THE GIESEKER-PETRI THEOREM AND IMPOSED RAMIFICATION

MELODY CHAN, BRIAN OSSERMAN, AND NATHAN PFLUEGER

ABSTRACT. We prove a smoothness result for spaces of linear series with prescribed ramification on twice-marked elliptic curves. In characteristic 0, we then apply the Eisenbud-Harris theory of limit linear series to deduce a new proof of the Gieseker-Petri theorem, along with a generalization to spaces of linear series with prescribed ramification at up to two points. Our main calculation involves the intersection of two Schubert cycles in a Grassmannian associated to almost-transverse flags.

1. INTRODUCTION

The classical Brill-Noether theorem states that if we are given $g, r, d \geq 0$, a general curve X of genus g carries a linear series $(\mathscr{L}, V)$ of projective dimension r and degree d if and only if the quantity

$$\rho(g, r, d) := g - (r+1)(r+g-d)$$

is nonnegative [GH80]. Moreover, in this case the moduli space $G_d^r(X)$ of such linear series has pure dimension ρ. This statement was generalized by Eisenbud and Harris to allow for imposed ramification: given marked points $P_1, \ldots, P_n \in X$, and sequences $0 \leq a_0^i < \cdots < a_r^i \leq d$ for $i = 1, \ldots, n$, consider the moduli space $G_d^r(X, (P_1, a_\bullet^1), \ldots, (P_n, a_\bullet^n)) \subseteq G_d^r(X)$ parametrizing linear series with vanishing sequence at least $a_\bullet^i$ at each of the P_i. Then Eisenbud and Harris used their theory of limit linear series to show in [EH86] that in characteristic 0, if $(X, P_1, \ldots, P_n)$ is a general n-marked curve of genus g, the dimension of $G_d^r(X, (P_1, a_\bullet^1), \ldots, (P_n, a_\bullet^n))$—if it is nonempty—is given by the generalized formula

$$\rho(g, r, d, a_\bullet^1, \ldots, a_\bullet^n) := g - (r+1)(r+g-d) - \sum_{i=1}^n \sum_{j=0}^r (a_j^i - j).$$

The condition for nonemptiness is still combinatorial, but becomes more complicated in this context.

This theorem fails in positive characteristic for $n \geq 3$, but is still true if $n \leq 2$. In this case, we also have a simple criterion for nonemptiness. To state it, we shift notation, supposing we have marked points $P, Q \in X$, and sequences $a_\bullet$, $b_\bullet$. We then introduce the following notation:

$$\widehat{\rho}(g, r, d, a_\bullet, b_\bullet) := g - \sum_{j: a_j + b_{r-j} > d-g} a_j + b_{r-j} - (d-g).$$

We summarize what was previously known about the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$, as follows.

Theorem 1.1. *Given (g, r, d) nonnegative integers, and sequences $0 \le a_0 < a_1 < \cdots < a_r \le d$, $0 \le b_0 < b_1 < \cdots < b_r \le d$, let (X, P, Q) be a twice-marked smooth projective curve of genus g over a field of any characteristic. Set $\rho = \rho(g, r, d, a_\bullet, b_\bullet)$ and set $\widehat{\rho} = \widehat{\rho}(g, r, d, a_\bullet, b_\bullet)$.*

Suppose that X and P, Q are general. Then $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is nonempty if and only if $\widehat{\rho} \ge 0$, and if nonempty, has pure dimension ρ. Furthermore, it is reduced and Cohen-Macaulay, and if $\widehat{\rho} \ge 1$, it is connected.

For the nonemptiness and dimension statements, see [Oss14]; for reducedness and connectedness, see [Oss]. The Cohen-Macaulayness statement follows from the construction of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ (see for instance the proof of Proposition 3.1 below) together with the Cohen-Macaulayness of relative Schubert cycles.

What has remained open until now is the question of the singularities of the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$. In the absence of marked points, Gieseker in 1982 used degenerations to prove a conjecture of Petri that if X is general, then the space $G_d^r(X)$ is also smooth [Gie82]. This proof was later simplified by Eisenbud and Harris [EH83] and Welters [Wel85] using ideas closely related to the theory of limit linear series. These proofs all relied on proving injectivity of the Petri map, by taking a hypothetical nonzero element of the kernel, and carrying out a careful analysis of how it would behave under degeneration. Recently, Jensen and Payne have given a different proof of the Gieseker-Petri theorem by tropical degeneration methods [JP14]. We also mention that in Proposition 3.2 of [CHT92], Ciliberto, Harris and Teixidor assert partial smoothness results in the case of a single ramification point via an argument building on the Eisenbud-Harris approach, but their statement omits a key hypothesis; see Remark 3.6 below.

In this paper, we give a new proof of the Gieseker-Petri theorem, and generalize it to the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$, proving that the singular locus of this space consists precisely of linear series with a certain type of excess vanishing. Our Gieseker-Petri theorem with imposed ramification, Theorem 1.4 below, generalizes two statements.

(1) In the absence of marked points, it reduces to the Gieseker Petri theorem, which holds for curves of any genus.
(2) With marked points allowed, but in genus 0, it reduces to the well-known characterization of the singular loci of Schubert varieties and Richardson varieties.

Indeed, in the case $g = 0$, a single ramification condition corresponds to a Schubert cycle in the Grassmannian $\mathrm{Gr}(r + 1, \mathcal{O}_{\mathbb{P}^1}(d))$, while a pair of ramification conditions similarly corresponds to a Richardson variety. These spaces are singular, and their singularities can be characterized precisely as loci with a specific type of excess vanishing. Our main theorem extends this characterization to all genera, and also deduces additional consequences on the geometry of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$. To state it, the following preliminary notation will be helpful.

Notation 1.2. If $(\mathcal{L}, V)$ is a $\mathfrak{g}_d^r$ on a smooth projective curve X, and D is an effective divisor on X, write

$$V(-D) = V \cap \Gamma(X, \mathcal{L}(-D)) \subseteq \Gamma(X, \mathcal{L}).$$

Thus, to say that $(\mathscr{L}, V)$ has vanishing sequence at least $a_\bullet$ at P is equivalent to saying that

$$\dim V(-a_j P) \geq r + 1 - j \tag{1.1}$$

for $j = 0, \ldots, r$.

We then make the following definition:

Definition 1.3. In the situation of Theorem 1.1, let

$$G_d^{r,\circ}(X, (P, a_\bullet), (Q, b_\bullet)) \subseteq G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$$

be the open subset consisting of $(\mathscr{L}, V)$ such that (1.1) holds with equality for all $j > 0$ such that $a_j > a_{j-1} + 1$, and the analogous condition holds for $(Q, b_\bullet)$.

We see that $G_d^{r,\circ}(X, (P, a_\bullet), (Q, b_\bullet))$ contains all linear series with precisely the prescribed vanishing at P and Q, but it also contains many linear series with more than the prescribed vanishing. For instance, if $a_\bullet = b_\bullet = (0, 1, \ldots, r)$ are both minimal, so that $G_d^r(X, (P, a_\bullet), (Q, b_\bullet)) = G_d^r(X)$, then we also have $G_d^{r,\circ}(X, (P, a_\bullet), (Q, b_\bullet)) = G_d^r(X)$. Our main theorem is then the following.

Theorem 1.4. *In the situation of Theorem 1.1, suppose further that we are in characteristic* 0. *Then the smooth locus of* $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ *is precisely equal to* $G_d^{r,\circ}(X, (P, a_\bullet), (Q, b_\bullet))$.

Furthermore, the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ *has singularities in codimension at least* 3, *is normal, and when* $\widehat{\rho} \geq 1$ *is irreducible.*

Thus, we are in particular giving a new proof of the Gieseker-Petri theorem (in characteristic 0). As an immediate consequence of Theorem 1.4, the twice-pointed Brill-Noether curves studied in [CLPT], as well as twice-pointed Brill-Noether surfaces [CP17, ACT17] are smooth. The singular locus of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ can also be described using hook removals in Young diagrams; see Remark 3.4.

Our proof proceeds by degenerating to a chain of elliptic curves, and studying the geometry of the corresponding moduli space of Eisenbud-Harris limit linear series. The key idea in this step is that although the space of limit linear series will be singular in codimension 1, after base change and blowup one can ensure that any given point of $G_d^{r,\circ}(X, (P, a_\bullet), (Q, b_\bullet))$ on the generic fiber will specialize to a smooth point of the limit linear series space of a chain of curves of genus 0 or 1. This is where the characteristic-0 hypothesis comes in. The case of genus 0 is well-known, so our main calculation is the following result, which does not depend on characteristic, concerning the case $g = 1$.

