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## Point Set Topology 點集拓樸

$\mathbb{R}^n$ $n \geq 1$ ordered n-tuple $x = (x_1, x_2, \cdots, x_n)$ $y = (y_1, y_2, \cdots, y_n)$

① $x = y \Leftrightarrow x_j = y_j \quad \forall\ 1 \leq j \leq n$

② $x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$

③ $ax = (ax_1, ax_2, \cdots, ax_n)$ $a \in \mathbb{R}$

④ $x - y = x + (-1) \cdot y$

⑤ $0 = (0, 0, \cdots, 0)$ $0 + x = x + 0 = x$

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$

Inner Product $g: \underset{x}{V} \times \underset{y}{V} \to \underset{(x \cdot y)}{\mathbb{R}}$

① $(x \cdot x) \geq 0$, $(x \cdot x) = 0 \Leftrightarrow x = 0$

② $(x \cdot y) = (y \cdot x)$

③ $(x + y \cdot z) = (x \cdot z) + (y \cdot z)$ $(ax \cdot y) = a(x \cdot y)$

$\mathbb{R}^n$ $M = (a_{ij})$. $n \times n$. $a_{ij} \in \mathbb{R}$ positive definite 正定 $(x \cdot y) = x^T M y$

if $M = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $(x \cdot y) = \sum_{i=1}^{n} x_i y_i$ std inner product on $\mathbb{R}^n$

Norm (length) $\|x\| = \sqrt{(x \cdot x)}$

Thm: $|(x \cdot y)| \leq \|x\| \cdot \|y\|$ (Cauchy inequality)

pf: let $t \in \mathbb{R}$ $x + ty \in \mathbb{R}^n$

$0 \leq (x + ty \cdot x + ty) = t^2 \|y\|^2 + 2(x \cdot y)t + \|x\|^2$ seem $t$ as a variable

① $\|y\| = 0$, $y = 0$ done

② $\|y\| > 0$, $\Delta = (2(x \cdot y))^2 - 4\|x\| \|y\| \leq 0 \Rightarrow |(x \cdot y)| \leq \|x\| \|y\|$

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lemma: $x, y \in \mathbb{R}^n$ $\|x+y\| \le \|x\| + \|y\|$ (triangular inequality)

pf: $\|x+y\|^2 = (x+y, x+y) = (x, x) + 2(x, y) + (y, y) \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$

Define the distance between $x$ and $y$ by $\|x-y\|$

$(V, \|\ \|)$ is seem as a <u>metric space</u>

$a \in \mathbb{R}^n, r > 0$ $B(a; r) = \{x \in \mathbb{R}^n \mid \|x-a\| < r\}$ n-ball

interior point 內點 $E \subseteq \mathbb{R}^n$ $a \in E$ a is called an interior point of E if $\exists r > 0$ s.t. $B(a; r) \subseteq E$

$E \subseteq \mathbb{R}^n$ is called <u>open</u> if $\forall a \in E$ is an interior point

$E \subseteq \mathbb{R}^n$, denote by $\mathring{E} = \operatorname{int} E = \{$all of the interior points of E$\}$ open $\Leftrightarrow E = \mathring{E}$

Thm ① $O_\alpha$ $\alpha \in \Lambda$ is open in $\mathbb{R}^n$ then $\bigcup_{\alpha \in \Lambda} O_\alpha$ is open

② $V_1, \cdots, V_k$ are open in $\mathbb{R}^n$ then $\bigcap_{j=1}^{k} V_j$ is open

Ex: $I_k = (\frac{1}{k}, \frac{1}{k}) \subseteq \mathbb{R}^1$ $k \in \mathbb{N}$ $\bigcap_{k=1}^{\infty} I_k = \{0\}$ is closed

pf: ① $p \in \bigcup_{\alpha \in \Lambda} O_\alpha$ $\therefore p \in O_\alpha$ for some $\alpha \in \Lambda$ $\therefore \exists r > 0$ s.t. s.t. $B(p; r) \subseteq O_\alpha \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha$

② $q \in \bigcap_{j=1}^{k} V_j$ $\therefore q \in V_j$ $j = 1, 2, \ldots, k$ $\exists r_j > 0$ st. $B(q; r_j) \subseteq V_j$

choose $r = \min\{r_1, r_2, \cdots, r_k\}$ $B(q; r) \subseteq B(q; r_j) \subseteq V_j$ $j = 1, 2, \ldots, k$ $B(q; r) \subseteq \bigcap_{j=1}^{k} V_j$

by the def: empty set $\emptyset$ is open

$\mathbb{R}^1$ $(a, b) = \{x \mid a < x < b\}$ is open $r = \frac{1}{2}\min\{p-a, b-p\}$ $B(p; r) \subseteq (a, b)$

Def: E is open in $\mathbb{R}$, $I \subseteq E$ is an open interval I is called a component interval if $\nexists$ open interval J s.t. $I \subsetneq J \subseteq E$

Thm: E is open in $\mathbb{R}$ $x \in E$ then $x$ belongs to exactly one component interval

pf: $\because x \in E$ $\exists (a, b)$ s.t. $x \in (a, b) \subseteq E$, consider $a(x) = \inf\{a \in E, (a, x) \subseteq E\}$ $b(x) = \sup\{b \in E, (x, b) \subseteq E\}$

$(a(x), b(x))$: component interval let $I_x = (a(x), b(x))$ consider $J_x$ is another component interval

$J_x \cup I_x \supseteq I_x$ $J_x \cup I_x \supseteq J_x$ $\Rightarrow$ $I_x = J_x$

Thm (Representation of open sets in $\mathbb{R}$) $\forall$ open set in $\mathbb{R}$ is a countable union of component interval

pf: $E \subseteq \mathbb{R}$ open, $x \in E \Rightarrow I_x$ comp $x \in I_x$ $E = \bigcup_{x \in E} I_x$ $I_x$: disjoint

$\varphi$: {component intervals of E} $\rightarrow \mathbb{Q}$ let $q \in I \rightarrow q \in \mathbb{Q}$ $\varphi$ is 1-1

$\because \mathbb{Q}$ is a countable set $\therefore$ the subset of $\mathbb{Q}$ is also countable done

Def: $F \subseteq \mathbb{R}^n$ is called to be a closed set if $\mathbb{R}^n \setminus F$ is open

Thm ① $F_\alpha$, $\alpha \in \Lambda$, is closed then $\bigcap_{\alpha\in\Lambda} F_\alpha$ is closed

② $F_1, \dots, F_N$ is closed then $\bigcup_{j=1}^{N} F_j$ is closed

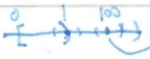
Ex: $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 1] = (0, 1]$ is not closed

pf: ① $(\bigcap_{\alpha\in\Lambda} F_\alpha)^c = \bigcup_{\alpha\in\Lambda} F_\alpha^c$ is open $\therefore \bigcap_{\alpha\in\Lambda} F_\alpha$ is closed

② $(\bigcup_{j=1}^{N} F_j)^c = \bigcap_{j=1}^{N} F_j^c$ is open $\therefore \bigcup_{j=1}^{N} F_j$ is closed

Def (adherent point 附著點) $E \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ is called an adherent point of E if $\forall r > 0$ $B(x; r) \cap E \neq \emptyset$

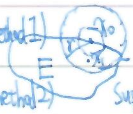
Def (accumulation point 聚集點 A.P.) $E \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ is called an accumulation point if $\forall r > 0$ $B(x; r) \cap (E \setminus \{x\}) \neq \emptyset$

Ex: $[0, 1) \cup \{100\}$  isolated point: $x \in E$ but $x$ is not an A.P.



Thm: Suppose E has an A.P. $x_0$, then each $B(x_0; r)$ contains infinitely many point of E

In particular, E has infinitely many point

pf: (method 1)  $\exists x_1 \in E$, $x_1 \neq x_0$ $r_1 = |x_0 - x_1| < r$ $B(x_0; r_1)$ $r_2 = |x_0 - x_2| < r_1$ ......



(method 2) Suppose there are finite many point $x_1 \cdots x_n$ and let $r_j = |x_j - x_0|$ $j = 1 \cdots n$

let $r = \frac{1}{2}\min\{r_1 \cdots r_n\}$ $B(x_0; r)$ contains no point $\rightarrow\leftarrow$

Thm: $E \subseteq \mathbb{R}^n$, E is closed $\Leftrightarrow$ E contains all its adherent point

(The closure of E, denoted by $\bar{E}$ means $\bar{E} = \{$all of the adherent points of E$\}$, $E \subseteq \bar{E}$)

pf: ($\Rightarrow$) If $p \in E^c$ open $\therefore \exists r > 0$ s.t. $B(p; r) \subseteq E^c$ $\therefore B(p; r) \cap E = \emptyset$ $\therefore p$ is not adherent point

($\Leftarrow$) If $q \in E^c$, then $q$ is not an adherent point $\therefore \exists r_0 > 0$ s.t. $B(q; r_0) \cap E = \emptyset$

$B(q; r_0) \subseteq E^c$ $\therefore q \in \mathring{E^c}$ $\therefore E^c$ is open $\therefore E$ is closed

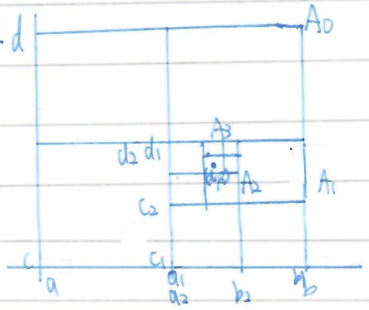
Def: Define the derive set $E'$ of E by $E' = \{$all A.P. of E$\}$ $\bar{E} = E \cup E'$

Thm: E is closed $\Leftrightarrow E = \bar{E} \Leftrightarrow E' \subseteq E$

Thm (Bolzano-Weierstrass) Every bounded infinite subset $E$ of $\mathbb{R}^n$ has an A.P. in $\mathbb{R}^n$

($E \subseteq \mathbb{R}^n$ is said to be bounded if $E \subseteq B(a;r)$ for some $a$, for some $r$)

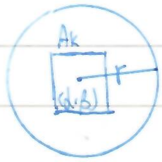
pf: $n=2$ d




$A_0 = [a,b]\times[c,d]$
$\cup\!|$
$A_1 = [a_1,b_1]\times[c_1,d_1]$ $\quad b_1-a_1=\frac{1}{2}(b-a)$ $\quad d_1-c_1=\frac{1}{2}(d-c)$
$\cup\!|$
$A_2 = [a_2,b_2]\times[c_2,d_2]$ $\quad b_k-a_k=\frac{1}{2^k}(b-a)$ $\quad d_k-c_k=\frac{1}{2^k}(d-c)$
$\cup\!|$
$A_k = [a_k,b_k]\times[c_k,d_k]$ $\quad \ldots$
$\cup\!|$
$\vdots$

$a \le a_1 \le \cdots \le a_k \le \cdots \le \alpha_1 \underset{\alpha}{=} \alpha_2 \le \cdots \le b_1 \le b$ bounded monotonic sequence $\Rightarrow$ converge

$c \le c_1 \le \cdots \le c_k \le \cdots \le \beta_1 \underset{\beta}{=} \beta_2 \le \cdots \le d_1 \le d$




$r>0$ $\quad b_k-a_k \le \frac{r}{10}$ $\quad d_k-c_k \le \frac{r}{10}$

$(\alpha,\beta)$ is an A.P.

Thm (Cantor intersection theorem) $E_k \subseteq \mathbb{R}^n$ $k=1,2,3,\ldots$

Assume: ① $E_1$ is bounded, $E_k \ne \emptyset$ $\forall k$, $E_k$ is closed $\forall k$ ② $E_1 \supseteq E_2 \supseteq \cdots$ then $\bigcap_{k=1}^{\infty} E_k \ne \emptyset$

Ex: $\bigcap_{k=1}^{\infty}[-\frac{1}{k},\frac{1}{k}] = \{0\} \ne \emptyset$

pf: If $E_k$ has finite elements, done.

We may assume $\#E_k = \infty$ $\forall k$ choose $x_k \in E_k$ $\forall k$ s.t. $x_j \ne x_k$ if $j \ne k$

$\therefore$ by Bolzano-Weierstrass $\{x_k\}$ has an A.P. $p$

$p \in E_1$ $p \in E_2 \cdots p \in E_k$ $\forall k$ $\because$ $E_k$ is closed set $\therefore$ $E_k$ contains all A.P.

$\therefore$ $p \in \bigcap_{k=1}^{\infty} E_k$ QED

Def : $\underset{\neq\emptyset}{E} \subseteq \mathbb{R}^n$ is called perfect if (i) E is closed (ii) Every point of E is an AP of E

Ex : $I = [a, b]$ ✓ $\quad I = [a, b] \cup \{p\}$ X

Thm If E is perfect then E is uncountable

pf : Assume E is countable, write $E = \{x_1, x_2, \dots x_k, \dots\}$ let $V_1 = B(x_1; r)$ for some $r > 0$

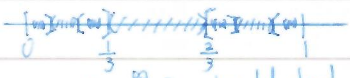
choose open neighborhood $V_2$ s.t. ① $x_1 \notin \overline{V_2} \subseteq V_1$ ② $V_2 \cap E$ has infinitely many point

continous this process $\overline{V_1} \supseteq V_1 \supseteq \overline{V_2} \supseteq V_2 \supseteq \cdots$ $x_k \notin \overline{V_{k+1}}$ $V_k \cap E$ has infinitely many point

consider $x_{k-1} \notin F_k = \overline{V_k} \cap E \neq \emptyset$ closed bounded. By Cantor intersection thm $\underset{\subseteq E}{\bigcap_{k=1}^{\infty} F_k \neq \emptyset}$

$\therefore x_\ell \in \bigcap_{k=1}^{\infty} F_k$ ($\rightarrow\leftarrow$) $\because x_\ell \notin \overline{V_{\ell+1}} \cap E = F_{\ell+1}$

Cantor set




| | # intervals | length | # removed open interval | length |
|---|---|---|---|---|
| $C_1$ | 2 | $\frac{1}{3}$ | $2-1$ | $\frac{1}{3}$ |
| $C_2$ | $2^2$ | $\frac{1}{3^2}$ | $2^2-1$ | |
| ⋮ | ⋮ | ⋮ | ⋮ | |
| $C_k$ | $2^k$ | $\frac{1}{3^k}$ | $2^k-1$ | |
| ⋮ | ⋮ | ⋮ | ⋮ | |

$C = \bigcap_{k=1}^{\infty} C_k$ bounded closed

given $\varepsilon > 0$ $p \in C$ $(p-\varepsilon, p+\varepsilon)$ $\therefore p \in C_k \ \forall k$ $\left(\frac{1}{3}\right)^k < \frac{\varepsilon}{2}$ $\therefore C$ is perfect set

removed length $\frac{1}{3} + 2 \cdot \left(\frac{1}{3}\right)^2 + 2^2\left(\frac{1}{3}\right)^3 + \cdots + 2^k\left(\frac{1}{3}\right)^{k+1} + \cdots = \frac{1}{3}\left(1 + \frac{2}{3} + \frac{2^2}{3^2} + \cdots + \left(\frac{2}{3}\right)^k + \cdots\right) = \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = 1$

Def : $\mathcal{F} = \{E_k\}_{k \in \Lambda}$ a collection subset of $\mathcal{F}$ is a cover of $A \subseteq \mathbb{R}^n$ if $A \subseteq \bigcup_{k \in \Lambda} E_k$

If $E_k$ is open for all $k \in \Lambda$ then $\mathcal{F}$ is called an open cover of A

Ex : $E = (0, 1)$ $E_k = \left(\frac{1}{k}, 1\right)$ $k = 2, 3, \dots$ $E = \bigcup_{k=2}^{\infty} E_k$ ,

lemma : let V be an open set in $\mathbb{R}^n$, $x_0 \in V$, then $\exists\, a = (a_1, \dots, a_n)$ $r > 0$ s.t. $a_j \in \mathbb{Q}$ $r \in \mathbb{Q}$ and $x_0 \in B(a; r) \subseteq V$

pf : $\because x_0$ is an interior point $\therefore \exists B(x_0; \rho) \subseteq V$, $\rho > 0$ $\quad \exists\, a \in B(x_0; \frac{\rho}{10})$ s.t. $a = (a_1, \dots, a_n)$ $a_j \in \mathbb{Q}$

chose a $r \in \mathbb{Q}$ s.t. $\frac{\rho}{10} < r < \frac{2}{10}\rho$ $\quad x_0 \in B(a; r) \subseteq B(x_0; \rho) \subseteq V$

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Thm (Lindelöf covering theorem): Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F} = \{V_\alpha\}_{\alpha\in\Lambda}$ be an open covering of E

Then $\mathcal{F}$ contains a countable subcovering of E

pf: let $x \in E \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$ $\therefore x \in V_\alpha$ some $\alpha\in\Lambda$ choose a $B(a;r) = B_x$ s.t. $a=(a_1,\dots,a_n)$ $a_j\in\mathbb{Q}$, $r\in\mathbb{Q}$

$x \in B_x = B(a;r) \subseteq V_\alpha$ $E \subseteq \bigcup_{x\in E} B_x \subseteq \bigcup_{\alpha'} V_{\alpha'}$ (choose one $B_x \subseteq V_{\alpha'}$ for some $\alpha'$)

Since $\{B(a,r)\}_{\mathbb{Q}^{n+1}}$ is countable it follows that in above statement Only at most countable $B_x$ appears

Thm (Heine-Borel) Let $E \subseteq \mathbb{R}^n$ be a bounded and closed subset and let $\mathcal{F} = \{V_\alpha\}_{\alpha\in\Lambda}$ be an open covering of E

Then $\exists$ a finite subcovering of $\mathcal{F}$ that also cover E

9/30 pf: by Lindelöf's thm $\exists$ a countable subcover $\{V_k\}_{k=1}^{\infty}$ of $\mathcal{F}$ that also covers E, i.e. $E \subseteq \bigcup_{k=1}^{\infty} V_k$

let $F_m = E \setminus (\bigcup_{k=1}^{m} V_k)$ closed. $F_{m+1} \subseteq F_m$ bounded, Assume $F_m \neq \emptyset$ $\forall m$

by Cantor intersection thm $\bigcap_{m=1}^{\infty} F_m \neq \emptyset$ $\therefore \exists p \in \bigcap_{m=1}^{\infty} F_m$ $\therefore p \in \bigcap_{m=1}^{\infty} F_m \subseteq E \subseteq \bigcup_{k=1}^{\infty} V_k$ $\therefore p \in V_k$ for some k $(\rightarrow\leftarrow)$

$\because p \in F_k = E \setminus (\bigcup_{j=1}^{k} V_j)$ $\therefore p \notin V_k$ $\Rightarrow F_m = \emptyset$ for some m $\therefore F_m = E \setminus (\bigcup_{k=1}^{m} V_k) \Rightarrow E \subseteq \bigcup_{k=1}^{m} V_k$

Ex: $(0,1)$ bounded, not closed open cover $(0,1)$, $\mathbb{R}$, $\{(0,1-\frac{1}{k})\}_{k=1}^{\infty}$

$[0,\infty)$ not bounded, closed open cover $\{(-1,n)\}_{n=1}^{\infty}$

Def: $E \subseteq \mathbb{R}^n$ is called compact if every open cover of E contains a finite subcover

Thm: $E \subseteq \mathbb{R}^n$. The following statement are equivalent ① E is compact ② E is bounded and closed

③ Every infinite subset of E has an A.P. in E

pf: ②⇒① by Heine-Borel thm

①⇒② $E \subseteq \bigcup_{k=1}^{\infty} B(0;k)$ by compactness of E $\therefore \exists m$ s.t. $E \subseteq \bigcup_{k=1}^{m} B(0;k) = B(0;m)$ ⇒ bounded

If E has an A.P. $y \notin E$ $\forall x \in E$ $\frac{1}{2}\|x-y\| = r_x > 0$ $E \subseteq \bigcup_{x\in E} B(x;r_x)$

by compactness $\therefore \exists x_1, \dots, x_l \in E$ s.t. $E \subseteq \bigcup_{j=1}^{l} B(x_j;r_j)$ $(\rightarrow\leftarrow)$

$\because$ choose $r = \frac{1}{2}\min\{r_1,\dots,r_l\}$ $B(y;r) \cap E = \emptyset$ $\Rightarrow y \in E$ ⇒ closed

②⇒③ by Bolzano-Weierstrass Every infinite subset of E has an AP in $\mathbb{R}^n$

by closeness of E $\therefore$ This AP is in E

③⇒② If E is not bounded $\therefore \exists x_k \in E$ s.t. $\|x_k\| > k$ $\forall k \in \mathbb{N}$, $\{x_k\}_{k=1}^{\infty}$ is an infinite subset of E

let y be an AP of $\{x_k\}_{k=1}^{\infty}$ $\|y-x_k\| \geq \|x_k\| - \|y\| > k - \|y\| > 1$ if k large $(\rightarrow\leftarrow)$ ⇒ bounded

let x be an AP of E $\therefore \exists x_k \in B(x;\frac{1}{k})$ $k\in\mathbb{N}$ s.t. $x_k \in E$ $x_k \neq x$ ( $\|x-x_k\| < \frac{1}{k} < \frac{d}{2}$ as k large )

$y \in E$, $y \neq x$. $\|y-x_k\| = \|y-x+x-x_k\| \geq \|y-x\| - \|x-x_k\| = d - \|x-x_k\| > \frac{d}{2} > 0$

$\{x_k\}$ has only one AP $x \in E$ ⇒ closed

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# Metric Space 度量空間

$(M,d)$ is a metric space if $d: M\times M \to \mathbb{R}$, $(x,y) \to d(x,y)$ s.t. ① $d(x,x)=0$ ② $d(x,y)>0$ if $x,y\in M$ $x\neq y$ ③ $d(x,y)=d(y,x)$

④ $d(x,y) \le d(x,z) + d(x,y)$ triangle inequality

$B_M(a;r) = \{x\in M \mid d(a,x) < r\}$ open ball with radius $r>0$

$p\in E \subseteq M$ $p$: interior point of $E$ if $B_M(p;r)\subseteq E$ for some $r>0$

Hausdorff(性) For $x\neq y$ in $E$ $\exists\, r=\frac{1}{3}d(x,y)>0$ $x\in B(x;r)$ $y\in B(y;r)$ $B(x;r)\cap B(y;r)=\emptyset$

Ex: ① $(\mathbb{R}^n, \|\ \|)$ ② $(M, d_M)$ metric space $S\subseteq M$ subset $d_S = d|_S : S\times S \to \mathbb{R}$

③ discrete metric $d(x,x)=0$ $\forall x\in M$ $d(x,y)=1$ $\forall x\neq y\in M$ every set is open and closed

④ $\mathbb{R}^2$ $d(x,y)=\max\{|x_1-x_2|, |y_1-y_2|\}$ $B_d(0;1)=$ 

⑤ $M=(a_{ij})=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn}\end{pmatrix}$ symmetric, positive define $\Leftrightarrow$ every eigenvalue $>0$

$\Rightarrow M=\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ $X^TMX=\lambda_1x_1^2+\cdots+\lambda_nx_n^2$ is a metric in $\mathbb{R}^n$

$\to$ $x\in S$ $B_{d_S}(x;r)=B_{d_M}(x;r)\cap S$

Thm: $(M,d)$ is a metric space, $S\subseteq M$ subspace ① then $X\subseteq S$, $X$ is open is $S \Leftrightarrow X=V\cap S$ $V$ is open in $M$ 6/4

pf: ($\Leftarrow$) Let $V$ be an open set in $M$ let $X=V\cap S$. If $p\in X \Rightarrow p\in V$, $\exists B(p;r)\subseteq V$

$B_S(p;r)=B(p;r)\cap S \subseteq V\cap S$ $\therefore X$ is open in $S$

($\Rightarrow$) Let $X$ be an open set in $S$. If $x\in X$ $\therefore \exists B_S(x;r_x)\subseteq X$ where $B_M(x;r_x)\cap S$

let $V=\bigcup_{x\in X} B_M(x;r_x)$ open in $M$ $\therefore X=V\cap S$

② $F\subseteq S$ $F$ is closed in $S \Leftrightarrow F=C\cap S$ $C$ is closed in $M$

($\Rightarrow$) $F$ is closed in $S$ $F=S-U$ ($U$ is open in $S$) $=S-(V\cap S)$ ($V$ is open in $M$) $=S-V=S\cap(M-V)=S\cap C$

($\Leftarrow$) $F=C\cap S$ ($C$: closed in $M$) $=S\cap(M-V)$ ($V$: open in $M$) $=S-V=S-(S\cap V)$ ($S\cap V$ open in $S$) $\therefore F$ closed in $S$

Thm: $(S,d)$ metric space, $M\subseteq S$, then $M$ is compact $\Leftrightarrow$ $M$ is sequentially compact (i.e. every infinite subset $T$ of $M$ has an AP in $M$)

pf: $(\Rightarrow)$ Assume $M$ is compact. Let $T=\{x_1,x_2,\dots\}\subseteq M$. Assume $T$ has no AP in $M$

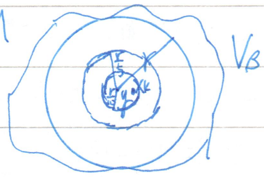
① If $y\notin T \Rightarrow \exists B(y;r_y)\cap T=\emptyset$ ② If $x_j\in T \Rightarrow \exists B(x_j;r_j)\cap T=\{x_j\}$

$M=\bigcup_{x\in M} B(x;r_x)$ (*) has no finite subcover

Thm: $(M,d)$ metric space. Assume $M$ is sequentially compact. Let $\mathcal{F}=\{V_\alpha\}_{\alpha\in\Lambda}$ be an open covering, then $\exists\ \delta>0$ wrt $\mathcal{F}$ s.t. if $E\subseteq M$ has diameter $<\delta$ $\exists$ a $V_\alpha$ in $\mathcal{F}$ that contains $E$ ($\delta$: Lebesgue number)

pf: $E\subseteq M$, $\operatorname{diam}E=\sup\{d(x,y)\mid x,y\in E\}$. If not $\forall\delta=\frac{1}{k}$ $\exists C_k\subseteq M$ s.t. $\operatorname{diam}C_k\le\frac{1}{k}$ and $C_k\not\subseteq V_\alpha$ $\forall\alpha\in\Lambda$ $k\in\mathbb{N}$

pick $x_k\in C_k$ $\forall k$. $\{x_k\}\subseteq M$ has an AP $y\in M\subseteq\bigcup_{\alpha\in\Lambda}V_\alpha$ $y\in V_\beta$ for some $\beta\in\Lambda$




since $V_\beta$ is open $\therefore \exists B(y;r)\subseteq V_\beta$ $\therefore x_k\in B(y;\frac{r}{10})$ and $\frac{1}{k}<\frac{r}{10}$

$\because x_k\in C_k$ $\operatorname{diam}C_k\le\frac{1}{k}<\frac{r}{10}$ $\Rightarrow C_k\subseteq B(y;r)\subseteq V_\beta$

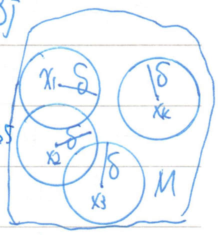
$(\Leftarrow)$ Assume $M$ is sequentially compact,

claim Given $\rho>0$ $\exists$ finitely many open ball $B_j$ $j=1,2,\dots N$ with radius $\rho$ s.t. $M\subseteq\bigcup_{j=1}^{N}B_j$

If not, choose $x_1\in M$ $\underbrace{B(x_1;\rho)}_{B_1}$ choose $x_2\notin B_1$ consider $B_2=B(x_2;\rho)$

choose $x_3\notin B_1\cup B_2$ consider $B_3=B(x_3;\rho)$ ...... continue this process

find $\{x_k\}$ infinite subset of $M$ $(\rightarrow\leftarrow)$ $\because$ $\{x_k\}$ has no AP $\because d(x_i,x_j)>\delta$ if $i\neq j$




Let $\mathcal{F}=\{V_\alpha\}_{\alpha\in\Lambda}$ be an open covering of $M$. Since $M$ is sequential compact

by previous thm, $\exists$ Lebesgue number $\delta>0$ consider $r=\frac{\delta}{3}$

$\exists B(x_j;r)$ $j=1,2,\dots,N$ s.t. $M=\bigcup_{j=1}^{N}B(x_j;r)$ $\operatorname{diam}B(x_j;r)\le\frac{2}{3}\delta<\delta$

$\exists V_{\alpha_j}$ $j=1,2\dots N$ s.t. $B(x_j;r)\subseteq V_{\alpha_j}$ $M=\bigcup_{j=1}^{N}B(x_j;r)\subseteq\bigcup_{j=1}^{N}V_{\alpha_j}$

Thm: $(M,d)$ metric space, $S\subseteq M$. If $S$ is compact, then $S$ is bounded and closed $(\not\Leftarrow)$

pf: the same method as $(\mathbb{R}^n,\|\ \|)$ just change the metric

Thm: $(M,d)$ metric space, $S\subseteq M$. If $S$ is compact then any closed subset is also compact.

pf: $F\subseteq S$ closed let $\{V_\alpha\}$ open cover of $F$ $\therefore S\subseteq F\cup F^c\subseteq\bigcup_{\alpha\in\Lambda}V_\alpha\cup F^c$

Since $S$ is compact $\therefore S\subseteq\bigcup_{j=1}^{N}V_{\alpha_j}\cup F^c$ $\therefore F\subseteq\bigcup_{j=1}^{N}V_{\alpha_j}$

$(M, d)$ metric space $S \subseteq M$ subset. $p \in M$ is called a boundary point if $\forall r > 0$ then $\begin{cases} B(p;r) \cap S \neq \emptyset \\ B(p;r) \cap (M-S) \neq \emptyset \end{cases}$

$p \in \overline{S} \cap \overline{(M-S)}$ $\quad \partial S = \{\text{All boundary point of } S\} = \overline{S} \cap \overline{(M-S)}$

Ex: $\mathbb{Q} \subseteq \mathbb{R}$ $\quad \mathring{\mathbb{Q}} = \emptyset$ $\quad \partial \mathbb{Q} = \mathbb{R}$

$I = (0, 1] \subseteq \mathbb{R}$ $\quad \partial I = \{0, 1\}$ $\quad I_1 = (0, 1) \subseteq \mathbb{R}$ $\; I_2 = [0, 1] \subseteq \mathbb{R} \Rightarrow \partial I = \partial I_1 = \partial I_2$

Topology: Continous deformation

Ch3 point set topology END

## Limit point. Continuity 極限點·連續性

$\{x_n\} \subseteq M$ $(M,d)$ metric space we say $\{x_n\}$ converges to $p$ if given $\varepsilon>0$ $\exists n_0 \in \mathbb{N}$ s.t. $d(x_n,p)<\varepsilon$ for $n \geq n_0$

Thm: Bounded monotonic sequence in $\mathbb{R}^1$ must converge

pf $a_n \nearrow$ $a_n \to \sup\{a_n\}$ $b_n \searrow$ $b_n \to \inf\{b_n\}$

Thm: $(M,d)$ metric space $\{x_n\}$ converge seq in $M$ $\Rightarrow$ $\{x_n\}$ converges to exactly one point

pf: If $\{x_n\} \to p$ $\{x_n\} \to q$ given $\varepsilon>0$ $\exists n_1, n_2 \in \mathbb{N}$ $d(x_n,p)<\varepsilon$ if $n \geq n_1$ $d(x_m,q)<\varepsilon$ if $m \geq n_2$

Let $n_0 = \max\{n_1,n_2\}$ For $n \geq n_0$ $0 \leq d(p,q) \leq d(p,x_n)+d(x_n,q) < 2\varepsilon$ $\therefore d(p,q)=0$ $\therefore p=q$

Thm: $(M,d)$ metric space $T=\{x_1,x_2,\dots\} \subseteq M$ If $\{x_n\}$ converges to $p$ then ① $T$ is bounded ② $p$ is an adherent point of $T$

pf: given $\varepsilon=1$ $\exists n_0 \in \mathbb{N}$ s.t. $d(x_n,p)<1$ for $n \geq n_0$ let $r=\max\{2\max\{d(p,x_1),\dots,d(p,x_{n_0-1})\},1\}$ $\{x_n\} \subseteq B(p;r)$ bounded

given $\varepsilon>0$ $\exists n_0 \in \mathbb{N}$ $x_n \in B(p;\varepsilon)$ $\to$ $p$ is an adherent point of $\{x_n\}$

Thm: $(M,d)$ metric space. $E \subseteq M$ If $p \in M$ is an AP of $E$, then $\exists$ a seq of pt. $\{x_n\} \subseteq E$ that converges to $p$

pf: $\because B(p;\frac{1}{k}) \cap E \neq \emptyset$ $k \in \mathbb{N}$ let $x_k \in B(p;\frac{1}{k}) \cap E$ $x_k \to p$

Thm: $(M,d)$ metric space then $x_n \to p$ iff every subseq $x_{k(n)} \to p$

pf: $(\Leftarrow)$ obvious

$(\Rightarrow)$ given $\varepsilon>0$ $\exists n_0 \in \mathbb{N}$ s.t. $d(x_n,p)<\varepsilon$ for $n \geq n_0$,

