Solutions to Practice Problems 1

Functions: Please state what the domain is for each of the following functions.

1. $f(x) = \dfrac{x+2}{x-1}$ 2. $g(x) = \sqrt{x-2}$ 3. $m(x) = \sqrt{2-x}$

4. $G(x) = \dfrac{1}{\sqrt{2-x}}$ 5. $h(x) = \dfrac{x-3}{x^2+3}$ 6. $W(x) = \dfrac{x^2+6x+8}{x+2}$

Solutions to Domain problems:
1. Need $x \neq 1$, so $\boxed{\text{Domain of } f \text{ is all real numbers except } 1}$
or if you prefer, $\{x \in \mathbb{R} | x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$

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2. Need $x - 2 \geq 0$, so $\boxed{\text{Domain of } g \text{ is } [2, \infty)}$ or if you prefer, $\{x \in \mathbb{R} | x \geq 2\}$

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3. Need $2 - x \geq 0$, so $x \leq 2$, so $\boxed{\text{Domain of } m \text{ is } (-\infty, 2]}$ or if you prefer, $\{x \in \mathbb{R} | x \leq 2\}$

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4. Need $2 - x \geq 0$, but also $2 - x \neq 0$, so $x < 2$, so $\boxed{\text{Domain of } G \text{ is } (-\infty, 2)}$ or if you prefer, $\{x \in \mathbb{R} | x < 2\}$

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5. Denom is never 0, so $\boxed{\text{Domain of } h \text{ is } (-\infty, \infty)}$ or if you prefer, $\mathbb{R}$

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6. Need $x \neq -2$, so $\boxed{\text{Domain of } f \text{ is all real numbers except } -2}$
or if you prefer, $\{x \in \mathbb{R} | x \neq -2\}$, or $(-\infty, -2) \cup (-2, \infty)$

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7. Let $g(x) = \dfrac{x+1}{x}$. Compute (and simplify, if possible) the following:

(a) $g(t-2) =$ (b) $\dfrac{g(2+h) - g(2)}{h} =$

Solution. (a) $g(t-2) = \dfrac{(t-2)+1}{t-2} = \boxed{\dfrac{t-1}{t-2}}$

(b) $$\frac{g(2+h) - g(2)}{h} = \frac{\left(\dfrac{(2+h)+1}{2+h}\right) - \dfrac{3}{2}}{h} \cdot \left(\frac{2(2+h)}{2(2+h)}\right) = \frac{(3+h)2 - 3(2+h)}{2h(2+h)}$$
$$= \frac{6 + 2h - 6 - 3h}{2h(h+2)} = \frac{-h}{2h(h+2)} = \boxed{\frac{-1}{2(h+2)}}$$

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8. Let $f(x) = \dfrac{1}{x+1} - \dfrac{1}{x}$. Compute (and simplify, if possible) the following:

(a) $f(t-1) =$ (b) $f\left(\dfrac{1}{t}\right) =$

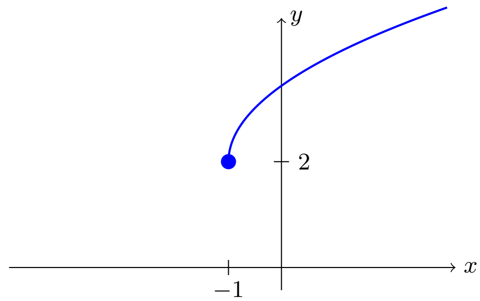
Solution. (a) $f(t-1) = \dfrac{1}{(t-1)+1} - \dfrac{1}{t-1} = \dfrac{1}{t} - \dfrac{1}{t-1} = \dfrac{(t-1)-t}{t(t-1)} = \boxed{\dfrac{-1}{t(t-1)}}$

(b) $$f\left(\frac{1}{t}\right) = \frac{1}{\left(\dfrac{1}{t}+1\right)} - \frac{1}{\left(\dfrac{1}{t}\right)} = \frac{t}{t\left(\dfrac{1}{t}+1\right)} - t = \frac{t}{t+1} - t = \frac{t - t(t+1)}{t+1} = \frac{t - t^2 - t}{t+1} = \boxed{\frac{-t^2}{t+1}}$$

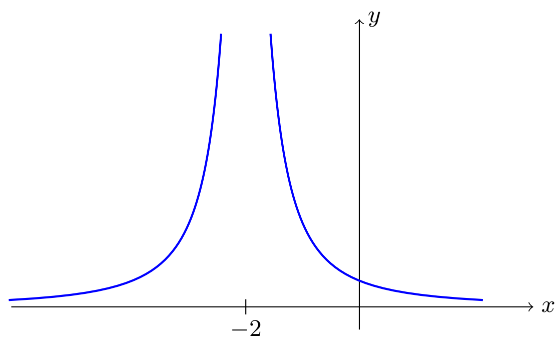
9. Graph the following functions using scaling, translation, etc.

(a) $y = 2 + \sqrt{x+1}$ (b) $y = \dfrac{3}{(x-2)^2}$ (c) $y = 2(x-1)^4 - 3$

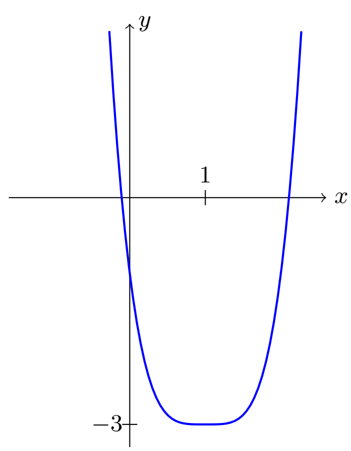
Solutions (a): This is $y = \sqrt{x}$ translated **left by 1** and then **up by 2**, so it looks like:



(b): This is $y = \dfrac{1}{x^2}$ translated **left by 2**, and then **stretched vertically by 3**, so it looks like:



(c): This is $y = x^4$ **stretched vertically by 2**, then translated **right by 1**, and then **down by 3**, so it looks like:



Limit Practice Problems: Evaluate the following limits. Always justify your work.