Theorem 1.5. *In the situation of Theorem 1.1, suppose that* $g = 1$, *and make the generality condition explicit as follows:* X *is arbitrary, and* P, Q *are such that* $P - Q$ *is not a torsion point of* $\mathrm{Pic}^0(X)$ *of order less than or equal to* d. *Then the space* $G_d^{r,\circ}(X, (P, a_\bullet), (Q, b_\bullet))$ *is smooth.*

The proof of Theorem 1.5 proceeds by consideration of the morphism

$$G_d^{r,\circ}(X, (P, a_\bullet), (Q, b_\bullet)) \to \mathrm{Pic}^d(X). \tag{1.2}$$

The main subtlety that needs to be addressed is that the map (1.2) is not smooth. The fibers are each described as an intersection of a pair of Schubert cycles in a Grassmannian. But in finitely many fibers, namely the ones above line bundles of the form $\mathcal{O}_X(aP + (d-a)Q)$ for $0 < a < d$, the pairs of flags defining the Schubert cycles are not transverse, but only almost-transverse (see Definition 2.7). We prove

Theorem 1.5 by first showing that in fibers, the tangent spaces have dimension at most 1 greater than expected, and then showing that at the points where the tangent space dimension jumps in the fiber, there cannot be any horizontal tangent vectors.

The statement on tangent spaces in fibers, which is Corollary 2.12 below, takes place entirely inside the Grassmannian, and may be of independent interest. Indeed, §2 is a study of tangent spaces of intersections of pairs of Schubert cycles, and we address the case of arbitrary pairs of flags in Theorem 2.10 and Remark 2.13. We then prove Theorems 1.4 and 1.5 in §3.

2. ALMOST-TRANSVERSE INTERSECTIONS OF SCHUBERT CYCLES

It is well known that the intersection of two Schubert varieties associated to transverse flags—commonly called a Richardson variety—is smooth on the open subset of points which are smooth in both Schubert varieties. In this section we consider intersections of pairs of Schubert varieties associated to not necessarily transverse flags. Our analysis recovers the usual smoothness statement in the transverse case, but our main purpose is to analyze the almost-transverse case in Corollary 2.12, where we characterize the smooth points and show that the dimension of the tangent space jumps only by 1 at the non-smooth points. While Schubert intersections and non-transverse flags have been studied by Vakil [Vak06] and Coskun [Cos09], those situations involved studying the flat limits of transverse intersections, rather than the direct analysis of the non-transverse intersections required in the present work.[1]

We fix k to be an algebraically closed field of any characteristic. Throughout this section, we will work entirely with k-valued (equivalently, closed) points. We index our complete flags by codimension, so that for a complete flag $P^\bullet$ in a d-dimensional vector space H,

$$0 = P^d \subset \cdots \subset P^1 \subset P^0 = H.$$

We fix further notation as follows.

Definition 2.1. Given a k-vector space H of finite dimension d and a complete flag $P^\bullet$ in H, if we are given also $a_\bullet = (a_0, \ldots, a_r) \in \mathbb{Z}^{r+1}$ with

$$0 \leq a_0 < \cdots < a_r < d,$$

we let $\Sigma_{P^\bullet, a_\bullet}$ be the **Schubert variety** defined as the closed subscheme of $\mathrm{Gr}(r + 1, H)$ given by the set of $\Lambda \in \mathrm{Gr}(r + 1, H)$ such that

$$(2.1) \qquad\qquad \dim(\Lambda \cap P^{a_i}) \geq r + 1 - i$$

for $i = 0, \ldots, r$.

More precisely, the conditions in (2.1) are determinantal, yielding a scheme structure on $\Sigma_{P^\bullet, a_\bullet}$ (which turns out to be reduced). In our notation, the codimension of $\Sigma_{P^\bullet, a_\bullet}$ is given by $\sum_{i=0}^{r}(a_i - i)$.

Definition 2.2. With $a_\bullet = (a_0, \ldots, a_r)$ an increasing sequence as above, say that an index i with $0 \leq i \leq r$ is **active** in $a_\bullet$ if $i > 0$ and $a_i > a_{i-1} + 1$, or $i = 0$ and $a_0 > 0$.

---

[1]More precisely, they work with closures of loci with prescribed behavior with respect to both flags; these are in particular irreducible, so are not the same thing as the intersection of Schubert cycles which we consider.

Definition 2.3. Let $\Sigma^{\circ}_{P^{\bullet},a_{\bullet}}$ be the open subscheme of $\Sigma_{P^{\bullet},a_{\bullet}}$ consisting of subspaces Λ for which for every active index i, the inequality in (2.1) is an equality.

Note that (2.1) is automatically an equality when $i = 0$, so in Definition 2.3 we can restrict to positive active choices of i.

We fix the following situation throughout this section.

Situation 2.4. Let H be a finite-dimensional k-vector space, and write $d := \dim H$. Fix complete flags $P^{\bullet}$, $Q^{\bullet}$ in H, and sequences $a_{\bullet}, b_{\bullet} \in \mathbb{Z}^{r+1}$ with $0 \leq a_0 < \cdots < a_r < d$ and $0 \leq b_0 < \cdots < b_r < d$.

Recall that we are indexing by codimension; thus $\operatorname{codim} P^i = \operatorname{codim} Q^i = i$. Note that for any $\Lambda \in \Sigma_{P^{\bullet},a_{\bullet}}$, the distinct subspaces in the collection $\Lambda \cap P^j$ form a complete flag in Λ; we denote the flag $\Lambda \cap P^{\bullet}$ by abuse of notation.

We have the following description of the tangent space at any point in $\Sigma_{P^{\bullet},a_{\bullet}}$. Tangent spaces to Schubert varieties are well understood [BL00], but for the sake of completeness, we provide a description in the particular case that we need of Grassmannian Schubert varieties.

Proposition 2.5.

(1) *Given $\Lambda \in \Sigma_{P^{\bullet},a_{\bullet}}$, let S be the set of active indices i such that $\dim \Lambda \cap P^{a_i} = r + 1 - i$. Then there is a canonical isomorphism of vector spaces*

$$T_{\Lambda}\Sigma_{P^{\bullet},a_{\bullet}} \cong \{\phi\colon \Lambda \to H/\Lambda : \phi(\Lambda \cap P^{a_i}) \subseteq (P^{a_i} + \Lambda)/\Lambda \text{ for } i \in S\}\,.$$

(2) *In particular, the smooth locus of $\Sigma_{P^{\bullet},a_{\bullet}}$ is precisely $\Sigma^{\circ}_{P^{\bullet},a_{\bullet}}$.*

Proof. By definition, $\Sigma_{P^{\bullet},a_{\bullet}}$ is the scheme-theoretic intersection of the following subschemes of $\operatorname{Gr}(r + 1, H)$ (for $i = 0, 1, \cdots, r$):

$$\Sigma_i = \{\Lambda \in \operatorname{Gr}(r + 1, H)\colon \dim(\Lambda \cap P^{a_i}) \geq r + 1 - i\}.$$

Define Σ°_i to be the open subscheme of Σ_i where equality holds. Note also that in fact $\Sigma_{P^{\bullet},a_{\bullet}}$ can be cut out as the intersection of the Σ_i over all active indices i: this is immediate set-theoretically, and is also true scheme-theoretically because whenever $a_{i+1} = a_i + 1$, the condition for a_{i+1} is obtained from that of a_i by adding a single row to the local matrix expression, and considering minors of size one larger. Thus, every minor occuring in the a_{i+1} condition can be expanded in terms of minors occuring in the a_i condition.

The first statement of the proposition then follows immediately from the following claim. For a fixed index i,

$$T_{\Lambda}\Sigma_i = \begin{cases} \{\phi\colon \Lambda \to H/\Lambda : \phi(\Lambda \cap P^{a_i}) \subseteq (P^{a_i} + \Lambda)/\Lambda\} & \text{if } \Lambda \in \Sigma^{\circ}_i \\ T_{\Lambda}\operatorname{Gr}(r + 1, H) & \text{otherwise,} \end{cases}$$

where we identify $T_{\Lambda}\operatorname{Gr}(r + 1, H)$ with $\operatorname{Hom}(\Lambda, H/\Lambda)$ as usual.