$x_{k(n)}$ is a subseq of $x_n$ given $n_0$ $\exists m_0 \in \mathbb{N}$ s.t. if $n \geq m_0 \Rightarrow k(n) \geq n_0$

given $\varepsilon>0$ $\exists m_0 \in \mathbb{N}$ s.t. for $n \geq m_0 \Rightarrow k(n) \geq n_0$ $d(x_{k(n)},p)<\varepsilon$

Cauchy sequence: $(M,d)$ metric space, $\{x_n\}$: seq in $M$ $\{x_n\}$ is called Cauchy seq if given $\varepsilon>0$ $\exists n_0 \in \mathbb{N}$ s.t. $d(x_n,x_m)<\varepsilon$ for $m,n \geq n_0$

Thm: $(M,d)$ metric space. Assume $\{x_n\}$ converges to $p$ in $M$ $\Rightarrow$ $\{x_n\}$ is Cauchy ($\not\Leftarrow$)

pf: given $\varepsilon>0$ $\exists n_0 \in \mathbb{N}$ s.t. $d(x_n,p)<\frac{\varepsilon}{2}$ for $n \geq n_0$

For $m,n \geq n_0$ $d(x_m,x_n) \leq d(x_m,p)+d(x_n,p) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

Ex: $[0,1]$ $x_n=\frac{1}{n} \to 0$ $(0,1]$ $x_n=\frac{1}{n}$ is Cauchy but not convergent

Thm: $\{x_n\} \subseteq \mathbb{R}^n$ If $\{x_n\}$ is Cauchy, then $\{x_n\}$ converges to $p$

pf: $\{x_n\}$ is bounded, given $\varepsilon=1$ $\exists n_0 \in \mathbb{N}$ s.t. $|x_m-x_n|<1$ for $m,n \geq n_0$

For $m>n_0$ $|x_m|=|x_m-x_{n_0}+x_{n_0}| \leq |x_m-x_{n_0}|+|x_{n_0}| < 1+|x_{n_0}|$ let $r=2\max\{|x_1|,\dots,|x_{n_0-1}|,1+|x_{n_0}|\}$ $\Rightarrow |x_k|<r$ $\forall k$

by Balzona-Weierstrass $\{x_n\}$ has an AP in $\mathbb{R}^n$ given $\varepsilon>0$ $\exists n_1 \in \mathbb{N}$ s.t. $|x_m-x_n|<\frac{\varepsilon}{2}$ for $m,n \geq n_1$

find $x_l \in B(p;\frac{\varepsilon}{2})$ $l>n_1$ $|p-x_l|<\frac{\varepsilon}{2}$ $n \geq n_1$ $l \geq n_1$ $|p-x_n|=|p-x_l+x_l-x_n| \leq |p-x_l|+|x_l-x_n| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

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Ex: $\{a_n\}$ in $\mathbb{R}^1$ s.t. $|a_{n+2}-a_{n+1}| \le \frac{1}{2}|a_{n+1}-a_n|$ $\forall n \Rightarrow \{a_n\}$ converges

$|a_{n+k}-a_n| = |a_{n+k}-a_{n+k-1}+a_{n+k-1} \cdots + a_{n+1}-a_n| \le |a_{n+k}-a_{n+k-1}| + \cdots + |a_{n+1}-a_n|$

$\left(|a_{n+1}-a_n| \le \frac{1}{2}|a_n-a_{n-1}| \le \left(\frac{1}{2}\right)^2 |a_{n-1}-a_{n-2}| \le \cdots \le \left(\frac{1}{2}\right)^{n-1}|a_2-a_1|\right)$

$\le |a_2-a_1|\left(\left(\frac{1}{2}\right)^{m+k-2} + \cdots + \left(\frac{1}{2}\right)^{m-1}\right) \le |a_2-a_1|\left(\frac{1}{2}\right)^{m-1}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\cdots\right) < 2|a_2-a_1|\left(\frac{1}{2}\right)^{m-1}$ Cauchy

Def: $(M,d)$ metric space. $(M,d)$ is said to be complete if every Cauchy seq in $M$ converges in $M$

Ex $(\mathbb{R}^n, |\;|)$ complete $(\mathbb{Q}, |\;|)$ incomplete $([0,1], |\;|)$ complete

$(M,d)$ metric space $S \subseteq M$ $(S,d)$ is called complete if $(S,d)$ is a complete metric space 10/11

Thm $(M,d)$ metric space $X \subseteq M$ compact subset $\Rightarrow (X,d)$ is a complete metric space

pf: let $\{x_n\}$ be a Cauchy seq in $X$. Since $X$ is compact, $\{x_n\}$ has an AP $p$ in $X$

given $\varepsilon>0$, $\exists n_0$ s.t. $d(x_j,x_k) < \frac{\varepsilon}{2}$ if $j,k \ge n_0$. choose a $x_m$ s.t. $m \ge n_0$ and $d(x_m,p) < \frac{\varepsilon}{2}$

$\therefore$ For $k \ge n_0$, $d(x_k,p) \le d(x_k,x_m) + d(x_m,p) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

$f:(S,d_S)\to(T,d_T)$ $A\subseteq S$ $p$ is an AP of $A$ in $S$ $b\in T$ $\lim\limits_{\substack{x\to p\\ x\in A}} f(x)=b$

given $\varepsilon>0$ $\exists \delta=\delta(\varepsilon,p)>0$ s.t. $d_T(f(x),b)<\varepsilon$ for $0<d_S(x,p)<\delta$ $x\in A$

Thm: $f:(S,d_S)\to(T,d_T)$ $A\subseteq S$ $p\in S$ is an AP of $A$ $b\in T$ then $\lim\limits_{x\to p} f(x)=b \Leftrightarrow$ If $\{x_n\}$ converges to $p$, $x_n\in A$ $\lim\limits_{n\to\infty} f(x_n)=b$

pf: $(\Rightarrow)$ $\lim\limits_{x\to p} f(x)=b$ Given $\varepsilon>0$ $\exists\delta>0$ s.t. $d_T(f(x),b)<\varepsilon$ if $0<d_S(x,p)<\delta$

If $\lim\limits_{n\to\infty} x_n=p$ For this $\delta$, $\exists n_0\in\mathbb{N}$ s.t. $d_S(x_n,p)<\delta$ if $n\ge n_0$. For $n\ge n_0$, $d_T(f(x_n),b)<\varepsilon$

$(\Leftarrow)$ If not $\exists \varepsilon_0>0$ s.t. $\forall\delta>0$ $\exists x_\delta\in A$ s.t. $0<d_S(x_\delta,p)<\delta$ and $d_T(f(x_\delta),b)\ge\varepsilon_0$

let $\delta=\frac{1}{k}$ $k\in\mathbb{N}$ $\exists x_k\in A$ $0<d_S(x_k,p)<\frac{1}{k}=\delta$ s.t. $d_T(f(x_k),b)\ge\varepsilon_0$ $\{x_k\}\subseteq A$ $x_k\to p$ $(\rightarrow\leftarrow)$

$\frac{f}{g}:(S,d_S)\to\mathbb{C}$ $f\pm g$ $\lambda f$ $f\cdot g$ $\frac{f}{g}$ $(g\ne0)$

Thm: $\frac{f}{g}(S,d_S)\to\mathbb{C}$ $A\subseteq S$ $p$. AP of $A$ $\lim\limits_{x\to p} f(x)=\alpha$ $\lim\limits_{x\to p} g(x)=\beta$

$\Rightarrow$ (i) $\lim\limits_{x\to p}(f(x)\pm g(x))=\alpha\pm\beta$ (ii) $\lim\limits_{x\to p}(\lambda\cdot f(x))=\lambda\alpha$ (iii) $\lim\limits_{x\to p}(f(x)\cdot g(x))=\alpha\cdot\beta$ (iv) $\lim\limits_{x\to p}\left(\frac{f(x)}{g(x)}\right)=\frac{\alpha}{\beta}$ if $\beta\ne0$

pf (iii) $|f(x)g(x)-\alpha\beta| = |f(x)g(x)-\alpha g(x)+\alpha g(x)-\alpha\beta| \le |f(x)-\alpha||g(x)|+|\alpha||g(x)-\beta| \le M_g|f(x)-\alpha|+|\alpha||g(x)-\beta| < (M_g+|\alpha|)\varepsilon$

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$\begin{matrix} f \\ g \end{matrix} : (S, d_S) \to \mathbb{R}^n$ $f = (f_1, f_2, \cdots, f_n)$ $g = (g_1, g_2, \cdots, g_n)$ component func define $f \pm g$ $\lambda \cdot f$ $f \cdot g$

$A \subseteq S$ $p$ : AP of $A$ $\lim_{x \to p} f(x) = \alpha = (a_1, \cdots, a_n)$ $\lim_{x \to p} g(x) = \beta = (\beta_1, \cdots, \beta_n) \Rightarrow \lim_{x \to p} (f(x) \pm g(x)) = \alpha + \beta$ $\lim_{x \to p} \lambda f(x) = \lambda \alpha$ $\lim_{x \to p} f(x) \cdot g(x) = \alpha \cdot \beta$

$f(x) \cdot g(x) - \alpha\beta = (f(x) - \alpha) \cdot (g(x) - \beta) + (f(x) - \alpha) \cdot \beta + (g(x) - \beta) \cdot \alpha$

$|f(x) \cdot g(x) - \alpha\beta| \leq \|f(x) - \alpha\| \|g(x) - \beta\| + \|\beta\| \|f(x) - \alpha\| + \|\alpha\| \|g(x) - \beta\| < \varepsilon \cdot \varepsilon + \|\beta\| \varepsilon + \|\alpha\| \varepsilon$

$f : (S, d_S) \to (T, d_T)$ $p \in S$ $f$ is continous at $p$ if given $\varepsilon > 0$ $\exists \delta = \delta(\varepsilon, p) > 0$ s.t. $d_T(f(x), f(p)) < \varepsilon$ for $d_S(x, p) < \delta$

If $f$ is conti at every point of $A$ then we say $f$ is conti on $A$

If $p$ is an isolated point of $S$ then $f$ is conti at $p$

Thm: $f$ is conti at $p \Leftrightarrow \lim_{n \to \infty} f(x_n) = f(p)$ for every seq $\{x_n\}$ converges to $p$

$(S, d_S) \xrightarrow{f} (T, d_T) \xrightarrow{g} (W, d_W)$ $p \in S$ $f$ is conti at $p$, $g$ is conti at $\underset{y_0}{f(p)}$ $\Rightarrow g \circ f$ is conti at $p$

given $\varepsilon > 0$ $\exists \delta_1 > 0$ s.t. $d_W(g(y_0), g(y)) < \varepsilon$ if $d_T(y, y_0) < \delta_1$ For this $\delta_1 > 0$ $\exists \delta > 0$ s.t. $d_T(f(x), \underset{y_0}{f(p)}) < \delta_1$ if $d_S(x, p) < \delta$

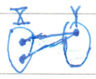
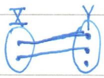
$\Rightarrow$ given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $d_W(g(f(p)), g(f(x))) < \varepsilon$ if $d_S(x, p) < \delta$

$\begin{matrix} f \\ g \end{matrix} : (S, d_S) \to \mathbb{C}$ $p \in S$ $f, g$ conti at $p \Rightarrow f \pm g$ $\lambda f$ $fg$ $\frac{f}{g}$ $(g(p) \neq 0)$ are also conti at $p$

$\begin{matrix} f \\ g \end{matrix} : (S, d_S) \to \mathbb{R}^n$ $p \in S$ $f, g$ conti at $p \Rightarrow f \pm g$ $\lambda f$ $f \cdot g$ are also conti at $p$

$f = (f_1, f_2, \cdots, f_n)$ is conti at $p \Leftrightarrow f_j$ is conti at $p$ $1 \leq j \leq n$

$f : \underset{\substack{\cup \\ E}}{X \to Y}$ $f^{-1}(E) = \{x \in X \mid f(x) \in E\}$ inverse image (preimage)

$f : X \to Y$, $A \subseteq X$, $B \subseteq Y$ $A \subseteq f^{-1}(f(A))$  $f(f^{-1}(B)) \subseteq B$ 

Thm: $f : (S, d_S) \to (T, d_T)$ $f$ is conti on $S \Leftrightarrow f^{-1}(V)$ is open in $S$ $\forall$ open subset $V$ of $T$ $f : \mathbb{R} \to \mathbb{R}$ $f(x) = c$ open $\not\Rightarrow$ open

$\Leftrightarrow f^{-1}(F)$ is closed in $S$ $\forall$ closed subset $F$ of $T$ $f : \mathbb{R} \to \mathbb{R}$ $f(x) = \tan^{-1} x$ closed $\not\Rightarrow$ closed

10/4 pf ($\Rightarrow$) $f$ is conti $V \subseteq T$ is open. show $f^{-1}(V)$ is open in $S$. If $p \in f^{-1}(V) \Rightarrow f(p) \in V$ $\therefore \exists B_T(f(p); \varepsilon) \subseteq V$

by conti of $f$ at $p$ $\exists \delta > 0$ s.t. $f(B_S(p; \delta)) \subseteq B_T(f(p), \varepsilon) \subseteq V$ $\therefore B_S(p; \delta) \subseteq f^{-1}(V)$

($\Leftarrow$) $f^{-1}(V)$ is open in $S$ $V \subseteq T$ open $p \in S$ $f(p) \in T$ given $\varepsilon > 0$ $B_T(f(p); \varepsilon) \underset{\text{open}}{\subseteq} T$ $\therefore f^{-1}(B_T(f(p); \varepsilon))$ is open in $S$

$p \in f^{-1}(B_T(f(p); \varepsilon))$ $\therefore \exists \delta > 0$ s.t. $B_S(p; \delta) \subseteq f^{-1}(B_T(f(p), \varepsilon))$, $f(B_S(p; \delta)) \subseteq B_T(f(p); \varepsilon)$

($\Rightarrow$) $f^{-1}(V)$ is open in $S$ $V \subseteq T$ open $\Rightarrow f^{-1}(F) = f^{-1}(T - F^c) = \underbrace{S - f^{-1}(F^c)}_{\text{closed}}$ $F \subseteq T$ closed

($\Leftarrow$) $f^{-1}(F)$ closed in $S$, $F \subseteq T$ closed $f^{-1}(V) = f^{-1}(T - V^c) = \underbrace{S - f^{-1}(V^c)}_{\text{open}}$ $V \subseteq T$ open

Thm: $f:(S,d_S)\to(T,d_T)$ $X\subseteq S$ compact subset. If $f$ is conti on $X$ then $f(X)$ is compact in $T$

pf: Let $F=\{O_\alpha\}_{\alpha\in\Lambda}$ be an open covering of $f(X)$ ∴ $\{f^{-1}(O_\alpha)\}_{\alpha\in\Lambda}$ is an open covering of $X$

by compactness of $X$ ∴ $\exists f^{-1}(O_{\alpha_1})\cdots f^{-1}(O_{\alpha_p})$ s.t. $X\subseteq\bigcup_{j=1}^{p} f^{-1}(O_{\alpha_j})$ $f(X)\subseteq\bigcup_{j=1}^{p} f(f^{-1}(O_{\alpha_j}))\subseteq\bigcup_{j=1}^{p} O_{\alpha_j}$

$f:S\to\mathbb{R}^k$ $f$: bounded if $\|f(x)\|<M$ $\forall x\in S$ for some $M$

Thm $f:[a,b]\to\mathbb{R}$ conti then $f$ attains both max and min

pf: ∵ $f([a,b])$ compact in $\mathbb{R}$ ∴ $f([a,b])$ bounded and closed in $\mathbb{R}$ ∴ $\sup f[a,b]=M$, $\inf f[a,b]=m$ exsit ⇒ $M\in f([a,b])$, $m\in f([a,b])$

⇒ $M=f(p)$ for some $p\in[a,b]$ $m=f(q)$ for some $q\in[a,b]$

Thm: $f:(S,d_S)\to(T,d_T)$ $X\subseteq S$ compact If $f$ is 1-1 on $X$ and $f$ conti on $X$ then $f^{-1}:f(X)\to X$ is conti

pf: $(f^{-1})^{-1}=f$ $C\subseteq X$ closed subset. $C$ compact. $f(C)$ compact. $f(C)$ closed

Topological equivalence (Homeomorphism) $f:(S,d_S)\to(T,d_T)$ 1-1, conti If $f^{-1}:f(S)\to S$. $S$ and $f(S)$ are homeomorphic

isometry 保距 $d_S(x,y)=d_T(f(x),f(y))$ $\forall x,y\in S$

Lemma: $f:[a,b]\to\mathbb{R}$ conti. If $p\in[a,b]$ $f(p)\neq0$ then $f$ does not change sign near $p$

pf: If $f(p)>0$ given $\varepsilon=\frac{f(p)}{10}$ $\exists\delta>0$ s.t. $|f(x)-f(p)|<\varepsilon=\frac{f(p)}{10}$ if $|x-p|<\delta$, $-\frac{f(p)}{10}<f(x)-f(p)<\frac{f(p)}{10}$, $\frac{9f(p)}{10}<f(x)<\frac{11f(p)}{10}$

Thm (Balzano) $f:[a,b]\to\mathbb{R}$ conti If $f(a)f(b)<0$ then $\exists c\in(a,b)$ s.t. $f(c)=0$

pf: may assume $f(a)>0$ $f(b)<0$ consider $E=\{x\in[a,b]\mid f(x)\geq0\}\subseteq[a,b]$ $E\neq\emptyset$ ($a\in E$) ∴ $\sup E=c$ exsits $a<c<b$

claim $f(c)=0$ $a$ $c$ $b$ If $f(c)>0$ →← ∵ $\sup E=c$ If $f(c)<0$ →← ∵ $\sup E=c$

Thm (Intermediate value thm) $f:[a,b]\to\mathbb{R}$ conti $f(a)<f(b)$ then for any $\lambda$ $f(a)<f(\lambda)<f(b)$ $\exists a<c<b$ s.t. $\lambda=f(c)$

pf: $g(x)=\lambda-f(x)$ $g(a)\cdot g(b)<0$ $\exists c\in(a,b)$ s.t. $g(c)=\lambda-f(c)=0$ ⇒ $f(c)=\lambda$ by Balzano thm

$f:[a,b]\to\mathbb{R}$ conti ⇒ $f([a,b])$ is also a closed interval by IVT $f([a,b])=[\inf_{[a,b]} f,\ \sup_{[a,b]} f]$

Connected 連通. $E \subseteq \mathbb{R}^n$ is called disconnected if $\exists$ nonempty open subsets $A, B$ of $E$ s.t. $A \cup B = E$, $A \cap B = \phi$

E is called connected if E is not disconnected

Def: $\varphi$ is a two-valued function on E, if $\varphi: E \to \underbrace{\{0,1\}}_{\text{discrete metric}}$ is conti

Thm: $E \subseteq \mathbb{R}^n$, E is connected iff every two-valued function on E is a constant function

pf ($\Rightarrow$) E: connected, $\varphi: E \to \{0,1\}$ two valued function let $\varphi^{-1}(\{0\}) = A$, $\varphi^{-1}(\{1\}) = B$ is open in E $A \cup B = E$, $A \cap B = \phi$

either $A = \phi$ or $B = \phi$ $\therefore \varphi$ single valued

($\Leftarrow$) If E is disconnected $\therefore E = A \cup B$, $A \cap B = \phi$ $A, B$ open in E, $A \neq \phi$, $B \neq \phi$

Define $\varphi: \begin{matrix} E \to \{0,1\} \\ x \mapsto 0 & \text{if } x \in A \\ x \mapsto 1 & \text{if } x \in B \end{matrix}$ $\therefore \varphi$ is conti ($\to\leftarrow$) $\because \varphi$ is a constant function

Thm: $f: (S, d_S) \to (T, d_T)$ $X \subseteq S$. If $X$ is connected and f is conti on $X$ $\Rightarrow f(X)$ is connect (※)

pf: let $\varphi: f(X) \to \{0,1\}$ two-valued function $\therefore \varphi \circ f: X \to \{0,1\}$ conti $\therefore \varphi \circ f$ is const function $\Rightarrow \varphi$ is const function

Thm: $E \subseteq \mathbb{R}$, $f: E \to \mathbb{R}$. Assume E is connected and f conti on E. If f takes the values a and b then f takes any value $\lambda$ between a and b

Thm: $(S, d_S)$: metric space, $E_\alpha \subseteq S$ connected subset for each $\alpha \in \Lambda$ If $\bigcap_{\alpha \in \Lambda} E_\alpha \neq \phi \Rightarrow \bigcup_{\alpha \in \Lambda} E_\alpha$ is connected

pf: let $p \in \bigcap_{\alpha \in \Lambda} E_\alpha$, $\varphi: \bigcup_{\alpha \in \Lambda} E_\alpha \to \{0,1\}$ two-valued $\varphi|_{E_\alpha} \equiv \varphi(p)$ $p \in E_\alpha$

$(X, d_X)$, $p \in X$ let $\{E_\alpha\}$ be a collection of all connected subset that contains p $\therefore \bigcup E_\alpha$: connected

$\bigcup E_\alpha$ is called connected component of $X$ that contains p

$\{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty} = E$, what is the connected component of E that contains 0 ?

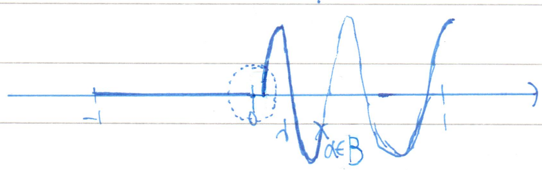
Arcwise connected. $E \subseteq \mathbb{R}^n$. E is arcwise connected if $a, b \in E$, then $\exists f: [0,1] \to E$ conti s.t. $f(0) = a$, $f(1) = b$

Thm: $E \subseteq \mathbb{R}^n$. E is arcwise connected $\Rightarrow$ E is connected

pf let $p \in E$ then $\forall x \in E$ have an arc called $A_x$, $A_x$ is connected $\bigcup_{x \in E} A_x = E$ and $p \in \bigcap_{x \in E} A_x$ $\Rightarrow$ E connected

Ex: $E = \{-1 \le x \le 0\} \cup \{\sin\frac{1}{x} \mid 0 < x \le 1\} \subseteq \mathbb{R}^2$ connected, not arcwise connected

Connected If $E = A \cup B$, $A, B$ open $A \cap B = \phi$ $0 \in E$ may assume $0 \in A$




$A|_\gamma \cup B|_\gamma = \gamma$ open in $\gamma$

$A|_\gamma \cap B|_\gamma = \phi$.

but $\gamma$ is arcwise connected $\Rightarrow \gamma$ is connected ($\to\leftarrow$)

If arcwise connected $\exists \varphi: [0,1] \to E$ $\varphi(0) = 0$ $\varphi(1) = \beta$ let $\varepsilon = \frac{1}{10}$ $\exists \delta > 0$ s.t. $0 \le t \le \delta \Rightarrow \|\varphi(t)\| < \varepsilon$ $\varphi(t) = \sin\frac{1}{t}$

Thm: $E \subseteq \mathbb{R}^n$ open If $E$ is connected then $E$ is arcwise connected

pf: Fix $x_0 \in E$. Let $A = \{x \in E \mid x \text{ can be arcwise connected to } x_0\}$ $B = \{x \in E \mid x \text{ cannot be arcwise connected to } x_0\}$

claim $A \cdot B$ open: If $p \in A$ can find an arc to $x_0$ $\because p \in E$ open $\therefore \exists B(p;r) \subseteq E$

then $\forall x \in B(p;r)$ we can use $\overline{x\text{-}p}$ to the arc then $B(p;r) \subseteq A$

If $q \in B$ cannot find an arc to $x_0$, $\forall x \in B(q;r)$ then $\overline{x\text{-}q}$ is the arc to $q$ $\therefore B(q;r) \subseteq B$

but $A \cap B = \emptyset$ $\because x_0 \in A$ $\therefore B = \emptyset$ $\Rightarrow$ $E$ is arcwise connected

Uniform Continuity: $f:(S, d_S) \to (T, d_T)$ $f$ is uniformly continuous on $S$, if $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon) > 0$ s.t. $d_T(f(x), f(y)) < \varepsilon$ if $d_S(x, y) < \delta$ 10/25

Ex: $f: (0,\infty) \to (0,\infty)$, $x \to \frac{1}{x}$ conti not uniform conti

$f: [0,1] \to \mathbb{R}$, $x \to x^2$ uniform conti $|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| \le 2|x-y|$ $\forall \varepsilon > 0$ $\exists \delta = \frac{\varepsilon}{2}$

Thm (Heine) $f: (T, d_T) \to (S, d_S)$ conti. Suppose $T$ is compact, then $f$ is uniformly conti on $T$ 10/28

pf: For $x \in T$. given $\varepsilon > 0$ $\exists \delta = \delta(\varepsilon, x) > 0$ s.t. $d_S(f(x), f(y)) < \varepsilon$ for $d_T(x, y) < \delta$
"$r_x$"

$T = \bigcup_{x \in T} B(x; \frac{r_x}{2})$ $\therefore T = \bigcup_{j=1}^{k} B(x_j; \frac{r_{x_j}}{2})$ choose $\delta_0 = \min\{\frac{r_{x_1}}{2} \cdots \frac{r_{x_k}}{2}\}$

If $\alpha, \beta \in T$ s.t. $d_T(\alpha, \beta) < \delta_0$ $\alpha \in T = \bigcup_{j=1}^{k} B(x_j; \frac{r_{x_j}}{2})$ $\therefore \alpha \in B(x_1; \frac{r_{x_1}}{2}) \subseteq B(x_1; r_{x_1})$

$d_T(\beta, x_1) \le d_T(\beta, \alpha) + d_T(\alpha, x_1) < \delta_0 + \frac{r_{x_1}}{2} \le \frac{r_{x_1}}{2} + \frac{r_{x_1}}{2} = r_{x_1}$, $\alpha, \beta \in B(x_1; r_{x_1})$

$d_S(f(\alpha), f(\beta)) \le d_S(f(\alpha), f(x_1)) + d_S(f(x_1), f(\beta)) < 2\varepsilon$

$f: (T, d_T) \to (T, d_T)$ $p \in T$ is called a fixed pt. of $f$ if $f(p) = p$

$f: (S, d_S) \to (T, d_T)$ is called a contraction map if $\exists\, 0 < c < 1$ s.t. $d_T(f(x), f(y)) \le c\, d_S(x, y)$ $\forall x, y \in S$

Thm: Let $(T, d_T)$ be a complete metric space and let $f: T \to T$ be a contraction map then $f$ has a unique fixed pt.

pf: Suppose $p, q$ are fixed pts, $d_T(p, q) = d_T(f(p), f(q)) \le c\, d_T(p, q)$, $0 < c < 1$ $(1-c)\, d_T(p, q) \le 0$ $\therefore p = q$

existence: pick $x_0 \in T$ let $x_1 = f(x_0)$ $x_2 = f(x_1) = f^2(x_0)$ $\cdots$ $x_k = f^k(x_0)$ $\Rightarrow \{x_k\}$

$d_T(x_{k+1}, x_k) = d_T(f(x_k), f(x_{k-1})) \le c\, d_T(x_k, x_{k-1}) \le c^k d_T(x_1, x_0)$

$d_T(x_{k+m}, x_k) \le d_T(x_{k+m}, x_{k+m-1}) + \cdots + d_T(x_{k+1}, x_k) \le (c^{k+m-1} + \cdots + c^k) d_T(x_1, x_0) = c^k(1 + \cdots + c^{m-1}) d_T(x_1, x_0) \le \frac{c^k}{1-c} d_T(x_1, x_0)$

$\therefore \{x_k\}$ is Cauchy $\therefore x_k \to p$ $\therefore f(p) = \lim_{k \to \infty} f(x_k) = \lim_{k \to \infty} x_{k+1} = p$

$f:(a,b)\to\mathbb{R}$ $p\in(a,b)$ If $\lim_{x\to p^+} f(x)$ exsits, then we say $f$ has right hand limit at $p$ denoted by $f(p^+)$

If $\lim_{x\to p^-} f(x)$ " , left " " $f(p^-)$

$f$ is conti at $p \Leftrightarrow f(p)=f(p^+)=f(p^-)$

$f$ is disconti at $p$ either ① $f(p^+)$ or $f(p^-)$ DNE ② $f(p^+)\neq f(p^-)$ ③ $f(p^+)=f(p^-)\neq f(p)$

②.③ disconti of the first kind . ① second kind ; ①.②: irremovable . ③ removable

first kind : left hand jump : $f(p)-f(p^-)$ . right hand jump . $f(p^+)-f(p)$ , jump $f(p^+)-f(p^-)$

$f:\mathbb{R}\to\mathbb{R}$ $x>y \Rightarrow f(x)\geq f(y)$ increasing $f(x)\leq f(y)$ decreasing
$>$ strictly increasing $<$ strictly decreasing } monotonic

Thm: $f:[a,b]\to\mathbb{R}$ increasing. For $c\in(a,b)$ then $f(c^+), f(c^-)$ exsits $f(c^-)\leq f(c)\leq f(c^+)$

pf: $E=\{f(x)\mid a<x<c\}$ , $f(x)\leq f(c)$ . let $\sup E = m$ claim $m=f(c^-)\leq f(c)$

given $\varepsilon>0$ $\exists x_1$ $a<x_1<c$ s.t. $m-\varepsilon<f(x_1)<m$ $m-\varepsilon<f(x_1)\leq f(x)\leq m$

$\exists\delta>0$ $\delta=c-x_1$ s.t. $c-\delta<x<c$ $c-\underbrace{(c-x_1)}_{x_1}<x<c$ $|m-f(x)|<\varepsilon$ $m=f(c^-)$

11/1 $\exists f:[0,1]\to\mathbb{R}$ $f$ increasing s.t. $f$ has a discontinuity at every rational pts.

chose a convergent series $\sum_{k=1}^{\infty} c_k<\infty$ $0<c_k<\infty$ $\forall k$ write $\mathbb{Q}\cap[0,1]=\{x_1,x_2,\cdots\}$ . Define $f(x)=\sum_{x_k\leq x} c_k$
$x_k \leftrightarrow c_k$

$f(x_0^-)=\sum_{x_k<x_0} c_k \neq f(x_0^+)=\sum_{x_k<x_0} c_k + \underset{x_k=x_0}{c_k}$ $\Rightarrow$ discontinuity is countable $\because$ $f: \begin{matrix} D\to\mathbb{Q} \\ \cancel{f(q)}\mapsto q\end{matrix}$ injective

Thm: $S\subseteq\mathbb{R}$ $f:S\to\mathbb{R}$ strickly increasing $\Rightarrow f^{-1}:f(S)\to S$ strickly increasing

pf: $x,y\in S$ $x<y$ $f(x)<f(y)$ $\therefore f$ is 1-1 , if $y_1,y_2\in f(S)$ $y_1<y_2$ $f(x_1)=y_1$ $f(x_2)=y_2$ $\therefore x_1<x_2$

Cor: $f:[a,b]\to\mathbb{R}$ conti, strickly increasing $\Rightarrow f^{-1}:[f(a),f(b)]\to[a,b]$ conti, strickly increasing

## $\boxed{\text{Derivative 微分}}$

$f:(a,b)\to\mathbb{R}$ $p\in(a,b)$ If $\lim\limits_{x\to p}\frac{f(x)-f(p)}{x-p}$ exsits we say $f$ is differentiable at $p$

the limit is called the derivation of $f$ at $p$ denoted by $f'(p)$

$f:(a,b)\to\mathbb{R}$ $p\in(a,b)$ If $f$ is diff at $p$, then for small $h$ $f(p+h)=f(p)+f'(p)h+hE(h)$ $\lim\limits_{h\to0}E(h)=0$

Thm: $f:(a,b)\to\mathbb{R}$ $p\in(a,b)$. If $f$ is diff at $p$, then $f(x)-f(p)=(x-p)f^*(p)$

where $f^*(x)$ is a function, which is conti at $p$ and $f^*(p)=f'(p)$

pf: define $f^*(x)=\begin{cases}\frac{f(x)-f(p)}{x-p} & x\neq p\\ f'(p) & x=p\end{cases}$, $f^*(x)$ is conti at $p$ $\because$ $\lim\limits_{x\to p}f^*(x)=\lim\limits_{x\to p}\frac{f(x)-f(p)}{x-p}=f'(p)=f^*(p)$

Thm: $f:(a,b)\to\mathbb{R}$ $p\in(a,b)$ If $f$ is diff at $p$, then $f$ is conti at $p$

pf: $\because$ $f$ is diff at $p$ $\therefore$ $f(x)-f(p)=(x-p)f^*(p)$ $\lim\limits_{x\to p}f(x)=f(p)$

Thm: $f,g:(a,b)\to\mathbb{R}$ $p\in(a,b)$ Assume $f,g$ are diff at $p$ $\Rightarrow$ $f\pm g$, $cf$, $fg$, $\frac{f}{g}$ $(g(p)\neq0)$ are diff at $p$

