Note This problem set is not yet complete. I will add some problems about deductions after we cover the relevant content in Friday's class.

1. Recall that we defined first-order languages in a fairly minimalist way, which includes only two logical connectives $\neg, \vee$, and one quantifier $\forall$. This problem shows how the other common symbols $\wedge, \rightarrow, \exists$ can be defined in terms of these, and confirms that the semantics are as we would expect.

   (a) For a formula α and variable u in a first-order language $\mathcal{L}$, we will write $(\exists u)(\alpha)$ as short-hand for the formula
   $$(\neg(\forall u)((\neg\alpha))).$$
   (More informally, we can also omit some parentheses and write simply $\neg\forall u\neg\alpha$). Let s be a vaf for an $\mathcal{L}$-structure $\mathcal{A}$ with universe A. Prove that $(\exists u)(\alpha)$ is true in $\mathcal{A}$ with s if and only if there is $a \in A$ such that α is true in $\mathcal{A}$ with $s[u|a]$.

   (b) For two formulas α, β, we will write $\alpha \wedge \beta$ as short-hand for the formula $(\neg((\neg\alpha) \vee (\neg\beta)))$ (or, less formally, $\neg(\neg\alpha \vee \neg\beta)$). Prove that $\alpha \wedge \beta$ is true in $\mathcal{A}$ with s if and only if both α and β are true in $\mathcal{A}$ with s.

   (c) For two formulas α, β, we will write $\alpha \rightarrow \beta$ as short-hand for the formula $((\neg\alpha) \vee \beta)$. Prove that if $\alpha \rightarrow \beta$ and α and both true in $\mathcal{A}$ with s, then so is β.

2. Prove that the $\mathcal{L}_{NT}$-formula $(\forall x)(=+x0x)$ (i.e. more informally, the formula $(\forall x)(x+0=x)$) logically implies the formula $=+000$ (i.e. more informally, the formula $0+0=0$).

3. Let $\mathcal{L}$ be a first-order language, and let $\mathcal{A}$ be a structure for $\mathcal{L}$.

   (a) Let ϕ be a sentence in $\mathcal{L}$ (i.e. a formula with no free variables). Prove that ϕ is true in $\mathcal{A}$ if and only if $(\neg\phi)$ is not true in $\mathcal{A}$.

   (b) Give an example to show that the conclusion of part (a) does not necessarily hold if ϕ has a free variable.

4+ {IOU: More problems to be posted after Friday's class}