Solutions to Practice Final B

1. Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 2} \frac{(x+1)^2-9}{x^2+4}$ (b) $\displaystyle\lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+g(x+1)}$, where $g(x)=x^2+7$.

(c) $\displaystyle\lim_{x\to 8} \frac{8-x}{\sqrt{x+1}-3}$ (d) $\displaystyle\lim_{x\to 6} \frac{x^2-4x-12}{|6-x|}$

Solutions. (a): $\displaystyle\lim_{x\to 2} \frac{(x+1)^2-9}{x^2+4} \overset{\text{DSP}}{=} \frac{0}{4+4} = \boxed{0}$

(b): $$\lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+g(x+1)} = \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+g(x+1)} = \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+(x+1)^2+7}$$
$$= \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+(x^2+2x+1+7)} = \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+(x^2+2x+8)} = \lim_{x\to 3^-} \frac{x^2-8x+15}{x^2-6x+9}$$
$$= \lim_{x\to 3^-} \frac{(x-5)(x-3)}{(x-3)(x-3)} = \lim_{x\to 3^-} \frac{x-5}{x-3} = \frac{-2}{0^-} = \boxed{+\infty}$$

(c): $$\lim_{x\to 8} \frac{8-x}{\sqrt{x+1}-3} = \lim_{x\to 8} \frac{8-x}{\sqrt{x+1}-3} \cdot \frac{\sqrt{x+1}+3}{\sqrt{x+1}+3} = \lim_{x\to 8} \frac{(8-x)(\sqrt{x+1}+3)}{(x+1)-9}$$
$$= \lim_{x\to 8} \frac{-(x-8)(\sqrt{x+1}+3)}{x-8} = \lim_{x\to 8} -(\sqrt{x+1}+3) \overset{\text{DSP}}{=} -(\sqrt{9}+3) = \boxed{-6}$$

(d): Because $|6-x|$ is piecewise, we look at both sides:

LHL: $\displaystyle\lim_{x\to 6^-} \frac{x^2-4x-12}{|6-x|} = \lim_{x\to 6^-} \frac{x^2-4x-12}{6-x} = \lim_{x\to 6^-} \frac{(x-6)(x+2)}{-(x-6)} = \lim_{x\to 6^-} -(x+2) \overset{\text{DSP}}{=} -8$

RHL: $\displaystyle\lim_{x\to 6^+} \frac{x^2-4x-12}{|6-x|} = \lim_{x\to 6^+} \frac{x^2-4x-12}{x-6} = \lim_{x\to 6^+} \frac{(x-6)(x+2)}{x-6} = \lim_{x\to 6^+} (x+2) \overset{\text{DSP}}{=} 8$

Since LHL$\neq$RHL, $\boxed{\text{the original limit DNE}}$

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2. Compute each of the following derivatives. Simplify numerical answers. Do not simplify your algebraically complicated answers.

(a) $f'\left(\frac{\pi}{12}\right)$, where $f(x)=\sec^2(2x)+\sin(4x)$ (b) $\displaystyle\frac{d}{dx} \ln\left(\frac{(x^2+1)^{4/7}\, e^{\tan x}}{\sqrt{1+\sqrt{x}}}\right)$

(c) $g'(x)$, where $g(x)=e^{\sqrt{x^2+7\cos x}}+\dfrac{1}{\sqrt{e^{x^2+7\cos x}}}$ (d) $\dfrac{dy}{dx}$, if $e^{xy^3}+\sin^3 x=\ln(xy)+\sin(e^9)$.