$(f\pm g)'(p)=f'(p)\pm g'(p)$ $(cf)'(p)=c\cdot f'(p)$ $(fg)'(p)=f'(p)g(p)+f(p)g'(p)$ $\left(\frac{f}{g}\right)'(p)=\frac{f'(p)g(p)-f(p)g'(p)}{g^2(p)}$

pf: $f(x)-f(p)=(x-p)f^*(x)$ $g(x)-g(p)=(x-p)g^*(x)$

$f(x)\cdot g(x)=(f(p)+(x-p)f^*(x))(g(p)+(x-p)g^*(x))=f(p)g(p)+(x-p)(f^*(x)g(p)+f(p)g^*(x)+(x-p)f^*(x)g^*(x))$

$f(x)\cdot g(x)-f(p)g(p)=(x-p)[f^*(x)g(p)+f(p)g^*(x)+(x-p)f^*(x)g^*(x))$ $x\to p$ $(fg)'(p)=f'(p)g(p)+f(p)g'(p)$

$\frac{f(x)}{g(x)}-\frac{f(p)}{g(p)}=\frac{f(x)g(p)-g(x)f(p)}{g(x)\cdot g(p)}=\frac{(f(x)-f(p))g(p)-f(p)(g(x)-g(p))}{g(x)\cdot g(p)}=\frac{(x-p)f^*(x)g(p)-(x-p)f(p)g^*(x)}{g(x)\cdot g(p)}=(x-p)\left(\frac{f^*(x)g(p)-f(p)g^*(p)}{g(x)\cdot g(p)}\right)$

Thm: $\underset{p}{(a,b)}\xrightarrow{f}\underset{f(p)\text{ is interior pt}}{(f(a,b))}\xrightarrow{g}\mathbb{R}$. Assume $f$ is diff at $p$, $g$ is diff at $f(p)$ $\Rightarrow$ $g\circ f$ is diff at $p$ $(g\circ f)'(p)=f'(p)g'(f(p))$

pf: $f(x)=f(p)+(x-p)f^*(x)$ $g(y)=g(f(p))+(y-f(p))g^*(y)$

$g(f(x))-g(f(p))=(f(x)-f(p))g^*(f(x))=(x-p)f^*(x)g^*(f(x))$ $\lim\limits_{x\to p}f^*(x)g^*(f(x))=f^*(p)\cdot g^*(f(p))=f'(p)g'(f(p))$

$f:(a,b)\to\mathbb{R}$ $p\in(a,b)$ $f$ is conti at $p$. $f$ has a righthand derivative if $\lim\limits_{x\to p^+}\frac{f(x)-f(p)}{x-p}$ exsits as a finite value or the limit is $\pm\infty$

Ex: $f(x)=\sqrt{x}$ $\lim\limits_{x\to0^+}\frac{1}{2\sqrt{x}}=\infty$

Thm: $f:(a,b)\to\mathbb{R}$ $p\in(a,b)$ If $f$ has a derivative at $p$, positive or $\infty$, then locally $f(x)>f(p)$ if $x>p$ $f(x)<f(p)$ if $x<p$

pf: If $f'(p)$ finite then $f(x)-f(p)=(x-p)f^*(p)$ $f^*(p)-f'(p)>0$

If $f(p)=\infty$. $\lim\limits_{x\to p}\frac{f(x)-f(p)}{x-p}=\infty$ $x\neq p$ $|x-p|<\delta$ $\frac{f(x)-f(p)}{x-p}>N$ $\Rightarrow$ $f(x)-f(p)\geq N(x-p)$ same sign

$f:(S,d_S)\to\mathbb{R}$ $a\in S$ If $f(a)\leq f(x)$ for $x\in S\cap B(a;\delta)$ some $\delta$ we say $f(a)$ is a local minimum at $a$

" $\geq$ " maximum "

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Thm: $(a,b)\to\mathbb{R}$ $p\in(a,b)$ Assume $f$ has a local minimum or maximum at $p$. If $f$ has a derivative at $p$ then $f'(p)=0$

pf: by last thm if $f'(p)\neq 0$ $f$ has no local min or max at $p$

1/4 Rolle's Thm: $f:[a,b]\to\mathbb{R}$ conti. $f$ has a derivative at every point in $(a,b)$. If $f(a)=f(b)$ then $\exists\, c\in(a,b)$ s.t. $f'(c)=0$

Ex: 

pf let $M=\max_{[a,b]} f$ $m=\min_{[a,b]} f$ If $\exists\, c\in(a,b)$ s.t. $f(c)=M$ or $m$ then $f'(c)=0$

otherwise max and min are attained at $a,b$ $\therefore M=m$ $\forall c\in(a,b)$ $f'(c)=0$

Mean Value Thm: $f[a,b]\to\mathbb{R}$ conti. $f$ has a derivative at every interior pt then $\exists\, c\in(a,b)$ s.t. $f(b)-f(a)=f'(c)(b-a)$

Ex: $f:[a,b]\to\mathbb{R}^2$, $t\to(\cos t,\sin t)$ $f(0)=(1,0)$ $f(2\pi)=(1,0)$ $f'(t)=(-\sin t,\cos t)\neq 0$

Generize MVT: $f,g[a,b]\to\mathbb{R}$ conti. $f,g$ has a derivative at every $c\in(a,b)$, then $\exists\, c\in(a,b)$ s.t. $(g(b)-g(a))f'(c)=g'(c)(f(b)-f(a))$

their derivative are not infinite simultaneously

pf: consider $h(x)=(f(b)-f(a))g(x)-f(x)(g(b)-g(a))$ $h(a)=f(b)g(a)-f(a)g(b)$ $h(b)=g(a)f(b)-f(a)g(b)$ $\Rightarrow h(a)=h(b)$

by Rolle's thm $\therefore \exists\, c\in(a,b)$ s.t. $h'(c)=0$ $(f(b)-f(a))g'(c)-f'(c)(g(b)-g(a))=0$

$f,g$ conti on $(a,b)$ and $f(a^+),f(b^-),g(a^+),g(b^-)$ exsit consider $\tilde{f}=\begin{cases} f(a^+) & x=a\\ f(b^-) & x=b\\ f(x) & a<x<b\end{cases}$ $\tilde{g}=\begin{cases} g(a^+) & x=a\\ g(b^-) & x=b\\ g(x) & a<x<b\end{cases}$ $\Rightarrow$ can use generize MVT

Cor: $f:[a,b]\to\mathbb{R}$, $f$ has a derivative at every $c\in(a,b)$. ① If $f'(c)>0$ or $\infty$ $\forall c\in(a,b)$ then $f$ is strickly increasing on $[a,b]$

② If $f'(c)<0$ or $-\infty$ $\forall c\in(a,b)$ then $f$ is strickly decreasing on $[a,b]$

③ If $f'(c)=0$ $\forall c\in(a,b)$ then $f$ is a const func on $[a,b]$

IVT $f:[a,b]\to\mathbb{R}$ conti $\lambda\in\mathbb{R}$ $f(a)<\lambda<f(b)\Rightarrow\exists\, c\in(a,b)$ s.t. $f(c)=\lambda$

IVT for derivative. $f:[a,b]\to\mathbb{R}$ conti. $f$ has a derivative at every $c\in(a,b)$ and assume $f'_+(a)$, $f'_-(b)$ exsit and are finite

then for any $\lambda$, $f'_+(a)<\lambda<f'_-(b)$, $\exists\, c\in(a,b)$ s.t. $f'(c)=\lambda$

pf: consider $g(x)=\begin{cases}\frac{f(x)-f(a)}{x-a} & x\neq a\\ f'_+(a) & x=a\end{cases}$ on $[a,b]$ conti $g(a)=f'_+(a)$ $g(b)=\frac{f(b)-f(a)}{b-a}$

$\therefore f'_+(a)<\lambda<\frac{f(b)-f(a)}{b-a}$ $\therefore \exists\, c_1\in(a,b)$ s.t. $h(c_1)=\lambda$ $\lambda=\frac{f(c_1)-f(a)}{c_1-a}=f'(c)$ $a<c<c_1$

consider $h(x)=\begin{cases}\frac{f(b)-f(x)}{b-x} & x\neq b\\ f'_-(b) & x=b\end{cases}$ on $[a,b]$ conti $\frac{f(b)-f(a)}{b-a}<\lambda<f'_-(b)$

$\therefore \exists\, c_2\in(a,b)$ s.t. $h(c_2)=\lambda$ $\lambda=\frac{f(b)-f(c_2)}{b-c_2}=f'(c)$ $c_2<c<b$

$f'_-(b)=\infty$, $f'_+(a)<\infty$ $f'_+(a)<c<\infty=f'_-(b)$ consider $g(x)=f(x)-cx$ $f'_+(b)=\infty=\lim_{x\to b^-}\frac{f(x)-f(b)}{x-b}$

choose $x_1$ closed to $b$ s.t. $\frac{f(x_1)-f(b)}{x_1-b}>c$ $f'(x_2)<\infty$

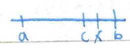
Thm: $f:[a,b]\to\mathbb{R}$ conti $f$ has a derivative at every $c\in(a,b)$. If $f'(c)\neq 0$ $\forall c\in(a,b)$ then $f$ is monotonic

Thm: $f(a,b)\to\mathbb{R}$ s.t. $f$ has a derivative on $(a,b)$ and $f'$ is monotonic then $f'$ is conti

Generize version: $f, g: [a,b] \to \mathbb{R}$ have finite n-th derivative on $(a,b)$. Assume $f^{(n-1)}, g^{(n-1)}$ are conti on $[a,b]$

let $c \in [a,b]$ and $x \in [a,b]$ $x \neq c$, then $\exists\, x_1$ between $x$ and $c$ s.t.

$$\left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k\right) g^{(n)}(x_1) = f^{(n)}(x_1)\left(g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!}\cdot(x-c)^k\right)$$

pf:  $t \in [c,x]$ $\quad F(t) = f(t) + \sum_{k=1}^{n-1} \frac{f^{(k)}(t)}{k!}(x-t)^k \qquad G(t) = g(t) + \sum_{k=1}^{n-1} \frac{g^{(k)}(t)}{k!}(x-t)^k$

$(F(x) - F(c))\, G'(x_1) = F'(x_1)\,(G(x) - G(c)) \qquad F^{(n)}(t) = \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} \qquad G^{(n)}(t) = \frac{g^{(n)}(t)}{(n-1)!}(x-t)^{n-1}$

$$\left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k\right) g^{(n)}(x_1) = f^{(n)}(x_1)\left(g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!}(x-c)^k\right)$$

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Vector value function $f: \underset{(a,b)}{\mathbb{R}} \to \mathbb{R}^n$ $n \geq 2$ $x \mapsto (f_1(x), \cdots, f_n(x))$, $(f \pm g)'(p) = f'(p) \pm g'(p)$, $(cf)'(p) = c\cdot f'(p)$

$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$, $\underset{t \to x}{\mathbb{R} \xrightarrow{u} \mathbb{R}} \xrightarrow{f} \mathbb{R}^n$ $x = u(t)$, $f(x) = f(u(t))$ $\quad f'(t) = f'(u(t))\cdot u'(t)$

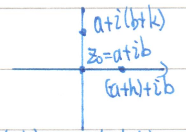
$\mathbb{C}$: complex plane, complex structure $J: \mathbb{R}^2 \to \mathbb{R}^2$ $\underbrace{\mathbb{R}\text{-linear, 1-1, onto}}_{\text{endomorphism}}$, $J \circ J = J^2 = -id$ $\quad \mathbb{C}$: field

$\mathbb{C} \simeq \mathbb{R}^2$ homeomorphism $\quad \mathbb{C} \simeq (\mathbb{R}^2, J)$

$f: \Omega \subseteq \mathbb{C} \to \mathbb{C}$ $\quad \Omega$: connected open $\quad \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$. If the limit exsits, we say $f$ is diff at $z_0$

If $f$ is diff at every point in $\Omega$, we say $f$ is holomorphic in $\Omega$, let $H(\Omega) = \{$all holomorphic func in $\Omega\}$

$f(z) - f(c) = (z-c) f^*(z)$ $\quad f^*$ conti at $c$ and $f^*(c) = f'(c)$ $\quad$ diff $\to$ conti




Assume $f$ is diff at $z_0 = a+ib$. $f(z) = u(z) + iv(z)$. $u, v$ real-valued

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{u(a+h,b) + iv(a+h,b) - (u(a,b) + iv(a,b))}{h} = \lim_{h \to 0} \frac{u(a+h,b) - u(a,b)}{h} + i\,\frac{v(a+h,b) - v(a+h,b)}{h} = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b)$$

$$f'(z_0) = \lim_{k \to 0} \frac{u(a,b+k) + iv(a,b+k) - (u(a,b) + iv(a,b))}{ik} = \frac{1}{i}\left(\frac{\partial u}{\partial y}(a,b) + i\frac{\partial v}{\partial y}(a,b)\right) = \frac{\partial v}{\partial y}(a,b) - i\frac{\partial u}{\partial y}(a,b) \quad \text{if diff}$$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$. Cauchy-Riemann equation

Def: $\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ Cauchy-Riemann operator. $f(z) = u(z) + iv(z) \in C^1$

$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \text{ iff } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Ex: $f(x,y) = u(x,y) + iv(x,y)$ $\quad u(x,y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ $\quad v(x,y) = \begin{cases} \frac{x^3 + y^3}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$\frac{\partial u}{\partial x}(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1 \qquad \frac{\partial u}{\partial y}(0,0) = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \to 0} \frac{-k}{k} = -1$

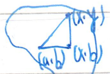
$\frac{\partial v}{\partial x}(0,0) = \lim_{h \to 0} \frac{v(h,0) - v(0,0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1 \qquad \frac{\partial v}{\partial y}(0,0) = \lim_{k \to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k \to 0} \frac{k}{k} = 1$

at $(0,0)$ Cauchy-Remann hold

$\lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{x + ix - 0}{x} = 1 + i \qquad \lim_{x = y \to 0} \frac{f(x,x) - f(0,0)}{x + ix} = \lim_{x \to 0} \frac{2ix}{x + ix} = \frac{2i}{1+i} = 1+i$ ... $\frac{i}{1+i} = \frac{1+i}{2}$ $\Rightarrow$ but cannot diff at $(0,0)$

Thm: $f: R \subseteq \mathbb{C} \to \mathbb{C}$ $f = u + iv$ $f \in H(\Omega)$, If $u \cdot v$ or $|f|$ is a const. then $f$ is a const func

pf: $u \equiv$ const. $u_x = v_y = 0$ $u_y = -v_x = 0$




$v(x,y) - v(a,b) = v(x,y) - v(x,b) + v(x,b) - v(a,b) = v_y(x,\bar{y})(y-b) + v_x(\bar{x},y)(x-a) = 0$

$|f| \equiv$ const. $u^2 + v^2 = |f|^2 =$ const $0 = 2uu_x + 2vv_x$ $0 = 2uu_y + 2vv_y$ $0 = uu_x - vu_y$ ① $0 = uu_y + vu_x$ ②

① $\times u$ − ② $\times v \Rightarrow u^2 u_x + v^2 u_x = 0$ $(u^2+v^2)u_x = 0$ $C \cdot u_x = 0$

(1) $C = 0 \Rightarrow u = 0$ $v = 0 \Rightarrow f = u + iv = 0$ (2) $C > 0 \Rightarrow u_x = 0$, $u_y = 0$ $u \equiv$ const $v \equiv$ const

## Bounded Variation 有限變量

Thm: $[a,b] \to \mathbb{R}$ increasing $a = x_0 < x_1 < \cdots < x_n = b$ , $f(x_j^+)$ , $f(x_j^-)$ $1 \le j \le n-1$ exsits

then $\sum_{j=1}^{n-1} \left( f(x_j^+) - f(x_j^-) \right) \le f(b) - f(a)$

pf: insert $\{y_n\}$ s.t. $y_j < x_j < y_{j+1}$ $f(y_{j+1}) \ge f(x_j^+)$ $f(y_j) \le f(x_j^-)$

$\sum_{j=1}^{n-1} \left( f(x_j^+) - f(x_j^-) \right) \le \sum_{j=1}^{n-1} \left( f(y_{j+1}) - f(y_j) \right) = f(y_n) - f(y_1) \le f(b) - f(a)$

Discontinuity $= \bigcup_{m=1}^{\infty} \{ \text{jump} \ge \frac{1}{m} \}$ jump $\ge \frac{1}{m}$ $\frac{k}{m} \le f(b) - f(a)$ $k \in \mathbb{N}$ $\Rightarrow$ discontinuity countable

$f: [a,b] \to \mathbb{R}$ partition $P$ of $[a,b]$ $P = \{ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \}$ $[x_{k-1}, x_k]$ $\Delta x_k = x_k - x_{k-1}$ $\|P\| = \max_k \{x_k\}$ 11/11

$\mathcal{P}[a,b] = \{$all partition of $[a,b]\}$ . The variation of $f$ wrt $P$ is $\Sigma(P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} |\Delta f_k|$

Def: $f: [a,b] \to \mathbb{R}$ : $f$ is said to be of bounded variation on $[a,b]$ if $\exists M > 0$ s.t. $\Sigma(P) \le M$ $\forall P \in \mathcal{P}[a,b]$

Thm: $f: [a,b] \to \mathbb{R}$ monotonic then $f$ is of B.V. and $V_f(a,b) \le f(b) - f(a)$ if $f$ is increasing

pf: $P = \{ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \}$ $\Sigma(P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = f(b) - f(a)$

Thm: $f: [a,b] \to \mathbb{R}$ If $f \in C^1([a,b])$, then $f$ is of B.V. on $[a,b]$ (key point $f'(c_k)$ bounded)

pf: $P = \{ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \}$ $\Sigma(P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} |f'(c_k)(x_k - x_{k-1})|$ $c_k \in (x_{k-1}, x_k)$ $\le C \sum_{k=1}^{n} (x_k - x_{k-1}) = C(b-a)$

Cor: If $f$ is of BV on $[a,b]$ then $f$ is bounded

pf: $P = \{ a < x < b \}$ $|f(a) - f(x)| + |f(x) - f(b)| \le M$ $|f(x)| = |f(x) - f(a) + f(a)| \le |f(x) - f(a)| + |f(a)| \le |f(a)| + M$

Ex: $g(x) = \begin{cases} x \cos \frac{\pi}{2x} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$ $\cos \frac{\pi}{2x} = \pm 1 \Leftrightarrow \frac{\pi}{2x} = k\pi \Leftrightarrow x = \frac{1}{2k}$ $\ge \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \cdots = \frac{1}{2} + 2\left( \frac{1}{2} + \frac{1}{3} + \cdots \right) = \infty$

Ex: $h(x) = \begin{cases} x^2 \cos \frac{\pi}{2x} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$ $h'(0) = \lim_{k \to 0} \frac{f(k) - f(0)}{k} = \lim_{k \to 0} k \cos \frac{\pi}{2k} = 0$ $x > 0$ $h'(x) = 2x \cos \frac{\pi}{2x} + x^2 \left( -\sin \frac{\pi}{2x} \right) \left( -\frac{\pi}{2x^2} \right) = 2x \cos \frac{\pi}{2x} + \frac{\pi}{2} \sin \frac{\pi}{2x}$

$|h'(x)| \le 2 + \frac{\pi}{2}$ $\therefore$ $h$ is of B.V. on $[0,1]$

Ex: $f(x) = x^{\frac{1}{3}}$ , $0 \le x \le 1$ $x > 0$ $f'(x) = \frac{1}{3} x^{-\frac{2}{3}} \to \infty$ as $x \to 0^+$ . B.V.

$f: [a,b] \to \mathbb{R}$ is Lipschitz if $|f(x) - f(y)| \le M|x - y|$ $\forall x, y \in [a,b]$ $\Sigma(P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \le M \sum_{k=1}^{n} (x_k - x_{k-1}) = M(b-a)$ BV

$f \in C^1[a,b] \Rightarrow f$ is Lipschitz

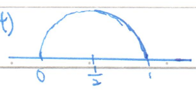
$f: [a,b] \to \mathbb{R}$ $f$ is Lipschitz of order $\alpha$ $0 < \alpha < 1$ if $|f(x) - f(y)| \le M|x - y|^\alpha$ $\forall x, y \in [a,b]$

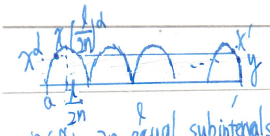
$f(x) = x^\alpha$ $0 < \alpha < 1$ on $[0,1]$ $|f(x) - f(0)| = x^\alpha = |x - 0|^\alpha$ $0 < x < y \le 1$ claim $|f(y) - f(x)| \le |y - x|^\alpha$ $y^\alpha - x^\alpha \le (y - x)^\alpha$

$1 - \left( \frac{x}{y} \right)^\alpha \le \left( 1 - \frac{x}{y} \right)^\alpha$ let $t = \frac{x}{y}$ $0 < t < 1$ $1 - t^\alpha \le (1 - t)^\alpha$ let $h(t) = (1-t)^\alpha - (1 - t^\alpha)$ show $h(t) \ge 0$

$h(0) = 1 - 1 = 0$ $h(1) = 0 - 0 = 0$ $0 < t < 1$ $h'(t) = -\alpha(1-t)^{\alpha-1} + \alpha t^{\alpha-1} = \alpha \left( \frac{1}{t^{1-\alpha}} - \frac{1}{(1-t)^{1-\alpha}} \right)$ $h'(t) = 0$ iff $t = \frac{1}{2}$

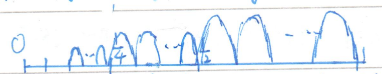
$h'(t)$ $\infty \searrow 0$ $0 < t < \frac{1}{2}$ $h'(t)$ $0 \searrow -\infty$ $\frac{1}{2} < t < 1$ $h(t)$





$|f(x)-f(y)| \le |x'-y|^{\alpha} \le |x-y|^{\alpha}$ $\quad \Sigma(p) \ge \left(\frac{\ell}{2n}\right)^{\alpha}(2n) = \ell^{\alpha}(2n)^{1-\alpha}$ $\quad 0<\alpha<1$

$n \in \mathbb{N}$ $2n$ equal subintervals



$\ge 1$

Thm: $f, g: [a,b] \to \mathbb{R}$ and of BV then $f \pm g$, $cf$, $f \cdot g$ are all of BV on $[a,b]$

pf: $\Sigma(p) = \sum_{k=1}^{n} |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})| = \sum_{k=1}^{n} |f(x_k)g(x_k) - f(x_{k-1})g(x_k) + f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})|$

$< \sum_{k=1}^{n} |g(x_k)||f(x_k)-f(x_{k-1})| + \sum_{k=1}^{n} |f(x_{k-1})||g(x_k)-g(x_{k-1})| \le \sum_{k=1}^{n} M_g |f(x_k)-f(x_{k-1})| + M_f |g(x_k)-g(x_{k-1})|$

$\le M_g \sum_{k=1}^{n} |f(x_k)-f(x_{k-1})| + M_f \sum_{k=1}^{n} |f(x_k)-f(x_{k-1})| = M_g \Sigma_f(p) + M_f \Sigma_g(p) \le C$

Ex: $g(x) = \begin{cases} x & 0<x\le 1 \\ 1 & x=0 \end{cases}$ $\quad \frac{1}{g(x)} = \begin{cases} \frac{1}{x} & 0<x\le 1 \\ 1 & x=0 \end{cases}$

Thm $f: [a,b] \to \mathbb{R}$ BV and $|f(x)| \ge m > 0$ $\forall x \in [a,b] \Rightarrow \frac{1}{f}$ is of BV

pf: $\Sigma(p) = \sum_{k=1}^{n} \left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right| = \sum_{k=1}^{n} \frac{|f(x_k)-f(x_{k-1})|}{|f(x_k) \cdot f(x_{k-1})|} \le \frac{1}{m^2} \sum_{k=1}^{n} |f(x_k)-f(x_{k-1})| \le \frac{1}{m^2} V_f(a,b)$

11/15 Thm: $f$ is of BV on $[a,b]$, $a<c<b \Rightarrow f$ is of BV on $[a,c]$, $[c,b]$ and $V_f(a,b) = V_f(a,c) + V_f(c,b)$

pf: $P = \{a = x_0 < \cdots < x_n = b\}$ let $P' = P \cup \{c\}$ $\because |f(x_k)-f(x_{k-1})| \le |f(x_k)-f(c)| + |f(c)-f(x_{k-1})|$ $\therefore \Sigma(p) \le \Sigma(p')$

$P' \cap [a,c] = P_1'$ $\quad P' \cap [c,b] = P_2'$ $\quad \Sigma(p) \le \Sigma(p') = \Sigma(P_1') + \Sigma(P_2') \le V_f(a,c) + V_f(c,b)$

let $P_1 \in \mathcal{P}[a,c]$ $P_2 \in \mathcal{P}[c,b]$ $\quad P = P_1 \cup P_2 \in \mathcal{P}[a,b]$ $\quad \Sigma(p_1) + \Sigma(p_2) = \Sigma(p) \le V_f(a,b)$

$V_f(a,c) + \Sigma(p_2) \le V_f(a,b)$ $\quad V_f(a,c) + V_f(c,b) \le V_f(a,b)$

Thm: $f$ is of BV on $[a,b]$ let $V(a) = V_f(a) = 0$ $\quad V(x) = V_f(x) = V_f(a,x)$, $a<x\le b$ then ① $V(x)$ is increasing ② $V(x)-f(x)$ is increasing

pf: $a<x<y\le b$ $\quad V(y)-V(x) = V_f(a,y) - V_f(a,x) = V_f(x,y) \ge 0$

$V(y)-f(y) - (V(x)-f(x)) = V(y)-V(x) - (f(y)-f(x)) = V_f(x,y) - (f(y)-f(x)) \ge 0$

Thm: $f$ is of BV on $[a,b] \Leftrightarrow f$ can be written as the difference of two increasing (strickly increasing) function

pf: $f(x) = V(x) - (V(x)-f(x))$ $\quad , \quad f(x) = (V(x)+g(x)) - (V(x) - f(x) + g(x))$

Thm: $f$ is of BV on $[a,b]$ then $f$ is conti at $c \in [a,b] \Leftrightarrow V_f$ is conti at $c$

pf: $a<c<x<b$ $\quad 0 \le |f(x)-f(c)| \le V(x)-V(c)$. If $f$ is conti at $c$ as $x \to c^+$ $\therefore f(c+) = f(c)$ $\quad x \to c^-$ $f(c-) = f(c)$

($\Rightarrow$) $f$ is conti at $c$, given $\varepsilon>0$ $\exists \delta>0$ s.t. $|f(x)-f(c)| < \varepsilon$ if $|x-c|<\delta$ $\quad \exists P_2 \in \mathcal{P}[a,b]$ s.t. $V(c,b) \le \Sigma(p_2) + \varepsilon$

$P_2 = \{c = x_0 < x_1 < \cdots < x_n = b\}$ $\quad |x_0 - x'| < \delta$ may assume $|x_0 - x_1| < \delta$

$V(c,b) \le \Sigma(p_2) + \varepsilon = |f(x_1)-f(x_0)| + \cdots + |f(x_{n-1})-f(x_n)| + \varepsilon < 2\varepsilon + \sum_{k=1}^{n} |f(x_k)-f(x_{k-1})| \le 2\varepsilon + V_f(x_1,b)$

$V(x_1) - V(c) = V_f(a,x_1) - V_f(a,c) = V_f(c,x_1) = V_f(c,b) - V_f(x_1,b) < 2\varepsilon$ $\therefore V(c+) = V(c)$ $\quad$ same reason $V(c-) = V(c)$

Curve $f:[a,b]\to\mathbb{R}^n$ conti  $f$: path  image: gragh

Def: $f(x)=(f_1(x),\cdots,f_m(x)):[a,b]\to\mathbb{R}^m$ conti  $P=\{a=x_0<x_1<\cdots<x_{n-1}<x_n=b\}\in\mathcal{P}[a,b]$

let $\Lambda(p)=\sum_{k=1}^{n}\|f(x_k)-f(x_{k-1})\| = \sum_{k=1}^{n}\sqrt{(f_1(x_k)-f_1(x_{k-1}))^2+\cdots+(f_m(x_k)-f_m(x_{k-1}))^2}$.

Define the length of this curve $C$ by $\Lambda_f(a,b)=\sup_{p}\Lambda(p)$.

then $C$ is said to be rectifiable if $\Lambda_f(a,b)<\infty$. otherwise, $C$ is said to be nonrectifiable

Ex: $f(x)=\begin{cases} x\cos\frac{\pi}{2x}, & 0<x\le 1 \\ 0, & x=0 \end{cases}$  nonvectitiable

Thm: If $f\in C'[a,b]$ then $L(a,b)=\int_a^b\|f'(x)\|\,dx$  (proved in Calculus)

If $f_1\cdots f_m$ of BV  $\sum_{k=1}^{n}|f_j(x_k)-f_j(x_{k-1})|\le\Lambda(p)\le\sum_{k=1}^{n}(|f_1(x_k)-f_1(x_{k-1})|+\cdots+|f_m(x_k)-f_m(x_{k-1})|)\Rightarrow V_{f_j}(a,b)\le\Lambda_j(a,b)\le(V_{f_1}(a,b)+\cdots+V_{f_m}(a,b))$

Thm: $f:[a,b]\to\mathbb{R}^n$ conti rectitiable $a<c<b$ then $\Lambda_f(a,b)=\Lambda_f(a,c)+\Lambda_f(c,b)$

pf: $P_1\in\mathcal{P}[a,c]$  $P_2\in\mathcal{P}[c,b]$  $P=P_1\cup P_2\in\mathcal{P}[a,b]$  $\Lambda(p_1)+\Lambda(p_2)=\Lambda(p)\le\Lambda_f(a,b)$  $\Lambda_f(a,c)+\Lambda_f(c,b)\le\Lambda_f(a,b)$

$P\in\mathcal{P}[a,b]$  $P'=P\cup\{c\}$  $P_1'=P'\cap[a,c]$  $P_2'=P'\cap[c,b]$  $\Lambda(p)\le\Lambda(p')=\Lambda(p_1')+\Lambda(p_2')\le\Lambda_f(a,c)+\Lambda_f(c,b)$. $\Lambda_f(a,b)\le\Lambda_f(a,c)+\Lambda_f(c,b)$

Thm: $f:[a,b]\to\mathbb{R}$ rectitiable let $s(a)=0$  $s(x)=\Lambda_f(a,x)$  $a<x\le b$. then ① $s(x)$ is increasing ② $s(x)$ is conti

pf: $a<x<y\le b$  $s(y)-s(x)=\Lambda_f(a,y)-\Lambda_f(a,x)=\Lambda_f(x,y)\ge 0$  "$=0$" iff $\Lambda_f(x,y)=0$ $\therefore f_1$: const $\cdots f_m$: const

$0\le s(y)-s(x)=\Lambda_f(x,y)\le\sum_{j=1}^{k}V_{f_j}(x,y)$  $y\to x$  $s(y)=s(x)$ 11/18

$f:[a,b]\to\mathbb{R}^k$ curve  $u(t)=[c,d]\to[a,b]$ conti, strickly monotonic, onto $g(t)=f(u(t)):[c,d]\to\mathbb{R}^k$, $f,g$ equivalent

$^{d}_{c}I \rightrightarrows I^{b}_{a}$ orientiion preserving  $^{d}_{c}I \times I^{b}_{a}$ oriention reversing

Thm: $\begin{matrix} f:[a,b]\to\mathbb{R}^k \\ g:[c,d]\to\mathbb{R}^k \end{matrix}$ curves both $f,g$ are 1-1 then $f,g$ are equivalent iff $f,g$ have the same gragh

pf: ($\Rightarrow$) OK ($\Leftarrow$) $g$: closed mapping, $g^{-1}$: conti  $u=g^{-1}\circ f:[a,b]\to[c,d]$ conti, onto, strictly monotonic

$g\circ u=g\circ g^{-1}\circ f=f$  by def $f,g$ equivalent

## Riemann-Stieltjes Integral 黎曼-斯蒂爾杰斯積分

Riemann integral: $\int_a^b f(x)\,dx$ . Riemann-Stieltjes integral $\int_a^b f(x)\cdot d\,g(x)$

Assume $f\cdot g\cdot \alpha\cdot \beta:[a,b]\to\mathbb{R}$ bounded, $\mathcal{P}[a,b]$ $P=\{a=x_0<x_1,\dots<x_n=b\}$ $\|P\|=\sup\{\Delta x_k=x_k-x_{k-1}\}$ norm

$P_1\subseteq P_2$ $P_2$ is finer than $P_1$.

