Applications of Linear Algebra in Sports Rankings

Math 22a Final Project

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Cover Letter

We, Nicholes Lopez and Elliot Chin, collaborated on this project, and thank Aris Zhu, Aneesa Roidad, and Caleb Miller for their feedback on the rough draft. Furthermore, we thank Caleb Miller and Dusty Grundmeier for answering questions on Massey matrices throughout the duration of reading and finals period.

Several pieces of feedback were especially helpful as we developed our final draft. At the request of our peer editors, we added a great deal of additional explanation for common sports terms, such as strength of schedule. We thank the peer editors for pointing out where our paper lacked clarity; because we were familiar with the terms, we weren't sure which ones warranted more explanation.

We also moved much of our work to theorem boxes to increase clarity throughout the paper, and added additional definitions for terms we had not previously defined. Especially for Massey matrices, we added a good deal of space and segmentation throughout the section to increase clarity.

Perhaps the largest edit we made was cutting a section on the Perron-Frobenius Theorem and its related ranking methods. We thank our peer editors for helping us recognize that the scope of the paper was already large enough, and that the additional proofs were unnecessary.

We also added an entirely new section on customizable team rankings with Massey matrices. This did not arise out of peer editing, but simply was a very cool application of the topics covered in our paper that we realized during reading period. The subsequently created web app makes the paper much more engaging to interact with and read.

Ultimately, work was divided equitably between both of us. Nicholas Lopez worked on Massey matrices, the comparison between ranking methods, the development of the Massey matrix web app, and general LaTeX formatting throughout the document. Elliot Chin worked on Colley matrices, developing the mathematical model of multi-variable Massey matrices, implementing feedback, the introduction and abstract, and applying Massey matrices to special teams. All other work was shared equally.

ABSTRACT

Ranking sports teams is important for coaches and players to make on and off the field decisions; commentators and entertainers to provide accurate perspectives and opinions; and fans to understand the competitive landscape of their chosen sports To accurately rank sports teams, large quantities of data must be processed and analyzed. Here, we describe two ranking methods rooted in linear algebra, which possess the ability to easily synthesize large data sets into comprehensible team ratings. Relevant background, theorems, and definitions are included with the typical MATH 22a student as the target audience. We test the two ranking methods against literature rankings, and present two novel applications of the Massey matrix system. First, we create an interactive model that allows the ranking of NFL teams based on customizable combinations of differential statistics. Second, we propose the use of the Massey matrix system for ranking NFL punt and punt return teams.

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1 Introduction

Team rankings and ratings have long been a fundamental aspect of sports culture, commentary, and strategy. On the field, coaches want to know the skill of their opponent, as it informs how to approach a given game. Off the field, a detailed model of the competitive landscape of the game yields strategic advantages for coaches in predicting postseason outcomes. For fans, team rankings are a source of endless debate and entertainment - after all some would say that the entire point of playing a game is to decide which team is better. TV personalities and sports talk shows make endless profits off of provocative opinions on which teams are best, and which teams can't keep up. And a whole industry - sports betting - is predicated on participants ability, or lack thereof, of formulating opinions about sports teams' relative strength.

Ranking and rating sports teams is thus an enormously important industry and field. However, this importance comes with vast complexity. Sports involves huge quantities of numerical data, involving results from hundreds if not thousands of matches or games. To process these large amounts of data, linear algebra is extremely useful. Such linear algebra-based mathematical methods come as part of a 21st-century shift in sports towards analytics and statistics. These methods extend beyond previous qualitative methods - such as the "eye test," which relies upon first-person visual observation. They also improve upon previous quantitative methods - such as basing ratings simply on win loss records - which were severely limited in accuracy until recent decades.

Linear algebra methods to rank sports teams tend to be more effective than simply ranking sports teams by win loss record because they can incorporate strength of schedule. Strength of schedule in sports refers to the skill of the opponents played. For example, in college football, teams frequently go undefeated during the regular season but miss playoffs because they did not play against difficult competition. Schools that play in strong divisions and schedule tough out of conference games, however, fare better even when they lose one or two games. By using matrices to take into account the full schedule of *who* a team played against, rather than the result of those games, we can gain a more accurate perspective of a given sports landscape.

Here, we investigate two matrix rating methods to investigate the question of how sports teams with different opponents can be accurately rated and ranked. Then, we apply these methods to the NFL to rate teams based on the results of the ongoing NFL season. We compare these results with literature and provide a way for readers to create their own team rankings based on what statistics they believe to be most important. Finally, we create novel rankings of NFL punt and punt return teams which, to our knowledge, are more comprehensive than current literature.

The first matrix rating method we cover is called the Colley matrix system, and is useful in its simplicity. By consider only binary game outcomes (ie. only wins and losses, with the exception of ties), the method remains rooted in its probabilistic origins. The Colley system, however, misses out on other potentially important information such as score differentials. The second matrix rating method we present incorporates these differentials under the assumption that a strong team will win by more than a weak team (and further, that this difference scales linearly with rating). This system is called the Massey matrix system, and incorporates the final score of games, rather than binary outcomes. Furthermore, although the Massey matrix system was designed to be used strictly with score differential, it can be used to rank teams based on any differential variable - that is, any variable where the sum of the variable across all teams after a game is zero. For example, the Massey matrix system can be used with turnover differential, because if the winner of a game had a +2 turnover differential, then the loser had a -2 turnover differential. We take advantage of this ability to construct customizable NFL team rankings.

Lastly, we seek to contribute a completely novel method of assessing, rating, and ranking special teams. The final section of this paper proposes a Massey system based model for ranking NFL teams' proficiency at punting and returning punts. Current special teams rankings have two deficiencies. First, they don't consider the strength of opponents; for a ranking model, returning a punt against a good punting team should hold more weight than returning a punt against a bad punting team. Second, they focus on singular stats, such as punt length of average return yards, which ignore factors such as field position, time on the clock, and blocked or fumbled punts. Our model uses aggregate differential statistics and Massey matrices to overcome these shortcomings of current special teams ranking methods.

1.1 Colley Matrices

An Overview Colley matrices constitute a ranking system originally developed for college football during the onset of computer ranking systems. It has remained relevant since due to its unbiased nature, simplification of game outcomes, concern for strength of schedule, and general output of reasonable results. Here, we will first summarize the Colley matrix system for non-sports fans; then provide an overview of the math, which boils down to fairly simple, yet interesting, linear algebra; finally, we will discuss the more complicated topic of Cholesky decomposition, a method to accelerate solving Colley matrices.