Solutions. 10. $\lim_{w \to 0} \dfrac{16}{w} = \dfrac{16}{0}$ so we check both sides:

LHL: $\lim_{w \to 0^-} \dfrac{16}{w} = \dfrac{16}{0^-} = -\infty$ RHL: $\lim_{w \to 0^+} \dfrac{16}{w} = \dfrac{16}{0^+} = +\infty$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

11. $\lim_{t\to 2} \frac{3-t}{t-2} = \frac{1}{0}$ so we check both sides:

$$\text{LHL: } \lim_{t\to 2^-} \frac{3-t}{t-2} = \frac{1}{0^-} = -\infty \qquad \text{RHL: } \lim_{t\to 2^+} \frac{3-t}{t-2} = \frac{1}{0^+} = +\infty$$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

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12. $\lim_{t\to 2} \frac{3-t}{(t-2)^2} = \frac{1}{0}$ so we check both sides:

$$\text{LHL: } \lim_{t\to 2^-} \frac{3-t}{(t-2)^2} = \frac{1}{0^+} = +\infty \qquad \text{RHL: } \lim_{t\to 2^+} \frac{3-t}{(t-2)^2} = \frac{1}{0^+} = +\infty$$

RHL=LHL= ∞, so $\boxed{\text{original limit diverges to } +\infty}$

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13. $\lim_{x\to 4} \frac{(x+2)^2}{x^2-3x-4} = \frac{6^2}{0}$ so we check both sides, noting that the denominator factors as $(x-4)(x+1)$

$$\text{LHL: } \lim_{x\to 4^-} \frac{(x+2)^2}{x^2-3x-4} = \frac{6^2}{(0^-)\cdot 5} = -\infty \qquad \text{RHL: } \lim_{x\to 4^+} \frac{(x+2)^2}{x^2-3x-4} = \frac{6^2}{(0^+)\cdot 5} = +\infty$$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

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14. $\lim_{x\to 4} \frac{x-4}{x^2-3x-4} = \lim_{x\to 4} \frac{x-4}{(x-4)(x+1)} = \lim_{x\to 4} \frac{1}{x+1} \overset{\text{DSP}}{=} \boxed{\frac{1}{5}}$

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15. $\lim_{x\to 4} \frac{x^2-2x-8}{x^2-3x-4} = \lim_{x\to 4} \frac{(x-4)(x+2)}{(x-4)(x+1)} = \lim_{x\to 4} \frac{x+2}{x+1} \overset{\text{DSP}}{=} \boxed{\frac{6}{5}}$

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16. $\lim_{x\to 6} \frac{x^2-4x-12}{x^2-3x-18} = \lim_{x\to 6} \frac{(x-6)(x+2)}{(x-6)(x+3)} = \lim_{x\to 6} \frac{x+2}{x+3} \overset{\text{DSP}}{=} \boxed{\frac{8}{9}}$

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17. $\lim_{x\to 1} \frac{x^2-4x-12}{x^2-3x-18} \overset{\text{DSP}}{=} \frac{1-4-12}{1-3-18} = \frac{-15}{-20} = \boxed{\frac{3}{4}}$

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18. $\lim_{x\to 0} \frac{x^2-4x-12}{x^2-3x-18} \overset{\text{DSP}}{=} \frac{-12}{-18} = \boxed{\frac{2}{3}}$

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19. $\lim_{x\to -3} \frac{x^2-4x-12}{x^2-3x-18} = \frac{9}{0}$ so we check both sides, noting that as we saw in #16 the function reduces to $\frac{x+2}{x+3}$:

$$\text{LHL: } \lim_{x\to -3^-} \frac{x+2}{x+3} = \frac{-1}{0^-} = +\infty \qquad \text{RHL: } \lim_{x\to -3^+} \frac{x+2}{x+3} = \frac{-1}{0^+} = -\infty$$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

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20. $\lim_{x\to -2} \frac{x^2-4x-12}{x^2-3x-18} \overset{\text{DSP}}{=} \frac{4+8-12}{4+6-18} = \frac{0}{-8} = \boxed{0}$

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21. $\lim_{x\to 0} \frac{x^2-4x-12}{x^2-7x} = \frac{-12}{0}$, so we check both sides, noting the the denominator factors as $x(x-7)$:

$$\text{LHL: } \lim_{x\to 0^-} \frac{x^2-4x-12}{x^2-7x} = \frac{-12}{(0^-)(-7)} = -\infty \qquad \text{RHL: } \lim_{x\to 0^+} \frac{x^2-4x-12}{x^2-7x} = \frac{-12}{(0^+)(-7)} = +\infty$$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

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22. $\lim_{x\to 0} \frac{x^2-4x}{x^2-7x} = \lim_{x\to 0} \frac{x-4}{x-7} \overset{\text{DSP}}{=} \frac{-4}{-7} = \boxed{\frac{4}{7}}$

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23. $\lim_{x\to 3} \dfrac{x^2-9}{|x-3|}$ is piecewise, so check both sides:

LHL: $\lim_{x\to 3^-} \dfrac{x^2-9}{|x-3|} = \lim_{x\to 3^-} \dfrac{(x-3)(x+3)}{-(x-3)} = \lim_{x\to 3^-} \dfrac{x+3}{-1} \overset{\text{DSP}}{=} -6$

RHL: $\lim_{x\to 3^+} \dfrac{x^2-9}{|x-3|} = \lim_{x\to 3^+} \dfrac{(x-3)(x+3)}{x-3} = \lim_{x\to 3^+} \dfrac{x+3}{1} \overset{\text{DSP}}{=} 6$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

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24. $\lim_{x\to 0} \dfrac{x^3+209x^2+200x}{|x|}$ is piecewise, so check both sides:

LHL: $\lim_{x\to 0^-} \dfrac{x^3+209x^2+200x}{|x|} = \lim_{x\to 0^-} \dfrac{x^3+209x^2+200x}{-x} = \lim_{x\to 0^-} \dfrac{x^2+209x+200}{-1} \overset{\text{DSP}}{=} -200$

RHL: $\lim_{x\to 0^+} \dfrac{x^3+209x^2+200x}{x} = \lim_{x\to 0^+} \dfrac{x^3+209x^2+200x}{x} = \lim_{x\to 0^+} \dfrac{x^2+209x+200}{1} \overset{\text{DSP}}{=} 200$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

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25. $\lim_{x\to -5} \dfrac{x^2+6x+5}{|x+5|}$ is piecewise, so check both sides:

LHL: $\lim_{x\to -5^-} \dfrac{x^2+6x+5}{|x+5|} = \lim_{x\to -5^-} \dfrac{(x+1)(x+5)}{-(x+5)} = \lim_{x\to -5^-} -(x+1) \overset{\text{DSP}}{=} 4$

RHL: $\lim_{x\to -5^+} \dfrac{x^2+6x+5}{|x+5|} = \lim_{x\to -5^+} \dfrac{(x+1)(x+5)}{x+5} = \lim_{x\to -5^+} (x+1) \overset{\text{DSP}}{=} -4$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

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26. $\lim_{t\to -1} \dfrac{200(t^2+6t+5)}{t^2+t} = \lim_{t\to -1} \dfrac{200(t+1)(t+5)}{t(t+1)} = \lim_{t\to -1} \dfrac{200(t+5)}{t} \overset{\text{DSP}}{=} \dfrac{200(4)}{-1} = \boxed{-800}$

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27. $\lim_{t\to 1} t^{300}+t^{200}+t^{100} \overset{\text{DSP}}{=} 1+1+1 = \boxed{3}$

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28. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1} \dfrac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)} = \lim_{x\to 1} \dfrac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4}$

$= \lim_{x\to 1} \dfrac{(x-1)(\sqrt{x+3}+2)}{x-1} = \lim_{x\to 1} \sqrt{x+3}+2 \overset{\text{DSP}}{=} \sqrt{4}+2 = \boxed{4}$

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29. $\lim_{x\to 9} \dfrac{9x-x^2}{3-\sqrt{x}} = \lim_{x\to 9} \dfrac{(9x-x^2)(3+\sqrt{x})}{(3-\sqrt{x})(3+\sqrt{x})} = \lim_{x\to 9} \dfrac{x(9-x)(3+\sqrt{x})}{9-x}$

$= \lim_{x\to 9} x(3+\sqrt{x}) \overset{\text{DSP}}{=} 9(3+\sqrt{9}) = 9(6) = \boxed{54}$

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30. $\lim_{x\to 1} \dfrac{\sqrt{x^2+8}-3}{x-1} = \lim_{x\to 1} \dfrac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(x-1)(\sqrt{x^2+8}+3)} = \lim_{x\to 1} \dfrac{(x^2+8)-9}{(x-1)(\sqrt{x^2+8}+3)}$

$= \lim_{x\to 1} \dfrac{x^2-1}{(x-1)(\sqrt{x^2+8}+3)} = \lim_{x\to 1} \dfrac{(x-1)(x+1)}{(x-1)(\sqrt{x^2+8}+3)} = \lim_{x\to 1} \dfrac{x+1}{\sqrt{x^2+8}+3}$

$\overset{\text{DSP}}{=} \dfrac{2}{\sqrt{9}+3} = \dfrac{2}{6} = \boxed{\dfrac{1}{3}}$

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31. $\lim_{x\to -4} \dfrac{x^2-3x-28}{x^2+4x} = \lim_{x\to -4} \dfrac{(x+4)(x-7)}{x(x+4)} = \lim_{x\to -4} \dfrac{x-7}{x} \overset{\text{DSP}}{=} \dfrac{-11}{-4} \boxed{\dfrac{11}{4}}$

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32. $\lim_{x\to 0} \dfrac{x^2-3x-28}{x^2+4x} = \dfrac{-28}{0}$ so we check both sides, noting that as we saw in #31 the function reduces to $\dfrac{x-7}{x}$:

LHL: $\lim_{x\to 0^-} \dfrac{x-7}{x} = \dfrac{-7}{0^-} = +\infty$ RHL: $\lim_{x\to 0^+} \dfrac{x-7}{x} = \dfrac{-7}{0^+} = -\infty$

RHL$\neq$LHL$=\infty$, so $\boxed{\text{original limit DNE}}$

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33. $\lim_{x\to 3} \dfrac{\frac{2}{x+3}-\frac{1}{3}}{x-3} = \lim_{x\to 3} \dfrac{\frac{2}{x+3}-\frac{1}{3}}{x-3}\left(\dfrac{3(x+3)}{3(x+3)}\right) = \lim_{x\to 3} \dfrac{6-(x+3)}{3(x-3)(x+3)}$