$f\cdot\alpha:[a,b]\to\mathbb{R}$ bounded. Riemann-Stieltjes sum of $f$ wrt $\alpha$ on $[a,b]$ $S(P,f,\alpha)=\sum_{k=1}^{n} f(t_k)\Delta\alpha_k$ $\Delta\alpha_k=\alpha(x_k)-\alpha(x_{k-1})$ $t_k\in[x_{k-1},x_k]$

Def: $f$ is Riemann integrable wrt $\alpha$ on $[a,b]$, if given $\varepsilon>0$ $\exists$ a partition $P_\varepsilon\in\mathcal{P}[a,b]$ and a number $A$ indep of $\varepsilon$

s.t. for any partition $P$ finer than $P_\varepsilon$, we have $|S(P,f,\alpha)-A|<\varepsilon$ $A=\int_a^b f(x)\,dg(x)$

(another version: $\exists\,\delta>0$ s.t. for any partition $P$ with $\|P\|<\delta$, then ) $\leftarrow$ more strong

Thm: $f\cdot\alpha:[a,b]\to\mathbb{R}$ bounded then $f\in\mathcal{R}(\alpha)\Leftrightarrow$ given $\varepsilon>0$ $\exists$ a partition $P_\varepsilon\in\mathcal{P}[a,b]$ s.t. for any two partitions $P\cdot Q$ finer than $P_\varepsilon$, than any two Riemann-Stieltjes sums $S(P,f,\alpha)$ and $S(Q,f,\alpha)$ correspond to $P,Q$ satisfy $|S(P,f,\alpha)-S(Q,f,\alpha)|<\varepsilon$

pf: ($\Rightarrow$) $f\in\mathcal{R}(\alpha)$, given $\varepsilon>0$, $\exists P_\varepsilon$ $\exists A$ (indep of $\varepsilon$) s.t. if $P\supseteq P_\varepsilon$ then $|S(P,f,\alpha)-A|<\varepsilon$

For any two $P,Q$ finer than $P_\varepsilon$ $|S(P,f,\alpha)-S(Q,f,\alpha)|\le|S(P,f,\alpha)-A|+|A-S(Q,f,\alpha)|<2\varepsilon$

(Cauchy condition for Riemann-Stieltjes integrable)

($\Leftarrow$) For each $k\in\mathbb{N}$, $k=1$, $\exists P_1$ s.t. $P',P''\supseteq P_1\Rightarrow|S(P',f,\alpha)-S(P'',f,\alpha)|<1$

$\exists P_k$ s.t. $P',P''\supseteq P_k\Rightarrow|S(P',f,\alpha)-S(P'',f,\alpha)|<\frac{1}{k}$ May assume $P_k\subseteq P_{k+1}$ $\forall k$

consider $\{S(P_k,f,\alpha)\}$, $k,k+m$ $|S(P_k,f,\alpha)-S(P_{k+m},f,\alpha)|<\frac{1}{k}$ It is Cauchy. $\lim_{k\to\infty}S(P_k,f,\alpha)=A$

given $\varepsilon>0$ $\exists k_0$ s.t. $\frac{1}{k_0}<\varepsilon$, $\exists P_{k_0}(=P_\varepsilon)$ for any $P\supseteq P_{k_0}$

$|S(P,f,\alpha)-A|=|S(P,f,\alpha)-S(P_{k_0},f,\alpha)+S(P_{k_0},f,\alpha)-A|\le|S(P,f,\alpha)-S(P_{k_0},f,\alpha)|+|S(P_{k_0},f,\alpha)-A|<\frac{1}{k_0}+\varepsilon<2\varepsilon$

Thm: $f\cdot g\cdot\alpha\cdot\beta:[a,b]\to\mathbb{R}$ bounded then ① $f\in\mathcal{R}(\alpha)$, $g\in\mathcal{R}(\alpha)\Rightarrow f+g\in\mathcal{R}(\alpha)$ and $\int_a^b f\,d\alpha+\int_a^b g\,d\alpha=\int_a^b(f+g)\,d\alpha$

② $f\in\mathcal{R}(\beta)\Rightarrow f\in\mathcal{R}(\alpha+\beta)$ and $\int_a^b f\,d(\alpha+\beta)=\int_a^b f\,d\alpha+\int_a^b f\,d\beta$

pf: $f\in\mathcal{R}(\alpha)$, $g\in\mathcal{R}(\alpha)$ given $\varepsilon>0$ $\exists P_\varepsilon'$ s.t. if $P_\varepsilon'\subseteq P$, $|S(P,f,\alpha)-\int_a^b f\,d\alpha|<\varepsilon$

$\exists P_\varepsilon''$ s.t. if $P_\varepsilon''\subseteq Q$, $|S(Q,f,\alpha)-\int_a^b g\,d\alpha|<\varepsilon$ let $P_\varepsilon=P_\varepsilon'\cup P_\varepsilon''$

For $P\supseteq P_\varepsilon$ $|S(P,f+g,\alpha)-\int_a^b f\,d\alpha-\int_a^b g\,d\alpha|=|S(P,f,\alpha)+S(P,g,\alpha)-\int_a^b f\,d\alpha-\int_a^b g\,d\alpha|<2\varepsilon$ $\int_a^b f\,d\alpha+\int_a^b g\,d\alpha=\int_a^b(f+g)\,d\alpha$

Thm: $f\cdot\alpha:[a\cdot b]\to\mathbb{R}$ bounded then ① If $a<c<b$ s.t. two of the integrals $\int_a^c f(x)d\alpha(x)$ $\int_c^b f(x)d\alpha(x)$ $\int_a^b f(x)d\alpha(x)$ exsit

then the often on a also b exsits, and $\int_a^c f(x)d\alpha(x)+\int_c^b f(x)d\alpha(x)=\int_a^b f(x)d\alpha(x)$

② If $f\in\mathcal{R}(\alpha)$ on $[a\cdot b]$ then $f\in\mathcal{R}(\alpha)$ on $[a\cdot c]$

pf: ① Assume $\int_a^c f(x)d\alpha(x)\cdot\int_c^b f(x)d\alpha(x)$ exsits,

by def, given $\varepsilon>0$ $\exists P_\varepsilon'\in P[a,c]$ s.t. $|S(P,f,\alpha)-\int_a^c f(x)d\alpha(x)|<\varepsilon$ $\forall$ partition $P$ finer than $P_\varepsilon'\in P[a\cdot c]$

" $\exists P_\varepsilon''\in P[c\cdot b]$ s.t. $|S(Q,f,\alpha)-\int_c^b f(x)d\alpha(x)|<\varepsilon$ $\forall$ " $Q$ " $P_\varepsilon''\in P[c\cdot b]$

let $P_\varepsilon=P_\varepsilon'\cup P_\varepsilon''\in P[a\cdot b]$ If $P\in P[a\cdot b]$ is finer than $P_\varepsilon$ $P\cap P[a\cdot c]=P'$ $P\cap P[c\cdot b]=P''$

$|S(P,f,\alpha)-\int_a^c f d\alpha-\int_c^b f d\alpha|=|S(P',f,\alpha)+S(P'',f,\alpha)-\int_a^c f d\alpha-\int_c^b f d\alpha|\le|S(P',f,\alpha)-\int_a^c f d\alpha|+|S(P'',f,\alpha)-\int_c^b f d\alpha|<2\varepsilon$

$\therefore\int_a^b f d\alpha=\int_a^c f d\alpha+\int_c^b f d\alpha$ the same reason as the others

② By Cauchy's Criterion given $\varepsilon>0$ $\exists P_\varepsilon\in P[a\cdot b]$ s.t. $|S(Q,f,\alpha)-S(Q',f,\alpha)|<\varepsilon$ if $Q\cdot Q'\in P[a\cdot b]$ are finer than $P_\varepsilon$

may assume $c\in P_\varepsilon$ let $\widetilde{P_\varepsilon}=P_\varepsilon\cap[a\cdot c]$, let $Q_1\cdot Q_2\in P[a\cdot c]$ be finer than $\widetilde{P_\varepsilon}$

$P_\varepsilon\cap[c\cdot b]=\widetilde{P_\varepsilon}'$ $\therefore Q_1\cup\widetilde{P_\varepsilon}'\cdot Q_2\cup\widetilde{P_\varepsilon}'\in P[a\cdot b]$ finer than $P_\varepsilon$

Also assume $t_k$ chosen in $[c\cdot b]$ are the same

$|S(Q_1,f,\alpha)-S(Q_2,f,\alpha)|=|S(Q_1,f,\alpha)+S(\widetilde{P_\varepsilon}',f,\alpha)-(S(Q_2,f,\alpha)+S(\widetilde{P_\varepsilon}',f,\alpha))|=|S(Q_1\cup\widetilde{P_\varepsilon}',f,\alpha)-S(Q_2\cup\widetilde{P_\varepsilon}',f,\alpha)|<\varepsilon$

$\int_a^c f d\alpha$ exsits on $[a\cdot b]$ Define $\int_a^b f d\alpha=-\int_b^a f d\alpha$ · $\int_a^a f d\alpha=0$, $\int_a^b f d\alpha+\int_b^c f d\alpha+\int_c^a f d\alpha=0$

Thm: $f\cdot\alpha:[a\cdot b]\to\mathbb{R}$ bounded, suppose $f\in\mathcal{R}(\alpha)$ on $[a\cdot b]$ then $\alpha\in\mathcal{R}(f)$. $\int_a^b f(x)d\alpha(x)+\int_a^b\alpha(x)df(x)=\alpha(x)f(x)\Big|_a^b$

pf: $\because f\in\mathcal{R}(\alpha)$, given $\varepsilon>0$ $\exists P_\varepsilon\in P[a\cdot b]$ s.t. $|S(P,f,\alpha)-\int_a^b f(x)d\alpha(x)|<\varepsilon$ $\forall P\in P[a\cdot b]$ finer than $P_\varepsilon$

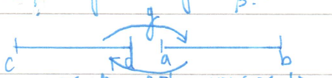
For such $P$ $S(P,\alpha,f)=\sum_{k=1}^n\alpha(t_k)(f(x_k)-f(x_{k-1}))$ consider: $\alpha(b)f(b)-\alpha(a)f(a)-S(P,\alpha,f)$

$|\alpha(b)f(b)-\alpha(a)f(a)-S(P,\alpha,f)-\int_a^b f d\alpha|=|\sum_{k=1}^n(\alpha(x_k)f(x_k)-\alpha(x_{k-1})f(x_{k-1}))-\sum_{k=1}^n\alpha(t_k)(f(x_k)-f(x_{k-1}))-\int_a^b f d\alpha|$

$=|\sum_{k=1}^n f(x_k)(\alpha(x_k)-\alpha(t_k))+\sum_{k=1}^n f(x_{k-1})(\alpha(t_k)-\alpha(x_{k-1}))-\int_a^b f d\alpha|<\varepsilon$ ($P'=\{a=x_0<x_1<\cdots<x_n=b\}\cup\{t_1\le t_2\le\cdots\le t_n\}$ finer than $P_\varepsilon$)

$f\cdot\alpha:[a\cdot b]\to\mathbb{R}$ bounded $f\in\mathcal{R}(\alpha)$ on $[a\cdot b]$, $g:[c\cdot d]\to[a\cdot b]$ conti, onto strickly increasing (decreasing)

let $\beta=\alpha\circ g$ $h=f\circ g$ $h{:}\ \beta{:}\ [c\cdot d]\to\mathbb{R}$ bounded on $[a\cdot b]$ If $f\in\mathcal{R}(\alpha)$ then $h\in\mathcal{R}(\beta)$ on $[c\cdot d]$ and $\int_a^b f d\alpha=\int_c^d h d\beta$




$P\in P[c\cdot d]$ $g(P)\in P[a\cdot b]$
$g^{-1}(P')\in P[c\cdot d]$ $P'\in P[a\cdot b]$

$\int_a^b f d\alpha$ $\int_c^d h d\beta$

$S(P,f,\alpha)\longrightarrow S(g^{-1}(P),h,\beta)$

Thm: $f\cdot\alpha:[a,b]\to\mathbb{R}$ bounded suppose $\alpha\in C'([a,b])$ and $f\in R(\alpha)$ then $\int_a^b f(x)\,d\alpha(x)=\int_a^b f(x)\,\alpha'(x)\,dx$

pf: given $\varepsilon>0$. $f\in R(\alpha)$ $\exists P_\varepsilon\in P[a,b]$ s.t. $|S(P,f,\alpha)-\int_a^b f\,d\alpha|<\varepsilon$ $\forall P\in P[a,b]$ finer than $P_\varepsilon$

$P=\{a=x_0<x_1<\cdots<x_n=b\}$ $S(P,f,\alpha)=\sum_{k=1}^n f(t_k)(\alpha(x_k)-\alpha(x_{k-1}))$ $t_k\in[x_{k-1},x_k]$ (Riemann-Stieltjes sum)

$S(P,f\alpha')=\sum_{k=1}^n f(s_k)\alpha'(s_k)\Delta x_k$ $(\Delta x_k=x_k-x_{k-1})$ $s_k\in[x_{k-1},x_k]$ (Riemann sum)

$\hookrightarrow=\sum_{k=1}^n f(t_k)\,\alpha'(c_k)(x_k-x_{k-1})$

$|S(P,f,\alpha)-S(P,f\alpha')|=|\sum_{k=1}^n f(t_k)(\alpha'(c_k)-\alpha'(t_k))(x_k-x_{k-1})|$ $\because\alpha\in C'([a,b])$ $\therefore\alpha'\in C([a,b])$ uniform conti on $[a,b]$

$\exists\delta$ $\exists P_\varepsilon'$ s.t. $\|P_\varepsilon'\|<\delta$ $\quad\leq\sum_{k=1}^n\varepsilon|f(t_k)|(x_k-x_{k-1})\leq\varepsilon\cdot M\cdot(b-a)$

$|S(P,f\alpha')-\int_a^b f\,d\alpha|=|S(P,f\alpha')-S(P,f,\alpha)+S(P,f,\alpha)-\int_a^b f\,d\alpha|\leq|S(P,f\alpha')-S(P,f,\alpha)|+|S(P,f,\alpha)-\int_a^b f\,d\alpha|<2\varepsilon$

11/25 $\int_a^b f(x)\,d\alpha(x)$ $\quad\alpha(x)=\begin{cases}\alpha(a) & a\leq x<c\\ \alpha(c) & x=c\\ \alpha(b) & c<x\leq b\end{cases}$

Thm: $f:[a,b]\to\mathbb{R}$ bounded. assume at least one of $f$ and $\alpha$ is conti from the left $c$. at least one of $f$ is conti from the right $c$

then $f\in R(\alpha)$ and $\int_a^b f(x)\,d\alpha(x)=f(c)(\alpha(c^+)-\alpha(c^-))$

pf: $P\in P[a,b]$. $c\in P$ $P=\{a=x_0<x_1<\cdots<x_{k-1}<x_k=c<x_{k+1}<\cdots<x_n=b\}$

$S(P,f,\alpha)=f(t_k)(\alpha(c)-\alpha(x_{k-1}))+f(t_{k+1})(\alpha(x_{k+1})-\alpha(c))=f(t_k)(\alpha(c)-\alpha(a))+f(t_{k+1})(\alpha(b)-\alpha(c))$

$I=f(c)(\alpha(b)-\alpha(a))=f(c)(\alpha(b)-\alpha(c)+\alpha(c)-\alpha(a))$. $S(P,f,\alpha)-I=(f(t_k)-f(c))(\alpha(c)-\alpha(a))+(f(t_{k+1})-f(c))(\alpha(b)-\alpha(c))$

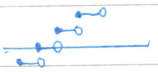
claim given $\varepsilon>0$ $\exists P_\varepsilon$ s.t. $P_\varepsilon\subseteq P$ $|S(P,f,\alpha)-I|\leq\varepsilon|\underset{\alpha(c^-)}{\alpha(c)-\alpha(a)}|+\varepsilon|\underset{\alpha(c^+)}{\alpha(b)-\alpha(c)}|$

① $f$ is conti at $c$ ② $\alpha$ is conti at $c$ ③ $f$ is conti from the right (left), $\alpha$ is conti from the left (right)

Ex: $\alpha(x)=0$ $x\neq0$ $\alpha(0)=-1$ $f(x)=1$ $-1\leq x\leq1$ $f\in R(\alpha)$ $\int_{-1}^1 f\,d\alpha=f(0)(\alpha(0^+)-\alpha(0^-))=0$ $\quad t_k\in[x_{k-1},0]$ $t_{k+1}\in[0,x_k]$

define $F(x)=\begin{cases}1 & x\neq0\\ 2 & x=0\end{cases}$ $S(P,F,\alpha)=F(t_k)(\alpha(x_k)-\alpha(x_{k-1}))+F(t_{k+1})(\alpha(x_{k+1})-\alpha(x_k))=F(t_k)(-1-0)+F(t_{k+1})(0-(-1))=F(t_{k+1})-F(t_k)=\begin{cases}1\\0\\-1\end{cases}$ DNE

$[a,b]$ $P=\{a=x_1<x_2<\cdots<x_n=b\}$ $\alpha:[a,b]\to\mathbb{R}$ bounded $\alpha$ is called a step func if $\alpha$ is a const in $(x_{k-1},x_k)$ $k=2,\dots,n$

Ex: $[x]$ 

Thm: $f\cdot\alpha:[a,b]\to\mathbb{R}$ $\alpha$: step func $a=x_1<x_2<\cdots<x_n=b$ $\alpha$ has a jump $\alpha_k$ at $x_k$ at least one of $f$ and $\alpha$ is conti at $x_k$ from the right and from the left then $\int_a^b f\,d\alpha=\sum_{k=1}^n f(x_k)\,\alpha_k$

Cor: $\sum_{k=1}^n a_k=\sum_{k=1}^n f(k)=\int_0^n f(x)\,d[x]$ Define: $f(0)=0$ $=a_k$ $k-1<x\leq k$ $1\leq k\leq n$

Euler's summation formula. Thm: $f \in C^1([a,b])$ then $\sum_{a<n\le b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)((x))dx + f(a)((a)) - f(b)((b))$ $\quad ((x)) = x - [x]$

If $a, b \in \mathbb{Z}$ $\sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\left(x - [x] - \frac{1}{2}\right)dx + \frac{f(a)+f(b)}{2}$

pf: $\because \int_a^b f(x)dx + \int_a^b x f'(x)\,dx = bf(b) - af(a)$. $\left(\sum_{a<n\le b} f(n) = \right) \int_a^b f(x)\,d[x] + \int_a^b [x] f'(x)\,dx = [b]f(b) - [a]f(a)$

$\sum_{a<n\le b} f(n) = \int_a^b -[x] f'(x)\,dx + [b]f(b) - [a]f(a) + \int_a^b f(x)dx + \int_a^b x f'(x)dx + af(a) - bf(b)$

$= \int_a^b f(x)\,dx + \int_a^b f'(x)((x))\,dx + f(a)((a)) - f(b)((b))$

$f, \alpha : [a,b] \to \mathbb{R}$ bounded $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ choose $t_k \in [x_{k-1}, x_k]$ form $S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k$ $\quad \Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$ Riemann-Stieltjes sum 11/29

let $M_k(f) = \sup_{[x_{k-1}, x_k]} f$ $\quad m_k(f) = \inf_{[x_{k-1}, x_k]} f$ $\quad m_k(f) \le M_k(f)$ $\quad U(P, f, \alpha) = \sum_{n=1}^{k} M_k(f)\Delta\alpha_k$ $\quad L(P, f, \alpha) = \sum_{n=1}^{k} m_k(f)\Delta\alpha_k$

Riemann integral $\alpha(x) = x$ $\quad \Delta x_k = x_k - x_{k-1} > 0$. Assume $\alpha(x)$ is increasing, $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) \ge 0$ $\quad \sum_{k=1}^{n} M_k(f)\Delta\alpha_k \ge \sum_{k=1}^{n} m_k(f)\Delta\alpha_k$

$U(P, f, \alpha) \ge S(P, f, \alpha) \ge L(P, f, \alpha)$

Thm: $\alpha \nearrow$, $P_1 \subseteq P_2$ two partitions, then ① $U(P_2, f, \alpha) \le U(P_1, f, \alpha)$. $L(P_2, f, \alpha) \ge L(P_1, f, \alpha)$, ② $\forall$ partition $P', P''$, $L(P', f, \alpha) \le U(P'', f, \alpha)$

pf: ① $\underset{[c,d]}{M}(f)(\alpha(d) - \alpha(c)) = \underset{[c,d]}{M}(f)(\alpha(d) - \alpha(e)) + \underset{[c,d]}{M}(f)(\alpha(e) - \alpha(c)) \ge \underset{[e,d]}{M}(f)(\alpha(d) - \alpha(e)) + \underset{[c,e]}{M}(f)(\alpha(e) - \alpha(c))$

② let $Q = P' \cup P''$ $\quad L(P', f, \alpha) \le L(Q, f, \alpha) \le U(Q, f, \alpha) \le U(P'', f, \alpha)$

Def: $\alpha \nearrow$, define $\overline{I}(f, \alpha) = \overline{\int_a^b} f\,d\alpha = \inf_{P \in P[a,b]} \{U(P, f, \alpha)\}$ the upper Riemann-Stieltjes integral

" $\quad \underline{I}(f, \alpha) = \underline{\int_a^b} f\,d\alpha = \sup_{P \in P[a,b]} \{L(P, f, \alpha)\}$ " lower "

Dirichlet function: $\underline{I} < \overline{I}$

Thm: $\alpha \nearrow$. $\underline{I} \le \overline{I}$

pf: given $\varepsilon > 0$ $\therefore \exists P \in P[a,b]$ s.t. $L(Q, f, \alpha) \le U(P, f, \alpha) \le \overline{I} + \varepsilon$ $\forall Q \in P[a,b]$ $\therefore \underline{I}(f, \alpha) \le U(P, f, \alpha) < \overline{I}(f, \alpha) + \varepsilon$ $\quad \varepsilon \to 0^+$

Thm: $\alpha \nearrow$, then $\overline{\int_a^b}(f+g)\,d\alpha \le \overline{\int_a^b} f\,d\alpha + \overline{\int_a^b} g\,d\alpha$ $\quad \underline{\int_a^b}(f+g)\,d\alpha \ge \underline{\int_a^b} f\,d\alpha + \underline{\int_a^b} g\,d\alpha$ $\quad \overline{\int_a^b} f\,d\alpha = \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha$. $\underline{\int_a^b} f\,d\alpha = \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$

pf: $P \in P[a,b]$ $\quad P \subseteq P \cup \{c\} = P' = P_1' \cup P_2'$ $\quad P_1' = P' \cap [a,c]$ $\quad P_2' = P' \cap [c,b]$

$U(P, f, \alpha) \ge U(P', f, \alpha) = U(P_1', f, \alpha) + U(P_2', f, \alpha) \ge \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha$ $\therefore$ by taking inf of $P \Rightarrow \overline{\int_a^b} f\,d\alpha \ge \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha$

let $Q_1 \in P[a,c]$ $\quad Q_2 \in P[c,b]$. $Q = Q_1 \cup Q_2 \in P[a,b]$

$U(Q_1, f, \alpha) + U(Q_2, f, \alpha) = U(Q, f, \alpha) \ge \overline{\int_a^b} f\,d\alpha$ Fixed $Q_2$ $\quad \overline{\int_a^c} f\,d\alpha + U(Q_2, f, \alpha) \ge \overline{\int_a^b} f\,d\alpha \Rightarrow \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha \ge \overline{\int_a^b} f\,d\alpha$

$f, g : [a,b] \to \mathbb{R}$ $\quad \max_{[a,b]}(f+g) \le \max_{[a,b]} f + \max_{[a,b]} g$ $\quad \min_{[a,b]}(f+g) \ge \min_{[a,b]} f + \min_{[a,b]} g$

Riemann Condition $f\cdot\alpha:[a,b]\to\mathbb{R}$ bounded. given $\varepsilon>0$ $\exists P_\varepsilon\in P[a,b]$ s.t. every refinement $P$ of $P_\varepsilon$, we have $0\le U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$

Thm: $\alpha\nearrow$, then the following three condition are equivalent ① $f\in R(\alpha)$ ② $f$ satisfies Riemann condition wrt $\alpha$ ③ $\bar{I}(f,\alpha)=\underline{I}(f,\alpha)$

12/6 pf: ①→②: given $\varepsilon>0$ $\exists P_\varepsilon$ s.t. for $P\ge P_\varepsilon$ $|S(P,f,\alpha)-\int_a^b f\,d\alpha|<\varepsilon$ $\quad S(P,f,\alpha)=\sum_{k=1}^n f(t_k)\Delta\alpha_k$ $\quad t_k\in[x_{k-1},x_k]$

$|\sum_{k=1}^n f(t_k)\Delta\alpha_k-\sum_{k=1}^n f(t_k')\Delta\alpha_k|<2\varepsilon$ $\quad U(P,f,\alpha)-L(P,f,\alpha)=\sum_{k=1}^n(M_k(f)-m_k(f))\Delta\alpha_k$

given $\varepsilon>0$ $\exists t_k,t_k'\in[x_{k-1},x_k]$ s.t. $M_k(f)-m_k(f)\le f(t_k)-f(t_k')+\varepsilon$ $\quad \overset{\wedge}{\sum_{k=1}^n}(f(t_k)-f(t_k'))\Delta\alpha_k+\sum_{k=1}^n\varepsilon\Delta\alpha_k<2\varepsilon+\varepsilon(\alpha(b)-\alpha(a))$

②→③: given $\varepsilon>0$ $\exists P_\varepsilon$ s.t. for $P\ge P_\varepsilon$ $\quad \bar{I}\le U(P,f,\alpha)\le L(P,f,\alpha)+\varepsilon\le\underline{I}+\varepsilon$ $\Rightarrow \bar{I}\le\underline{I}$ and $\underline{I}\le\bar{I}$ $\Rightarrow \underline{I}=\bar{I}$

③→①: $\bar{I}=\underline{I}$ given $\varepsilon>0$ $\exists P_1$ s.t. for $P\ge P_1$ $U(P,f,\alpha)\le\bar{I}+\varepsilon$

$\qquad \exists P_2 \qquad Q\ge P_2 \quad \underline{I}-\varepsilon\le L(Q,f,\alpha)$

let $P_\varepsilon=P_1\cup P_2$. for $P\ge P_\varepsilon$ $\quad \underline{I}-\varepsilon\le L(P,f,\alpha)\le S(P,f,\alpha)\le U(P,f,\alpha)\le\bar{I}+\varepsilon$

$I=\bar{I}=\underline{I}$ $\quad -\varepsilon\le S(P,f,\alpha)-I\le\varepsilon$ $\Rightarrow$ $|S(P,f,\alpha)-I|<\varepsilon$ $\quad (I=\int_a^b f\,d\alpha)$

Thm: $\alpha\nearrow$. $f,g\in R(\alpha)$ If $f(x)\le g(x)$ $\forall x\in[a,b]$ then $\int_a^b f(x)\,d\alpha(x)\le\int_a^b g(x)\,d\alpha(x)$

pf: $\because f,g\in R(\alpha)$ we can choose the same $t_k$ $\therefore f(t_k)\le g(t_k)$ $\quad \sum_{k=1}^n f(t_k)\Delta\alpha_k\le\sum_{k=1}^n g(t_k)\Delta\alpha_k$ $\to \int_a^b f\,d\alpha\le\int_a^b g\,d\alpha$

Thm: $\alpha\nearrow$. $f\in R(\alpha)\Rightarrow|f|\in R(\alpha)$ and $|\int_a^b f(x)\,d\alpha(x)|\le\int_a^b|f(x)|\,d\alpha(x)$

pf: $f\le|f|$ $\;-f\le|f|$ $\Rightarrow$ the inequality holds, only need to show that $|f|\in R(\alpha)$

check Riemann Condition $\quad 0\le U(P,|f|,\alpha)-L(P,|f|,\alpha)=\sum_{k=1}^n(M_k(|f|)-m_k(|f|))\Delta\alpha_k$

$\because\ ||f(t_k)|-|f(t_k')||\le|f(t_k)-f(t_k')|$ $\quad t_k,t_k'\in[x_{k-1},x_k]$ $\quad \overset{\wedge}{\sum_{k=1}^n}(M_k(f)-m_k(f))\Delta\alpha_k<\varepsilon$ $\quad$(given $\varepsilon>0$ $\exists P_\varepsilon$ s.t. $P\ge P_\varepsilon$)

12/9 Thm: $\alpha\nearrow$. $f\in R(\alpha)\Rightarrow f^2\in R(\alpha)$

pf: Riemann Condition $\quad 0\le U(P,f^2,\alpha)-L(P,f^2,\alpha)=\sum_{k=1}^n(M_k(f^2)-m_k(f^2))\Delta\alpha_k=\sum_{k=1}^n(M_k^2(|f|)-m_k^2(|f|))=\sum_{k=1}^n(M_k|f|+m_k|f|)(M_k|f|-m_k|f|)\Delta\alpha_k$

$M=\sup_{[a,b]}|f|$ $\quad m=\inf_{[a,b]}|f|$ $\qquad \le 2M\sum_{k=1}^n(M_k|f|-m_k|f|)\Delta\alpha_k=2M(U(P,|f|,\alpha)-L(P,|f|,\alpha))<2M\cdot\frac{\varepsilon}{2M}=\varepsilon$

Thm: $\alpha\nearrow$. $f,g\in R(\alpha)\Rightarrow f\cdot g\in R(\alpha)$

pf: $\because (f+g)^2,f^2,g^2\in R(\alpha)$ $\quad\therefore \frac{(f+g)^2-f^2-g^2}{2}=fg\in R(\alpha)$

$\alpha$: integrator is of BV $f\in(\alpha)$ write $\alpha=\alpha_1-\alpha_2$ $\alpha_1,\alpha_2\nearrow$ choose $\alpha=V_\alpha-(V_\alpha-\alpha)$

Thm: $\alpha$ is of BV on $[a,b]$. $f\in R(\alpha)$ Let $V(x)=V_\alpha(a,x)$ $a<x\le b$ $V_\alpha(a)=0$ Then $f\in R(V)$

pf: given $\varepsilon>0$ $\exists P_\varepsilon$ s.t. for $P\ge P_\varepsilon$ then $\left|\sum_{k=1}^{n}(f(t_k)-f(t_k'))\Delta\alpha_k\right|<\varepsilon$ $V(b)=V_\alpha(a,b)\le\Sigma(p)+\varepsilon$ $t_k,t_k'\in[x_{k-1},x_k]$

show (i) $\sum_{k=1}^{n}(M_k(f)-m_k(f))|\Delta\alpha_k|<\varepsilon$ (ii) $\sum_{k=1}^{n}(M_k(f)-m_k(f))(\Delta V_k-|\Delta\alpha_k|)<\varepsilon$ $\Rightarrow$ (i)+(ii) $0\le\sum_{k=1}^{n}(M_k(f)-m_k(f))\Delta V_k<2\varepsilon$

pf (ii): $\Delta V_k-|\Delta\alpha_k|=V(x_k)-V(x_{k-1})-|\alpha(x_k)-\alpha(x_{k-1})|\ge0$.

