Goal
Reference: §

Goal: Exploring Improper Integrals, for both Type I (unbounded domain) and Type II (unbounded range). We will need IBP, Complete the Square, Partial Fractions, and some u-sub here. We may also need L'Hôpital's Rule to finish a few of the limits at hand.

Examples to study first

Example Evaluate $\displaystyle\int_{-\infty}^{7} \frac{1}{x^2-6x+25}\,dx$.

Solution

$$\begin{aligned}
\int_{-\infty}^{7} \frac{1}{x^2-6x+25}\,dx &= \lim_{t\to-\infty}\int_{t}^{7} \frac{1}{x^2-6x+25}\,dx \overset{\substack{\text{complete}\\\text{square}}}{=} \lim_{t\to-\infty}\int_{t}^{7} \frac{1}{(x-3)^2+16}\,dx \\
&= \lim_{t\to-\infty}\int_{t-3}^{4} \frac{1}{u^2+16}\,du \qquad \boxed{\begin{aligned} u &= x-3 \\ du &= dx \end{aligned}} \\
&= \lim_{t\to-\infty} \frac{1}{4}\arctan\left(\frac{u}{4}\right)\Bigg|_{t-3}^{4} = \lim_{t\to-\infty}\frac{1}{4}\left(\arctan\left(\frac{4}{4}\right) - \arctan\left(\frac{t-3}{4}\right)\right) \\
&= \lim_{t\to-\infty}\frac{1}{4}\left(\arctan(1) - \arctan\overset{\nearrow -\infty}{\left(\frac{t-3}{4}\right)}\right) = \frac{1}{4}\left(\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{4}\left(\frac{3\pi}{4}\right) \\
&= \boxed{\frac{3\pi}{16}}
\end{aligned}$$

Example Evaluate $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$.

Solution This is a Type 2 improper integral.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{s\to 0^+}\int_s^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{s\to 0^+}\int_s^1 (\ln x)x^{-\frac{1}{2}}\,dx$$

$$\stackrel{IBP}{=} \lim_{s\to 0^+} 2\sqrt{x}\ln x\Big|_s^1 - 2\int_s^1 \frac{1}{\sqrt{x}}\,dx \qquad \boxed{\begin{array}{ll} u=\ln x & dv = x^{-\frac{1}{2}}dx \\ du = \frac{1}{x}\,dx & v = 2\sqrt{x}\end{array}}$$

$$= \lim_{s\to 0^+} 2\sqrt{x}\ln x\Big|_s^1 - 4\sqrt{x}\Big|_s^1 = \lim_{s\to 0^+} 2\overset{0}{\cancel{\ln 1}} - 4\overset{1}{\cancel{\sqrt{1}}} - \left(2\sqrt{s}\ln s^{0\cdot(-\infty)} - 4\overset{0}{\cancel{\sqrt{s}}}\right)$$

$$\stackrel{*}{=} \boxed{-4}$$

* L'H Finish: $\displaystyle\lim_{s\to 0^+}\sqrt{s}\ln s \stackrel{0\cdot(-\infty)}{=} \lim_{s\to 0^+}\frac{\ln s}{\frac{1}{\sqrt{s}}}^{\frac{-\infty}{\infty}} \stackrel{\text{L'H}}{=} \lim_{s\to 0^+}\frac{\frac{1}{s}}{-\frac{1}{2s^{\frac{3}{2}}}} = \lim_{s\to 0^+} -2\sqrt{s} = 0$

Example Evaluate $\displaystyle\int_0^6 \frac{8}{x^2-4x-12}\,dx$.

Solution

$$\int_0^6 \frac{8}{x^2-4x-12}\,dx = \int_0^6 \frac{8}{(x-6)(x+2)}\,dx = \lim_{t\to 6^-}\int_0^t \frac{8}{(x-6)(x+2)}\,dx$$

$$\stackrel{\text{PFD}}{=} \lim_{t\to 6^-}\int_0^t \frac{1}{x-6} - \frac{1}{x+2}\,dx = \lim_{t\to 6^-} \ln|x-6| - \ln|x+2|\Big|_0^t$$

$$= \lim_{t\to 6^-} \overset{-\infty}{\cancel{\ln|\overset{0^+}{\cancel{t-6}}|}} - \ln|t+2| - (\ln|-6| - \ln 2)$$

$$= -\infty - \ln 8 - \ln 6 + \ln 2 = \boxed{-\infty} \quad \text{(the integral diverges)}$$

PFD algebra:
$$\frac{8}{(x-6)(x+2)} = \frac{A}{x-6} + \frac{B}{x+2}$$
Clearing the denominator yields:
$8 = A(x+2) + B(x-6) = Ax + 2A + Bx - 6B = (A+B)x + (2A-6B)$
so that $A+B=0$ and $2A-6B=8$.
Solving gives $B=-A$ and $2A+6A=8,$ hence $A=1$ and $B=-1$.

Problems to hand in

Compute each of the following Integrals. Simplify when possible.

1. $\displaystyle\int_{-\infty}^{0} \frac{1}{3-4x}\, dx$
2. $\displaystyle\int_{1}^{\infty} \frac{1}{(2x+1)^3}\, dx$
3. $\displaystyle\int_{2}^{\infty} \frac{x}{e^{3x}}\, dx$
4. $\displaystyle\int_{e}^{\infty} \frac{\ln x}{x^3}\, dx$
5. $\displaystyle\int_{e}^{\infty} \frac{1}{x(\ln x)^2}\, dx$
6. $\displaystyle\int_{e}^{\infty} \frac{1}{x \ln x}\, dx$
7. $\displaystyle\int_{-\infty}^{7} \frac{1}{x^2-4x+29}\, dx$
8. $\displaystyle\int_{0}^{5} \frac{6}{x^2-4x-5}\, dx$
9. $\displaystyle\int_{0}^{e^5} \frac{1}{x\left[25+(\ln x)^2\right]} dx$
10. $\displaystyle\int_{1}^{2} \frac{1}{x \ln x}\, dx$
11. $\displaystyle\int_{0}^{1} x \ln x\, dx$