$= \lim_{x\to 3} \dfrac{-(x-3)}{3(x-3)(x+3)} = \lim_{x\to 3} \dfrac{-1}{3(x+3)} \overset{\text{DSP}}{=} = \dfrac{-1}{3(6)} = \boxed{-\dfrac{1}{18}}$

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34. $\lim_{x\to 1} \dfrac{x^2-1}{|x-1|}$ is piecewise, so check both sides:

LHL: $\lim_{x\to 1^-} \dfrac{x^2-1}{|x-1|} = \lim_{x\to 1^-} \dfrac{(x-1)(x+1)}{-(x-1)} = \lim_{x\to 1^-} \dfrac{x+1}{-1} \overset{\text{DSP}}{=} -2$

RHL: $\lim_{x\to 1^+} \dfrac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \dfrac{(x-1)(x+1)}{x-1} = \lim_{x\to 1^+} \dfrac{x+1}{1} \overset{\text{DSP}}{=} 2$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

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35. $\lim_{x\to -5} \dfrac{x^2+6x+5}{|x+5|}$ is piecewise, so check both sides:

LHL: $\lim_{x\to -5^-} \dfrac{x^2+6x+5}{|x+5|} = \lim_{x\to -5^-} \dfrac{(x+1)(x+5)}{-(x+5)} = \lim_{x\to -5^-} -(x+1) \overset{\text{DSP}}{=} 4$

RHL: $\lim_{x\to -5^+} \dfrac{x^2+6x+5}{|x+5|} = \lim_{x\to -5^+} \dfrac{(x+1)(x+5)}{x+5} = \lim_{x\to -5^+} (x+1) \overset{\text{DSP}}{=} -4$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

(And yes, this is the same problem as #25.)

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36. $\lim_{x\to -1} \dfrac{x^2+3x+2}{(x+1)^2} = \lim_{x\to -1} \dfrac{(x+1)(x+2)}{(x+1)^2} = \lim_{x\to -1} \dfrac{x+2}{x+1} = \dfrac{1}{0}$, so we check both sides:

LHL: $\lim_{x\to -1^-} \dfrac{x+2}{x+1} = \dfrac{1}{0^-} = -\infty$ RHL: $\lim_{x\to -1^+} \dfrac{x+2}{x+1} = \dfrac{1}{0^+} = +\infty$

RHL$\neq$LHL, so $\boxed{\text{original limit DNE}}$

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37. $\lim_{x\to 7^-} \dfrac{7-x}{|x-7|} = \lim_{x\to 7^-} \dfrac{-(x-7)}{-(x-7)} = \lim_{x\to 7^-} 1 \overset{\text{DSP}}{=} \boxed{1}$

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38. $\lim_{x\to 0^-} \dfrac{x}{x-|x|} = \lim_{x\to 0^-} \dfrac{x}{x-(-x)} = \lim_{x\to 0^-} \dfrac{x}{2x} = \lim_{x\to 0^-} \dfrac{1}{2} \overset{\text{DSP}}{=} \boxed{\dfrac{1}{2}}$

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39. $\lim_{x\to 2^+} \dfrac{2-x}{|x-2|} = \lim_{x\to 2^+} \dfrac{-(x-2)}{x-2} = \lim_{x\to 2^+} -1 \overset{\text{DSP}}{=} \boxed{-1}$

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40. Let $G(u) = u^2+u$. Compute $\lim_{u\to 2} \dfrac{u^2-2u}{G(u-3)}$

Solution. $\lim_{u\to 2} \dfrac{u^2-2u}{G(u-3)} = \lim_{u\to 2} \dfrac{u^2-2u}{(u-3)^2+(u-3)} = \lim_{u\to 2} \dfrac{u(u-2)}{u^2-5u+6} = \lim_{u\to 2} \dfrac{u(u-2)}{(u-3)(u-2)}$

$= \lim_{u\to 2} \dfrac{u}{u-3} = \dfrac{2}{-1} = \boxed{-2}$

41. Let $h(y) = y^2 - 3$. Compute $\lim\limits_{x\to -2} \dfrac{x+2}{h(2x) - h(x+6)}$

Solution.

$$\lim_{x\to-2}\frac{x+2}{h(2x)-h(x+6)} = \lim_{x\to-2}\frac{x+2}{((2x)^2-3)-((x+6)^2-3)} = \lim_{x\to-2}\frac{x+2}{(4x^2-3)-(x^2+12x+36-3)}$$
$$= \lim_{x\to-2}\frac{x+2}{4x^2-3-x^2-12x-33} = \lim_{x\to-2}\frac{x+2}{3x^2-12x-36} = \lim_{x\to-2}\frac{x+2}{3(x^2-4x-12)}$$
$$= \lim_{x\to-2}\frac{x+2}{3(x-6)(x+2)} = \lim_{x\to-2}\frac{1}{3(x-6)} = \boxed{-\frac{1}{24}}$$

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42. Let $g(x) = \sqrt{x}$. Compute $\lim\limits_{s\to 1} \dfrac{g(s^2+8)-3}{s-1}$

Solution. $$\lim_{s\to1}\frac{g(s^2+8)-3}{s-1} = \lim_{s\to1}\frac{\sqrt{s^2+8}-3}{s-1} = \lim_{s\to1}\frac{(\sqrt{s^2+8}-3)}{(s-1)}\cdot\frac{(\sqrt{s^2+8}+3)}{(\sqrt{s^2+8}+3)}$$
$$= \lim_{s\to1}\frac{s^2+8-9}{(s-1)(\sqrt{s^2+8}+3)} = \lim_{s\to1}\frac{s^2-1}{(s-1)(\sqrt{s^2+8}+3)} = \lim_{s\to1}\frac{(s-1)(s+1)}{(s-1)(\sqrt{s^2+8}+3)}$$
$$= \lim_{s\to1}\frac{s+1}{\sqrt{s^2+8}+3} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