Rating sports teams is universally important across nearly every sport. Whether for determining postseason rankings or predicting future matchups, a standardized method of rating teams is frequently necessary. Colley matrices are one such team-rating method, developed for college football. In college football, teams do not play a representative sample of the competition, making determining their true rating difficult. In other words, it is hard to compare two teams that may play opponents of wildly different skill. This is compounded by the intrinsic randomness of any sport. Thus, accounting for strength of schedule (the average strength of a team's opponents) is key: a team that plays many good teams and wins is likely better than a team that plays many bad teams and wins. At the same time, however, strength of schedule is often overvalued in qualitative analysis, as evidenced by the frequently criticized college football playoff selection committee. An effective rating system must offer reasonable estimates that take into account opponent strength, but do not weight it too heavily.

The Colley matrix system, as presented in the next section, seeks to do exactly that. It combines win loss records with a measure of strength of schedule to produce a system of equations; solving this system gives team ratings.

Calculating Colley Matrices We make two underlying assumptions to inform the Colley matrix system. First, absent other information, sports teams ratings are random. Second, absent other information, a team that beats another team is the better team and has a higher rating. Observe the exceptions to these assumptions: if we have a lot of data on a sports team, we will rate it less randomly, and may be forced to produce contradictory scenarios where a worse team beats a better team. The assumptions, however, are useful in that they are simplifying, and allow the Colley matrix system to be easy to understand and implement. Greater complexity will arise when we consider the linear algebra implications of easier implementation.

To further simplify, let us set bounds for the ratings of a given sports team. Teams will be rated from 0 (absolute worst) to 1 (absolute best). Per our assumption, a random team that we know nothing about is assumed to have a random rating: its average rating is thus $\frac{1}{2}$.

Now consider two teams, both of which we have no information about and thus consider them as having random ratings per our first assumption. Now, suppose team 1 beats team 2. All this tells us is that the rating of team 1 is higher than the rating of team 2, per our second assumption. What, then, should we expect the rating of team 1 to be?

This simple question can be solved in many ways: with a probability density function, integrals, symmetry, geometrically. We will solve it with symmetry. Both teams started with an average value of $\frac{1}{2}$, and wins and losses should theoretically be symmetric, so the rating of team 1, r_1 and of team 2 r_2 should sum to 1. Additionally, because r_1 is randomly distributed but greater than r_2, its expected value should be halfway between its max (1) and min (r_2). So $r_1 + r_2 = 1$ and $r_1 = (1 - r_2)/2$. This gives us that $r_1 = \frac{2}{3}$ and $r_2 = \frac{1}{3}$.

This simple result helps develop an intuition to how rating emerge organically from game results. To calculate r_1 after team 1 has beaten $n-1$ teams, simply calculate

$$r_1 = \int_0^1 \cdots \int_0^1 \max(x_1, ..., x_n) dx_n ... dx_1$$

This corresponds to the rating of team 1 because per assumption 2, we assume that team 1 is the best team out of n teams. Further, per assumption 1, we assume that all n teams have randomly distributed ratings. Taking the multiple integral of the max function of the rating of each team gives the average maximum rating of the n teams across all possible ratings of each team. Similar integrals can be used to find a team that has one loss and $n-2$ wins, and so on. For example, such a team would have a rating of

$$\int_0^1 \cdots \int_0^1 \max2(x_1, ..., x_n) dx_n ... dx_1$$

where max2($x_1, ..., x_n$) is the second-highest value out of $x_1, ..., x_n$. The generalized formula is as follows, and is left as an exercise to the reader.

Theorem 1: Definition of Rating

The rating of team i can be expressed as $r_i = \frac{1+w}{2+g}$ where w is wins and g is games.

We now are able to proceed to rating calculations which involve strength of schedule, the novel portion of Colley matrices (compared to simple win loss metrics).[1] We require a few key observations. First,

$$w = \frac{w-l}{2} + \frac{g}{2}$$

with w as wins, l as losses, and g as games. The reader may easily verify that this is true. Second (also verifiable via the definition of addition),

$$\frac{g}{2} = \underbrace{\frac{1}{2} + ... + \frac{1}{2}}_{g \text{ times}}$$

[1] the following derivation is from Colley, no date

Finally, each of the above values of $\frac{1}{2}$ are just the expected value of the rating of random opponents. What if, we instead replace them with the expected value of known opponents? Going back to our original example, this is analogous to having team 1 beat team 2, but knowing some history of team 2. This gives us a new equation for wins for a team that has played teams $t_1, ..., t_j$:

$$w = \frac{w-l}{2} + r_1 + ... + r_j$$

Substituting this new definition of wins into the rating definition gives the rating equation

Theorem 2: Definition of Strength of Schedule Adjusted Rating

The strength of schedule adjusted rating of team i can be expressed as $r_i = \frac{1+\frac{w-l}{2}+r_1+...+r_j}{2+g}$

Let us consider the two-team situation from above:

$$r_1 = \frac{1+.5+.5}{3} = \frac{2}{3}$$

$$r_2 = \frac{1-.5+.5}{3} = \frac{1}{3}$$

This result makes sense, and matches our previous probability analysis. However, we have yet to take into account strength of schedule. Now that we know r_1 and r_2, we can implement them into our above equations. Sports fans reading may notice a discrepancy: it feels odd to take into account strength of schedule after a single game, when the strength of schedule is determined by the result of that game. However, this is the smallest possible situation in which Colley matrices can be applied. Taking into account rating is necessary as we apply Colley matrices to teams who have played multiple, different opponents.

An even more important reason why we must recalculate ratings using strength of schedule is that our old ratings no longer make mathematical sense. We have, based on our strength of schedule adjusted ratings, that $\frac{1.5+r_2}{3} = \frac{1.833}{3}$ However, this doesn't equal the ratings we just calculated. $\frac{2}{3}$. Instead, let's recalculate rankings, incorporating the ratings of $\frac{1}{3}$ and $\frac{2}{3}$

into the strength of schedule adjusted ratings:

$$r_1 = \frac{1 + .5 + .333}{3} = \frac{11}{18}$$

$$r_2 = \frac{1 - .5 + .666}{3} = \frac{7}{18}$$

Recalculating again gives

$$r_1 = \frac{1 + .5 + .388}{3} = \frac{17}{27}$$

$$r_2 = \frac{1 - .5 + .611}{3} = \frac{10}{27}$$

These ratings will eventually converge, as evidenced by the decreasing difference, which alternates in sign, between subsequent iterations of recalculation. However, it is difficult to see what the ratings converge to. To do so, we will construct a system of equations. These equations are specific examples of the strength of schedule adjusted ratings.

$$r_1 = \frac{1.5 + r_2}{3}$$

$$r_2 = \frac{.5 + r_1}{3}$$

Which rearranges to

$$3r_1 - r_2 = \frac{3}{2}$$

$$3r_2 - r_2 = \frac{1}{2}$$

The result is that $r_1 = \frac{5}{8}$ and $r_2 = \frac{3}{8}$. We abstain from showing a matrix equation due to the simplicity of the above equation.