To prove this claim, one may work on an affine open subset of $\operatorname{Gr}(r + 1, H)$, as follows. Choose a basis of H extending a basis of Λ; then an affine neighborhood of Λ is given by the set of $(r + 1) \times d$ matrices whose first $(r + 1)$ columns form the identity matrix (where the point in $\operatorname{Gr}(r + 1, H)$ is given by taking the span of the rows). More precisely, for any k-algebra R, we may identify the R-points of this open subscheme with R-valued matrices whose first $r + 1$ columns form the identity matrix. In particular, taking $R = k[\epsilon]$, the tangent space $T_{\Lambda}\operatorname{Gr}(r + 1, H)$ is identified with matrices in block form $(I \ \epsilon M)$, where M is a matrix of values of k;

the matrix M then determines an element of $\mathrm{Hom}(\Lambda, H/\Lambda)$. Now, we may further assume that the chosen basis of H also includes a basis of P^{a_i} as a subset. Then the R-points of Σ_i consist of those matrices such that the submatrix consisting of all columns not corresponding to the basis of P^{a_i} has rank at most i. Assuming that we order our basis of Λ so that a basis of $\Lambda \cap P^{a_i}$ comes at the end, the submatrix in question has the form

$$\begin{pmatrix} I & A \\ 0 & B \end{pmatrix},$$

where the size of the identity matrix in the upper left is $\dim(\Lambda/(\Lambda \cap P^{a_i}))$. Therefore, lying in Σ_i corresponds to the condition that

$$\mathrm{rk}(B) \leq i - (r+1) + \dim(\Lambda \cap P^{a_i}).$$

Now, specialize to the case $R = k[\epsilon]$, and consider a tangent vector to $\mathrm{Gr}(r+1, H)$ at Λ. The submatrix B is a multiple of ϵ. Therefore all 2×2 and larger minors of B are guaranteed to vanish. Thus in the case $\dim(\Lambda \cap P^{a_i}) > r + 1 - i$ (i.e. $\Lambda \notin \Sigma_i^\circ$), all tangent vectors to $\mathrm{Gr}(r+1, H)$ at Λ are also tangent vectors to Σ_i at Λ. On the other hand, when $\Lambda \in \Sigma_i^\circ$, a tangent vector to $\mathrm{Gr}(r+1, H)$ at Λ is a tangent vector to Σ_i if and only if the matrix B vanishes entirely. This condition can be made intrinsic by observing that, if $\phi \colon \Lambda \to H/\Lambda$ is the linear map encoding a tangent vector, then B is a matrix representation for the linear map $\Lambda \cap P^{a_i} \to H/(P^{a_i} + \Lambda)$ induced by ϕ. Therefore it follows that, in the case $\Lambda \in \Sigma_i^\circ$, ϕ described a tangent vector to Σ_i if and only if $\phi(\Lambda \cap P^{a_i}) \subseteq (P^{a_i} + \Lambda)/\Lambda$. This proves the claim, and the first statement of the proposition.

The second statement follows by direct computation of the codimension imposed by the conditions on the tangent space in the first part. If we have $i \in S$, let i_n denote the next (greater) element of S, setting $i_n = r + 1$ if i is maximal in S. By starting from the condition imposed at the maximal element of S, and inductively working downwards, one computes that the codimension of the tangent space is given by

$$\sum_{i \in S} (i_n - i)(a_i - i).$$

Each term of this sum is always less than or equal to $\sum_{j=i}^{i_n - 1}(a_j - j)$, with equality if and only if there are no actives indices strictly between i and i_n. The proposition follows. $\qquad\square$

Corollary 2.6. *Given $\Lambda \in \Sigma_{P^\bullet, a_\bullet}^\circ$, there is a canonical isomorphism of vector spaces*

$$T_\Lambda \Sigma_{P^\bullet, a_\bullet} \cong \left\{ \phi \colon \Lambda \to H/\Lambda : \phi(\Lambda \cap P^{a_i}) \subseteq (P^{a_i} + \Lambda)/\Lambda \text{ for active } i = 0, \dots, r \right\}.$$

Following Definition 4.1 of [CP17], we define:

Definition 2.7. Two complete flags $P^\bullet$ and $Q^\bullet$ are called **almost-transverse** if there exists an index $t \in \{1, \dots, d-1\}$ such that

$$\dim P^i \cap Q^{d-i} = \begin{cases} 0 & \text{if } i \neq t, \\ 1 & \text{if } i = t. \end{cases}$$

More generally, we have the following statement, which is easy to check:

Proposition 2.8. *There is a unique permutation $\sigma \in S_d$ associated to the flags $P^\bullet$ and $Q^\bullet$ with the property that there exists a basis $e_1, \ldots, e_d$ for H satisfying*

$$e_i \in P^{i-1} \smallsetminus P^i, \quad and \quad e_i \in Q^{\sigma(i)-1} \smallsetminus Q^{\sigma(i)}.$$

Such a basis can also be characterized by the property that for all indices i and j, $P^i \cap Q^j$ is spanned by $\{e_1, \ldots, e_d\} \cap P^i \cap Q^j$. In particular, if $\dim P^i \cap Q^j = 1$ then $P^i \cap Q^j$ contains one of the e_ℓ.

Definition 2.9. We refer to a basis as in Proposition 2.8 as a $(P^\bullet, Q^\bullet)$-**basis**.

Thus, following the notation of Proposition 2.8, we have that $P^\bullet = Q^\bullet$ if and only if $\sigma = \mathrm{id}$, $P^\bullet$ and $Q^\bullet$ are transverse if and only if $\sigma = \omega := (d, d-1, \ldots, 1)$, and $P^\bullet$ and $Q^\bullet$ are almost-transverse if and only if σ is the composition of ω with an adjacent transposition.

We set $\rho = \rho(1, r, d, a_\bullet, b_\bullet)$, so that

$$\rho - 1 = (r+1)(d-r-1) - \sum_{i=0}^{r}(a_i - i) - \sum_{i=0}^{r}(b_i - i) = \rho(0, r, d-1, a_\bullet, b_\bullet)$$

is precisely the expected dimension of $\Sigma_{P^\bullet, a_\bullet} \cap \Sigma_{Q^\bullet, b_\bullet}$. Also recall the definition of the complete flags $\Lambda \cap P^\bullet$ and $\Lambda \cap Q^\bullet$ from Situation 2.4.

Theorem 2.10. *Given $\Lambda \in \Sigma^\circ_{P^\bullet, a_\bullet} \cap \Sigma^\circ_{Q^\bullet, b_\bullet}$, let $\sigma \in S_{r+1}$ denote the permutation associated to $\Lambda \cap P^\bullet$ and $\Lambda \cap Q^\bullet$ in Λ by Proposition 2.8. Given any $j \in \{0, \ldots, r\}$, let*

$$\mu(j) = \max\{i \leq j : i \text{ is active in } a_\bullet\},$$

setting $\mu(j) = 0$ if no such i exists. Set $m(j) = a_{\mu(j)}$. Similarly, let

$$\nu(j) = \max\{i \leq \sigma(j) : i \text{ is active in } b_\bullet\},$$

setting $\nu(j) = 0$ if no such j exists. Set $n(j) = b_{\nu(j)}$. Then

$$(2.2) \qquad \dim T_\Lambda(\Sigma_{P^\bullet, a_\bullet} \cap \Sigma_{Q^\bullet, b_\bullet}) = \rho - 1 + \sum_{j=0}^{r} \mathrm{codim}_H(P^{m(j)} + Q^{n(j)} + \Lambda).$$

Proof. Let $\lambda_0, \ldots, \lambda_r$ be a $(\Lambda \cap P^\bullet, \Lambda \cap Q^\bullet)$-basis for Λ. Then for any i active in $a_\bullet$, respectively $b_\bullet$, have

$$(2.3) \qquad \Lambda \cap P^{a_i} = \langle \lambda_i, \ldots, \lambda_r \rangle, \qquad \Lambda \cap Q^{b_i} = \langle \lambda_{\sigma^{-1}(i)}, \ldots, \lambda_{\sigma^{-1}(r)} \rangle.$$