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43. Let $f(t) = \dfrac{1}{t}$. Compute $\lim\limits_{t\to 2} \dfrac{f(t-1)-2f(t)}{t^2-4}$

Solution. $$\lim_{t\to2}\frac{f(t-1)-2f(t)}{t^2-4} = \lim_{t\to2}\frac{\left(\frac{1}{t-1}-\frac{2}{t}\right)}{t^2-4} = \lim_{t\to2}\frac{\left(\frac{1}{t-1}-\frac{2}{t}\right)}{t^2-4}\left(\frac{t(t-1)}{t(t-1)}\right)$$
$$= \lim_{t\to2}\frac{t-2(t-1)}{t(t-1)(t^2-4)} = \lim_{t\to2}\frac{-(t-2)}{t(t-1)(t-2)(t+2)} = \lim_{t\to2}\frac{-1}{t(t-1)(t+2)} \stackrel{\text{DSP}}{=} \frac{-1}{1\cdot2\cdot4} = \boxed{-\frac{1}{8}}$$

Derivatives: Use the **limit definition of the derivative** to compute these derivatives:

44. $f(x) = -4x - x^2 - 3$ Find $f'(x)$

45. $g(x) = \dfrac{-3}{x}$ Find $g'(x)$

46. $R(x) = x^3$ Find $R'(x)$

47. $G(x) = \dfrac{1}{x^2}$ Find $G'(x)$

48. $f(x) = \sqrt{x-7}$ Find $f'(x)$

49. $g(x) = \sqrt{7-3x}$ Find $g'(x)$

Solutions to Derivative problems:

44. $$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0}\frac{(-4(x+h)-(x+h)^2-3)-(-4x-x^2-3)}{h} =$$
$$= \lim_{h\to0}\frac{-4x-4h-x^2-2xh-h^2-3+4x+x^2+3}{h} = \lim_{h\to0}\frac{-4h-2xh-h^2}{h}$$
$$= \lim_{h\to0}\frac{h(-4-2x-h)}{h} = \lim_{h\to0} -4-2x-h \stackrel{\text{DSP}}{=} \boxed{-4-2x}$$

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45. $$g'(x) = \lim_{h\to0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to0}\frac{\frac{-3}{x+h}-\left(\frac{-3}{x}\right)}{h}\cdot\left(\frac{x(x+h)}{x(x+h)}\right) = \lim_{h\to0}\frac{-3x+3(x+h)}{hx(x+h)}$$
$$= \lim_{h\to0}\frac{3h}{hx(x+h)} = \lim_{h\to0}\frac{3}{x(x+h)} \stackrel{\text{DSP}}{=} \boxed{\frac{3}{x^2}}$$

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46. $$R'(x) = \lim_{h\to0}\frac{R(x+h)-R(x)}{h} = \lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0}\frac{(x+h)^3-x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 \overset{\text{DSP}}{=} \boxed{3x^2}$$

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47. $$G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \cdot \left(\frac{x^2(x+h)^2}{x^2(x+h)^2}\right)$$

$$= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{x^2 - (x^2 + 2hx + h^2)}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{-2hx - h^2}{h(x+h)^2x^2}$$

$$= \lim_{h\to 0} \frac{-2x - h}{(x+h)^2x^2} \overset{\text{DSP}}{=} \frac{-2x}{(x^2)x^2} = \boxed{-\frac{2}{x^3}}$$

---

48. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h) - 7} - \sqrt{x-7}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h) - 7} - \sqrt{x-7}}{h} \cdot \left(\frac{\sqrt{(x+h) - 7} + \sqrt{x-7}}{\sqrt{(x+h) - 7} + \sqrt{x-7}}\right) = \lim_{h\to 0} \frac{(x+h-7) - (x-7)}{h(\sqrt{(x+h) - 7} + \sqrt{x-7})}$$

$$= \lim_{h\to 0} \frac{x + h - 7 - x + 7}{h(\sqrt{(x+h) - 7} + \sqrt{x-7})} = \lim_{h\to 0} \frac{h}{h(\sqrt{(x+h) - 7} + \sqrt{x-7})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{(x+h) - 7} + \sqrt{x-7}} \overset{\text{DSP}}{=} \boxed{\frac{1}{2\sqrt{x-7}}}$$

---

49. $$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\sqrt{7 - 3(x+h)} - \sqrt{7-3x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{7 - 3(x+h)} - \sqrt{7-3x}}{h} \cdot \left(\frac{\sqrt{7 - 3(x+h)} + \sqrt{7-3x}}{\sqrt{7 - 3(x+h)} + \sqrt{7-3x}}\right) = \lim_{h\to 0} \frac{(7 - 3(x+h)) - (7-3x)}{h(\sqrt{7 - 3(x+h)} + \sqrt{7-3x})}$$

$$= \lim_{h\to 0} \frac{7 - 3x - 3h - 7 + 3x}{h(\sqrt{7 - 3(x+h)} + \sqrt{7-3x})} = \lim_{h\to 0} \frac{-3h}{h(\sqrt{7 - 3(x+h)} + \sqrt{7-3x})}$$

$$= \lim_{h\to 0} \frac{-3}{\sqrt{7 - 3(x+h)} + \sqrt{7-3x}} \overset{\text{DSP}}{=} \boxed{\frac{-3}{2\sqrt{7-3x}}}$$

