Linear optimization

(P) $\max\{c^T x \mid Ax = b,\ x \geq 0\}$ $\qquad A \in \mathbb{R}^{m \times n}$

(D) $\min\{b^T y \mid A^T y \geq c\} = \min\{b^T y \mid A^T y - s = c,\ s \geq 0\}$

Theorem (Weak Duality)

If x is feasible for P, y feasible for D, then

$$b^T y \geq c^T x$$

Theorem (Strong Duality)

For each pair of linear programs, one of the folling is true:

1. (P), (D) are both infeasible
2. (P) unbounded, (D) infeasible
3. P) infeasible, (D) unbounded
4. (P) feasible, (D) feasible

$$\Rightarrow \quad b^T y^* = c^T x^*$$ ☺

Thm (Complementary Slackness)

When x^*, y^* are optimal solutions of P, D, then

$$x_i^* s_i^* = 0 \quad , i = 1, \ldots, n$$

Moral: Solving LP's results to solving the following system of equations and inequalities

$$\boxed{\begin{array}{ccc} Ax = b, & A^T y - s = c, & x^T s = 0 \\ x \geq 0, & s \geq 0 & \end{array}}$$

Simplex Method (1947 George Dantzig)

Satisfy $Ax=b$, $A^Ty-S=c$, $x\geq 0$

Defn: X is a Basic Feasible Solution if $\exists$ a submatrix A_B of A with rank m (invertible) such that

$$X_j = 0 \text{ if } j \notin B$$

$$\left(\text{therefore } A_B X_B = b, \quad X_B = A_B^{-1} b\right)$$

Lemma: If there exists an optimal solution of (P), then there exists an optimal BFS of (P).
(Assume bounded solution)

proof: Need 2 things:

1. Every BFS = an extreme point of (P)
   and vice versa
2. Every point in P can be written as a convex combination of extreme points + direction of unboundedness.

hence $x^* = d + \sum \alpha_i V_i$

• note that $c^Td=0$ since otherwise we could scale in that direction

then $c^Tx^* = \sum \alpha_i c^T V_i \leq \max c^T V_i \Rightarrow c^Tx^* = c^TV_i$

$\square$

Representation

$$b = Ax = A_B X_B + A_N X_N \Rightarrow X_B = A_B^{-1}b - A_B^{-1}A_N X_N$$

$$\begin{aligned} z = c^Tx &= c_B^T X_B + c_N^T X_N \\ &= c_B^T A_B^{-1} b + (c_N^T - \underbrace{\boxed{c_B^T A_B^{-1}}}_{= y} A_N) X_N \end{aligned}$$

$$c_j - y a_j = s_j$$

Pivot Rule: Select index from those non-basic variables that correspond to

$$s_j < 0$$

i.e. $c_j - y^T a_j = -s_j > 0$.

Many choices for pivot rules!!!

Entering variable: Call a_j the column vector of the entering variable $x_j = 0$.

increase x_j slowly by $t > 0$

$$x_B^{new} = x_B^{old} - t(A_B^{-1} a_j)$$

$\Rightarrow$ eventually one other variable will vanish

$\rightarrow$ this variable leaves the basis

$\rightarrow$ OR no variable vanishes $\Rightarrow$ unbounded direction.

How to find first BFS? $Ax = b$, $x \geq 0$, $b \geq 0$ (otherwise multiply by -1)

Set $\bar{A} = (A, I_m)$, $\bar{x} = (x, x_{n+1}, \dots, x_{n+m})$

Solve $\bar{A}\bar{x} = b$, $\bar{x} \geq 0$, using $\bar{x}_0 = (0, b)$

i.e.

$$\max \; -(x_{n+1} + \dots + x_{n+m})$$
$$\text{s.t.} \quad \bar{A}\bar{x} = b$$

if $\max = 0$ $\Rightarrow$ we found an initial BFS

if $\max < 0$ $\Rightarrow$ original problem is infeasible.

Open Problem: Can you find a pivot rule s.t. the number of pivot rules necessary to get an optimal solution is polynomial in n & m?

Can you bound the diameter of a graph in n & m?

Interior Point Methods (1984 Karmarker)

Theorem: For all $\lambda > 0$, $\star$ has a unique real solution $(x^*(\lambda), y^*(\lambda), s^*(\lambda))$ s.t.

