Geometric Independet Set Problem

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Chapter

Intersection graph

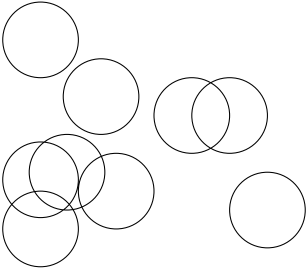
- Introduction
- Independent set of IG
- PTAS

M(W)IS in Unit Disk Graphs

MIS for Rectangles

Intersection graph

- Given an arrangement A of geometric objects



Intersection graph

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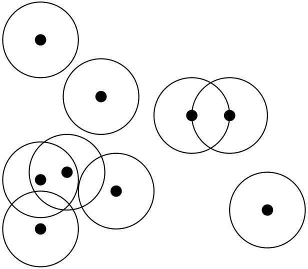

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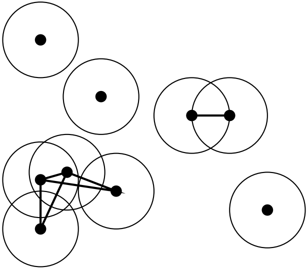

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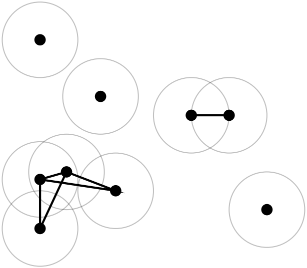

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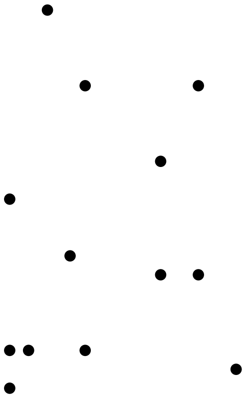
  - Path between two vertices on IG of geometric shapes corresponds to a path between two points without leaving perimeter of shapes (e.g. cheapest way from A to B with fee on borders)
  - Independent set (non-adjacent vertices) of IG of geometric shapes corresponds to non-intersecting subset of the shapes in the arrangement

Map labelling

- Given a 2D map with various points of interest – which labels should be drawn?



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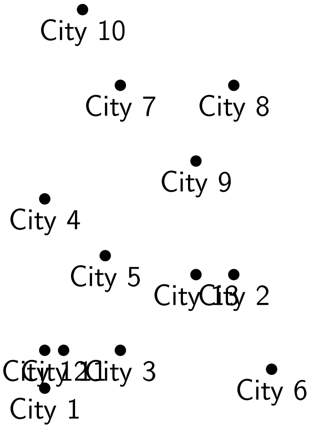

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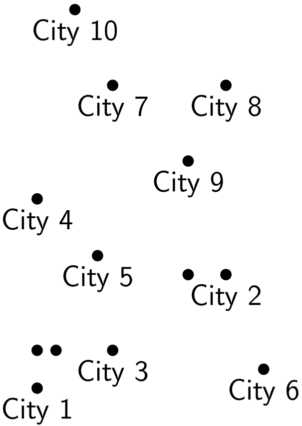
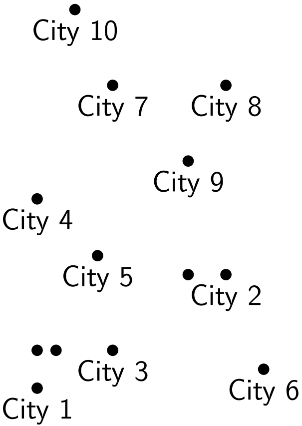


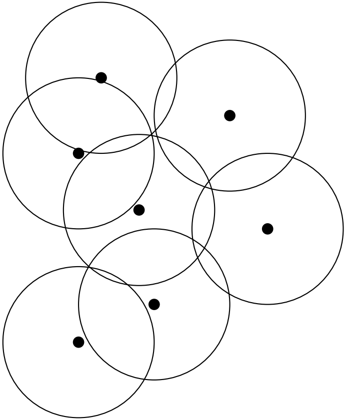
Map labelling

- Given a 2D map with various points of interest – which labels should be drawn?
- Labels must not intersect each other
- Variants of the problem:
  - Labels with size-constraints (e.g.: uniform height)
  - Labels are allowed in a radius around corresponding datapoint



Construction of wireless network [Chamaret et al., 1997]

- Maximize area with reception



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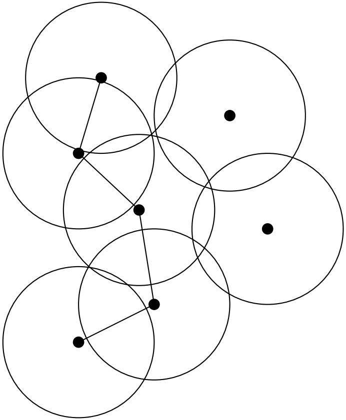
- Maximize area with reception
- Building base stations may be expensive $\rightarrow$ overlap is inefficient.



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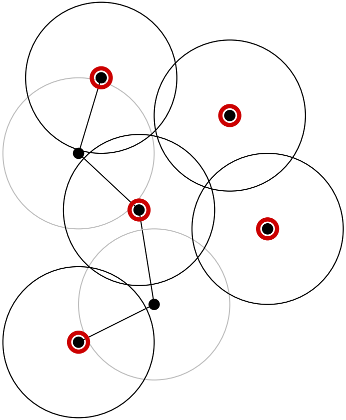

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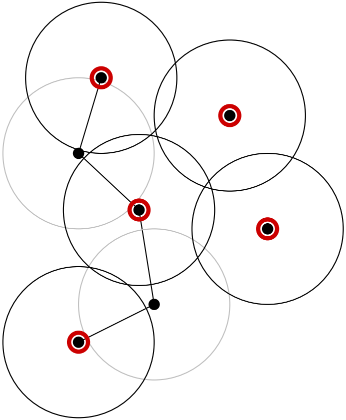
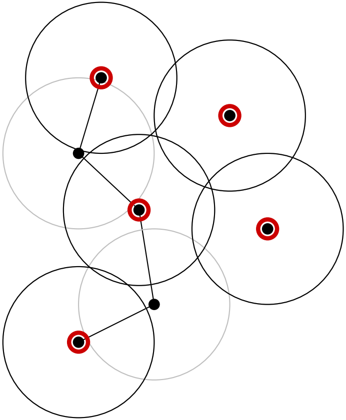


Construction of wireless network [Chamaret et al., 1997]

- Maximize area with reception
- Building base stations may be expensive → overlap is inefficient.
- Can also be used for assigning frequencies
  - More or less a graph coloring problem
  - Smallest number of necessary frequencies (≙ colors) is the chromatic number [McDiarmid and Müller, 2011] which is related to IS.



Polynomial time approximation scheme

- Finding IS of graph $\mathcal{NP}$- hard — therefore finding approximative solutions in polynomial time would already be nice.
- Some random estimate rather uninteresting—approximation should have some quality estimate.

Polynomial time ρ – approximation scheme

Finds solution with a solution quality that is at least $\frac{S_{OPT}}{\rho}$ in polynomial time (for fixed $\rho > 1$)

Chapter

Intersection graph

M(W)IS in Unit Disk Graphs

- Problem description
- Unweighted MIS
- Weighted MIS

MIS for Rectangles

PTAS for M(W)IS problem in UDG [Nieberg et al., 2004]

- Geometric objects are only disks with radius 1
- Two variants of the problem – with or without geometric representation
- If geometric representation is known seperation alongside some grid is possible which allows more efficient approaches (shifting strategy [Fonseca et al., 2014])
- Finding geometric representation from intersection graph is $\mathcal{NP}$- hard [Hlineny and Kratochvil, 2001]

Unweighted MIS of UDG

Given: Some graph $G = (E, V)$ which is a UDG iff there is a mapping $f : V \to \mathbb{R}^2$ s.t.:

$$(u, v) \in E \leftrightarrow ||f(u) - f(v)|| \leq 2 \qquad \forall u, v \in V, (u \neq v)$$

Desired: Subset $I \subset V$ s.t. $|I| \cdot \rho \geq \alpha(G)$
$\alpha(G)$ . . . size maximum IS

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- Idea of algorithm
  1. Take subsets of graph with bounded size
  2. Calculate MIS of subsets
  3. Combine MIS of subsets to get IS of whole graph

Step 1: Subsets of finite size

Define the following sets for some arbitary node $v_0 \in V$

$$N_r := \{w | w \in V, w \text{ has distance at most } r \text{ from } v_0\}$$

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and calculate MIS $I_r \subset N_r$ for $r = 0, 1, \dots \bar{r}$ where $\bar{r}$ is defined as the smallest r s.t.

$$I_{r+1} > \rho |I_r|$$

does not hold.

Bounding size of $I_{\bar{r}}$

$$\forall w \in N_r : \quad ||f(v_0) - f(w)|| \leq 2r$$

therefore it's possible to draw a circle with radius $R = 2r + 1$ and centerpoint v_0 that contains all disks in I_r and

$$|I_r| \leq \pi R^2 / \pi = O(r^2).$$

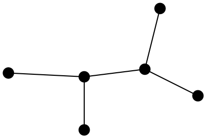
By definition of $\bar{r}$

$$|I_r| > \rho |I_{r-1}| > \ldots > \rho^r |I_0| = \rho^r$$

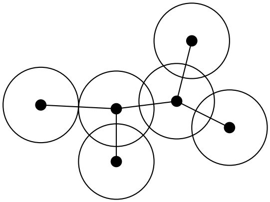
also holds. Combining these two results:

$$\rho^r < |I_r| \leq O(r^2)$$

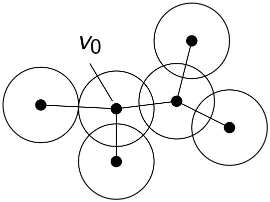
$\implies$ constant (depending only on ρ) bound on $\bar{r}$



UDG



UD arrangement & UDG



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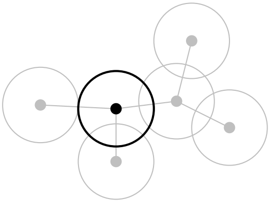
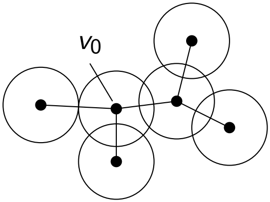


N_0



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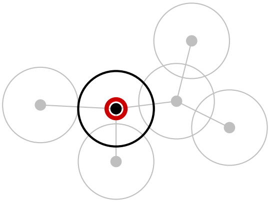
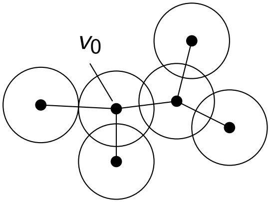


N_0 (black) and I_0 (red)



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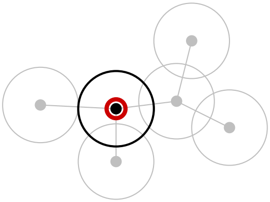

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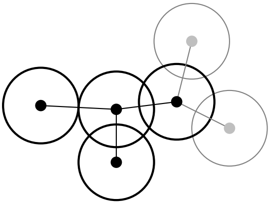
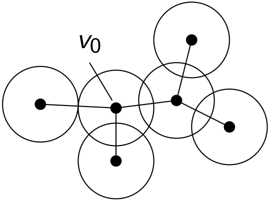


N_1



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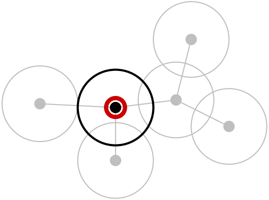

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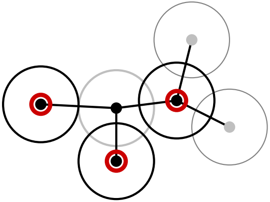

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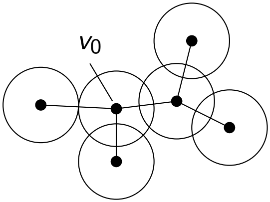

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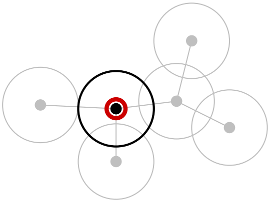

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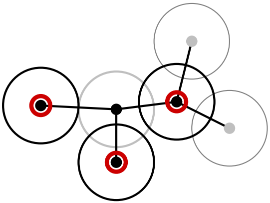

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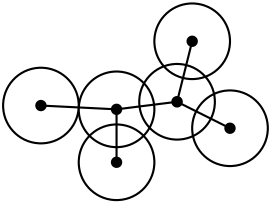
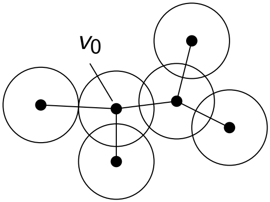


N_2



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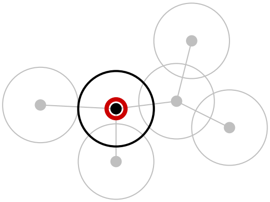

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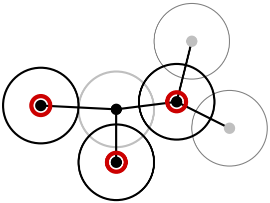

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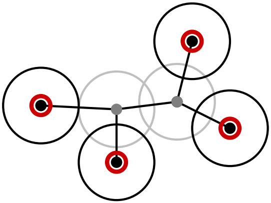
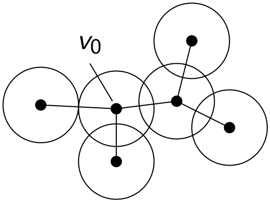


N_2 and I_2



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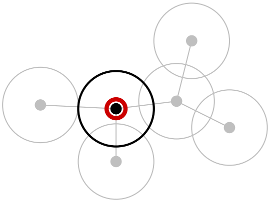

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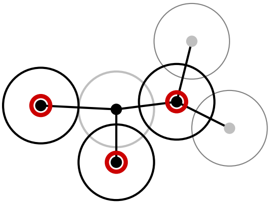

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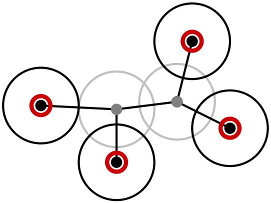

N_2 and I_2, $\bar{r} = 1$

From the definition of N_r it follows that for the subgraph $G' = G[N_{\bar{r}+1}]$ the maximum independet set size is bounded:

$$\alpha(G') \leq \rho|I_{\bar{r}}|.$$

$H = G \setminus G'$ has no vertices adjacent to vertices in $N_{\bar{r}}$ and therefore a ρ-approximate IS for H (I_H) combined with $I_{\bar{r}}$ yields a ρ-approximate IS for G.

$$\alpha(G) \leq \alpha(H) + \alpha(G') \leq \rho|I_H \cup I_{\bar{r}}|$$

Algorithm

1. Define sets N_r for arbitary node $v \in V$ and calculate independent sets I_r until stopping criterion reached
   - Remember that r has constant bound $\implies$ determining I_r is possible in $O(n^{C^2})$.
2. Repeat step 1. for subgraph $H = G \setminus N_{\bar{r}+1}$ until $H = \varnothing$.
3. Output independent set $I = \bigcup I_{\bar{r}}$ where $I_{\bar{r}}$ are alle the IS found in step 1.

Algorithm for maximum weighted independet set problem

When defining sets N_r start with v such that

$$\omega(v) \geq \omega(v') \forall v' \in V \qquad (= \text{argmax}\, \omega(v))$$

and use

$$\omega(I_{r+1}) > \rho\omega(I_r)$$

as stopping criterion (for $\bar{r}$) to get bound on the weight of the IS of the subsets.

Chapter

Intersection graph

M(W)IS in Unit Disk Graphs

MIS for Rectangles

- MIS for arbitrary rectangles
- MIS for unit height rectangles
- Take away points

MIS for arbitrary rectangles [Agarwal et al., 1998]

- Geometric objects are rectangles in $\mathbb{R}^2$ with parallel edges
- Desired: Biggest subset of those rectangles that don't intersect each other
- $\log n$ – approximate algorithm for this case with $O(n \log n)$ runtime:
  - Divide and concquer: Split into subsets and calculate independet sets for them
  - Combine IS of subsets.

Maximum independet set for arbitrary rectangles

Given: Set S of rectangles.

- Sort edges by x and y coordinates — $O(n \log n)$

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  4. Calculate MIS of S_M with greedy approach (outlined on blackboard) — $O(n)$

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  5. Take either $I_L \cup I_R$ or I_M

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Overall runtime: $O(n \log n)$

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- Map labels often have similar/identical height
- Use height to split problem in small subproblems that don't depend on each other
- Apply greedy scheme from before for subproblems

MIS for unit height rectangles (UHR)

Draw $m+1$ horizontal lines that fulfill the following criteria:

- y - distance > 1
- Each line intersects ≥ 1 rectangles
- Each rectangle is intersected by 1 line.

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Take union of either odd or even subsets

Take away points

For difficult problems use all available information about objects at hand to make problem solvable (or at least get good approximate solutions):

- 1st algorithm (UDG): minimum area of a independet set of geometric shapes used to get an upper bound on size of subsets
- 2nd algorithm (greedy heuristic): reduce dimensionality of problem by line that intersects all rectangles
- 3rd algorithm (UHR): decompose problem into subproblems that can be solved easily

References I

Agarwal, P. K., van Kreveld, M., and Suri, S. (1998).
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- Robustness would mean: Determine in polynomial runtime whether runtime will be polynomial. Is this possible?
- Actually possible in this case: If $|I_r > (2r+1)^2$ is found $\implies$ graph is not UDG.

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- Algorithm described solves n independet subproblems optimally and discards results of $\frac{n}{2}$

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- Solving subproblems of k lines each only every $\frac{1}{k}$-th subproblem has to be discarded $\rightarrow \left(1+\frac{1}{k}\right)-$ approximation.