---

50. Find an equation for the tangent line to the graph of $f(x) = \dfrac{1}{x-1}$ at the point $(0, -1)$.

Solution. First, we find the slope $f'(0)$:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{h-1} - \left(\frac{1}{-1}\right)}{h} \cdot \left(\frac{h-1}{h-1}\right) = \lim_{h\to 0} \frac{1 + (h-1)}{h(h-1)}$$

$$= \lim_{h\to 0} \frac{h}{h(h-1)} = \lim_{h\to 0} \frac{1}{h-1} \overset{\text{DSP}}{=} -1$$

So the tangent line goes through $(0, -1)$ with slope -1. By point-slope, the equation is:

$$y - (-1) = (-1)(x - 0), \qquad \text{i.e.} \qquad \boxed{y = -x - 1}$$

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51. Find an equation for the tangent line to the graph of $g(x) = \dfrac{1}{x+1}$ at the point $\left(1, \dfrac{1}{2}\right)$.

Solution. First, we find the slope $g'(1)$:

$$g'(1) = \lim_{h\to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{h+2} - \left(\frac{1}{2}\right)}{h} \cdot \left(\frac{2(h+2)}{2(h+2)}\right) = \lim_{h\to 0} \frac{2 - (h+2)}{2h(h+2)}$$

$$= \lim_{h\to 0} \frac{-h}{2h(h+2)} = \lim_{h\to 0} \frac{-1}{2(h+2)} \overset{\text{DSP}}{=} -\frac{1}{4}$$

So the tangent line goes through $\left(1, \frac{1}{2}\right)$ with slope $-\frac{1}{4}$. By point-slope, the equation is:

$$y - \frac{1}{2} = -\frac{1}{4}(x-1), \qquad \text{i.e.} \qquad \boxed{y = -\frac{x}{4} + \frac{3}{4}}$$

---

52. Find an equation for the tangent line to the graph of $y = \dfrac{3}{x} + 1$ when $x = 1$.

Solution. Let $f(x) = \dfrac{3}{x} + 1$. First, we find the slope $f'(1)$:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\left(\frac{3}{1+h} + 1\right) - \left(\frac{3}{1} + 1\right)}{h} \cdot \left(\frac{h+1}{h+1}\right) = \lim_{h\to 0} \frac{3 - 3(h+1)}{h(h+1)}$$

$$= \lim_{h\to 0} \frac{-3h}{h(h+1)} = \lim_{h\to 0} \frac{-3}{h+1} \overset{\text{DSP}}{=} -3$$

We also have $f(1) = \dfrac{3}{1} + 1 = 4$. So the tangent line goes through $(1, 4)$ with slope -3. By point-slope, the equation is:

$$y - 4 = -3(x-1) \qquad \text{i.e.} \qquad \boxed{y = -3x + 7}$$

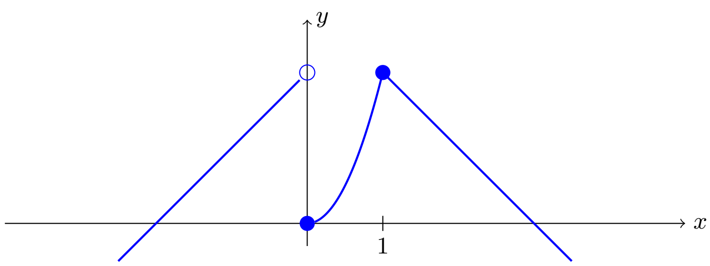
Piecewise-defined functions Answer the questions (and **justify** your answers) about each of the following piecewise defined functions.

53. Let $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x \le 1 \\ 3-x & \text{if } x > 1 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$$\lim_{x\to 2} f(x) = \qquad \lim_{x\to 1} f(x) = \qquad \lim_{x\to 0} f(x) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



For the limits:

$\lim_{x\to 2} f(x) = \lim_{x\to 2} 3 - x = \boxed{1}$

And since that's $f(2)$ also, that means f is continuous at $x = 2$

$\lim_{x\to 1} f(x)$: we check both sides:

LHL: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^2 \overset{\text{DSP}}{=} 2$

RHL: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3 - x \overset{\text{DSP}}{=} 2$

So $\boxed{\lim_{x\to 1} f(x) = 2}$ since RHL = LHL.

And since that's $f(1)$ also, that means f is continuous at $x = 1$

$\lim\limits_{x\to 0} f(x)$: we check both sides:

LHL: $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x + 2 \overset{\text{DSP}}{=} 2$

RHL: $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2x^2 \overset{\text{DSP}}{=} 0$

So $\boxed{\lim\limits_{x\to 1} f(x) \text{ DNE}}$ since RHL $\neq$ LHL

The various pieces of the function are continuous away from the break points. So the only place where f is discontinuous is at $\boxed{x = 0}$

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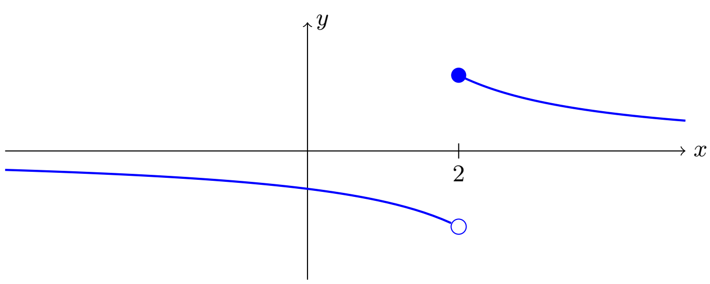
54. Let $g(x) = \begin{cases} \dfrac{1}{x-4} & \text{if } x < 2 \\ \dfrac{1}{x} & \text{if } x \geq 2 \end{cases}$

