1. Denote by N the list of axioms for number theory on page 68 our our textbook. Denote by $\mathcal{H}$ the *Henkin structure* of N (as defined in class). Refamiliarize yourself with the notation in that section if necessary, especially the "canonical term" $\overline{n}$. You may freely use anything proved in that section, e.g. Lemma 2.8.4, without proving it again.

   (a) Prove that for every variable-free term t in $\mathcal{L}_{NT}$, there exists a canonical term $\overline{n}$ such that $N \vdash t = \overline{n}$.

   (b) Prove that the universe of $\mathcal{H}$ may be identified with the set $\mathbb{N}$ of natural numbers.

   (c) Prove that for any sentence σ, $\mathcal{N} \models \sigma$ if and only if $N \vdash \sigma$. Here $\mathcal{N}$ denotes the standard structure on $\mathcal{L}_{NT}$. So the Henkin construction faithfully captures the "usual" semantics of number theory (remember that this is by no means guaranteed!).

   Hint In part (a), use induction on term structure. Part (b) will follow quickly, once you check that all canonical terms give distinct equivalence classes. In part (c), use induction on complexity of the sentence.

2. (Generalization of constants) Prove the following facts, which we will make use of to (but not prove) in class.

   (a) Suppose that $(\phi_1, \phi_2, \cdots, \phi_n)$ is a deduction in a first-order language $\mathcal{L}$ from a set Σ of sentences. Suppose that c is a constant symbol that does appear anywhere in Σ, and that u is a variable that does not appear anywhere in the deduction. Prove that

   $$((\phi_1)^c_u, (\phi_2)^c_u, \cdots, (\phi_n)^c_u)$$

   is also a valid deduction from the same set Σ. Note that we officially only defined the notation ϕ^x_t for substitution of *variables* x, but we can define it exactly the same way for constants; simply regard every occurence of c as "free."

   Hint Induct on n. Consider three cases: the last formula could belong to Σ, it could be an axiom, or it could result from a rule of inference.

   (b) Deduce from (a) that if $\Sigma \vdash \phi$, and c is a constant that does not occur anywhere in Σ, then there exists a variable u (that does not occur anywhere in ϕ) such that $\Sigma \vdash \forall u(\phi^c_u)$. Furthermore, there is a deduction of this sentence from Σ in which c does not appear anywhere. (This fact is usually called "generalization of constants.")

   (c) Suppose that c is a constant symbol that does not occur anywhere in Σ nor, in formulas ϕ, ψ. Assume also that $\Sigma, \phi^u_c \vdash \psi$ for some variable u. Prove that

   $$\Gamma, \exists u(\phi) \vdash \psi.$$

   The traditional terminology for this fact is the "existential instantiaion" rule, or "rule EI."

   Hint Some of the facts above are mentioned in the book's proof of Completeness. It is fine to follow the proof as it is written in the book in those cases, but be sure to carefully specify any details left out in the book's exposition.

3. Suppose that Σ_1, Σ_2 are two sets of sentences such that no structure is a model for both Σ_1 and Σ_2. Prove that there exists a sentence ϕ such that $\mathcal{A} \models \phi$ for any model $\mathcal{A}$ of Σ_1, and $\mathcal{A} \models \neg\phi$ for any model $\mathcal{A}$ of Σ_2. You may assume the completeness and compactness theorems in your proof (the compactness theorem will be stated in class soon, or you can look it up in the book).