$\sum_{k=1}^{n}(M_k(f)-m_k(f))(\Delta V_k-|\Delta\alpha_k|)\le2M\sum_{k=1}^{n}(\Delta V_k-|\Delta\alpha_k|)=2M\left(\sum_{k=1}^{n}\Delta V_k-\sum_{k=1}^{n}|\Delta\alpha_k|\right)=2M\left(V(b)-\sum_{k=1}^{n}|\Delta\alpha_k|\right)\le2M\varepsilon$

pf (i): $\sum_{k=1}^{n}(M_k(f)-m_k(f))|\Delta\alpha_k|=\sum_{k\in A}+\sum_{k\in B}$ $(A=\{k|\Delta\alpha_k\ge0\}$ $B=\{k|\Delta\alpha_k<0\})$

$=\sum_{k\in A}(M_k(f)-m_k(f))\Delta\alpha_k+\sum_{k\in B}(M_k(f)-m_k(f))(-\Delta\alpha_k)$ $\left(\sum_{k\in B}(m_k(f)-M_k(f))\Delta\alpha_k\right)$

$(M_k(f)-m_k(f)<f(t_k)-f(t_k')+\varepsilon$ $\exists t_k,t_k'\in[x_{k-1},x_k]$, $m_k(f)-M_k(f)+\varepsilon>f(t_k)-f(t_k')$ $f(t_k')-f(t_k)+\varepsilon>M_k(f)-m_k(f))$

$\le\sum_{k\in A}(f(t_k)-f(t_k')+\varepsilon)\Delta\alpha_k+\sum_{k\in B}(f(t_k')-f(t_k)+\varepsilon)(-\Delta\alpha_k)=\sum_{k=1}^{n}(f(t_k)-f(t_k'))\Delta\alpha_k+\varepsilon\sum_{k=1}^{n}|\Delta\alpha_k|<\varepsilon+\varepsilon V(b)$

Thm: $\alpha$ is of BV on $[a,b]$. $f\in R(\alpha)$ let $V(x)=V_\alpha(a,x)$ $a<x\le b$ $V(a)=0$ If $[c,d]\subseteq[a,b]\Rightarrow f\in R(V)$ on $[c,d]$

pf: $\alpha\nearrow$ $f\in R(\alpha)$ $[a,b]\Rightarrow f\in R(\alpha)$ $[a,c]$ $a<c<b$

If $P_\varepsilon=\{a=x_0<x_1<\cdots<x_n=b\}$ $P\ge P_\varepsilon$ $0\le U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$

$0\le U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$ $P_\varepsilon\cap[a,c]=P_\varepsilon'$ $P'\ge P_\varepsilon'$ on $[a,c]$ $P'\cup P_\varepsilon$

$0\le U(P',f,\alpha)-L(P',f,\alpha)\le U(P'\cup P_\varepsilon,f,\alpha)-L(P'\cup P_\varepsilon,f,\alpha)<\varepsilon$ on $[a,b]$

Thm: $\alpha\nearrow$ $f\in R(\alpha)$ $g\in R(\alpha)$ on $[a,b]$. define $F(x)=\int_a^x f(t)\,d\alpha(t)$ $G(x)=\int_a^x g(t)\,d\alpha(t)$ $a\le x\le b$

then $f\in R(G)$, $g\in R(F)$ and $f\cdot g\in R(\alpha)$. $\int_a^b f(x)g(x)\,d\alpha(x)=\int_a^b f(x)\,dG(x)=\int_a^b g(x)\,dF(x)$

pf: $P$: partition $S(P,f,G)=\sum_{k=1}^{n}f(t_k)\Delta G_k=\sum_{k=1}^{n}f(t_k)(G(x_k)-G(x_{k-1}))=\sum_{k=1}^{n}f(t_k)\left(\int_a^{x_k}g(t)\,d\alpha(t)-\int_a^{x_{k-1}}g(t)\,d\alpha(t)\right)=\sum_{k=1}^{n}f(t_k)\cdot\int_{x_{k-1}}^{x_k}g(t)\,d\alpha(t)$

$\left|S(P,f,G)-\int_a^b f(t)\cdot g(t)\,d\alpha(t)\right|=\left|\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}f(t_k)g(t)\,d\alpha(t)-\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}f(t)\cdot g(t)\,d\alpha(t)\right|=\left|\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}(f(t_k)-f(t))g(t)\,d\alpha(t)\right|$

$\le\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}|f(t_k)-f(t)||g(t)|\,d\alpha(t)\le M_g\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}|f(t_k)-f(t)|\,d\alpha(t)\le M_g\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}(M_k(f)-m_k(f))\Delta\alpha_k<M_g\varepsilon$

Thm: $f\in C([a,b])$ and $\alpha$ is of BV on $[a,b]$ then $f\in R(\alpha)$ on $[a,b]$

pf: check assume $\alpha\nearrow$ $0\le U(P,f,\alpha)-L(P,f,\alpha)=\sum_{k=1}^{n}(M_k(f)-m_k(f))\Delta\alpha_k<\varepsilon\sum_{k=1}^{n}\Delta\alpha_k=\varepsilon(\alpha(b)-\alpha(a))$

(by Hiene thm $f$ is uniformly conti given $\varepsilon>0$ $\exists\delta>0$ s.t. $|f(x)-f(y)|<\varepsilon$ if $|x-y|<\delta$ $\therefore\|P_\varepsilon\|<\delta$)

Thm: $\alpha \nearrow$ on $[a,b]$ $a<c<b$. Suppose both $\alpha$ and $f$ are disconti from the right (or left) then $f \notin R(\alpha)$

($\exists\, \varepsilon_0>0$ for any $\delta>0$ $\exists\, x, y$ $c<x<c+\delta$ s.t. $|f(x)-f(c)| \geq \varepsilon_0$, $c<y<c+\delta$ s.t. $|\alpha(y)-\alpha(c)| \geq \varepsilon_0$)

pf: $P=\{a=x_0<x_1<\cdots<\overset{x_k}{\overset{\shortparallel}{c}}<\cdots<x_n=b\}$ $t_k \in [x_{k-1}, x_k]$ $S(P,f,\alpha)=\sum_{k=1}^{n} f(t_k)\Delta\alpha_k$

$0 \leq U(P,f,\alpha)-L(P,f,\alpha)=\sum_{k=1}^{n}(M_k(f)-m_k(f))\Delta\alpha_k$ $\Delta\alpha_k \geq 0$ $\sum_{k=1}^{n}(M_k(f)-m_k(f))\Delta\alpha_k \geq (M_{k+1}(f)-m_{k+1}(f))(\alpha(x_{k+1})-\alpha(x_k)) \geq \varepsilon_0^2$

Thm: $\alpha \nearrow$ on $[a,b]$ $f \in R(\alpha)$ $M=\sup\limits_{[a,b]} f$ $m=\inf\limits_{[a,b]} f$ then $\exists\, m \leq c \leq M$ s.t. $\int_a^b f(x)\,d\alpha = c\int_a^b d\alpha = c(\alpha(b)-\alpha(a))$

In particular, if $f \in C([a,b])$ then $\exists\, a \leq x_0 \leq b$ s.t. $\int_a^b f(x)\,d\alpha(x) = f(x_0)(\alpha(b)-\alpha(a)) = f(x_0)\int_a^b d\alpha$ (IVT)

pf: $m(\alpha(b)-\alpha(a)) \leq L(P,f,\alpha) \leq \int_a^b f(x)\,d\alpha \leq U(P,f,\alpha) \leq M(\alpha(b)-\alpha(a))$

① $\alpha(b)=\alpha(a)$ done ② $\alpha(b)>\alpha(a)$ $m \leq \frac{\int_a^b f(x)\,d\alpha}{\alpha(b)-\alpha(a)} \leq M$ let $c=\frac{\int_a^b f(x)\,d\alpha}{\alpha(b)-\alpha(a)}$

Thm: $f \nearrow$; $\alpha$ conti on $[a,b]$ then $\exists\, x_0 \in [a,b]$ s.t. $\int_a^b f(x)\,d\alpha(x) = f(a)\int_a^{x_0} d\alpha + f(b)\int_{x_0}^b d\alpha$

pf: $\int_a^b f(x)\,d\alpha + \int_a^b \alpha(x)\,df = f(b)\alpha(b)-f(a)\alpha(a)$ $\int_a^b \alpha(x)\,df = \alpha(x_0)\int_a^b df = \alpha(x_0)(f(b)-f(a))$

$\int_a^b f(x)\,d\alpha = \alpha(b)f(b)-\alpha(a)f(a)-\alpha(x_0)f(b)+\alpha(x_0)f(a) = f(a)(\alpha(x_0)-\alpha(a))+f(b)(\alpha(b)-\alpha(x_0))$

If $f(x) \geq 0$ we can let $f(a)=0$ $\int_a^b f(x)\,d\alpha(x) = B\int_{x_0}^b d\alpha$ (Bounet's Thm)

Thm: Assume $\alpha$ is of BV on $[a,b]$, $f \in R(\alpha)$, then ($F(x)=\int_a^x f(t)\,d\alpha(t)$)

① $F$ is also of BV on $[a,b]$ ② $F$ is conti at where $\alpha$ is conti

③ $F$ is differentiable at where $\alpha$ is differentiable and $f$ is conti, the derivative $F'(x)$ satisfies $F'(x)=f(x)\cdot\alpha'(x)$

pf: may assume $\alpha \nearrow$, $a \leq x<y \leq b$ $F(y)-F(x)=\int_a^y f(t)\,d\alpha(t)-\int_a^x f(t)\,d\alpha(t)=\int_x^y f(t)\,d\alpha(t)=c\int_x^y d\alpha = c(\alpha(y)-\alpha(x))$ $\underset{[x,y]}{m_f} \leq c \leq \underset{[x,y]}{M_f}$

$|F(y)-F(x)| \leq |c|(\alpha(y)-\alpha(x)) \leq A(\alpha(y)-\alpha(x))$ $\Sigma(P) \leq A\sum_{k=1}^{n}\Delta\alpha_k \leq A(\alpha(b)-\alpha(a))$

$\frac{F(y)-F(x)}{y-x} = c\cdot\frac{(\alpha(y)-\alpha(x))}{y-x}$ $\underset{[x,y]}{m_f}<c<\underset{[x,y]}{M_f}$ $y \to x$ $\frac{F(y)-F(x)}{y-x}=f(x)\cdot\alpha'(x)$

Thm $f \in R$ $g \in R$ on $[a,b]$ let $F(x)=\int_a^x f(t)\,dt$ $G(x)=\int_a^x g(t)\,dt \Rightarrow F, G \in C([a,b])$, $f \in R(G)$, $g \in R(F)$ $fg \in R$

and $\int_a^b f(t)g(t)\,dt = \int_a^b f\,dG = \int_a^b g\,dF$

Thm: $f \in R$ on $[a,b]$ $g: [a,b] \to \mathbb{R}$ bounded s.t. $g'(x)=f(x)$ on $(a,b)$ Assume $g(b)-g(a)=\underset{(\text{ie } g(b^-),\, g(a^+) \text{ exsit})}{g(b^-)-g(a^+)}$ then $\int_a^b f(x)\,dx=\int_a^b g'(x)\,dx=g(b)-g(a)$

pf: $P=\{a=x_0<\cdots<x_n=b\}$ $g(b)-g(a)=g(x_n)-g(x_0)=\sum_{k=1}^{n}(g(x_k)-g(x_{k-1}))$ $g(x_1)-g(x_0)=g(x_1)-g(a^+)$ $g(x_n)-g(x_{n-1})=g(b^-)-g(x_{n-1})$

$\therefore g(b)-g(a)=g(x_1)-g(a^+)+\sum_{k=2}^{n-1}(g(x_k)-g(x_{k-1}))+g(b^-)-g(x_{n-1}) = \sum_{k=1}^{n} g'(t_k)(x_k-x_{k-1}) = \sum_{k=1}^{n} f(t_k)\Delta x_k$ $(x_{k-1}<t_k<x_k) \to \int_a^b f(t)\,dt$

Thm: $f \in R$ on $[a,b]$ $\alpha \in C([a,b])$ s.t. $\alpha' \in R$ on $[a,b]$ then $\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx$

Thm: $g \in C^1([c,d])$ $f \in C(g([c,d]))$ define $F(y)=\int_{g(c)}^{y} f(t)\,dt$ $y \in g([c,d])$

then $\forall\, x \in [c,d]$ $\int_c^x f(g(t))g'(t)\,dt = F(g(x))$ exsit when $x=d$ $\int_c^d f(g(t))g'(t)\,dt = \int_{g(c)}^{g(d)} f(t)\,dt$

Thm: $g\in C([a,b])$, $f\nearrow$ on $[a,b]$ let $A\le f(a^+)$ $B\ge f(b^-)$ then $\exists x_0\in[a,b]$ s.t. $\int_a^b f(x)\cdot g(x)dx = A\int_a^{x_0} g(x)dx + B\int_{x_0}^b g(x)dx$

In particular $f(x)\ge 0$ $\forall x\in[a,b]$ then $\int_a^b f(x)\cdot g(x)dx = B\int_{x_0}^b g(x)dx$ (Bonnet's Thm)

Thm $f(x,y)\in C([a,b]\times[c,d])$ $\alpha$ is of BV on $[a,b]$ Define $F(y)=\int_a^b f(x,y)d\alpha(x)$ then $F(y)\in C([c,d])$ and $\lim_{y\to y_0}F(y)=F(y_0)=\int_a^b f(x,y_0)d\alpha(x)$

pf: may assume $\alpha\nearrow$. $F(y)-F(y_0)=\int_a^b (f(x,y)-f(x,y_0))d\alpha(x)$ given $\varepsilon>0$ $\exists\delta>0$ s.t. $|f(x,y)-f(x',y')|<\varepsilon$ if $|(x,y)-(x',y')|<\delta$

For $|y-y_0|<\delta$. $|F(y)-F(y_0)| < \int_a^b |f(x,y)-f(x,y_0)|\,d\alpha(x) \le \varepsilon\int_a^b d\alpha(x) = \varepsilon(\alpha(b)-\alpha(a))$

Thm: $f\in C([a,b]\times[c,d])$. $g\in R$ on $[a,b]$ Define $F(y)=\int_a^b f(x,y)g(x)dx$ then $F\in C([a,b])$ and $\lim_{y\to y_0}F(y)=\int_a^b f(x,y_0)d\alpha(x)$

pf: let $G(x)=\int_a^x g(t)dt$ $\quad\int_a^b f(x,y)g(x)dx=\int_a^b f(x,y)\,dG(x)$

Thm: $\alpha$ is of BV on $[a,b]$ let $Q=[a,b]\times[c,d]$ For each $y\in[c,d]$ assume $\int_a^b f(x,y)d\alpha(x)$ exsits and $\frac{\partial f}{\partial y}(x,y)\in C(Q)$

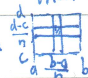
then $\frac{\partial}{\partial y}\int_a^b f(x,y)d\alpha(x)=\int_a^b\frac{\partial}{\partial y}f(x,y)\,d\alpha(x)$ $\quad y\in(c,d)$

pf: $y_0\in(c,d)$ $\frac{F(y)-F(y_0)}{y-y_0}=\frac{\int_a^b f(x,y)d\alpha(x)-\int_a^b f(x,y_0)d\alpha(x)}{y-y_0}=\int_a^b\frac{f(x,y)-f(x,y_0)}{y-y_0}d\alpha(x)=\int_a^b\frac{\partial f}{\partial y}(x,\bar y(x))\,d\alpha(x)$ $\quad y_0<\bar y(x)<y$, $y<\bar y(x)<y_0$

$\lim_{y\to y_0}\frac{F(y)-F(y_0)}{y-y_0}=\lim_{y\to y_0}\int_a^b\frac{\partial}{\partial y}f(x,\bar y(x))\,d\alpha(x)=\int_a^b\frac{\partial f}{\partial y}(x,y_0)\,d\alpha(x)$

Thm: $f\in C(\underbrace{[a,b]\times[c,d]}_{Q})$. $\begin{matrix}\alpha(x)\\ \beta(y)\end{matrix}$ is of BV on $\begin{matrix}[a,b]\\ [c,d]\end{matrix}$ Define $F(y)=\int_a^b f(x,y)d\alpha(x)$ $\quad G(x)=\int_c^d f(x,y)\,d\beta(y)$

then $F\in R(\beta)$. $G\in R(\alpha)$ $\int_c^d\left(\int_a^b f(x,y)d\alpha(x)\right)d\beta(y)=\int_c^d F(y)d\beta(y)=\int_a^b G(x)d\alpha(x)=\int_a^b\left(\int_c^d f(x,y)d\beta(y)\right)d\alpha(x)$ (Fubini)

pf:  given $\varepsilon>0$ $\exists\delta>0$ s.t. $|f(x,y)-f(x',y')|<\varepsilon$ if $|(x,y)-(x',y')|<\delta$

$A=\int_c^d\left(\int_a^b f(x,y)d\alpha(x)\right)d\beta(y)=\sum_{j=1}^n\left(\int_a^b f(x,\bar y_j)d\alpha(x)\right)(\beta(y_j)-\beta(y_{j-1}))=\sum_{j=1}^n\left(\sum_{i=1}^n f(\bar x_i,\bar y_j)(\alpha(x_i)-\alpha(x_{i-1}))\right)(\beta(y_j)-\beta(y_{j-1}))$

$B=\int_a^b\left(\int_c^d f(x,y)d\beta(y)\right)d\alpha(x)=\sum_{i=1}^n\left(\int_c^d f(x_i^*,y)d\beta(y)\right)(\alpha(x_i)-\alpha(x_{i-1}))=\sum_{i=1}^n\left(\sum_{j=1}^n f(x_i^*,y_j^*)(\beta(y_j)-\beta(y_{j-1}))\right)(\alpha(x_i)-\alpha(x_{i-1}))$

$|A-B|=\left|\sum_{i=1}^n\sum_{j=1}^n\left(f(\bar x_i,\bar y_j)-f(x_i^*,y_j^*)\right)(\alpha(x_i)-\alpha(x_{i-1}))(\beta(y_j)-\beta(y_{j-1}))\right|\le\varepsilon(\alpha(b)-\alpha(a))(\beta(d)-\beta(c))$

Thm (Lebesgue) $f:[a,b]\to\mathbb{R}$ bounded $f\in R\Leftrightarrow D=\{x\in[a,b]\mid f$ is disconti at $x\}$ has measured zero

Def: $E\subseteq\mathbb{R}$ is said to be of measure zero if given $\varepsilon>0$ $\exists$ an countable open covering $\{I_k\}_{k=1}^\infty$ $(a_k,b_k)$ of $E$ s.t. $\sum_{k=1}^\infty(b_k-a_k)<\varepsilon$, $\bigcup_{k=1}^\infty(a_k,b_k)\supseteq E$

Ex: $\{p\}$ $p\in\mathbb{R}$ measure zero $\because$ $p\in(p-\frac{\varepsilon}{3},p+\frac{\varepsilon}{3})$

Ex: $E_k$: measure zero $\forall k$ $E=\bigcup_{k=1}^\infty E_k$ is of measure zero. given $\varepsilon>0$ $\exists\bigcup(a_i^{(k)},b_i^{(k)})\supseteq E_k$ s.t. $\sum_{i=1}^\infty(b_i^{(k)}-a_i^{(k)})<\frac{\varepsilon}{2^k}$

$\bigcup_{k=1}^\infty\bigcup_{i=1}^\infty(a_i^{(k)},b_i^{(k)})\supseteq\bigcup_{k=1}^\infty E_k$ $\quad\sum_{k=1}^\infty\sum_{i=1}^\infty(b_i^{(k)}-a_i^{(k)})\le\sum_{k=1}^\infty\frac{\varepsilon}{2^k}=\varepsilon$

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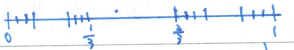
Cor: Countable union of measure zero sets is of measure zero

pf: above the Cor

Cor: $\mathbb{Q}$ has measure zero

Cantor set  $C_1 \supseteq C_2 \supseteq \cdots$ Cantor set $C=\bigcap_{n=1}^{\infty} C_n$ closed·bounded $\Rightarrow$ compact

$C_k$: $2^k$ closed interval with length $\frac{1}{3^k}$ $\quad 2^k\cdot\frac{1}{3^k}=\left(\frac{2}{3}\right)^k\to 0$ is of measure zero

Cantor set is perfect set (closed and $S=S'$) $\Rightarrow$ uncountable

Def: $E$ is an interval in $\mathbb{R}$, $T\subseteq E$ $\; f:E\to\mathbb{R}$ bounded define the oscillation on $T$ by $\Omega_f(T)=\sup\{|f(x)-f(y)| \mid x,y\in T\}$

If $p\in E$, define $\omega_f(p)=\lim_{\delta\to0^+}\Omega_f(B(p;\delta)\cap E)$

Thm: $\omega_f(p)=0 \iff f$ is conti at $p$

pf: $(\Leftarrow)$ given $\varepsilon>0$ $\exists\delta=\delta(\varepsilon,p)>0$ s.t. $|f(x)-f(y)|<\varepsilon$ if $|x-y|<\delta$

consider $x\in B(p;\delta)\cap E$ then $|f(x)-f(p)|<\varepsilon$ $\Rightarrow$ $\omega_f(p)=0$

Thm: $f:[a,b]\to\mathbb{R}$ bounded If $\omega_f(x)<\varepsilon$ $\forall x\in[a,b]$ Then $\exists\delta>0$ s.t. if $T$ is an interval with length $|T|<\delta$ then $\Omega_f(T)<\varepsilon$

pf: $\because \omega_f(x)<\varepsilon$ $\exists\delta_x>0$ s.t. $\Omega_f(B(x;\delta_x)\cap[a,b])<\varepsilon$ $\therefore [a,b]\subseteq\bigcup_{x\in[a,b]}B(x;\frac{\delta_x}{2})$ by Heine-Borel thm $[a,b]\subseteq\bigcup_{j=1}^{N}B(x_j;\frac{\delta_{x_j}}{2})$

$\therefore$ let $\delta=\min\{\frac{\delta_{x_1}}{2},\dots,\frac{\delta_{x_N}}{2}\}$ let $T$ be an interval with $|T|<\delta$ let $T\cap B(x_i;\frac{\delta_{x_i}}{2})\neq\phi$ $\; |T|<\frac{\delta_{x_i}}{2}$

$\therefore T\subseteq B(x_i;\delta_{x_i})$ $\therefore \Omega_f(T)<\varepsilon$

Thm $f:[a,b]\to\mathbb{R}$ bounded then $\{x\in[a,b] \mid \omega_f(x)\geq\varepsilon\}$ is closed

pf: $\{x\in[a,b] \mid \omega_f(x)<\varepsilon\}$ is open $\because \forall p\in B(x;\delta)$ $\omega_f(p)<\varepsilon$.

pf of Lebesgue: $(\Rightarrow)$ If $f\in R$ write $D=\bigcup_{k=1}^{\infty}D_k=\bigcup_{k=1}^{\infty}\{x\in[a,b] \mid \omega_f(x)\geq\frac{1}{k}\}$, suppose $D$ is not of measure zero

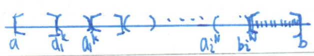
$\therefore D_k$ is not of measure zero for some $k$ $\therefore \exists\varepsilon_0>0$ for every open covering $\{(a_k,b_k)\}_{k=1}^{\infty}$ of $D_k$ $\sum_{k=1}^{\infty}(b_k-a_k)\geq\varepsilon_0$

For any partition $P\in\mathcal{P}[a,b]$ $0\leq U(P,f)-L(P,f)=\sum_{j=1}^{n}(M_j(f)-m_j(f))\Delta x_j\geq\sum_{D_k\cap(x_{j-1},x_j)\neq\phi}(M_j(f)-m_j(f))\Delta x_j\geq\sum_{D_k\cap(x_{j-1},x_j)\neq\phi}\frac{1}{k}\Delta x_j\geq\frac{\varepsilon_0}{k}>0$

$(\Leftarrow)$ $\therefore D_k$ is of measure zero $\forall k$, let $\varepsilon=\frac{1}{k}$ $\exists\{(a_i^k,b_i^k)\}_{k=1}^{\infty}$ s.t. $D_k\subseteq\bigcup_{i=1}^{\infty}(a_i^k,b_i^k)$, $\sum_{k=1}^{\infty}(b_i^k-a_i^k)<\frac{1}{k}$

$\because D_k$ is compact $\therefore D_k\subseteq\bigcup_{i=1}^{N}(a_i^k,b_i^k)$ $\sum_{k=1}^{N}(b_i^k-a_i^k)<\frac{1}{k}=\varepsilon$

$[a,b]-[\bigcup_{i=1}^{N}(a_i^k,b_i^k)]$ (closed compact) $=\bigcup[\ ,\ ]$ If $p\in[\ ,\ ]$ $\omega(p)<\frac{1}{k}\Rightarrow\exists\delta>0$ s.t. $[\alpha,\beta]$ $\beta-\alpha<\delta\Rightarrow\Omega_f[\alpha,\beta]<\frac{1}{k}$

let $\forall[\ ]$ seperate into $|\ |<\delta$ 

$0\leq U(P,f)-L(P,f)\leq(M-m)\frac{1}{k}+\frac{1}{k}(b-a)=\frac{1}{k}(M-m+b-a)<\varepsilon$

Thm: ① $f$ is of BV on $[a,b]\Rightarrow f\in R$ on $[a,b]$ $\quad$ pf: $f=f_1-f_2$ $\; f_1\nearrow, f_2\nearrow$ monotonic function disconti is of measure zero

② $f\in R$ on $[a,b]\Rightarrow f\in R$ on $[c,d]\subseteq[a,b]$. $|f|$, $f^2$ $f\cdot g\in R$ on $[a,b]$

③ $f\in R$ $g\in R$ on $[a,b]\Rightarrow\frac{f}{g}\in R$ on $[a,b]$ if $|g|\geq c>0$

④ $D_f=D_g\Rightarrow f\in R\Leftrightarrow g\in R$ $\quad$ ⑤ $g\in R$ on $[a,b]$ $m\leq g(x)\leq M$ $\phi:[m,M]\to\mathbb{R}$ conti $\Rightarrow\phi\circ g\in R$ on $[a,b]$

# Infinite Sequence and Series 無窮數列與級數

$a_1 \cdots a_n \in \mathbb{C}$ $\{a_n\}$ converges to a point $p \in \mathbb{C}$ if given $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $|a_n - p| < \varepsilon$ if $n \geq n_0$

$\{a_n\}$ diverges if it is not converge $(\mathbb{R}^2, J) \cong \mathbb{C}$

① $\{a_n\}$ is converge $\Rightarrow$ $\{a_n\}$ bounded ② $\{a_n\}$ converges to $p$ $\Rightarrow$ $p$ is unique

③ $\{a_n\}$ is converge $\Leftrightarrow$ given $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $|a_j - a_k| < \varepsilon$ if $j, k \geq n_0$

$\{a_n\}$ diverges to $+\infty$ given $M > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $a_n \geq M$ if $n \geq n_0$ ($a_n$: real)

" $-\infty$ " $a_n \leq -M$ if $n \geq n_0$ ($a_n$: real)

Ex: $a_n = (-1)^n$ oscillate

limit superior (inferior) $\{a_n\}$ $a_n \in \mathbb{R}$ If $M \in \mathbb{R}$ satisfies

① given $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $a_n < M + \varepsilon$ $(a_n > M - \varepsilon)$ for all $n \geq n_0$ ② given $\varepsilon > 0$ $m \in \mathbb{N}$ then $\exists n > m$ s.t. $a_n > M - \varepsilon$ $(a_n < M + \varepsilon)$

then we say $M$ is the limit superior (inferior) of $\{a_n\}$, denote by $\limsup_{n\to\infty} a_n = M$ or $\overline{\lim_{n\to\infty}}\, a_n = M$ (largest subsequential limit)

Ex: $a_n = (-1)^n (3 + \frac{1}{n})$ $\overline{\lim_{n\to\infty}}\, a_n = 3$ $\underline{\lim_{n\to\infty}}\, a_n = -3$

If $\{a_n\}$ is not bounded above $\Rightarrow$ $\overline{\lim_{n\to\infty}}\, a_n = \infty$ Ex: $a_{2n} = 2n$ $a_{2n+1} = 0$

If $\{a_n\}$ is bounded above and has no finite limit superior then $\limsup_{n\to\infty} a_n = -\infty$ $(\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty}(-a_n))$

extended real number system $\overline{\mathbb{R}}$ or $\mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$

Thm: $\{a_n\}$: a seq in $\mathbb{R}$ ① $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ ② $\lim_{n\to\infty} a_n = L$ exsits $\Leftrightarrow$ $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$

③ $\lim_{n\to\infty} a_n = \infty \ (-\infty)$ $\Leftrightarrow$ $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty \ (-\infty)$

④ If $a_n \leq b_n \ \forall n$ $\Rightarrow$ $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$, $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$

Thm: Bounded monotonic seq converges

series $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ partial sum $S_n = \sum_{k=1}^{n} a_k$ $\sum_{k=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} S_n$ exsits

Thm: $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ conv then $\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n)$ converges $\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$ $\alpha, \beta \in \mathbb{R}$

pf: $\sum_{k=1}^{n} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k$

Telescoping series $\sum_{n=1}^{\infty} (b_{n+1} - b_n)$ converges $\Leftrightarrow$ $\lim_{x\to\infty} b_n$ exsits 12/27

pf: $S_m = \sum_{n=1}^{m} (b_{n+1} - b_n) = b_{m+1} - b_1$

Thm: $\sum_{n=1}^{\infty} a_n$ converge $\Leftrightarrow$ given $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $\left|\sum_{j=n}^{n+b} a_j\right| < \varepsilon$ if $n \geq n_0$

pf: $\sum_{n=1}^{\infty} a_n$ conv $\Leftrightarrow$ $\lim_{m\to\infty} S_m$ exsits $S_m = \sum_{n=1}^{m} a_n$, $\{S_m\}$ cauchy $\therefore$ given $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $|S_n - S_{n+k}| < \varepsilon$ if $n \geq n_0$

$p:\mathbb{N}\to\mathbb{N}$, $n\to p(n)$, $p(n)<p(m)$ if $n<m$, $\sum_{j=1}^{\infty}a_j$ define $b_1=a_1+\cdots+a_{p(1)}$ $b_2=a_{p(1)+1}+\cdots+a_{p(2)}$ $\cdots$ $b_n=a_{p(n-1)+1}+\cdots+a_{p(n)}$

$\sum_{n=1}^{\infty}b_n$ is obtained from $\sum_{j=1}^{\infty}a_n$ by inserting parentheses $(a_1+\cdots+a_{p(1)})+(a_{p(1)+1}+\cdots+a_{p(2)})+\cdots+(a_{p(n-1)+1}+\cdots+a_{p(n)})+\cdots$

$\sum_{j=1}^{\infty}a_n$ 〃 $\sum_{n=1}^{\infty}b_n$ by removing parentheses

Thm: If $\sum_{j=1}^{\infty}a_j$ conv to L then any series $\sum_{n=1}^{\infty}b_n$ obtained from $\sum_{j=1}^{\infty}a_j$ by inserting parentheses

pf: $t_m=\sum_{n=1}^{m}b_n=\sum_{j=1}^{p(m)}a_j=S_{p(m)}$ $\{S_n\}\to L$

Ex: $\sum_{n=1}^{\infty}(1-1)=(1-1)+(1-1)+\cdots$ remove the parentheses $1-1+1-1\cdots$

Thm: let $\sum_{n=1}^{\infty}b_n$ be obtained from $\sum_{j=1}^{\infty}a_j$ by inserting parentheses. Suppose $\sum_{n=1}^{\infty}b_n$ converge to L and $\exists M>0$ s.t. $p(n+1)-p(n)\le M$ $\forall n$ and that $\lim_{j\to\infty}a_j=0$ then $\sum_{j=1}^{\infty}a_j$ also conv to L

pf: let $L=\sum_{n=1}^{\infty}b_n$, $t_m=\sum_{n=1}^{m}b_n$ given $\varepsilon>0$ $\exists n_0\in\mathbb{N}$ $|t_m-L|<\varepsilon$ $|a_j|<\varepsilon$ $j\ge n_0$

For $k\ge p(n_0)$ $\exists\, l\ge n_0$ s.t. $p(l)\le k<p(l+1)$

$$\left|\sum_{j=1}^{k}a_j-L\right|=\left|\sum_{j=1}^{p(l)}a_j+a_{p(l)+1}+\cdots+a_k-L\right|\le\left|\sum_{j=1}^{p(l)}a_j-L\right|+|a_{p(l)+1}|+\cdots+|a_k|=\left|\sum_{n=1}^{l}b_n-L\right|+|a_{p(l)+1}|+\cdots+|a_k|<\varepsilon+\varepsilon\cdot M=\varepsilon(1+M)$$

Alternating series $a_n>0$ $\forall n$ $\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4\cdots$

Thm: $a_n>0$ $a_n\searrow 0$ then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ conv to S, $0<(-1)^n(S-S_n)<a_{n+1}$

pf: let $b_n=a_{2n-1}-a_{2n}>0$ $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}(a_{2n-1}-a_{2n})$ $0<t_m=(a_1-a_2)+\cdots+(a_{2m-1}-a_{2m})=a_1-\underset{\vee\,0}{(a_2-a_3)}-\cdots-\underset{\vee\,0}{(a_{2m-2}-a_{2m-1})}-\underset{\vee\,0}{a_{2m}}<a_1$

$0<(-1)^n(S-S_n)=(-1)^n\left((-1)^{n+1+1}a_{n+1}-(-1)^{n+3}a_{n+2}\cdots\right)=a_{n+1}-a_{n+2}+a_{n+3}-a_{n+4}+\cdots=a_{n+1}-(a_{n+2}-a_{n+3})-\cdots<a_{n+1}$

Def: $\sum_{n=1}^{\infty}a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty}|a_n|$ converges

Thm: $\sum_{n=1}^{\infty}a_n$ converge if it converges absolutely

pf: $\left|\sum_{n=m}^{m+k}a_n\right|\le\sum_{n=m}^{m+k}|a_n|<\varepsilon$ given $\varepsilon>0$ $\exists n_0\in\mathbb{N}$ s.t. if $n\ge n_0$

Ex: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ cov but $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\cdot\frac{1}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$ diverge

Thm: $\sum_{n=1}^{\infty}a_n$ conditionally conv $a_n^+=\frac{|a_n|+a_n}{2}\ge 0$ $a_n^-=\frac{|a_n|-a_n}{2}\ge 0$ then $\sum_{n=1}^{\infty}a_n^+$, $\sum_{n=1}^{\infty}a_n^-$ diverge

pf: if not $\sum_{n=1}^{\infty}\frac{|a_n|+a_n}{2}+\sum_{n=1}^{\infty}\left(\frac{-a_n}{2}\right)=\sum_{n=1}^{\infty}\frac{|a_n|}{2}$ conv $(\to\leftarrow)$

$c_n=a_n+ib_n$ $a_n,b_n\in\mathbb{R}$ $\sum_{n=1}^{\infty}c_n=\sum_{n=1}^{\infty}a_n+i\sum_{n=1}^{\infty}b_n$

$b_n>0$ $a_n=O(b_n)$ $a_n$ is big oh of $b_n$, ie $\exists M>0$ s.t. $|a_n|\le Mb_n$ $\forall n$

