Mini Course: Matrix Eigendecomposition

$A \rightsquigarrow \det(A-\lambda I)=0$

$A=\begin{pmatrix}1&0&1\\-1&2&1\\0&0&2\end{pmatrix} \quad \det(A-\lambda I)=\det\begin{pmatrix}1-\lambda&\cdot&1\\-1&2-\lambda&1\\\cdot&\cdot&2-\lambda\end{pmatrix}$

$=(2-\lambda)^2(1-\lambda)=0 \rightsquigarrow \lambda_1=1$

$\lambda_{2/3}=2 \quad (\times 2)$

$S=(C_1\ C_2\ C_3)$

$C=\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \quad \boxed{AC=\lambda_1 C_1}$

$AC=\begin{pmatrix}1&\cdot&1\\-1&2&1\\\cdot&\cdot&2\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}\alpha+\gamma\\-\alpha+2\beta+\gamma\\2\gamma\end{pmatrix}=\lambda\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$

$\lambda=1 \rightsquigarrow \alpha+\gamma=\alpha \rightsquigarrow \gamma=0$

$-\alpha+2\beta=\beta \rightsquigarrow \alpha=\beta \quad \boxed{C_1=\begin{pmatrix}\alpha\\\alpha\\0\end{pmatrix}}$

$\lambda=2 \rightsquigarrow -\alpha+\cancel{2\beta}+\gamma=\cancel{2\beta} \rightsquigarrow \alpha=\gamma \quad C_{2/3}=\begin{pmatrix}\alpha\\\beta\\\alpha\end{pmatrix}$

$\boxed{C_2=\begin{pmatrix}\alpha\\0\\\alpha\end{pmatrix},\ C_3=\begin{pmatrix}0\\\beta\\0\end{pmatrix}}$

$A=\begin{pmatrix}2&4&3\\-4&-6&-3\\3&3&1\end{pmatrix} \quad \det(A-\lambda I)=0$

$\det\begin{pmatrix}2-\lambda&4&3\\-4&-6-\lambda&-3\\3&3&1-\lambda\end{pmatrix}$

$=(2-\lambda)\left((-6-\lambda)(1-\lambda)+9\right)=\cdots$

$\Rightarrow \boxed{\lambda=1,-2,-2}$

$\lambda \quad AC=\lambda C$

$\lambda=1 \rightsquigarrow \begin{pmatrix}2\alpha+4\beta+3\gamma\\-4\alpha-6\beta-3\gamma\\3\alpha+3\beta+\gamma\end{pmatrix}=\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$

$\alpha+4\beta+3\gamma=0 \rightsquigarrow 3\beta+3\gamma=0$

$\rightsquigarrow \beta=-\gamma$

$-4\alpha-7\beta-3\gamma=0$

$3\alpha+3\beta=0 \rightsquigarrow \alpha=-\beta$

$$C_1 = \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \end{pmatrix}$$

$\lambda = -2$ $(\times 2)$

$$\begin{aligned} 4\alpha + 4\beta + 3\gamma &= 0 \\ -4\alpha - 4\beta - 3\gamma &= 0 \\ 3\alpha + 3\beta + 3\gamma &= 0 \rightsquigarrow \alpha + \beta + \gamma = 0 \\ &\gamma = -\alpha - \beta \\ 4\alpha + 4\beta + 3\alpha + 3\beta = 0 &\rightsquigarrow \alpha = -\beta \end{aligned}$$

$$C = \begin{pmatrix} \alpha \\ -\alpha \\ 0 \end{pmatrix}$$

$\nexists S$ ⚠ NUMERICS

POWERS

$$A^m = \overbrace{A \cdot A \cdots A}^{m} \qquad S^{-1} A S = \operatorname{diag}(\ldots) = \Delta$$

$$A = S \Delta S^{-1}$$

$$A^m = S\Delta \underbrace{S^{-1} \cdot S}_{I} \Delta \underbrace{S^{-1} \cdots S}_{I} \Delta S^{-1}$$

$$= S \Delta \cdot \Delta \cdot \Delta \cdots \Delta \cdot S^{-1}$$

$$= S \Delta^m S^{-1} \qquad \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} = \begin{pmatrix} \alpha^2 & \\ & \beta^2 \end{pmatrix}$$

