Παραδείγματα ασκήσεων (ολοκληρώματα)

Παράδειγμα

- Να υπολογιστεί το $\int_1^0 (3x^2 + x - 5)\, dx$

- Να υπολογιστεί το $\int_{-1}^1 (1 - |x|)\, dx$

- Να βρεθεί το εμβαδόν την επιφάνειας στο 1 τεταρτημόριο που φράσσεται από τις ευθείες $\mathrm{y} = x$, $\mathrm{x} = 2,$ την καμπύλη $\mathrm{y} = \frac{1}{x^2}$ και το άξονα x

$$\int_1^0 3x^2 + x - 5\,dx = -\underbrace{\int_0^1 3x^2 + x - 5\,dx}\quad 1$$

$$\int 3x^2 + x - 5\,dx = \int 3x^2\,dx + \int x\,dx + \int(-5)\,dx$$

$$= 3\cdot\frac{x^3}{3} + \frac{x^2}{2} - 5x + C = x^3 + \frac{x^2}{2} - 5x + C$$

$$1 = -\left(\left[x^3 + \frac{x^2}{2} - 5x(+C)\right]_0^1\right) = -\left(\left(1 + \frac{1}{2} - 5 + \cancel{C}\right) - \cancel{(C)}\right) =$$

$$= -\left(1 + \frac{1}{2} - 5\right) = \frac{7}{2}$$

$$\int_{-1}^{1} (1-|x|)\,dx \qquad (-1, 1) \to (-1, 0) \cup (0, 1)$$

$$= \int_{-1}^{0} (1-(-x))\,dx + \int_{0}^{1} (1-x)\,dx$$

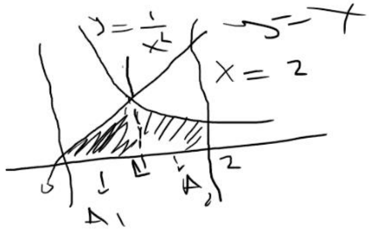
$$= \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} (1-x)\,dx$$

$$\int 1+x\,dx = \int 1\,dx + \int x\,dx = x + \frac{x^2}{2} + C$$

$$\int_{-1}^{0} 1+x\,dx = \left[x + \frac{x^2}{2}\right]_{-1}^{0} = [0+0] - \left[-1+\frac{1}{2}\right] = \frac{1}{2} \qquad A$$

$$\int_{0}^{1} 1-x\,dx = \left[x - \frac{x^2}{2}\right]_{0}^{1} = \frac{1}{2} \qquad B$$

$$\int_{-1}^{1} (1-|x|)\,dx = \frac{1}{2} + \frac{1}{2} = 1$$



$$f_1(x) = x$$
$$g_1(x) = 0$$

$$A_1: \int_0^1 f_1(x) - g_1(x)\,dx = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$A_2: \quad f_2(x) = \frac{1}{x^2}, \quad g_2(x) = 0$$

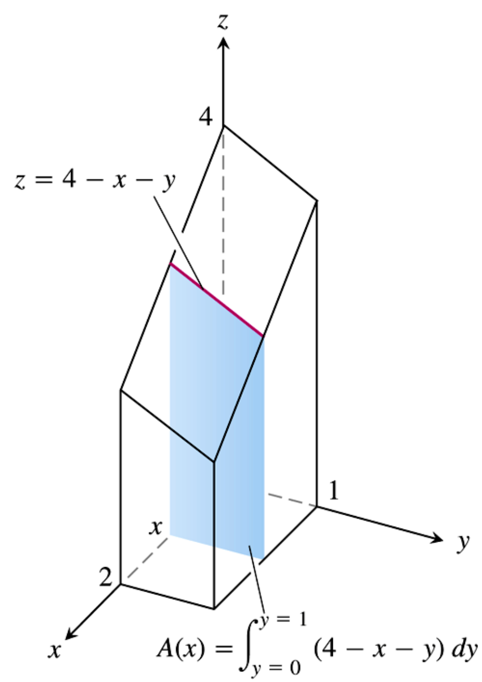
$$\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

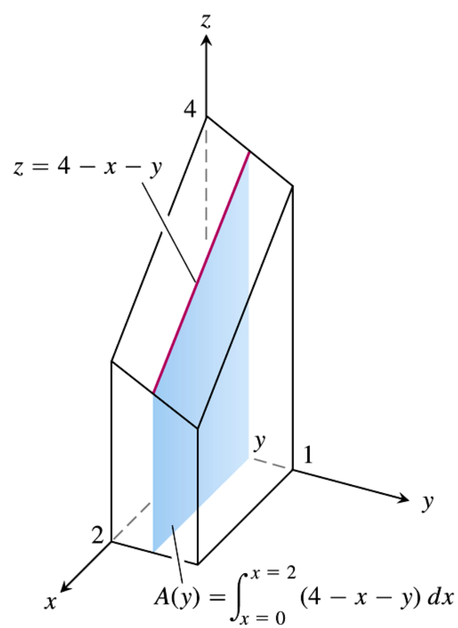
$$A: \quad A_1 + A_2 = \int_0^1 \dots dx + \int_1^2 \dots dx = \int_0^2 \dots dx$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Παράδειγμα

Να βρεθεί ο όγκος του στερεού που βρίσκεται ανάμεσα στα επίπεδα $z = 4 - x - y$ και $\mathrm{x}y$ $(\mathrm{z} = 0)$ στην περιοχή $R: 0 \le x \le 2, 0 \le y \le 1$





$z = 4 - x - y$, xy-επίπεδο, $R: \ 0 \le x \le 2, \ 0 \le y \le 1$

$$V = \int_{x=0}^{x=2} A(x)\,dx \quad , \ \forall x \ A(x) = \int_{y=0}^{y=1} f(x,y)\,dy = \int_{y=0}^{y=1} 4 - x - y \, dy$$

$$V = \int_{x=0}^{x=2} \Big(\overbrace{\int_{y=0}^{y=1} 4 - x - y \, dy}^{A(x)} \Big) dx = \int_{x=0}^{x=2} \Big[4y - xy - \frac{y^2}{2} \Big]_0^1 dx$$

$$= \int_{x=0}^{x=2} \Big[\Big(4 - x - \frac{1}{2}\Big) - 0 \Big] dx = \int_{x=0}^{x=2} 4 - x - \frac{1}{2} \, dx$$

$$= \int_{x=0}^{x=2} \frac{7}{2} - x \, dx = \Big[\frac{7}{2}x - \frac{x^2}{2} \Big]_{x=0}^{x=2} = \frac{7}{2} \cdot 2 - \frac{4}{2} = 5$$

$$A(y) = \int_{x=0}^{x=2} 4 - x - y \, dx = \left[4x - \frac{x^2}{2} - xy\right]_0^2 =$$

$$= 6 - 2y$$

$$V = \int_{y=0}^{y=1} A(y)\, dy = \int_{y=0}^{y=1} 6 - 2y \, dy = 6y - y^2 \Big]_{y=0}^{y=1} = 5$$