Sketch the graph. Find the numbers at which g is discontinuous. Evaluate:

$$\lim_{x\to 1} g(x) = \qquad\qquad \lim_{x\to 2} g(x) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



For the limits:

$\lim\limits_{x\to 1} g(x) = \lim\limits_{x\to 1} \dfrac{1}{x-4} = \boxed{-\dfrac{1}{3}}$

$\lim\limits_{x\to 2} g(x)$: we check both sides:

LHL: $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} \dfrac{1}{x-4} \overset{\text{DSP}}{=} -\dfrac{1}{2}$

RHL: $\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+} \dfrac{1}{x} \overset{\text{DSP}}{=} \dfrac{1}{2}$

So $\boxed{\lim\limits_{x\to 2} g(x) \text{ DNE}}$ since RHL $\neq$ LHL

The various pieces of the function are continuous away from the break points. (Note that $1/x$ is discontinuous at $x = 0$, but g isn't given by that formula there. Similarly for $1/(x-4)$ at $x = 4$.) So the only place where g is discontinuous is at $\boxed{x = 2}$

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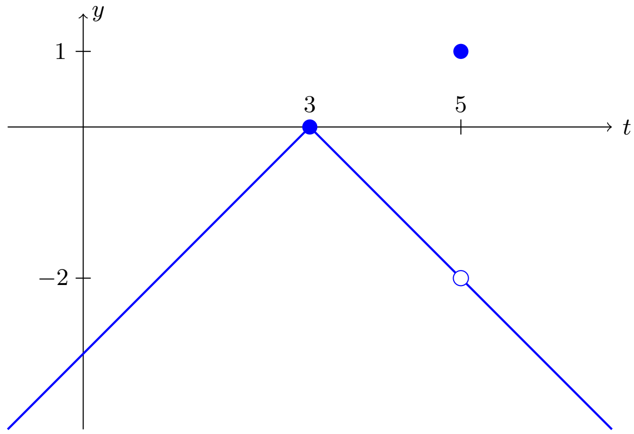
55. Let $f(t) = \begin{cases} t-3 & \text{if } t \leq 3 \\ 3-t & \text{if } 3 < t < 5 \\ 1 & \text{if } t = 5 \\ 3-t & \text{if } t > 5 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$$\lim_{t\to 3} f(t) = \qquad \lim_{t\to 0} f(t) = \qquad \lim_{t\to 5} f(t) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



For the limits:
$\lim_{t\to 3} f(t)$: we check both sides:

LHL: $\lim_{t\to 3^-} f(t) = \lim_{t\to 3^-} t - 3 \overset{\text{DSP}}{=} 0$

RHL: $\lim_{t\to 3^+} f(t) = \lim_{t\to 3^+} 3 - t \overset{\text{DSP}}{=} 0$

So $\boxed{\lim_{t\to 3} f(t) = 0}$ since RHL = LHL. And since that's $f(3)$ also, that means f is continuous at $t = 3$.

$\lim_{t\to 0} f(t) = \lim_{t\to 0} t - 3 = \boxed{-3}$

$\lim_{t\to 5} f(t) = \lim_{t\to 0} 3 - t = \boxed{-2}$

However, that is **not** equal to $f(5) = -2$, so f is discontinuous at $t = 5$.

The various pieces of the function are continuous away from the break points.
So the only place where f is discontinuous is at $\boxed{t = 5}$

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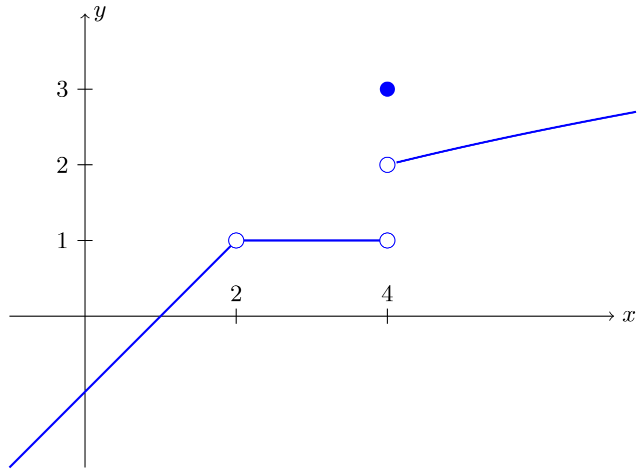
56. Let $H(x) = \begin{cases} x - 1 & \text{if } x < 2 \\ 1 & \text{if } 2 < x < 4 \\ 3 & \text{if } x = 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

Sketch the graph. Find the numbers at which H is discontinuous. Evaluate:

$$\lim_{x\to 0} H(x) = \qquad \lim_{x\to 2} H(x) = \qquad \lim_{x\to 4} H(x) = \qquad H(4) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



For the limits/value:

$\lim_{x\to 0} H(x) = \lim_{x\to 0} x - 1 \overset{\text{DSP}}{=} \boxed{-1}$

$\lim_{x\to 2} H(x)$: we check both sides:

LHL: $\lim_{x\to 2^-} H(x) = \lim_{x\to 2^-} x - 1 \overset{\text{DSP}}{=} 1$

RHL: $\lim_{x\to 2^+} H(x) = \lim_{x\to 2^+} 1 \overset{\text{DSP}}{=} 1$

So $\boxed{\lim_{x\to 2} H(x) = 1}$ since RHL = LHL. But $H(2)$ is undefined, so H is discontinuous at $x = 2$.