$a_n=o(b_n)$ little 〃, ie, $\lim_{n\to\infty}\frac{a_n}{b_n}=0$

No.
Date: / /

Comparison. $a_n > 0$ $b_n > 0$ $\forall n$

① $a_n \le b_n$ $\forall n$ $\sum b_n$ conv $\Rightarrow \sum a_n$ conv $\quad \sum a_n$ div $\Rightarrow \sum b_n$ conv

② $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L > 0$ then $\sum a_n$ conv $\Leftrightarrow \sum b_n$ conv

pf: choose $0 < \varepsilon < L$ $\quad 0 < L-\varepsilon < \frac{a_n}{b_n} < L+\varepsilon \quad (L-\varepsilon)b_n < a_n < (L+\varepsilon)b_n$

Ex: $\sum\limits_{k=0}^{\infty} x^k = \frac{1}{1-x}$ $\quad |x|<1$ conv $\quad |x| \ge 1$ div

Ex: $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{2^k}+\frac{1}{3^k}$ $\begin{cases} a_{2k} = \frac{1}{3^k} \\ a_{2k-1} = \frac{1}{2^k} \end{cases}$ ratio test $\frac{a_{2k}}{a_{2k-1}} = \left(\frac{2}{3}\right)^k \to 0 \quad \frac{a_{2k-1}}{a_{2k-2}} = \frac{1}{3}\left(\frac{3}{2}\right)^k \to \infty \quad k\to\infty$

root test $|a_n|^{\frac{1}{n}}$ $\quad |a_{2k}|^{\frac{1}{2k}} = \frac{1}{\sqrt{3}} \quad |a_{2k-1}|^{\frac{1}{2k-1}} = \left(\frac{1}{2}\right)^{\frac{k}{2k-1}} \to \frac{1}{\sqrt{2}}$

Integral test $f: [1,\infty) \to \mathbb{R}$ positive decreasing. let $s_n = \sum\limits_{k=1}^{n} f(k)$ $t_n = \int_1^n f(x)dx$, let $d_n = s_n - t_n$, then 12/30

(Cauchy test: $a_n > 0$ $a_n \ge a_{n+1}$ $\forall n$ $\sum\limits_{n=1}^{\infty} a_n$ converge $\Leftrightarrow \sum\limits_{k=1}^{\infty} 2^k a_{2^k}$ converge)

① $0 < f(n+1) \le d_{n+1} \le d_n \le f(1)$ ② $\lim\limits_{n\to\infty} d_n$ exsits ③ $\sum\limits_{n=1}^{\infty} f(n)$ converge $\Leftrightarrow \{t_n\}$ converge ④ $0 \le d_k - \underbrace{\lim\limits_{n\to\infty} d_n}_{D} \le f(k)$

pf ① $0 < f(n+1) = s_{n+1} - s_n \le s_{n+1} - t_{n+1} = d_{n+1} \le d_n \le f(1)$

④ $d_n - d_{n+1} = s_n - t_n - (s_{n+1} - t_{n+1}) = t_{n+1} - t_n - (s_{n+1} - s_n) = \int_n^{n+1} f(x)dx - f(n+1)$ $\begin{cases} \ge f(n+1)\int_n^{n+1} dx - f(n+1) = f(n+1) - f(n+1) = 0 \\ \le f(n)\int_n^{n+1} dx - f(n+1) = f(n) - f(n+1) \end{cases}$

$\sum\limits_{n=k}^{\infty} (d_n - d_{n+1}) \le \sum\limits_{n=k}^{\infty} (f(n) - f(n+1)) \Rightarrow d_k - \lim\limits_{n\to\infty} d_n \le f(k)$

Ex: $f(x) = \frac{1}{x}$ $[1,\infty)$ $\quad s_n = \sum\limits_{k=1}^{n} f(k) = \sum\limits_{k=1}^{n} \frac{1}{k}$ $\quad t_n = \int_1^n \frac{1}{x}dx = \log n$ $\quad d_n = s_n - t_n = \sum\limits_{k=1}^{n} \frac{1}{k} - \log n$ $\quad \lim\limits_{n\to\infty} d_n = \lim\limits_{n\to\infty}\left(\sum\limits_{k=1}^{n} \frac{1}{k} - \log n\right) = \gamma$ {Euler const}

$d_n - \gamma \le f(n)$ $\quad \sum\limits_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}dx - \gamma \le f(n)$ $\quad \sum\limits_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O\left(\frac{1}{n}\right)$ $\quad (\log n = O(\frac{1}{n}))$

$\sum\limits_{k=1}^{\infty} 2^k \cdot a_{2^k} = \sum\limits_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \sum\limits_{k=1}^{\infty} 1 \to \infty$

Ratio test · Root test : $\sum\limits_{n=1}^{\infty} a_n$ $a_n \ne 0$ $a_n \in \mathbb{C}$ $\quad \liminf\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r \quad \limsup\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = R \quad \limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$

① $R<1$ : converge ② $r>1$ diverge ③ $r \le 1 \le R$ inconclusive $\quad$ ① $\rho<1$ : converge ② $\rho>1$ : diverge ③ $\rho=1$ : inconclusive

root
① $\rho < \rho_1 < 1$ $\exists n_0 \in \mathbb{N}$ s.t. $n \ge n_0$ $|a_n|^{\frac{1}{n}} < \rho_1$ $\quad |a_n| < \rho_1^{\,n}$ $\quad \sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{n_0-1} |a_n| + \sum\limits_{n=n_0}^{\infty} |a_n| \le \sum\limits_{n=1}^{n_0-1} |a_n| + \rho_1^{\,n_0} \frac{1}{1-\rho}$

② $\rho > 1 \Rightarrow$ infinitely many $|a_n|^{\frac{1}{n}} \ge 1 \Rightarrow |a_n| \ge 1 \Rightarrow \lim\limits_{n\to\infty} a_n \ne 0$ ③ Ex: $\rho = 1$ $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ · $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ $\quad \lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1 \quad \lim\limits_{n\to\infty} \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1$

ratio
① $R<1 \Rightarrow R<R'<1$ $\exists n_0$ s.t. $n \ge n_0$ $\left|\frac{a_{n+1}}{a_n}\right| < R' \Rightarrow |a_{n+1}| < R'|a_n|$ $\quad |a_{n_0+k}| < (R')^k |a_{n_0}|$ $\quad \sum\limits_{k=n_0}^{\infty} |a_n| \le |a_{n_0}| \frac{1}{1-R'}$

② $r>1 \Rightarrow r > 1+\varepsilon > 1$ $\exists n_0$ s.t. $n \ge n_0$ $\left|\frac{a_{n+1}}{a_n}\right| > 1+\varepsilon \Rightarrow |a_{n+1}| > (1+\varepsilon)|a_n|$ $\quad |a_{n_0+k}| > (1+\varepsilon)^k |a_{n_0}|$ $\quad \lim\limits_{n\to\infty} |a_n| \ge |a_{n_0}| > 0$

③ $r \le 1 \le R$ : Ex $\frac{1}{n}$ 、 $\frac{1}{n^2}$ ratio $=1$

$\{a_n\}, \{b_n\}$ : $\sum_{n=1}^{\infty} a_n b_n$ let $\sum_{k=1}^{n} a_k = A_n$ $A_0 = 0$

$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1} = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1})$ (Abel summation formula)

Thm (Dirichlet) Suppose $\sum_{n=1}^{\infty} a_n$ has bounded partial sum and $b_n \searrow 0$ then $\sum_{n=1}^{\infty} a_n b_n$ converge Ex: $\sum_{n=1}^{\infty} \underbrace{(-1)^{n+1}}_{a_n} \underbrace{\frac{1}{n}}_{b_n}$

pf: $\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1})$ $\lim_{n\to\infty} |A_n||b_{n+1}| \le C \lim_{n\to\infty} |b_{n+1}| = 0$

$\sum_{k=1}^{n} |A_k (b_k - b_{k+1})| \le C \sum_{k=1}^{n} (b_k - b_{k+1}) = C(b_1 - b_{n+1}) \to C b_1$ as $n \to \infty$

Thm (Abel) Suppose $\sum_{n=1}^{\infty} a_n$ converges and $\{b_n\}$ is a monotonic convergent seq then $\sum_{n=1}^{\infty} a_n b_n$ converges

pf: $\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1})$ $\lim_{n\to\infty} A_n b_{n+1} = \left(\sum_{n=1}^{\infty} a_n\right) L$ $L = \lim_{n\to\infty} b_n$

$\sum_{k=1}^{n} |A_k (b_k - b_{k+1})| \le C \sum_{k=1}^{n} (b_k - b_{k+1}) = C(b_1 - b_{n+1}) \to C(b_1 - L)$

Fourier series $[0, 2\pi]$ $f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$

Lemma: $x \neq 2m\pi$ $m \in \mathbb{Z}$, $\sum_{k=1}^{n} e^{ikx} = 1 + e^{ix} + e^{i2x} + \cdots + e^{inx} - 1 = \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}} \cdot e^{i(\frac{n+1}{2})x}$

pf: $= \frac{e^{i(n+1)x} - 1}{e^{ix} - 1} - 1 = \frac{e^{i(n+1)x} - e^{ix}}{e^{ix} - 1} = \frac{e^{i(\frac{n}{2}+1)x}}{e^{i\frac{x}{2}}} \cdot \frac{e^{i\frac{n}{2}x} - e^{-i\frac{n}{2}x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}} = e^{i(\frac{n+1}{2})x} \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}$

$\sum_{k=1}^{n} \cos kx = \frac{\sin\frac{n}{2}x \cdot \cos\frac{n+1}{2}x}{\sin\frac{x}{2}} \Rightarrow \left|\sum_{k=1}^{n} \cos kx\right| \le \frac{1}{|\sin\frac{x}{2}|}$ $x \neq 2m\pi$

$\sum_{k=1}^{n} \sin kx = \frac{\sin\frac{n}{2}x \cdot \sin\frac{n+1}{2}x}{\sin\frac{x}{2}} \Rightarrow \left|\sum_{k=1}^{n} \sin kx\right| \le \frac{1}{|\sin\frac{x}{2}|}$ $x \neq 2m\pi$

$\sum_{k=1}^{n} e^{i(2k-1)x} = e^{-ix} \sum_{k=1}^{n} e^{i2kx} = e^{-ix} \frac{\sin nx}{\sin x} \cdot e^{i(n+1)x} = \frac{\sin nx}{\sin x} e^{inx}$

$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin nx}{\sin x} \cos nx = \frac{\sin 2nx}{\sin x}$ $x \neq m\pi$, $\sum_{k=1}^{n} \sin(2k-1)x = \frac{\sin nx}{\sin x} \sin nx = \frac{\sin^2 nx}{\sin x}$ $x \neq m\pi$

Rearrangement: $\phi: \mathbb{N} \to \mathbb{N}$ 1-1, onto, $\sum_{k=1}^{n} a_n$, define $b_n = a_{\phi(n)}$, $b_{\phi^{-1}(n)} = a_n$, $\sum_{n=1}^{\infty} b_n$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$

Thm: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ also converge absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n = S$

pf: $\sum_{n=1}^{m} |b_n| = \sum_{n=1}^{m} |a_{\phi(n)}| \le \sum_{n=1}^{N} |a_n| \le \sum_{n=1}^{\infty} |a_n| < \infty$ $N = \max\{\phi(1), \cdots, \phi(m)\}$

given $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $\sum_{n=n_0+1}^{\infty} |a_n| < \frac{\varepsilon}{2}$, $|S - \sum_{n=1}^{n_0} a_n| < \frac{\varepsilon}{2}$, find $\mathbb{N}$ s.t. $\{1, 2, \dots, n_0\} \subseteq \{\phi(1), \dots, \phi(m)\}$

$\left|\sum_{n=1}^{m} b_n - S\right| \le \left|\sum_{n=1}^{m} b_n - S_{n_0}\right| + |S_{n_0} - S| = \left|\sum_{n=1}^{m} a_{\phi(n)} - \sum_{n=1}^{n_0} a_n\right| + |S_{n_0} - S| \le \sum_{n=n_0+1}^{\infty} |a_n| + |S_{n_0} - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thm (Riemann) Suppose $\sum_{n=1}^{\infty} a_n$ converges conditionally let $-\infty \le a \le b \le \infty$, then $\exists$ rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$, $t_n = \sum_{k=1}^{n} b_k$

s.t. $\limsup_{n\to\infty} t_n = b$, $\liminf_{n\to\infty} t_n = a$

pf: $\sum_{n=1}^{\infty} a_n = L$ $\sum_{n=1}^{\infty} |a_n| = \infty$, $\lim_{n\to\infty} a_n = 0$, let $b > 0$, has infinitely many term of $a_n > 0$ and $a_n < 0$

Subseries: $\sum_{n=1}^{\infty} a_n$ $f: \mathbb{N} \to \mathbb{N}$ 1-1, define $b_n = a_{f(n)}$, $\sum_{n=1}^{\infty} b_n$ is a subseries of $\sum_{n=1}^{\infty} a_n$

Thm: $\sum_{n=1}^{\infty} a_n$ converge absolutely then any subseries $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ also converge absolutely

pf: $\left|\sum_{n=1}^{m} b_n\right| = \left|\sum_{n=1}^{m} a_{f(n)}\right| \le \left|\sum_{k=1}^{\infty} a_k\right| < \infty$

Thm: $\sum_{n=1}^{\infty} a_n$ converges absolutely, for each $k \in \mathbb{N}$ let $f_k: \mathbb{N} \to \mathbb{N}$ 1-1, Assume $\bigcup_{k=1}^{\infty} f_k(\mathbb{N}) = \mathbb{N}$, $f_k(\mathbb{N}) \cap f_j(\mathbb{N}) = \emptyset$ (disjoint)

define $b_k(n) = a_{f_k(n)}$ then ① $\sum_{n=1}^{\infty} b_k(n)$ converge absolutely and $\sum_{k=1}^{\infty} b_k(n) = S_k$

② $\sum_{k=1}^{\infty} S_k$ converge absolutely and $\sum_{k=1}^{\infty} S_k = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} b_k(n) = \sum_{n=1}^{\infty} a_n = S$

pf ② $\sum_{k=1}^{m} |S_k| \le \sum_{k=1}^{m} \sum_{n=1}^{\infty} |a_{f_k(n)}| \le \sum_{n=1}^{\infty} |a_n| < \infty$

given $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $\sum_{k=n_0+1}^{\infty} |a_k| < \frac{\varepsilon}{2}$ $|S - \sum_{k=1}^{n_0} a_k| < \frac{\varepsilon}{2}$ $\{1, 2, \dots, n_0\} \subseteq \bigcup_{k=1}^{r} f_k(\mathbb{N})$ $r$: large

$\sum_{k=1}^{r} S_k = \sum_{n=1}^{\infty} a_{f_1(n)} + \cdots + \sum_{n=1}^{\infty} a_{f_r(n)}$ find $m > r$

$\left|\sum_{k=1}^{m} S_k - \sum_{k=1}^{\infty} a_k\right| \le \left|\sum_{k=1}^{m} S_k - \sum_{n=1}^{n_0} a_n\right| + \left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{n_0} a_n\right| \le \sum_{k=n_0+1}^{\infty} |a_k| + \left|\sum_{n=1}^{n_0} a_n - \sum_{n=1}^{\infty} a_n\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

1 st midterm exam: March 21 (T)   2 nd midterm: April 28 (F)   Final: June 9 (F)

Double sequence: $f: \mathbb{N}\times\mathbb{N} \to \mathbb{C}$ or $\mathbb{R}$, $(m,n) \mapsto f(m,n)$

Def: The double sequence $\{f(m,n)\}$ is said to converge to $a$, if given $\varepsilon>0$ $\exists N>0$ s.t. $|f(m,n)-a|<\varepsilon$ for $m>N, n>N$

Thm: Let $\{f(m,n)\}$ be a double seq, suppose $\lim_{n\to\infty} f(m,n) = a_m$ exsits for each $m$ and $\lim_{m,n\to\infty} f(m,n) = A$ then $\lim_{m\to\infty}(\lim_{n\to\infty} f(m,n)) = A$

pf: given $\varepsilon>0$ $\exists N>0$ s.t. $|f(m,n)-A|<\varepsilon$ for $m>N, n>N$

$\therefore$ If $m>N$ same $\varepsilon$ $\exists N_2>0$ s.t. $|a_m - f(m,n)|<\varepsilon$ if $n\geq \underset{\text{may assume}}{\underline{N_2>N}}$

$|a_m - A| = |a_m - f(m,n) + f(m,n) - A| \leq |a_m - f(m,n)| + |f(m,n)-A| < \varepsilon+\varepsilon = 2\varepsilon$

Ex: $f(m,n) = \frac{mn}{m^2+n^2}$  $\lim_{n\to\infty} f(m,n) = 0$, $\lim_{m\to\infty} f(m,n) = 0 \Rightarrow \lim_{m\to\infty}(\lim_{n\to\infty} f(m,n)) = \lim_{n\to\infty}(\lim_{m\to\infty} f(m,n)) = 0$

$m=n$  $f(m,n) = \frac{m^2}{2m^2} = \frac{1}{2}$   $m=2n$  $f(m,n) = \frac{2n^2}{(2n)^2+n^2} = \frac{2}{5}$   $\lim_{m,n\to\infty} f(m,n)$ DNE

Def: let $\{f(m,n)\}$ be a double seq, let $S(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q} f(m,n)$

We say the double series converge to $s$ if $\lim_{p,q\to\infty} S(p,q) = s$

Thm: If $f(m,n)\geq 0$ $\forall m,n$ then $\sum_{m,n} f(m,n)$ converges $\Leftrightarrow$ $\{S(p,q)\}$ is bounded (above)

pf: ($\Leftarrow$) $\{S(n,n)\}$  $S(n+1,n+1)\geq S(n,n)$  $\therefore S(n,n)\nearrow A<\infty$

claim: $\sum_{m,n} f(m,n) = \lim_{p,q\to\infty} S(p,q) = A$, given $\varepsilon>0$ $\exists n_0>0$ s.t. $|S(n,n)-A|<\varepsilon$ if $n\geq n_0$

For $m>n_0, n>n_0$ $|S(m,n)-A| \leq |S(n,n)-A| < \varepsilon$.

Thm: Suppose $\sum_{m,n}|f(m,n)|$ converge then $\sum_{m,n} f(m,n)$ converge

Def: let $g: \mathbb{N}\to\mathbb{N}\times\mathbb{N}$ 1-1 and onto  define $G: \mathbb{N}\to\mathbb{C}$, $G(n)\to f(g(n))$  $G(n)$ is called an arrangement of the double seq $f(m,n)$

Thm: let $\{f(m,n)\}$ be a double seq $G(n)$ be an arrangement of $\{f(m,n)\}$

then ① $\sum G(n)$ converges absolutely $\Leftrightarrow$ $\sum_{m,n} f(m,n)$ converge absolutely

pf: let $T_k = \sum_{n=1}^{k}|G(n)| = |G(1)|+\cdots+|G(k)|$  $S(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q}|f(m,n)| \Rightarrow$ each $k$, $T_k \leq S(p,q)$ for some $p,q$,

each $p,q$, $S(p,q)\leq T_{k'}$ for some $k'$

Assume $\sum_{m,n} f(m,n)$ converges absolutely with sum $\sum_{m,n} f(m,n) = S$, ② $\sum_{n=1}^{\infty} G(n) = S = \sum_{m,n} f(m,n)$

pf: $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} G(n) = S'$   $\sum_{m,n} f(m,n) = S$

let $t_k = \sum_{n=1}^{k} G(n)$  $T = \sum_{m,n}|f(m,n)| = \lim_{p,q\to\infty} S(p,q) = \lim_{p,q\to\infty}\sum_{m=1}^{p}\sum_{n=1}^{q}|f(m,n)|$

given $\varepsilon>0$ $\exists N>0$ s.t. $T - S(p,q) < \varepsilon$ for $p>N, q>N$   known $\lim_{k\to\infty} t_k = S'$  claim $\lim_{k\to\infty} t_k = S$

$\exists M>0$ s.t. if $k>M$ $f(m,n)\in\{G(1),\cdots,G(k)\}$ $m,n\leq N+1$

$|t_k - S| \leq |t_k - S(N+1,N+1)| + |S - S(N+1,N+1)| \leq (T - S(N+1,N+1)) + (T - S(N+1,N+1)) < 2\varepsilon$

Thm: Given $\sum_{m,n} f(m,n)$, suppose $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|f(m,n)|$ converges then

① $\sum_{m,n} f(m,n)$ converge absolutely ② $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)$, $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n)$ absolutely converge and $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{m,n} f(m,n)$

Thm: $\sum a_n \cdot \sum b_n$ converge absolutely then $\sum_{m,n} a_m b_n$ converge absolutely $\sum_{m,n} a_m b_n = (\sum_{m=1}^{\infty} a_m)(\sum_{n=1}^{\infty} b_n)$

pf: $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_m b_n| = \sum_{m=1}^{\infty}\left(|a_m|\left(\sum_{n=1}^{\infty}|b_n|\right)\right) = \left(\sum_{m=1}^{\infty}|a_m|\right)\left(\sum_{n=1}^{\infty}|b_n|\right) < \infty$

Define $c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{\alpha+\beta=n} a_\alpha b_\beta$ $\sum_{n=0}^{\infty} c_n$ is called the Cauchy product of $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$

Thm (Mertens) Suppose $\sum_{n=0}^{\infty} a_n$ converges absolutely and $\sum_{n=0}^{\infty} b_n$ converges then the Cauchy product $\sum_{n=0}^{\infty} c_n$ conv $\sum_{n=0}^{\infty} c_n = (\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n)$

pf: $\sum_{n=0}^{\infty}|a_n| = C$ let $B = \sum_{n=0}^{\infty} b_n$ let $R_m = B - \sum_{n=0}^{m} b_n$ let $A = \sum_{n=0}^{\infty} a_n$

$\sum_{n=0}^{k} c_n = (a_0 b_0) + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_{k-1} + \cdots + a_{k-1} b_0) + (a_0 b_k + \cdots + a_k b_0)$

$= a_0 B_k + a_1 B_{k-1} + \cdots + a_{k-1} B_1 + a_k B_0 = a_0(B - R_k) + a_1(B - R_{k-1}) + \cdots + a_{k-1}(B - R_1) + a_k(B - R_0)$

$= (a_0 + \cdots + a_k)B - (a_0 R_k + a_1 R_{k-1} + \cdots + a_{k-1} R_1 + a_k R_0) = A_k B - (a_0 R_k + a_1 R_{k-1} + \cdots + a_{k-1} R_1 + a_k R_0)$

$k \to \infty$ $k > N$ $\underbrace{a_0 R_k + a_1 R_{k-1} + \cdots + a_N R_{k-N}}_{\mathrm{I}} + \underbrace{a_{N+1} R_{k-(N+1)} + \cdots + a_k R_0}_{\mathrm{II}}$ $A_k B = AB$

$|\mathrm{II}| \le |a_{N+1}||R_{k-N-1}| + \cdots + |a_k||R_0| \le M(|a_{N+1}| + \cdots + |a_k|) < M\varepsilon$ (given $\varepsilon$ $\exists N$)

$|\mathrm{I}| \le |a_0||R_k| + \cdots + |a_N||R_{k-N}| < \varepsilon(|a_0| + \cdots + |a_N|) < \varepsilon \cdot C$

Cesàro sum: $\sum_{n=1}^{\infty} a_n$ $S_m = \sum_{n=1}^{m} a_n$ $m = 1, 2, \cdots$ Define $\sigma_n = \frac{S_1 + \cdots + S_n}{n}$

If $\lim_{n\to\infty} \sigma_n = L$ exsits then we say $\sum_{n=1}^{\infty} a_n$ is Cesaro summable ((C,1)-summable) to the Cesaro sum $L$

Ex: $1-1+1-1+1-1+\cdots$ $S_m = \sum_{n=1}^{m}(-1)^{n+1} = \begin{cases} 1 & m: \text{odd} \\ 0 & m: \text{even} \end{cases}$ $\sigma_n = \frac{S_1 + \cdots + S_n}{n} = \begin{cases} \frac{n}{2}/n & n: \text{even} \\ (\frac{n-1}{2}+1)/n & n: \text{odd} \end{cases} = \begin{cases} \frac{1}{2} \\ \frac{n+1}{2n} \end{cases} \to \frac{1}{2}$

Thm: Suppose $\sum_{n=1}^{\infty} a_n = A$, then $\sum_{n=1}^{\infty} a_n$ is Cesaro summable, and $\sum_{n=1}^{\infty} a_n = A$ (C,1)

pf: $\sigma_n = \frac{S_1 + \cdots + S_n}{n}$ consider $\sigma_n - A = \frac{S_1 + \cdots + S_n - nA}{n} = \frac{(S_1 - A) + \cdots + (S_n - A)}{n} = \underbrace{\frac{(S_1 - A) + \cdots + (S_N - A)}{n}}_{\mathrm{I}} + \underbrace{\frac{(S_{N+1} - A) + \cdots + (S_n - A)}{n}}_{\mathrm{II}}$ $n > N$

$A - S_k = \sum_{n=k+1}^{\infty} a_n$ given $\varepsilon > 0$ $\exists N > 0$ s.t. $|A - S_k| < \varepsilon$ if $k > N$. $|\mathrm{II}| \le \frac{|A - S_{N+1}| + \cdots + |A - S_n|}{n} \le \frac{(n-N)\varepsilon}{n} < \varepsilon$

$|\mathrm{I}| \le \frac{|A - S_1| + \cdots + |A - S_N|}{n} < \varepsilon$ if $n$ is large

Ex: $a_n = (-1)^{n+1} n$ $\sum_{n=1}^{\infty} a_n = 1 - 2 + 3 - 4 + \cdots$ $S_{2k} = -k$ $S_{2k-1} = S_{2k-2} + a_{2k-1} = -k + 1 + (2k-1) = k$

$\sigma_n = \frac{S_1 + \cdots + S_n}{n}$ $\sigma_{2k} = \frac{S_1 + S_2 + \cdots + S_{2k-1} + S_{2k}}{2k} = 0$ $\sigma_{2k+1} = \frac{S_{2k+1}}{2k+1} = \frac{k+1}{2k+1}$ $\limsup \sigma_n = \frac{1}{2}$ $\liminf \sigma_n = 0$

Ex: $|z| = 1$ $\sum_{n=0}^{\infty} z^n$, If $z \ne 1$ $S_k = \sum_{n=0}^{k-1} z^n = \frac{1 - z^k}{1 - z} = \frac{1}{1-z} - \frac{z^k}{1-z}$

$\sigma_n = \frac{S_0 + \cdots + S_{n-1}}{n} = \frac{1}{1-z} - \frac{1}{1-z} \cdot \frac{1}{n} \sum_{k=0}^{n-1} z^k = \frac{1}{1-z} - \frac{1}{1-z} \cdot \frac{1}{n} \cdot \frac{1 - z^n}{1 - z} \to \frac{1}{1-z}$

If $z \ne 1$ $|z| = 1$ $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, (C,1) let $z = -1$ $\frac{1}{(1-(-1))} = \frac{1}{2}$

Abel summability: consider (#) $a_0+a_1x+\cdots+a_kx^k+\cdots$, Assume (#) conv for $0<x<1$ to $\sigma(x)$

If $\lim_{x\to1^-}\sigma(x)=S$ exsits, then we say $\sum_{n=0}^{\infty}a_n$ is Abel summable to $S$

Thm (Abel limit thm): If $\sum_{n=0}^{\infty}a_n=L$ exsits, then $\sum_{n=0}^{\infty}a_n$ is Abel summable to $L$

Thm: If $\sum_{n=0}^{\infty}a_n$ is Cesàro summable to $S$ then $\sum_{n=0}^{\infty}a_n$ is Abel summable to $S$

Ex: $1-1+1-1+\cdots$ consider: $1-x+x^2-x^3+\cdots=\frac{1}{1-(-x)}=\frac{1}{1+x}$ $\lim_{x\to1^-}\frac{1}{1+x}=\frac{1}{2}$

product: $\{u_n\}_{n=1}^{\infty}$, $u_n\in\mathbb{R}$ or $\mathbb{C}$, $P_m=\prod_{n=1}^{m}u_n$ mth partial product $u_n$: the factor of the product

Def: ① If there are infinitely many $u_n=0$ then we say the product diverges to zero

② Assume $u_n\neq0$ $\forall n$, ie $P_m\neq0$ $\forall m$ (i) If $\{P_m\}$ conv to a $P\neq0$ then we say the product conv to $P$

(ii) If $\{P_m\}$ conv to zero, then we say the product diverge to zero

③ Assume $u_n\neq0$ for $n\geq N$ then the conv or divergence is define in ② for $\prod_{n=N}^{\infty}u_n$ and the value of product is $u_1\cdots u_{N-1}\cdot\prod_{n=N}^{\infty}u_n$

④ the product diverges if does not conv as in ② and ③

Ex: $\prod_{n=1}^{\infty}(1+\frac{1}{n})=\prod_{n=1}^{\infty}(\frac{n+1}{n})=\lim_{m\to\infty}\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{m+1}{m}=\lim_{m\to\infty}m+1=+\infty$ div

$\prod_{n=2}^{\infty}(1-\frac{1}{n})=\prod_{n=2}^{\infty}(\frac{n-1}{n})=\lim_{m\to\infty}\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{m-1}{m}=\lim_{m\to\infty}\frac{1}{m}=0$ div

Thm (Cauchy condition): $\prod_{n=1}^{\infty}u_n$ conv iff given $\varepsilon>0$ $\exists N>0$ s.t. for $n>N$, $k\in\mathbb{N}$ $|u_{n+1}\cdot u_{n+2}\cdots u_{n+k}-1|<\varepsilon$

pf: ($\Rightarrow$) $\prod_{n=1}^{\infty}u_n$ conv. may assume $u_n\neq0$ $\forall n$ $\therefore P_m=\prod_{n=1}^{m}u_n\neq0$ $\forall m$ $\therefore \lim_{m\to\infty}P_m=P\neq0$

$\therefore \exists M>0$ s.t. $|P_m|\geq M$ $|P|\geq M$ $\forall m$, given $\varepsilon>0$ $\exists N\in\mathbb{N}$ s.t. $n>N$ $|P_{n+k}-P_n|<\varepsilon M$

$\frac{|P_{n+k}-P_n|}{|P_n|}<\frac{\varepsilon M}{|P_n|}$ $|u_{n+1}\cdots u_{n+k}-1|<\varepsilon$

($\Leftarrow$) $\varepsilon<1$ $k=1$ $\exists N>0$ $n>N$ $|u_{n+1}-1|<\varepsilon$ $u_n\neq0$

given $\varepsilon=\frac{1}{2}$ $\exists N_0$ let $q_n=u_{N_0+1}\cdots u_n$, $|q_n-1|<\frac{1}{2}$ $\Rightarrow \frac{1}{2}<|q_n|<\frac{3}{2}$

given $\varepsilon>0$ $\exists N=N(\varepsilon)$ for $n>N$ $|u_{n+1}\cdots u_{n+k}-1|<\varepsilon$ $\left|\frac{u_{N_0+1}\cdots u_n\cdot u_{n+1}\cdots u_{n+k}}{u_{N_0+1}\cdots u_n}-1\right|<\varepsilon$

$\left|\frac{q_{n+k}}{q_n}-1\right|<\varepsilon$ $|q_{n+k}-q_n|<\varepsilon|q_n|<\varepsilon\frac{3}{2}$ $\{q_n\}$ Cauchy $q_n\to L\neq0$

Thm: Suppose $a_n\geq0$ then $\prod_{n=1}^{\infty}(1+a_n)$ conv $\Leftrightarrow$ $\sum_{n=1}^{\infty}a_n$ conv

pf: $x\geq0$ $1+x\leq e^x$ $a_1+\cdots+a_m\leq P_m=(1+a_1)\cdots(1+a_m)\leq e^{a_1+\cdots+a_m}$

Def: $\prod_{n=1}^{\infty}(1+a_n)$ conv abs if $\prod_{n=1}^{\infty}(1+|a_n|)$ conv

Thm: Suppose $\prod_{n=1}^{\infty}(1+a_n)$ conv abs then $\prod_{n=1}^{\infty}(1+a_n)$ conv

pf: $|(1+a_{n+1})\cdots(1+a_{n+k})-1|<(1+|a_{n+1}|)\cdots(1+|a_{n+k}|)-1<\varepsilon$

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Thm: Suppose $a_n \geq 0$ then $\prod_{n=1}^{\infty}(1-a_n)$ conv $\Leftrightarrow \sum_{n=1}^{\infty} a_n$ conv

pf: ($\Leftarrow$) $\sum_{n=1}^{\infty} a_n$ conv $\Rightarrow \prod_{n=1}^{\infty}(1+a_n)$ conv $\Rightarrow \prod_{n=1}^{\infty}(1-a_n)$ conv

($\Rightarrow$) $\prod_{n=1}^{\infty}(1-a_n)$ conv $\Rightarrow \lim_{n\to\infty} a_n = 0$ If $\sum_{n=1}^{\infty} a_n = \infty$ may assume $0 \leq a_n < \frac{1}{2}$ $\forall n$ $\therefore 1-a_n \geq \frac{1}{2}$

$(1-a_n)(1+a_n) = 1-a_n^2 \leq 1$ $(1-a_1)\cdots(1-a_n) \leq \frac{1}{(1+a_1)\cdots(1+a_n)} \to 0$ as $n\to\infty$ div ($\to\leftarrow$)

Riemann-Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ $s\in\mathbb{R}$ $s>1$

2/24 Thm: For $s>1$ $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{\text{primes}} \frac{1}{1-(\frac{1}{p_k})^s}$

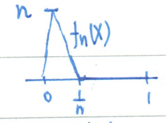
pf: $P_m = \prod_{p_1\cdots p_m}\left(\frac{1}{1-(\frac{1}{p_k})^s}\right) = \prod_{p_1\cdots p_m}\left(1+\frac{1}{p_k^s}+\frac{1}{p_k^{2s}}+\cdots\right) = 1+\cdots+\frac{1}{p_1^{a_1 s}\cdots p_m^{a_m s}}+\cdots$

$\zeta(s) - P_m = \sum_{\substack{n:\text{ has no} \\ \text{prime factor } p_1\cdots p_m}} \frac{1}{n^s} \leq \sum_{n>p_m}\frac{1}{n^s} \to 0$ as $m\to\infty$ ($P_m\to\infty$), $\sum_{k=1}^{m}\left(\frac{1}{p_k^s}+\frac{1}{p_k^{2s}}+\cdots\right) \leq \sum_{n=1}^{\infty}\frac{1}{n^s} = \zeta(s)$

## Sequence of Functions

Ex: $f_n(x) = x^n \quad 0 \le x \le 1 \qquad f_n(x) \xrightarrow[n\to\infty]{} f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

Ex: $f_n(x) = \frac{x^{2n}}{1+x^{2n}} \quad x \in \mathbb{R} \qquad f_n(x) \xrightarrow[n\to\infty]{} f(x) = \begin{cases} 0 & -1<x<1 \\ \frac{1}{2} & x = \pm 1 \\ 1 & x<-1,\ x>1 \end{cases}$

Ex: 


$\int_0^1 f_n(x)\,dx = \frac{1}{2} \qquad f_n(x) \to f(x) \equiv 0 \qquad \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \frac{1}{2} \ne 0 = \int_0^1 f(x)\,dx$

Ex: $f_n(x) = \frac{\sin x}{\sqrt{n}} \to f(x) \equiv 0 \qquad \left|\frac{\sin nx}{\sqrt{n}} - 0\right| \le \frac{1}{\sqrt{n}} \to 0 \qquad f_n'(x) = \sqrt{n}\cos x \quad f'(x) \equiv 0 \quad f_n'(0) = \sqrt{n} \to \infty$

$x = \frac{q}{p}(2\pi) \quad p, q \in \mathbb{N} \quad n \to \infty \quad n = pk \quad k \to \infty \quad f_{pk}'(x) = \sqrt{pk}\cdot\cos\left((p\cdot k)\frac{q}{p}\cdot(2\pi)\right) = \sqrt{pk}\cdot\cos((qk)(2\pi)) = \sqrt{pk} \to \infty$

pointwise convergence at $x$: given $\varepsilon>0\ \exists N = N(\varepsilon, x) \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ if $n \ge N$

uniform convergence: given $\varepsilon > 0\ \exists N = N(\varepsilon) \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ if $n \ge N\ \forall x \in S$

$\{f_n(x)\}$ uniformly bounded $|f_n(x)| < M \quad \forall n\ \forall x \in S$

Thm: $f_n$: conv to $f$ uniformly on $S$ let $p \in S$, assume $f_n$ is conti at $p$ then $f$ is cont at $p$

pf: given $\varepsilon > 0\ \exists n_0$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ if $n \ge n_0\ \forall x$

choose $n = n_0$ same $\varepsilon>0\ \exists \delta > 0$ s.t. $|f_{n_0}(p) - f_{n_0}(x)| < \frac{\varepsilon}{3}$ if $|x-p|<\delta$

$\therefore$ same $\varepsilon > 0\ \exists \delta>0$ for $|x-p|<\delta \quad |f(x) - f(p)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(p)| + |f_n(p) - f(p)| < \varepsilon$

$\sum_{n=1}^{\infty} f_n(x) \to f(x)$, $S_m(x) = \sum_{n=1}^{m} f_n(x) \to f(x)$

$f_n(x) \to f(x)$ uniform $\Leftrightarrow$ (Cauchy condition) given $\varepsilon>0\ \exists n_0$ s.t. $|f_j(x) - f_k(x)| < \varepsilon$ for $j, k \ge n_0\ \forall x$

($\Rightarrow$) given $\varepsilon > 0\ \exists n_0$ s.t. $n \ge n_0 \quad |f_n(x) - f(x)| < \varepsilon\ \forall x \quad |f_j(x) - f_k(x)| \le |f_j(x) - f(x)| + |f(x) - f_k(x)| < 2\varepsilon$

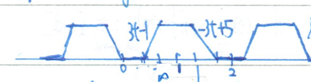
($\Leftarrow$) $\{f_n(x)\}$ Cauchy seq $\forall x \quad f_n(x) \to f(x) \quad \forall \varepsilon>0\ \exists n_0$ s.t. $n \ge n_0 \quad |f_n(x) - f_{n+k}(x)| < \varepsilon \quad \forall k \in \mathbb{N}\ \forall x$

let $k \to \infty \quad \therefore |f_n(x) - f(x)| \le \varepsilon \quad \forall x \in S$

Thm: Suppose $0 \le |f_n(x)| \le M_n\ \forall n, x$, If $\sum_{n=1}^{\infty} M_n$ conv then $\sum_{n=1}^{\infty} f_n(x)$ conv uniformly

pf: $\left|\sum_{j=n}^{n+k} f_j(x)\right| \le \sum_{j=n}^{n+k} |f_j(x)| \le \sum_{j=n}^{n+k} M_j < \varepsilon$

Space filling curve: construct a conti map $f = (f_1, f_2) : [0,1] \to [0,1]\times[0,1]$ s.t. $f$ is onto




$\phi(t) = \phi(t+2)$
$0 \le \phi \le 1$

define $f(t) = (f_1(t), f_2(t)) = \left(\sum_{n=1}^{\infty} \frac{\phi(3^{2n-2}t)}{2^n}, \sum_{n=1}^{\infty} \frac{\phi(3^{2n-1}t)}{2^n}\right)$

$0 \le f_1(t) \le \sum_{n=1}^{\infty} \frac{1}{2^n} = 1, \quad 0 \le f_2(t) \le 1 \qquad \left|\frac{\phi(3^{2n-2}t)}{2^n}\right| \le \frac{1}{2^n} \Rightarrow f_1, f_2$ conti on $[0,1]$

$(a,b) \in [0,1]\times[0,1] \quad a = \sum_{n=1}^{\infty}\frac{a_n}{2^n} \quad b = \sum_{n=1}^{\infty} \frac{b_n}{2^n} \quad$ let $c = 2\sum_{n=1}^{\infty} \frac{c_n}{3^n} \quad c_{2n-1} = a_n \quad c_{2n} = b_n \quad c \le 2\sum_{n=1}^{\infty}\frac{1}{3^n} = 2\cdot\frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}} = 1$

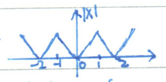
claim: $f(c) = (a,b)$: suffice to show $\phi(3^k c) = c_{k+1} \quad \phi(3^{2n-2}\cdot c) = c_{2n-1} = a_n \quad \phi(3^{2n-1}\cdot c) = c_{2n} = b_n$

$\phi(3^k C) = \phi(3^k\cdot 2\cdot(\sum_{n=1}^{k}\frac{c_n}{3^n} + \frac{c_{k+1}}{3^{k+1}} + \frac{c_{k+2}}{3^{k+2}}+\cdots)) = \phi(2\cdot(\frac{c_{k+1}}{3}+\frac{c_{k+2}}{3^2}+\cdots)) = c_{k+1}$

$c_{k+1}=0$ $\quad 2(\frac{c_{k+2}}{3^2}+\cdots) \le 2(\frac{1}{3^2}+\frac{1}{3^3}+\cdots) = \frac{2}{9}\cdot\frac{1}{1-\frac13} = \frac13$

$c_{k+1}=1$ $\quad =\phi(\frac23 + 2(\frac{c_{k+2}}{3^2}+\cdots)) = 1 = c_{k+1}$

3/3 Thm: There exsits a conti func on $\mathbb{R}$ which is nowhere differentiable

pf:  $\phi(x+2)=\phi(x)$ $\quad |\phi(x)-\phi(y)|\le|x-y|$ Lipchitz $\quad |\phi(x)|\le 1$ $\forall x\in\mathbb{R}$

define $f(x)=\sum_{n=0}^\infty (\frac34)^n\phi(4^n x) \le \sum_{n=0}^\infty(\frac34)^n<\infty$ conti func on $\mathbb{R}$

Fix $x\in\mathbb{R}$, For each $m\in\mathbb{N}$, let $\delta_m=\pm\frac12\cdot\frac1{4^m}$ the sign is chosen so that ∄ int between $4^m x$ and $4^m(x+\delta_m)$

$(4^m(x+\delta_m)=4^m x+4^m\delta_m$, $\quad |4^m\delta_m| = 4^m\cdot\frac12\cdot\frac1{4^m}=\frac12)$; $\quad |\delta_m|=\frac12\cdot\frac1{4^m}\to0$ as $m\to\infty$

consider $\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| = \frac{|\sum_{n=0}^\infty(\frac34)^n(\phi(4^n(x+\delta_m))-\phi(4^n x))|}{|\delta_m|}$ $\qquad 4^n(x+\delta_m)-4^n x = \pm 4^n\cdot\frac12\cdot\frac1{4^m}$ even num if $n>m$

$= \frac{|\sum_{n=0}^m (\frac34)^n(\phi(4^n(x+\delta_m))-\phi(4^nx))|}{|\delta_m|} \ge \frac{1}{\cancel{|\delta_m|}}\left((\frac34)^m 4^m\cancel{|\delta_m|} - \sum_{n=0}^{m-1}(\frac34)^n 4^n \cancel{|\delta_m|}\right)$

$=3^m-\sum_{n=0}^{m-1}3^n = 3^m-\frac12 3^m+\frac12 = \frac12(3^m+1)\to\infty$

Thm: Let $\alpha$ be of BV on $[a,b]$ Assume ① $f_n\in R(\alpha)$ $n=1,2,\cdots$ ② $f_n$ conv uniformly to $f$ on $[a,b]$

let $g_n(x)=\int_a^x f_n(t)\,d\alpha(t)$ $a\le x\le b$, then ① $f\in R(\alpha)$ ② $g_n(x)$ conv uniformly to $g(x)=\int_a^x f(t)\,d\alpha(t)$

i.e. $\lim_{n\to\infty} g_n(x)=g(x)$ $\quad \lim_{n\to\infty}\int_a^x f_n(t)\,d\alpha(t)=\int_a^x\lim_{n\to\infty}f_n(t)\,d\alpha(t)$

pf: may assume $\alpha\nearrow$ $f$ is bounded write $f=(f-f_N)+f_N$, $\sup f-\inf f\le \sup(f-f_N)-\inf(f-f_N)+\sup f_N-\inf f_N$

$U(P,f,\alpha)-L(P,f,\alpha)\le U(P,f-f_N,\alpha)-L(P,f-f_N,\alpha)+U(P,f_N,\alpha)-L(P,f_N,\alpha) < \varepsilon(\alpha(b)-\alpha(a))+\varepsilon$

given $\varepsilon>0$ $\exists N$ s.t. $|f(x)-f_n(x)|<\varepsilon$ if $n\ge N$ $\quad\therefore U(P,f-f_N,\alpha)-L(P,f-f_N,\alpha)<\varepsilon(\alpha(b)-\alpha(a))$.

For this $N$ $\exists P_\varepsilon$ s.t. if $P\ge P_\varepsilon$ then $|g_n(x)-g(x)|=\left|\int_a^x(f_n(t)-f(t))\,d\alpha(t)\right|\le\int_a^x|f_n(t)-f(t)|\,d\alpha(t)$

$<\varepsilon\int_a^x d\alpha(t)\le\varepsilon(\alpha(b)-\alpha(a))\to0$

Ex: $f_n(x)=x^n$, on $[0,1]$ $\quad f_n(x)\to f(x)=\begin{cases}0, & 0\le x<1\\ 1, & x=1\end{cases}$ $\quad \int_0^1 f_n(x)dx=\int_0^1x^ndx=\frac1{n+1}\to0$ $\quad \int_0^1f(x)dx=0$

Def: $\{f_n\}$ is boundedly conv if $\{f_n\}$ is uniformly bounded and $f_n$ conv pointwise

Thm: $\{f_n\}$ is boundly conv to $f$ on $[a,b]$ ① Assume $f_n, f\in R$, let $P=\{a=x_0<\cdots<x_m=b\}$

② let $[c,d]\subseteq[a,b]$ assume $[c,d]\cap P=\emptyset$ then $f_n$ conv uniformly on $[c,d]$ then $\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^bf(x)dx=\int_a^b\lim_{n\to\infty}f_n(x)\,dx$

Thm: Suppose ① $f_n(x)$ is diff with finite derivative on $(a,b)$ $n=1,2,\cdots$ ② $f_n(x_0)$ conv for some $x_0$

③ $f_n'(x)$ conv uniformly to $g(x)$ on $(a,b)$

then (i) $f_n(x)$ conv uniform to $f(x)$ on $(a,b)$ (ii) $f'(x)=g(x)$ on $(a,b)$ $f'(x) = (\lim_{n\to\infty} f(x))' = \lim_{n\to\infty} f_n'(x)$

pf: pick $c\in(a,b)$ define $g_n(x) = \begin{cases} \frac{f_n(x)-f_n(c)}{x-c} & x\neq c \\ f_n'(c) & x=c\end{cases}$ ∴ ① $g_n \in C(a,b)$ ② $g_n$ conv uniformly $g_n(c)=f_n'(c)$ conv

$x\neq c$ $g_n(x)-g_m(x) = \frac{f_n(x)-f_n(c)}{x-c} - \frac{f_m(x)-f_m(c)}{x-c} = \frac{(f_n(x)-f_m(x))-(f_n(c)-f_m(c))}{x-c} \overset{MVT}{=} f_n'(\bar{x}) - f_m'(\bar{x})$ $c<\bar{x}<x$

∴ $\lim_{n\to\infty} g_n(x) = G(x) \in C(a,b)$ $\lim_{x\to c} G(x) = G(c)$, to show $f_n(x)$ conv uniformly we choose $c=x_0$

∴ $f_n(x) = f_n(x_0) + (x-x_0)\, g_n(x)$ done ∴ $f_n(x)\to f(x)\in C(a,b)$

∴ $G(x) = \lim_{n\to\infty} g_n(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & x\neq c\\ g(c) & x=c\end{cases}$ ∴ $\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = G(c)=g(c)$

Thm {Weierstrass M-test} If $|f_k(x)|\le M_k$ $\forall x, k$ $\sum_{k=1}^{\infty} M_k$ conv $\Rightarrow$ $\sum_{k=1}^{\infty} f_k(x)$ conv unif 3/7

pf: given $\varepsilon>0$ $\exists n_0$ s.t. $|f_n(x)+\cdots+f_{n+k}(x)| \le |f_n(x)|+\cdots+|f_{n+k}(x)| \le |M_n|+\cdots+|M_{n+k}| < \varepsilon$ $n\ge n_0$.

Thm (Dirichlet) let $F_n(x)=\sum_{k=1}^{n} f_k(x)$. Suppose $\{F_n(x)\}$ is unif bounded on a set E

let $\{g_n(x)\}$ be a seq of func on E s.t. $g_k(x)\ge g_{k+1}(x)$ $\forall x\in E$ $\forall k$ and $g_k(x)\to 0$ unif

then $\sum_{k=1}^{\infty} f_k(x)g_k(x)$ conv unif on E

pf: $S_m(x) = \sum_{k=1}^{m} f_k(x)g_k(x) = \sum_{k=1}^{m} (F_k(x)-F_{k-1}(x))g_k(x) = \sum_{k=1}^{m} F_k(x)g_k(x) - \sum_{k=1}^{m} F_{k-1}(x)g_k(x) = \sum_{k=1}^{m} F_k(x)(g_k(x)-g_{k+1}(x)) + F_m(x)g_{m+1}(x)$

$S_n(x)-S_{n+p}(x) = \sum_{k=1}^{n} F_k(x)(g_k(x)-g_{k+1}(x)) + F_n(x)g_{n+1}(x) - \left(\sum_{k=1}^{n+p} F_k(x)(g_k(x)-g_{k+1}(x)) + F_{n+p}(x)g_{n+p+1}(x)\right)$

$= \sum_{k=n+1}^{n+p} F_k(x)(g_k(x)-g_{k+1}(x)) + F_n(x)g_{n+1}(x) - F_{n+p}(x)g_{n+p+1}(x)$

$|S_n(x)-S_{n+p}(x)| \le \sum_{k=n+1}^{n+p}|F_k(x)|(g_k(x)-g_{k+1}(x)) + |F_n(x)|g_{n+1}(x) + |F_{n+p}(x)|g_{n+p+1}(x) \le C(g_{n+1}(x) - \cancel{g_{n+p+1}(x)} + g_{n+1}(x) + \cancel{g_{n+p+1}}) \to 0$

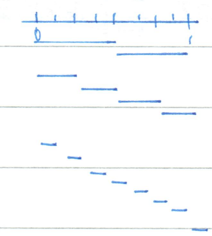
Thm: $f(m,n)$ double seq $g_n(m)=f(m,n)$, assume $\lim_{n\to\infty} g_n(m)=g(m)$ unif $\lim_{m\to\infty} g(m)=A$ then $\lim_{m,n\to\infty} f(m,n)=A$

pf: given $\varepsilon>0$ $\exists N_1>0$ s.t. $|g_n(m)-g(m)|<\frac{\varepsilon}{2}$ if $n\ge N_1$ $\exists N_2>0$ s.t. $m\ge N_2$ $|g(m)-A|<\frac{\varepsilon}{2}$

$|\underbrace{f(m,n)}_{g_n(m)}-A| \le |\underbrace{f(m,n)}_{g_n(m)} - g(m)| + |g(m)-A| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

Convergence in the mean: $f_n\in R$ on $[a,b]$ $f\in R$ on $[a,b]$ we say $f_n$ conv in the mean to f if $\lim_{n\to\infty}\int_a^b |f_n(x)-f(x)|^2dx=0$

Ex: [0,1]



$\underset{n\to\infty}{\text{l.i.m}}\, f_n = 0$ $\lim_{n\to\infty}\int_0^1|f_n(x)-0|^2dx = \lim_{k\to\infty}\frac{1}{2^k}=0$

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Thm: $\underset{n\to\infty}{\text{l.i.m.}} f_n = f$ on $[a,b]$ $g\in R$ on $[a,b]$ let $h_n(x)=\int_a^x f_n(t)g(t)dt$ $h(x)=\int_a^x f(t)g(t)\,dt$ then $h_n\to h$ unif

pf: $|h_n(x)-h(x)|\le\left(\int_a^x|f_n(t)-f(t)|^2dt\right)^{\frac12}\left(\int_a^x|g(t)|^2dt\right)^{\frac12}\le\left(\int_a^b|f_n(t)-f(t)|^2dt\right)^{\frac12}\left(\int_a^b|g(t)|^2dt\right)^{\frac12}<\varepsilon\left(\int_a^b|g(t)|^2dt\right)$

Thm: $\underset{n\to\infty}{\text{l.i.m}} f_n=f$ $\underset{n\to\infty}{\text{l.i.m.}} g_n=g$ on $[a,b]$ let $h_n(x)=\int_a^x f_n(t)g_n(t)dt$ $h(x)=\int_a^x f(t)g(t)dt$ then $h_n\to h$ unif

pf: $f_n(t)g_n(t)-f(t)g(t)=(f_n(t)-f(t))(g_n(t)-g(t))+f(t)(g_n(t)-g(t))+g(t)(f_n(t)-f(t))$

$h_n(x)-h(x)=\int_a^x(f_n(t)g_n(t)-f(t)g(t))dt=\int_a^x(f_n(t)-f(t))(g_n(t)-g(t))dt+\int_a^x f(t)\underbrace{(g_n(t)-g(t))dt}_{\to 0}+\int_a^x g(t)\underbrace{(f_n(t)-f(t))}_{\to 0}dt$

Power Series $a_0+\sum_{k=1}^{\infty}a_k(z-z_0)^k\to\sum_{k=0}^{\infty}a_k(z-z_0)^k$ $a_0\cdot a_n\cdot z\cdot z_0\in\mathbb{C}$ disc of conv,

Thm: $\sum_{k=1}^{\infty}a_k(z-z_0)^k$ let $\lambda=\overline{\lim}|a_k|^{\frac1k}$ $r=\frac1\lambda$

then $\sum_{k=0}^{\infty}a_k(z-z_0)^k$ conv at every $z\in B(z_0;r)$. div at every $z\notin\overline{B(z_0;r)}$, con unif on every compact set of $B(z_0;r)$

3/10 pf: $\overline{\lim}|a_n(z-z_0)^n|^{\frac1n}=\overline{\lim}|a_n|^{\frac1n}|z-z_0|=\frac{|z-z_0|}{r}$

$z\in k$ $|z-z_0|\le\rho<r$ $\sum_{n=0}^{\infty}|a_n(z-z_0)^n|\le\sum_{n=0}^{\infty}|a_n|\rho^n$ . $\overline{\lim}(|a_n|\rho^n)^{\frac1n}=\frac{\rho}{r}<1$

$\overset{\to f(z)}{\sum_{k=0}^{\infty}a_n(z-z_0)^n}$ conv on $B(z_0;r)$ $f\in C(B(z_0;r))$ , $B(z_1;R)\subseteq B(z_0;r)$ $f(z)=\sum_{n=0}^{\infty}b_n(z-z_1)^n$

$\sum_{n=0}^{\infty}a_n(z-z_0)^n=\sum_{n=0}^{\infty}a_n(z-z_1+z_1-z_0)=\sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\binom{n}{k}(z-z_1)^k(z_1-z_0)^{n-k}$

$\left[\sum_{n=0}^{\infty}|a_n|\sum_{k=0}^{n}\binom{n}{k}|z-z_1|^k|z_1-z_0|^{n-k}=\sum_{n=0}^{\infty}|a_n|\left(|z-z_1|+|z_1-z_0|+z_0-z_0\right)^n,\ |z-z_1|+|z_1-z_0|<R+|z_1-z_0|\le r\right]$

$=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}a_n\binom{n}{k}(z-z_1)^k(z_1-z_0)^{n-k}=\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}a_n\binom{n}{k}(z-z_0)^{n-k}\right)(z-z_1)^k$

Ex: $f(x)=\frac{1}{1-x}$ $x=1$. singularity

$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n=\sum_{n=0}^{\infty}b_n(z-z_1)^n=b_0+b_1(z-z_1)+\sum_{n=2}^{\infty}b_n(z-z_1)^n$

$\frac{f(z)-f(z_1)}{z-z_1}=b_1+\sum_{n=2}^{\infty}b_n(z-z_1)^{n-1}$ $\lim_{z\to z_1}\frac{f(z)-f(z_1)}{z-z_1}=b_1=\sum_{n=1}^{\infty}n\cdot a_n(z_1-z_0)^{n-1}=f'(z_1)\Rightarrow f'(z)=\sum_{n=1}^{\infty}na_n(z-z_0)^{n-1}$

$f^{(k)}(z)=\sum_{n=k}^{\infty}a_n\cdot n\cdot(n-1)(n-2)\cdots(n-k+1)(z-z_0)^{n-k}$ $f^{(k)}(z_0)=a_k k!$ $a_k=\frac{f^{(k)}(z_0)}{k!}$

$f(z)=\sum_{n=0}^{\infty}a_nz^n$ $B(0;r)$ $g(z)=\sum_{n=0}^{\infty}b_nz^n$ $B(0;R)$ $z\in B(0;r)\cap B(0;R)$ $f(z)g(z)=\sum_{n=0}^{\infty}c_nz^n$ $c_n=\sum_{j+k=n}a_jb_k$

$f(z)=\sum_{n=0}^{\infty}a_nz^n$ $B(0;r)$ $f^2(z)=\sum_{n=0}^{\infty}a_n(2)z^n$ $a_n(2)=\sum_{i+k=n}a_ia_k$ $f^p(z)=\sum_{n=0}^{\infty}a_n(p)z^n$ $a_n(p)=\sum_{i_1+\cdots+i_p=n}a_{i_1}\cdots a_{i_p}$

Thm: $f(z)=\sum_{n=0}^{\infty}a_nz^n$ on $B(0;r)$ $g(z)=\sum_{n=0}^{\infty}b_nz^n$ on $B(0;R)$ . If $\sum_{n=0}^{\infty}|b_n||z|^n<r$ $z\in B(0;R)$ then $f(g(z))=\sum_{k=0}^{\infty}c_kz^k$

pf: $g(z)^p=\sum_{n=0}^{\infty}b_n(p)z^n$ $b_n(p)=\sum_{i_1+\cdots+i_p=n}b_{i_1}\cdots b_{i_p}$ , $f(g(z))=\sum_{p=0}^{\infty}a_p(g(z))^p=\sum_{p=0}^{\infty}a_p\sum_{n=0}^{\infty}b_n(p)z^n=\sum_{n=0}^{\infty}\left(\sum_{p=0}^{\infty}a_pb_n(p)\right)z^n$

$|b_n(p)|\le\sum_{i_1+\cdots+i_p=n}|b_{i_1}|\cdots|b_{i_p}|=B_n(p)$ $\left(\sum_{n=0}^{\infty}|b_n|z^n\right)^p=\sum_{n=0}^{\infty}B_n(p)z^n$

$\sum_{p=0}^{\infty}|a_p|\sum_{n=0}^{\infty}|b_n(p)||z|^n\le\sum_{p=0}^{\infty}|a_p|\sum_{n=0}^{\infty}B_n(p)|z|^n=\sum_{p=0}^{\infty}|a_p|\left(\sum_{n=0}^{\infty}|b_n|\cdot|z|^n\right)^p<\infty$

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Thm: $f(z)=a_0+a_1z+\cdots+a_nz^n+\cdots$ $B(0;r)$ $a_0\neq 0$, then $\frac{1}{f(z)}=g_0+g_1z+\cdots+g_nz^n+\cdots$ $B(0;\delta)$ and $a_0g_0=1$

pf: may assume $a_0=1$, $\frac{1}{f(z)}=\frac{1}{1-(1-f(z))}$ $\quad 1-f(z)=-a_1z-a_2z^2-\cdots$ $\quad (\exists\,\delta>0$ s.t. $z\in B(0;\delta))$

$|a_1z|+|a_2z^2|+\cdots<1$, $\frac{1}{1-\omega}=1+\omega+\omega^2+\cdots$ $\quad =1+g(z)+g^2(z)+\cdots$

$\sum_{n=0}^{\infty}a_n(x-x_0)^n$ given $f\in C^\infty(I)$ I: open interval $x_0\in I$ $\quad f\sim\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

Ex: $f(x)=\begin{cases}e^{-\frac{1}{x^2}} & x\neq 0\\ 0 & x=0\end{cases}$ $\quad f^{(k)}(0)=0$ $\quad \sum_{k=1}^{\infty}\frac{f^{(k)}(0)}{k!}\cdot x^k\equiv 0$

Ex: $f(x)=\sum_{n=0}^{\infty}e^{-n}\cdot\cos n^2x\in C^\infty$ $\quad \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$

If $x\neq 0$ $\quad \frac{|f^{(2k)}(0)|}{(2k)!}x^{2k}=\frac{\sum_{n=0}^{\infty}e^{-n}n^{4k}}{(2k)!}\geq\frac{(n^2x)^{2k}}{(2k)^{2k}}e^{-n}$ $\quad$ let $n=2k$ $\quad \frac{(2k)^{4k}\cdot x^{2k}}{(2k)^{2k}}e^{-2k}=\left(\frac{2kx}{e}\right)^{2k}\to\infty$

$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k+\frac{f^{(n)}(\bar{x})}{n!}(x-c)^n$ $\quad \lim_{n\to\infty}\frac{f^{(n)}(\bar{x})}{n!}(x-c)^n\neq 0$ $\quad$ 3/14

Thm: I: open $c\in I$ $x\in I$ $f\in C^{n+1}(I)$ then $f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k+E_n(x)$ $\quad E_n(x)=\frac{1}{n!}\int_c^x(x-t)^nf^{(n+1)}(t)\,dt$

pf: $n=1$ $\quad f(x)=f(c)+f'(c)(x-c)+E_1(x)$ $\quad \therefore E_1(x)=f(x)-f(c)-f'(c)(x-c)=\int_c^x(f'(t)-f'(c))\,dt$

$\int_c^x(f'(t)-f'(c))\,dt=\int_c^x(f'(t)-f'(c))\,d(t-x)=\cancel{(f'(t)-f'(c))(t-x)\Big|_c^x}-\int_c^x(t-x)f''(t)\,dt=\int_c^x(x-t)f''(t)\,dt$

assume $E_n$ is done consider $E_{n+1}$, $E_{n+1}(x)=E_n(x)-\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}=\frac{1}{n!}\int_c^x(x-t)^nf^{(n+1)}(t)\,dt-\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$

$\frac{1}{n!}\int_c^x(x-t)^nf^{(n+1)}(t)\,dt-\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}=\frac{1}{n!}\int_c^xf^{(n+1)}(t)\,d\left(-\frac{(x-t)^{n+1}}{n+1}\right)-\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$

$=\frac{1}{n!}\left(-\frac{1}{n+1}(x-t)^{n+1}f^{(n+1)}(t)\right)\Big|_c^x+\frac{1}{(n+1)!}\int_c^x(x-t)^{n+1}f^{(n+2)}(t)\,dt-\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$

$=\cancel{\frac{1}{(n+1)!}(x-c)^{n+1}f^{(n+1)}(c)}+\frac{1}{(n+1)!}\int_c^x(x-t)^{n+1}f^{(n+2)}(t)\,dt-\cancel{\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}}$

$E_n(x)=\frac{1}{n!}\int_c^x(x-t)^nf^{(n+1)}(t)\,dt=\frac{1}{n!}\int_1^0(x-c)^nu^nf^{(n+1)}(x+(c-x)u)\,du=\frac{(x-c)^{n+1}}{n!}\int_0^1u^nf^{(n+1)}(x+(c-x)u)\,du$

Thm (Bernstein) let $f\in C^\infty[b,b+r]$ $r>0$. Assume $\underline{f^{(k)}(x)\geq 0}$ $\forall\,k=0,1,2\ldots$ $\forall\,x\in[b,b+r]$

then for $b\leq x<b+r$, $f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(b)}{k!}(x-b)^k$

pf: assume $b=0$, $f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+E_n(x)$, $E_n=\frac{x^{n+1}}{n!}\int_0^1u^nf^{(n+1)}(x-xu)\,du$

$\frac{E_n(x)}{x^{n+1}}=\frac{1}{n!}\int_0^1u^nf^{(n+1)}(x(1-u))\,du\leq\frac{E_n(r)}{r^{n+1}}\leq\frac{f(r)}{r^{n+1}}$ $\Rightarrow E_n(x)\leq x^{n+1}\frac{f(r)}{r^{n+1}}=\left(\frac{x}{r}\right)^{n+1}f(r)\to 0$ as $n\to\infty$

$a\in\mathbb{R}$ $\quad (1+x)^a=\sum_{k=0}^{\infty}\binom{a}{k}x^k$ $-1<x<1$ $\quad \binom{a}{k}=\frac{a(a-1)\cdots(a-k+1)}{k!}$ $\quad f(x)=\frac{1}{(1-x)^a}=(1-x)^{-a}$ $\quad a>0$ $\quad -1\leq x<1$

$f'(x)=(-a)(-1)(1-x)^{-a-1}=a(1-x)^{-a-1}$ $\quad f''=a(a+1)(1-x)^{-a-2}$ $\quad f^{(k)}=a(a+1)\cdots(a+k-1)(1-x)^{-a-k}$

$-1\leq x\leq 1-\varepsilon$ $\quad f=\frac{1}{(1-x)^a}=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=\sum_{k=0}^{\infty}\binom{-a}{k}x^k$

$(1+x)^{-a}=\sum_{k=0}^{\infty}\binom{-a}{k}(-1)^kx^k=\sum_{k=0}^{\infty}\binom{-a}{k}x^k$ $\quad (1+x)^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$ $\alpha<0$

