Reinforcement Learning

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Machine Learning

Outline

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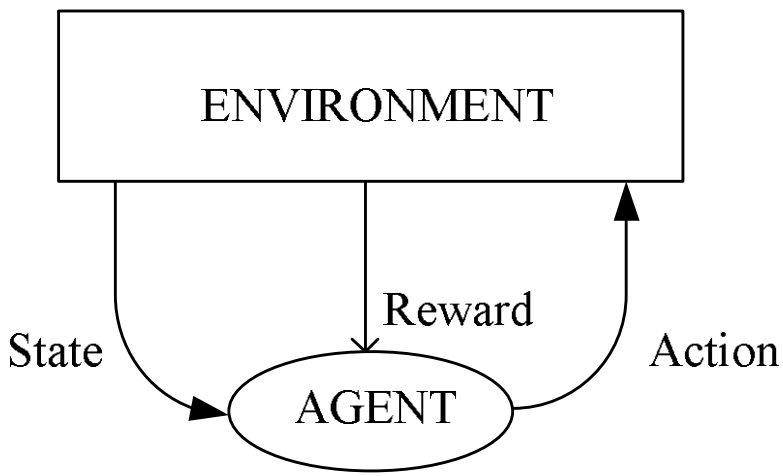
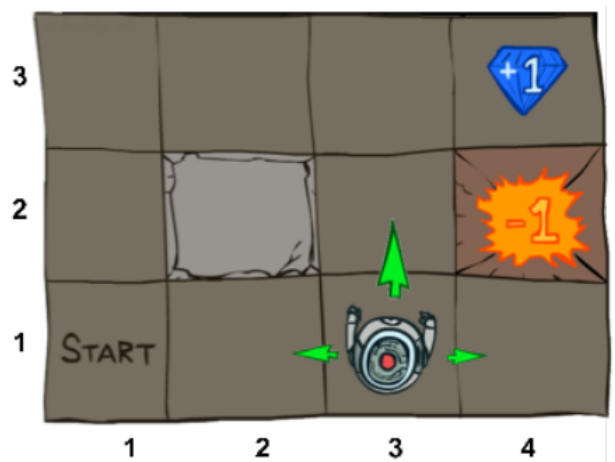
Reinforcement Learning I

- An agent sees ***states*** $s^{(t)}$'s of an environment, takes ***actions*** $a^{(t)}$'s, and receives ***rewards*** $R^{(t)}$'s (or penalties)





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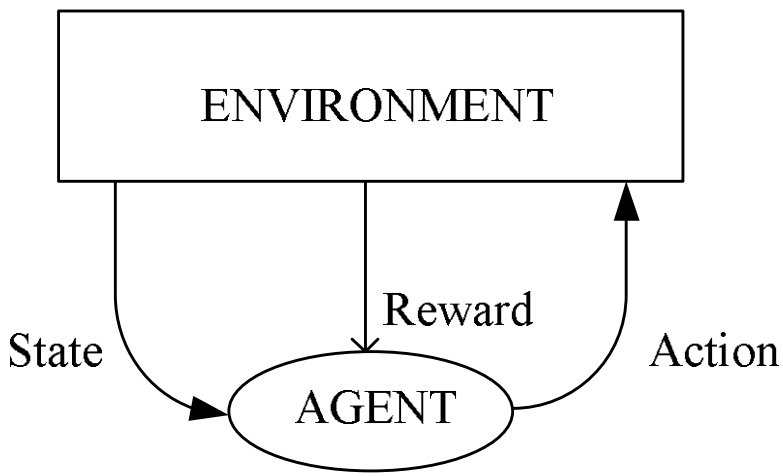

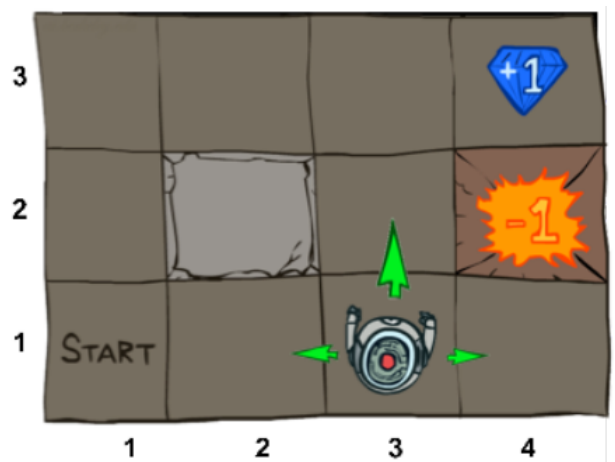

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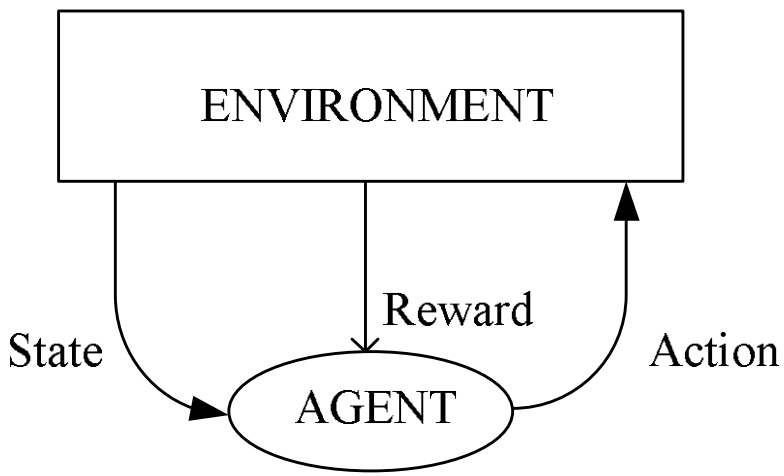

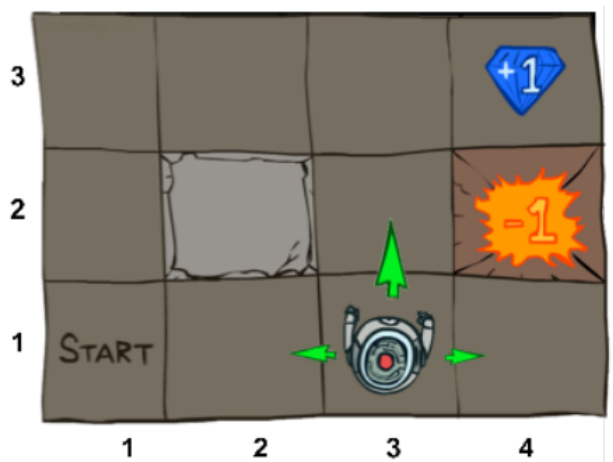

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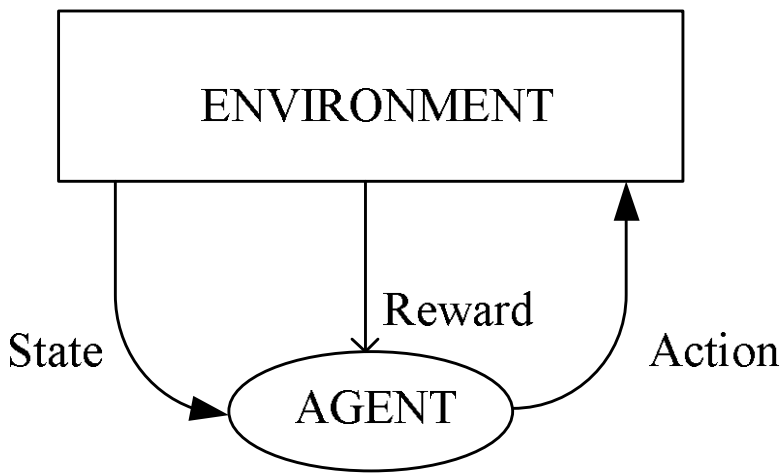
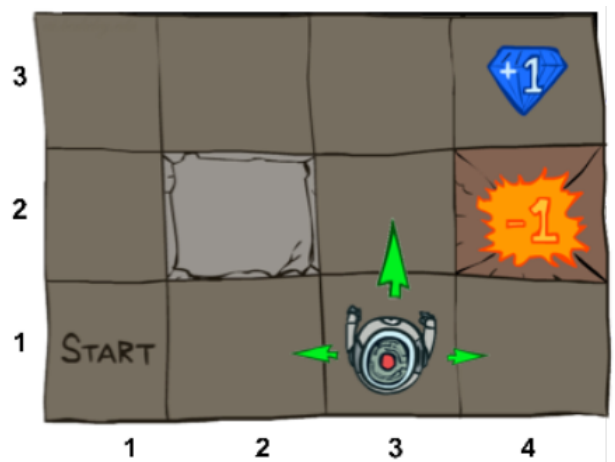
  - Reward $R^{(t)}$ depends on $(s^{(t)}, a^{(t)}, s^{(t+1)})$





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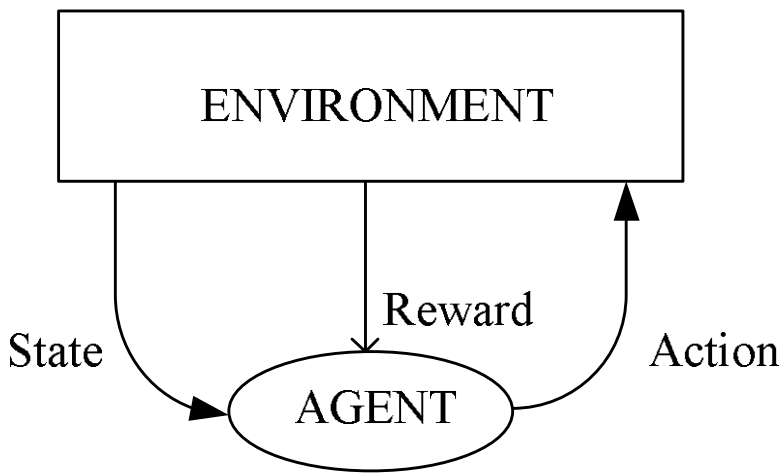
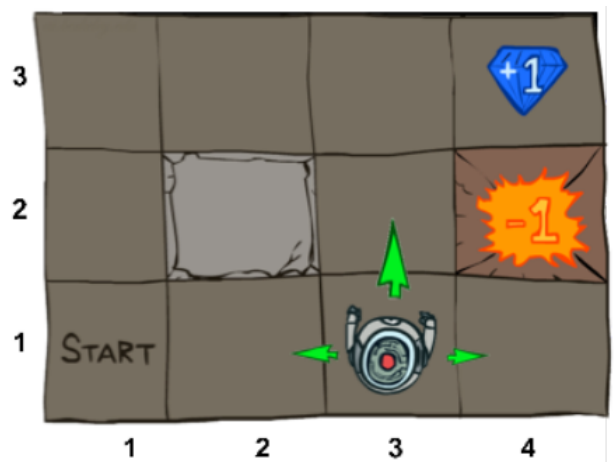
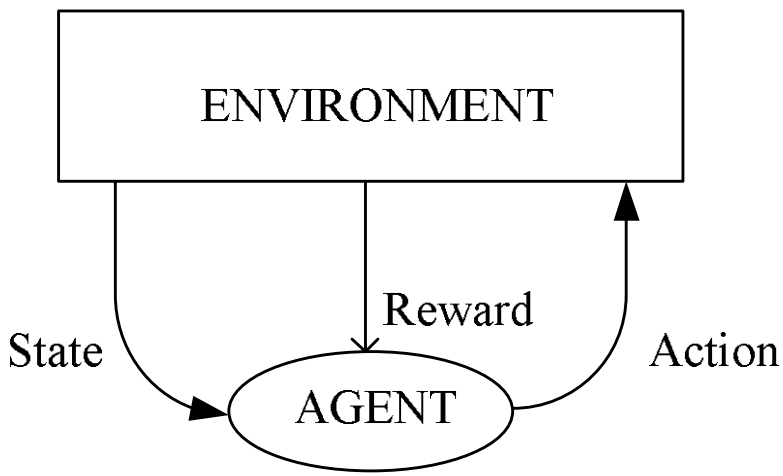
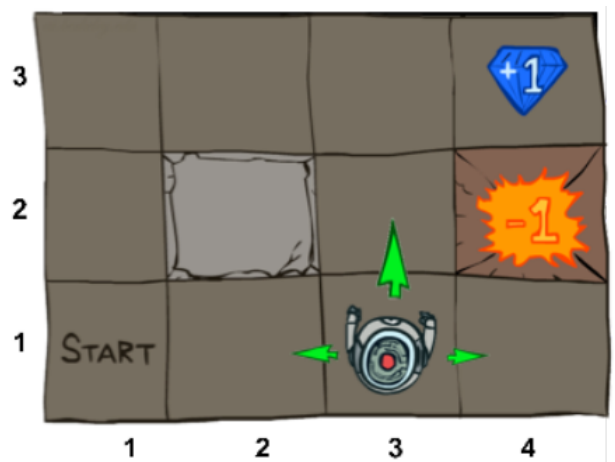




Reinforcement Learning II

- Goal: to learn the best ***policy*** $\pi^*(s^{(t)}) = a^{(t)}$ that maximizes the total reward $\sum_t R^{(t)}$
- Training:
  1. Perform trial-and-error runs
  2. Learn from the experience





Applications

- Sequential decision making and control problems

Kohl and Stone, 2004

Ng et al, 2004

Tedrake et al, 2005

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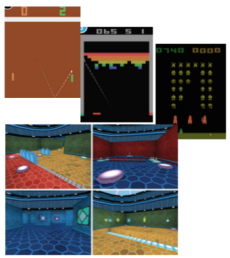

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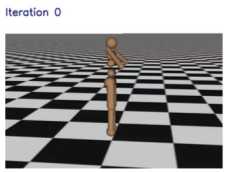

Schulman et al,
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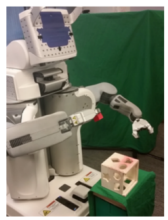

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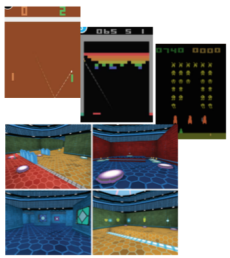

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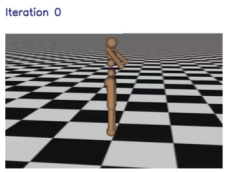

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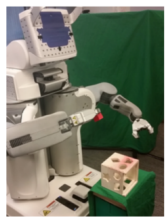

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- From machine learning to AI that changes the world

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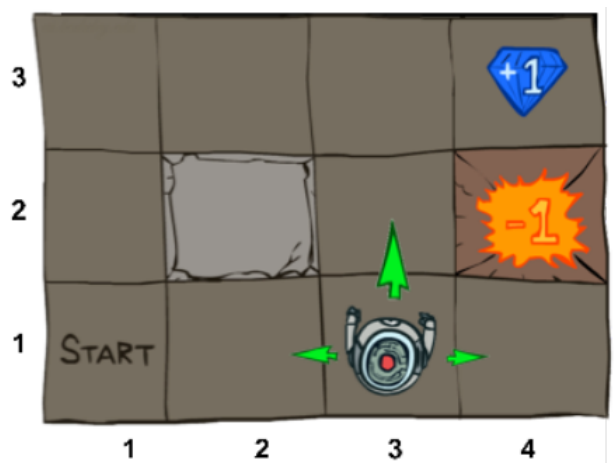
  2. with correct labels $\boldsymbol{y}^{(i)}$'s
- In RL, neither holds
- $\boldsymbol{s}^{(t+1)}$ may depend on $\boldsymbol{s}^{(t)}$ and $\boldsymbol{a}^{(t)}$
- No "what to predict" ($\boldsymbol{y}^{(i)}$'s), just "how good a prediction is" ($R^{(t)}$'s)
  - $R^{(t)}$'s are also called the ***critics***



Outline

Markov Processes

- Random process $\{\mathbf{s}^{(t)}\}_t$ is a collection of time-indexed random variables
  - A classic way to model dependency between input samples $\{\boldsymbol{s}^{(t)}\}_t$
- A random process is called a ***Markov process*** if it satisfies the ***Markov property***:

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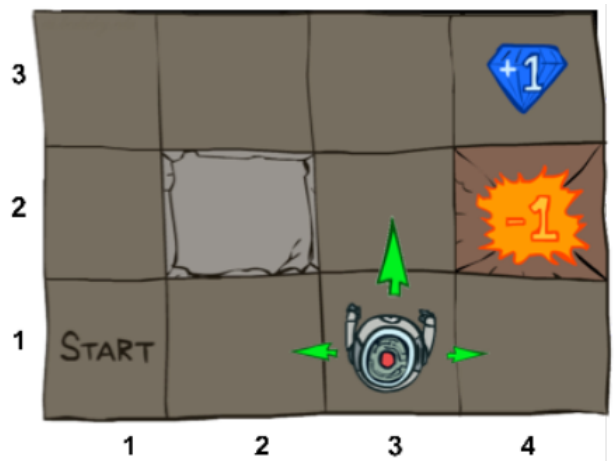
- For those who knows Markov process:**

	States are fully observable	**States are partially observable**
Transition is autonomous	Markov chains	Hidden Markov models
Transition is controlled	***Markov decision processes (MDP)***	Partially observable MDP

Markov Decision Process I

- A Markov decision process (MDP) is defined by
  - $\mathbb{S}$ the state space; $\mathbb{A}$ the action space
  - Start state $\boldsymbol{s}^{(0)}$
  - $\mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})$ the ***transition distribution*** controlled by actions; fixed over time t
  - $R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') \in \mathbb{R}$ (or simply $R(\boldsymbol{s}')$) the deterministic reward function
  - $\gamma \in [0,1]$ is the ***discount factor***
  - $H \in \mathbb{N}$ the ***horizon***; can be infinite



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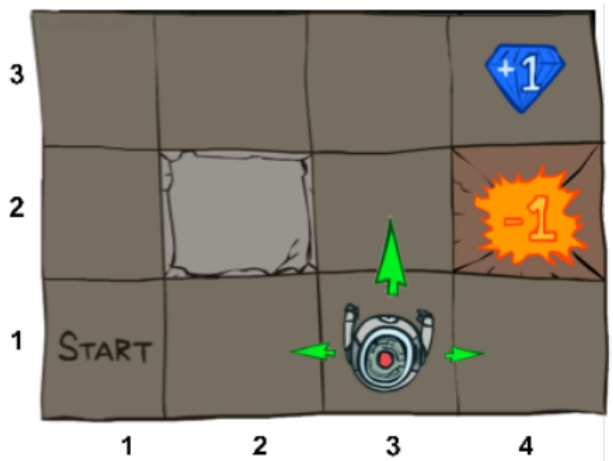
- An absorbing/terminal state transit to itself with probability 1



Markov Decision Process II

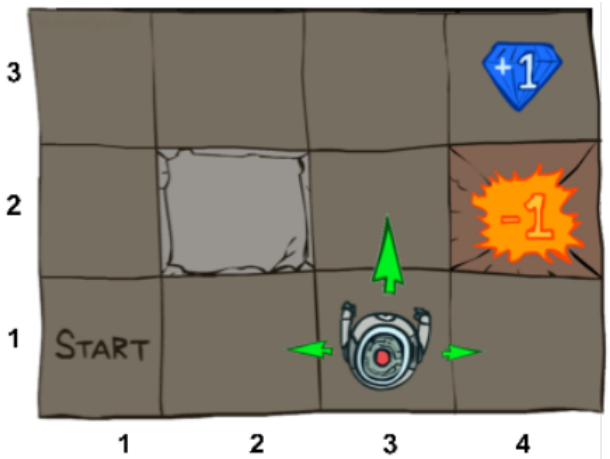
- Given a policy $\pi(s) = a$, an MDP proceeds as follows:

$$s^{(0)} \xrightarrow{a^{(0)}} s^{(1)} \xrightarrow{a^{(1)}} \cdots \xrightarrow{a^{(H-1)}} s^{(H)},$$

with the accumulative reward

$$R(s^{(0)}, a^{(0)}, s^{(1)}) + \gamma R(s^{(1)}, a^{(1)}, s^{(2)}) + \cdots + \gamma^{H-1} R(s^{(H-1)}, a^{(H-1)}, s^{(H)})$$

MDP



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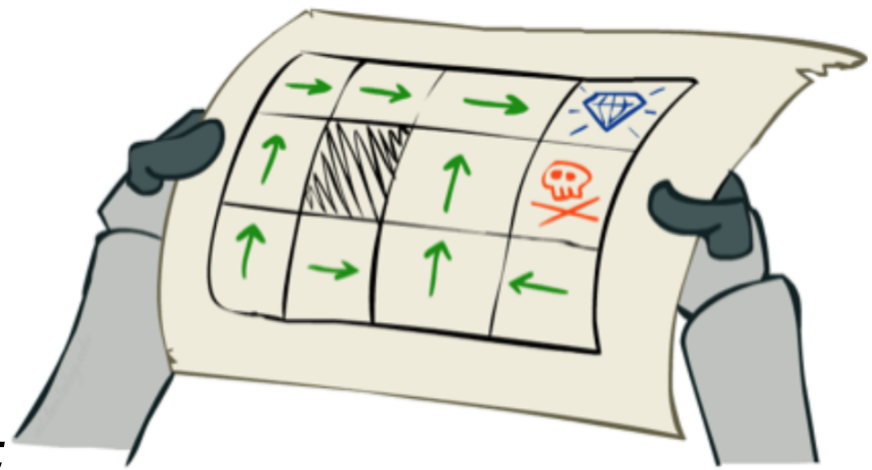

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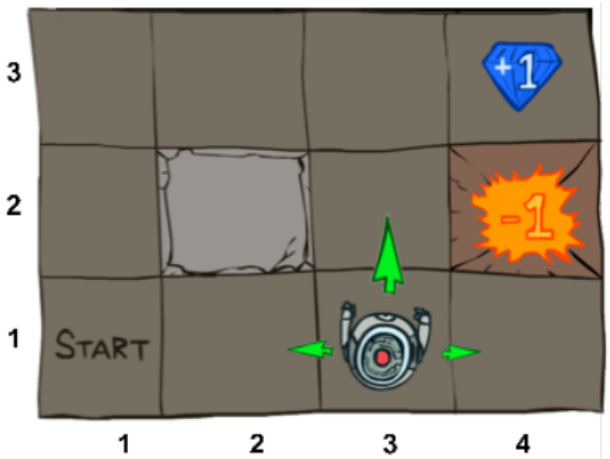
- *To accrue rewards as soon as possible (prefer a shot path)*
- *Different accumulative rewards in different trials*

MDP



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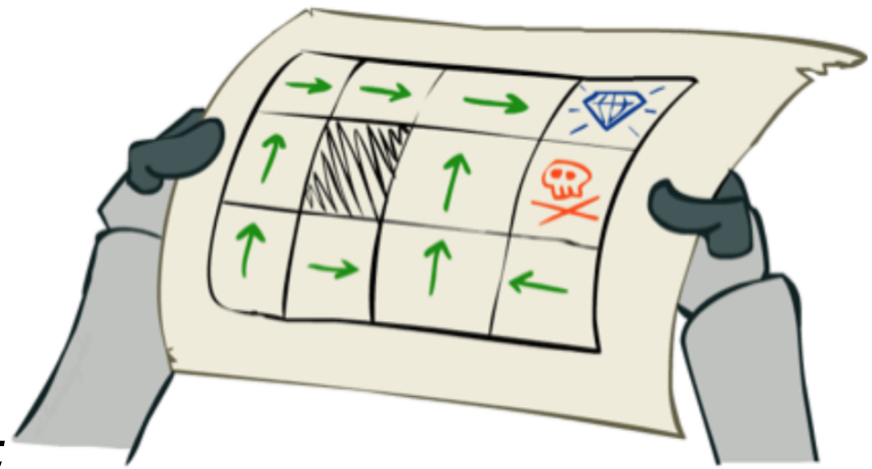

Goal

- Given a policy π, the expected accumulative reward collected by taking actions following π can be express by:

$$V_\pi = \mathrm{E}_{\mathbf{s}^{(0)},\cdots,\mathbf{s}^{(H)}}\left(\sum_{t=0}^{H}\gamma^t R(\mathbf{s}^{(t)},\pi(\mathbf{s}^{(t)}),\mathbf{s}^{(t+1)});\pi\right)$$

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- How?

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Optimal Value Function

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$$\pi^*(\boldsymbol{s}) = \arg\max_{\boldsymbol{a}} \sum_{\boldsymbol{s}'} \mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})[R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma V^{*(H-1)}(\boldsymbol{s}')], \forall \boldsymbol{s}$$

  in $O(|\mathbb{S}||\mathbb{A}|)$ time

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  in $O(|\mathbb{S}||\mathbb{A}|)$ time
- How to obtain $V^{*(H-1)}(\boldsymbol{s})$ for each $\boldsymbol{s}$?

Dynamic Programming

$$V^{*(h)}(\boldsymbol{s}) = \max_{\pi} \mathrm{E}_{\mathbf{s}^{(1)},\cdots,\mathbf{s}^{(H)}} \left(\sum_{t=0}^{h} \gamma^t R(\mathbf{s}^{(t)}, \pi(\mathbf{s}^{(t)}), \mathbf{s}^{(t+1)}) | \mathbf{s}^{(0)} = \boldsymbol{s}; \pi \right)$$

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- $h = 1$:

$$V^{*(1)}(\boldsymbol{s}) = \max_{\boldsymbol{a}} \sum_{\boldsymbol{s}'} \mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})[R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma V^{*(0)}(\boldsymbol{s}')], \forall \boldsymbol{s}$$

- $h = 0$:

$$V^{*(0)}(\boldsymbol{s}) = 0, \forall \boldsymbol{s}$$

Algorithm: Value Iteration (Finite Horizon)

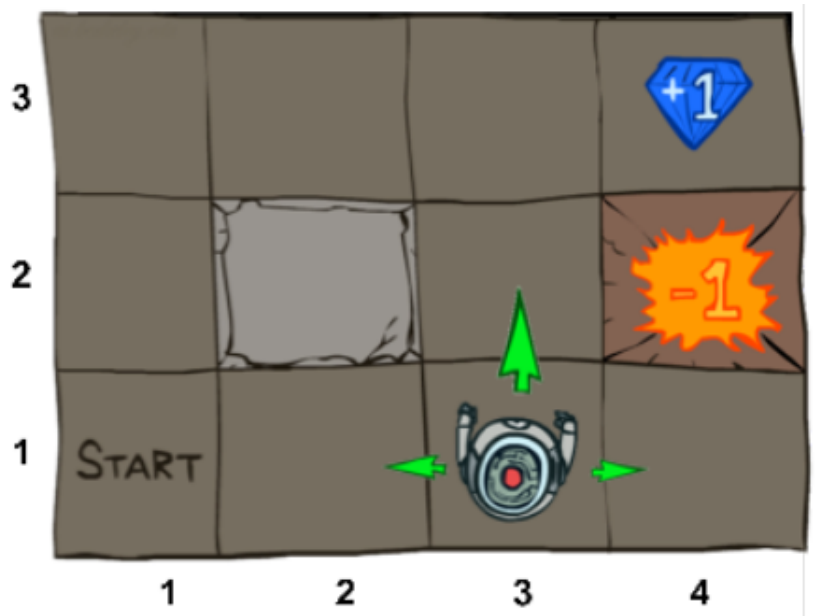
Input: MDP $(\mathbb{S}, \mathbb{A}, \mathrm{P}, R, \gamma, H)$
Output: $\pi^*(s)$'s for all s's

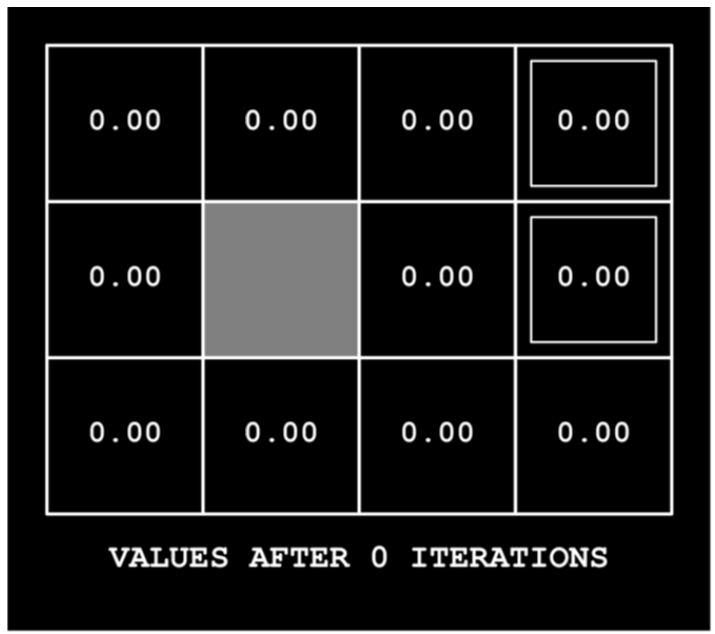
For each state s, initialize $V^*(s) \leftarrow 0$;
for $h \leftarrow 0$ **to** $H-1$ **do**
  foreach s **do**
    $V^*(s) \leftarrow \max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$;
  end
end
foreach s **do**
  $\pi^*(s) \leftarrow \arg\max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$;
end

Algorithm 1: Value iteration with finite horizon.

Example

- MDP settings:
  - Noise of transition probability $\mathrm{P}(s'|s;a)$: 0.2
  - γ: 0.9





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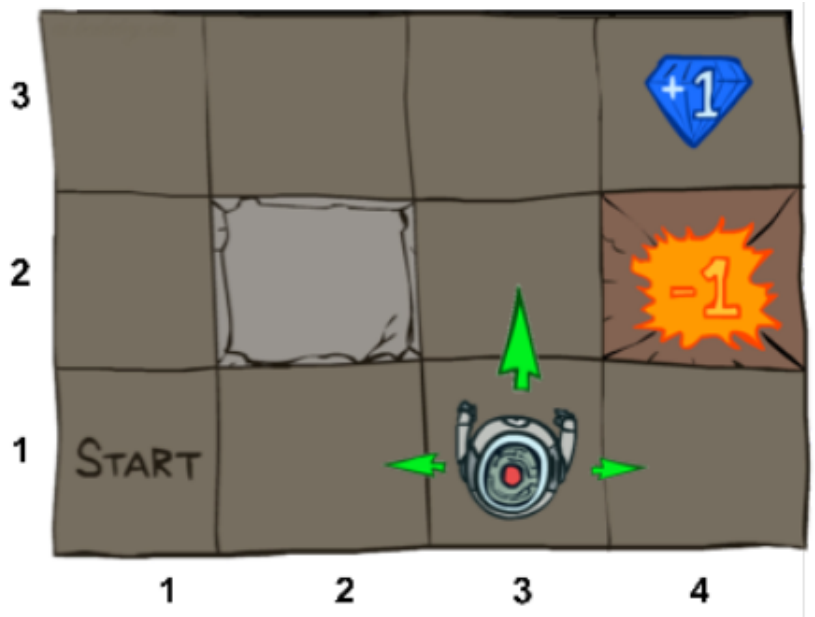

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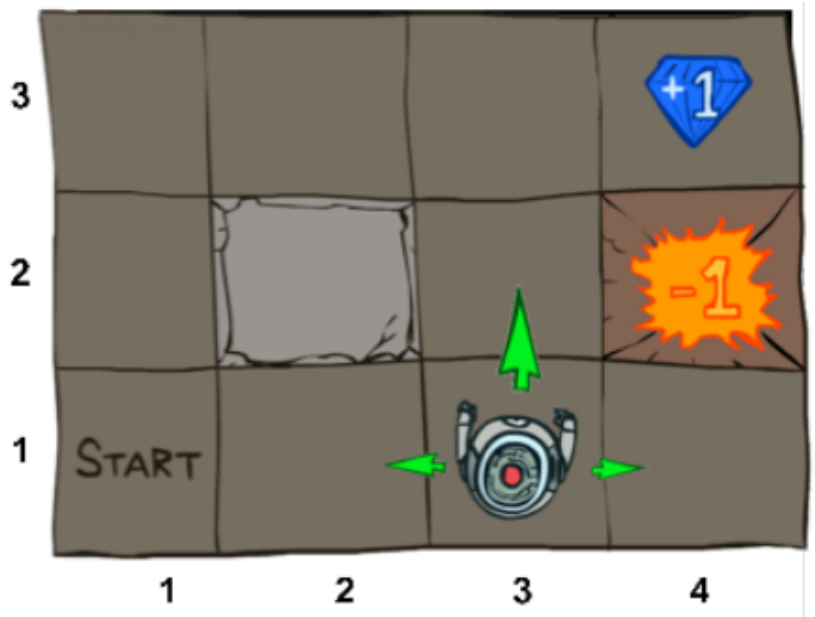

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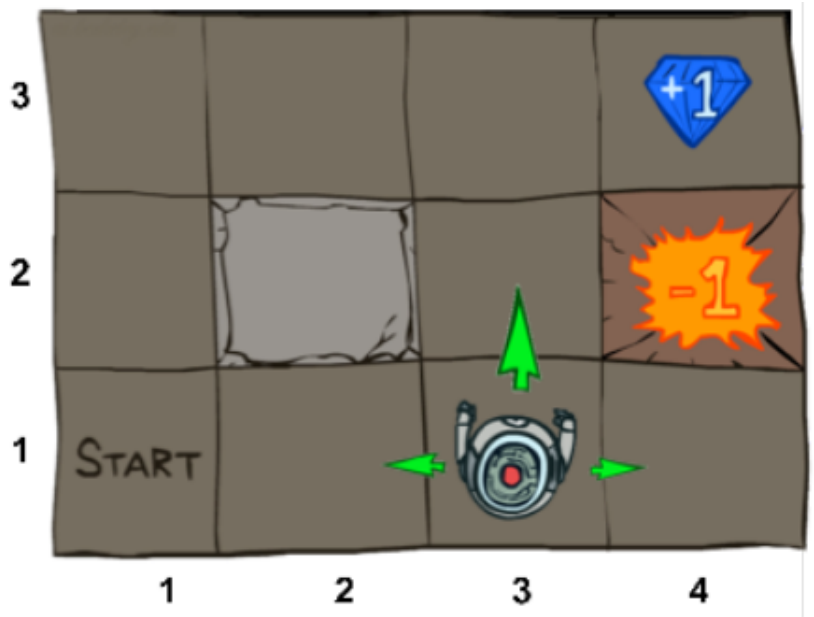

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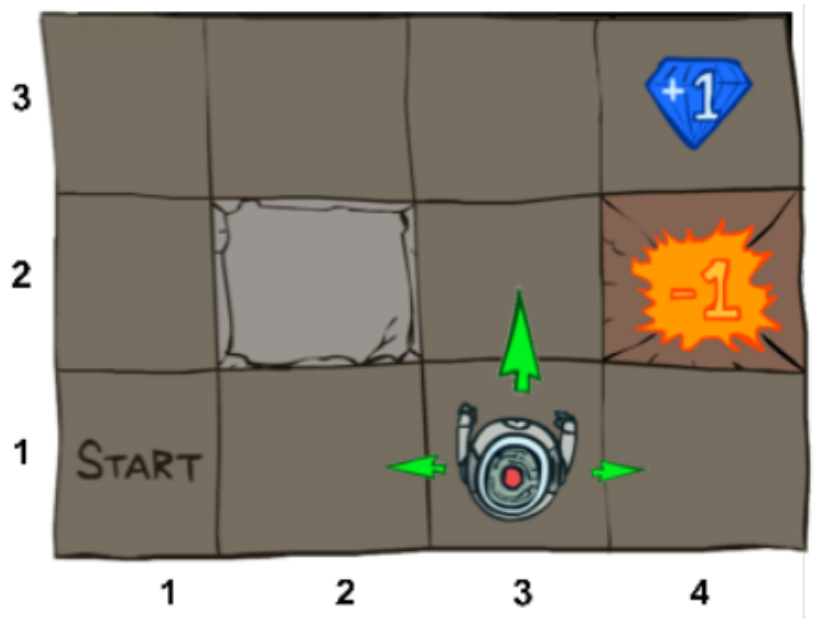

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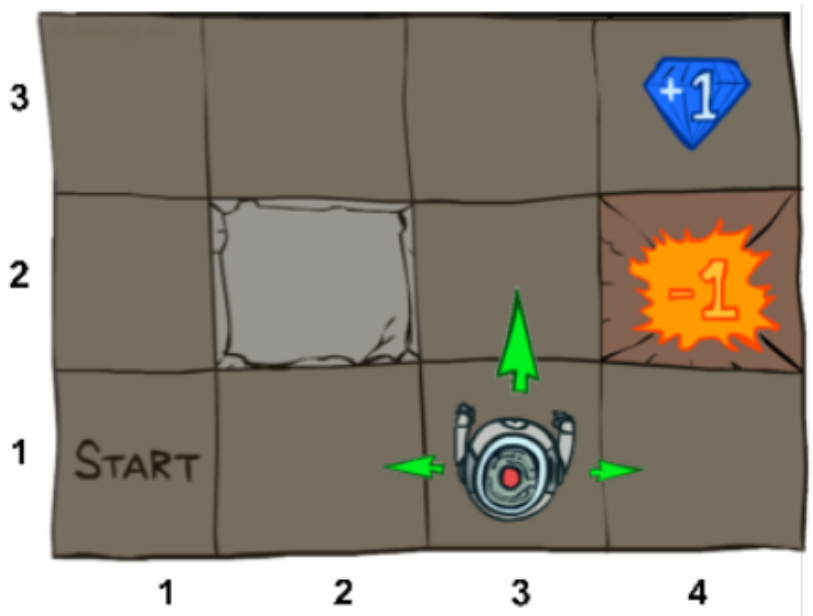

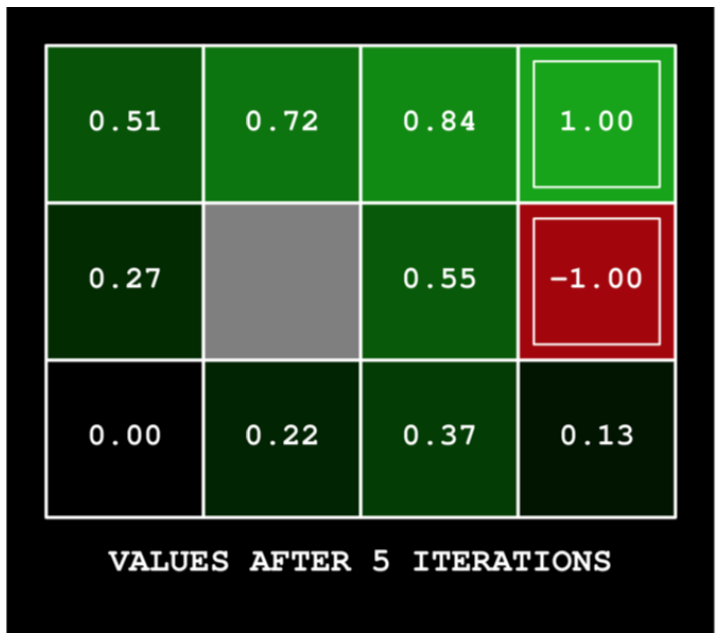

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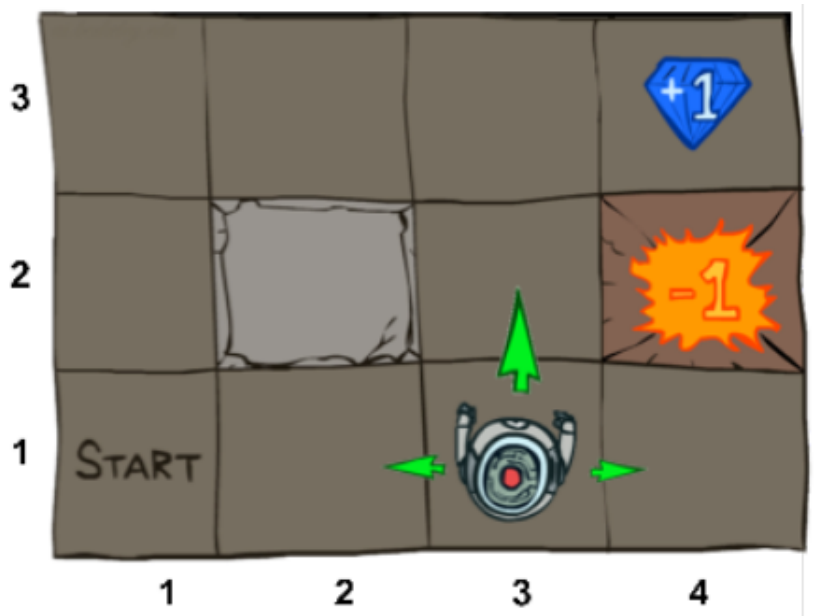

Infinite Horizon & Bellman Optimality Equation

- Recurrence of optimal values:

$$V^{*(h)}(\boldsymbol{s}) = \max_{a} \sum_{\boldsymbol{s}'} \mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})[R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma V^{*(h-1)}(\boldsymbol{s}')], \forall \boldsymbol{s}$$

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- When $h \to \infty$, π^* is
  - ***Stationary***: the optimal action at a state s is the same at all times (efficient to store)
  - ***Memoryless***: independent with $s^{(0)}$

Algorithm: Value Iteration (Infinite Horizon)

Input: MDP $(\mathbb{S}, \mathbb{A}, \mathrm{P}, R, \gamma, H \to \infty)$
Output: $\pi^*(s)$'s for all s's

For each state s, initialize $V^*(s) \leftarrow 0$;
repeat
  foreach s **do**
    $V^*(s) \leftarrow \max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$;
  end
until $V^*(s)$*'s converge*;
foreach s **do**
  $\pi^*(s) \leftarrow \arg\max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$;
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Algorithm 2: Value iteration with infinite horizon.

Convergence

Theorem

Value iteration converges and gives the optimal policy π^* when $H \to \infty$.

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$$\begin{aligned} V^*(\boldsymbol{s}) - V^{*(H)}(\boldsymbol{s}) &= \gamma^{H+1} R(\boldsymbol{s}^{(H+1)}, \boldsymbol{a}^{(H+1)}, \boldsymbol{s}^{(H+2)}) \\ &\quad + \gamma^{H+2} R(\boldsymbol{s}^{(H+2)}, \boldsymbol{a}^{(H+2)}, \boldsymbol{s}^{(H+3)}) + \cdots \\ &\leq \gamma^{H+1} R_{\max} + \gamma^{H+2} R_{\max} + \cdots \\ &= \frac{\gamma^{H+1}}{1-\gamma} R_{\max} \end{aligned}$$

  - Goes to 0 as $H \to \infty$
  - Hence, $V^{*(H)}(\boldsymbol{s}) \xrightarrow{H \to \infty} V^*(\boldsymbol{s})$

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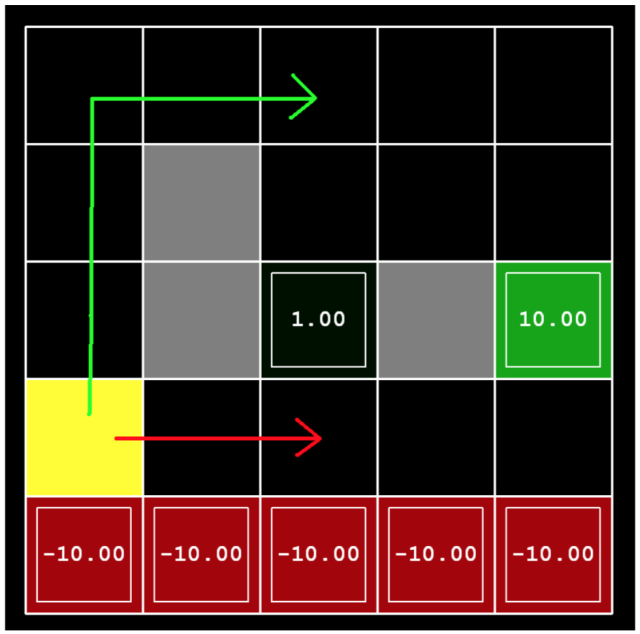
  - Goes to 0 as $H \to \infty$
  - Hence, $V^{*(H)}(\boldsymbol{s}) \xrightarrow{H \to \infty} V^*(\boldsymbol{s})$
- Assumed that $R(\cdot) \geq 0$; still holds if rewards can be negative
  - by using $\max |R(\cdot)|$ and bounding from both sides

Effect of MDP Parameters

- How does the noise of $\mathrm{P}(s'|s;a)$ and γ affect π^*?



1. $\gamma = 0.99$, noise $= 0.5$
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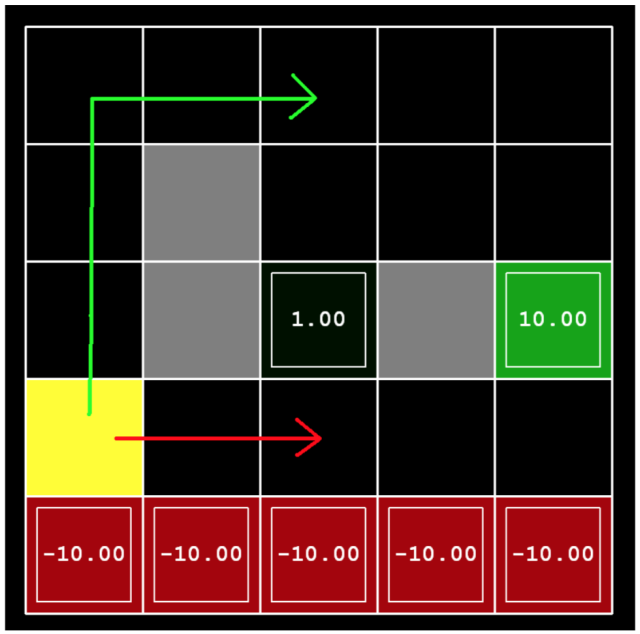

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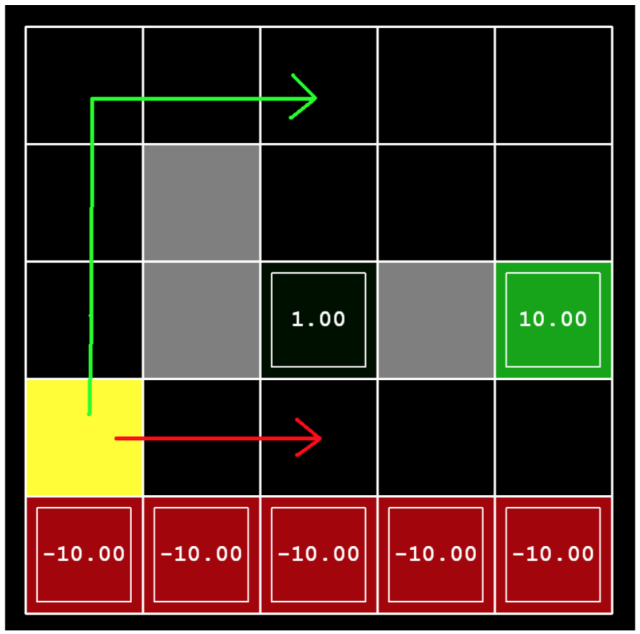

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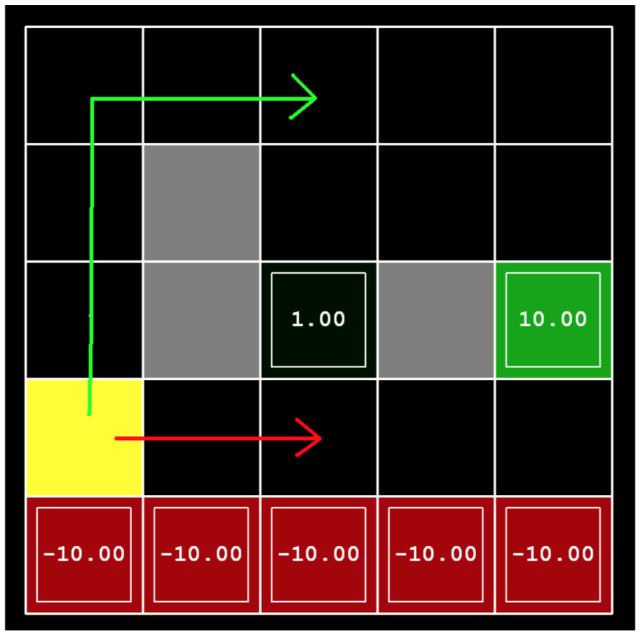

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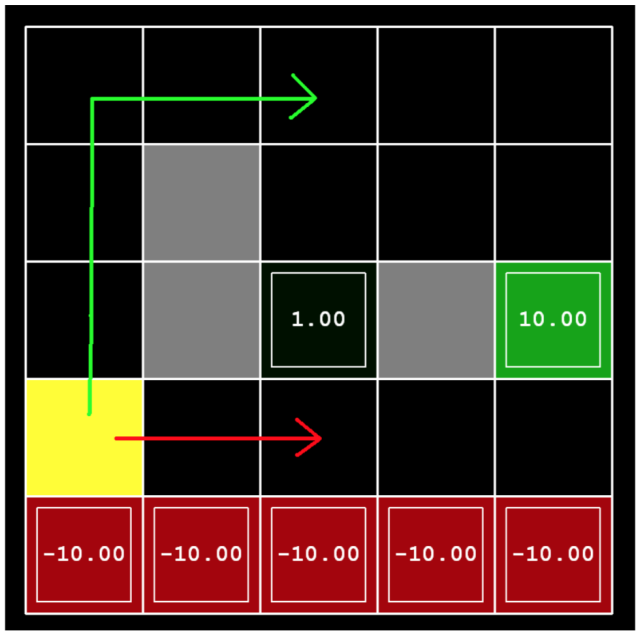

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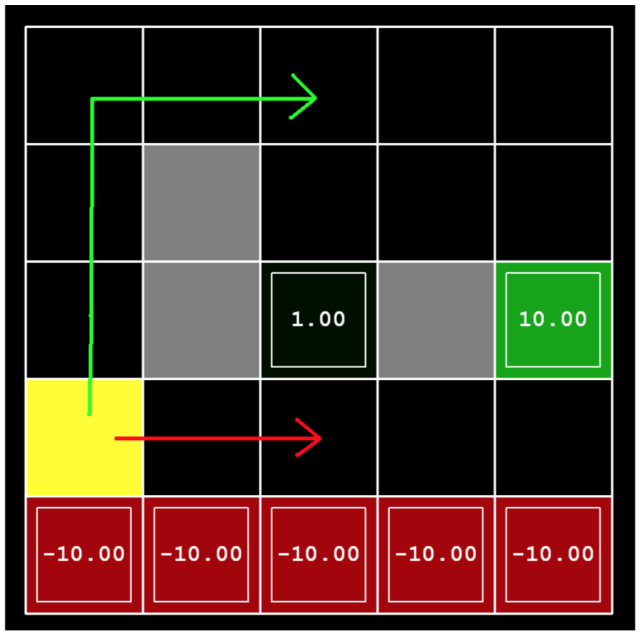

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Outline

Goal

- Given an MDP $(\mathbb{S}, \mathbb{A}, \mathrm{P}, R, \gamma, H \to \infty)$
- Expected accumulative reward collected by taking actions following a policy π:

$$V_\pi = \mathrm{E}_{\mathbf{s}^{(0)}, \cdots, \mathbf{s}^{(H)}} \left(\sum_{t=0}^{\infty} \gamma^t R(\mathbf{s}^{(t)}, \pi(\mathbf{s}^{(t)}), \mathbf{s}^{(t+1)}); \pi \right)$$

- Goal: to find the optimal policy

$$\pi^* = \arg\max_\pi V_\pi$$

Algorithm: Policy Iteration (Simplified)

- Given a π, define its ***value function***:

$$V_\pi(\boldsymbol{s}) = \mathrm{E}_{\mathbf{s}^{(1)},\cdots}\left(\sum_{t=0}^{\infty}\gamma^t R(\mathbf{s}^{(t)}, \pi(\mathbf{s}^{(t)}), \mathbf{s}^{(t+1)}) | \mathbf{s}^{(0)} = \boldsymbol{s}; \pi\right)$$

  - Expected accumulative reward when starting from state $\boldsymbol{s}$ and acting based on π

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Input: MDP $(\mathbb{S},\mathbb{A},\mathrm{P},R,\gamma,H\to\infty)$
Output: $\pi(\boldsymbol{s})$'s for all $\boldsymbol{s}$'s

For each state $\boldsymbol{s}$, initialize $\pi(\boldsymbol{s})$ randomly;
repeat
  Evaluate $V_\pi(\boldsymbol{s}),\forall\boldsymbol{s}$;
  Improve π such that $V_\pi(\boldsymbol{s})$, $\forall\boldsymbol{s}$, becomes higher;
until $\pi(\boldsymbol{s})$*'s converge*;

Algorithm 4: Policy iteration.

Bellman Expectation Equation

- How to evaluate the value function of a given π?

$$V_\pi(\boldsymbol{s}) = \mathrm{E}_{\mathbf{s}^{(1)},\cdots}\left(\sum_{t=0}^{\infty}\gamma^t R(\mathbf{s}^{(t)},\pi(\mathbf{s}^{(t)}),\mathbf{s}^{(t+1)})|\mathbf{s}^{(0)}=\boldsymbol{s};\pi\right)$$

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M1 Solve the system of linear equations

- $|\mathbb{S}|$ linear equations and $|\mathbb{S}|$ variables
- Time complexity: $O(|\mathbb{S}|^3)$

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M1 Solve the system of linear equations

- $|\mathbb{S}|$ linear equations and $|\mathbb{S}|$ variables
- Time complexity: $O(|\mathbb{S}|^3)$

M2 Dynamic programming (just like value iteration):

- Initializes $V_\pi(s)=0,\forall s$
- $V_\pi(s) \leftarrow \sum_{s'}\mathrm{P}(s'|s;\pi(s))[R(s,\pi(s),s')+\gamma V_\pi(s')],\forall s$

Policy Improvement

- How to find $\hat{\pi}$ such that $V_{\hat{\pi}}(\boldsymbol{s}) \geq V_{\pi}(\boldsymbol{s})$ for all $\boldsymbol{s}$?

Policy Improvement

- How to find $\hat{\pi}$ such that $V_{\hat{\pi}}(s) \geq V_{\pi}(s)$ for all s?
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$$\hat{\pi}(s) \leftarrow \arg\max_{a} \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V_{\pi}(s')]$$

- Why?

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&\cdots \\
&\leq \mathrm{E}_{s',s'',\ldots}\left(R(\mathbf{s},\hat{\pi}(\mathbf{s}),\mathbf{s}') + \gamma R(\mathbf{s}',\hat{\pi}(\mathbf{s}'),\mathbf{s}'') + \cdots | \mathbf{s}^{(0)} = s;\hat{\pi}\right) \\
&= V_{\hat{\pi}}(s), \forall s
\end{aligned}$$

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  until $V_\pi(s)$*'s converge*;
  foreach s **do**
    $\pi(s) \leftarrow \arg\max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V_\pi(s')]$;
  end
until $\pi(s)$*'s converge*;

Algorithm 5: Policy iteration.

Convergence

Theorem

Policy iteration converges and gives the optimal policy π^* when $H \to \infty$.

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Theorem

Policy iteration converges and gives the optimal policy π^* when $H \to \infty$.

- Convergence: in every step the policy improves
- Optimal policy: at convergence, $V_\pi(\boldsymbol{s})$, $\forall \boldsymbol{s}$, satisfies the Bellman optimality equation

$$V^*(\boldsymbol{s}) = \max_a \sum_{\boldsymbol{s}'} \mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})[R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma V^*(\boldsymbol{s}')]$$

Outline

Difference from MDP

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Kohl and Stone, 2004

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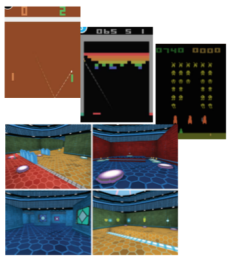
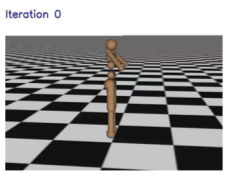
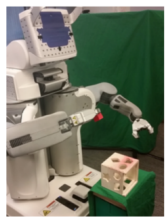


Mnih et al 2013 (DQN)
Mnih et al, 2015 (A3C)

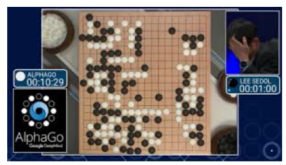
Silver et al, 2014 (DPG)
Lillicrap et al, 2015 (DDPG)



Schulman et al,
2016 (TRPO + GAE)



Levine*, Finn*, et
al, 2016
(GPS)



Silver*, Huang*, et
al, 2016
(AlphaGo)

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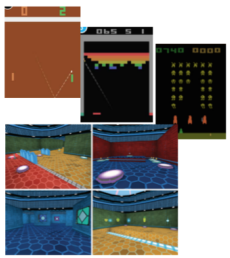

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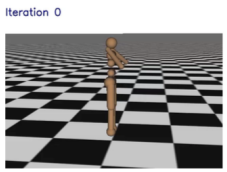

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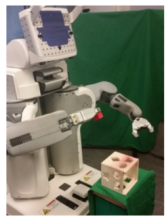

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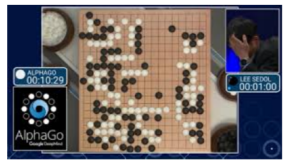

Silver*, Huang*, et
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(AlphaGo)

- Unknown reward function $R(s,a,s')$

Exploration vs. Exploitation

- What would you do to get some rewards?

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1. Perform actions randomly to ***explore*** $\mathrm{P}(s'|s;a)$ and $R(s,a,s')$ first
   - Collect samples so to estimate $\mathrm{P}(s'|s;a)$ and $R(s,a,s')$

Exploration vs. Exploitation

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   - Collect samples so to estimate $\mathrm{P}(s'|s;a)$ and $R(s,a,s')$
2. Then, perform actions to ***exploit*** the learned $\mathrm{P}(s'|s;a)$ and $R(s,a,s')$
   - π^* can be computed/planned "in mind" using value/policy iterations

Model-based RL using Monte Carlo Estimation

1. Use some ***exploration policy*** π' to perform one or more ***episodes***/***trails***
   - Each episode records samples of $\mathrm{P}(s'|s;a)$ and $R(s,a,s')$ from start to terminal state

$$s^{(0)} \xrightarrow{\pi'(s^{(0)})} s^{(1)} \xrightarrow{\pi'(s^{(1)})} \cdots \xrightarrow{\pi'(s^{(H-1)})} s^{(H)}$$

   and

$$R(s^{(0)},\pi'(s^{(0)}),s^{(1)}) \to \cdots \to R(s^{(H-1)},\pi(s^{(H-1)}),s^{(H)})$$

2. Estimate $\mathrm{P}(s'|s;a)$ and $R(s,a,s')$ using the samples and update the ***exploitation policy*** π
   - $\hat{\mathrm{P}}(s'|s;a) = \dfrac{\#\text{ times the action } a \text{ takes state } s \text{ to state } s'}{\#\text{ times action } a \text{ is taken in state } s}$
   - $\hat{R}(s,a,s') =$ average of reward values received when a takes s to s'
3. Repeat from Step 1, but gradually mix π into π'
   - Mix-in strategy? E.g., ε-greedy (more on this later)

Problems of Model-based RL

- There may be lots of $\mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})$ and $R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}')$ to estimate
- If $\mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})$ is low, there may be too few samples to have a good estimate
- Low $\mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})$ may also lead to a poor estimate of $R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}')$
  - when $R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}')$ depends on $\boldsymbol{s}'$

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- We estimate $\mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})$'s and $R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}')$'s in order to compute $V^*(\boldsymbol{s})/V_\pi(\boldsymbol{s})$ and solve π^*
- Why not estimate $V^*(\boldsymbol{s})/V_\pi(\boldsymbol{s})$ directly?

Outline

Model-based vs. Model-Free

- How to estimate $\mathrm{E}(f(\mathbf{x})|\boldsymbol{y}) = \sum_{\boldsymbol{x}} \mathrm{P}(\boldsymbol{x}|\boldsymbol{y}) f(\boldsymbol{x})$ given samples $(\boldsymbol{x}^{(1)}, \boldsymbol{y}^{(1)}), (\boldsymbol{x}^{(2)}, \boldsymbol{y}^{(2)}), \cdots$?

Model-based vs. Model-Free

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- Model-based estimation:
  1. For each $\boldsymbol{x}$, estimate $\hat{\mathrm{P}}(\boldsymbol{x}|\boldsymbol{y}) = \frac{\mathsf{count}(\boldsymbol{x}^{(i)}=\boldsymbol{x} \text{ and } \boldsymbol{y}^{(i)}=\boldsymbol{y})}{\mathsf{count}(\boldsymbol{y}^{(j)}=\boldsymbol{y})}$
  2. $\hat{\mathrm{E}}(f(\mathbf{x})|\boldsymbol{y}) = \sum_{\boldsymbol{x}} \hat{\mathrm{P}}(\boldsymbol{x}|\boldsymbol{y}) f(\boldsymbol{x})$

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  - Why does it work?

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  - Why does it work? Because samples appear in right frequencies

Challenges of Model-Free RL

- Value iteration: start from $V^*(s) \leftarrow 0$
  1. Iterate $V^*(s) \leftarrow \max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$ until converge
  2. Solve $\pi^*(s) \leftarrow \arg\max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$
- Policy iteration: start from a random π, repeat until converge:
  1. Iterate $V_\pi(s) \leftarrow \sum_{s'} \mathrm{P}(s'|s;\pi(s))[R(s,\pi(s),s') + \gamma V_\pi(s')]$ until converge
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     ☹ ***Not easy to estimate*** $V^*(s) = \max_\pi \mathrm{E}\left(\sum_t \gamma^t R^{(t)} | \mathbf{s}^{(0)} = s; \pi\right)$
  2. Solve $\pi^*(s) \leftarrow \arg\max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$
     ☹ ***Need model to solve***
- Policy iteration: start from a random π, repeat until converge:
  1. Iterate $V_\pi(s) \leftarrow \sum_{s'} \mathrm{P}(s'|s;\pi(s))[R(s,\pi(s),s') + \gamma V_\pi(s')]$ until converge
     ☺ ***Can estimate*** $V_\pi(s) = \mathrm{E}\left(\sum_t \gamma^t R^{(t)} | \mathbf{s}^{(0)} = s; \pi\right)$ ***using MC est.***
  2. Solve $\pi(s) \leftarrow \arg\max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V_\pi(s')]$
     ☹ ***Need model to solve***

Q Function for π

- Value function for a given π:

$$V_\pi(\boldsymbol{s}) = \mathrm{E}_{\mathbf{s}^{(1)},\cdots}\left(\sum_{t=0}^{\infty}\gamma^t R(\mathbf{s}^{(t)},\pi(\mathbf{s}^{(t)}),\mathbf{s}^{(t+1)})\,|\,\mathbf{s}^{(0)}=\boldsymbol{s};\pi\right)$$

  with recurrence (Bellman expectation equation):

$$V_\pi(\boldsymbol{s}) = \sum_{\boldsymbol{s}'}\mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\pi(\boldsymbol{s}))[R(\boldsymbol{s},\pi(\boldsymbol{s}),\boldsymbol{s}')+\gamma V_\pi(\boldsymbol{s}')],\forall \boldsymbol{s}$$

  - Maximum expected accumulative reward when starting from state $\boldsymbol{s}$ and acting based on π onward

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  - Maximum expected accumulative reward when starting from state $\boldsymbol{s}$ and acting based on π onward

- Define ***Q function*** for π:

$$Q_\pi(\boldsymbol{s},\boldsymbol{a}) = \mathrm{E}_{\mathbf{s}^{(1)},\ldots}\left(R(\boldsymbol{s},\boldsymbol{a},\mathbf{s}^{(1)})+\sum_{t=1}^{\infty}\gamma^t R(\mathbf{s}^{(t)},\pi(\mathbf{s}^{(t)}),\mathbf{s}^{(t+1)});\boldsymbol{s},\boldsymbol{a},\pi\right)$$

  such that $V_\pi(\boldsymbol{s}) = Q_\pi(\boldsymbol{s},\pi(\boldsymbol{s}))$ with recurrence:

$$Q_\pi(\boldsymbol{s},\boldsymbol{a}) = \sum_{\boldsymbol{s}'}\mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})[R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}')+\gamma Q_\pi(\boldsymbol{s}',\pi(\boldsymbol{s}'))],\forall \boldsymbol{s},\boldsymbol{a}$$

  - Maximum expected accumulative reward when starting from state $\boldsymbol{s}$, taking action $\boldsymbol{a}$, and then acting based on π

Algorithm: Policy Iteration based on Q_π

Input: MDP $(\mathbb{S}, \mathbb{A}, \mathrm{P}, R, \gamma, H \to \infty)$

Output: $\pi(s)$'s for all s's

For each state s, initialize $\pi(s)$ randomly;

repeat

~~For each state s, initialize $V_\pi(s) \leftarrow 0$~~ Initialize $Q_\pi(s,a) = 0, \forall s, a$;

repeat

foreach s *and* a **do**

~~$V_\pi(s) \leftarrow \sum_{s'} \mathrm{P}(s'|s;\pi(s))[R(s,\pi(s),s') + \gamma V_\pi(s')]$~~

$Q_\pi(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma Q_\pi(s', \pi(s'))]$;

end

until ~~$V_\pi(s)$'s~~ $Q_\pi(s,a)$'s *converge*;

foreach s **do**

~~$\pi(s) \leftarrow \arg\max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V_\pi(s')]$~~

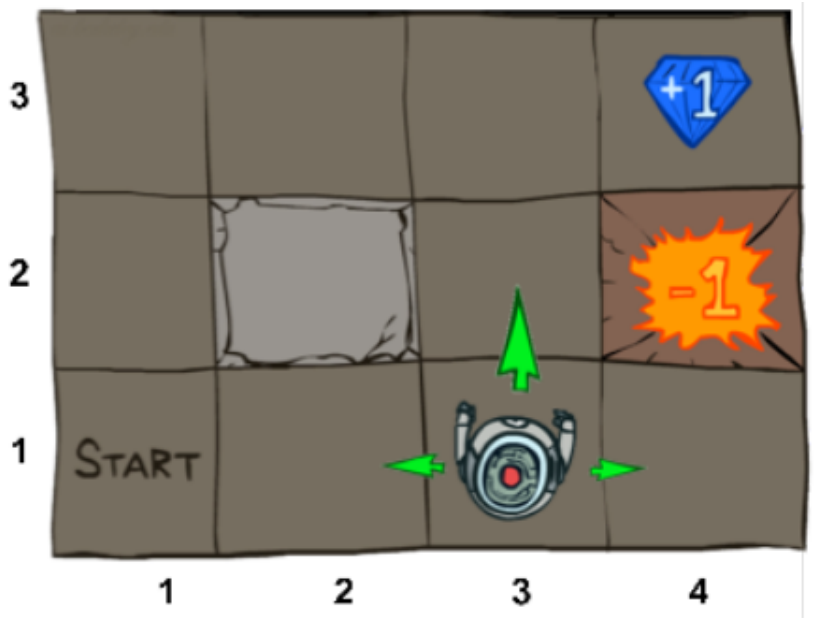
$\pi(s) \leftarrow \arg\max_a Q_\pi(s,a)$;

end

until $\pi(s)$'s *converge*;

Algorithm 6: Policy iteration.

Example


3
2
1
START
+1
-1
1
2
3
4


0.64
0.74
0.85
1.00
0.57
0.57
-1.00
0.49
0.43
0.48
0.28
VALUES AFTER 100 ITERATIONS


0.59
0.57
0.64
0.53
0.67
0.60
0.74
0.67
0.77
0.66
0.85
0.57
1.00
0.57
0.51
0.51
0.46
0.57
0.53
-0.60
0.30
-1.00
0.49
0.45
0.41
0.44
0.40
0.43
0.42
0.40
0.48
0.40
0.29
0.41
-0.65
0.28
0.13
0.27
Q-VALUES AFTER 100 ITERATIONS

Policy Iteration and Model-Free RL

- Policy iteration: start from a random π, repeat until converge:
  1. Iterate $Q_\pi(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;\pi(s))[R(s,\pi(s),s') + \gamma Q_\pi(s',\pi(s'))]$ until converge
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- RL: start from a random π for both exploration & exploitation, repeat until converge:
  1. Create episodes using π and get estimates $\hat{Q}_\pi(s,a), \forall s,a$
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- Problem: π improves little after running lots of trials
- Can we improve π right after each action?

Outline

Policy Iteration Revisited

- Policy iteration: start from a random π, repeat until converge:
  1. Iterate $Q_\pi(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;\pi(s))[R(s,\pi(s),s') + \gamma Q_\pi(s',\pi(s'))]$ until converge
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Policy Iteration Revisited

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     - ☺ ***Can estimate*** $Q_\pi(s,a) = \mathrm{E}\left(R^{(0)} + \sum_{t=1} \gamma^t R^{(t)}\right)$ ***using MC est.***
     - ☺ ***Can also estimate*** $Q_\pi(s,a)$ ***based on the recurrence***
  2. Solve $\pi(s) \leftarrow \arg\max_a Q_\pi(s,a)$
     - ☺ ***No need for model to solve***

Temporal Difference Estimation I

$$Q_\pi(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;\pi(s))[R(s,\pi(s),s') + \gamma Q_\pi(s',\pi(s'))]$$

- Given the samples

$$s^{(0)} \xrightarrow{a^{(0)}} s^{(1)} \xrightarrow{a^{(1)}} \cdots \xrightarrow{a^{(H-1)}} s^{(H)}$$

and

$$R(s^{(0)},a^{(0)},s^{(1)}) \to \cdots \to R(s^{(H-1)},a^{(H-1)},s^{(H)})$$

- ***Temporal difference (TD) estimation*** of $Q_\pi(s,a)$:
  1. $\hat{Q}_\pi(s,a) \leftarrow$ randon value, $\forall s,a$
  2. Repeat until converge ***for each action*** $a^{(t)}$:

$$\hat{Q}_\pi(s^{(t)},a^{(t)}) \leftarrow \hat{Q}_\pi(s^{(t)},a^{(t)}) + \eta\left[(R(s^{(t)},a^{(t)},s^{(t+1)}) + \gamma\hat{Q}_\pi(s^{(t+1)},\pi(s^{(t+1)}))) - \hat{Q}_\pi(s^{(t)},a^{(t)})\right]$$

Temporal Difference Estimation I

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$$s^{(0)} \xrightarrow{a^{(0)}} s^{(1)} \xrightarrow{a^{(1)}} \cdots \xrightarrow{a^{(H-1)}} s^{(H)}$$

and

$$R(s^{(0)},a^{(0)},s^{(1)}) \rightarrow \cdots \rightarrow R(s^{(H-1)},a^{(H-1)},s^{(H)})$$

- ***Temporal difference (TD) estimation*** of $Q_\pi(s,a)$:
  1. $\hat{Q}_\pi(s,a) \leftarrow$ randon value, $\forall s,a$
  2. Repeat until converge ***for each action*** $a^{(t)}$:

$$\hat{Q}_\pi(s^{(t)},a^{(t)}) \leftarrow \hat{Q}_\pi(s^{(t)},a^{(t)}) + \eta\left[(R(s^{(t)},a^{(t)},s^{(t+1)}) + \gamma\hat{Q}_\pi(s^{(t+1)},\pi(s^{(t+1)}))) - \hat{Q}_\pi(s^{(t)},a^{(t)})\right]$$

- $\hat{Q}_\pi(s,a)$ can be updated ***on the fly*** during an episode

Temporal Difference Estimation I

$$Q_\pi(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;\pi(s))[R(s,\pi(s),s') + \gamma Q_\pi(s',\pi(s'))]$$

- Given the samples

$$s^{(0)} \xrightarrow{a^{(0)}} s^{(1)} \xrightarrow{a^{(1)}} \cdots \xrightarrow{a^{(H-1)}} s^{(H)}$$

and

$$R(s^{(0)},a^{(0)},s^{(1)}) \rightarrow \cdots \rightarrow R(s^{(H-1)},a^{(H-1)},s^{(H)})$$

- ***Temporal difference (TD) estimation*** of $Q_\pi(s,a)$:
  1. $\hat{Q}_\pi(s,a) \leftarrow$ randon value, $\forall s,a$
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     $$\hat{Q}_\pi(s^{(t)},a^{(t)}) \leftarrow \hat{Q}_\pi(s^{(t)},a^{(t)}) + \eta \left[(R(s^{(t)},a^{(t)},s^{(t+1)}) + \gamma \hat{Q}_\pi(s^{(t+1)},\pi(s^{(t+1)}))) - \hat{Q}_\pi(s^{(t)},a^{(t)}) \right]$$
- $\hat{Q}_\pi(s,a)$ can be updated ***on the fly*** during an episode
- η the "learning rate"?

Temporal Difference Estimation II

$$\begin{aligned}\hat{Q}_\pi(s,a) \leftarrow \quad & \hat{Q}_\pi(s,a) + \eta \left[(R(s,a,s') + \gamma \hat{Q}_\pi(s', \pi(s'))) - \hat{Q}_\pi(s,a) \right] \\ & = \eta (R(s,a,s') + \gamma \hat{Q}_\pi(s', \pi(s'))) + (1-\eta) \hat{Q}_\pi(s,a)\end{aligned} ,$$

Temporal Difference Estimation II

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- ***Exponential moving average***:

$$\bar{x}_n = \frac{x^{(n)} + (1-\eta) x^{(n-1)} + (1-\eta)^2 x^{(n-2)} + \cdots}{1 + (1-\eta) + (1-\eta)^2 + \cdots},$$

where $\eta \in [0,1]$ is the "forget rate"

- Recent samples are exponentially more important

Temporal Difference Estimation II

$$\begin{aligned}\hat{Q}_\pi(s,a) \leftarrow \quad & \hat{Q}_\pi(s,a) + \eta \left[(R(s,a,s') + \gamma \hat{Q}_\pi(s', \pi(s'))) - \hat{Q}_\pi(s,a) \right] \\ & = \eta (R(s,a,s') + \gamma \hat{Q}_\pi(s', \pi(s'))) + (1-\eta) \hat{Q}_\pi(s,a)\end{aligned},$$

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  - Recent samples are exponentially more important
- Since $1/\eta = 1 + (1-\eta) + (1-\eta)^2 + \cdots$, we have $\bar{x}_n = \eta x^{(n)} + (1-\eta) \bar{x}_{n-1}$ [Proof]

Temporal Difference Estimation II

$$\begin{aligned}\hat{Q}_\pi(s,a) \leftarrow \quad & \hat{Q}_\pi(s,a) + \eta\left[(R(s,a,s') + \gamma\hat{Q}_\pi(s',\pi(s'))) - \hat{Q}_\pi(s,a)\right] \\ & = \eta(R(s,a,s') + \gamma\hat{Q}_\pi(s',\pi(s'))) + (1-\eta)\hat{Q}_\pi(s,a)\end{aligned},$$

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  where $\eta \in [0,1]$ is the "forget rate"
  - Recent samples are exponentially more important
- Since $1/\eta = 1 + (1-\eta) + (1-\eta)^2 + \cdots$, we have $\bar{x}_n = \eta x^{(n)} + (1-\eta)\bar{x}_{n-1}$ [Proof]
- As long as η gradually decreases to 0:
  - $\hat{Q}_\pi(s,a)$ degenerates to average of accumulative rewards
  - $\hat{Q}_\pi(s,a)$ converges to $Q_\pi(s,a)$ (more on this later)

Algorithm: SARSAR

Input: $\mathbb{S}$, $\mathbb{A}$, and γ
Output: $\pi^*(s)$'s for all s's

For each state s and a, initialize $Q_\pi(s,a)$ arbitrarily;
foreach *episode* **do**
  Set s to initial state;
  repeat
    Take action $a \leftarrow \arg\max_{a'} Q_\pi(s,a)$;
    Observe s' and reward $R(s,a,s')$;
    $Q_\pi(s,a) \leftarrow Q_\pi(s,a) + \eta\left[(R(s,a,s') + \gamma Q_\pi(s',\pi(s'))) - Q^*(s,a)\right]$;
    $s \leftarrow s'$;
  until s *is terminal state*;
end

Algorithm 7: State-Action-Reward-State-Action (SARSA).

Algorithm: SARSAR

Input: $\mathbb{S}$, $\mathbb{A}$, and γ
Output: $\pi^*(\boldsymbol{s})$'s for all $\boldsymbol{s}$'s

For each state $\boldsymbol{s}$ and $\boldsymbol{a}$, initialize $Q_\pi(\boldsymbol{s},\boldsymbol{a})$ arbitrarily;
foreach *episode* **do**
  Set $\boldsymbol{s}$ to initial state;
  repeat
    Take action $\boldsymbol{a} \leftarrow \arg\max_{\boldsymbol{a}'} Q_\pi(\boldsymbol{s},\boldsymbol{a})$;
    Observe $\boldsymbol{s}'$ and reward $R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}')$;
    $Q_\pi(\boldsymbol{s},\boldsymbol{a}) \leftarrow Q_\pi(\boldsymbol{s},\boldsymbol{a}) + \eta\left[(R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma Q_\pi(\boldsymbol{s}',\pi(\boldsymbol{s}'))) - Q^*(\boldsymbol{s},\boldsymbol{a})\right]$;
    $\boldsymbol{s} \leftarrow \boldsymbol{s}'$;
  until $\boldsymbol{s}$ *is terminal state*;
end

Algorithm 8: State-Action-Reward-State-Action (SARSA).

- Policy improves each time when deciding the next action $\boldsymbol{a}$

Convergence

Theorem

SARSA converges and gives the optimal policy π^* almost surely if

1) π is GLIE (Greedy in the Limit with Infinite Exploration);

2) η small enough eventually, but not decreasing too fast.

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  - At each step, we choose $\boldsymbol{a} \leftarrow \arg\max_{\boldsymbol{a}'} Q_\pi(\boldsymbol{s}, \boldsymbol{a})$

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- Infinite exploration: all $(\boldsymbol{s},\boldsymbol{a})$ pairs are visited infinite times
  - $\boldsymbol{a} \leftarrow \arg\max_{\boldsymbol{a}'} Q_\pi(\boldsymbol{s},\boldsymbol{a})$ cannot guarantee this
  - Need a better way, e.g., ε-greedy policy

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  - $\boldsymbol{a} \leftarrow \arg\max_{\boldsymbol{a}'} Q_\pi(\boldsymbol{s},\boldsymbol{a})$ cannot guarantee this
  - Need a better way, e.g., ε-greedy policy
- Furthermore, η should satisfy: $\sum_t \eta^{(t)} = \infty$ and $\sum_t \eta^{(t)2} < \infty$
  - E.g., $\eta^{(t)} = O(\frac{1}{t})$
  - $\sum_t \frac{1}{t}$ is a harmonic series known to diverge
  - $\sum_t (\frac{1}{t})^p$, $p > 1$, is a p-series converging to Riemann zeta $\zeta(p)$

Outline

General RL Steps

- Repeat until converge:
  1. Use some exploration policy π' to run episodes/trails
  2. Estimate targets (e.g., $\mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})$ and $R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}')$, $Q_\pi(\boldsymbol{s},\boldsymbol{a})$, or $Q^*(\boldsymbol{s},\boldsymbol{a})$) and update the exploitation policy π
  3. Gradually mix π into π'

General RL Steps

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  1. Use some exploration policy π' to run episodes/trails
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  3. Gradually mix π into π'
- Goals of mix-in strategy:
  - Infinite exploration with π'
  - π' is greedy/exploitative in the end

ε-Greedy Strategy

- At every time step, flip a coin
  - With probability ε, act randomly (explore)
  - With probability $(1-\varepsilon)$, compute/update the exploitation policy and act accordingly (exploit)

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ε-Greedy Strategy

- At every time step, flip a coin
  - With probability ε, act randomly (explore)
  - With probability $(1-\varepsilon)$, compute/update the exploitation policy and act accordingly (exploit)
- Gradually decrease ε over time
- At each time step, the action is at either of two extremes
  - Exploration or exploitation
- "Soft" policy between the two extremes?

Softmax Policy

- Idea: perform action $\boldsymbol{a}$ more often if $\boldsymbol{a}$ gives more accumulative rewards
- E.g., in Q-learning, choose $\boldsymbol{a}$ based on the softmax function:

$$\mathrm{P}(\boldsymbol{a}|\boldsymbol{s}) = \frac{\exp(Q^*(\boldsymbol{s},\boldsymbol{a})/t)}{\sum_{\boldsymbol{a}'}(\exp Q^*(\boldsymbol{s},\boldsymbol{a}')/t)}$$

  - Converts $Q^*(\boldsymbol{s},\boldsymbol{a})$'s to probabilities

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  - Converts $Q^*(\boldsymbol{s},\boldsymbol{a})$'s to probabilities
- t starts from a large value (exploration), and decreases over time (exploitation)

Exploration Function

- Idea: to explore areas with fewest samples
- E.g., in each step of Q-learning, define an exploration function

$$f(q,n) = q + K/n,$$

where

- q the estimated Q-value
- n the number of samples for the estimate
- K some positive constant

Exploration Function

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$$f(q,n) = q + K/n,$$

  where
  - q the estimated Q-value
  - n the number of samples for the estimate
  - K some positive constant
- Instead of:

$$Q^*(\boldsymbol{s},\boldsymbol{a}) \leftarrow Q^*(\boldsymbol{s},\boldsymbol{a}) + \eta \left[(R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma \max_{\boldsymbol{a}'} Q^*(\boldsymbol{s}',\boldsymbol{a}')) - Q^*(\boldsymbol{s},\boldsymbol{a}) \right]$$

- Use f when updating $Q^*(\boldsymbol{s},\boldsymbol{a})$:

$$Q^*(\boldsymbol{s},\boldsymbol{a}) \leftarrow Q^*(\boldsymbol{s},\boldsymbol{a}) + \eta \left\{ [R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma \max_{\boldsymbol{a}'} f(Q^*(\boldsymbol{s}',\boldsymbol{a}'), \mathsf{count}(\boldsymbol{s}',\boldsymbol{a}'))] - Q^*(\boldsymbol{s},\boldsymbol{a}) \right\}$$

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- Exploit once exploring enough

Outline

Value Iteration Revisited

- Value iteration: start from $V^*(s) \leftarrow 0$
  1. Iterate $V^*(s) \leftarrow \max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$ until converge
     ☹ ***Not easy to estimate*** $V^*(s) = \max_\pi \mathrm{E}\left(\sum_t \gamma^t R^{(t)} | \mathbf{s}^{(0)} = s; \pi\right)$
  2. Solve $\pi^*(s) \leftarrow \arg\max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$
     ☹ ***Need model to solve***

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     ☹ ***Need model to solve***
- Q-version that helps sample-based estimation?

Optimal Q Function

- Optimal value function:

$$V^*(\boldsymbol{s}) = \max_{\pi} \mathrm{E}_{\mathbf{s}^{(1)},\cdots} \left(\sum_{t=0}^{\infty} \gamma^t R(\mathbf{s}^{(t)}, \pi(\mathbf{s}^{(t)}), \mathbf{s}^{(t+1)}) \,|\, \mathbf{s}^{(0)} = \boldsymbol{s}; \pi \right)$$

  with recurrence

$$V^*(\boldsymbol{s}) = \max_{\boldsymbol{a}} \sum_{\boldsymbol{s}'} \mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})[R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma V^*(\boldsymbol{s}')], \forall \boldsymbol{s}$$

  - Maximum expected accumulative reward when starting from state $\boldsymbol{s}$ and acting optimally onward

Optimal Q Function

- Optimal value function:

$$V^*(\boldsymbol{s}) = \max_{\pi} \mathrm{E}_{\mathbf{s}^{(1)},\cdots} \left(\sum_{t=0}^{\infty} \gamma^t R(\mathbf{s}^{(t)}, \pi(\mathbf{s}^{(t)}), \mathbf{s}^{(t+1)}) \,|\, \mathbf{s}^{(0)} = \boldsymbol{s}; \pi \right)$$

  with recurrence

$$V^*(\boldsymbol{s}) = \max_{\boldsymbol{a}} \sum_{\boldsymbol{s}'} \mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})[R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma V^*(\boldsymbol{s}')], \forall \boldsymbol{s}$$

  - Maximum expected accumulative reward when starting from state $\boldsymbol{s}$ and acting optimally onward

- Q^* ***function***:

$$Q^*(\boldsymbol{s},\boldsymbol{a}) = \max_{\pi} \mathrm{E}_{\mathbf{s}^{(1)},\cdots} \left(R(\boldsymbol{s},\boldsymbol{a},\mathbf{s}^{(1)}) + \sum_{t=1}^{\infty} \gamma^t R(\mathbf{s}^{(t)}, \pi(\mathbf{s}^{(t)}), \mathbf{s}^{(t+1)}); \boldsymbol{s},\boldsymbol{a},\pi \right)$$

  such that $V^*(\boldsymbol{s}) = \max_{\boldsymbol{a}} Q^*(\boldsymbol{s},\boldsymbol{a})$ with recurrence:

$$Q^*(\boldsymbol{s},\boldsymbol{a}) = \sum_{\boldsymbol{s}'} \mathrm{P}(\boldsymbol{s}'|\boldsymbol{s};\boldsymbol{a})[R(\boldsymbol{s},\boldsymbol{a},\boldsymbol{s}') + \gamma \max_{\boldsymbol{a}'} Q^*(\boldsymbol{s}',\boldsymbol{a}')], \forall \boldsymbol{s}$$