Solutions. (a): $f'(x)=2\sec(2x)\sec(2x)\tan(2x)2+4\cos(4x)$, so

$$f'\left(\frac{\pi}{12}\right) = 4\sec\left(\frac{2\pi}{12}\right)\sec\left(\frac{2\pi}{12}\right)\tan\left(\frac{2\pi}{12}\right)+4\cos\left(\frac{4\pi}{12}\right) = 4\sec^2\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{6}\right)+4\cos\left(\frac{\pi}{3}\right)$$

$$= 4 \cdot \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \frac{1}{\sqrt{3}} + 4\left(\frac{1}{2}\right) = \boxed{\frac{16}{3\sqrt{3}} + 2}$$

[Note: we used $\sec x = \dfrac{1}{\cos x}$ and $\sec\dfrac{\pi}{6} = \dfrac{1}{\cos\frac{\pi}{6}} = \dfrac{1}{\sqrt{3}/2} = \dfrac{2}{\sqrt{3}}$ in this computation.]

(b): $$\frac{d}{dx}\ \ln\left(\frac{(x^2+1)^{4/7}\ e^{\tan x}}{\sqrt{1+\sqrt{x}}}\right) = \frac{d}{dx}\left[\ln\left((x^2+1)^{4/7}\right) + \ln e^{\tan x} - \ln\sqrt{1+\sqrt{x}}\right]$$
$$= \frac{d}{dx}\left[\frac{4}{7}\ln(x^2+1) + \tan x - \frac{1}{2}\ln(1+\sqrt{x})\right]$$
$$= \frac{4}{7}\left(\frac{1}{x^2+1}\right)\cdot 2x + \sec^2 x - \frac{1}{2}\left(\frac{1}{1+\sqrt{x}}\right)\cdot\left(\frac{1}{2\sqrt{x}}\right) = \boxed{\frac{8x}{7(x^2+1)} + \sec^2 x - \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}}$$

(c): We have $g(x) = e^{\sqrt{x^2+7\cos x}} + e^{-(x^2+7\cos x)/2}$, so

$$g'(x) = e^{\sqrt{x^2+7\cos x}}\frac{1}{2}(x^2+7\cos x)^{-1/2}(2x-7\sin x) + e^{-(x^2+7\cos x)/2}\cdot\left(-\frac{1}{2}\right)(2x-7\sin x)$$
$$= \boxed{\frac{1}{2}(2x-7\sin x)\left[(x^2+7\cos x)^{-1/2}e^{\sqrt{x^2+7\cos x}} - e^{-(x^2+7\cos x)/2}\right]}$$

(d): Applying implicit differentiation:

$$e^{xy^3}\left(x3y^2\frac{dy}{dx} + y^3\right) + 3\sin^2 x\cos x = \frac{1}{xy}\left(x\frac{dy}{dx} + y\right) + 0$$

$$3x^2y^3e^{xy^3}\frac{dy}{dx} + xy^4e^{xy^3} + 3xy\sin^2 x\cos x = x\frac{dy}{dx} + y$$

$$3x^2y^3e^{xy^3}\frac{dy}{dx} - x\frac{dy}{dx} = y - xy^4e^{xy^3} - 3xy\sin^2 x\cos x$$

$$\left(3x^2y^3e^{xy^3} - x\right)\frac{dy}{dx} = y - xy^4e^{xy^3} - 3xy\sin^2 x\cos x$$

$$\boxed{\frac{dy}{dx} = \frac{y - xy^4e^{xy^3} - 3xy\sin^2 x\cos x}{3x^2y^3e^{xy^3} - x}}$$

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3. Compute each of the following integrals.

(a) $\displaystyle\int_{\pi/18}^{\pi/9}\tan(3x)\ dx$ (b) $\displaystyle\int\frac{\left(x^{7/2}+1\right)^2}{\sqrt{x}}\ dx$ (c) $\displaystyle\int_e^{e^4}\frac{3}{x\sqrt{\ln x}}\ dx$ (d) $\displaystyle\int\frac{1}{x^2\ e^{1/x}}\ dx$

Solutions. (a): $\displaystyle\int_{\pi/18}^{\pi/9}\tan(3x)\ dx = \int_{\pi/18}^{\pi/9}\frac{\sin(3x)}{\cos(3x)}\ dx$ [$u = \cos(3x)$, $du = -3\sin(3x)\,dx$,