Παράδειγμα

Να βρεθεί ο όγκος του στερεού που βρίσκεται ανάμεσα στην επιφάνεια $z = 16 - x^2 - y^2$ και xy $(\mathrm{z} = 0)$ στην περιοχή $R: 0 \leq x \leq 2, 0 \leq y \leq 2$

$$V = \iint_R f(x,y)\,dA = \int_{x=0}^{2}\int_{y=0}^{2}(16 - x^2 - y^2)\,dy\,dx$$

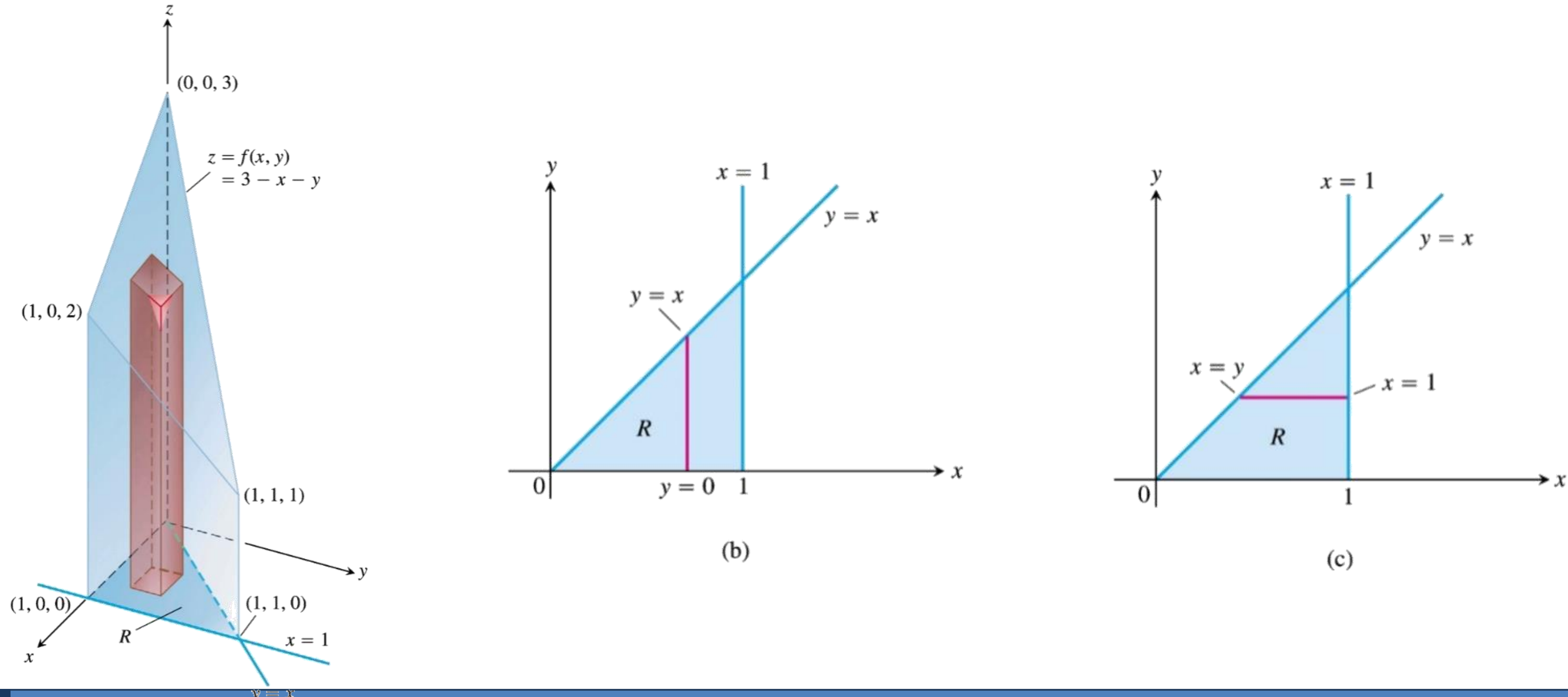
$$= \int_{x=0}^{2}\left(\left[16y - x^2 y - \frac{y^3}{3}\right]_{y=0}^{2}\right)dx =$$

οχι y

$$= \int_{x=0}^{2}\left(\frac{88}{3} - 2x^2\right)dx = \left[\frac{88}{3}x - \frac{2x^3}{3}\right]_0^2 = \frac{160}{3} = 53{,}33$$

Παράδειγμα

Να βρεθεί ο όγκος του πρίσματος, του οποίου η βάση είναι το τρίγωνο που ορίζεται από τον άξονα x και τις γραμμές $y = x$ και $x = 1$ και φράσσεται από την $z = f(x, y) = 3 - x - y$



(a) (b) (c)

