\to 0}\frac{-h}{h(4+h)} = \lim_{h\to 0}\frac{-1}{4+h} = \boxed{-\frac{1}{4}}$

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5. Find an equation for the tangent line to the graph of $y=\ln(x^2+1)$ at the point where $x=2$.

Solution. With $f(x)=\ln(x^2+1)$, we have $f'(x)=(x^2+1)^{-1}(2x)$, and hence $f'(2)=4/5$.

Since $f(2)=\ln 5$, the equation of the tangent line is $y-\ln 5 = \frac{4}{5}(x-2)$, i.e.,

$$\boxed{y = \frac{4}{5}x + \left(\ln 5 - \frac{8}{5}\right)}$$

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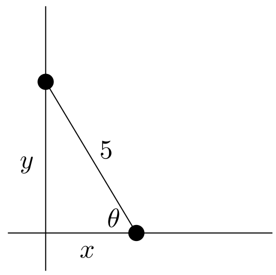
6. A ladder 5 meters long is leaning against a vertical wall. The base of the ladder starts to slide away from the wall along the (horizontal) ground, and so the top of the ladder starts to slide down the wall. At the moment when the top of the ladder is 4 meters above the ground, it is sliding down the wall at 1 meter per second. How fast is the angle between the ladder and the ground increasing (or decreasing) at that moment?

Solution. Here's the picture:



We have $y = 5\sin\theta$. Differentiating gives $y' = 5\theta'\cos\theta$.

At the key moment, we have $y = 4$, and therefore the horizontal leg has length 3. Thus, at that moment, $\cos\theta = 3/5$. Also, we have $y' = -1$ at that moment, so $-1 = 5\theta' \cdot (3/5)$, and hence $\theta' = -1/3$.

That is, the angle is decreasing at $\boxed{1/3 \text{ radians per second}}$

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7. Find the absolute maximum and absolute minimum values of the function

$$g(x) = (x^2 - 3)e^x$$

on the interval $[0, 4]$.

Solution. Since g is continuous on this closed interval, we use the Closed Interval Method.
$g'(x) = 2xe^x + (x^2 - 3)e^x = (x^2 + 2x - 3)e^x = (x + 3)(x - 1)e^x$, which is always defined.
Setting $g'(x) = 0$ gives $x = 1$ or $x = -3$, but -3 is not in the domain. So the only critical point is $x = 1$. Testing gives $g(0) = -3, \quad g(1) = -2e, \quad \text{and} \quad g(4) = 13e^4$.
Since $e \approx 2.7$, the $\boxed{\text{absolute minimum value is } g(1) = -2e}$
and the $\boxed{\text{absolute maximum is } g(4) = 13e^4}$

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8. Let $F(x) = 3x^4 + 2x^3 - 3x^2 - 5$. Find all of the critical numbers of F, and classify each of them as local maximum, local minimum, or neither.

Solution. We have $F'(x) = 12x^3 + 6x^2 - 6x = 6x(2x^2 + x - 1) = 6x(2x - 1)(x + 1)$, which is **always defined**.

Solving $F' = 0$ gives $x = 0, \dfrac{1}{2}, -1$. Our F' chart is:

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1/2)$	$(1/2, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Thus, F has **local minima at** $x = -1$ **and** $x = \dfrac{1}{2}$, and a **local maximum at** $x = 0$

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9. Let $f(x) = \dfrac{3x^3 + 9x^2 + 10x}{(x + 1)^3}$. Take my word for it that

$$f'(x) = \frac{-2(x - 5)}{(x + 1)^4}, \qquad \text{and} \qquad f''(x) = \frac{6(x - 7)}{(x + 1)^5}.$$

Sketch the graph of $y = f(x)$, clearly indicating **horizontal and vertical asymptotes**, **local extrema**, **inflection points**, and **intervals of increase and decrease and of concavity**. Please do **NOT** try to draw your graph to scale.