$\sin(3x)\,dx = -\dfrac{1}{3}\,du$; $x = \dfrac{\pi}{18} \Rightarrow u = \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$, $x = \dfrac{\pi}{9} \Rightarrow u = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$]

$$= -\frac{1}{3}\int_{\sqrt{3}/2}^{1/2}\frac{du}{u} = \ln|u|\Big|_{\sqrt{3}/2}^{1/2} = -\frac{1}{3}\left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)\right) = -\frac{1}{3}\ln\left(\frac{1/2}{\sqrt{3}/2}\right) = -\frac{1}{3}\ln\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{3}\ln\left(\sqrt{3}\right) = \boxed{\frac{1}{6}\ln(3)}$$

(b): $\displaystyle\int \frac{\left(x^{7/2}+1\right)^2}{\sqrt{x}}\,dx = \int x^{-1/2}\left(x^7+2x^{7/2}+1\right)dx = \int x^{13/2}+2x^3+x^{-1/2}\,dx$

$\displaystyle = \boxed{\frac{2}{15}x^{15/2}+\frac{1}{2}x^4+2x^{1/2}+C}$

(c): $\displaystyle\int_e^{e^4} \frac{3}{x\sqrt{\ln x}}\,dx \qquad [u=\ln x,\ du=\frac{1}{x}\,dx;\ x=e \Rightarrow u=1,\ x=4 \Rightarrow u=4]$

$\displaystyle = 3\int_1^4 \frac{1}{\sqrt{u}}\,du = 3\int_1^4 u^{-\frac{1}{2}}\,du = 6\sqrt{u}\Big|_1^4 = 6\left(\sqrt{4}-\sqrt{1}\right) = 6(2-1) = \boxed{6}$

(d): $\displaystyle\int \frac{1}{x^2\ e^{1/x}}\,dx \qquad [u=\frac{1}{x},\ du=-\frac{1}{x^2}\,dx]$

$\displaystyle = -\int \frac{1}{e^u}\,dx = -\int e^{-u}\,dx = e^{-u}+C = \boxed{e^{-1/x}+C}$ Or if you prefer, $\displaystyle\frac{1}{e^{1/x}}+C$

4. Let $f(x) = \dfrac{x^2+1}{x-3}$. Calculate $f'(x)$ in two different ways:

(a) Using the Quotient Rule.

(b) Using the **limit definition** of the derivative.

Solutions. (a): $\displaystyle f'(x) = \frac{(x-3)(2x)-(x^2+1)(1)}{(x-3)^2} = \frac{2x^2-6x-x^2-1}{(x-3)^2} = \boxed{\frac{x^2-6x-1}{(x-3)^2}}$

(b): $\displaystyle f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{(x+h)^2+1}{(x+h)-3}-\frac{x^2+1}{x-3}}{h}$

$\displaystyle = \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{((x+h)^2+1)(x-3)-(x^2+1)(x+h-3)}{(x+h-3)(x-3)}\right)$

$\displaystyle = \lim_{h\to 0}\frac{(x^2+2xh+h^2+1)(x-3)-(x^2+1)(x+h-3)}{h(x+h-3)(x-3)}$

$\displaystyle = \lim_{h\to 0}\frac{x^3+2x^2h+xh^2+x-3x^2-6xh-3h^2-3-(x^3+x^2h-3x^2+x+h-3)}{h(x+h-3)(x-3)}$

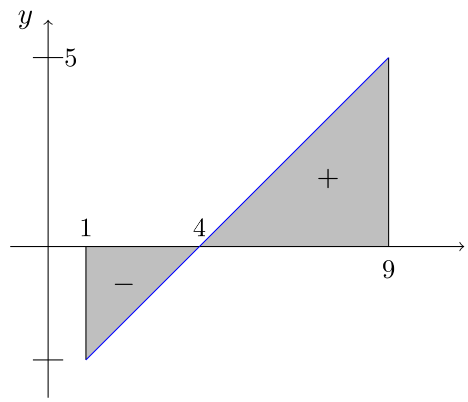
$\displaystyle = \lim_{h\to 0}\frac{x^2h+xh^2-6xh-3h^2-h}{h(x+h-3)(x-3)} = \lim_{h\to 0}\frac{h(x^2+xh-6x-3h-1)}{h(x+h-3)(x-3)}$