1.) $x^*(\lambda)$ is an optimal solution to
$$\max\left(c^T x + \sum_{i=1}^{n} \lambda \log(x_i) \;\middle|\; Ax=b,\ x \geq 0\right)$$

2.) In the limit, the point $(x^*(0), y^*(0), s^*(0))$ is an optimal solution to (P).

Defn: The set of all real solutions of $\star$ as $\lambda \to 0$ is the central path

$$\boxed{\star \quad Ax=b,\quad x_i s_i = \lambda,\quad A^T y - s = c,\quad x \geq 0,\ s \geq 0}$$

Steps of IPM: 1. Set $\lambda = 1$, solve $*$, find initial point (x, y, s) (or close enough point)

Given $\epsilon > 0$,

while $\lambda > \epsilon$ REPEAT

$$\lambda \to \left(1 - \frac{1}{2\sqrt{n}}\right)\lambda$$

$$(x, y, s) \to (x + \Delta x, y + \Delta y, s + \Delta s)$$

$$A(x + \Delta x) = b \qquad A^T(y + \Delta y) - (s + \Delta s) = c$$

$$(x_i + \Delta x_i)(s_i + \Delta s_i) = \lambda$$

$$s_i + \Delta s_i \geq 0 \qquad x_i + \Delta x_i \geq 0$$

Cheat: $A(\Delta x) = 0, \quad A^T(\Delta y) - \Delta s = 0$

$$s_i \Delta x_i + x_i \Delta s_i = \hat{\lambda}$$

LP Applications

- game theory
- combinatorial optimization
  - maxflow / min cut
  - help Integer Programs
  - Approximation Algorithms

game theory:

2-person, 0-sum game

win for one person = loss for other person

player 1 "cops"

"robbers" Player 2

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 4 & -10 & -10 \\ 2 & -8 & 5 & -8 \\ 3 & -12 & -12 & 9 \end{array}$$

pay off for player 1 = −(payoff for player 2)

$\Delta X = \{x \mid \sum x_i = 1\}$ $\Delta y = \{y \mid \sum y_i = 1\}$

Mixed strategy: Police patrol site i with probability x_i

robbers attack site i with probability y_i

$\sum x_i = 1$ $\sum y_i = 1$

Expected pay off: $\sum_{i,j} a_{ij} x_j y_i = y^T A x$ ← Police want to maximize this

Police try to find $\bar{x} \in \Delta x$ s.t.

$$\bar{x} = \underset{x \in \Delta x}{\arg\max} \left(\min_{y \in \Delta y} y^T A x \right)$$

$\Rightarrow$ Police can guarantee a payoff of at least

$$\text{Payoff} \geq \max_x \left(\min_y y^T A x \right)$$

Similarly, let $\bar{y} = \underset{y \in \Delta y}{\arg\min} \left(\max_{x \in \Delta x} (y^T A x) \right)$

Then $\max_x \min_y y^T A x \leq \bar{y}^T A \bar{x} \leq \min_y \max_x y^T A x$

Thm (Von-Neuman)

$$\max_x \min_y y^T A x = \min_y \max_x y^T A x$$

<u>Corollary</u>: $\bar{x}, \bar{y}$ is an equilibrium point

i.e. $\bar{y}^T A x \leq \bar{y}^T A \bar{x}$

and $y^T A \bar{x} \geq \bar{y}^T A \bar{x}$

$\Rightarrow$ no one has incentive to move.

<u>Pseudo-Proof of Theorem</u>:

Consider $\max_{x \in \Delta x} \min_{y \in \Delta y} y^T A x$

If we fix a particular x^*, then $\min_y y^T A x^* = \min_i (Ax^*)_i$

("Put all chips in one bucket")

<u>LP formulation</u>:

$v = \max z$

(Primal) s.t. $z \leq a_i x$ $\quad$ a_i = i^{th} row of A, it rows of A.

