Course:	COMPUTER APPLICATIONS IN PHYSICS-II(**CAP-II**)
Course No.:	MPC 43
Semester:	M.SC. SEM-IV (2021)
Topics:	Integration, Diff. Eqn. & Roots of Equations
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Write C++ Programmes to implement the tasks as described below.

Integration

1. Integrate: $\int_0^{\frac{\pi}{2}} \frac{x^2 dx}{sin^2 x}, \quad Ans. :\pi ln2$
2. Integrate: $\int_0^{\infty} \frac{e^{-qx}}{\sqrt{x}} dx,\ q > 0, \quad Ans. : \sqrt{\frac{\pi}{q}}$
3. Integrate: $\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{qx^{pq-1}}{1-x^q} \right) dx,\ q > 0, \quad Ans. : ln\ q$
4. Integrate: $\int_0^{\infty} \frac{\sin(x)}{x} dx, \quad Ans. : \frac{\pi}{2}$
5. Integrate: $\int_0^{\frac{2}{\pi}} x^2 \sin(1/x) dx, \quad Ans. : .0585676$
6. Integrate: $\int_0^1 \frac{x^p - x^{-p}}{1+x} dx,\ p^2 < 1, \quad Ans. : \frac{1}{p} - \frac{\pi}{\sin(p\pi)}$
7. Integrate: $\int_{-1}^1\ dx x^{-\frac{2}{n}}, n > 2, \quad Ans. : \frac{2n}{n-2}$
8. Integrate $I = \int_0^1 \frac{dx}{1+x^2}$ by using Trapezoidal as well as Simpson-$\frac{1}{3}$ rules with an accuracy of the order of 10^{-5}. Find the minimum number of subintervals required to achieve the specified accuracy. Ans. $I = \frac{\pi}{4}$.
9. Integrate $I = \int_0^1 dx e^{x^2}$ by using Trapezoidal rule. Choose the total number of subintervals $n = 1000$ and plot error vs. n. (Ans.:I=1.46265)
10. Find $(\triangle x)^2 = \langle x^2 \rangle - (\langle x \rangle)^2$ for the one dimensional simple harmonic oscillator.

11. The integral representation of Beta Function $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ can be expressed in many different, but, equivalent forms. Some of the expressions are given below. Evaluate the corresponding integrals.

$$\begin{aligned}
B(m,n) &= \int_0^1 u^{m-1}(1-u)^{n-1}du, \\
B(m+1,n+1) &= \int_0^\infty \frac{u^m du}{(1+u)^{m+n+2}}, \\
B(m+1,n+1) &= 2\int_0^{\frac{\pi}{2}} \cos^{2m+1}\theta \sin^{2n+1}\theta d\theta
\end{aligned}$$

12. The error function erf(x) is defined as,

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Evaluate erf(x) by using Simpson's $\frac{1}{3}$ formula for 20 distinct points within the interval $0 \leq x \leq 2$ and plot it as a function of x.

13. One of the Fresnel integrals is defined as,

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt$$

Evaluate $C(x)$ by using Simpson's $\frac{1}{3}$ formula for 20 distinct points within the interval $0 \leq x \leq 2$ and plot it as a function of x.

14. One of the Fresnel integrals is defined as,

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

Evaluate $S(x)$ by using Simpson's $\frac{1}{3}$ formula for 20 distinct points within the interval $0 \leq x \leq 2$ and plot it as a function of x.

15. The Euler's integral is defined as,

$$\Gamma(p) = \int_0^\infty e^{-x}x^{p-1}dx, \ p > 0$$

Evaluate $\Gamma(p)$ by using Simpson's $\frac{1}{3}$ formula for 20 distinct points within the interval $0 \leq p \leq 2$ and plot it as a function of p.

Monte Carlo Method of Integration

1. Use the Monte Carlo Method of integration to find the area of (i) a circle, (ii) an ellipse.

2. Use the Monte Carlo Method of integration to find the volume of (i) a sphere, (ii) an ellipsoid, (iii) a tetrahedron, (iv) a spherical shell with inner and outer radii as r_i and r_o, respectively.

3. Use the Monte Carlo Method of integration to find the volume of a 10-dimensional sphere (S^9).

4. Evaluate: (i) $I = \int_0^{\frac{3\pi}{2}} sin(x)dx$, (ii) $I = \int_{x=-5}^{5} \int_{y=-3}^{7} (x^2 - 3xy + y^2)dxdy$

Roots of equations, Fractal and Zeroes of functions

1. Find the roots of the equation $x^2 - 3\ x + 2 = 0$ by using bi-section as well as Newton-Raphson method.

2. Find both the roots of the equation $x^2 - 2\ n\ x - 3\ n^2 = 0$ for all prime numbers between $n = 1$ to 1000 and write the results in an output file.

3. Use 1000 random numbers between 0 and $2n$ as the guess/initial values for finding the roots of the equation $x^2 - 2\ n\ x - 3\ n^2 = 0$ for a fixed integer n by using Newton-Raphson(NR) method. Tabulate the random guess values corresponding to each root in an output file.

4. Find the values of b and c within the ranges $-20 \leq b \leq 20, -20 \leq c \leq 20$ and with the least interval $h = .1$ so that the equation $x^2 + bx + c = 0$ has only real roots. Plot 'b vs. c' within the specified ranges.

5. Find all the roots of the equation: (i) $z^4 + 5z^2 + 4 = 0$, (ii) $sin\ z = z$.

6. Take the guess/initial values x_0, y_0, for finding the roots of the equation $z^n - 1 = 0, n \in N$ by using Newton-Raphson method, from the region defined by $-a \leq x \leq a, -b \leq y \leq b,\ a, b \in R$. Plot $y\ vs.\ x$ for those values of x_0, y_0 for which the root converges to $z = 1$.