$\displaystyle = \lim_{h\to 0}\frac{x^2+xh-6x-3h-1}{(x+h-3)(x-3)} \overset{\text{DSP}}{=} \boxed{\frac{x^2-6x-1}{(x-3)^2}}$

5. Compute $\displaystyle\int_1^9 x-4\,dx$ using each of the following **three** different methods:

(a) Area interpretations of the definite integral,
(b) Fundamental Theorem of Calculus,
(c) Riemann Sums and the limit definition of the definite integral.

Solutions. Here's a sketch:



(a) Area Above x-axis $= \frac{1}{2}$(base) (height) $= \frac{1}{2}(5)(5) = \frac{25}{2}$

Area Below x-axis $= \frac{1}{2}$(base) (height) $= \frac{1}{2}(3)(3) = \frac{9}{2}$

$\int_1^9 x - 4\, dx = \frac{25}{2} - \frac{9}{2} = \frac{16}{2} = \boxed{8}$

(b): $\int_1^9 x - 4\, dx = \frac{x^2}{2} - 4x\Big|_1^9 = \left(\frac{81}{2} - 36\right) - (\frac{1}{2} - 4) = \frac{80}{2} - 32 = 40 - 32 = \boxed{8}$

(c): $\Delta x = \frac{9-1}{n} = \frac{8}{n}$ and $x_i = 1 + i\Delta x = 1 + \frac{8i}{n}$. So:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(1 + \frac{8i}{n}\right)\frac{8}{n} = \sum_{i=1}^{n}\left(1 + \frac{8i}{n} - 4\right)\frac{8}{n} = \frac{64}{n^2}\sum_{i=1}^{n} i - \frac{24}{n}\sum_{i=1}^{n} 1$$

$$= \frac{64}{n^2}\cdot\left(\frac{n(n+1)}{2}\right) - \frac{24}{n}\cdot n = 32\left(1 + \frac{1}{n}\right) - 24$$

Thus, $\int_1^9 x - 4\, dx = \lim_{n\to\infty} 32\left(1 + \frac{1}{n}\right) - 24 \overset{\text{DSP}}{=} 32(1+0) - 24 = \boxed{8}$

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6. Find the equation of the tangent line to $y = \cos(\ln(x+1)) + \ln(\cos x) + e^{\sin x} + \sin(e^x - 1)$ at the point where $x = 0$.

Solution. We have $y' = -\sin(\ln(x+1))\left(\frac{1}{x+1}\right) + \frac{1}{\cos x}(-\sin x) + e^{\sin x}\cos x + \cos(e^x - 1)e^x$

So $y'(0) = -\sin(\ln(0+1))\left(\frac{1}{0+1}\right) + \frac{1}{\cos 0}(-\sin 0) + e^{\sin 0}\cos 0 + \cos(e^0 - 1)e^0$

$= 0 + 0 + 1 + 1 = 2.$ So the tangent line has slope 2.

And it goes through the point $(0, y(0)) = (0, 2)$, because
$y(0) = \cos(\ln(0+1)) + \ln(\cos 0) + e^{\sin 0} + \sin(e^0 - 1) = \cos 0 + \ln 1 + e^0 + \sin 0 = 1+0+1+0 = 2$

Thus, the tangent line has equation $y - 2 = 2(x - 0)$, i.e., $\boxed{y = 2x + 2}$

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7. Let $\boldsymbol{f(x) = \frac{x}{e^x} = xe^{-x}}$.

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.
Take my word that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Solution. **Domain and V.A.**: $f(x)$ has domain $(-\infty, \infty)$ so No Vertical Asymptotes.