• Για $x=0$ έως $x=1$, $y=0$ έως $y=x$

$$V = \int_0^1 \left(\int_0^x (3 - x - y)\, dy \right) dx =$$

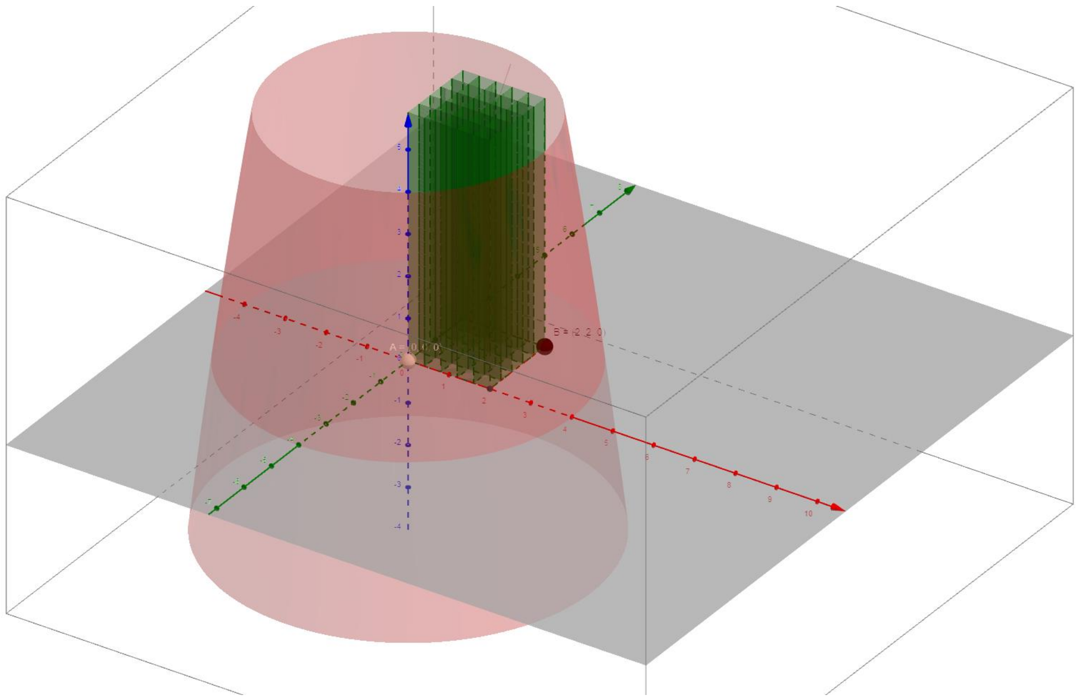
$$\int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx =$$

$$\int_0^1 \left[\left(3x - x^2 - \frac{x^2}{2} \right) - \left(3 \cdot 0 - x \cdot 0 - \frac{0^2}{2} \right) \right] dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \frac{3x^2}{2} - \frac{x^3}{2} \Big]_0^1 = 1$$

Παράδειγμα

Να βρεθεί ο όγκος του στερεού που βρίσκεται ανάμεσα στην επιφάνεια $z = 16 - x^2 - y^2$ και xy ($\mathrm{z} = 0$) στην περιοχή R: $0 \leq x \leq 2, 0 \leq y \leq 2$



$$V = \iint_R f(x,y)\,dA = \int_{x=0}^{2}\int_{y=0}^{2} (16 - x^2 - y^2)\,dy\,dx$$

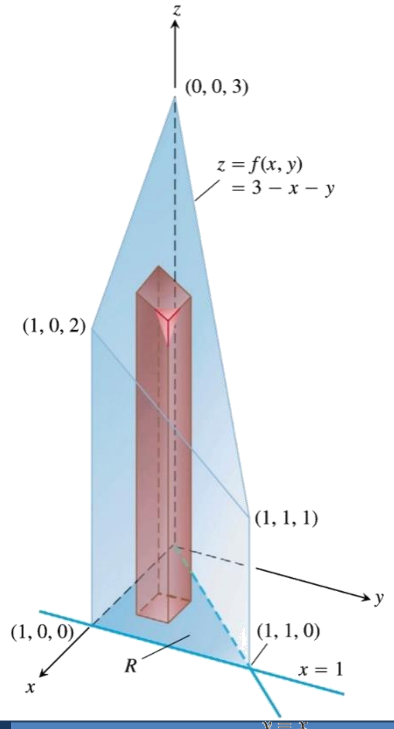
$$= \int_{x=0}^{2} \left(\left[16y - x^2 y - \frac{y^3}{3} \right]_{y=0}^{2} \right) dx =$$

Δοχι y

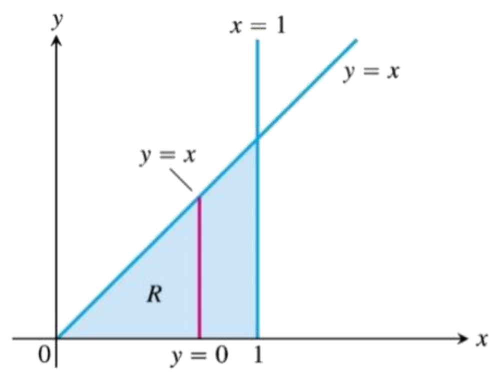
$$= \int_{x=0}^{2} \left(\frac{88}{3} - 2x^2 \right) dx = \left[\frac{88}{3}x - \frac{2x^3}{3} \right]_0^2 = \frac{160}{3} = 53{,}33$$

Παράδειγμα

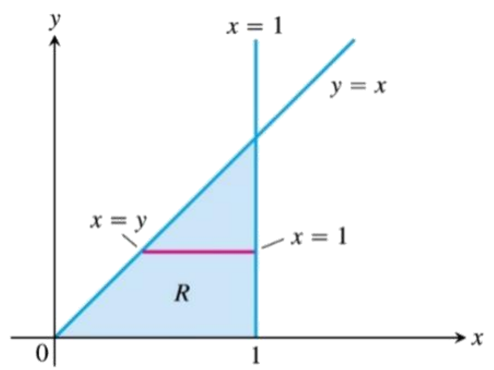
Να βρεθεί ο όγκος του πρίσματος, του οποίου η βάση είναι το τρίγωνο που ορίζεται από τον άξονα x και τις γραμμές $y = x$ και $x = 1$ και φράσσεται από την $z = f(x, y) = 3 - x - y$



(a)



(b)



(c)

Για $x=0$ έως $x=1$, $y=0$ έως $y=x$

$$V=\int_0^1\left(\int_0^x(3-x-y)\,dy\right)dx=$$

$$\int_0^1\left[3y-xy-\frac{y^2}{2}\right]_{y=0}^{y=x}dx=$$

$$\int_0^1\left[\left(3x-x^2-\frac{x^2}{2}\right)-\left(3\cdot 0-x\cdot 0-\frac{0^2}{2}\right)\right]dx$$

$$=\int_0^1\left(3x-\frac{3x^2}{2}\right)dx=\frac{3x^2}{2}-\frac{x^3}{2}\Big]_0^1=1$$

