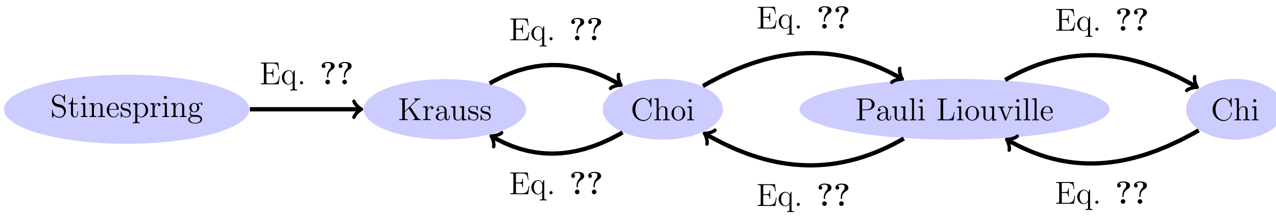
1 Converting between channel representations



1. Krauss operators $\rightarrow$ Pauli Liouville matrix.

   Input: $\{K_i\}_{k=1}^{r}$, where each of K_i are 2×2 matrices. Output: Γ, a 4×4 real matrix.

   The entries of Γ are given by the following expression.

   $$\Gamma_{i,j} = \mathsf{Tr}(\mathcal{E}(P_i)P_j) \tag{1}$$

   where $\mathcal{E}$ is the CPTP map described by the input Krauss operators, as in (Eq. **??**) and P_i is one of the Pauli matrices $\{I, X, Y, Z\}$. So, the process matrix can be expressed in terms of the Krauss operators as

   $$\Gamma_{i,j} = \sum_{k=1}^{r} \mathsf{Tr}(K_k P_i K_k^{\dagger} P_j). \tag{2}$$

2. Krauss operators $\rightarrow$ Choi Matrix

   Input: $\{K_i\}_{k=1}^{r}$, where each of K_i are 2×2 matrices. Output: J, a 4×4 complex matrix.

   The Choi matrix of a CPTP map $\mathcal{E}$ (specified by the input Krauss operators) is the result of applying $\mathcal{E}$ on the first qubit of a bell state, i.e,

   $$\begin{aligned} J &= \frac{1}{4} \sum_{i,j=0}^{1} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j| && (3) \\ &= \frac{1}{4} \sum_{i,j=0}^{1} \sum_{k=1}^{r} K_k |i\rangle\langle j| K_k^{\dagger} \otimes |i\rangle\langle j| && (4) \end{aligned}$$

3. χ matrix $\rightarrow$ Pauli Liouville Matrix

   Input: χ, a 4×4 complex matrix. Output: Γ, a 4×4 real matrix.

   The action of a CPTP map $\mathcal{E}$ on an input state ρ is specified using its χ matrix as

   $$\mathcal{E}(\rho) = \sum_{l,m} \chi_{l,m} P_l \rho P_m. \tag{5}$$

   Using the definition of the process matrix Γ, corresponding to the CPTP map $\mathcal{E}$, in (Eq. **??**), we have

   $$\Gamma_{i,j} = \sum_{l,m} \mathsf{Tr}(P_l P_i P_m P_j) \chi_{l,m}.$$

Defining a 16×16 matrix Ω, such that $\Omega_{4\times i+j,4\times l+m} = \mathsf{Tr}(P_l P_i P_m P_j)$, we have

$$|\Gamma\rangle\rangle = \Omega|\chi\rangle\rangle, \tag{6}$$

where $|\Gamma\rangle\rangle$ and $|\chi\rangle\rangle$ are the (column) vectorized forms of Γ and χ respectively.

4. Pauli Liouville Matrix $\rightarrow$ χ matrix.

   We can easily obtain $|\chi\rangle\rangle$ by inverting the relation in Eq. **??**, giving

   $$|\chi\rangle\rangle = \Omega^{-1}|\Gamma\rangle\rangle. \tag{7}$$

5. Choi Matrix $\rightarrow$ Krauss operators.

   Input: J, a 4×4 complex matrix. Output: $\{K_i\}_{k=1}^{r}$, where each of K_i are 2×2 matrices.

   Eq. **??** can be expressed as

   $$J = \sum_{k=1}^{r} |K_k\rangle\rangle\langle\langle K_k|, \tag{8}$$

   where $|K_k\rangle\rangle$ is the (column) vectorized form of the Krauss operator $\frac{1}{2}K_k$. Performing a singular value decomposition of J, we find $J = \sum_k \lambda_k|\lambda_k\rangle\langle\lambda_k|$. Hence the Krauss operator K_k can be constructed from un-vectorizing $2|\lambda_k\rangle$.

6. Pauli Liouville Matrix $\rightarrow$ Choi Matrix

   Input: Γ, a 4×4 real matrix. Output: J, a 4×4 complex matrix.

