STAT430 Homework #8: Due Friday, April 19, 2019.

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0. We will continue with **Chapter 9** on estimation.

1. (a). Suppose $Y_1, Y_2, \ldots, Y_n$ are iid Geometric with probability θ, where $0 < \theta < 1$. Show that $\sum_{i=1}^{n} Y_i$ is sufficient for θ.

Answer:

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \theta(1-\theta)^{y_i - 1} \\
&= \theta^n \prod (1-\theta)^{y_i}(1-\theta)^{-1} 1_{(y_i \geq 1)} \\
&= \left(\frac{\theta^n}{(1-\theta)^n}(1-\theta)^{\sum y_i}\right)\left(\prod 1_{(y_i \geq 1)}\right)
\end{aligned}
$$

So, $u = \sum y_i$ is sufficient for θ

1(b). Find $\hat{\theta}_n$, the method of moments estimator of θ. Is $\hat{\theta}_n$ a function of the sufficient statistic?

Answer:

1(c). Show that $\hat{\theta} \xrightarrow{P} \theta$ as $n \to \infty$.

Answer:

1(d). Use the delta method to show that a large-sample normal approximation for $\hat{\theta}_n$ is

$$
\hat{\theta}_n \text{ is approximately } N\left(\theta, \frac{\theta^2(1-\theta)}{n}\right) = N\left(\theta, \sigma_{\hat{\theta}}^2\right).
$$

Answer:

1(e). Give an estimator $\hat{\sigma}_{\hat{\theta}}^2$ for the theoretical variance $\sigma_{\hat{\theta}}^2$ from the delta method. Prove that

$$
\frac{\hat{\sigma}_{\hat{\theta}}^2}{\sigma_{\hat{\theta}}^2} \xrightarrow{P} 1,
$$

then show that

$$
Q_n = \frac{\hat{\theta}_n - \theta}{\sqrt{\hat{\sigma}_{\hat{\theta}}^2}}
$$

is an asymptotic pivotal value for θ: a function of θ and the data that is asymptotically $N(0,1)$.

Answer:

1(f). Suppose $\theta = 0.2$ and $n = 75$. The book parameterizes a geometric random variable as the trial number of the first success $(1, 2, 3,\ldots)$, while R parameterizes a geometric random variable as the number of failures before the first success $(0,1,2,\ldots)$. Both parameterizations are common. Use the follow bit of R code to simulate 10,000 realizations of $\bar{Y}$ (each the average of 75 Geometric(0.2) random realizations, using the book's parameterization).

```
set.seed(4302019)
num_realizations <- 10000
n <- 75
theta <- 0.2
Y <- rgeom(n* num_realizations, prob = theta) +1 # add one to go from number of failures to trial number
YY <- matrix(Y, n, num_realizations)
Y_bar <- apply(YY, MAR = 2, FUN = "mean") # 1 for rows, 2 for columns
mean(Y_bar)
```

[1] 4.997267

Use the simulated $\bar{Y}$ values to create 10,000 realizations of $\hat{\theta}_n$. Compute the empirical variance of your 10,000 realizations and compare to the theoretical asymptotic variance, $\sigma_{\hat{\theta}}^2$. Plot the histogram of your 10,000 realizations and add the theoretical pdf of $N\left(\theta, \sigma_{\hat{\theta}}^2\right)$ using the dnorm function in R.

Answer:

2. Complete **Exercise 9.37** of the text.

Answer:

$$
\begin{aligned}
L(p) &= \prod p^{x_i}(1-p)^{1-x_i}\mathbf{1}_{(0<p<1)}\mathbf{1}_{(x_i=0,1)} \\
&= p^{\sum x_i}(1-p)^{\sum(1-x_i)} \\
&= (\frac{p}{1-p})^{\sum x_i}(1-p^{-\sum x_i})
\end{aligned}
$$

So, $\sum x_i$ is a sufficient stat for p by factorization.

3. If $Y_1, Y_2, \ldots, Y_n$ are iid Exponential(β), show that $\sum_{i=1}^{n} Y_i$ is sufficient for β, and hence that $\bar{Y} = n^{-1}\sum_{i=1}^{n} Y_i$ is sufficient for β.