Solution. f has a vertical asymptote at $x = -1$ but is continuous everywhere else. For horizontal asymptotes,

$$\lim_{x\to\infty} \frac{3x^3 + 9x^2 + 10x}{(x+1)^3} = \lim_{x\to\infty} \frac{3 + 9x^{-1} + 10x^{-2}}{(1 + x^{-1})^3} = 3, \text{ and similarly } \lim_{x\to-\infty} \frac{3x^3 + 9x^2 + 10x}{(x+1)^3} = 3.$$

Thus, f has a horizontal asymptote of $y = 3$ on both sides.

f' is defined everywhere except $x = -1$, and f' is zero at $x = 5$. The f' chart is

x	$(-\infty, -1)$	$(-1, 5)$	$(5, \infty)$
$f'(x)$	$+$	$+$	$-$
$f(x)$	$\nearrow$	$\nearrow$	$\searrow$

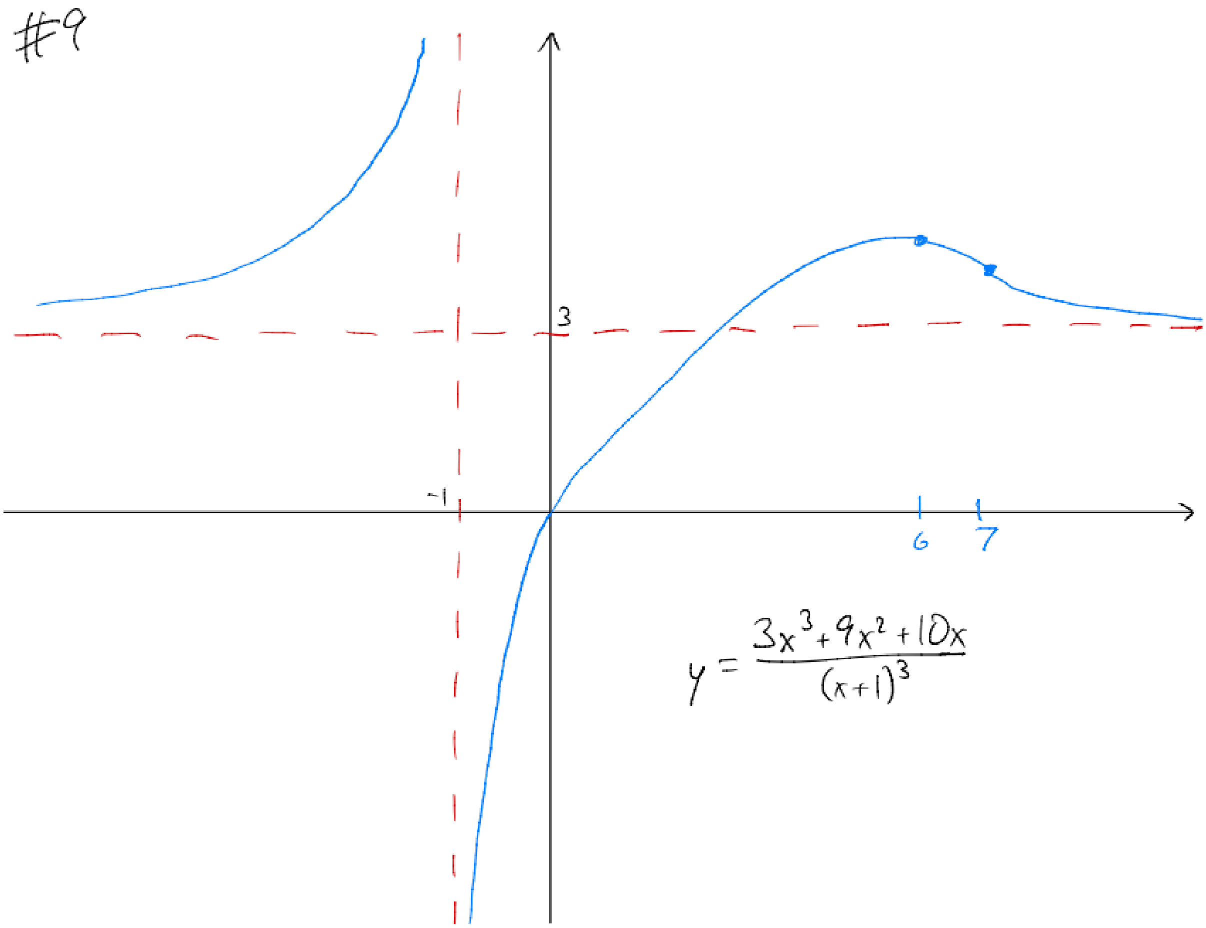
Note there is a local max at $x = 5$. [But don't forget $x = -1$ is a vertical asymptote.]

