Goal Practice with Inverse Trigonometric Functions, and more review of limits (no L'Hôpital's Rule yet)
Reference Stewart §1.6

Examples to study first

Example The following two formulas are called "a-rules." You should learn them, and you can use them freely in your solutions (just write "a-rule" so that your reader knows what you're doing; see the next two examples for how this can be presented).

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C.$$

These formula is valid for any choice of a. How would you prove/derive these rules?

Solution There are at least three ways, which I'll just summarize here for now. We saw one technique in some examples in class, using the substitution $u = x/a$. You can also differentiate the right hand side of the equation, using either the chain rule or implicit differentiation.

Example Evaluate $\displaystyle\int_e^{e^3} \frac{1}{x\left(3+(\ln x)^2\right)}\,dx$

Solution Make the substitution $\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$ to obtain

$$\int_e^{e^3} \frac{1}{x\left(3+(\ln x)^2\right)}\,dx = \int_{\ln e}^{\ln e^3} \frac{1}{3+u^2}\,du \overset{\text{a-rule}}{=} \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_1^3$$

$$= \frac{1}{\sqrt{3}}\left(\arctan\left(\overset{\sqrt{3}}{\cancel{\frac{3}{\sqrt{3}}}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\left(\frac{2\pi}{6} - \frac{\pi}{6}\right) = \boxed{\frac{\pi}{6\sqrt{3}}}.$$

Example Evaluate $\displaystyle\int \frac{e^{3x}}{4+e^{3x}}\,dx$.

Solution $$\int \frac{e^{3x}}{4+e^{3x}}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \boxed{\frac{1}{3}\ln\left|4+e^{3x}\right| + C}$$

where we have made the substitution: $\boxed{\begin{aligned} u &= 4+e^{3x} \\ du &= 3e^{3x}\,dx \\ \frac{1}{3}du &= e^{3x}\,dx \end{aligned}}$

Example Evaluate $\displaystyle\int \frac{e^{3x}}{4+e^{6x}}$.

Solution Substitute $\boxed{\begin{aligned} u &= e^{3x} \\ du &= 3e^{3x}\,dx \\ \frac{1}{3}du &= e^{3x}\,dx \end{aligned}}$ to obtain

$$\int \frac{e^{3x}}{4+e^{6x}}\,dx = \int \frac{e^{3x}}{4+(e^{3x})^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{4+u^2}\,du \overset{\text{a-rule}}{=} \frac{1}{3}\left(\frac{1}{2}\arctan\left(\frac{u}{2}\right)\right) + C = \boxed{\frac{1}{6}\arctan\left(\frac{e^{3x}}{2}\right) + C}$$

Note The little note “a-rule” above the equals sign above is a handy and concise way to explain what fact you’re using to justify an equation. We’ll use this type of shorthand often, e.g writing $\overset{\text{l'H}}{=}$ for equations justified by L’Hôpital’s rule.

Problems to hand in

Compute each of the following Integrals. Simplify.

1. $\displaystyle\int_2^{2\sqrt{3}} \frac{1}{\sqrt{16-x^2}}\,dx$
2. $\displaystyle\int_0^{\ln 3} \frac{e^x}{3+e^{2x}}\,dx$
3. $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$
4. $\displaystyle\int_4^{4\sqrt{3}} \frac{1}{16+x^2}\,dx$
5. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$
6. $\displaystyle\int \frac{x^2}{x^2+4}\,dx$
7. $\displaystyle\int \frac{2x^2+5}{x^2+1}\,dx$
8. $\displaystyle\int \frac{1}{(1+x^2)\left(5+(\arctan x)^2\right)}\,dx$
9. $\displaystyle\int_3^{9} \frac{1}{\sqrt{x}\,(x+9)}\,dx$
10. $\displaystyle\int \frac{x^2+x+1}{x^2+4}\,dx$

Compute each of the following Limits. Simplify.

11. $\displaystyle\lim_{x\to 5^+} \frac{1}{x-5}$

12. $\displaystyle\lim_{x\to 5^-} \frac{1}{x-5}$

13. $\displaystyle\lim_{x\to 8^+} \ln|x-8|$

14. $\displaystyle\lim_{x\to 8^-} \ln|x-8|$

15. $\displaystyle\lim_{x\to 3^+} e^{\frac{2}{x-3}}$

16. $\displaystyle\lim_{x\to 3^-} e^{\frac{2}{x-3}}$

17. $\displaystyle\lim_{x\to \infty} \ln\left(1-\arctan\left(\frac{5}{x^4}\right)\right)$

18. $\displaystyle\lim_{x\to \infty} \ln\left(\frac{\pi}{2}-\arctan x\right)$

19. $\displaystyle\lim_{x\to 4^-} \ln|\ln|x-4||$

20. $\displaystyle\lim_{x\to 0^+} \arctan\left(\frac{\ln x}{5}\right)$