H.A.: Because $\lim_{x\to\infty} f(x) = 0$, we have a horizontal asymptotes at $y = 0$ on the right. (But no H.A. on the left, because $\lim_{x\to-\infty} f(x) = -\infty$.)

First Derivative Information:
$f'(x) = xe^{-x}(-1) + e^{-x} = -(x-1)e^{-x}$, which is always defined.

Solving $f' = 0$ gives $x = 1$ as the only critical number. The f' chart is

x	$(-\infty, 1)$	$(1, \infty)$
$f'(x)$	$+$	$-$
$f(x)$	$\nearrow$	$\searrow$

Therefore, f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$ with local max at $x = 1$.

Second Derivative Information:
$f''(x) = e^{-x}(-1) + (-x+1)e^{-x}(-1) = e^{-x}(-1 + x - 1) = (x-2)e^{-x}$, which is always defined.

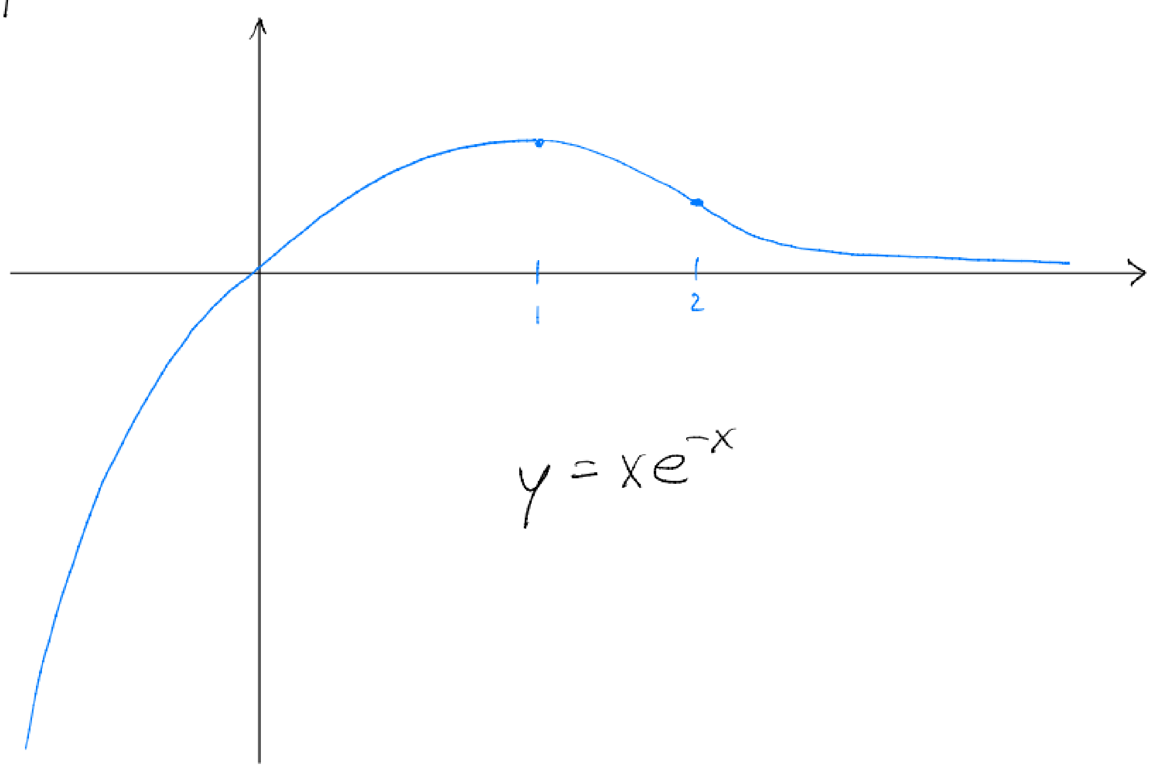
Solving $f'' = 0$ gives $x = 2$. So our f'' chart is:

x	$(-\infty, 2)$	$2, \infty)$
$f''(x)$	$-$	$+$
$f(x)$	$\cap$	$\cup$

Therefore, f is concave down on $(-\infty, 2)$, and f is concave up on $(2, \infty)$, with an inflection point at $x = 2$.