Answer:

3) $Y_1, \ldots, Y_n$ iid $\exp(\beta)$, show that

$\qquad \Sigma\, Y_i$ is sufficient for β, and hence

$\qquad \bar{Y} = \frac{1}{n}\Sigma\, Y_i$ is suff. for β.

$$f(y_i|\beta) = \beta e^{-y_i\beta} \qquad \beta > 0$$

$$L(\beta) = \prod \beta\, e^{-y_i\beta}\, \frac{1}{(y_i > 0)}\, \frac{1}{(\beta > 0)}$$

$$= \beta^n \prod\left((e^{-\beta})^{y_i}\right) \prod 1_{(y_i > 0)} \prod 1_{(\beta > 0)}$$

$$= \underbrace{\beta^n\, (e^{-\beta})^{\Sigma y_i}}_{g(u|\beta)} \overbrace{\prod 1_{(y_i > 0)}}\quad \underbrace{\prod 1_{(\beta > 0)}}_{h(\cdot)}$$

$\qquad\qquad \Sigma\, y_i$ is suff for β

$\qquad$ and $\quad \bar{Y} = \dfrac{\Sigma y_i}{n}$ is also suff for β.

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4. Complete **Exercise 9.52** of the text.

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4) Exercise 9.52

$$Y_1, \ldots, Y_n$$

$$f(y|\theta) = \begin{cases} \dfrac{3y^2}{\theta^3} & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}$$

Show $Y_{(n)} = \max(Y_1, \ldots, Y_n)$ is suff for θ

$$L(\theta) = \prod f(y_i|\theta) \, \mathbb{1}_{(0 \le y_i \le \theta)}$$

$$= \prod \left(\frac{3y_i^2}{\theta^3} \, \mathbb{1}_{(0 \le y_i \le \theta)} \right)$$

$$= \prod \frac{1}{\theta^3} \prod 3y_i^2 \prod \mathbb{1}_{(0 \le y_i \le \theta)} \, \mathbb{1}_{(0 \le y_i)} \, \mathbb{1}_{(y_i \le \theta)}$$

$$= \left(\frac{1}{\theta^3} \right)^n \prod \mathbb{1}_{(y_i \le \theta)} \underbrace{\left(\prod 3y_i^2 \, \mathbb{1}_{(0 \le y_i)} \right)}_{h(\cdot)}$$

$$= \underbrace{\left(\frac{1}{\theta^3} \right)^n \mathbb{1}_{(y_{(n)} \le \theta)}}_{g(u|\theta)} \Updownarrow \left(\underbrace{h(\cdot)}_{h(\cdot)} \right)$$

$u = Y_{(n)}$ so $Y_{(n)}$ is suff for θ
by factorization Criterion.

Answer:

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5. Complete **Exercise 9.70** of the text.

Answer:

5) Exercise 9.70 (MOME)

Suppose $Y_1, \ldots, Y_n$ constitute a random sample from a poisson distribution with mean λ. Find the method-of-moments est. of λ.

$\cancel{E(Y_i|\lambda)} =$

$$\mu_1' = E(Y_i) = \lambda = m_1' = \bar{Y} = \frac{\sum y_i}{n}$$

$$\hat{\lambda} = \bar{Y}$$

6. Complete **Exercise 9.74(a)** of the text.

Answer:

6) Exercise 9.74a

$Y_1, \ldots Y_n$ constitute a random sample from the probability density function given by

$$f(y|\theta) = \begin{cases} \left(\frac{2}{\theta^2}\right)(\theta-y) & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}$$

a) Find an estimator for θ by using Method of moments

$$\mu_1' = E(Y_i) \stackrel{\text{set}}{=} \bar{Y}$$

$$E(Y_i) = \int_0^\theta y_i (f(y|\theta)) dy$$

$$= \int_0^\theta y \left(\frac{2}{\theta^2}\right)(\theta-y) dy$$

$$= \frac{2}{\theta^2} \int_0^\theta \theta y - y^2 dy$$

$$= \frac{2}{\theta^2} \left(\frac{\theta}{2}y^2 - \frac{1}{3}y^3 \right) \Big|_0^\theta$$

$$= \frac{2}{\theta^2} \left(\frac{\theta^3}{2} - \frac{\theta^3}{3} \right)$$

$$= \theta - \frac{2}{3}\theta$$

$$= \frac{1}{3}\theta \;\;\Longleftarrow$$

So $\bar{Y} = \frac{1}{3}\theta$

$$\hat{\theta}_{MOME} = 3\bar{Y}$$

Optional problems: Any of the sufficiency problems in Section 9.4, except 9.39 (showing sufficiency by checking conditional distribution, instead of by factorization criterion), are good for practicing writing down likelihoods and using the factorization criterion to determine sufficient statistics.