In other words, given any j, and any i that is active in $a_\bullet$, we have $\lambda_j \in \Lambda \cap P^{a_i}$ if and only if $i \leq j$. Similarly, for any i that is active in $b_\bullet$, we have $\lambda_j \in \Lambda \cap Q^{b_i}$ if and only if $i \leq \sigma(j)$. By Corollary 2.6, we have isomorphisms

$$T_\Lambda(\Sigma_{P^\bullet, a_\bullet} \cap \Sigma_{Q^\bullet, b_\bullet}) \cong \{\phi \colon \Lambda \to H/\Lambda \ : \ \phi(\lambda_j) \in (P^{m(j)} + \Lambda)/\Lambda \cap (Q^{n(j)} + \Lambda)/\Lambda\}$$

$$\cong \bigoplus_{j=0}^{r} \mathrm{Hom}\left(\langle \lambda_j \rangle, (P^{m(j)} + \Lambda)/\Lambda \cap (Q^{n(j)} + \Lambda)/\Lambda\right).$$

We are thus reduced to computing the dimensions $(P^{m(j)} + \Lambda)/\Lambda \cap (Q^{n(j)} + \Lambda)/\Lambda$, which are equal to

$$\dim(P^{m(j)} + \Lambda) + \dim(Q^{n(j)} + \Lambda) - \dim(P^{m(j)} + Q^{n(j)} + \Lambda) - \dim \Lambda.$$

Moreover, the first two terms are determined by the fact that $\dim P^{m(j)} \cap \Lambda = r + 1 - \mu(j)$ and $\dim Q^{n(j)} \cap \Lambda = r + 1 - \nu(j)$, by the assumption that $\mu(j)$ and $\nu(j)$ are active or are equal to 0. Furthermore note by definition of μ and ν that

$$a_j - j = m(j) - \mu(j), \qquad b_{\sigma(j)} - \sigma(j) = n(j) - \nu(j).$$

A straightforward calculation then produces (2.2). $\qquad\square$

We observe that the well-known case of transverse flags follows immediately from Theorem 2.10.

Corollary 2.11. *If $P^\bullet$ and $Q^\bullet$ are transverse, then* $\dim T_\Lambda(\Sigma_{P^\bullet,a_\bullet} \cap \Sigma_{Q^\bullet,b_\bullet}) = \rho - 1$ *for all* $\Lambda \in \Sigma^\circ_{P^\bullet,a_\bullet} \cap \Sigma^\circ_{Q^\bullet,b_\bullet}$.

Proof. Following the notation of the proof of Theorem 2.10, for each j we have $\lambda_j \in P^{m(j)} \cap Q^{n(j)}$ by construction. Since $P^\bullet$ and $Q^\bullet$ are transverse, it follows that $m(j) + n(j) < d$, so $P^{m(j)} + Q^{n(j)} + \Lambda = P^{m(j)} + Q^{n(j)} = H$. $\qquad\square$

More importantly, we can also deduce the desired statement in the almost-transverse case.

Corollary 2.12. *Given* $\Lambda \in \Sigma^\circ_{P^\bullet,a_\bullet} \cap \Sigma^\circ_{Q^\bullet,b_\bullet}$, *suppose $P^\bullet$ and $Q^\bullet$ are almost-transverse, with $t + t' = d$ such that* $\dim P^t \cap Q^{t'} = 1$.
Suppose first that $t = a_i$ for i active in $a_\bullet$, that $t' = b_{i'}$ for i' active in $b_\bullet$, and that

$$P^t \cap Q^{t'} \subseteq \Lambda \subseteq P^t + Q^{t'}.$$

Then

$$\dim T_\Lambda(\Sigma_{P^\bullet,a_\bullet} \cap \Sigma_{Q^\bullet,b_\bullet}) = \rho.$$

If those conditions do not all hold, then

$$\dim T_\Lambda(\Sigma_{P^\bullet,a_\bullet} \cap \Sigma_{Q^\bullet,b_\bullet}) = \rho - 1.$$

Proof. Let $\lambda_0, \ldots, \lambda_r$ be a $(\Lambda \cap P^\bullet, \Lambda \cap Q^\bullet)$-basis for Λ. First suppose that $t = a_i$ for i active in $a_\bullet$, and $t' = b_{i'}$ for i' active in $b_\bullet$, and that $P^t \cap Q^{t'} \subseteq \Lambda \subseteq P^t + Q^{t'}$. We will deduce that $\dim T_\Lambda(\Sigma_{P^\bullet,a_\bullet} \cap \Sigma_{Q^\bullet,b_\bullet}) = \rho$.

We have that $\Lambda \cap P^t$ and $\Lambda \cap Q^{t'}$ are elements in the flags $\Lambda \cap P^\bullet$ and $\Lambda \cap Q^\bullet$ respectively with intersection $\Lambda \cap P^t \cap Q^{t'}$ of dimension 1. Proposition 2.8 implies that $P^t \cap Q^{t'} = \langle \lambda_j \rangle$ for a unique $j \in \{0, \ldots, r\}$. By Theorem 2.10 it is enough to show that for each $j' \in \{0, \ldots, r\}$,

$$\operatorname{codim}_H(P^{m(j')} + Q^{n(j')} + \Lambda) = \begin{cases} 1 & \text{if } j' = j, \\ 0 & \text{if } j' \neq j. \end{cases}$$

Now, for $j' \neq j$, the fact that $\lambda_{j'} \in P^{m(j')} \cap Q^{n(j')}$ and $P^\bullet$ and $Q^\bullet$ are almost-transverse implies that either $m(j') + n(j') < d$, or that $m(j') = t$ and $n(j') = t'$. But the latter case cannot be, since then both $\lambda_j, \lambda_{j'} \in P^t \cap Q^{t'}$, contradicting that $\dim P^t \cap Q^{t'} = 1$. Therefore $m(j') + n(j') < d$ and

$$P^{m(j')} + Q^{n(j')} = P^{m(j')} + Q^{n(j')} + \Lambda = H,$$

as desired.

Next, to show that $\operatorname{codim}_H(P^{m(j)} + Q^{n(j)} + \Lambda) = 1$, we claim that $m(j) = t$ and $n(j) = t'$. Recall that $a_i = t$ and $a_{i'} = t'$. By assumption, i is active in $a_\bullet$ and $\lambda_j \in \Lambda \cap Q^{a_i}$, so $i \leq j$ by (2.3). We want to show that i is the largest active index

in $a_\bullet$ with $i \leq j$. Indeed, if l is active in $a_\bullet$ with $i < l \leq j$, then $\lambda_j \in P^{a_l} \cap Q^{b_{i'}}$. But now $a_l > a_i$, so $a_l + b_{i'} > a_i + b_{i'} = t + t' = d$. Therefore $P^{a_l} \cap Q^{b_{i'}} = 0$, contradiction. A similar argument shows $n(j) = t'$. Therefore,

$$\operatorname{codim}_H(P^{m(j)} + Q^{n(j)} + \Lambda) = \operatorname{codim}_H(P^t + Q^{t'} + \Lambda) = 1,$$

since $P^t + Q^{t'}$ is a hyperplane in H, and Λ is contained in it by assumption.

It remains to show that if the conditions in the statement of Corollary 2.12 do not all hold, then $\dim T_\Lambda(\Sigma_{P^\bullet,a_\bullet} \cap \Sigma_{Q^\bullet,b_\bullet}) = \rho - 1$. We prove the contrapositive. Suppose that $\dim T_\Lambda(\Sigma_{P^\bullet,a_\bullet} \cap \Sigma_{Q^\bullet,b_\bullet}) > \rho - 1$. By Theorem 2.10, there is an index j such that $\operatorname{codim}_H(P^{m(j)} + Q^{n(j)} + \Lambda) > 0$. Again, given that $\lambda_j \in P^{m(j)} \cap Q^{n(j)}$ and that $P^\bullet$ and $Q^\bullet$ are almost-transverse, it follows that either $m(j) + n(j) < d$ or that $m(j) = t$ and $n(j) = t'$. But $m(j) + n(j) < d$ would imply $P^{m(j)} + Q^{n(j)} = H$, contradicting the codimension statement. So $m(j) = t$ and $n(j) = t'$, implying that $t = a_i$ and $t' = b_{i'}$ for active indices i and i' in $a_\bullet$ and $b_\bullet$ respectively. (It is not possible that $m(j) = 0$ or $n(j) = 0$, since $\operatorname{codim}_H P^{m(j)} + Q^{n(j)} + \Lambda > 0$.) Furthermore,

$$\langle \lambda_j \rangle = P^t \cap Q^{t'} \subseteq \Lambda \subseteq P^t + Q^{t'}$$

where the last containment holds again by the codimension assumption.