$\sum x_i = 1$

$x_i \geq 0$

$\Rightarrow$ (Dual) $\min \gamma$

s.t. $\gamma \geq y a^i$

$\sum y_i = 1$

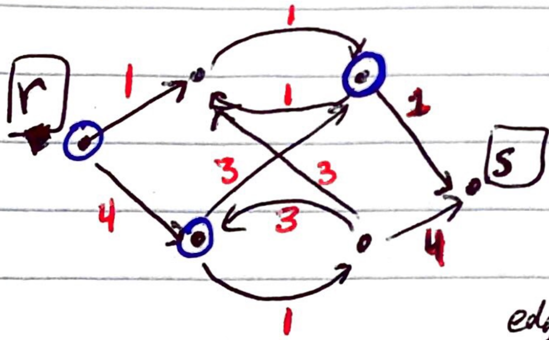
$y \geq 0$

Dual equivalent to

$$\min_y \max_x y^T A x.$$

Since $opt(Primal) = opt(dual)$ $\Rightarrow$ Result $\square$

Max-flow / min-cut Problem



Consider max flow from left to right

max flow ≥ min cut.

edges leaving ⊙ is 4

Theorem: Given a network/directed graph with vertices r, s, the maximum flow from r to s is exactly equal to the min cut between r & s.

Defn: A r-s cut is a partition of the vertices

$$V = R \sqcup S$$

s.t. $r \in R$, $s \in S$, $S \cap R = \emptyset$

Defn: For a vertex v, $\delta^+(v)$ = set of all edges leaving v.

$\delta^-(v)$ = set of all edges entering v.

LP formulation:
- let x_e for all $e \in E$ be the amount of flow along the edge.
- for a vertex v, $x(\delta^+(v)) = \sum_{e \in \delta^+(v)} x_e$

and $x(\delta^-(v)) = \sum_{e \in \delta^-(v)} x_e$

Max flow

$$\max \quad x(\delta^+(r)) - x(\delta^-(r))$$

s.t.

$$x(\delta^+(v)) = x(\delta^{-}(v)) \quad \forall v \neq r, s$$

$$x_e \geq 0, \quad x_e \leq c_e$$

Min cut

$$\min \quad \sum_{e \in E} c_e y_e$$

s.t.

$$z_r = 1, \quad z_s = 0$$

$$z_w - z_v + y_{vw} \geq 0 \quad \forall e \in E$$

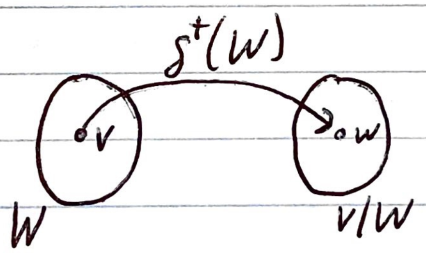
$$y_e \geq 0, \quad z_v \in \mathbb{R}$$

Complementary Slackness

Let x^*, y^*, z^* be optimal solutions

then $y_e^*(c_e - x_e^*) = 0, \quad (z_w - z_v + z_{vw}) \cdot x_{vw} = 0$

? $W = \{v \mid z_v^* \geq 1\}$, any edge in $\delta^+(W)$.

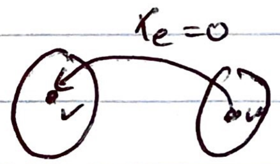


$z_v^* \geq 1, \quad z_w^* < 1$

$(z_w^* - z_v^* + y_e^*) \geq 0 \Rightarrow y_e^* > 0 \Rightarrow x_e^* = c_e$

"edge is at full capacity"

new case



for any edge in $e \in \delta^-(W)$, $e \in V/W$

$z_w^* - z_v^* + y_e^* \geq z_w^* - z_v^* > 0$

$z_v^* < 1, \quad z_w^* \geq 1 \Rightarrow x_e = 0$

$\Rightarrow$ Capacity of the cut is equal to the flow value. $\square$

LP Solves IP

Theory of Totally Unimodular Matrices

Defn: A is called Totally Unimodular
if every square submatrix has determinant $\pm 1, 0$.

Thm: $A = T.U.$ then
$$\min\{c^T x \mid Ax = b,\ x \geq 0\}$$
always has integral optimal solutions for every integral vector b.

Proof: Let x^* be the opt solution.
Then $\exists$ A_B $(m \times m)$ s.t. $x_B^* = A_B^{-1} b$, $x_N = 0$.
A_B^{-1} is integral.
$\Rightarrow$ by Cramers rule, x_B^* is integral.

Thm: Let A be a $\pm 1, 0$ matrix. If every column of A has at most one $+1$ and one -1, then A is $T.U.$

Proof: Let B be a submatrix of A.
if B has a 0 column (or row) then $\det(B) = 0$.
if B has a column w/ exactly 1, or -1, then expand about this column (induction)
if B has all 1 & -1 columns, then sum of rows is 0.
$\Rightarrow$ linearly dependent $\Rightarrow \det(B) = 0$.