Now that we have a fundamental understanding of Colley matrices, let's examine a larger example where matrices are helpful to represent the large system of equations.

First, we would like to well define the matrix we would like to solve.

Theorem 3: Definition of a Colley Matrix

Consider teams $t_1, ..., t_k$, where t_i, with $1 \leq i \leq k$, has played n_i games against opposing teams. Further, w_i and l_i refer to the number of wins and losses of t_i. The resulting Colley Matrix equation is

$$\begin{bmatrix} 2+n_1 & a_{12} & a_{13} & \dots & a_{1k} \\ a_{21} & 2+n_2 & a_{23} & \dots & a_{2k} \\ \vdots & & \ddots & & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \dots & 2+n_k \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix} 1+\frac{w_1-l_1}{2} \\ 1+\frac{w_2-l_2}{2} \\ \vdots \\ 1+\frac{w_k-l_k}{2} \end{bmatrix}$$

With $a_{pq} = \begin{cases} 0 & \text{if } t_p \text{ did not play } t_q; \\ -x & \text{if } t_p \text{ played } t_q \text{ x times.} \end{cases}$

Proof of Theorem 3. We have from above that, if t_i has played against $t_1, ..., t_j$, then $r_i = \frac{1+\frac{w_i-l_i}{2}+r_1+...+r_j}{2+n_i}$.

Manipulating yields $r_i(2+n_i) - r_1 - ... - r_j = 1 + \frac{w_i-l_i}{2}$ for $1 \leq i \leq k$

We now have a system of equations with coefficient $(2+n_i)$ corresponding to r_i and -1 corresponding to each team that t_i has played once. If t_i played a team multiple times, the coefficient -1 would be multiplied by the number of times then played. $1 + \frac{w_i-l_i}{2}$ is the corresponding constant for the equation.

We have a similar equation for every $1 \leq i \leq k$. Thus, we have k variables and k equations, and can situate the equations in a matrix as shown in Theorem 1. ■

As an example, we will rank NFL teams[2]. Our Colley matrix equation is of the form $Ax = b$. Matrix A, x, and b can be found in Appendix A

Solving: Cholesky Decomposition The above linear equation of $Ax = b$ is tough to solve. A is a 32×32 matrix, which is quite sizable. A computer could crunch the numbers

[2] This is real data accurate through Week 11 of the 2021 NFL Season. Source: Stathead

easily, but finding and understanding an optimized solution will be useful. This is even more important when Colley matrices are applied to other team sports, such as college football, which has over a hundred teams. That's a 10,000 entry matrix!

To optimize the solving process, we will use Cholesky decomposition.[3] Cholesky decomposition is a decomposition of a matrix A into LL^*, where L is a lower triangular matrix with real, positive entries on the diagonal. L^* refers to the complex conjugate of L, where $L^*_{ij} = \overline{L_{ji}}$. Because we will deal with real matrices in this paper, $L^* = L^T$ for our purposes. Cholesky decomposition then allows $Ax = b$ to be solved easily because triangular matrices can be solved simply. Cholesky decomposition, however, requires A to be hermitian and positive-definite.

Theorem 4: Definition of Hermitian

$n \times n$ matrix A is hermitian if any element $a_{ij} = \overline{a_{ij}}$. If A is real, this is equivalent to A being symmetric; that is $a_{ij} = a_{ji}$.

Theorem 5: Definition of Positive-Definite

A, a real $n \times n$ matrix, is positive-definite if $x^T Ax > 0$ for all nonzero $x \in \mathbb{R}^n$.

We start by proving that A, a Colley matrix, satisfies this definition.

Theorem 6: Colley matrices are symmetric and positive definite

If A is an $n \times n$ Colley matrix as defined in Theorem 1, then $A = A^T$ and $x^T Ax > 0$ for all nonzero $x \in \mathbb{R}^n$

Proof of Theorem 6. First we will prove symmetry; that is, that $A = A^T$. It is evident from the example above that A is symmetric, but we seek to verify that this is true for all $n \times n$ Colley matrices. Every value a_{pq} of A with $p \neq q$ is equal to -1 multiplied by the number

[3]The following derivation is inspired by Horn and Johnson, 2013, pg. 441

of times t_p played t_q. Because this final operation is commutative; that is, t_p plays t_q the same number of times that t_q plays t_p, $a_{pq} = a_{qp}$. So $A = A^T$.

Next we will prove that A is positive-definite. For any $x \in \mathbb{R}^n$, x represents the vector of ratings of sports teams. We seek to show that $x^T Ax > 0$ for all nonzero $x \in \mathbb{R}^n$. Observe first that the diagonal of A is composed of $2 + n_i$ for $1 \leq i \leq k$. We can thus decompose A as

$$A = 2I_k + \begin{bmatrix} n_1 & a_{12} & a_{13} & \dots & a_{1k} \\ a_{21} & n_2 & a_{23} & \dots & a_{2k} \\ \vdots & & \ddots & & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \dots & n_k \end{bmatrix}$$

With $a_{pq} = \begin{cases} 0 & \text{if } t_p \text{ did not play } t_q; \\ -x & \text{if } t_p \text{ played } t_q \text{ x times.} \end{cases}$

Note that $A - 2I_k$ can further be decomposed by observing that it is the sum of many individual $k \times k$ matrices that each represent an individual game. That is, if c games were or have been played in a season, then

$$A - 2I_k = \Sigma G_{rs}$$

where G_{rs} represents the matrix of a single game between t_r and t_s.

$$G_{rs} = \begin{bmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & -1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & -1 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{bmatrix}$$

In other words, for element g_{ij} of G_{rs} with $1 \le i \le k$ and $1 \le j \le k$, we have that

$$g_{ij} = \begin{cases} 1 & \text{if } i = j = r \text{ or } i = j = s; \\ -1 & \text{if } i = r, j = s \text{ or } i = s, j = s \\ 0 & \text{if } i \neq 0 \text{ and } j \neq 0. \end{cases}$$

We now return to the equation for positive definiteness

$$x^T Ax = x^T(2I_k + \Sigma G_{rs})x$$

$$= x^T(2I_k)x + \Sigma x^T G_{rs} x$$

The left side simplifies easily:

$$x^T(2I_k)x = x^T 2x = 2x^T x > 0 \text{ because } x \neq 0$$

Now consider $x^T G_{rs} x$ for any $1 \le r \le k$ and $1 \le s \le k$. x is a column vector with indexed entries. Then,

$$G_{rs}x = \begin{bmatrix} 0 \\ \vdots \\ x_r - x_s \\ \vdots \\ x_s - x_r \\ 0 \end{bmatrix}$$

where $(G_{rs}x)_r = x_r - x_j$ and $(G_{rs}x)_s = x_s - x_r$. This follows from the structure of G_{rs}, which is primarily made up of zeros. Then,

$$x^T(G_{rs}x) = x_r(x_r - x_s) + x_s(x_s - x_r) = x_r^2 + x_s^2 - 2x_r x_s = (x_r - x_s)^2 \ge 0$$

Thus, $x^T Ax$ is equal to the sum of a positive number ($x^T 2I_k x$)) and multiple nonnegative numbers ($\Sigma x^T G_{rs} x$) and is thus positive. Thus, A is symmetric and positive definite. ■

Our next step is to prove that A can be represented as LL^T with L as a lower triangular matrix.