$$= S \operatorname{diag}(\lambda_1^m, \ldots, \lambda_m^m) S^{-1}$$

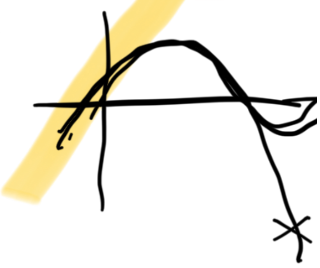
FUNCTIONS

$$f(x) \rightsquigarrow f(A)$$

QM $\quad i\hbar \frac{\partial}{\partial t} \psi(t) = H \psi(t)$

$$\rightsquigarrow \psi(t) = e^{-\frac{i}{\hbar} H t} \psi(t=0)$$

TAYLOR: $\quad f(x) = \sum_{i=0}^{\infty} a_i x^i$



$$\exp(x) = \sum_{i=0}^{\infty} \frac{x^n}{n!}$$

$$f(A) = \sum_{i=0}^{\infty} a_i A^i$$

$$f(A) = \sum_{i=0}^{\infty} a_i A^i$$
$$= \sum a_i (S \Delta S^{-1})^i$$
$$= \sum a_i (S \Delta^i S^{-1})$$
$$= S \left(\sum a_i \Delta^i\right) S^{-1}$$
$$= S \left(\sum a_i \operatorname{diag}(\lambda_1^i, \dots, \lambda_n^i)\right) S^{-1}$$
$$= S \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_n)) S^{-1}$$

SESSION 3

SINGULAR VALUE DECOMPOSITION (SVD)

$M \in \mathbb{C}^m_n \qquad M = U \Sigma V^*$

U, V unitary $UU^* = U^*U = I$

Σ diagonal

M REAL, U,V ORTHOGONAL

$$M = U\Sigma V^T$$

↳ σ is SV if $M\vec{v} = \sigma\vec{u}$

↳ $M \Rightarrow \Sigma$

$\not\Rightarrow U, V \ , \ \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$

$$MM^* = V\Sigma \underbrace{U^* U}\Sigma^* V^* = V(\Sigma\Sigma^*)V^*$$

$$\underbrace{M^*M} = V\Sigma^* V^* V\Sigma U^* = \underbrace{U(\Sigma^*\Sigma)U^*}$$

V are eigenvectors of MM^*

U $\underline{M^*M}$

↳ $\underline{M^*M = MM^*}$ AND positive semi-def M

$\rightarrow M = UDU^*$ ~>

DIFFERENTIAL $\underline{\text{EQ}}$

LINEAR SYSTEM

$$\frac{\partial x}{\partial t} = ax + by \qquad \begin{pmatrix}\partial_t x \\ \partial_t y\end{pmatrix} = A\begin{pmatrix}x \\ y\end{pmatrix}$$

$$\frac{\partial y}{\partial t} = cx + dy$$

$$A = S\Delta S^{-1} \qquad \begin{pmatrix}\partial_t x \\ \partial_t y\end{pmatrix} = S\Delta S^{-1}\begin{pmatrix}x \\ y\end{pmatrix}$$

$$\underbrace{S^{-1}\begin{pmatrix}\partial_t x \\ \partial_t y\end{pmatrix}} = \Delta \underbrace{S^{-1}\begin{pmatrix}x \\ y\end{pmatrix}}$$

$$\begin{pmatrix}\partial_t u \\ \partial_t v\end{pmatrix} = \Delta \begin{pmatrix}u \\ v\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\partial_t u \\ \partial_t v\end{pmatrix} = \begin{pmatrix}\lambda_1 u \\ \lambda_2 v\end{pmatrix} \Rightarrow \begin{pmatrix}u \\ v\end{pmatrix} = \begin{pmatrix}e^{\lambda_1 t} u_0 \\ e^{\lambda_2 t} v_0\end{pmatrix}$$

$$\underline{\begin{pmatrix}x \\ y\end{pmatrix}} = S\begin{pmatrix}u \\ v\end{pmatrix} = S\begin{pmatrix}e^{\lambda_1 t} & \\ & e^{\lambda_2 t}\end{pmatrix} S^{-1}\underline{\begin{pmatrix}x_0 \\ y_0\end{pmatrix}}$$

u v
- ↳ λ real, >1, expansion
- ↳ λ real, <1, squeeze
- ↳ λ imaginary → rotation
- ↳ λ complex → growth + oscillation