Summarizing, we have shown that the *only* way that $\dim T_\Lambda(\Sigma_{P^\bullet,a_\bullet} \cap \Sigma_{Q^\bullet,b_\bullet}) > \rho - 1$ is for all the conditions in the statement of Corollary 2.12 to hold, in which case we have already proved that the dimension is exactly ρ. $\square$

Remark 2.13. For arbitrary flags $P^\bullet$ and $Q^\bullet$ and $\Lambda \in \Sigma^\circ_{P^\bullet,a_\bullet} \cap \Sigma^\circ_{Q^\bullet,b_\bullet}$, let $\tau \in S_d$ be the associated permutation from Proposition 2.8 (maintaining other notation as in Theorem 2.10). Then the extent to which the dimension of the tangent space at Λ of $\Sigma_{P^\bullet,a_\bullet} \cap \Sigma_{Q^\bullet,b_\bullet}$ exceeds $\rho - 1$ can be bounded in terms of τ as follows. We have:

$$(2.4) \qquad \dim T_\Lambda(\Sigma_{P^\bullet,a_\bullet} \cap \Sigma_{Q^\bullet,b_\bullet}) \leq (\rho - 1) + \operatorname{inv}(\omega\tau).$$

Here ω denotes the decreasing permutation $(d, d-1, \cdots, 1)$, and $\operatorname{inv}(\omega\tau)$ denotes the inversion number of $\omega\tau$, i.e. the number of $i < j$ with $\omega\tau(i) > \omega\tau(j)$.[2] We briefly sketch a proof of this more general inequality.

Using the second part of Proposition 2.8, it follows that for each i, j, we have

$$\dim P^i \cap Q^j = \#\{i' \geq i : \tau(i') \geq j\}.$$

From this it follows that $\dim P^i \cap Q^j > \dim P^{i+1} \cap Q^j$ if and only if $\tau(i) \geq j$. Now, since we have $P^{a_i} \cap Q^{b_{\sigma(i)}} \neq P^{a_i+1} \cap Q^{b_{\sigma(i)}}$, we find that $b_{\sigma(i)} \leq \tau(a_i)$. Note also that for all j, $m(j) \leq a_j$ and $n(j) \leq b_{\sigma(j)}$. Then:

---

[2]If one views S_d as a Coxeter group with reflections being the adjacent transpositions, then $\operatorname{inv}(\omega\tau)$ is also the Coxeter length of $\omega\tau$.

$$\sum_{j=0}^{r} \operatorname{codim}(\Lambda + P^{m(j)} + Q^{n(j)}) \;\leq\; \sum_{j=0}^{r} \operatorname{codim}(P^{m(j)} + Q^{n(j)})$$

$$\leq\; \sum_{j=0}^{r} \operatorname{codim}(P^{a_j} + Q^{b_{\sigma(j)}})$$

$$\leq\; \sum_{j=0}^{r} \operatorname{codim}(P^{a_j} + Q^{\tau(a_j)})$$

$$\leq\; \sum_{j=0}^{d-1} \operatorname{codim}(P^{j} + Q^{\tau(j)}).$$

Using that $\dim(P^j \cap Q^{\tau(j)}) = \#\{j' \geq j : \tau(j') \geq \tau(j)\}$, we compute that $\sum_{j=0}^{d-1} \operatorname{codim}(P^j + Q^{\tau(j)}) = \operatorname{inv}(\omega\tau)$, and the inequality (2.4) follows from (2.2). When $P^\bullet$ and $Q^\bullet$ are almost-transverse, $\operatorname{inv}(\omega\tau) = 1$, and the precise statement in Corollary 2.12 can be deduced from characterizing the equality cases of the four inequalities above in the case where $\omega\tau$ is equal to an adjacent transposition.

3. Linear series in positive genera

We begin with a proposition that will show that our smoothness result, Theorem 1.4 to be proved below, is sharp.

Proposition 3.1. *In the situation of Theorem 1.1, every point of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ in the complement of $G_d^{r,\circ}(X, (P, a_\bullet), (Q, b_\bullet))$ is singular.*

Proof. This is a consequence of the standard construction of the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$: we let $\widetilde{\mathscr{L}}$ be a Poincaré line bundle on $X \times \operatorname{Pic}^d(X)$. Take a sufficiently ample[3] effective divisor D on X with support disjoint from P and Q, and write $\widetilde{D} = D \times \operatorname{Pic}^d(X)$. Writing $p \colon X \times \operatorname{Pic}^d(X) \to \operatorname{Pic}^d(X)$ for projection, let G be the relative Grassmannian $\operatorname{Gr}(r+1, p_*\widetilde{\mathscr{L}}(\widetilde{D}))$, equipped with structure map $\pi \colon G \to \operatorname{Pic}^d(X)$. Let $\widetilde{\mathscr{V}} \hookrightarrow \pi^* p_* \widetilde{\mathscr{L}}(\widetilde{D})$ denote the universal subbundle. Then $G_d^r(X)$ is cut out in G by the condition that the induced map

$$\widetilde{\mathscr{V}} \hookrightarrow \pi^* p_* \left(\widetilde{\mathscr{L}}(\widetilde{D})|_{\widetilde{D}} \right)$$

vanishes identically. Because we have chosen D to have support disjoint from P and to be sufficiently ample, the space $G_d^r(X, (P, a_\bullet))$ is cut out by imposing the additional Schubert condition that the maps

$$\widetilde{\mathscr{V}} \hookrightarrow \pi^* p_* \left(\widetilde{\mathscr{L}}(\widetilde{D})|_{a_j P} \right)$$

have rank at most j for each j. Imposing the analogous condition at Q, we obtain $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ as an intersection of three conditions: a determinantal condition (in fact a complete intersection), and two relative Schubert cycles. It is routine to check that for $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ to have dimension $\rho(g, r, d, a_\bullet, b_\bullet)$, as asserted by Theorem 1.1, these three conditions must intersect in the maximal

---

[3]Precisely, of degree strictly greater than $2g - 2$

codimension. Given that we know from Theorem 1.1 that $G^r_d(X, (P, a_\bullet), (Q, b_\bullet))$ does in fact have dimension $\rho(g, r, d, a_\bullet, b_\bullet)$, it then further follows that in order for $G^r_d(X, (P, a_\bullet), (Q, b_\bullet))$ to be smooth at any point, that point must lie in the smooth locus of each of the three conditions, and in particular of the two Schubert cycles.

But we claim that $G^{r,\circ}_d(X, (P, a_\bullet), (Q, b_\bullet))$ consists precisely of the points of $G^r_d(X, (P, a_\bullet), (Q, b_\bullet))$ which lie in the smooth locus of both relative Schubert cycles. Indeed, each relative Schubert cycle is nothing but a locally constant family of Schubert varieties over the base G, so we are done by the standard characterization of the smooth locus of a Schubert variety (see the second part of Proposition 2.5). $\qquad\square$

We now use our calculations in Grassmannians in §2 to complete the proof of our main theorem, beginning with the case of genus 1 in Theorem 1.5.

Proof of Theorem 1.5. Set $\rho = \rho(1, r, d, a_\bullet, b_\bullet)$. We may assume $d > 0$, as otherwise the result is trivial. Thus, $G^r_d(X)$ is a Grassmannian bundle over $\operatorname{Pic}^d(X)$, with the fiber over a line bundle $\mathscr{L}$ being canonically identified with $\operatorname{Gr}(r+1, \Gamma(X, \mathscr{L})) \cong \operatorname{Gr}(r+1, d)$. The condition imposed by requiring vanishing sequence at least $a_\bullet$ at P then gives a Schubert cycle in each fiber, corresponding to the complete flag determined by vanishing order at P. The codimension of spaces in the flag corresponds precisely to vanishing order except over the point $\mathscr{L} \cong \mathscr{O}_X(dP)$, where no sections vanish to order precisely $d-1$, and vanishing to order d imposes codimension only $d-1$. Consequently, there are two possibilities for $G^r_d(X, (P, a_\bullet))$. First, if $a_r < d$, it is a relative Schubert cycle of codimension $\sum_j (a_j - j)$ in $G^r_d(X)$, Cohen-Macaulay and flat over $\operatorname{Pic}^d(X)$. Or, if $a_r = d$, it is supported entirely over $\mathscr{L} \cong \mathscr{O}_X(dP)$ (even scheme-theoretically), and is still a Schubert cycle, but of codimension $(\sum_j (a_j - j)) - 1$. The same analysis applies to $G^r_d(X, (Q, b_\bullet))$, so we find that every fiber of

$$(3.1) \qquad\qquad G^r_d(X, (P, a_\bullet), (Q, b_\bullet)) \to \operatorname{Pic}^d(X)$$

is an intersection of a pair of Schubert cycles. The basic properties of the map (3.1) are analyzed for instance in Lemma 2.1 of [Oss14] and Proposition 2.1 of [Oss]; we review the main points of this analysis in order to carry out the necessary tangent space analysis.

