STAT430 Homework #7: Due Friday, April 12, 2019.

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0. We will continue with **Chapter 9** on estimation.
1. (a). Complete **Exercise 9.2** of the text.

Answer:

Part a)

$$\begin{aligned}
B(\hat{\mu}_1) &= E(\hat{\mu}_1) - \mu \\
&= E(\frac{Y_1 + Y_2}{2}) - \mu \\
&= \frac{1}{2}(E(Y_1) + E(Y_2)) - \mu \\
&= \frac{1}{2}(\mu + \mu) - \mu \\
&= 0
\end{aligned}$$

$$\begin{aligned}
B(\hat{\mu}_2) &= E(\hat{\mu}_2) - \mu \\
&= E(\frac{1}{4}Y_1 + \frac{Y_2 + \cdots + Y_{n-1}}{2(n-2)} + \frac{1}{4}Y_n) - \mu \\
&= \frac{E(Y_1)}{4} + E(\frac{(n-2)Y_1}{2(n-2)} + \frac{E(Y_n)}{4}) - \mu \\
&= \frac{\mu}{4} + \frac{\mu}{2} + \frac{\mu}{4} - \mu \\
&= 0
\end{aligned}$$

$$\begin{aligned}
B(\hat{\mu}_2) &= E(\hat{\mu}_2) - \mu \\
&= E(\bar{Y}) - \mu \\
&= \mu - \mu \\
&= 0
\end{aligned}$$

Part b)

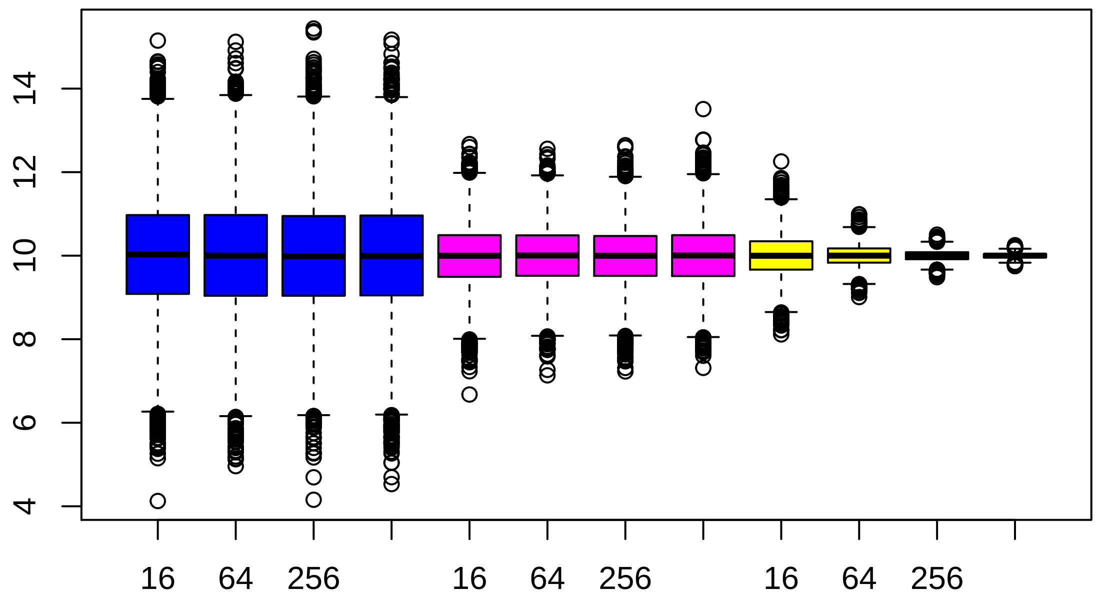
$$
\begin{aligned}
eff(\hat{\mu}_3, \hat{\mu}_2) &= \frac{V(\hat{\mu}_2)}{V(\hat{\mu}_3)} \\
&= \frac{V(\frac{1}{4}Y_1 + \frac{Y_2+\cdots+Y_{n-1}}{2(n-2)} + \frac{1}{4}Y_n)}{V(\bar{Y})} \\
&= \frac{\frac{1}{16}V(Y_1) + \frac{V(Y_2+\cdots+Y_{n-1})}{4(n-2)^2} + \frac{1}{16}V(Y_n)}{V(\bar{Y})} \\
&= \frac{\frac{1}{16}V(Y_1) + \frac{V(Y_2)+\cdots+V(Y_{n-1})}{4(n-2)^2} + \frac{1}{16}V(Y_n)}{V(\bar{Y})} \\
&= \frac{\frac{1}{16}\sigma^2 + \frac{\sigma^2+\cdots+\sigma^2}{4(n-2)^2} + \frac{1}{16}\sigma^2}{\sigma^2/n} \\
&= n/16 + n^2/(4(n-2)^2) + n/16 \\
&= n/8 + n^2/(4(n-2)^2)
\end{aligned}
$$

$$
\begin{aligned}
eff(\hat{\mu}_3, \hat{\mu}_1) &= \frac{V(\hat{\mu}_1)}{V(\hat{\mu}_3)} \\
&= \frac{V(\frac{Y_1+Y_2}{2})}{V(\bar{Y})} \\
&= \frac{\frac{1}{4}(V(Y_1)+V(Y_2))}{\sigma^2/n} \\
&= \frac{\sigma^2+\sigma^2}{4\sigma^2/n} \\
&= \frac{n}{2}
\end{aligned}
$$