Για y από 0 έως 1, $x: y \to 1$

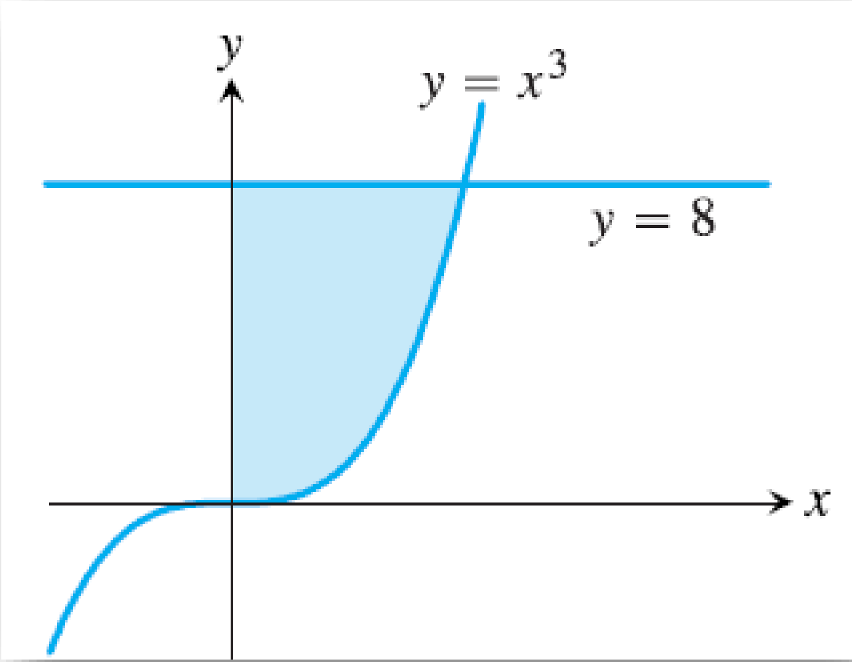
$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_y^1 dy$$

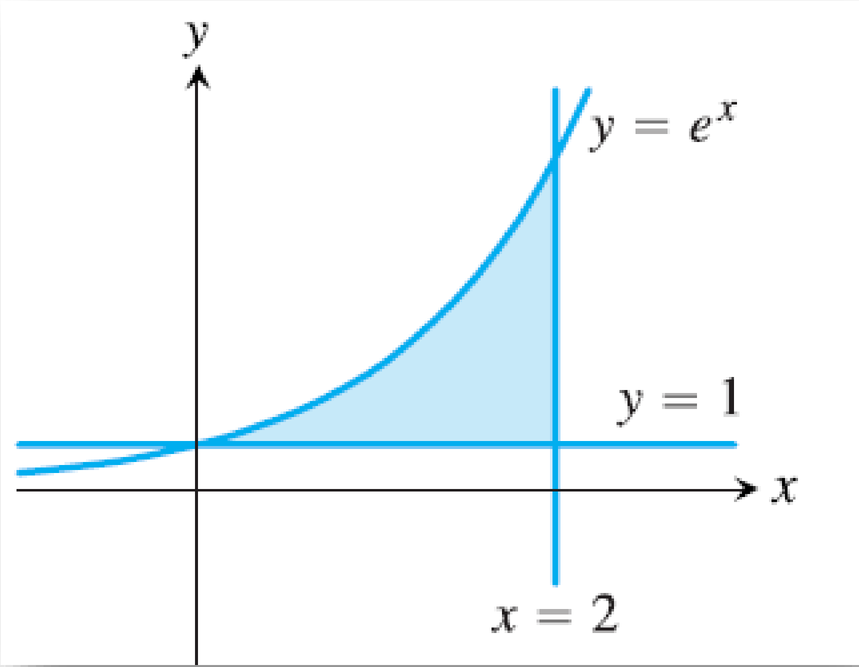
$$= \int_0^1 \left[3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y \cdot y \right] dy$$

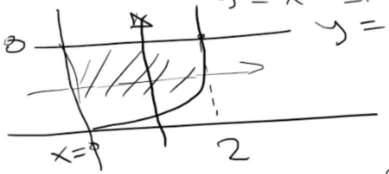
$$= \int_0^1 \left(\frac{5}{2} - 4y - \frac{3}{2} y^2 \right) dy = \left. \frac{5}{2} y - 2y^2 - \frac{y^3}{2} \right]_{y=0}^{y=1} = L$$

Παράδειγμα

Να γραφεί το διαδοχικό ολοκλήρωμα $\iint_R dA$ για το στο χωρίο R χρησιμοποιώντας κατακόρυφες και οριζόντιες διατομές







$$\iint_R dA = \int_0^2 \int_{x^3}^{8} dy\, dx = \int_0^2 y\Big]_{x^3}^{8} dx = \int_0^2 (8 - x^3)\, dx$$

$$= 8x - \frac{x^4}{4}\Big]_0^2 = 12$$

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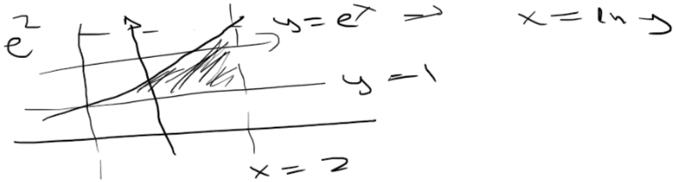
$$\iint_R dA = \int_0^8 \int_0^{y^{1/3}} dx\, dy = \int_0^8 y^{1/3}\, dy = \frac{3y^{4/3}}{4}\Big]_0^8 = \frac{3 \cdot 16}{4} = 12$$



$$\iint_R dA = \int_0^2 \int_1^{e^x} dy\, dx = \int_0^2 y\Big]_1^{e^x} dx = \int_0^2 e^x - 1\, dx =$$

$$= \left[e^x - x\right]_0^2 = e^2 - 3$$

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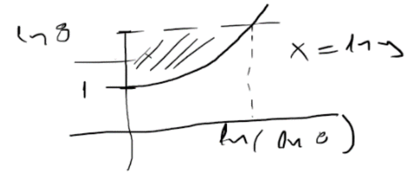
$$\iint_R dA = \int_1^{e^2} \int_{\ln y}^2 dx\, dy = \int_1^{e^2} x\Big]_{\ln y}^2 dy = \int_1^{e^2} 2 - \ln y\, dy =$$

$$= \left[2y - (y \ln y - y)\right]_1^{e^2} = e^2 - 3$$

Παράδειγμα

Να σχεδιαστεί το χωρίο ολοκλήρωσης και να υπολογιστεί το ολοκλήρωμα $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} dx dy$

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$$



$$\int_1^{\ln 8}\left[e^{x+y}\right]_0^{\ln y} dy = \int_1^{\ln 8}\left(e^{\ln y + y} - e^{y}\right) dy$$

$$= \int_1^{\ln 8} e^{\ln y}\cdot e^{y} - e^{y}\,dy = \int_1^{\ln 8} y e^{y} - e^{y}\,dy$$

$$= \int_1^{\ln 8} y e^{y}\,dy - \int_1^{\ln 8} e^{y}\,dy = \left. y e^{y} - e^{y}\right]_1^{\ln 8} - \left. e^{y}\right]_1^{\ln 8} = \boxed{8\ln 8 + 16 - e}$$