Theorem 7: Cholesky Decomposition Works on Colley Matrices

A symmetric, positive-definite matrix $k \times k$ A can be written as LL^T with L as a $k \times k$ lower triangular matrix.

Proof of Theorem 7. First, because A is symmetric, the Real Spectral Theorem states that A has a diagonal matrix D, with respect to a orthonormal eigenbasis of A. Furthermore, the values along the diagonal of D will be positive per Lay 407. There thus exists a matrix $D^{\frac{1}{2}}$ where the values along the diagonal of $D^{\frac{1}{2}}$ are the square roots of the values along the diagonal of D. This can be conceptualized as the "square root" of D.

Let P be the change of basis matrix from the orthonormal eigenbasis that corresponds to D and the standard basis. $P^{-1} = P^T$ because the eigenbasis is orthonormal. So

$$A = PDP^{-1} = PD^{\frac{1}{2}}D^{\frac{1}{2}}P^{-1} = PD^{\frac{1}{2}}D^{\frac{1}{2}}P^T = PD^{\frac{1}{2}}D^{\frac{1}{2}^T}P^T = PD^{\frac{1}{2}}(PD^{\frac{1}{2}})^T$$

Let us designate $B = PD^{\frac{1}{2}}$. Thus, $A = BB^T$. Observe that B is a diagonal matrix written in a different basis, and thus must have linearly independent columns. The same must be true of B^T, because the row space has the same dimension as the columns space. Thus, we can use QR decomposition (from Lay 359): $B^T = QR$ where the columns of Q form an orthonormal basis for $\mathbb{R}^k$ and R is a $k \times k$ uppertriangular matrix. Then, $B = (QR)^T = R^T Q^T$. Further, because Q has orthonormal columns, $Q^T Q = I_k$ by Lay 345. Thus,

$$A = BB^T = R^T Q^T QR = R^T R$$

R is upper triangular, so R^T must be lower triangular. Thus, any symmetric positive definite A can be decomposed into LL^T with L as a lower triangular matrix. ■

To conclude this section, let's use Cholesky decomposition to solve our example ranking of NFL teams. Via MATLAB, we can easily perform Cholesky decomposition on our matrix. We get the matrix L in Appendix A.

It is much easier to solve $LL^Tx = b$ than $Ax = b$ because forward and back substitution can be used. The resulting ratings are as follows, and are a table representation of x:

ARI	0.73345565294926762	LAR	0.50120143899441294
GB	0.71749638856521214	MIN	0.49632420641084846
BAL	0.70283853890466153	NO	0.47331390547341828
TB	0.69599908416412093	WSH	0.46676058012687405
KC	0.67938235507907141	IND	0.46172226485116052
LAC	0.67916508424371502	ATL	0.45061204215902617
DAL	0.65917587320239968	PHI	0.44364335193416027
NE	0.62466593496888989	MIA	0.41283511182501215
TEN	0.61212952551483435	NYG	0.4105800218777183
CIN	0.58648312447534257	CHI	0.40920074940753842
LV	0.57077658574205326	CAR	0.40639630744210942
BUF	0.56483968236896398	SEA	0.35148362892807561
DEN	0.52334209014913324	NYJ	0.31397433489361337
SF	0.51436935281697138	HOU	0.21064609275120694
CLE	0.50456498354733559	JAX	0.20594221055249728
PIT	0.5018486010082861	DET	0.11483089467206418

1.2 Massey Matrices

An Overview Massey Matrices are another method developed for use in college football rankings. It is once again vital to stress the need for such a system. In environments such as NCAA Division 1 football, there are over 100 teams, yet the regular season lasts for usually 12 games per team. Thus, it is impossible for each team to play each other or even a representative sample (ie. a significant fraction of the total number) of teams. Massey's Method was introduced in 1997 to determine the rankings at the end of the season and has since played an official role in the selection process. The method is based on the theory of least squares (least squares is an optimization method that minimizes the sum of the squared difference between expected and actual result), using both the number of games played and point differentials [4] across the games. Two important advantages to using Massey Matrices are the automatic incorporation of ties, useful for the NFL and Premier League, for example, and the use of point differentials, an important factor in college football. Often times, point differential can be the deciding factor in the rankings of teams. If two teams, for example, finish with a 11-1 season playing similar teams, the team with the greatest margin of victories overall is almost always the commonly accepted favorite.

Calculating Massey Matrices Massey's Method hinges upon attempting satisfy a single equation for every team and every game.[5]

Theorem 8: Fundamental Equation of Massey's Method

Massey's method ultimate goal is to satisfy the equation $r_i - r_j = y$ as well as possible, where r_i and r_j are the rating (not ranking) values of teams i and j, respectively, and y is the point differential in a given game played between the two teams. The rating value is calculated for each team, and the teams are ordered by these values to obtain the determined rankings.

Consider this is done for each game across a number of teams. With m total games played between n teams, applying the given formula for each game will yield a $m \times n$ matrix, denoted X.

[4] The point differential is the margin of victory between the winner and loser of a game. For example, a game ending with the score 42-28 would have a point differential of 14 for the winner and -14 for the loser.

[5] The following derivation is inspired by Heroux, 2020 and Pilkington, 2015

Theorem 9: Massey's Matrix Equation

Incorporating the fundamental equation of Massey's method to every single team and game yields the $m \times n$ matrix equation $Xr = y$, where $M_{ij} = 1$ if team j won game i, $M_{ij} = -1$ if team j lost game i, r_j is the rating of team j, and y_i is the absolute value of the point differential of game i.

In almost any scenario, there will be more games played than teams considered in the ranking system. (Consider the NCAA FBS regular season, which has over 1500 games across 130 teams.) Thus, our matrix system will have more equations than unknowns, and it will almost certainly have no solution. Thus, the method of least squares is implemented.

To solve for y, we will use the Method of Least Squares. Begin by applying left multiplication of X^T to both sides of our equation, yielding

$$X^T X r = X^T y$$

Let $M = X^T X$ and $p = X^T y$. Observe that p is now the total point differential for each team. We now attempt to solve the more reasonable equation

$$Mr = p$$

M is a square $n \times n$ matrix with the total number of games played by each team along the diagonal. The other matrix entries, M_{ij} for $i \neq j$, are the negative value of the number of games each team i played against team j.

