2014)

$$f(x,y) = \begin{cases} x^2 \cdot \sin\frac{1}{x^2+y^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$$

α) Διαφορίσιμη στο (0,0)

$\cdot\ \frac{\partial f}{\partial x}(0,0) = \frac{f((0,0)+t(1,0)) - (0,0)}{t} = \frac{f(t,0)}{t} = \frac{t^2 \sin\frac{1}{t^2}}{t} = t\cdot\sin\frac{1}{t^2}$

$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to0}\frac{f(0,t)}{t} = \lim_{t\to0}\frac{0\cdot\sin\frac{1}{0+t^2}}{t} = 0$

$\cdot\ \lim_{h\to(0,0)} \frac{f(h_1,h_2) - (0\ 0)\cdot\binom{h_1}{h_2}}{\|h\|} = \lim_{h\to0}\frac{h_1^2\cdot\sin\frac{1}{h_1^2+h_2^2}}{\sqrt{h_1^2+h_2^2}} =$

Αρα $g(h_1,h_2) = \lim_{h\to0}\frac{h_1^2\cdot\sin\frac{1}{h_1^2+h_2^2}}{\|h\|} \leq \frac{|h_1^2|}{\sqrt{h_1^2+h_2^2}}\cdot\left|\sin\frac{1}{h_1^2+h_2^2}\right| \leq \frac{1}{2}\frac{h_1^2}{\sqrt{h_1^2+h_2^2}}\left|\sin\frac{1}{h_1^2+h_2^2}\right|$

$= \frac{1}{2}\cdot\sqrt{h_1^2+h_2^2}\left|\sin\frac{1}{h_1^2+h_2^2}\right|$

$\cdot\ \cancel{g(0,h_2)=0}$

$\cdot\ g(h_1,0) =$

Αρα $r\to0$ αφου $\left|\sin\frac{1}{h_1^2+h_2^2}\right| \leq 1$ αλλ $\lim_{h\to0} r = 0$

Αρα διαφορ στο (0,0).

β)

$\frac{\partial f}{\partial x}(0,0) = 2x\cdot\sin\frac{1}{x^2+y^2} - \frac{2x^3\cdot\cos\left(\frac{1}{x^2+y^2}\right)}{(x^2+y^2)^2} = 0$

$\frac{\partial f}{\partial y}(0,0) = \frac{-2x^2\cdot y\cdot\cos\left(\frac{1}{x^2+y^2}\right)}{(x^2+y^2)^2} = 0$

Αρα $z-0 = 0(x-0)+0(y-0) \Rightarrow z=0$

γ) Συνεχώς διαφ στο (0,0)

$\cdot\ f(x,0) = x^2\sin\frac{1}{x^2}$, $f(0,y) = 0\cdot\sin\frac{1}{y^2}$

$\cancel{\lim_{h\to0} f}$

Θέμα 2º

$f(x,y) = x^3 - 2y^2 - 4xy + x$

$\frac{\partial f}{\partial x} = 3x^2 - 4y + 1$

$\frac{\partial f}{\partial y} = -4y - 4x$

$\Rightarrow$ $3x^2 - 4y + 1 = 0 \Rightarrow 3x^2 + 4x + 1 = 0$

$4y = 4x \Rightarrow y = -x$

$\Delta = 16 - 12 = 4$

$x_{1,2} = \frac{-4 \pm 2}{6}$ $\to x_1 = -1$, $\to x_2 = -1/3$

Αν $x_1 = -1 \Rightarrow y = 1$

$x_2 = -1/3 \to y = 1/3$

Άρα $(-1, 1)$ και $(-1/3, 1/3)$ κρ. σημεία.

$\frac{\partial^2 f}{\partial x^2} = 6x$

$\frac{\partial^2 f}{\partial y^2} = -4$

$\frac{\partial^2 f}{\partial x \partial y} = -4$

$H = \begin{vmatrix} 6x & -4 \\ -4 & -4 \end{vmatrix} = -24x - 16$

$H(-1,1) = 8 > 0$

$\frac{\partial^2 f}{\partial x^2}(-1,1) = -6 < 0$ Άρα τ. μεγ.

$H(-1/3, 1/3) = -8 < 0$ άρα σάγμα.

Θέμα 3ο |

$a_1, a_2, \ldots, a_n \in \mathbb{R}$

μεγ/ελαχ της $f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$

$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^2 : x_1^2 + x_2^2 + \ldots + x_n^2 = 1\}$

Άρα $f(x_1, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$

$g(x_1, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2$

Από Θ. Lagrange.

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= \lambda \cdot \frac{\partial g}{\partial x_1} \\ \frac{\partial f}{\partial x_2} &= \lambda \cdot \frac{\partial g}{\partial x_2} \\ &\vdots \\ \frac{\partial f}{\partial x_n} &= \lambda \cdot \frac{\partial g}{\partial x_n} \end{aligned}\right\} \Rightarrow \begin{aligned} a_1 &= \lambda \cdot 2x_1 \Rightarrow x_1 = \frac{a_1}{2\lambda} \\ a_2 &= \lambda \cdot 2x_2 \Rightarrow x_2 = \frac{a_2}{2\lambda} \\ &\vdots \\ a_n &= \lambda \cdot 2x_n \Rightarrow x_n = \frac{a_n}{2\lambda} \end{aligned}$$