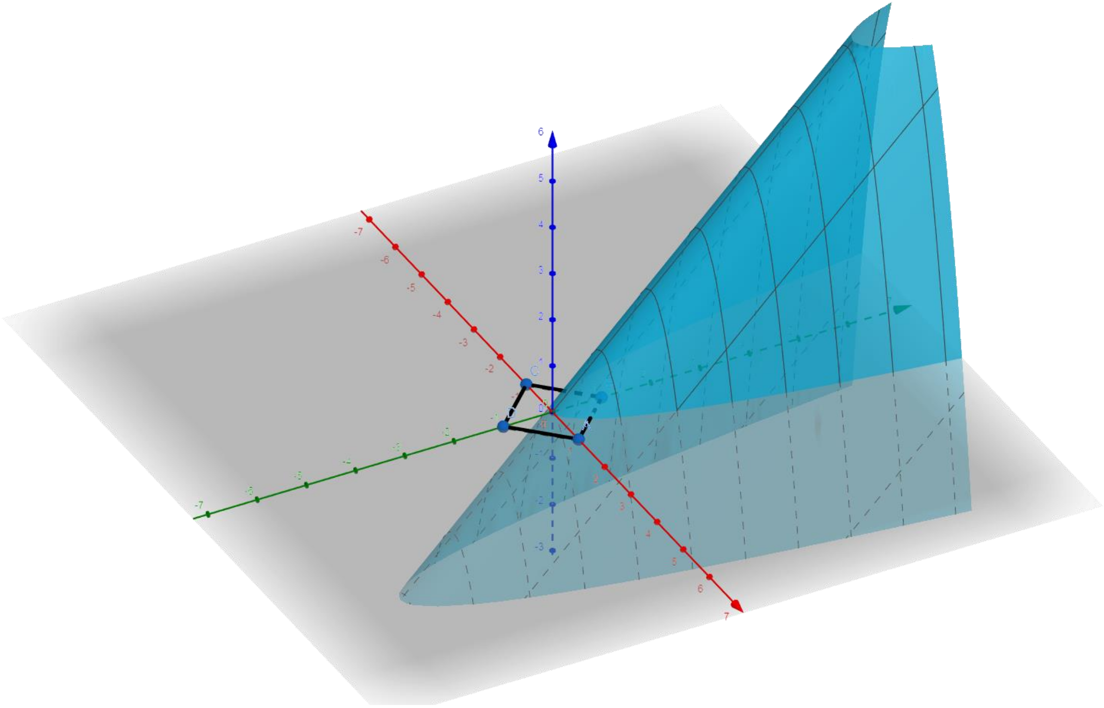
$*$

$$* \int u v'\,dx = u\cdot v - \int u'\cdot v\,dx$$

$$\int_1^{\ln 8} y (e^{y})'\,dy = \left. y e^{y} - \int e^{y}\,dy\right]_1^{\ln 8} = \left. y e^{y} - e^{y}\right]_1^{\ln 8}$$

Παράδειγμα

Να υπολογιστεί το ολοκλήρωμα $\iint_R (y - 2x^2)\,dA$ στο χωρίο R που φράσσεται από το τετράγωνο $|x| + |y| = 1$



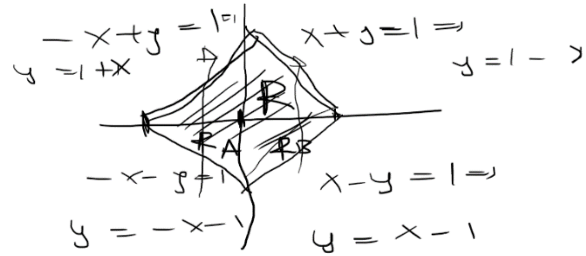
$$\iint_R y - 2x^2\,dA \qquad R: |x|+|y| = 1$$

$x<0,\ y>0 \Rightarrow R_1 \triangleq -x+y=1$

$x<0,\ y<0 \to R_2: -x-y=1$

$x>0,\ y>0 \to R_3 \quad x+y=1$

$x>0,\ y<0 \to R_4 \quad x-y=1$



$$V = \overset{R_A}{\int_{-1}^{0}\int_{-x-1}^{x+1} f(x,y)\,dy\,dx} + \overset{R_B}{\int_{0}^{1}\int_{x-1}^{1-x} f(x,y)\,dy\,dx}$$

$$= \int_{-1}^{0}\int_{-x-1}^{x+1} y-2x^2\,dy\,dx + \int_{0}^{1}\int_{x-1}^{1-x} y-2x^2\,dy\,dx =$$

$$= \int_{-1}^{0}\left[\frac{y^2}{2}-2x^2y\right]_{-x-1}^{x+1}dx + \int_{0}^{1}\left[\frac{y^2}{2}-2x^2y\right]_{x-1}^{1-x}dx =$$

$$= \int_{-1}^{0}\left[\frac{1}{2}(x+1)^2 - 2x^2(x+1) - \frac{1}{2}(-x-1)^2 + 2x^2(-x-1)\right]dx + \int_{0}^{1}\frac{1}{2}(1-x)^2 - 2x^2\cdot(1-x) - \frac{1}{2}\cdot(x-1)^2 + 2x^2(x-1)\,dx$$