$\square$

Prop: A = T.U. $\iff$ $\begin{bmatrix} A \\ I \end{bmatrix}$ is T.U. $\iff$ $\begin{bmatrix} A \\ I \end{bmatrix}$ is unimodular.

Vertex Cover:

- Find a subset of vertices which touch all edges.

Approximate Solution:

given weights c_v, $x_v \in \{0,1\}$

$$\min \sum_{v \in V} c_v x_v$$

$$\text{s.t.} \quad x_v + x_w \geq 1 \quad \forall\ (v,w) \in E$$

$$0 \leq x_v \leq 1 \quad \forall\ v \in V.$$

Solve using LP to get x^*.

- If $x^* \in \mathbb{Z}^n$, done!
- If not, let $\boxed{S = \{v \mid x_v^* \geq 1/2\}}$

Claim 1: S is a vertex cover

Claim 2: $\boxed{\text{weight}(S) \leq 2\,\text{weight}(S^*)}$

$$= 2 \sum_{v \in V} c_v x_v^*$$

$\square$

Ellipsoid Method (1979) Kachiyan

input: a convex set $S \subseteq \mathbb{R}^n$
given to you by a seperation oracle
& ϵ s.t. $vol(S) > \epsilon$.

Output: A point $x \in S$ or "$S = \emptyset$."

Idea: Assume $\exists$ an ellipsoid

$$E_{M, z_0} = \{x \in \mathbb{R}^n \mid (x - z_0)^T M (x - z_0) \leq 1\}$$

$M =$ positive definite

$S \subseteq E_{M, z_0}$, $z_0 =$ center of E

* Check if $z_0 \in S$.
→ if $z_0 \notin S$, find new ellipsoid that encompases $H_k^+ \cap E_k$.

Construct $E_{M_{k+1}, z_{k+1}}$ & Repeat while

$$vol(E_{M_k, z_k}) > vol(S).$$

lemma: one can explicitly construct from $E_{M_k z_k}$ & H_k,
the exact equations for $E_{M_{k+1} z_{k+1}}$

with minimum volume &

$$vol(E_{M_{k+1} z_{k+1}}) < vol(E_{M_k, z_k}) \cdot e^{\frac{-1}{2(n+1)}}$$

After k Steps

$$\mathrm{vol}(E_{M_k, z_k}) < \mathrm{vol}(E_{M_0, z_0})\, e^{-\frac{k}{2(n+1)}}$$

Note: $\mathrm{vol}(S) < \mathrm{vol}(E_{M_0, z_0})\, e^{-k/2(n+1)}$

$$\Rightarrow \quad k \le 2(n+1) \ln\left(\frac{\mathrm{vol}(E_{M_0, z_0})}{\mathrm{vol}(S)}\right)$$

Also: If $P = \{y \mid A^T y \le b\}$, $A = n \times n$ matrix,

if $P \ne \emptyset$, then it must contain a ball of radius

$2^{-7\phi^3}$ where $\phi = O(m \cdot n \cdot \max(\log_2 A_{ij}, \log_2 b_i))$

History: 1989 (Renegar, Gonzaga, Roos | Vial)

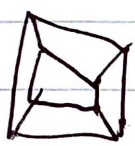
find an optimal Interior Point Method in $O(n^3 L)$

2000 (Deza, Terlaky) best possible ever.

Simplex Counterexample (exponential)

Klee-Minty Cube

$0 \le x_1 \le 1, \quad \epsilon x_{k-1} \le x_k \le 1 - \epsilon x_{k-1}, \quad k = 2, \ldots, n$



Other algorithms for LP:

1.) Criss-Cross Method
2.) Perceptron Algorithm
3.) Motzkin Relaxation Algorithm
4.) Vempala Style Algorithms

Open: Prove any of these are polynomial

Curvature

Total curvature = length of the Gauss map

Curve $\gamma \rightarrow \dfrac{\dot{\gamma}}{\|\dot{\gamma}\|}$ tangent vector

- Gauss map takes point on curve to tangent vector pointing on the sphere

Open: $2\pi n \overset{?}{\geq}$ total curvature

Strongly Polynomial $= O(n^k)$ k constant

* Holy grail for LP *