$x_1^2 + x_2^2 + \ldots + x_n^2 = 1 \Rightarrow \frac{a_1^2}{4\lambda^2} + \frac{a_2^2}{4\lambda^2} + \ldots + \frac{a_n^2}{4\lambda^2} = 1 \Rightarrow$

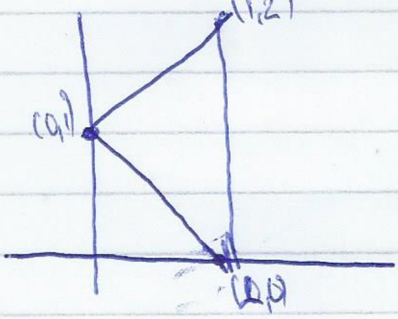
$\Rightarrow a_1^2 + a_2^2 + \ldots + a_n^2 = 4\lambda^2 \Rightarrow \lambda = \frac{a_1 + a_2 + \ldots + a_n}{2}$ αυτό αν $\lambda \neq 0$

Άρα $x_1 = \frac{a_1}{\frac{2a_1 + \ldots}{2}} = \frac{a_1}{a_1 + a_2 + \ldots + a_1}$

$x_2 = \frac{a_2}{a_1 + a_2 + \ldots + a_n}$

Άρα $\max f(x_1, x_2, x_n) = a_1^2$, $\min f = a_n^2$

Θέμα 4ο



Άρα: $y = \lambda x + b$

Για $(0,1)$: $1 = b$
$(1,2)$: $2 = \lambda + b \Rightarrow \lambda = 1$ $\Big\}$ $\underline{y = x+1}$

$(0,1)$: $1 = b$
$(1,0)$: $0 = \lambda + b \Rightarrow \lambda = -1$ $\Big\}$ $y = -x$

Άρα

$$\int_0^1 \int_{-x}^{x+1} x \cdot y \, dy dx =$$

$$\int_{-x}^{x+1} x \cdot y \cdot dy = x \cdot \int_{-x}^{x+1} y \, dy = x \cdot \frac{y^2}{2}\Big|_{-x}^{x+1} = x\left(\frac{(x+1)^2}{2} - \frac{(-x)^2}{2}\right) =$$

$$= x \cdot \left(\frac{x^2+2x+1}{2} - \frac{x^2}{2}\right) = \frac{x^3+2x^2+x}{2} - \frac{x^3}{2}$$

$$\int_0^1 \frac{x^3+2x^2+x}{2} dx - \int_0^1 \frac{x^3}{2} dx = \frac{17}{24} - \frac{1}{8} = \frac{17-3}{24} = \frac{14}{24} = \boxed{\frac{7}{12}}$$

$$\frac{1}{2} \cdot \int_0^1 x^3+2x^2+x \, dx = \frac{1}{2}\left[\frac{x^4}{4} + 2 \cdot \frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 =$$

$$= \frac{1}{2} \cdot \left(\frac{1}{4} + \frac{2}{3} + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{3+8+6}{12}\right) = \frac{17}{24}$$

$$\frac{1}{2} \cdot \int_0^1 x^3 dx = \frac{1}{2} \cdot \frac{x^4}{4}\Big|_0^1 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

Θέμα 5ο

$k = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le R^2 \ , x^2+y^2 \le z^2 \cdot \tan^2 a \ , z \ge 0\}$

Σφαιρικές $\quad d\rho \, d\phi \, d\theta$

$x = \rho \cdot \cos\phi \cdot \sin\theta$

$y = \rho \cdot \sin\phi \cdot \sin\theta$

$z = \rho \cdot \cos\theta$

$d = \rho^2 \cdot \sin\theta$

Άρα $\rho^2 \le R^2$

$\cdot \ \rho^2 \cdot \cos^2\phi \cdot \sin^2\theta + \rho^2 \cdot \sin^2\phi \cdot \sin^2\theta \le \rho^2 \cdot \cos^2\theta \cdot \tan^2 a \Rightarrow$

$\Rightarrow \sin^2\theta \le \cos^2\theta \cdot \tan^2 a \Rightarrow$

$\Rightarrow \dfrac{\tan^2\theta}{\tan^2 a} \le 1 \ {}^{\tan^2 a} \Rightarrow \dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2}$

$$\int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^{R} \rho^2 \cdot \sin\theta \, d\rho \, d\phi \, d\theta = \ldots$$

Θέμα 6ο

$B = \{(x,y) \in \mathbb{R}^2 : 3x^2 + 4y^2 \le 12 \ , x \ge 0\}$

$\cdot \ 3x^2+4y^2 \le 12 \Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{3} \le 1 \Rightarrow \dfrac{4r^2 \cdot \cos^2\phi}{4} + \dfrac{3q^2 \cdot \sin^2\phi}{3} \le 1 \Rightarrow$

Άρα έλλειψη με

$x = 2r \cdot \cos\phi$ $\qquad \Rightarrow r^2 \le 1 \Rightarrow 0 \le r \le 1$

$y = \sqrt{3} \cdot r \cdot \sin\phi$

και $d = 2\sqrt{3}\, r$

$\cdot \ x \ge 0 \Rightarrow 2r \cdot \cos\phi \ge 0 \Rightarrow \cos\phi = 0 \Rightarrow \phi \ge \pi/2$

Άρα
$$\int_0^{\pi/2} \int_0^1 2r \cdot \cos\phi \cdot 3 \cdot r^2 \sin^2\phi \cdot 2\sqrt{3} \cdot r \, dr \, d\phi = \ldots$$