Here is the resulting sketch:

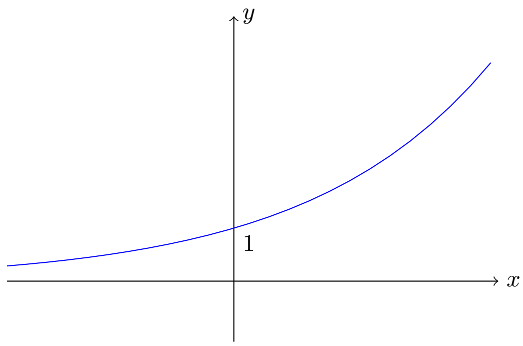
#7

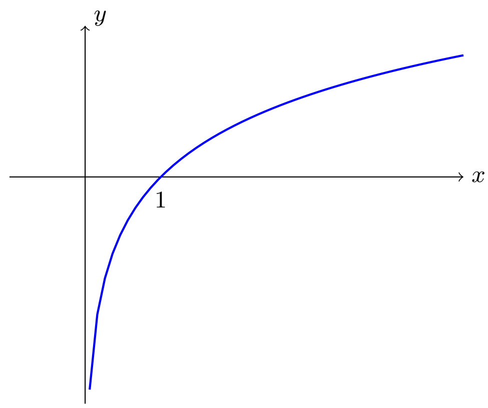


8.
(a) State the definition of **continuity** of $f(x)$ **at** $x = a$.
(b) Carefully sketch the graphs of $y = e^x$ and $y = \ln x$.
(c) Use the definition of continuity from part (a) to find which k-value makes $f(x)$ continuous at $x = 1$ for $f(x) = \begin{cases} e^x & \text{if } x \leq 1 \\ \ln x + k & \text{if } x > 1 \end{cases}$
(d) Using your k value from part (c) above, sketch $f(x)$. Is this piece-wise defined function $f(x)$ continuous on $(-\infty, \infty)$? Explain.

Solutions. (a): To say f is continuous at $x = a$ means that
$\boxed{\lim_{x \to a} f(x) = f(a)}$

(b): Here are $y = e^x$ and $y = \ln x$, respectively:





(c): For $\lim_{x \to 1} f(x)$ to exist, we need

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1).$$

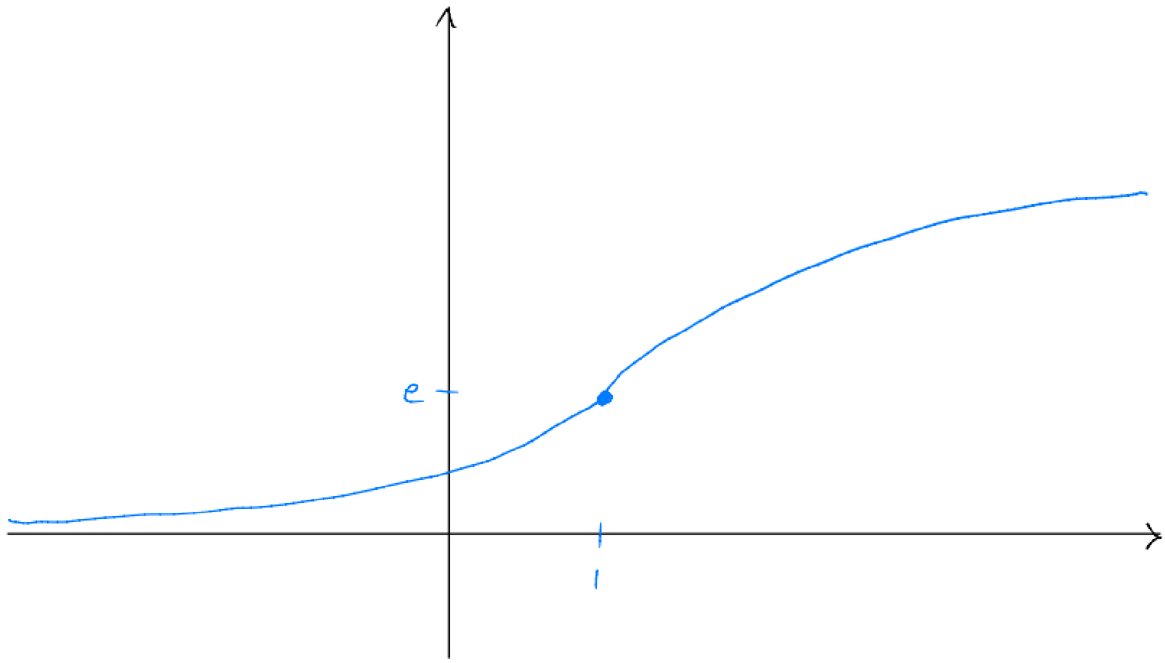
The RHL and LHL are:

RHL: $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \ln x + k \overset{\text{DSP}}{=} \ln 1 + k = k$
and
LHL: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} e^x = e$

So we need $k = e$, which gives $\lim_{x \to 1} f(x) = e$. Since we also have $f(1) = e^1 = e$, the choice of $k = e$ gives $\lim_{x \to 1} f(x) = f(1)$, making f continuous at $x = 1$.

(d): Here is a sketch:

9(d)



We have that $f(x)$ is continuous at all $x < 1$ because on that interval, we have $f(x) = e^x$, which is continuous.

We also have that $f(x)$ is continuous at all $x > 1$ because on that interval, we have $f(x) = \ln x$, which is continuous on $(0, \infty)$ and hence on the smaller interval $(1, \infty)$.

Finally, by part (c), f is continuous at $x = 1$. so $\boxed{\text{yes, } f \text{ is continuous on } (-\infty, \infty)}$

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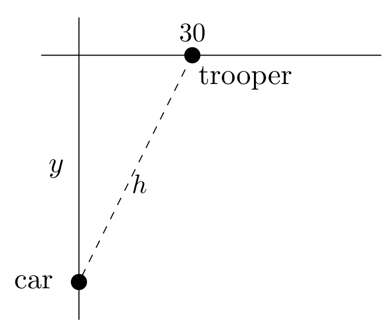
9. A state trooper is parked 30 meters east of a road that runs north-south. He spots a speeding car and (using his radar gun) determines that the car's distance to him is decreasing at 32 meters per second at the moment when the car is at a point 50 meters from him. (That is, 50 meters along the diagonal from him to the car.) How fast is the car actually going at that moment?

Solution. Here's the **Picture**:



Variables:
y =North-South distance from car to point on road, in m
h =diagonal distance from trooper to car, in m

(And t =time, in sec)

Main $\boxed{\textbf{Equation:}}$ $y^2 + 30^2 = h^2$

$\boxed{\textbf{Differentiate}}$ (implicitly, w.r.t. time): $2y\dfrac{dy}{dt} = 2h\dfrac{dh}{dt}$

$\boxed{\textbf{Use key moment info:}}$

At the key moment, we have $h = 50$ m and we are told that $\dfrac{dh}{dt} = -32\text{m/sec}$

Also, plugging $h = 50$ into the original equation gives $y^2 + 30^2 = 50^2$,
i.e., $y^2 = 2500 - 900$, i.e., $y^2 = 1600$, so $y = \pm 40$. Must be $y = 40\,\text{m}$.