Similarly, f'' is undefined at -1 and zero at 7, with chart

x	$(-\infty, -1)$	$(-1, 7)$	$(7, \infty)$
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cup$

Note there is an inflection point at $x = 7$. [But don't forget $x = -1$ is a vertical asymptote.]

A sketch is shown below.



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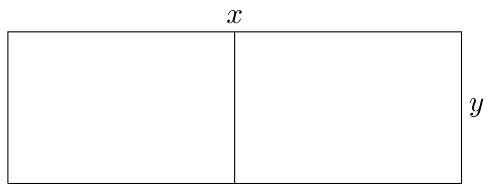
10. A farmer needs to fence off a rectangular field of area of 2000 m^2 and then divide the rectangle into two pens with an extra middle fence running parallel to two of the sides. The outside fencing costs \$20 per meter, while the middle fencing costs \$10 per meter. What should the dimensions of the field be to minimize the cost of the fence?

Solution. Here is the picture:



The area is xy. The outer fence has length $2x + 2y$, costing $20x + 20y$ dollars, and the inner fence has length y, costing $5y$ dollars. So the total cost of the fence is $20x + 25y$, which we set to 1000. Solving for y gives $y = 40 - 4x/5$. So the area is $A(x) = 40x - (4/5)x^2$, which we wish to maximize on the domain $[0, 50]$. (The 0 is because $x \geq 0$, and the 50 is because $y \geq 0$, so that $40 \geq (4/5)x$, and hence $x \leq 50$.)

Let's use the Closed Interval Method. We have $A'(x) = 40 - (8/5)x$, which is **always defined**.

Solving $A' = 0$ gives $(8/5)x = 40$, i.e., $x = 25$, which *is* in the domain. For testing, let's write $A(x) = 4x(10 - x/5)$. We have:

$$A(0) = 0, \quad A(25) = 100 \cdot (10 - 5) = 500, \quad \text{and} \quad A(50) = 0.$$

Thus, the maximum are is $\boxed{\text{500 square feet}}$
[It's attained when the two fences are length 25 feet, and the three fences are length 20 feet.]

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11. Compute the integral $\displaystyle\int_0^3 x^2 - 1\, dx$ directly from the definition, i.e., **as a limit of Riemann sums**.

Solution. Let $f(x) = x^2 - 1$. Chopping the interval $[0, 3]$ into n equal pieces gives $\Delta x = 3/n$, with i-th chop point $x_i = 0 + i\Delta x = 3i/n$. Thus, the n-th right-endpoint Riemann sum is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 1\right]\frac{3}{n} = \sum_{i=1}^{n} \frac{27i^2}{n^3} - \frac{3}{n} = \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{3}{n}\sum_{i=1}^{n} 1$$
$$= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{3}{n}n = \frac{9}{2}(1 + n^{-1})(2 + n^{-1}) - 3.$$

Therefore, $\displaystyle\int_0^3 x^2 - 1\, dx = \lim_{n\to\infty} \frac{9}{2}(1 + n^{-1})(2 + n^{-1}) - 3 = \frac{9}{2}(1)(2) - 3 = 9 - 3 = \boxed{6}$

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12. Find a function $f(x)$ such that $f'(x) = \dfrac{x^2 - 1}{x}$ with $f(1) = 2$.

Solution. We have $f'(x) = x - \dfrac{1}{x}$, so $f(x) = \dfrac{1}{2}x^2 - \ln|x| + C$.

Thus, $2 = f(1) = \dfrac{1}{2} - \ln 1 + C = C + \dfrac{1}{2}$, and hence $C = \dfrac{3}{2}$.

That is, $\boxed{f(x) = \dfrac{1}{2}x^2 - \ln|x| + \dfrac{3}{2}}$