Unfortunately, the matrix M is not invertible.

Proof. First, observe that in a closed system of teams, the total point differential of all games will always add to zero. The victor of each match receives a positive score addend, while the loser receives the negative value addend. Since this continues for each game played throughout those considered, the sum of p is always zero.

Consider p to be a $n \times 1$ matrix. Since $\sum_{i=1}^{n} p_{i1} = 0$, it is given that $-p_{n1} = p_{11} + \cdots + p_{(n-1)1}$. Put simply, the total point differential of one team in a set of games is the negative value of the sum of every other team's point differential for the set of games. Thus, by setting the final, or any single value, to zero, no data is lost.

Now, consider M, an $n \times n$ matrix. We will prove that the sum of each column's entries amounts to 0. Each column is a single team's list of games, where M_{ij} represents the total number of games played by a specific team when $i = j$, and M_{ij} otherwise represents the negative value number of times each team j has played opponent i. (Note $1 \leq i \leq n$ and $1 \leq j \leq n$.) Thus, adding the entries along a single column, you add the total number of total games a team has played with the negative values of each specific opponent played. Thus, utilizing the associative property of addition to generalize to any column of M the sum of the entries along a column can be shown as the following:

$$0 = \text{total number of games} +$$

$$\underbrace{(-\text{games played against opponent } 1 + \cdots + - \text{ games played against opponent } n)}_{-\text{total number of games}}$$

Without loss of generality, the same goes for any other row in the matrix, as all rows and columns follow the same format.

Consider each row in M to be a vector. Denote each row of M as $v_1, \ldots, v_n$. As proven above, the sum of each entry along a column of M sums to 0. This can be denoted as $M_{1i} + \cdots + M_{ni} = 0$ for any $1 \leq i \leq n$. Thus, $v_n = -(v_1 + v_{n-1})$. Therefore, the rows of M are not linearly independent, and Rank $M < n$, so M is not invertible. ■

To make M invertible, we set the final row in the matrix M is changed to a row of all ones, and the final entry of p to zero. That is,

$$M_{ni} = 1 \text{ for } 1 \leq i \leq n$$

$$p_n = 0$$

These new components are denoted $\overline{M}$ and $\overline{p}$, respectively. Now, we have that $\overline{M}$ is invertible, meaning $\overline{M}r = \overline{p}$ has a solution. Additionally, the solution, r, will have an average value of 0. We can solve for r by applying $\overline{M}^{-1}$ to both sides of the equation $\overline{M}r = \overline{p}$ through left multiplication. With our solution, r, isolated, we can order the ratings to obtain the final result, the Massey Rankings.

Theorem 10: $\overline{M}$ is Invertible and Gives Normalized Results

The last row of the matrix M can be changed to a row of 1s, and the final entry of p can be changed to 0 without any effect on the results no matter which team is placed in the last row. This ensures the ratings of all teams sum to zero and ensures that M is invertible.

Proof. First, no data is lost in the replacement. Assume the final row is simply removed from M. Given the number of games each team has played, it is easily to fill in the entries along the final row to account for these "missing" values. Put simply, the non-diagonal entries are the negative sum of the entries in the column above it. In fact, the same goes for the diagonal entry, as the total games played is the negative of the sum of the negative values of each time a team has played each of their opponents.

We will now discuss the significance of the values used in the change. Changing the last row of M to ones and the last value of p to 0 yields the equation $\overline{M}r = \overline{p}$. Thus, this data normalization ensures the sum of all total rating values is 0. This allows for an easy analysis of the results. The power of teams can easily be inferred by how far positive or negative they are. Because the sum of the rating values is 0, the mean of the rating values is also 0. With an always-constant mean rating, further simple analysis can be made once the results are obtained.

Finally, let us show that $\overline{M}r = \overline{p}$ has a solution by showing that $\overline{M}$ is invertible. We will prove this case for $n = 3$; proving larger $\overline{M}$ are invertible is left as an exercise to the reader.

Consider any 3×3 matrix $\overline{M}$. The matrix will be of the structure

$$\begin{bmatrix} -a-b & a & b \\ a & -a-c & c \\ 1 & 1 & 1 \end{bmatrix}$$

We operate under the assumption that a and b are nonnegative integers, and $-a-b \neq 0$ and $-a-c \neq 0$, because if a team has played 0 games they cannot be ranked.

To show that $\overline{M}$ is invertible, we will row reduce. Because $-a-b \neq 0$, we can keep the first row at the top and row reduce to

$$\begin{bmatrix} -a-b & a & b \\ 0 & -a-c-\frac{a^2}{-a-b} & c-\frac{ab}{-a-b} \\ 0 & 1-\frac{a}{-a-b} & 1-\frac{b}{-a-b} \end{bmatrix}$$

Observe that if the bottom right 2×2 submatrix has linearly independent columns, then $\overline{M}$ is invertible, because the 2×2 submatrix can be row reduced.

Multiplying the bottom right 2×2 submatrix by $-a-b$ yields

$$\begin{bmatrix} ab+ac+cb & -ac-cb-ab \\ -2a-b & -2b-a \end{bmatrix}$$

If the left column and right column are linearly independent, then the right column can be expressed as a linear combination of the left column. Observe that the top left entry is equal to the additive inverse of the top right entry. Thus, we multiply the left column by -1. We now have that

$$-ab-ac-cb = -ac-cb-ab$$

and

$$2a+b = -2b-a$$

Thus,

$$3a = -3b$$

So, $a = b = 0$. However, this is not possible because then $-a - b = 0$ and team 1 could be removed from the rankings. Thus, we have reached a contradiction, and the left and right column are linearly independent.

Thus, the bottom right 2×2 submatrix is invertible, so $\overline{M}$ is invertible. ■

An Example We will now apply Massey Matrices to the same subject detailed above, NFL teams through Week 11 of the 2021 regular season. Consider the equation $Xr = y$, where X is a 165×32 matrix, one equation (row) for each team and one column per team (there were 165 games played through the first 11 weeks of the season).[6] The vector y is the score differential of each individual game. X^T is then applied by left multiplication to both sides of the equation, and we now have the equation $Mr = p$. Apply the operation changing the finals rows of M and p, and we have the equation $\overline{M}r = \overline{p}$, whose entries are defined in Appendix A.

We will now solve for the value of r by applying $\overline{M}^{-1}$ to both sides of the equation $\overline{M}r = \overline{p}$ through left multiplication. Calculating matrix multiplication, we solve for the entries of r, defined in Appendix A.