First, we see that in most fibers of (3.1), the relevant Schubert cycles are associated to transverse flags: the only way in which the flags fail to be transverse is if $\mathscr{L} \cong \mathscr{O}_X(aP + (d-a)Q)$ for some $a \in \{1, \ldots, d-1\}$, which is unique by genericity of P and Q; then the conditions of vanishing to order a at P and $d-a$ at Q intersect in dimension 1 instead of dimension 0. Thus, on fibers of (3.1) over points not of the form $\mathscr{O}_X(aP + (d-a)Q)$ for $0 \leq a \leq d$, we have that the Schubert indexing matches vanishing sequences, and the flags are transverse, so the standard theory (see for instance Corollary 2.11) gives us that (on these fibers) the space $G^{r,\circ}_d(X, (P, a_\bullet), (Q, b_\bullet))$ is smooth (of relative dimension $(r+1)(d-r-1) - \sum_j (a_j - j) - \sum_j (b_j - j) = \rho - 1$) over $\operatorname{Pic}^d(X)$, and hence smooth of relative dimension ρ over $\operatorname{Spec} k$. Similarly, it is easily verified that we still obtain Richardson varieties over $\mathscr{O}_X(aP + (d-a)Q)$ for $0 < a < d$ unless a occurs in $a_\bullet$ and $d-a$ occurs in $b_\bullet$. Thus, we obtain the desired statement in these cases. On the other hand, if $a = 0$ or $a = d$, we have transverse intersection of flags, but a potential difference in indexing. In the case $a = d$, the difference in

indexing arises only in that imposing vanishing order d at P is a codimension $d-1$ condition. Thus, this can only affect the final term in the vanishing sequence, which is irrelevant for determining membership in $G_d^{r,\circ}(X,(P,a_\bullet),(Q,b_\bullet))$. We conclude that—whether or not $a_r = d$—the fiber of $G_d^{r,\circ}(X,(P,a_\bullet),(Q,b_\bullet))$ over $\mathscr{O}_X(dP)$ precisely corresponds to the open subset addressed in Corollary 2.11, and hence has the desired smoothness property. The case that $a = 0$ is the same, with Q in place of P.

It thus remains to analyze the fibers with $\mathscr{L} = \mathscr{O}_X(aP+(d-a)Q)$, for $0 < a < d$, and with a occurring in $a_\bullet$ and $d - a$ occurring in $b_\bullet$. Our hypothesis on $P - Q$ implies that $\mathscr{O}_X(aP + (d - a)Q) \not\cong \mathscr{O}_X(a'P + (d - a')Q)$ for any $a \neq a'$, so even in these cases, our flags in $\Gamma(X,\mathscr{L})$ are almost-transverse. In the case that $a = a_j$ and $d - a = b_{r-j}$ for some j, then $G_d^r(X,(P,a_\bullet),(Q,b_\bullet))$ is supported (scheme-theoretically) in the given fiber, and one checks (see the proof of Proposition 2.1 of [Oss]) that the nonempty fiber can still be described as a Richardson variety, by replacing a_j with $a - 1$, and changing the choice of codimension-a subspace in the first flag. Because of this modification to the flag, we will still have that $G_d^{r,\circ}(X,(P,a_\bullet),(Q,b_\bullet))$ corresponds precisely to the open subset treated in Corollary 2.11, and is hence smooth (but this time of dimension ρ).

Finally, we consider the case that $a_j + b_{r-j} < d$ for all j, but we have $\mathscr{L} = \mathscr{O}_X(aP + (d - a)Q)$, and $a = a_j$ and $d - a = b_{j'}$ for some j, j' with $j + j' > r$; in particular, we must have $0 < a < d$. See Example 3.5 for what is essentially the smallest nontrivial example of this case. In this situation, the given fiber of $G_d^{r,\circ}(X,(P,a_\bullet),(Q,b_\bullet))$ over $\mathrm{Pic}^d(X)$ may be singular or even reducible, but at least it is pure of dimension $\rho - 1$: see the proof of Proposition 2.1 of [Oss]. Moreover, as we have observed, the fiber is an intersection of Schubert cycles associated to almost-transverse flags, so we can invoke Corollary 2.12 to conclude that the fiber of $G_d^{r,\circ}(X,(P,a_\bullet),(Q,b_\bullet))$ has tangent space dimension equal to $\rho-1$ or ρ everywhere. Moreover, the latter occurs precisely at linear series $(\mathscr{L},V)$ satisfying the following conditions:

(1) V contains a section s vanishing to order a at P and $d - a$ at Q;
(2) V is contained in the linear span of the spaces of sections vanishing to order a at P and order $d - a$ at Q;
(3) there is some j with $a = a_j$ and j active in $a_\bullet$ in the sense of Definition 2.2;
(4) there is some j' with $d - a = b_{j'}$ and j' active in $b_\bullet$.

Thus, in order to complete the proof of the theorem, we will prove that the space $G_d^{r,\circ}(X,(P,a_\bullet),(Q,b_\bullet))$ is smooth of dimension ρ at every point of the given fiber by showing that if $(\mathscr{L},V)$ is a point at which the tangent space of the fiber has dimension ρ, then every tangent vector of the total space at $(\mathscr{L},V)$ is in fact vertical. Accordingly, given $(\mathscr{L},V)$ satisfying the four conditions above, suppose $(\widetilde{\mathscr{L}},\widetilde{V})$ is a first-order deformation of $(\mathscr{L},V)$, and let $s \in V$ be a section vanishing to order a at P and $d - a$ at Q. We claim that s has a lift $\tilde{s} \in \widetilde{V}$ which vanishes (scheme-theoretically) to order a at P. Indeed, in the notation of the proof of Proposition 3.1, recall that on $G_d^r(X,(P,a_\bullet),(Q,b_\bullet))$ we have a map of vector bundles

$$(3.2) \qquad \Phi\colon \widetilde{\mathscr{V}} \hookrightarrow \pi^* p_* \left(\widetilde{\mathscr{L}}(\tilde{D})|_{a_j P} \right)$$

which has rank at most j. The assumption that $(\mathscr{L},V)$ lies in the open subset $G_d^{r,\circ}(X,(P,a_\bullet),(Q,b_\bullet))$, together with the fact that j is active in $a_\bullet$, says that on

every point in an open neighborhood of $(\mathscr{L}, V)$, the map (3.2) has rank *exactly* j. In this situation, a standard argument shows that $\ker \Phi$ is locally free, and that the restriction map $(\ker \Phi)|_{(\tilde{\mathscr{L}}, \tilde{V})} \to (\ker \Phi)|_{(\mathscr{L}, V)}$ is surjective. A sketch of this standard argument is as follows. The cokernel of Φ must be locally free (see e.g. [Eis95, §16.7]), and hence also the image and kernel. Therefore, the short exact sequences $0 \to \ker \Phi \to \tilde{\mathscr{V}} \to \operatorname{im} \Phi \to 0$ and $0 \to \operatorname{im} \Phi \to \pi^* p_* \left(\tilde{\mathscr{L}}(\tilde{D})|_{a_j P} \right) \to \operatorname{cok} \Phi \to 0$ remain exact after base change, and hence taking kernels commutes with base change.

From surjectivity of the restriction map above, we deduce that our section s admits a lift $\tilde{s}$ that also vanishes (scheme-theoretically) to order a at P. Similarly, s must have another lift which vanishes (scheme-theoretically) to order $d - a$ at Q; since it is another lift of s, it can be expressed as $\tilde{s} + \epsilon v$ for some $v \in V$.

