# On Curvature and Evolute for Curves Generated by a Rolling Circle

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**Resumo**

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## 1 Introduction

Consider a parametrized curve in an oriented coordinate system, $(O,\, \underline{\mathbf{i}},\, \underline{\mathbf{j}})$:

$$\underline{\mathbf{r}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

with $t$ in some interval, $I \subset \mathbb{R}$. We will assume that the functions $x(t)$ and $y(t)$ are two times diferentiable in $I$, ensuring the existence of the *velocity*, $\underline{\mathbf{v}}(t)$, and *acceleration*, $\underline{\mathbf{a}}(t)$, vectors:

$$\underline{\mathbf{v}}(t) = \underline{\mathbf{r}}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \qquad \underline{\mathbf{a}}(t) = \underline{\mathbf{r}}''(t) = \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix}$$

The (scalar) velocity is given by:

$$v(t) = |\,\underline{\mathbf{r}}'(t)| = \sqrt{x'^2 + y'^2}$$

A point of the curve where: $v(t) \neq 0$ is said to be *regular*, in such a point we defined the unit tangent vector:

$$\underline{\mathbf{t}}(t) = \frac{\underline{\mathbf{r}}'(t)}{|\,\underline{\mathbf{r}}'(t)|} = \frac{1}{\sqrt{x'^2 + y'^2}} \begin{pmatrix} x' \\ y' \end{pmatrix},$$

as well as it's positive normal:

$$\underline{\mathbf{n}}(t) = \widehat{\underline{\mathbf{t}}}(t) = \frac{1}{\sqrt{x'^2 + y'^2}} \begin{pmatrix} -y' \\ x' \end{pmatrix}$$

Before we continue, we note that for a vector function of fixed length: $\underline{\mathbf{a}}(t) \cdot \underline{\mathbf{a}}(t) = l^2$, it's derivative is always perpendicular to itself:

$$\underline{\mathbf{a}}(t) \cdot \underline{\mathbf{a}}'(t) = 0$$

In the following, we shall also need the unity vectors:

$$\underline{\mathbf{e}}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \qquad \underline{\mathbf{f}}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}; \qquad \underline{\mathbf{p}}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \qquad \underline{\mathbf{q}}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

Note that:

$$\widehat{\underline{\mathbf{e}}}(t) = \underline{\mathbf{f}}(t), \qquad \widehat{\underline{\mathbf{f}}}(t) = -\underline{\mathbf{e}}(t); \qquad \widehat{\underline{\mathbf{p}}}(t) = \underline{\mathbf{q}}(t), \qquad \widehat{\underline{\mathbf{q}}}(t) = -\underline{\mathbf{p}}(t)$$

and:

$$\underline{\mathbf{e}}'(t) = \underline{\mathbf{f}}(t), \qquad \underline{\mathbf{f}}'(t) = -\underline{\mathbf{e}}(t); \qquad \underline{\mathbf{p}}'(t) = -\underline{\mathbf{q}}(t), \qquad \underline{\mathbf{q}}'(t) = -\underline{\mathbf{p}}(t)$$

# 2 Curvatura

As may found in any textbook on Diferential Geometry, the *Curvature* of a curve, in a regular point, is the function:

$$\kappa = \kappa(t) = \frac{[\underline{\mathbf{r}}'(t) \times \underline{\mathbf{r}}''(t)]}{|\underline{\mathbf{r}}'(t)|^3} = \frac{x'y'' - x''y'}{|\underline{\mathbf{r}}'(t)|^3}$$

In the following we denote:

$$D(t) = x'y'' - x''y = [\underline{\mathbf{r}}'(t) \times \underline{\mathbf{r}}''(t)] = \widehat{\underline{\mathbf{r}}}'(t) \cdot \underline{\mathbf{r}}''(t)$$

Geometrically, the value of $D(t)$ is the (signed) area of the parallellogram spanned by $\underline{\mathbf{r}}'$ and $\underline{\mathbf{r}}''$, being positive when $\underline{\mathbf{r}}''(t)$ points in the same direction as $\underline{\mathbf{n}}$. Moreover, $D(t) = 0$, is equivalente to $(\underline{\mathbf{r}}', \underline{\mathbf{r}}'')$ being linearly dependent.

In points where $\kappa(t) \neq 0$, the curvature ratio is given by: $\rho(t) = \kappa(t)^{-1}$, and the curvature vector is:

$$\underline{\rho}(t) = \rho\,\underline{\mathbf{n}} = \frac{|\underline{\mathbf{r}}'(t)|^3}{D(t)} \cdot \frac{\widehat{\underline{\mathbf{r}}}'(t)}{|\underline{\mathbf{r}}'(t)|} = \frac{|\underline{\mathbf{r}}'(t)|^2}{D(t)}\,\widehat{\underline{\mathbf{r}}}'(t) = \nu(t)\,\widehat{\underline{\mathbf{r}}}'(t),$$

where we have introduced the quantity:

$$\nu(t) = \frac{|\underline{\mathbf{r}}'(t)|^2}{D(t)}$$

Finally, the formula for calculating the *Center of Curvature*, is:

$$\underline{\mathbf{c}}(t) = \underline{\mathbf{r}}(t) + \rho\,\underline{\mathbf{n}}$$

The curve traced by the centers of curvature, is termed the curve's *Evolute*. In the following, we will calculate the evolute for some curve families.

# 3 Cycloids and Trochoids

Letting a circle of ratio $a$ roll on the $x$-axis, results in a curve, called a Cycloid.

**Definition 1.** ***Cycloids***.

*By a Cycloid, with ratio $a$ and phase $\gamma$, $C(a, \gamma)$, we understand the curve parametrisized by:*

$$\underline{\mathbf{r}}(t) = at\,\underline{\mathbf{i}} + a\,\underline{\mathbf{j}} - a\,\underline{\mathbf{p}}(t+\gamma)$$

*The Cycloid $C(1, -\pi/3)$ is shown in Fig. 1.*

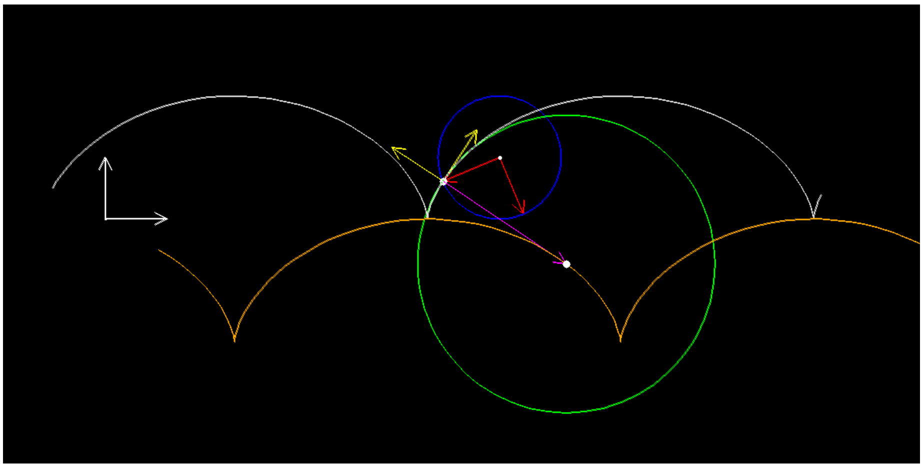

Figura 1: **Cycloid**. The Cycloid $C(1, -\pi/3)$, (grey curve). Included is, calculated using numerical derivation: the Cycloid's evolute (orange curve), the curvature vecor (magenta), the osculating circle (green) as well as the coordinate systems: World (white), rolling circle (red) and the accompanying, ($\underline{\mathbf{r}}$, $\underline{\mathbf{t}}$, $\underline{\mathbf{n}}$) (yellow). In the figures to follow, we shall adopt this colouration scheme.

Fig. 1 seems to indicate, that: The Evolute of the Cycloid, is another Cycloid. We shall prove this in the remainder of this section.

Scaling the rolling vector, $b\,\underline{\mathbf{p}}(t)$, $b \in \mathbb{R}$, we obtain another curve, called a Trochoid. As for the Cycloid, we allow for a phase, $\gamma$, thus parametrizising:

**Definition 2.** ***Trochoids***.

*By a $\gamma$-fased Trochoid, $T(a, b, \gamma)$, we understand the curve parameterized by:*

$$\underline{\mathbf{r}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} at - b\sin(t+\gamma) \\ a - b\cos(t+\gamma) \end{pmatrix} =$$

$$a\begin{pmatrix} t \\ 1 \end{pmatrix} - b\begin{pmatrix} \sin(t+\gamma) \\ \cos(t+\gamma) \end{pmatrix} =$$

$$at\,\underline{\mathbf{i}} + a\,\underline{\mathbf{j}} - b\,\underline{\mathbf{p}}(t+\gamma)$$

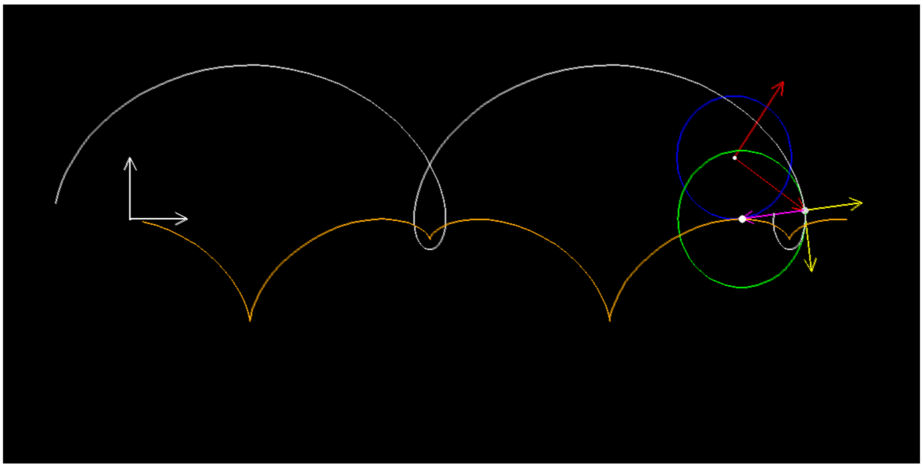

Figura 2: **Trochoid**. The Cycloid $T(1, 0.5, -\pi/3)$

The Trochoid $T(a, -a, \gamma)$, is a rephased Cycloid: $C(a, \gamma+\pi)$. Omitting the phase parameter, we assume it 0: $T(a, b) = T(a, b, 0)$ and $C(a) = C(a, \gamma)$. Before continuing, we note that the vector $\underline{\mathbf{p}}(t)$, behaves similarly to the vectors $\underline{\mathbf{e}}$ and $\underline{\mathbf{f}}$ in the previous section. It's orthogonal complement, $\underline{\mathbf{q}}$, is:

$$\underline{\mathbf{q}}(t) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

Omitting the tedious parameter, $t$, clearly: $\widehat{\underline{\mathbf{q}}} = -\underline{\mathbf{p}}$, and for the derivatives:

$$\underline{\mathbf{p}}' = -\underline{\mathbf{q}}, \qquad \underline{\mathbf{q}}' = \underline{\mathbf{p}}$$

Considering the family of Trochoids, we shall calculate the curvature, $\kappa(t)$, the curvature radius, $\rho(t)$, as well as the curvature vector, $\rho\,\underline{\mathbf{n}}$ and the parametrization of the Trochoid evolute, $\underline{\mathbf{c}} = \underline{\mathbf{r}} + \rho\,\underline{\mathbf{n}}$. In fact, we will show that the evolute of a Cycloid, is a translated Cycloid, whereas this is *not* the case for the general Trochoid, $b \neq 0$.

Deriving once:

$$\underline{\mathbf{r}}' = a\,\underline{\mathbf{i}} + b\,\underline{\mathbf{q}}$$

Note, that $\underline{\mathbf{r}}'(t) = 0$, implies $\sin t = 0$, that is: $t = p\pi$, $p \in \mathbb{Z}$. This satisfied, $\cos p\pi = (-1)^p = -a/b$. That is, for $b = a$ (the Cycloid), the irregular points are at $t = (2n+1)\pi$. In the case $b = -a$, which is another Cycloid, the irregular points are at $t = 2n\pi$. In the case of a 'true' Trochoid, $b \neq \pm a$, all points are regular.

The second derivative is:

$$\underline{\mathbf{r}}'' = b\,\underline{\mathbf{p}}$$

Taking the orthogonal complement of the derivate:

$$\widehat{\underline{\mathbf{r}}'} = a\,\underline{\mathbf{j}} - b\,\underline{\mathbf{p}}$$

For the determinant:

$$D(t) = \widehat{\underline{\mathbf{r}}'}(t) \cdot \underline{\mathbf{r}}''(t) = \left(a\,\underline{\mathbf{j}} - b\,\underline{\mathbf{p}}\right) \cdot b\,\underline{\mathbf{p}} =$$

$$ab\,\underline{\mathbf{j}}\cdot\,\underline{\mathbf{p}}-b^2=b\,(a\cos t-b)$$

Investigating, $D(t)=0$:

$$a\cos t=b\quad\Leftrightarrow\quad\cos t=\frac{b}{a},$$

Thus $D(t)$ has has two roots (counting multiplicity), if and only if, $b\in[-a,a]$, given by:

$$\delta_1=\mathrm{Arccos}\left(\frac{b}{a}\right)\qquad\qquad\delta_2=\pi-\delta_1$$

This leads us to suspect to see diferent Trochoid curvature behaviour, in the cases:

1. $|b|>a$: $D(t)$ has fixed sign; Type 1 Trochoids.
2. $|b|=a$: $D(t)$ has a zero, but does not change sign. Cycloids or degenerated Trochoids.
3. $|b|>a$: $D(t)$ changes sign twice; Type 2 Trochoids.

Calculating the squared velocity:

$$\underline{\mathbf{r}}'\cdot\underline{\mathbf{r}}'=\left(a\,\underline{\mathbf{i}}+b\,\underline{\mathbf{q}}\right)\cdot\left(a\,\underline{\mathbf{i}}+b\,\underline{\mathbf{q}}\right)=$$

$$a^2+b^2+2ab\,\underline{\mathbf{i}}\cdot\,\underline{\mathbf{q}}=a^2+b^2-2ab\cos t$$

The Trochoid curvature is:

$$\kappa(t)=\frac{D(t)}{|\,\underline{\mathbf{r}}'(t)|^3}=\frac{b\,(a\cos t-b)}{(a^2+b^2-2ab\cos t)^{3/2}}$$

For the ratio:

$$\nu(t)=\frac{|\,\underline{\mathbf{r}}'(t)|^2}{D(t)}=\frac{a^2+b^2-2ab\cos t}{b\,(a\cos t-b)}$$

Therefore, the curvature vector, $\rho\,\underline{\mathbf{n}}$, is:

$$\rho\,\underline{\mathbf{n}}=\nu(t)\,\widehat{\underline{\mathbf{r}}'}=\frac{a^2+b^2-2ab\cos t}{b\,(a\cos t-b)}\left(a\,\underline{\mathbf{j}}-b\,\underline{\mathbf{p}}\right)$$

Thus, we obtain for the Trochoid evolute:

$$\underline{\mathbf{c}}(t)=\,\underline{\mathbf{r}}(t)+\rho\,\underline{\mathbf{n}}(t)=at\,\underline{\mathbf{i}}+a\,\underline{\mathbf{j}}-b\,\underline{\mathbf{p}}+\nu(t)\left(a\,\underline{\mathbf{j}}-b\,\underline{\mathbf{p}}\right)=at\,\underline{\mathbf{i}}+(1+\nu(t))\left(a\,\underline{\mathbf{j}}-b\,\underline{\mathbf{p}}\right)$$

We observe the common factor:

$$1+\nu(t)=1+\frac{a^2+b^2-2ab\cos t}{b\,(a\cos t-b)}=$$

$$\frac{a^2+b^2-2ab\cos t+ba\cos t-b^2}{b\,(a\cos t-b)}=\frac{a^2-ab\cos t}{b\,(a\cos t-b)}=-\frac{a}{b}\cdot\frac{a-b\cos t}{b-a\cos t}$$

Finally we may write the Trochoid evolute:

$$\underline{\mathbf{c}}(t)=at\,\underline{\mathbf{i}}-\frac{a}{b}\cdot\frac{a-b\cos t}{b-a\cos t}\left(a\,\underline{\mathbf{j}}-b\,\underline{\mathbf{p}}\right)$$

For this curve to be another Trochoid, the appearing factor should be constant:

$$\varphi(t) = \frac{a - b\cos t}{b - a\cos t}$$

Deriving:

$$\varphi'(t) = \frac{b\cos t - a\cos t}{(b - a\cos t)^2} = \frac{(b-a)\cos t}{(b - a\cos t)^2} = 0$$

Which is satisifed, if and only if, $a = b$, being the special case of a Cycloid, $b = a$ obtaining the Cycloid evolute:

$$\underline{\mathbf{c}}(t) = at\,\underline{\mathbf{i}} - \frac{1-\cos t}{1-\cos t}\left(a\,\underline{\mathbf{j}} - a\,\underline{\mathbf{p}}\right) = at\,\underline{\mathbf{i}} - \left(a\,\underline{\mathbf{j}} - a\,\underline{\mathbf{p}}\right)$$

This is a translated Cycloid, with the same half-axis and the phase $\pi$ *ahead* of the original. We state this in a theorem:

**Theorem 1.** *The evolute of the Cycloid $C(a)$ is the translated Cycloid $\underline{\mathbf{R}}_0 + C(a,\pi)$, whereas the evolute of a 'true' Trochoid $T(a,b)$ is* not *a Trochoid.*

# 4 Epitrochoids and Hypotrochoids

**Definition 3.** ***Epicycloids and Hypocycloids.***
*Consider a fixed circle of radius $a > 0$, centered in the origin, and a rolling circle of radius $b > 0$. Rolling the latter on the outside of the fixed circle, results in a curve called an* Epicycloid*, with parametrization:*

$$\underline{\mathbf{r}}(t) = (a+b)\,\underline{\mathbf{e}} - b\,\underline{\mathbf{e}}(\omega t + \gamma)$$

*As for the Trochoids, we have included a phase parameter, and introduced the angular velocity of rolling circle, $\omega = (a+b)/b$. We write $E(a,b,\gamma)$.*
*Changing the sign of the last term in this parametrization, the circle are rolling on the inside of the fixed circe and we arrive at a socalled* Hypocycloid*, $H(a,b,\gamma)$. Allowing negative b's, we may write: $H(a,b,\gamma) = E(a,-b,\gamma)$.*

As to be seen in the remainder of this section, these curves presents striking similarities with the Cycloids and Trochoids. In fact, we shall prove that the evolute of an Epicycloid, respectively a Hypocycloid, is again an Epicycloid, respectively a Hypocycloid.

**Definition 4.** ***Epictrochoids and Hypotrochoids.***
*As in the case of Cycloids and Trochoids, we extend the description to include Trochoid like curves, with $c \in \mathbb{R}$:*

$$\underline{\mathbf{r}}(t) = (a+b)\,\underline{\mathbf{e}} - c\,\underline{\mathbf{e}}(\omega t + \gamma)$$

*For brevity, we write: $E(a,b,c,\gamma)$, respectively $H(a,b,c,\gamma)$.*

We will also prove that the evolute of an Epitrochoid, respectively a Hypotrochoid, is *not* an Epitrochoid, respectively a Hypotrochoid.

Dado circumferência fixo de raio $R$ e circumferência de raio $r$, rolando sem deslizar dentro deste, ou seja com velocidadade angular:

$$\omega = \frac{R+r}{r} = 1+\eta$$

Epicicloid:

$$\underline{\mathbf{r}}(t) = (R+r)\,\underline{\mathbf{e}} - r\,\underline{\mathbf{e}}_\omega$$

Derivando:

$$\underline{\mathbf{r}}'(t) = (R+r)\,\underline{\mathbf{f}} - r\omega\,\underline{\mathbf{f}}_\omega$$

$$\underline{\mathbf{r}}''(t) = -(R+r)\,\underline{\mathbf{e}} + r\omega^2\,\underline{\mathbf{e}}_\omega$$

Transversor da primeira derivada:

$$\widehat{\underline{\mathbf{r}}'}(t) = -(R+r)\,\underline{\mathbf{e}} + r\omega\,\underline{\mathbf{e}}_\omega$$

Vetor de curvatura:

$$\rho\,\underline{\mathbf{n}} = \frac{v(t)^2}{D(t)}\,\widehat{\underline{\mathbf{r}}'}(t) = \frac{v(t)^2}{D(t)}\,\left[-(R+r)\,\underline{\mathbf{e}} + r\omega\,\underline{\mathbf{e}}_\omega\right]$$

Centro de Curvatura:

$$\underline{\mathbf{c}}(t) = (R+r)\,\underline{\mathbf{e}} - r\,\underline{\mathbf{e}}_\omega + \frac{v(t)^2}{D(t)}\,\left[-(R+r)\,\underline{\mathbf{e}} + r\omega\,\underline{\mathbf{e}}_\omega\right] =$$

$$(R+r)\left[1-\frac{v(t)^2}{D(t)}\right]\underline{\mathbf{e}} - r\left[1-\omega\frac{v(t)^2}{D(t)}\right]\underline{\mathbf{e}}_\omega$$

Velocidade:

$$v(t)^2 = (R+r)^2 + r^2\omega^2 - 2(R+r)r\omega\;\underline{\mathbf{f}}\cdot\underline{\mathbf{f}}_\omega$$

Determinante:

$$D(t) = [\underline{\mathbf{r}}'(t)\times\underline{\mathbf{r}}''(t)] = \widehat{\underline{\mathbf{r}}'}(t)\cdot\underline{\mathbf{r}}''(t) =$$

$$\left[-(R+r)\,\underline{\mathbf{e}} + r\omega\,\underline{\mathbf{e}}_\omega\right]\cdot\left[-(R+r)\,\underline{\mathbf{e}} + r\omega^2\,\underline{\mathbf{e}}_\omega\right] =$$

$$(R+r)^2 + (r\omega)^2\omega - \left\{(R+r)r\omega^2 + r\omega(R+r)\right\}\,\underline{\mathbf{e}}\cdot\underline{\mathbf{e}}_\omega$$

No Epicicloide, $r\omega = R+r$:

$$v(t)^2 = (R+r)^2 + (R+r)^2 - 2(R+r)(R+r)\;\underline{\mathbf{f}}\cdot\underline{\mathbf{f}}_\omega = 2(R+r)^2\left\{1-\underline{\mathbf{f}}\cdot\underline{\mathbf{f}}_\omega\right\}$$

$$D(t) = (R+r)^2 + (R+r)^2\omega - \{(R+r)(R+r)\omega + (R+r)(R+r)\}\;\underline{\mathbf{e}}\cdot\underline{\mathbf{e}}_\omega =$$

$$(R+r)^2(1+\omega)\left\{1-\underline{\mathbf{e}}\cdot\underline{\mathbf{e}}_\omega\right\}$$

Juntando e usando o fato $\underline{\mathbf{e}}\cdot\underline{\mathbf{e}}_\omega = \underline{\mathbf{f}}\cdot\underline{\mathbf{f}}_\omega = \cos\eta t$:

$$\frac{v(t)^2}{D(t)} = \frac{2(R+r)^2\left\{1 - \underline{\mathbf{f}} \cdot \underline{\mathbf{f}}_\omega\right\}}{(R+r)^2(1+\omega)\left\{1 - \underline{\mathbf{e}} \cdot \underline{\mathbf{e}}_\omega\right\}} =$$

$$\frac{2}{1+\omega} = \frac{2r}{r+r\omega} = \frac{2r}{R+2r}$$

Calculamos a evoluta do Epicicloide:

$$\underline{\mathbf{r}}_c = \underline{\mathbf{r}} + \frac{v(t)^2}{D(t)}\widehat{\underline{\mathbf{r}}'} =$$

$$(R+r)\,\underline{\mathbf{e}} - r\,\underline{\mathbf{e}}_\omega + \frac{v(t)^2}{D(t)}\left[-(R+r)\,\underline{\mathbf{e}} + r\omega\,\underline{\mathbf{e}}_\omega\right] =$$

$$(R+r)\left\{1 - \frac{v(t)^2}{D(t)}\right\}\underline{\mathbf{e}} - r\left\{1 - \omega\frac{v(t)^2}{D(t)}\right\}\underline{\mathbf{e}}_\omega$$

Finalmente obtemos:

$$1 - \frac{v(t)^2}{D(t)} = 1 - \frac{2r}{R+2r} = \frac{R+2r-2r}{R+2r} = \frac{R}{R+2r}$$

E:

$$1 - \omega\frac{v(t)^2}{D(t)} = 1 - \frac{R+r}{r}\frac{2r}{R+2r} = \frac{1}{R+2r}\left[R+2r-2(R+r)\right] = -\frac{R}{R+2r}$$

A evoluta do Epicicloide:

$$\underline{\mathbf{r}}_c(t) = (R+r)\frac{R}{R+2r}\,\underline{\mathbf{e}} + r\frac{R}{R+2r}\,\underline{\mathbf{e}}_\omega$$

Ou seja, novamente um Epicicloide, com:

$$R' = \frac{R}{R+2r}R, \qquad r' = -\frac{R}{R+2r}r$$