$$= -4\int_{-1}^{0} x^3 + x^2\,dx + 4\int_{0}^{1} x^3 - x^2\,dx$$

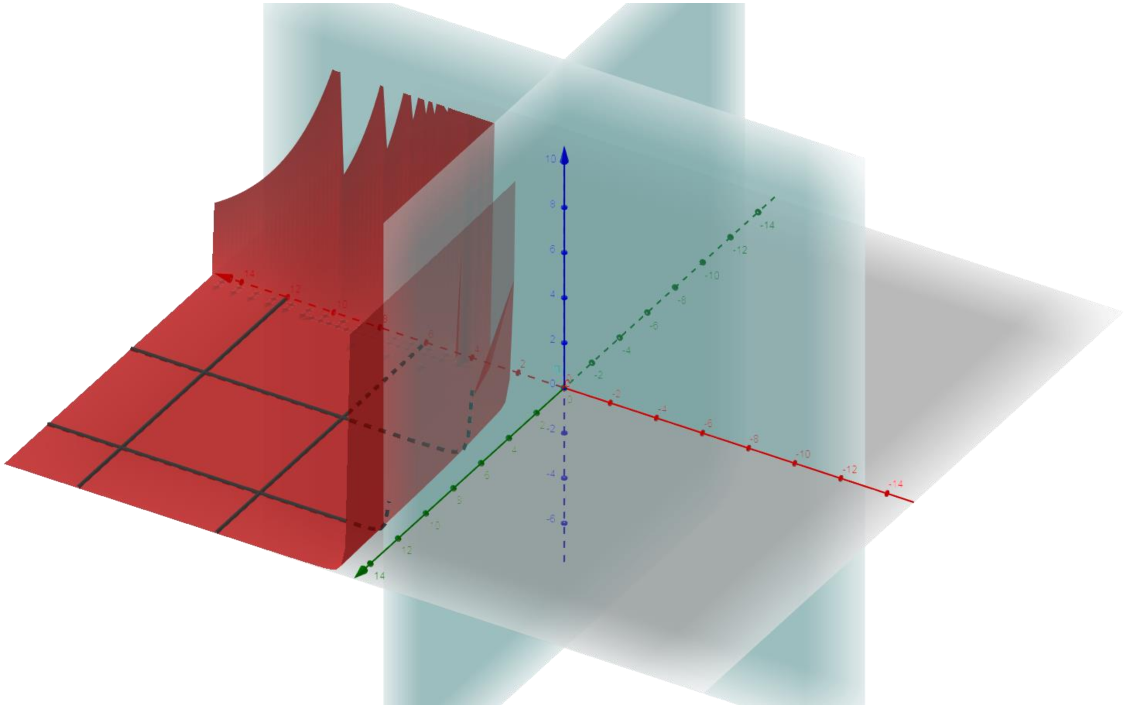
$$= -4\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{0} + 4\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_{0}^{1} = -\frac{2}{3}$$

Γενικευμένα Ολοκληρώματα

Να υπολογιστούν τα παρακάτω γενικευμένα διπλά ολοκληρώματα

- $\int_1^\infty \int_{e^{-x}}^{1} \frac{1}{x^3 y}\, dy dx$

- $\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{\frac{2}{3}}}\, dy dx$



$$\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y}\,dy\,dx$$

$$= \int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3}\cdot\frac{1}{y}\,dy\,dx = \int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3}\cdot y^{-1}\,dy\,dx$$

$$= \int_1^\infty \left[\frac{1}{x^3}\ln y\right]_{e^{-x}}^1 dx = \int_1^\infty \left(\frac{1}{1^3}\cdot\ln 1 - \frac{1}{x^3}\ln e^{-x}\right)dx$$

$$= \int_1^\infty 0 - \frac{-x}{x^3}\,dx = \int_1^\infty \frac{x}{x^3}\,dx \overset{x\neq 0}{=} \int_1^\infty \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1$$

$$\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}}\, dy\, dx = \int_0^3 \int_0^1 \frac{x^2}{(y-1)^{2/3}}\, dx\, dy$$

$$= \int_0^3 \int_0^1 \frac{1}{(y-1)^{2/3}} \cdot x^2\, dx\, dy = \int_0^3 \frac{1}{(y-1)^{2/3}} \cdot \left[\frac{x^3}{3}\right]_0^1 dy =$$

$$= \frac{1}{3} \int_0^3 \frac{1}{(y-1)^{2/3}}\, dy = \frac{1}{3} \lim_{b\to 1^-} \int_0^b \frac{1}{(y-1)^{2/3}}\, dy + \frac{1}{3} \lim_{b\to 1^+} \int_b^3 \frac{dy}{(y-1)^{2/3}}$$

$$= \frac{1}{3} \lim_{b\to 1^-} \int_0^b (y-1)^{-2/3}\, dy + \frac{1}{3} \lim_{b\to 1^+} \int_b^3 (y-1)^{-2/3}\, dy$$

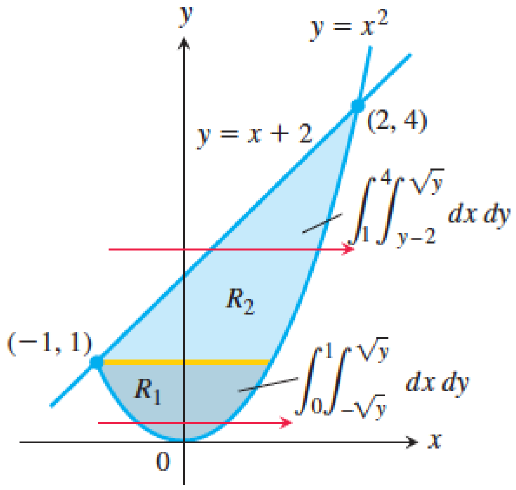
$$= \frac{1}{3} \lim_{b\to 1^-} \left[3(y-1)^{1/3}\right]_0^b + \frac{1}{3} \lim_{b\to 1^+} \left[3(y-1)^{1/3}\right]_b^3$$

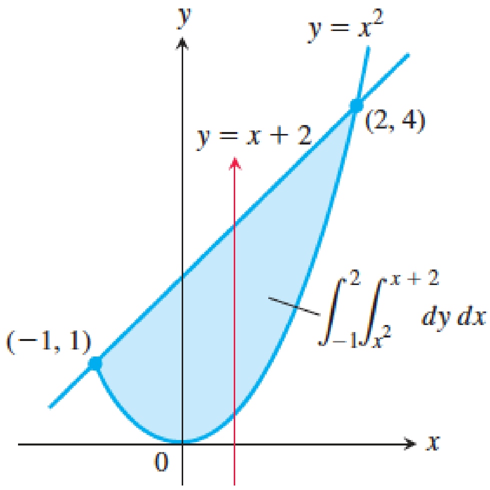
$$= \lim_{b\to 1^-} \left[(b-1)^{1/3} - (-1)^{1/3}\right] + \lim_{b\to 1^+} \left[2^{1/3} - (b-1)^{1/3}\right] =$$

$$= 1 + \sqrt[3]{2}$$

Παράδειγμα

Να υπολογιστεί το εμβαδόν του χωρίου που περικλείεται από την παραβολή $y = x^2$ και την ευθεία $y = x + 2$





$y = x^2$ και $y = x + 2$

$$\left.\begin{array}{l} y = x^2 \\ y = x+2 \end{array}\right\} \Rightarrow x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x_{1,2} = 2, -1$$

Τομή: $(2,4)$ και $(-1,1)$

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} 1\, dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy$$

$$= \int_0^1 \sqrt{y} - (-\sqrt{y})\, dy + \int_1^4 \sqrt{y} - y + 2\, dy$$

$$= \underbrace{\int_0^1 2\sqrt{y}\, dy}_{R_1} + \underbrace{\int_1^4 \sqrt{y} - y + 2\, dy}_{R_2}$$