$\lim_{x\to 4} H(x)$: we check both sides:

LHL: $\lim_{x\to 4^-} H(x) = \lim_{x\to 4^-} 1 \overset{\text{DSP}}{=} 1$

RHL: $\lim_{x\to 4^+} H(x) = \lim_{x\to 4^+} \sqrt{x} \overset{\text{DSP}}{=} 2$

So $\boxed{\lim_{x\to 2} H(x) \text{ DNE}}$ since RHL $\neq$ LHL.

From the formula, $\boxed{H(4) = 3}$

The various pieces of the function are continuous away from the break points.
So the only places where H is discontinuous are at $\boxed{x = 2, 4}$

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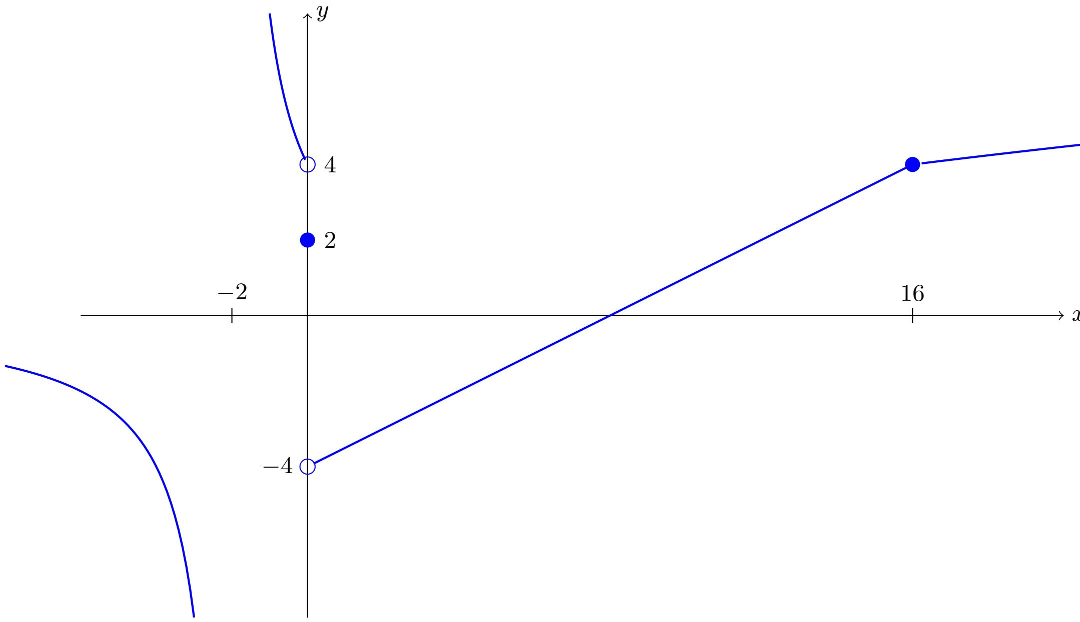
57. Let $h(x) = \begin{cases} \dfrac{8}{x+2} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \dfrac{1}{2}x - 4 & \text{if } 0 < x \leq 16 \\ \sqrt{x} & \text{if } x > 16 \end{cases}$

Sketch the graph. Find the numbers at which h is discontinuous. Evaluate:

$$\lim_{x\to -2} h(x) = \qquad \lim_{x\to 0} h(x) = \qquad \lim_{x\to 16} h(x) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



(To the left of $x = 16$, that is actually a (very slightly) curved graph; it's just that $y = \sqrt{x}$ gets pretty flat out there.)

For the limits:
$\lim_{x\to-2} h(x) = \frac{8}{0}$, so we check both sides:

LHL: $\lim_{x\to-2^-} h(x) = \lim_{x\to-2^-} \frac{8}{x+2} = \frac{8}{0^-} = -\infty$

RHL: $\lim_{x\to-2^+} h(x) = \lim_{x\to-2^+} \frac{8}{x+2} = \frac{8}{0^+} = +\infty$

So $\boxed{\lim_{x\to-2} h(x) \text{ DNE}}$ since RHL $\neq$ LHL.

$\lim_{x\to0} h(x)$: we check both sides:

LHL: $\lim_{x\to0^-} h(x) = \lim_{x\to0^-} \frac{8}{x+2} = \frac{8}{2} = 4$

RHL: $\lim_{x\to0^+} h(x) = \lim_{x\to0^+} \frac{1}{2}x - 4 \overset{\text{DSP}}{=} -4$

So $\boxed{\lim_{x\to0} h(x) \text{ DNE}}$ since RHL $\neq$ LHL.

$\lim_{x\to16} h(x)$: we check both sides:

LHL: $\lim_{x\to16^-} h(x) = \lim_{x\to16^-} \frac{1}{2}x - 4 \overset{\text{DSP}}{=} 8 - 4 = 4$

RHL: $\lim_{x\to16^+} h(x) = \lim_{x\to16^+} \sqrt{x} \overset{\text{DSP}}{=} \sqrt{16} = 4$

So $\boxed{\lim_{x\to16} h(x) = 4}$ since RHL = LHL. And since $h(16) = 4$ also, that means h is continuous at $x = 16$.

The various pieces of the function are continuous away from the breakpoints and the point $x = -2$ where the first piece is undefined.
So the only places where $h(x)$ is discontinuous are at $\boxed{x = 0, 16}$