A cannot be diagonalized

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \lambda = 1 \ (\times 2)$$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad A\vec{v} = \vec{v}$$

$\vec{v}$ eigenvectors is solution: $\vec{v}_0 e^{\lambda t}$

& $\vec{v} \cdot t \cdot e^{\lambda t} + \vec{w} e^{\lambda t}$

where $(A - \lambda I)\vec{w} = \vec{v}$

$$\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \beta \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\vec{v}$ solut°.

$$\alpha \vec{v} e^{\lambda t} + \beta (\vec{v} t e^{\lambda t} + \vec{w} e^{\lambda t})$$

$$t = 0 \quad \alpha \vec{v} + \beta \vec{w} = \vec{x_0}$$

$$\begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$x_0 \vec{v} e^{\lambda t} + y_0 (\vec{v} t e^{\lambda t} + \vec{w} e^{\lambda t})$$

QUANTUM MECHANICS

$H \rightarrow$ Hamiltonian

Schrödinger eq.

$$i\hbar \frac{\partial}{\partial t} \Psi = H \Psi$$

constant; WAVE FUNCTION

- Ψ is vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ Quant. INFOR.

- Ψ is complex function / continuous

$$H = \frac{-p^2}{2m} + V(x) \longrightarrow \text{pot. En.}$$

$\hookrightarrow$ kinetic En.

$$= \frac{-\hbar^2}{2m} \nabla^2 + V(x)$$

$$\nabla^2 \longrightarrow \frac{\partial^2}{\partial x^2}$$

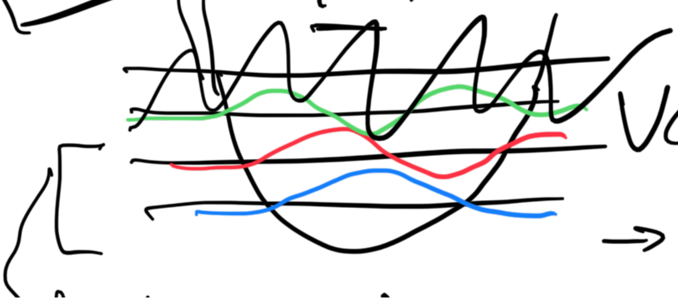
$\rightarrow$ MATRIX / OP.

$$i\hbar \frac{\partial}{\partial t} \Psi = H \Psi \longrightarrow \text{any}$$

$$\boxed{\begin{array}{c} H = H^* \\ \text{HERMITIAN} \end{array}}$$

if $H \underline{\Psi} = E \underline{\Psi}$ $\rightarrow$ real, eigenvalue; $\rightarrow$ eigenstate

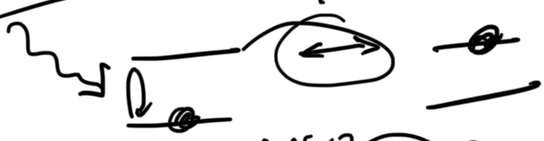
$$\Psi(t) = e^{-\frac{i}{\hbar}Et} \, \underline{\Psi}(t=0)$$



$V(x) = \frac{1}{2} m\omega^2 x^2$

$\rightarrow E_n = (n + \frac{1}{2})\hbar\omega$

$\psi \hookrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$



$\cancel{H = \delta |ee\rangle\langle ee|}$

INF 2

$$H = \begin{pmatrix} \delta & \Omega & \Omega & 0 \\ \Omega & 0 & 0 & \Omega \\ \Omega & 0 & 0 & \Omega \\ 0 & \Omega & \Omega & 0 \end{pmatrix} \rightarrow \text{LASER}$$

Eig $\hookrightarrow$

$\cancel{2\Omega}$

$\boxed{\delta}$

$\lambda = 2$ - - -

$\lambda = -2$ - - -

$\lambda = 0$ - - - $\rightarrow$ X

$$H = -\frac{\hbar^2}{2m} \nabla^2 + V(x)$$

$$\frac{\partial^2}{\partial x^2} f(x) \simeq \frac{1}{dx^2} \left(f(x+dx) - 2f(x) + f(x-dx) \right)$$

$$\frac{1}{dx^2} \begin{pmatrix} \ddots & & & \\ 1 & -2 & 1 & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ x-dx \\ x \\ x+dx \\ \vdots \end{pmatrix}$$