Plugging these values into the derivative equation above,
we have $2(40)\dfrac{dy}{dt} = 2(50)(-32)$, i.e., $\dfrac{dy}{dt} = -\dfrac{50 \cdot 32}{40} = -5 \cdot 8 = -40\text{m/s}$,
which means that y is decreasing at 40 m/s, i.e., the car is going $\boxed{40\,\text{m/sec}}$

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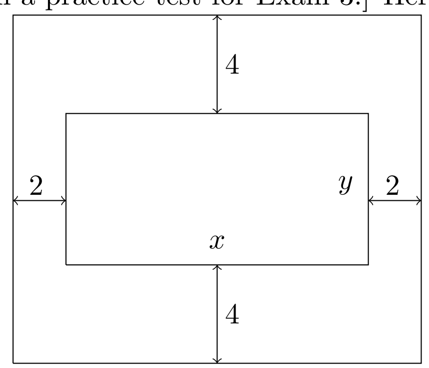
10. A rectangular poster is to contain $50\,\text{in}^2$ of printed matter with margins of 4 inches at each of the top and bottom, and margins of 2 inches on each side. What are the height and width of the poster fitting those requirements that has the smallest possible area?

Solution. [Yes, this was also on a practice test for Exam 3.] Here's the diagram:



The printed matter inside the poster is a rectangle; call this smaller rectangle's width x and its height y. Taking the margins into account, the full poster has width $x + 4$ and height $y + 8$.

So the printed area is $50 = xy$, which means $y = 50/x$.

Meanwhile, the full poster has area $(x + 4)(y + 8) = (x + 4)(50/x + 8) = 8x + 82 + 200/x$.
We have $x > 0$ and $y > 0$, which gives just $x > 0$. [Note that $x = 0$ is impossible to get $xy = 50$. And $y > 0$ gives only $50/x > 0$, which is the same as $x > 0$.]

So we must minimize $f(x) = 8x + 82 + 200x^{-1}$ on $(0, \infty)$.

We compute $f'(x) = 8 - 200x^{-2}2$, which is always defined on the original domain.

Setting $f'(x) = 0$ gives $8x^2 = 200$, so $x^2 = 25$, and so $x = \pm 5$; but $-5 \notin (0, \infty)$, meaning that the only critical point is $x = 5$. Our f' chart is:

x	$(0, 5)$	$(5, \infty)$
$f'(x)$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$

So by FDTAE, f has an absolute minimum at $x = 5\,\text{in}$. That gives $y = 50/5 = 10\,\text{in}$.

So the best poster therefore has width $x + 4 = 9\,\text{in}$, and height $y + 8 = 18\,\text{in}$.

That is, the best poster is $\boxed{9\,\text{in wide by } 18\,\text{in high}}$

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11. Find the absolute maximum and minimum values of the function

$$g(x) = 3x - x\ln x$$

on the interval $[1, e^4]$.

Solution. We have $g'(x) = 3 - \ln x - x \cdot x^{-1} = 2 - \ln x$, which is defined everywhere on the interval $[1, e^4]$.

Solving $g'(x) = 0$ gives $\ln x = 2$, so $x = e^2$, which is in the interval.

Using CIM, testing this critical number and the endpoints gives

$$g(1) = 3, \qquad g(e^2) = 3e^2 - e^2 \cdot 2 = e^2, \qquad g(e^4) = 3e^4 - e^4 \cdot 4 = -e^4.$$

Note that $e^2 > 2^2 = 4$, so:
$\boxed{\text{the maximum value is } e^2\text{, and the minimum is } -e^4}$

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12. Consider an object moving on the number line such that its velocity at time t seconds is $\boldsymbol{v(t) = 4 - t^2}$ feet per second. Also assume that the position of the object at one second is $\frac{5}{3}$.

(a) Compute the acceleration function $a(t)$.

(b) Compute the position function $s(t)$.

Solutions. (a): $a(t) = v'(t) = \boxed{-2t}$

(b): $s(t)$ is an antiderivative of $v(t)$, so $s(t) = 4t - \frac{t^3}{3} + C$ for some constant C.

We have $s(1) = \frac{5}{3}$, so $\frac{5}{3} = s(1) = 4 - \frac{1}{3} + C$, so $C = \frac{5}{3} + \frac{1}{3} - 4 = -2$.

Thus, $s(t) = \boxed{4t - \frac{t^3}{3} - 2}$