   Combining the expression for the bell state in the Pauli basis,

   $$\frac{1}{4}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) = \frac{1}{4}\sum_{i=1}^{4} P_i \otimes P_i^T,$$

   where $P_i \in \{I, X, Y, Z\}$ is a Pauli matrix, with Eq. **??**, we find

   $$J = \frac{1}{4}\sum_i \mathcal{E}(P_i) \otimes P_i^T.$$

   Applying the definition of the Pauli Liouville representation of $\mathcal{E}$ in Eq. **??**, we find

   $$J = \sum_{i,j} \Gamma_{i,j} P_j \otimes P_i^T. \tag{9}$$

7. Choi Matrix $\rightarrow$ Pauli Liouville Matrix.

   Input: J, a 4×4 complex matrix. Output: Γ, a 4×4 real matrix.

   Multiplying on either sides of Eq. **??** by $P_a \otimes P_b^T$, we find

   $$J \cdot (P_a \otimes P_b^T) = \sum_{i,j} \Gamma_{i,j} P_j P_a \otimes P_i^T P_b^T.$$

   Taking the trace of the above equation yields

   $$\Gamma_{b,a} = \mathsf{Tr}(J \cdot (P_a \otimes P_b^T)). \tag{10}$$

8. Strinespring dilation $\rightarrow$ Krauss operators.

   The action of any CPTP map $\mathcal{E}$ on an input state ρ can be derived from unitary dynamics U on a larger Hilbert space, that comprises of the system ρ and its environment, which is initially in some state $|0\rangle_E\langle 0|_E$. Under the unitary evolution described by U, we find

$$\rho \otimes |0\rangle_E\langle 0|_E \rightarrow U\left[\rho \otimes |0\rangle_E\langle 0|_E\right] U^\dagger.$$

   Since the state of the environment is essentially unknown, we must discard it resulting in a partial trace over the environment basis, denoted by $\{|e_k\rangle_E\}$, resulting in

$$\begin{aligned} \rho \otimes |0\rangle_E\langle 0|_E \rightarrow & \mathsf{Tr}_E\left(U\left[\rho \otimes |0\rangle_E\langle 0|_E\right] U^\dagger\right) \\ = & \rho_{ab} \sum_{k=1}^{r} \langle e_k|U|e_0\rangle \rho \langle e_0|U^\dagger|e_k\rangle. \end{aligned} \tag{11}$$

   The above expression is the Krauss representation of the channel where the Krauss operators are $\{\langle e_k|U|e_0\rangle\}_{k=1}^{r}$.

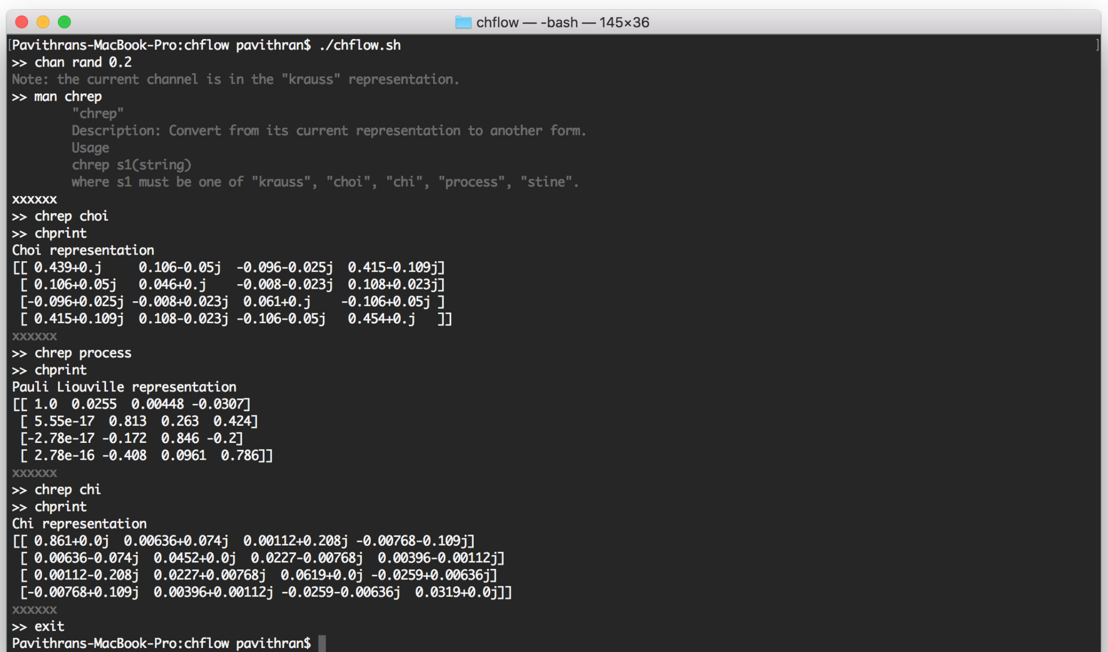


Figure 1: The `chrep` command in chflow can be used to change channel representations. The usage of the command is `chrep <representation>`, where `<representation>` should be replaced by one of {`krauss`, `process`, `choi`, `chi`, `stine`}, which results in the current channel being stored in the specified representation.