Ordering these values from largest to smallest, we arrive at our final Massey Rankings for the 2021 NFL Season as of Week 11:

1. ARI
2. DAL
3. BUF
4. TB
5. NE
6. KC
7. IND
8. LAR
9. TEN
10. PHI
11. SF
12. GB
13. MIN
14. WSH
15. LAC
16. CIN
17. SEA
18. BAL
19. CAR
20. DEN
21. CLE
22. PIT
23. NO
24. NYG

[6] data from Stathead, 2021

25. LV	27. CHI	29. HOU	31. DET
26. MIA	28. JAX	30. ATL	32. NYJ

These rankings are very fitting for a Week 11 assessment and hold reliable through common perceptions of the strength of NFL teams as of November 30, 2021.

1.3 Comparison between ranking methods

We will now contrast the results of these methods with other strategies commonly used and compare the findings against current standings as of December 9, 2021.

A graph of various ranking methods plotted against the most recent ESPN Power Ranking can be seen in Appendix B. Some important analysis can be made from the data. Compared against the Week 13 ESPN Power Rankings for the 2021 NFL season, the following data from Week 11 is evident:[7]

	Total Difference of Ratings	Accuracy Performance
ESPN Power Rankings	92	91.02%
FiveThirtyEight Rankings	82	91.99%
Colley Matrix Rankings	92	91.02%
Massey Matrix Rankings	136	86.91%

In this table, the total difference of ratings is the sum of the absolute value of the difference between each team's rating for a given method and the ESPN Week 13 Power rankings. Accuracy performance is computed as $1 - \frac{\text{total difference of rankings}}{\text{total possible rankings to assign}}$.
Because there are 32 teams that must be placed in 32 spots, there are 1024 total possible rankings to be assigned.

It is worth noting that the Colley and Massey Methods are purely quantitative and did not directly react to significant injuries and other factors that are not included in the calculations.

[7]data from Stathead, 2021; 538, 2021; and NFL Nation, 2021

Furthermore, the data assumes ESPN's rankings are the most accurate baseline. Thus, it is vital to see that Colley Matrices were able to match ESPN's own prediction system, as judged by ESPN's own basis. Therefore, it is conclusive that both Colley and Massey matrices provide reliable, significant rankings with real-world use. These Week 11 predictions will continue to be compared with data as the season progresses.

2 Applying Linear Algebra to Rank NFL Special Teams

Massey matrices are most useful because of their simplicity and ability to take into account point differentials. This makes the Massey matrix system a prime candidate for ranking NFL special teams. First, special teams analysis is a relatively young field. Unlike ranking NFL teams, which often involves complex analytics and qualitative rankings, there is little current literature that quantifies the abilities of NFL special teams. This means that the field is ripe for new insights, even via relatively simple analytics methods. Additionally, special teams plays rarely have binary outcomes; there exists a spectrum of possibilities which range from very favorable to very unfavorable for each team. Thus, special teams' ability cannot be quantified by a simple win loss record.

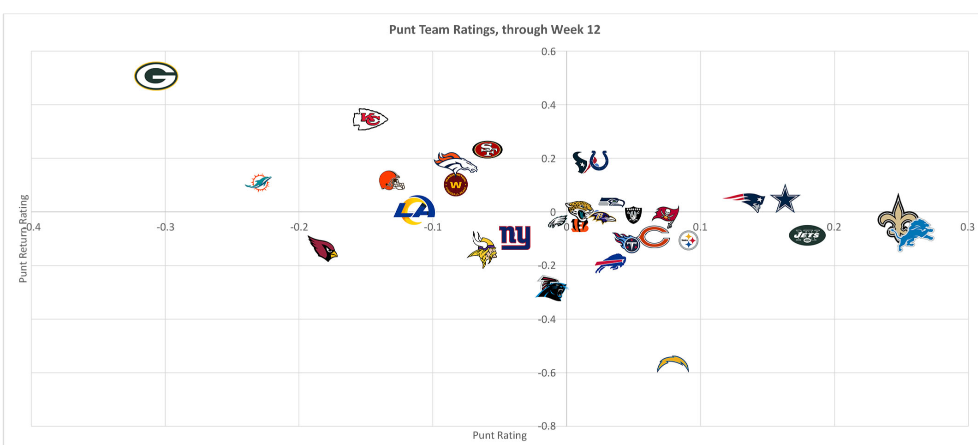
Here, we specifically choose to examine the ability of special teams punt and punt return teams. There exist metrics such as punt return yards and average punt distance which seek to quantify the ability of special teams, but these don't correlate well with skill. For example, a punter that punts 30 yards from the 40 yard line to the 10 yard line would have a statistically short punt, even though such a 30 yard punt would be incredibly effective. Similarly, blocked punts are some of the most effective plays in the game for punt return teams, but don't show readily in statistics in a way that is easily incorporated with return yardage statistics.

Instead, we use an aggregate statistic called expected points added, which quantifies the number of expected points a given play yields a team. For example, if a team is expected to score 21 points in a game, but scores a kick return touchdown on the first play of the game, that kick return would yield them about 5 expected points added (about 2 points are expected on a normal drive, so 7-2=5 is the number of expected points *added*). For further information on the statistic, a great resource can be found here.

Thus, we seek to contribute new analysis to the field of sports rankings by using Massey matrices to rank special teams, where where point differential is replaced by expected points added (EPA) on a given punt. Classifying point differential in this way accounts for the net result a punt play has on a game, whether or not the result is lauded by conventional statistics. Additionally, we modify the statistics in two key ways. First, we separate punt and punt return teams as two separate teams in the rankings, meaning we rank 64 teams in total. Second, we normalize punt EPA such that the average punt is net neutral for both the punting and returning teams. This is because punts, on average, are bad for the punting team because coaches are risk-averse and punt when they should not. We normalize average punt EPA so that punt teams are not penalized for these coaching decisions.

Through week 12, we end up incorporating 1381 punt plays into the Massey matrix model, creating a 1381 by 64 X and following the procedures above to calculate final rankings. An Excel spreadsheet with the step by step calculations is included in the Appendix.[8]

Ultimately, we get the following results from Massey matrices of punt and punt return team ratings. Interestingly, there appears to be a negative correlation between punt and punt return



[8]data from nflfastR, 2021

ranking. We cross reference our ratings with third party ratings to assess the efficacy of our method. The teams that have highest punt ratings correlate with low opponent punt return yards. We rate GB, KC, SF, DEN, and HOU as having the best punt teams; these teams are rated 6, 3, 2, 9, and 27 in opponent punt return yards. The teams that have highest punt return ratings, however, don't correlate very well with high average yards per punt return. We rate DET, NO, NYJ, DAL, and NE as having the best punt return teams; the primary punt returner on these teams rank 11, 12, 5, 18, and 6,

The discrepancy between our ratings and traditional statistics implies a significant value-add by Massey matrices to the field of special teams ratings. These new special teams ratings should be tested further and compared to future NFL results, but have potential applications in betting markets, in advising special teams coordinators, and in informing casual sports fans.