7. Find all the roots of the following equations: (i) $x^3 - 5x = 0$, (ii) $x^3 - 2x + 2 = 0$, (iii) $x^2 - \alpha x + \alpha^2, \alpha \neq 0$. State the problem, if any, for the seed values $x_0 = 0, 1$. Give an explanation and a remedy of the problem.

8. Show that the roots of the following set of equations correspond to the zeroes of the Nth order Hermite polynomial:

$$x_i + \sum_{j(\neq i)=1}^{N} \frac{1}{x_i - x_j} = 0, \ i = 1, 2, \ldots N.$$

Differential equation

1. Solve the following equation by using the Runge-Kutta fourth-order formula:

$$\ddot{y} + b\dot{y} + cy = 0, \ y(0) = 1, \dot{y}(0) = 0, \ \dot{y} \equiv \frac{dy}{dt}, \ \delta \equiv b^2 - 4c$$

   for (i) $b > 0, \delta < 0$, (ii) $b < 0, \delta < 0$, (iii) $b > 0, \delta = 0$ and (iv) $b < 0, \delta = 0$.
   Plot $y(t)$ vs t for all the cases within a suitable range of t.

2. Solve the following equation by using the Runge-Kutta fourth-order formula:

$$\ddot{y} = y(y^2 - 1), \ \dot{y} \equiv \frac{dy}{dt}$$

   for (i) $y(0) = 0, \dot{y}(0) = 0$, (ii) $y(0) = 1, \dot{y}(0) = 1$, (iii) $y(0) = 0, \dot{y}(0) = -1$ and (iv) $y(0) = 1, \dot{y}(0) = 0$.
   Plot $y(t)$ vs t for all the cases within a suitable range of t.

3. Solve the following equation by using the Runge-Kutta fourth-order formula:

$$\ddot{y} + \sin y = 0, \ \dot{y} \equiv \frac{dy}{dt}$$

   for (i) $y(0) = 1, \dot{y}(0) = 0$, (ii) $y(0) = 0, \dot{y}(0) = 1$, (iii) $y(0) = 0, \dot{y}(0) = -1$ and (iv) $y(0) = 1, \dot{y}(0) = 0$.
   Plot $y(t)$ vs t for all the cases within a suitable range of t.

4. Solve the following equation by using the Runge-Kutta fourth-order formula:
$$\ddot{y} + \omega_0^2 y = \cos(\omega t), \quad \dot{y} \equiv \frac{dy}{dt},$$
with $\omega \sim \omega_0$ for (i) $y(0) = 1, \dot{y}(0) = 0$. Plot $y(t)$ vs t for all the cases within a suitable range of t.

5. Driven damped harmonic oscillator:
$$\ddot{y} + b\dot{y} + \omega_0^2 y = a\cos(\omega t), \quad a, b, \omega, \omega_0 \in R$$
Consider different limiting cases of the above equation to study beats, resonances, damping etc.

6. Kepler Problem and its Generalizations:
$$\frac{d^2u}{d\theta^2} + \omega u = \omega_0 + \omega_1 u^2; \quad Q \equiv \frac{1}{2\omega_0}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - u.$$
Choose $Q < 0$ and find the orbits(polar plot of $r \equiv \frac{1}{u}$ vs. θ) for the following cases:
(i) Standard Kepler Problem: $\omega = 1, \omega_1 = 0$,
(ii) Precession of orbits: $\omega \neq 1, \omega_1 = 0$,
(iii) Generalized Kepler Problem: $\omega \neq 1, \omega_1 \neq 0$.
Repeat the above computations for $Q = 0$ and $Q > 0$.

7. Coupled equations:
$$\dot{x}_a = \sum_{b=1}^{N} M_{ab} x_b, \;\; M_{ab} \equiv \omega_0 \delta_{ab} + \omega_+ \delta_{a\ a+1} + \omega_- \delta_{a\ a-1}, \;\; x_{N+1} = x_1$$

8. The following sets of equations exhibit chaos for suitable ranges of the real parameters a, b and c.
Find the time-development of $X(t), Y(t)$ and $Z(t)$ and check sensitivity of the solutions to the initial conditions.
Plot (i) $Y\ vs.\ X$, (ii) $Z\ vs.\ X$ and (iii) $Z\ vs.\ Y$.

(a) Damped Driven Non-linear Pendulum(DDNP):

$$\dot{X} = Y,\ \dot{Y} = -bY - sinX + a\ cosZ,\ \dot{Z} = c$$

(Check that the decoupled second order equation in terms of $X = x, Y = \dot{x}, Z = ct$ indeed corresponds to a DDNP.)

(b) Lorenz Attractor(Canonical values: $a = 28, b = 8/3, c = 10$):

$$\dot{X} = -c(X - Y),\ \dot{Y} = aX - Y - XZ,\ \dot{Z} = b(XY - Z)$$

(c) Chua's Circuit:

$$\dot{X} = a(Y - X) - f(X),\ \dot{Y} = b\left[a(X - Y) + Z\right],\ \dot{Z} = -c(Y + dZ),\ d \in R,$$

where $f(-X) = -f(X)$ and is defined for positive X as follows:

$$\begin{aligned} &f(X) = -X\ for\ X < 1, \\ &f(X) = -1 - .636(X - 1)\ for\ 1 < X < 10, \\ &f(X) = 10(X - 10) - 6.724\ for\ X > 10 \end{aligned}$$