R_1: $2 \cdot \frac{2}{3} y^{3/2} \Big]_0^1 = \frac{4}{3} y^{3/2} \Big]_0^1 = \frac{4}{3}$

R_2: $\frac{2}{3} y^{3/2} - \frac{y^2}{2} + 2y \Big]_1^4 = \frac{19}{6}$

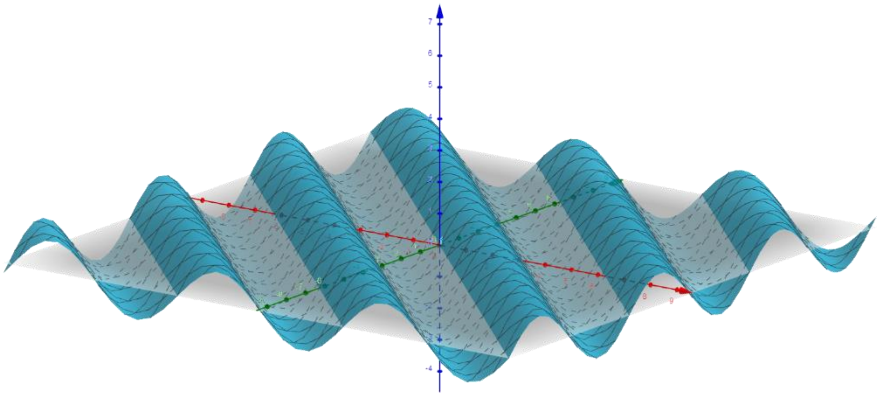
$A_{ολ}$: $R_1 + R_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2} x + 2 - x^2 \, dx$$

$$= \frac{x^2}{2} - 2x - \frac{x^3}{3} \Big]_{-1}^{2} = \frac{9}{2}$$

Παράδειγμα

- Ποια είναι η μέση τιμή της $\mathrm{f}(x, y) = \sin(x + y)$ για $0 \leq x \leq \pi\ ,\ 0 \leq y \leq \pi$



$$\text{avg } f(x,y) = \sin(x+y), \quad \begin{array}{l} 0 \le x \le \pi \\ 0 \le y \le \pi \end{array}$$

$$A = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \sin(x+y)\, dy\, dx =$$

$$= \frac{1}{\pi^2} \int_0^{\pi} -\cos(x+y) \Big]_0^{\pi} dx =$$

$$= \frac{1}{\pi^2} \Big[-\sin(x+\pi) + \sin(x) \Big]_0^{\pi} = \cdots = 0$$

Παράδειγμα

Να υπολογιστούν τα ολοκλήρωμα

- $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dzdydx$
- $\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz} dxdydz$

$$\int_0^1 \int_0^{3-3x} \underbrace{\int_0^{3-3x-y} 1 \, dz}\, dy \, dx$$

$$\int_0^1 \int_0^{3-3x} z \Big]_0^{3-3x-y} dy \, dx$$

$$= \int_0^1 \underbrace{\int_0^{3-3x} 3-3x-y \, dy}\, dx = \int_0^1 3y - 3xy - \frac{y^2}{2} \Big]_0^{3-3x} dx$$

$$= \int_0^1 (3-3x)(3-3x) - \frac{1}{2}(3-3x)^2 dx$$

$$= \frac{9}{2}\int_0^1 1-2x+x^2 \, dx = \frac{9}{2}\left[x - 2\frac{x^2}{2} + \frac{x^3}{3} \right]_0^1 = \frac{3}{2}$$

$$\int_1^e \int_1^{e^2} \underbrace{\int_1^{e^3} \frac{1}{xyz}\, dx}\, dy\, dz$$

$$= \int_1^e \int_1^{e^2} \left[\frac{1}{yz} \cdot \ln x\right]_1^{e^3} dy\, dz = \int_1^e \int_1^{e^2} \frac{1}{yz} \ln e^3 \, dy\, dz$$

$$= \int_1^e \underbrace{\int_1^{e^2} \frac{3}{yz}\, dy}\, dz = \int_1^e \frac{3}{z} \ln y \Big]_1^{e^2} dz$$

$$= \int_1^e \frac{3}{z} \ln e^2 \, dz = \int_1^e \frac{3}{z} \cdot 2\, dz = 6 \int_1^e \frac{1}{z}\, dz$$

$$= 6 \ln z \Big]_1^e = 6$$

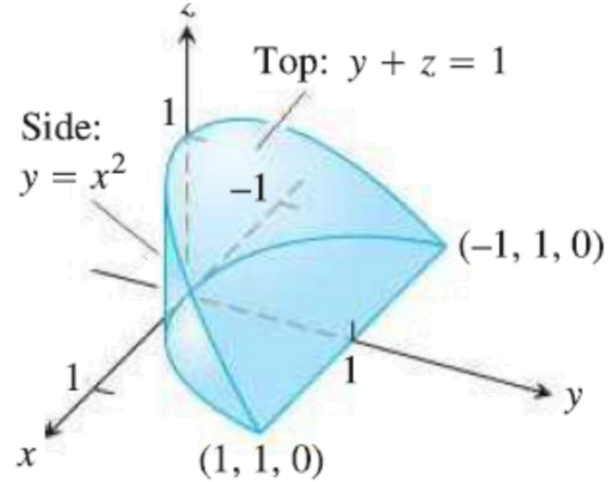
Παράδειγμα

- Έστω το χωρίο ολοκλήρωσης για το ολοκλήρωμα $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dzdydx$

  Να γραφεί το ολοκλήρωμα με

  – $dy\ dz\ dx$

  – $dy\ dx\ dz$



- Να υπολογιστεί το $\int_{0}^{1}\int_{0}^{1}\int_{x^2}^{1} 12xze^{zy^2}\ dy\ dx\ dz$

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx =$$

$$\int_{-1}^{1}\int_{x^2}^{1} z\Big]_{0}^{1-y} dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} (1-y)\,dy\,dx =$$

$$= \int_{-1}^{1} y - \frac{y^2}{2}\Big]_{x^2}^{1} dx = \int_{-1}^{1} \left(1-\frac{1}{2}\right) - \left(x^2 - \frac{x^4}{2}\right) dx$$

$$= \int_{-1}^{1} \frac{1}{2} - x^2 + \frac{x^4}{2}\,dx = \frac{1}{2}x - \frac{x^3}{3} + \frac{1}{2}\frac{x^5}{5}\Big]_{-1}^{1} = 0{,}533\ldots$$