Now, recall our hypothesis that V is contained in the span of sections vanishing to order at least a at P and at least $d - a$ at Q. Write $v = v_1 + v_2$, where v_1 vanishes to order at least a at P and v_2 vanishes to order at least $d - a$ at Q. But then $\tilde{s} + \epsilon v_1$ still vanishes to order $d - a$ at Q, and also vanishes to order a at P. This forces $\widetilde{\mathscr{L}}$ to be the trivial deformation of $\mathscr{L}$, yielding the desired verticality assertion and the theorem. $\qquad\square$

To conclude the proof of our main theorem, we need to make use of the Eisenbud-Harris theory of limit linear series. We first set up notation for our reducible curves, and recall the relevant definitions.

Situation 3.2. Fix g, d, n. Let $Z_1, \ldots, Z_n$ be smooth projective curves, with (distinct) points P_i, Q_i on Z_i for each i, and let X_0 be the nodal curve obtained by gluing Q_i to P_{i+1} for $i = 1, \ldots, n - 1$.

Definition 3.3. Given r, d, a **limit linear series** of dimension r and degree d on X_0 consists of a tuple $(\mathscr{L}^i, V^i)$ of linear series of dimension r and degree d on the Z_i, satisfying the following condition: if $a_\bullet^i, b_\bullet^i$ are the vanishing sequences of $(\mathscr{L}^i, V^i)$ at P_i and Q_i respectively, then we require

$$(3.3) \qquad\qquad b_j^i + a_{r-j}^{i+1} \geq d$$

for all $i = 1, \ldots, n - 1$ and $j = 0, \ldots, r$. If (3.3) is an equality for all i, j, we say that the limit linear series is **refined**.

The space of all such limit linear series on X_0 is denoted by $G_d^r(X_0)$. If we have sequences $a_\bullet$ and $b_\bullet$, we also have the closed subscheme $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet)) \subseteq G_d^r(X_0)$ consisting of limit linear series such that, following the above notation, we have $a_\bullet^1 \geq a_\bullet$ and $b_\bullet^n \geq b_\bullet$. Finally, denote by $G_d^{r,\circ}(X_0, (P_1, a_\bullet), (Q_n, b_\bullet)) \subseteq G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ the open subscheme consisting of refined limit linear series which further satisfy

$$\dim V^1(-a_j P_1) = r + 1 - j \quad \text{for } j > 0 \text{ active in } a,$$
$$\dim V^n(-b_j Q_n) = r + 1 - j \quad \text{for } j > 0 \text{ active in } b.$$

We comment that these last two conditions can be re-expressed purely in terms of $a_\bullet^1$ and $b_\bullet^n$ as follows: for all $j > 0$ active in a, we require $\#\{j' : a_{j'}^1 \geq a_j\} = r + 1 - j$, and similarly for b.

We are now ready to prove our main smoothness result.

Proof of Theorem 1.4. In Situation 3.2, observe that we can decompose the limit linear series space $G_d^{r,\circ}(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ into disjoint open subsets according to the vanishing sequences at each node, i.e. according to the possible values in the left hand side of the equalities (3.3). Then each such open subset is almost a product over i of spaces of the form $G_d^{r,\circ}(Z_i, (P_i, a_\bullet^i), (Q_i, b_\bullet^i))$. In fact, it is an open subset of this product, since the refinedness condition completely fixes the vanishing sequences at the nodes. If further each Z_i has genus 0 or 1, and for the Z_i of genus 1 we suppose that $P_i - Q_i$ is not m-torsion for any $m \leq d$, then we know that each $G_d^{r,\circ}(Z_i, (P_i, a_\bullet^i), (Q_i, b_\bullet^i))$ is smooth. Indeed, the genus-1 case is Theorem 1.5, while the genus-0 case is well known, but follows in particular immediately from Corollary 2.11 taking into account that $\dim \Gamma(\mathbb{P}^1, \mathcal{O}(d)) = d + 1$, so there is a shift of 1 in the value of d. We thus conclude that $G_d^{r,\circ}(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ is also smooth, of dimension $\rho(g, r, d, a_\bullet, b_\bullet)$.

Now, fix $n = g$ and suppose each Z_i has genus 1. Let B be the spectrum of a discrete valuation ring, and $\pi : X \to B$ be a flat, proper family of curves of genus g, with X regular, the generic fiber X_η smooth, and the special fiber isomorphic to X_0. Further assume that π has sections P, Q, specializing to P_1 and Q_n respectively on X_0.

Suppose that we have a closed point of $G_d^{r,\circ}(X_\eta, (P_\eta, a_\bullet), (Q_\eta, b_\bullet))$. After a base change, we may assume that the corresponding linear series is defined on X_η, and that all ramification points are also rational over the base field. Blow up the nodes in X_0 as necessary to resolve any singularities of X resulting from the base change,[4] and finally, blow up P_1 and Q_n as necessary so that no generic ramification point distinct from P or Q limits to P_1 or Q_n in the special fiber. Denote the resulting family by $\pi' : X' \to B'$, and the special fiber by X_0', and write P' and Q' (respectively, P_1' and Q_n') for the resulting sections of π' and their restrictions to X_0'. Then X_0' is obtained by X_0 by base extension and insertion of chains of genus-0 curves at the nodes and at P_1 and Q_n. By construction, none of the ramification points on X_η can specialize to nodes of X_0', so by Proposition 2.5 of [EH86] (and using the characteristic 0 hypothesis), the extension of the given linear series is a refined limit linear series on X_0'. Moreover, by the same argument, the ramification at P_1' and at Q_n' must be precisely equal to the ramification at P_η and Q_η, so that the induced limit linear series lies in $G_d^{r,\circ}(X_0', (P_1', a_\bullet), (Q_n', b_\bullet))$. But as we have discussed above, this space is smooth. According to Proposition 2.7 of [CHT92] (see also Theorem 3.4 of [Oss] for the situation with imposed ramification) there is a flat relative moduli space recovering linear series on the generic fiber and (refined) limit linear series on the special fiber.[5] Alternatively, the more recent construction of [MO16] gives an independent approach which extends to all limit linear series. Either way, it follows that the original point of $G_d^{r,\circ}(X_\eta, (P_\eta, a_\bullet), (Q_\eta, b_\bullet))$ must have been smooth as well.

Now, since the spaces we are considering are in general not proper, the condition that $G_d^{r,\circ}$ is smooth is not open in families. However, the condition does define a constructible subset of $\mathcal{M}_{g,2}$, and the generic fibers of the possible families π as

---

[4]The preceding constitutes an alternative for the argument of Theorem 2.6 of [EH86], avoiding invocation of the stable reduction theorem.

[5]Proposition 2.7 of [CHT92] claims incorrectly to apply to all limit linear series in the special fiber, but it only applies to the refined limit linear series, since it uses the construction of Eisenbud and Harris [EH86], which contains only the refined limit linear series. However, this is enough for our purposes.

above correspond to a Zariski-dense subset, so we conclude the main smoothness statement of the theorem. The fact that the remaining points are not smooth is Proposition 3.1.

The statement on codimension of singularities follows from the observation that a point in the complement of $G_d^{r,\circ}(X, (P, a_\bullet), (Q, b_\bullet))$ is a union of closed subvarieties of the form $G_d^r(X, (P, a_\bullet'), (Q, b_\bullet'))$, where $a_\bullet' \geq a$ and $b_\bullet' \geq b$ are sequences such that for some $j > 0$ active in $a_\bullet$, we have $a_{j-1}' \geq a_j$, implying that $a_{j-1}' \geq a_{j-1} + 2$ and $a_j' \geq a_j + 1$ and hence $\sum a_\bullet' \geq \sum a_\bullet + 3$; or analogously for $b_\bullet$. We then conclude normality from the Cohen-Macaulayness and Serre's criterion, and the irreducibility statement follows immediately from the connectedness in the case $\widehat{\rho} \geq 1$. $\qquad\square$

Remark 3.4. The singular locus of $G_d^r(X, (P, a_\bullet))$ can also be described as follows. Given $j > 0$ with $a_j > a_{j-1} + 1$, let $m \geq 0$ be largest such that $a_{j+m} = a_j + m$. Let

$$a_\bullet^{(j)} = (a_0, \ldots, a_{j-2}, a_j, a_j + 1, \ldots, a_{j+m} + 1, \ldots, a_r).$$

Then

$$G_d^r(X, (P, a_\bullet))^{\mathrm{sing}} = \bigcup_{\substack{j > 0 \\ a_j > a_{j-1}+1}} G_d^r(X, (P, a_\bullet^{(j)})).$$

In terms of Young diagrams, let $\alpha_\bullet = (a_i - i)_{i=0}^r$ and let $\mathrm{Young}(a_\bullet)$ be the complement of the partition $\alpha_\bullet$ in a $(r+1) \times (d+r)$ box. Then $\{\mathrm{Young}(a_\bullet^{(j)})\}$ consists of all diagrams obtained from $\mathrm{Young}(a_\bullet)$ by removing hooks.