3 Customizable Massey Matrices

Clearly, there are boundless limits for the use of Massey Matrices. To emphasize this perspective and show Massey Matrices in a new light, we created a web application to allow users to create custom NFL rankings. The application uses data[9] across multiple fields, from record to penalty differential. The web application can be found at http://nfl-rank.herokuapp.com/, and the source code can be found at https://github.com/nrlopez03/nfl-rank. It is important to note that was done entirely from scratch. As far as we are aware, this has never been made before in such a fashion. The calculations are made real-time, following Massey's Method exactly. No packages for matrix multiplication, data interpolation, or other shortcuts were used. All math is done entirely using matrices represented as nested lists in Python, and the data is displayed through HTML and JavaScript.

The program allows users to set weightings for such a factor. As a default, the record is set to 100% of the weighting and must always have some non-zero weighting. Users may alter

[9] data from Stathead, 2021

the weightings, which are applied to unique, individual y columns that contain data from every single game played through Week 12 of the NFL season. These y columns are added together to contain the resultant final y column. The rest of the calculations use Massey's Method and the method of least squares to determine the ranking values for the unique parameters.

4 References

Annette Pilkington. Colley's method. - university of notre dame, 2015. URL https://www3.nd.edu/~apilking/Math10170/Information/Lectures%202015/Topic8Colley.pdf.

Amanda Harsy. Predictive modeling and analysis of sports using linear algebra-based models, 2021. URL https://www.lewisu.edu/sure/pdf/2021/SURE_2021_Application1__Harsy_Predictive_Modeling_of_Sports_Ranking_Systems.pdf.

Charles Hepler Pamini Thangarajah and Peter Zizler. Ranking in professional sports: An application of linear algebra for computer science students, 2016. URL https://www.researchgate.net/publication/301793656_Ranking_in_Professional_Sports_An_Application_of_Linear_Algebra_for_Computer_Science_Students.

Hannah Cairns. A short proof of perron's theorem, 2021. URL https://pi.math.cornell.edu/~web6720/Perron-Frobenius_Hannah%20Cairns.pdf.

James P. Keener. The perron-frobenius theorem and the ranking of football teams. *SIAM Review*, 35(1):80–93, 1993. ISSN 00361445. URL http://www.jstor.org/stable/2132526.

Michael Heroux. Massey's method, 2020. URL https://maherou.github.io/Teaching/files/CS317/masseyMethod.pdf.

Mark McClure. Numerical ranking of sports teams, no date. URL https://www.marksmath.org/visualization/eigenbrackets/.

NFL Nation. Nfl power rankings week 14: 1-32 poll, plus players who need to step it up in last month, Dec 2021. URL https://www.espn.com/nfl/story/_/page/NFLpowerranking1207/nfl-power-rankings-week-14-1-32-poll-plus-players-need-step-last-month.

Roger A. Horn and Charles R. Johnson. Matrix analysis, isbn 0521839408, 2013.

Sebastian Carl et al. nflfastr play by play data, 2021. URL https://www.nflfastr.com/.

Sthathead. Stathead football data, 2021. URL https://stathead.com/all/.

Wesley Colley. Colley's bias free college football ranking method, no date. URL https://www.colleyrankings.com/matrate.pdf.

538. 2021 nfl predictions, 2021. URL https://projects.fivethirtyeight.com/2021-nfl-predictions/?ex_cid=rrpromo.

Appendices

Appendix A Matrices

Matrices A, x, and b:

$$
A = \begin{bmatrix}
13 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & -1 & 0 \\
0 & 12 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 12 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\
-1 & -1 & 0 & 0 & 13 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 12 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 12 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & -1 & 13 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 12 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 12 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 12 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 13 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 12 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 13 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -2 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 12 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 13 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 12 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 12 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 13 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 13 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 12 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 12 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 13 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & -1 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 12 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 12 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 12 & 0 & -1 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 13 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 12
\end{bmatrix}
$$

$$x = \begin{bmatrix} ARI \\ ATL \\ BAL \\ BUF \\ CAR \\ CHI \\ CIN \\ CLE \\ DAL \\ DEN \\ DET \\ GB \\ HOU \\ IND \\ JAX \\ KC \\ LV \\ LAR \\ LAC \\ MIA \\ MIN \\ NE \\ NO \\ NYG \\ NYJ \\ PHI \\ PIT \\ SF \\ SEA \\ TB \\ TEN \\ WSH \end{bmatrix}, \text{ and } b = \begin{bmatrix} 4.5 \\ 0.5 \\ 3.5 \\ 2.5 \\ 0 \\ -0.5 \\ 2.5 \\ 1 \\ 2.5 \\ 1.5 \\ -4 \\ 4 \\ -2.5 \\ 1 \\ -2.5 \\ 2.5 \\ 1.5 \\ 1.5 \\ 2.5 \\ 0 \\ 0.5 \\ 3 \\ 0.5 \\ -0.5 \\ -1.5 \\ 0 \\ 1 \\ 1.5 \\ -1 \\ 3.5 \\ 3 \\ 0 \end{bmatrix}$$

Matrix L:

$$L = \begin{bmatrix}
3.6 & 0.0 \\
0.0 & 3.5 & 0.0 \\
0.0 & 0.0 & 3.5 & 0.0 \\
0.0 & 0.0 & 0.0 & 3.5 & 0.0 \\
-0.3 & -0.3 & 0.0 & 0.0 & 3.6 & 0.0 \\
0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 3.5 & 0.0 \\
0.0 & 0.0 & -0.3 & 0.0 & 0.0 & -0.3 & 3.4 & 0.0 \\
-0.3 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.3 & 3.6 & 0.0 \\
0.0 & -0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & 3.4 & 0.0 \\
0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.3 & 3.4 & 0.0 \\
0.0 & 0.0 & -0.3 & 0.0 & 0.0 & -0.3 & -0.3 & -0.3 & 0.0 & -0.1 & 3.4 & 0.0 \\
-0.3 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.3 & -0.1 & 0.0 & 0.0 & -0.4 & 3.6 & 0.0 \\
-0.3 & 0.0 & 0.0 & -0.3 & -0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 3.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -0.3 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & 3.6 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & -0.3 & 0.0 & -0.1 & -0.3 & -0.3 & 3.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -0.3 & -0.3 & 0.0 & 0.0 & 0.0 & -0.3 & -0.3 & -0.1 & -0.1 & -0.3 & -0.1 & -0.1 & -0.1 & 3.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -0.3 & 0.0 & 0.0 & -0.3 & -0.3 & -0.1 & 0.0 & -0.3 & -0.1 & -0.1 & 0.0 & 0.0 & -0.1 & -0.3 & 3.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.3 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.1 & -0.3 & -0.3 & -0.1 & 0.0 & -0.1 & 3.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.3 & -0.1 & -0.1 & 0.0 & 0.0 & 0.0 & 0.0 & -0.4 & -0.4 & 0.0 & 3.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.3 & -0.3 & -0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.4 & -0.4 & -0.1 & -0.4 & -0.1 & -0.1 & 3.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.3 & 0.0 & -0.3 & 0.0 & -0.3 & 0.0 & -0.3 & -0.3 & -0.3 & -0.1 & -0.4 & -0.4 & -0.1 & 0.0 & -0.1 & -0.1 & -0.1 & -0.1 & -0.4 & -0.1 & 3.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & -0.3 & -0.3 & -0.1 & 0.0 & 0.0 & -0.3 & 0.0 & -0.1 & -0.1 & 0.0 & 0.0 & -0.4 & -0.4 & -0.2 & 3.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & -0.1 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.1 & -0.4 & 3.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & -0.3 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.4 & -0.3 & -0.1 & -0.1 & -0.1 & -0.1 & -0.4 & 3.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.3 & 0.0 & -0.3 & -0.3 & 0.0 & -0.3 & 0.0 & -0.1 & -0.3 & 0.0 & 0.0 & -0.1 & -0.3 & -0.1 & 0.0 & -0.1 & 0.0 & 0.0 & -0.4 & -0.1 & -0.7 & -0.1 & -0.2 & 3.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & -0.3 & -0.3 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.4 & 0.0 & -0.4 & -0.1 & -0.2 & -0.2 & -0.4 & -0.3 & -0.2 & 3.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -0.3 & 0.0 & -0.3 & -0.3 & -0.3 & 0.0 & -0.3 & -0.4 & -0.4 & -0.1 & 0.0 & -0.1 & -0.1 & -0.1 & -0.4 & -0.1 & -0.4 & -0.1 & -0.2 & -0.1 & -0.1 & -0.1 & -0.2 & 3.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.6 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & 0.0 & -0.1 & 0.0 & 0.0 & -0.3 & -0.4 & -0.1 & -0.3 & -0.4 & -0.1 & -0.1 & -0.1 & -0.5 & 0.0 & -0.1 & -0.2 & -0.1 & -0.1 & -0.1 & -0.1 & -0.4 & -0.2 & 3.3 & 0.0 & 0.0 & 0.0 \\
-0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & 0.0 & -0.3 & -0.4 & 0.0 & 0.0 & 0.0 & -0.4 & 0.0 & -0.1 & -0.4 & -0.1 & -0.3 & -0.1 & -0.1 & -0.1 & -0.4 & -0.6 & 3.3 & 0.0 & 0.0 \\
0.0 & -0.3 & 0.0 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & 0.0 & -0.3 & -0.1 & -0.4 & -0.4 & -0.5 & -0.2 & -0.5 & -0.2 & -0.2 & -0.2 & 3.3 & 0.0 \\
-0.3 & 0.0 & 0.0 & -0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.6 & -0.4 & -0.3 & 0.0 & -0.4 & -0.1 & -0.2 & -0.1 & -0.1 & -0.3 & -0.1 & -0.5 & -0.2 & -0.2 & -0.3 & -0.6 & -0.3 & 3.3 & 0.0 \\
0.0 & -0.3 & 0.0 & -0.3 & -0.3 & 0.0 & 0.0 & 0.0 & -0.1 & -0.3 & 0.0 & -0.3 & -0.1 & 0.0 & -0.1 & -0.3 & -0.1 & 0.0 & -0.4 & -0.1 & -0.1 & -0.1 & -0.4 & -0.5 & -0.2 & -0.3 & -0.2 & -0.2 & -0.2 & -0.6 & -0.3 & 3.2
\end{bmatrix}$$

Matrices $\overline{M}$, r, and $\overline{p}$:

$$\overline{M} = \begin{bmatrix}
11 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & -1 & 0 \\
0 & 10 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 10 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\
-1 & -1 & 0 & 0 & 11 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 10 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 10 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & -1 & 11 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 10 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 10 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 11 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 11 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -2 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 10 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 11 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 10 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 11 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 11 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 11 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & -1 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 10 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 10 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 10 & 0 & -1 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 11 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

$$r = \begin{bmatrix} ARI \\ ATL \\ BAL \\ BUF \\ CAR \\ CHI \\ CIN \\ CLE \\ DAL \\ DEN \\ DET \\ GB \\ HOU \\ IND \\ JAX \\ KC \\ LV \\ LAR \\ LAC \\ MIA \\ MIN \\ NE \\ NO \\ NYG \\ NYJ \\ PHI \\ PIT \\ SF \\ SEA \\ TB \\ TEN \\ WSH \end{bmatrix}, \text{ and } \overline{p} = \begin{bmatrix} 108 \\ -110 \\ 17 \\ 119 \\ 6 \\ -77 \\ 52 \\ -7 \\ 79 \\ 17 \\ -113 \\ 33 \\ -121 \\ 64 \\ -103 \\ 31 \\ -39 \\ 44 \\ -5 \\ -68 \\ 13 \\ 123 \\ -37 \\ -57 \\ -142 \\ 37 \\ -12 \\ 24 \\ -15 \\ 87 \\ 37 \\ 0 \end{bmatrix}$$

$$r = \begin{bmatrix} ARI \\ ATL \\ BAL \\ BUF \\ CAR \\ CHI \\ CIN \\ CLE \\ DAL \\ DEN \\ DET \\ GB \\ HOU \\ IND \\ JAX \\ KC \\ LV \\ LAR \\ LAC \\ MIA \\ MIN \\ NE \\ NO \\ NYG \\ NYJ \\ PHI \\ PIT \\ SF \\ SEA \\ TB \\ TEN \\ WSH \end{bmatrix} = \begin{bmatrix} 9.991107906 \\ -10.45033504 \\ 0.250572435 \\ 8.785076047 \\ 0.222556081 \\ -7.467843002 \\ 1.145907139 \\ -0.746179675 \\ 9.330448608 \\ -0.100553902 \\ -10.49203983 \\ 3.257646883 \\ -9.793245162 \\ 4.584833878 \\ -8.78469864 \\ 4.903302945 \\ 1.848595716 \\ 4.200668047 \\ -4.850625454 \\ -6.885280598 \\ 2.819427453 \\ 7.96079999 \\ -2.228466256 \\ -4.519514046 \\ -12.50375311 \\ 3.341994453 \\ -2.023435681 \\ 0.385659262 \\ 3.300843687 \\ 8.278212425 \\ 3.738665345 \\ 2.499652086 \end{bmatrix}$$

Massey Matrices for special teams.

Appendix B Figures

The following section contains figures referenced throughout the project. As with all data and calculations used in the project, high-resolution copies and the underlying data can be provided upon request.

Week 11 Predictions plotted against ESPN Week 13 Power Rankings, 2021 NFL Season: