The Provability Logic G

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Chapter

Incompleteness

- 1931: Paper by Gödel "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme"[1]
  - Incompleteness of Peano Arithmetic was shown by enumerating formulas of PA which enabled arguing about them in PA itself
  - Predicate $\mathrm{Prov}(x)$ – formula A s.t. $\ulcorner A \urcorner = x$ is provable in PA. More formally $\mathrm{Prov}(x) \equiv \exists p \, \mathrm{Proof}(p, x)$ where $\mathrm{Proof}(p, x)$ formalizes the notion that p is the GN of the formula that is a proof of the formula with GN x.
  - Constructed formula that asserts its own unprovability.

Incompleteness Theorems

Theorem (First Incompleteness Theorem)

Every consistent and reasonably expressive arithmetic system contains sentences that are neither provable nor refutable.

Theorem (Second Incompleteness Theorem)

No consistent and moderately expressive arithmetic system can prove its own consistency.

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$$\neg A \equiv A \supset \bot : \qquad \Box(\Box\bot \supset \bot) \supset \Box\bot$$

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  is a fixed point of $A(p)$ and (almost) uniquely defined.
- For $A(p)$ as defined above the fixed point B is $\neg\Box\bot$ and:

$$G \vdash \neg\Box\bot \equiv \neg\Box(\neg\Box\bot)$$

Chapter

The Logic G – Axioms

- Distribution axiom

$$\Box(A \supset B) \supset (\Box A \supset \Box B)$$

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- Transitivity (consequence of Löb's theorem $[(B \wedge \Box B)/A]$)

$$\Box B \supset \Box\Box B$$

Tableau rules

- Start with K4 (transitive) tableau
  - If α is true in Γ then α_1 and α_2 are true in Γ.
  - If β is true in Γ then β_1 or β_2 are true in Γ.
  - If π is true in Γ then π_0 is true in some Γ^*.
  - If ν is true in Γ then ν_0 is true in all Γ^*.
  - If ν is true in Γ then ν is true in all Γ^*.

α		β		
$TA \wedge B$	TA, TB	$FA \wedge B$	FA	FB
$FA \vee B$	FA, FB	$TA \vee B$	FA	FB
$FA \supset B$	TA, FB	$TA \supset B$	FA	TB

ν	$F\Diamond A$	$FA, F\Diamond A$
ν	$T\Box A$	$TA, T\Box A$
π	$T\Diamond A$	TA
π	$F\Box A$	FA

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Examples

On the blackboard:

With K4 Tableau: $\Box A$

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With K4 Tableau: $\Box A$

With G Tableau: $\Box(\Box A \supset A) \supset \Box A$

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- Local compactness does not hold for G

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- Global compactness not known (at least neither Fittings nor Google knew)

References

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Derivation of Löb's theorem

Assume $T \not\models A$

$T \not\models \mathrm{Prov}(\ulcorner A \urcorner) \supset A$

$T \cup \{\neg A\} \not\models \bot$

$T \cup \{\neg A\} \not\models \neg\,\mathrm{Prov}_{T \cup \neg A}(\ulcorner \bot \urcorner)$

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$T \not\models \neg A \supset \neg\,\mathrm{Prov}_{T}(\ulcorner A \urcorner)$

$T \not\models \mathrm{Prov}_{T}(\ulcorner A \urcorner) \supset A$