1(b). The following code simulates 10,000 realizations of each of the three estimators $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\mu}_3$ at each of four sample sizes, n = 16, 64, 256, 1024. One of the three estimators is consistent for μ. Which one? Prove your claim.

```r
set.seed(4302019)
nreps <- 10000
mu_hat1 <- c() # initialization
mu_hat2 <- c()
mu_hat3 <- c()
mu <- 10
sigma <- 2
for(i in 2:5){
  n <- 4 ^ i
  Y <- rnorm(n * nreps, mean = mu, sd = sigma)
  YY <- matrix(Y, n, nreps)
  hat1 <- c(0.5, 0.5, rep(0, n-2))
  hat2 <- c(0.25, rep(1 / (2 * (n-2)), n-2), 0.25)
  hat3 <- rep(1 / n, n)
  tmp1 <- rbind(hat1) %*% YY
  # this is matrix multiplication of the 1 x n vector hat1 times the n x nreps matrix YY
  tmp2 <- rbind(hat2) %*% YY
```

```
  tmp3 <- rbind(hat3) %*% YY
  mu_hat1 <- rbind(mu_hat1, tmp1)
  mu_hat2 <- rbind(mu_hat2, tmp2)
  mu_hat3 <- rbind(mu_hat3, tmp3)
}
boxplot(t(rbind(mu_hat1, mu_hat2, mu_hat3)),
        names = paste(rep(4 ^ (2:5), 3)),
        col = c(rep("blue", 4), rep("magenta", 4), rep("yellow", 4)))
```



Answer:

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2. Complete **Exercise 9.8** of the text. LaTeX starter for a $N(\mu, \sigma 2)$ density:

$$\frac{\partial \ln f(y)}{\partial \mu} = \frac{\partial}{\partial \mu} \ln \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ \frac{-(y-\mu)^2}{2\sigma^2} \right\} \right] = \frac{\partial}{\partial \mu} \left\{ -\frac{1}{2} \ln(2\pi\sigma^2) - \frac{(y-\mu)^2}{2\sigma^2} \right\}.$$

Answer:

Part a)

Finding the second partial derivative of $\ln f(y)$ with respect to μ gives us:

$$\frac{\partial^2 \ln f(y)}{\partial \mu^2} = -1/\sigma^2$$

So,

$$\begin{aligned} I(\mu) &= (nE(-(-1/\sigma^2)))^{-1} \\ &= \sigma^2/n \end{aligned}$$

So, $Var(\bar{Y}) = \sigma^2/2 = I(\mu)$.

Part b)

Finding the second partial derivative of $\ln p(y)$ with respect to λ gives us:

$$\frac{\partial^2 \ln f(y)}{\partial \lambda^2} = -y/\lambda^2$$

So,

$$\begin{aligned} I(\mu) &= (nE(-(-y/\lambda^2)))^{-1} \\ &= \lambda/n \end{aligned}$$

So, $Var(\bar{Y}) = \sigma^2/2 = I(\lambda)$.

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3. Let $Y_1, Y_2, \ldots, Y_n$ denote an iid sample from Uniform$(\theta, \theta+1)$. (a) Show that $\hat{\theta} = \bar{Y} - 1/2$ is unbiased for θ; and (b) Show that $\hat{\theta}$ converges in mean square to θ (and hence is consistent for θ) as $n \to \infty$.

Answer:

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4. Complete **Exercise 9.24** of the text.

Answer:

Part a)

$$\sum_{i=1}^{n} Y_i^2 \sim \chi_n^2$$

Part b)

$$\sum_{i=1}^{n} Y_i^2/n = W_n$$

$E(W_n) = n/n = 1$

$V(W_n) = \frac{1}{n^2}(2n) = 2/n$

Since $E(W_n) = 1$ as $n \to \infty$ and $V(W_n) = 0$ as $n \to \infty$,

W_n converges by Mean Square to 1 $\implies$ W_n converges in Probability to 1.

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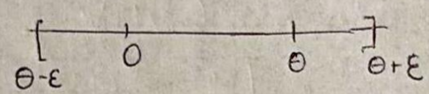
5. Complete **Exercise 9.26** of the text.

Answer:

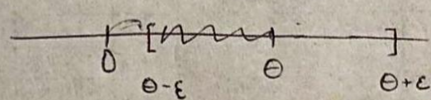
a)

$\varepsilon > \theta$

If $\varepsilon > \theta$, verify that $P(\theta - \varepsilon \le Y_{(n)} \le \theta + \varepsilon) = 1$



$\varepsilon < \theta$



$Y_{(n)} \le \theta \le \theta + \varepsilon$

so

$\theta - \varepsilon \le Y_{(n)} \le \theta$

$F(\theta) - F(\theta - \varepsilon)$

$1 - \underbrace{\left(\frac{\theta-\varepsilon}{\theta}\right)^n}_{<1}$

$\lim_{n \to \infty} 1 - \left(\frac{\theta-\varepsilon}{\theta}\right)^n = 1$

b) $\varepsilon > \theta$ always 1

$\varepsilon < \theta$ converges to 1

$\forall \theta$ $\lim_{n \to \infty} P(|Y_{(n)} - \theta| \leq \varepsilon) = 1$

Optional problems: Good optional review problems are 9.15 through 9.25; most have some combination of expectation computation, variance computation, and consistency (usually established by showing mean square consistency).