  - Maximum expected accumulative reward when starting from state $\boldsymbol{s}$, taking actin $\boldsymbol{a}$, and then acting optimally onward

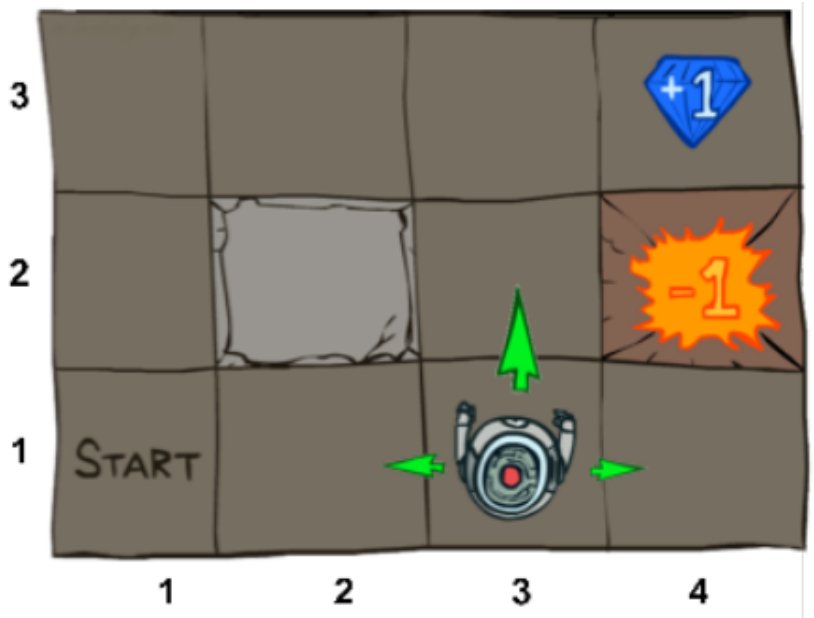
Algorithm: Q-Value Iteration

Input: MDP $(\mathbb{S}, \mathbb{A}, \mathrm{P}, R, \gamma, H \to \infty)$
Output: $\pi^*(s)$'s for all s's

~~For each state s, initialize $V^*(s) \leftarrow 0$~~ Initialize $Q^*(s,a) = 0, \forall s, a$;
repeat
  foreach s *and* a **do**
    ~~$V^*(s) \leftarrow \max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$~~
    $Q^*(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma \max_{a'} Q^*(s',a')]$;
  end
until ~~$V^*(s)$'s~~ $Q^*(s,a)$'s *converge*;
foreach s **do**
  ~~$\pi^*(s) \leftarrow \arg\max_a \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma V^*(s')]$~~
  $\pi^*(s) \leftarrow \arg\max_a Q^*(s,a)$;
end

Algorithm 9: Q-Value iteration with infinite horizon.

Example


3
2
1
START
+1
-1
1
2
3
4


0.64
0.74
0.85
1.00
0.57
0.57
-1.00
0.49
0.43
0.48
0.28
VALUES AFTER 100 ITERATIONS


0.59
0.67
0.77
0.57
0.64
0.60
0.74
0.66
0.85
1.00
0.53
0.67
0.57
0.57
0.57
0.51
0.51
0.53
-0.60
-1.00
0.46
0.30
0.49
0.40
0.48
-0.65
0.45
0.41
0.43
0.42
0.40
0.29
0.28
0.13
0.44
0.40
0.41
0.27
Q-VALUES AFTER 100 ITERATIONS

Q-Value Iteration

- Value iteration: start from $Q^*(s,a) \leftarrow 0$
  1. Iterate $Q^*(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma \max_{a'} Q^*(s',a')]$ until converge
  2. Solve $\pi^*(s) \leftarrow \arg\max_a Q^*(s,a)$

Q-Value Iteration

- Value iteration: start from $Q^*(s,a) \leftarrow 0$
  1. Iterate $Q^*(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma \max_{a'} Q^*(s',a')]$ until converge

     ☹ ***Still not easy to estimate*** $Q^*(s,a) = \max_\pi \mathrm{E}\left(R^{(0)} + \sum_t \gamma^t R^{(t)}\right)$

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     ☺ ***No need for model to solve***

Q-Value Iteration

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  1. Iterate $Q^*(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma \max_{a'} Q^*(s',a')]$ until converge
     - ☹ ***Still not easy to estimate*** $Q^*(s,a) = \max_\pi \mathrm{E}\left(R^{(0)} + \sum_t \gamma^t R^{(t)}\right)$
     - ☺ ***But we can estimate*** $Q^*(s,a)$ ***based on the recurrence now!***
  2. Solve $\pi^*(s) \leftarrow \arg\max_a Q^*(s,a)$
     - ☺ ***No need for model to solve***

Temporal Difference Estimation

$$Q^*(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma \max_{a'} Q^*(s',a')]$$

- Given an exploration policy π' and samples

$$s^{(0)} \xrightarrow{a^{(0)}} s^{(1)} \xrightarrow{a^{(1)}} \cdots \xrightarrow{a^{(H-1)}} s^{(H)}$$

and

$$R(s^{(0)},a^{(0)},s^{(1)}) \rightarrow \cdots \rightarrow R(s^{(H-1)},a^{(H-1)},s^{(H)})$$

- ***Temporal difference (TD) estimation*** of $Q^*(s,a)$ for exploitation policy π:
  1. $\hat{Q}^*(s,a) \leftarrow$ randon value, $\forall s,a$
  2. Repeat until converge ***for each action*** $a^{(t)}$:

$$\hat{Q}^*(s^{(t)},a^{(t)}) \leftarrow \hat{Q}^*(s^{(t)},a^{(t)}) + \eta \left[(R(s^{(t)},a^{(t)},s^{(t+1)}) + \gamma \max_a \hat{Q}^*(s^{(t+1)},a)) - \hat{Q}^*(s^{(t)},a^{(t)}) \right]$$

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$$Q^*(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma \max_{a'} Q^*(s',a')]$$

- Given an exploration policy π' and samples

$$s^{(0)} \xrightarrow{a^{(0)}} s^{(1)} \xrightarrow{a^{(1)}} \cdots \xrightarrow{a^{(H-1)}} s^{(H)}$$

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- $\hat{Q}^*(s,a)$ can be updated ***on the fly*** during exploration

Temporal Difference Estimation

$$Q^*(s,a) \leftarrow \sum_{s'} \mathrm{P}(s'|s;a)[R(s,a,s') + \gamma \max_{a'} Q^*(s',a')]$$

- Given an exploration policy π' and samples

$$s^{(0)} \xrightarrow{a^{(0)}} s^{(1)} \xrightarrow{a^{(1)}} \cdots \xrightarrow{a^{(H-1)}} s^{(H)}$$

and

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- $\hat{Q}^*(s,a)$ can be updated ***on the fly*** during exploration
- η the "forget rate" that needs to gradually decreases

Algorithm: Q-Learning

Input: $\mathbb{S}$, $\mathbb{A}$, and γ
Output: $\pi^*(s)$'s for all s's

For each state s and a, initialize $Q^*(s,a)$ arbitrarily;
foreach *episode* **do**
  Set s to initial state;
  repeat
    Take action a from s using some exploration policy π' derived from Q^* (e.g., ε-greedy);
    Observe s' and reward $R(s,a,s')$;
    $Q^*(s,a) \leftarrow$
    $Q^*(s,a) + \eta\left[(R(s,a,s') + \gamma \max_{a'} Q^*(s',a')) - Q^*(s,a)\right]$;
    $s \leftarrow s'$;
  until s *is terminal state*;
end

Algorithm 10: Q-learning.

Convergence

Theorem

Q-learning converges and gives the optimal policy π^* if

1) π' has explored enough;

2) η small enough eventually, but not decreasing too fast.

Convergence

Theorem

Q-learning converges and gives the optimal policy π^* if

1) π' has explored enough;

2) η small enough eventually, but not decreasing too fast.

- π' has explored enough:
  - All states and actions are visited infinitely often
  - Does ***not*** matter how π' selects actions!

Convergence

Theorem

Q-learning converges and gives the optimal policy π^* if

1) π' has explored enough;

2) η small enough eventually, but not decreasing too fast.

- π' has explored enough:
  - All states and actions are visited infinitely often
  - Does ***not*** matter how π' selects actions!
- η satisfies $\sum_t \eta^{(t)} = \infty$ and $\sum_t \eta^{(t)2} < \infty$

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- η satisfies $\sum_t \eta^{(t)} = \infty$ and $\sum_t \eta^{(t)2} < \infty$
  - E.g., $\eta^{(t)} = O(\frac{1}{t})$
  - $\sum_t \frac{1}{t}$ is a harmonic series known to diverge
  - $\sum_t (\frac{1}{t})^p$, $p > 1$, is a p-series converging to Riemann zeta $\zeta(p)$

Outline

Off-Policy vs. On-Policy RL

- General RL steps: repeat until converge:
  1. Use some exploration policy π' to run episodes/trails
  2. Use smaples to update the exploitation policy π
  3. Gradually mix π into π'

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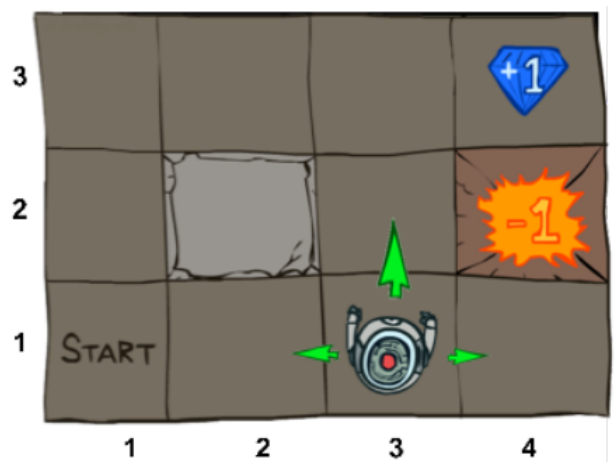
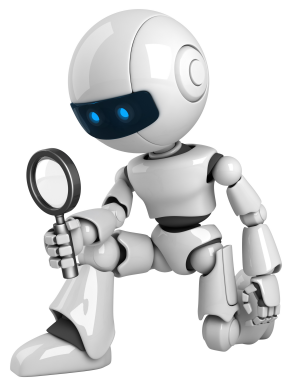
  1. Use some exploration policy π' to run episodes/trails
  2. Use smaples to update the exploitation policy π
  3. Gradually mix π into π'
- ***Off-policy***: π updated toward a greedy policy ***independent with*** π'
  - Q-learning:
    $Q^*(s,a) \leftarrow Q^*(s,a) + \eta\left[(R(s,a,s') + \gamma \max_{a'} Q^*(s',a')) - Q^*(s,a)\right]$
- ***On-policy***: π updated to improve (and ***depends on***) π'
  - SARSA: $Q_\pi(s,a) \leftarrow Q_\pi(s,a) + \eta\left[(R(s,a,s') + \gamma Q_\pi(s',\pi(s'))) - Q^*(s,a)\right]$



π'



π

Practical Results

- SARSA has the capability to ***avoid the mistakes due to exploration***
  - E.g., the maze-with-cliff problem
- But Q-learning has the capability to continue learning while changing the exploration policy

Remarks

- Space complexity for SARSA/Q-learning: $O(|\mathbb{S}||\mathbb{A}|)$
  - Store $Q^*(\boldsymbol{s},\boldsymbol{a})$'s or $Q_\pi(\boldsymbol{s},\boldsymbol{a})$'s for all $(\boldsymbol{s},\boldsymbol{a})$ combinations

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  - Store $Q^*(\boldsymbol{s},\boldsymbol{a})$'s or $Q_\pi(\boldsymbol{s},\boldsymbol{a})$'s for all $(\boldsymbol{s},\boldsymbol{a})$ combinations
- Why not train a (deep) regressor for $Q^*(\boldsymbol{s},\boldsymbol{a})$'s or $Q_\pi(\boldsymbol{s},\boldsymbol{a})$'s?
  - Reduced space thanks to the generalizability of the regressor