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\, dy\, dx$$

$dy\, dz\, dx$: $\int_{-1}^{1}\int_{0}^{z(x)}\int^{y(z,x)} dy\, dz\, dx$

$y = x^2$ A

$y + z = 1$ B $\Rightarrow$

A $z = 1 - y \overset{B}{\Rightarrow}$

$z = 1 - x^2$

$$= \int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} dy\, dz\, dx$$

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} dy\, dz\, dx$$

$$= \int_{-1}^{1}\int_{0}^{1-x^2} y\Big]_{x^2}^{1-z} dz\, dx = \int_{-1}^{1}\int_{0}^{1-x^2} 1-z-x^2\, dz\, dx$$

$$= \int_{-1}^{1} z - \frac{z^2}{2} - x^2 z\Big]_{0}^{1-x^2} dx = \int_{-1}^{1} (1-x^2) - \frac{1}{2}(1-x^2)^2 - x^2(1-x^2)\, dx$$

$$= \ldots\ldots = \frac{1}{2}\int_{-1}^{1} 1-2x^2+x^4\, dx =$$

$$\frac{1}{2}\Big[x - 2\frac{x^3}{3} + \frac{x^5}{5}\Big]_{-1}^{1} = 0,533$$

$$\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$$

$$\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{1-y} dz\,dx\,dy$$

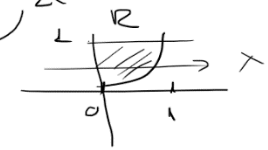
$$\int_0^1 \int_0^1 \int_{x^2}^1 12xz\, e^{2y^2}\, dy\, dx\, dz \qquad (A)\ \int_{y=a}^{y=b} \int_{x(y_1)}^{x(y_2)} \int_{z(x_1,y_1)}^{z(x_2,y_2)}$$

$$\left[\int_0^1 \int_{x^2}^1 \int_0^1 f(x,y,z)\, dx\, dy\, dz\right]$$

$z : 0 \to 1$

$x : 0 \to 1$

$y : x^2 \to 1$



$x(y,z) = \sqrt{y}$

$y : 0 \to 1$

$x : 0 \to \sqrt{y}$

$z : 0 \to 1$

$$(A) : \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12\underline{x}z\, e^{2y^2}\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 12z\, e^{2y^2} \cdot \frac{x^2}{2} \Big]_0^{\sqrt{y}} dy\, dz$$

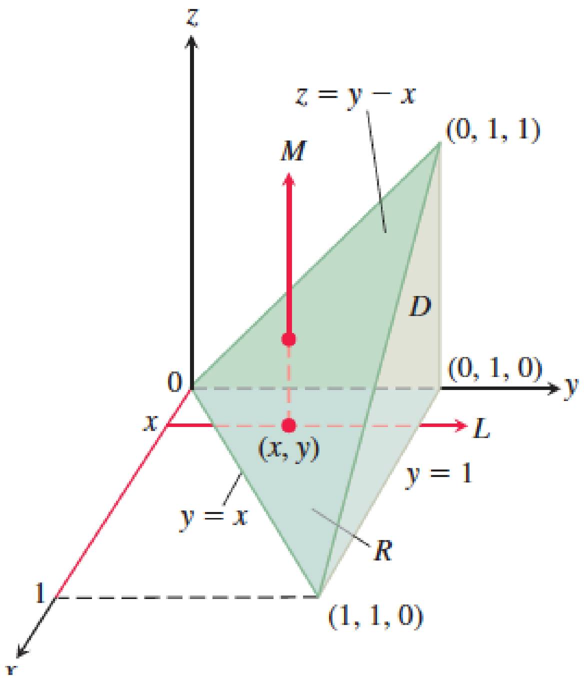
$$= \int_0^1 \int_0^1 6yz\, e^{2y^2}\, dy\, dz = 6 \int_0^1 \int_0^1 zy\, \underline{e^{2y^2}}\, dy\, dz$$

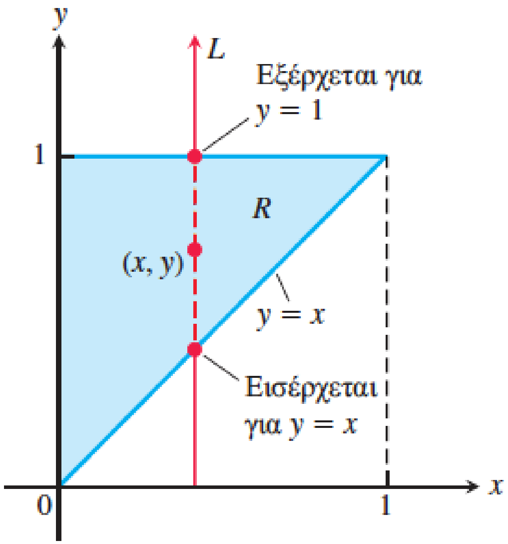
$$= 6 \int_0^1 \int_0^1 \frac{1}{2} \left(e^{2y^2}\right)' dy\, dz = 3 \int_0^1 e^{2y^2} \Big]_0^1 dz$$

$$= 3 \int_0^1 e^2 - 1\, dz = 3\left[e^2 - z\right]_0^1 = 3e - 6$$

Παράδειγμα

Να υπολογιστεί ο όγκος του τετράεδρου με άκρα τις κορυφές (0,0,0), (1,1,0), (0,1,0), (0,1,1).





Επιλ 2 : $\vec{AB}$: $(0,0,0) \to (0,1,1)$

$\vec{A\Gamma}$: $(0,0,0) \to (1,1,0)$

$\vec{AB}$: $(0-0)i + (1-0)j + (1-0)k = j+k$

$\vec{A\Gamma}$: $(1-0)i + (1-0)j + (0-0)k = i+j$

$$\vec{AB} \times \vec{A\Gamma} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -i+j-k$$

(1) και

(2) $(0,0,0)$

$$-1(x-0) + 1\cdot(y-0) - 1\cdot(z-0) = 0 \Rightarrow -x+y-z=0 \Rightarrow$$

$$z = y - x$$

$z: 0 \to y-x$

$y: x \to 1$

$x: 0 \to 1$

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\,dy\,dx =$$

$$= \int_0^1 \int_x^1 (y-x)\,dy\,dx = \int_0^1 \left[\frac{y^2}{2} - xy \right]_{y=x}^{y=1} dx$$

$$= \int_0^1 \frac{1}{2} - x + \frac{1}{2}x^2\,dx = \left. \frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \right]_{x=0}^{x=1}$$

$$= \frac{1}{6}$$