In the twice-marked case, it is immediate from Theorem 1.4 that the singular locus of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is

$$\left(G_d^r(X, (P, a_\bullet))^{\mathrm{sing}} \cap G_d^r(X, (Q, b_\bullet))\right) \cup \left(G_d^r(X, (P, a_\bullet)) \cap G_d^r(X, (Q, b_\bullet))^{\mathrm{sing}}\right).$$

These descriptions generalize the well-known characterization of Grassmannian Schubert singularities, and Richardson singularities, in terms of hook removals [BL00, Theorem 9.3.1].

Example 3.5. We provide here essentially the smallest interesting example in the $g = 1$ case, exhibiting a fiber of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet)) \to \mathrm{Pic}^d(X)$ that is not a Richardson variety, and is in fact reducible. Let $a_\bullet = b_\bullet = (0, 2)$, let $\mathscr{L} = \mathscr{O}_X(2P + 2Q)$, and consider the fiber of $G_4^1(X, (P, a_\bullet), (Q, b_\bullet))$ over $\mathscr{L}$. This fiber has two irreducible components Z_1 and Z_2, each isomorphic to $\mathbb{P}^2$, meeting along a $\mathbb{P}^1$. It may be described as the variety of lines in $\mathbb{P}^3$ that meet two fixed lines that themselves intersect at a point.

In this situation, the vertical tangent space at a point in $Z_1 \cap Z_2 \cong \mathbb{P}^1$ has dimension jumping up to 3. Now, the fiber of $G_4^{1,\circ}(X, (P, a_\bullet), (Q, b_\bullet))$ over $\mathscr{L}$ is obtained from that of $G_4^1(X, (P, a_\bullet), (Q, b_\bullet))$ by removing two points of $Z_1 \cap Z_2$. Those two points correspond to the space of sections of $\mathscr{L}$ vanishing to order at least 2 at P, respectively the space of sections of $\mathscr{L}$ vanishing to order at least 2 at Q. Then Theorem 1.5 asserts that on $Z_1 \cap Z_2$, except for at those two points, $G_4^1(X, (P, a_\bullet), (Q, b_\bullet))$ has no horizontal tangent vectors.

Remark 3.6. Proposition 3.2 of [CHT92] asserts that for a general marked curve, the space $G_d^r(X, (P, a_\bullet))$ is connected when $\rho > 0$, and is smooth at all points corresponding to complete linear series. However, Proposition 3.1 as well as the results in [Oss] refute both of these statements; see Example 3.7 below for counterexamples. No argument is provided in [CHT92] for connectedness, but an argument

is provided for the smoothness assertion. The authors have indicated to us that upon further examination, their argument appears correct but implicitly uses that $h^0(L(-a_iP)) = r + 1 - i$ for all i, which is consistent with the criterion we provide in Theorem 1.4.

Example 3.7. The space $G^r_d(X, (P, a_\bullet))$ need not be smooth on the locus of complete linear series. For instance, suppose that $g \geq 11$ is odd, and set $d = \frac{g+1}{2} + 3$, and $r = 1$, with $a_\bullet = 0, 2$. Then $\rho = 4$, and we see that $G^r_d(X, (P, a_\bullet))$ consists entirely of complete linear series when X is general, since ρ becomes negative if r is increased to 2. However, the locus of linear series with a double base point at P is nonempty, and is contained in the singular locus by Proposition 3.1.

The assertion that $G^r_d(X, (P, a_\bullet))$ is connected is true when $\widehat{\rho} > 0$ by the main theorem of [Oss], but it can fail when $\widehat{\rho} = 0$ even with a single ramification point. For instance, consider the case that $g = 3$, $r = 2$, $d = 6$, and $a_\bullet = 0, 3, 5$. According to Remark 4.6 of [Oss], we can compute the number of connected components of $G^r_d(X, (P, a_\bullet))$ for X general by counting the number of connected components of the space of limit linear series on our chain of genus-1 curves. We see that in order to get a valid sequence (i.e., nonnegative and without repetitions) for $b^3_\bullet$, we must have $b^3_\bullet$ of the form $0, 1, *$, and for each i we must have some j with $a^i_j + b^i_{r-j} = d$, with $j = 1$ occurring exactly once and $j = 2$ occurring exactly twice. This leads to two possibilities: we could have $a^2_\bullet = *, 3, 6$ and $a^3_\bullet = *, 4, 6$, or we could have $a^2_\bullet = *, 4, 5$ and $a^3_\bullet = *, 4, 6$. We see that both possibilities can occur in valid limit linear series, and that we obtain two distinct connected components of the limit linear series space. Indeed, in each case the underlying line bundles on each Z_i are uniquely determined, but they are distinct on Z_1 and Z_2. In the first case we must have $\mathscr{O}_{Z_1}(3P_1 + 3Q_1)$ on Z_1 and $\mathscr{O}_{Z_2}(6P_2)$ on Z_2, while in the second case we must have $\mathscr{O}_{Z_1}(5P_1 + Q_1)$ on Z_1 and $\mathscr{O}_{Z_2}(4P_2 + 2Q_2)$ on Z_2.

References

[ACT17] Dave Anderson, Linda Chen, and Nicola Tarasca, *K-classes of Brill-Noether loci and a determinantal formula*, preprint, 2017.

[BL00] Sara Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, Progress in Mathematics, vol. 182, Birkhauser, 2000.

[CHT92] Ciro Ciliberto, Joe Harris, and Montserrat Teixidor i Bigas, *On the endomorphisms of* $\mathrm{Jac}(W^1_d(C))$ *when $\rho = 1$ and C has general moduli*, Classification of irregular varieties (Trento, 1990), Lecture Notes in Mathematics, vol. 1515, Springer, 1992, pp. 41–67.

[CLPT] Melody Chan, Alberto López Martin, Nathan Pflueger, and Montserrat Teixidor i Bigas, *Genera of Brill-Noether curves and staircase paths in Young tableaux*, Transactions of the AMS **370** (2018) 2405-3439.

[Cos09] Izzet Coskun, *A Littlewood-Richardson rule for two-step flag varieties*, Inventiones Mathematicae **176** (2009), no. 2, 325–395.

[CP17] Melody Chan and Nathan Pflueger, *Euler characteristics of Brill-Noether varieties*, Transactions of the AMS, to appear.

[EH83] David Eisenbud and Joe Harris, *A simpler proof of the Gieseker-Petri theorem on special divisors*, Inventiones Mathematicae **74** (1983), 269–280.

[EH86] ———, *Limit linear series: Basic theory*, Inventiones Mathematicae **85** (1986), no. 2, 337–371.

[Eis95] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, 1995.

[GH80] Phillip Griffiths and Joseph Harris, *On the variety of special linear systems on a general algebraic curve*, Duke Mathematical Journal **47** (1980), 233–272.

[Gie82] David Gieseker, *Stable curves and special divisors: Petri's conjecture*, Inventiones Mathematicae **66** (1982), 251–275.

[JP14]   David Jensen and Sam Payne, *Tropical independence I: shapes of divisors and a proof of the Gieseker-Petri theorem*, Algebra Number Theory **8** (2014) 2043-2066.

[MO16]   John Murray and Brian Osserman, *Linked determinantal loci and limit linear series*, Proceedings of the AMS **144** (2016), no. 6, 2399–2410.

[Oss]   Brian Osserman, *Connectedness of Brill-Noether loci via degenerations*, International Mathematics Research Notices (2018).

[Oss14]   _____ , *A simple characteristic-free proof of the Brill-Noether theorem*, Bulletin of the Brazilian Mathematical Society **45** (2014), no. 4, 807–818, special issue in honor of Steven Kleiman and Aron Simis.

[Vak06]   Ravi Vakil, *A geometric Littlewood-Richardson rule*, Annals of Mathematics **164** (2006), no. 2, 371–422.

[Wel85]   Gerald Welters, *A theorem of Gieseker-Petri type for Prym varieties*, Annales Scientifiques de l'École Normale Supérieure **18** (1985), no. 4, 671–683.