Q&C

Thm (Abel's limit Thm) Suppose $\sum_{n=0}^{\infty} a_n$ conv, let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ $-1<x<1$ then $\lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$

pf: $S = \sum_{k=0}^{\infty} a_k$ $S_n = \sum_{k=0}^{n} a_k$ $-1<x<1$ . $\sum_{n=0}^{\infty} a_n x^n = (1-x)\cdot\frac{1}{(1-x)}\sum_{n=0}^{\infty} a_n x^n = (1-x)(\sum_{n=0}^{\infty} x^n)(\sum_{n=0}^{\infty} a_n x^n) = (1-x)\sum_{n=0}^{\infty}(a_0+\cdots+a_n)x^n = (1-x)\sum_{n=0}^{\infty} S_n x^n$

$\sum_{n=0}^{\infty} a_n x^n - S = (1-x)\sum_{n=0}^{\infty} S_n x^n - (1-x)\sum_{n=0}^{\infty} S x^n = (1-x)\sum_{n=0}^{\infty}(S_n - S)x^n = \underbrace{(1-x)\sum_{n=0}^{n_0}(S_n-S)x^n}_{\mathrm{I}} + \underbrace{(1-x)\sum_{n=n_0+1}^{\infty}(S_n-S)x^n}_{\mathrm{II}}$

$|\mathrm{II}| \leq (1-x)\sum_{n=n_0+1}^{\infty}|S_n - S|x^n < \varepsilon(1-x)\sum_{n=n_0+1}^{\infty} x^n = \varepsilon(1-x)\cdot\frac{x^{n_0+1}}{(1-x)} \leq \varepsilon$

$|\mathrm{I}| \leq (1-x)\sum_{n=0}^{n_0}|S_n - S|x^n < \varepsilon$ if $x \to 1$ $\because \sum_{n=0}^{n_0}|S_n - S|x^n$ finite term

3/17 Thm: If $\Sigma a_n \cdot \Sigma b_n \cdot \Sigma c_n$ conv then $\sum_{n=0}^{\infty} c_n = (\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n)$ $c_n = \sum_{j+k=n} a_j b_k$

pf: $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty} c_n x^n$ $-1<x<1$ let $x \to 1^-$ $(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty} c_n$

Thm: Suppose $\sum_{n=0}^{\infty} a_n$ is Cesàro summable to L then $\sum_{n=0}^{\infty} a_n$ is also Abel summable to L

pf: let $S_n = \sum_{k=0}^{n} a_k$ $\sigma_{n+1} = \frac{S_1+\cdots+S_n}{n+1} \to L$ , Assume $\sum_{n=0}^{\infty} a_n x^n \to f(x)$ conv on $(-1,1)$

show $\lim_{x\to 1^-} f(x) = L$ , $\sum_{n=0}^{\infty} a_n x^n - L = (1-x)\sum_{k=0}^{\infty} x^k (\sum_{k=0}^{\infty} a_k x^k - L) = (1-x)\sum_{k=0}^{\infty}(a_0+\cdots+a_k)x^k - (1-x)\sum_{k=0}^{\infty} L x^k = (1-x)\sum_{k=0}^{\infty}(S_k - L)x^k$

$(1-x)\sum_{k=0}^{\infty}(S_k - L)x^k = (1-x)^2\sum_{n=0}^{\infty} x^n(\sum_{k=0}^{\infty}(S_k-L)x^k) = (1-x)^2\sum_{k=0}^{\infty}((S_0-L)+\cdots+(S_k-L))x^k = (1-x)^2\sum_{k=0}^{\infty}((S_0+\cdots+S_k)-(k+1)L)x^k$

$= (1-x)^2\sum_{k=0}^{\infty}(k+1)(\sigma_{k+1}-L)x^k = \underbrace{(1-x)^2\sum_{k=0}^{n_0-1}(k+1)(\sigma_{k+1}-L)x^k}_{\mathrm{I}} + \underbrace{(1-x)^2\sum_{k=n_0}^{\infty}(k+1)(\sigma_{k+1}-L)x^k}_{\mathrm{II}}$

$|\mathrm{II}| \leq (1-x)^2\varepsilon\sum_{k=n_0}^{\infty}(k+1)x^k = (1-x)^2\varepsilon(\sum_{k=n_0}^{\infty} x^{k+1})' = \varepsilon(1-x)^2(\frac{x^{n_0+1}}{1-x})' = \varepsilon\cancel{(1-x)^2}\frac{(n_0+1)x^{n_0}(1-x)+x^{n_0+1}}{\cancel{(1-x)^2}}$

$= \varepsilon(n_0+1)x^{n_0}(1-x) + \varepsilon x^{n_0+1} \overset{x\to 1^-}{<} \varepsilon+\varepsilon = 2\varepsilon$ , $|\mathrm{I}| \leq \varepsilon$ if $x \to 1^-$

Thm (Tauber) $f(x) = \sum_{n=0}^{\infty} a_n x^n$ conv on $(-1,1)$ . If $\lim_{x\to 1^-} f(x) = S$ and $a_n = o(\frac{1}{n})$ (ie $\lim_{n\to\infty} n a_n = 0$) then $\sum_{n=0}^{\infty} a_n = S$

pf: let $S_m = \sum_{n=0}^{m} a_n$ estimate $S_m - S = S_m - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^n - S$

$S_m - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{m} a_n - \sum_{n=0}^{m} a_n x^n - \sum_{n=m+1}^{\infty} a_n x^n = \sum_{n=0}^{m} a_n(1-x^n) - \sum_{n=m+1}^{\infty} a_n x^n = \underbrace{\sum_{n=0}^{m} a_n(1-x)(1+x+\cdots+x^{n-1})}_{\mathrm{I}} - \underbrace{\sum_{n=m+1}^{\infty} a_n x^n}_{\mathrm{II}}$

$|\mathrm{I}| \leq \sum_{n=0}^{m}|a_n|(1-x)n = (1-x)\sum_{n=0}^{m} n|a_n| = (1-x)(m+1)\sigma_{m+1}$ , $\lim_{n\to\infty} n|a_n| = 0$ $\lim n a_n = 0$ $\therefore \sigma_{m+1} = \frac{\sum_{n=0}^{m} n|a_n|}{m+1} \to 0$

$|\mathrm{II}| \leq \sum_{n=m+1}^{\infty}|a_n|x^n \leq \varepsilon\sum_{n=m+1}^{\infty}\frac{x^n}{n} \leq \frac{\varepsilon}{m+1}\sum_{n=m+1}^{\infty} x^n = \frac{\varepsilon}{m+1}x^{m+1}\cdot\frac{1}{1-x} < \frac{\varepsilon}{m+1}\cdot\frac{1}{\frac{1}{n}} < \varepsilon$

(given $\varepsilon>0$ $\exists n_0$ for $n>n_0$ ① $n|a_n|<\varepsilon$ ($|a_n|<\frac{\varepsilon}{n}$) ② let $x_n = 1-\frac{1}{n}$ $|f_n(x)-S|<\varepsilon$ ③ $\sigma_n<\varepsilon$)

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## Multiple Differential Calculus 重微分

Directional derivative: $f: U \underset{\mathbb{R}^n}{\overset{open}{\subseteq}} \to \mathbb{R}$ $\vec{c} \in U$ $\vec{u}$: vector If $\lim_{t\to 0} \frac{f(\vec{c}+t\vec{u})-f(\vec{c})}{t}$ exsits, denoted by $f'(\vec{c};\vec{u})$

$\vec{u} = \vec{u}_k$: unit coordinate vector $f'(\vec{c};\vec{u}) = D_k f(\vec{c}) = \frac{\partial f}{\partial x_k}(\vec{c})$

$f: U \underset{\mathbb{R}^n}{\subseteq} \to \mathbb{R}^m: x \mapsto f(x) = (f_1(x), \dots, f_m(x))$ $f'(\vec{c};\vec{u}) = (f_1'(\vec{c};\vec{u}), \dots, f_m'(\vec{c};\vec{u}))$

If: $f: \mathbb{R}^n \to \mathbb{R}^m$ is linear, $\lim_{t\to 0} \frac{f(\vec{c}+t\vec{u})-f(\vec{c})}{t} = \lim_{t\to 0} \frac{\cancel{f(\vec{c})}+\cancel{t}f(\vec{u})-\cancel{f(\vec{c})}}{\cancel{t}} = f(\vec{u})$

$f: \mathbb{R}\to\mathbb{R}$ $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exsits ($f: \mathbb{R}^n \to \mathbb{R}$ not work $\overset{?}{\underset{0}{\phantom{x}}}$)

$\Updownarrow$

For $|h|$ small $f(x+h) = f(x) + \underbrace{ch}_{\text{linear}} + hE(h)$ $E(h) = o(h)$ $\lim_{h\to 0} E(h) = 0$

Def: $f: U \underset{open}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$ $c \in U$ $f$ is differentiable at $c$ if $\exists\, T_c: \mathbb{R}^n \to \mathbb{R}^m$ linear s.t. $f(c+v) = f(c) + T_c(v) + \|v\| E(v)$

for small $v$ $(c+v \in U)$ and $\lim_{\|v\|\to 0} E(v) = 0$, $T_c$ is called total derivative

Thm: $f: U \to \mathbb{R}^m$ is diff at $c$ then $f'(c;u)$ exsits $\forall u$ and $f'(c;u) = T_c(u)$

pf: $f'(c;u) \equiv \lim_{t\to 0}\frac{f(c+tu)-f(c)}{t} = \lim_{t\to 0}\frac{1}{t}\left(\cancel{f(c)} + T_c(tu) + \|tu\|E(tu) - \cancel{f(c)}\right) = \lim_{t\to 0}\frac{tT_c(u) + |t|\|u\|E(tu)}{t}$

$= T_c(u) + \lim_{t\to 0}\frac{|t|}{t}\|u\|E(tu) = T_c(u)$

Thm: $T: \mathbb{R}^n \to \mathbb{R}^m$ linear, then $\lim_{\|u\|\to 0} Tu = 0$

pf: $u = u_1e_1 + \dots + u_ne_n$ $Tu = u_1Te_1 + \dots + u_nTe_n \to 0$ $u_1^2 + \dots + u_n^2 \to 0$

Thm: $f: U \to \mathbb{R}^m$ is diff at $c$ then $f$ is conti at $c$

pf: $\lim_{\|v\|\to 0} f(c+v) = \lim_{\|v\|\to 0}(f(c) + T_c(v) + \|v\|E(v)) = f(c)$ by def of conti QED

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Thm: $f: \mathbb{R}^n \to \mathbb{R}^m$ diff at $c$ $v = (v_1, \dots, v_n) = \sum_{k=1}^n v_k u_k$ $u_k$: unit coordinate vector $\Rightarrow f'(c)(v) = f'(c)\left(\sum_{k=1}^n v_k u_k\right) = \sum_{k=1}^n v_k f'(c) u_k = \sum_{k=1}^n v_k D_k f(c)$

In particular $m=1$ $f'(c)(v) = \sum_{k=1}^n v_k D_k f(c) = v_1 \frac{\partial f}{\partial x_1}(c) + \dots + v_n \frac{\partial f}{\partial x_n}(c) = \nabla f(c)\cdot v$

$f: \mathbb{C}\to\mathbb{C}$ $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$ if exsits $\Rightarrow \frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0)$, $\frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$ Cauchy Riemann ($f(z) = u(z) + iv(z)$)

Thm: $f: \mathbb{C}\to\mathbb{C}$ $f(z) = u(z) + iv(z)$. Suppose $u$ and $v$ are diff at $z_0$ and $u$, $v$ satisfy Cauchy Riemann equation

then $f$ is diff at $z_0$ in complex sense

pf: $\frac{f(z)-f(z_0)}{z-z_0} = \frac{u(z)+iv(z)-u(z_0)-iv(z_0)}{z-z_0}$, $z - z_0 = h + ik$

$$\begin{cases} u(z) = u(x,y) = u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)h + \frac{\partial u}{\partial y}(x_0,y_0)k + |(h,k)|E_1(h,k) \\ v(z) = v(x,y) = v(x_0,y_0) + \frac{\partial v}{\partial x}(x_0,y_0)h + \frac{\partial v}{\partial y}(x_0,y_0)k + |(h,k)|E_2(h,k) \end{cases}$$

$\lim_{(h,k)\to 0} E_j(h,k) = 0$ $j = 1, 2$

$f(z) - f(z_0) = \frac{\partial u}{\partial x}(z_0)h + \frac{\partial u}{\partial y}(z_0)k + |(h,k)|E_1 + i\left(\frac{\partial v}{\partial x}(z_0)h + \frac{\partial v}{\partial y}(z_0)k + |(h,k)|E_2\right) = \frac{\partial u}{\partial x}(z_0)h - \frac{\partial v}{\partial x}(z_0)h + i\frac{\partial v}{\partial x}(z_0)h + i\frac{\partial u}{\partial x}(z_0)k + |(h,k)|(E_1+E_2)$

$= \left(\frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)\right)(h+ik) + |(h,k)|(E_1+E_2)$

$\therefore \frac{f(z)-f(z_0)}{z-z_0} = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) + \frac{|(h,k)|}{h+ik}(E_1+E_2)$ $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) = f_x(z_0)$

$T:\mathbb{R}^n\to\mathbb{R}^m$ linear $u_1\cdots u_n$ basis for $\mathbb{R}^n$ $e_1,\cdots,e_m$ basis for $\mathbb{R}^m$, $v\in\mathbb{R}^n$ $v=v_1u_1+\cdots+v_nu_n=\sum_{k=1}^n v_k u_k$ $T(v)=\sum_{k=1}^n v_k T(u_k)$

$Tu_k = t_{1k}e_1+\cdots+t_{mk}e_m = \begin{pmatrix}t_{1k}\\ \vdots\\ t_{mk}\end{pmatrix}$ $T(v)=v_1\begin{pmatrix}t_{11}\\ \vdots\\ t_{m1}\end{pmatrix}+\cdots+v_n\begin{pmatrix}t_{1n}\\ \vdots \\ t_{mn}\end{pmatrix} = \begin{pmatrix}t_{11}&\cdots&t_{1n}\\ \vdots&\ddots&\vdots\\ t_{m1}&\cdots&t_{mn}\end{pmatrix}\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$

$$\begin{tikzcd} \underset{u_i}{\mathbb{R}^n} \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & \underset{e_i}{\mathbb{R}^m} \arrow[r, "S"] & \underset{w_i}{\mathbb{R}^p} \end{tikzcd}$$

$S\circ T u_k = S(Tu_k) = S(\sum_{j=1}^m t_{jk}e_j)=\sum_{j=1}^m t_{jk}S(e_j)=\sum_{j=1}^m t_{jk}\sum_{i=1}^p s_{ij}w_i = \sum_{i=1}^p(\sum_{j=1}^m s_{ij}t_{jk})w_i$

$m(S\circ T)=\sum_{j=1}^m s_{ij}t_{jk}=(s_{ij})(t_{jk})=m(S)\cdot m(T)$

$f:\mathbb{R}^n\to\mathbb{R}^m$ diff at $c$, $f'(c)=T_c:\underset{u_i}{\mathbb{R}^n}\to\underset{e_i}{\mathbb{R}^m}$ $f'(c)(\sum_{k=1}^n v_ku_k)=\sum_{k=1}^n v_k f'(c)u_k=\sum_{k=1}^n v_k \begin{pmatrix}\frac{\partial f_1}{\partial x_k}\\ \vdots \\ \frac{\partial f_m}{\partial x_k}\end{pmatrix}(c) = \begin{pmatrix}\frac{\partial f_1}{\partial x_1}&\cdots&\frac{\partial f_1}{\partial x_n}\\ \vdots&&\vdots\\ \frac{\partial f_m}{\partial x_1}&\cdots&\frac{\partial f_m}{\partial x_n}\end{pmatrix}(c)\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$

$=\begin{pmatrix}\nabla f_1(c)\\ \vdots\\ \nabla f_m(c)\end{pmatrix}\begin{pmatrix}v_1\\ \vdots \\ v_n\end{pmatrix} = \sum_{j=1}^m(\nabla f_j(c)\cdot v)e_j = f'(c)(v)\in\mathbb{R}^m$

Jacobian matrix $\|f'(c)(v)\|=\|\sum_{j=1}^m(\nabla f_j(c)\cdot v)e_j\|\le\sum_{j=1}^m|\nabla f_j(c)\cdot v|\le\sum_{j=1}^m\|\nabla f_j(c)\|\|v\| = \overbrace{(\sum_{j=1}^m\|\nabla f_j(c)\|)}^{M}\|v\| \Rightarrow \|f'(c)(v)\|\le M\|v\|$

Thm (Chain Rule): $\underset{a}{\mathbb{R}^n}\xrightarrow{f}\underset{f(a)=b}{\mathbb{R}^m}\xrightarrow{g}\mathbb{R}^p$, suppose $f$ is diff at $a$, $g$ is diff at $b$ then $g\circ f$ is diff at $a$. $(g\circ f)'(a)=g'(b)\circ f'(a)$

pf: $h\in\mathbb{R}^n$, $|h|$ small $f(a+h)=f(a)+f'(a)h+|h|E_f(h)$ $\lim_{h\to0}E_f(h)=0$, $k\in\mathbb{R}^m$ $|k|$ small $g(b+k)=g(b)+g'(b)k+|k|E_g(k)$ $\lim_{k\to0}E_g(k)=0$

$f(a+h)-f(a)=f'(a)h+|h|E_f(h)=k$.

$g(f(a+h))=g(f(a))+g'(b)(f'(a)\cdot h+|h|E_f(h))+|k|E_g(k)=g(f(a))+g'(b)\circ f'(a)h+|h|g'(b)(E_f(h))+|k|E_g(k)$

$=g(f(a))+g'(b)\circ f'(a)h+|h|\underbrace{\{g'(b)(E_f(h))+\frac{|k|}{|h|}E_g(k)\}}_{E(h)}$

show: $\lim_{h\to0}E(h)=0$ $h\to0$ $E_f(h)\to0$ $\therefore \lim_{h\to0}g'(b)(E(h))=0$ $k=f'(a)h+\|h\|E_f(h)$

$\|k\|\le\|f'(a)h\|+\|h\|\|E_f(h)\|\le M\|h\|+\|h\|\|E_f(h)\|$ $\therefore \frac{\|k\|}{\|h\|}\le M+\|E_f(h)\|$

$\underset{x_i}{\mathbb{R}^n}\xrightarrow{f}\underset{y_i}{\mathbb{R}^m}\xrightarrow{g}\underset{z_i}{\mathbb{R}^p}$ $a\overset{f}{\mapsto} f(a)=b\overset{g}{\mapsto} g(b)$ $g\circ f:\mathbb{R}^n\to\mathbb{R}^p$ $x\mapsto g(f(x))=(g_1(f(x)),\dots,g_p(f(x)))$

$\left(\frac{\partial g_i(f(x))}{\partial x_k}\right)_{i=1}^{p} = \left(\frac{\partial g_i}{\partial y_l}\right)_{i=1\ l=1}^{p\ \ m}\circ\left(\frac{\partial f_l}{\partial x_k}\right)_{l=1\ k=1}^{m\ \ n}$ $\frac{\partial g_i}{\partial x_k}(a)=\left(\frac{\partial g_i}{\partial y_1},\dots,\frac{\partial g_i}{\partial y_m}\right)(b)\cdot\begin{pmatrix}\frac{\partial f_1}{\partial x_k}\\ \vdots\\ \frac{\partial f_m}{\partial x_k}\end{pmatrix}(a)=\frac{\partial g_i}{\partial y_1}(b)\frac{\partial f_1}{\partial x_k}(a)+\cdots+\frac{\partial g_i}{\partial y_m}(b)\frac{\partial f_m}{\partial x_k}(a)$

Thm: let $f$, $\frac{\partial f}{\partial x}$ be conti on $[a,b]\times[c,d]$ $p,q:[c,d]\to[a,b]$ diff. define $F(y)=\int_{q(y)}^{p(y)}f(x,y)\,dx$ $y\in[c,d]$

then for $y\in(c,d)$ $F'(y)=\int_{q(y)}^{p(y)}\frac{\partial f}{\partial y}(x,y)\,dx+f(p(y),y)p'(y)+f(q(y),y)q'(y)$

pf: let $G(x_1,x_2,x_3)=\int_{x_1}^{x_2}f(t,x_3)\,dt$ $x_1,x_2\in[a,b]$ $x_3\in[c,d]$, $F(y)=G(q(y),p(y),y)$ use chain rule

3/31 Thm: $E\subseteq\mathbb{R}^n$ open, $f:E\to\mathbb{R}^m$ $f$ is diff on $E$. let $a,b\in E$ s.t. $L(a,b)\subseteq E$ $L(a,b)=\{(1-t)a+tb\mid t\in[0,1]\}$

then $\forall c\in\mathbb{R}^m$, $\exists z\in L(a,b)$ s.t. $c\cdot(f(b)-f(a))=c\cdot\{f'(z)(b-a)\}$

pf: $\exists\delta>0$ s.t. $(1-t)a+tb\in E$ $-\delta<t<1+\delta$, define $g(t)=f((1-t)a+tb)\cdot c$, $g:(-\delta,1+\delta)\to\mathbb{R}$

$g(1)-g(0)=g'(t_0)(1-0)=g'(t_0)$ $c\cdot(f(b)-f(a))=c\cdot(f'((1-t_0)a+t_0b)\cdot(b-a))$

$|c\cdot(f(b)-f(a))|=|c\cdot\{f'(z)(b-a)\}|\le\|c\|\|f'(z)(b-a)\|\overset{\|c\|=1}{=}\|f'(z)(b-a)\|\le M\|b-a\|$ $\sum_{k=1}^m\|\nabla f_k(z)\|=M$

If $f(b)-f(a)=0$ done, If $f(b)-f(a)\ne0$ choose $c=\frac{f(b)-f(a)}{\|f(b)-f(a)\|}$ $\frac{|(f(b)-f(a))(f(b)-f(a))|}{\|f(b)-f(a)\|}\le M\|b-a\|$ $\|f(b)-f(a)\|\le M\|b-a\|$

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Thm: $E \subseteq \mathbb{R}^n$ open connected $f: E \to \mathbb{R}^m$ diff on $E$ If $f'(x)=0$ $\forall x \in E$ then $f$ is a const function

pf: choose $a, b \in E$. $L(a,b) \subseteq E$ $\forall c \in \mathbb{R}^m$ $z \in L(a,b)$ $c(f(b)-f(a)) = \underbrace{f'(z)}_{0}(b-a) \cdot c$ $\therefore f(b)-f(a)=0$

if $L(a,b) \not\subseteq E$ $\because E$: open connected in $\mathbb{R}^n$ $\therefore E$ arcwise-connected $\Rightarrow$ can use polygonal connect $a, b$

Thm: $E \subseteq \mathbb{R}^n$ open, $f: E \to \mathbb{R}$, $f(x_1, \dots, x_n)$. suppose $\frac{\partial f}{\partial x_1}(c)$ exsits $c \in E$ and $\frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$ are conti at $c$

then $f$ is diff at $c$ (general criteria of diff)

pf: For $|h|$ small $f(c+h) = f(c) + T_c(c) + hE(h)$ $\lim_{h \to 0} E(h) = 0$ $\|h\| = \lambda$ $h = \lambda y$ $\|y\|=1$ $y = u_1 y_1 + \dots + u_n y_n$

$f(c+h) - f(c) = f(c+h) - f(c+\lambda(y_1 u_1 + \dots + y_{n-1} u_{n-1})) + f(c+\lambda(y_1 u_1 + \dots + y_{n-1} u_{n-1})) - f(c+\lambda(y_1 u_1 + \dots + y_{n-2} u_{n-2})) + \dots + f(c+\lambda(y_1 u_1)) - f(c)$

$0<\theta_n<1$

$= \frac{\partial f}{\partial x_n}(c+\lambda(y_1 u_1 + \dots + y_{n-1} u_{n-1}) + \theta_n(\lambda y_n u_n))(\lambda y_n) + \frac{\partial f}{\partial x_{n-1}}(c+\lambda(y_1 u_1 + \dots + y_{n-2} u_{n-2}) + \theta_{n-1}(\lambda y_{n-1} u_{n-1}))(\lambda y_{n-1}) + \dots + \frac{\partial f}{\partial x_1}(c)\lambda y_1 + \lambda y_1 E_1(\lambda)$

$= \frac{\partial f}{\partial x_n}(c+\lambda(y_1 u_1 + \dots + y_{n-1} u_{n-1}) + \theta_n(\lambda y_n u_n))(\lambda y_n) - \frac{\partial f}{\partial x_n}(c)\lambda y_n + \frac{\partial f}{\partial x_n}(c)\lambda y_n + \dots + \frac{\partial f}{\partial x_1}(c)\lambda y_1 + \lambda y_1 E_1(\lambda)$

$= \frac{\partial f}{\partial x_1}(c)(\lambda y_1) + \lambda y_1 E_1(\lambda) + \frac{\partial f}{\partial x_2}(c)\lambda y_2 + \dots + \frac{\partial f}{\partial x_n}(c)\lambda y_n + \lambda y_2 E_2(\lambda) + \dots + \lambda y_n E_n(\lambda)$

$= \left(\frac{\partial f}{\partial x_1}(c) + \dots + \frac{\partial f}{\partial x_n}(c)\right)h + \underbrace{\lambda(y_1 E_1(\lambda) + \dots + y_n E_n(\lambda))}_{\|h\| \; E(h)}$

$g: \mathbb{R} \to \mathbb{R}$ $g(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = g(x) + g(y)$ can diff at 0

$f: (x_1, \dots, x_n)$ $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$ exsits $\frac{\partial^2 f}{\partial x_1 \partial x_2} \neq \frac{\partial^2 f}{\partial x_2 \partial x_1}$ $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$

Thm: $D \subseteq \mathbb{R}^2$ open $c \in D$ $f: D \to \mathbb{R}$ suppose $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exsit on $B(c,r) \subseteq D$ and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are diff at $c$ then $\frac{\partial^2 f}{\partial x \partial y}(c) = \frac{\partial^2 f}{\partial y \partial x}(c)$

pf: assume $c=0$, consider $\Delta = f(h,h) - f(h,0) - f(0,h) + f(0,0)$ $|h|$ small

$\Delta = (f(h,h) - f(h,0)) - (f(0,h) - f(0,0)) = G(h) - G(0) = \underline{\underline{\left(\frac{\partial f}{\partial x_1}(\theta_1 h, h) - \frac{\partial f}{\partial x_1}(\theta_1 h, 0)\right)h}} *$ $G(y) = f(y,h) - f(y,0)$

$= \left(\cancel{\frac{\partial f}{\partial x_1}(0,0)} + \cancel{\frac{\partial^2 f}{\partial x_1 \partial x_1}(0,0)\theta_1 h} + \frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0)h + |(\theta_1 h, h)|E_1(h)\right)h - \left(\cancel{\frac{\partial f}{\partial x_1}(0,0)} + \cancel{\frac{\partial^2 f}{\partial x_1 \partial x_1}(0,0)\theta_1 h} + \cancel{\frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0) \cdot 0} + |(\theta_1 h, h)|E_2(h)\right)h$

$= \frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0)h^2 + h^2 E(h)$

$\Delta = H(h) - H(0) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0)h^2 + h^2 \tilde{E}(h)$ $H(y) = f(h,y) - f(0,y)$

$\lim_{h \to 0} \frac{\Delta}{h^2} = \lim_{h \to 0}\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0) + E(h)\right) = \lim_{h \to 0}\left(\frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0) + \tilde{E}(h)\right) \Rightarrow \frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0)$

Thm: If $\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are conti at $c$ then $\frac{\partial^2 f}{\partial x \partial y}(c) = \frac{\partial^2 f}{\partial y \partial x}(c)$

pf $* = \frac{\partial^2 f}{\partial x_2 \partial x_1}(\theta_1 h, \theta_2 h)h^2$ $\frac{\Delta}{h^2} = \frac{\partial^2 f}{\partial x_1 \partial x_2}(\tilde{\theta}_1 h, \tilde{\theta}_2 h) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(\theta_1 h, \theta_2 h)$

# Implicit Functions & Extremum Problems 隱函數與極值

Thm: $B=B(a;r)$ ball in $\mathbb{R}^n$ $\overline{B}=B\cup\partial B$ let $f\in C(\overline{B})$ $f:\overline{B}\to\mathbb{R}^n$ assume ① $\frac{\partial f_i}{\partial x_k}$ exsits in $B$ $\forall j.k$

② $J_f(x)=\det\left(\frac{\partial f_i}{\partial x_k}\right)(x)\neq 0$ $\forall x\in B$ ③ $f(x)\neq f(a)$ $\forall x\in\partial B$ then $f(B)$ contains an open ball centered at $f(a)$

pf: define $g(x)=\|f(x)-f(a)\|$ $x\in\partial B$ continuous on $\partial B$, $\partial B$ is compact $\therefore \exists\, \bar{x}\in\partial B$ s.t. $g(\bar{x})=m>0$

claim $B(f(a);\frac{m}{2})\subseteq f(B)$ ie $\forall y\in B(f(a);\frac{m}{2})$ $\exists x\in B$ s.t. $y=f(x)$

Fix $y\in B(f(a);\frac{m}{2})$ Define $h(x)=\|f(x)-y\|$, $x\in\overline{B}$, $h(a)=\|f(a)-y\|<\frac{m}{2}$

$w\in\partial B$ $h(w)=\|f(w)-y\|=\|f(w)-f(a)+f(a)-y\|\geq\|f(w)-f(a)\|-\|f(a)-y\|\geq m-\|f(a)-y\|>\frac{m}{2}$

$\therefore \exists$ minimum $c\in B$ $\therefore h^2(x)$ has a min at $c$ $\|f(x)-y\|^2=\sum_{j=1}^n (f_j(x)-y_j)^2$

$\frac{\partial h^2}{\partial x_k}(c)=0=\sum_{j=1}^n 2(f_j(c)-y_j)\frac{\partial f_j}{\partial x_k}(c)$, $\det\left(\frac{\partial f_i}{\partial x_k}(c)\right)\neq 0$ $\therefore f_j(c)=y_j$ $\forall j$ $\therefore y=f(c)$

$f: \underset{\text{open}\subseteq\mathbb{R}^n}{S}\to\mathbb{R}^n$ open mapping $f(V)$ is open if $V\subseteq S$ is open

Thm: $f:\overset{\text{open}}{\Omega}\subseteq\mathbb{C}\to\mathbb{C}$ holomorphic if $f$ is not a constant func then $f$ is open

Thm: $f:\overset{\text{open}}{S}\subseteq\mathbb{R}^n\to\mathbb{R}^n$ $f\in C(S)$ and $\frac{\partial f_i}{\partial x_k}(x)$ exsits and is finite $\forall x\in S$ $j.k$

If $f$ is 1-1 on $S$ and $J_f(x)\neq 0$ $\forall x\in S$ then $f$ is an open mapping

pf: $V\underset{\text{open}}{\subseteq} S$ show $f(V)$ is open If $y\in f(V)$ $\exists a\in V$ s.t. $y=f(a)$ by Thm $f(V)$ is open

Thm: $f:\overset{\text{open}}{S}\subseteq\mathbb{R}^n\to\mathbb{R}^n$, $f\in C(S)$ $\frac{\partial f_i}{\partial x_k}(x)$ conti on $S$ $\forall j.k$. If $J_f(a)\neq 0$ $a\in S$ then $\exists B(a;r)\subseteq S$ s.t. $f$ is 1-1 on $B(a;r)$

pf: $\frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)}=\begin{pmatrix}\nabla f_1(x)\\ \vdots\\ \nabla f_n(x)\end{pmatrix}$ $x\in S$, consider $\tilde{J}_f=\det\begin{pmatrix}\nabla f_1(z_1)\\ \vdots\\ \nabla f_n(z_n)\end{pmatrix}$ $z_1,\dots,z_n\in S$ define on $\mathbb{R}^{n^2}$

If $z_1=z_2=\dots=z_n=a$ $\therefore \exists B(a;r)\subseteq S$ s.t. $\tilde{J}_f\neq 0$ for $z_1\dots z_n\in B(a;r)$

claim $f$ is 1-1 on $B(a;r)$. If not $\therefore \exists x\neq y$ $x.y\in B(a;r)$ s.t. $f(x)=f(y)$ $\left(\begin{matrix}f_j(x)=f_j(y)\\ j=1.2\cdots n\end{matrix}\right)$

$f_j:S\subseteq\mathbb{R}^n\to\mathbb{R}$ $0=f_j(y)-f_j(x)=\nabla f_j(t_j)(y-x)$ $0=\frac{\partial f_j}{\partial x_1}(t_j)(y_1-x_1)+\dots+\frac{\partial f_j}{x_n}(t_j)(y_n-x_n)$

$\because \det\begin{pmatrix}\nabla f_1(t_1)\\ \vdots\\ \nabla f_n(t_n)\end{pmatrix}\neq 0$ $\therefore$ By Crammer's rule $x_1=y_1,\dots,x_n=y_n$ $\therefore x=y$ ($\to\leftarrow$)

Ex: $f:\mathbb{C}\to\mathbb{C}$, $z\mapsto e^z$ $f'(z)=e^z$ $J_f(z)=|f'(z)|^2=|e^z|^2=e^{2x}>0$ $f(z+2\pi i)=f(z)$

Thm: $f:\overset{\text{open}}{S}\subseteq\mathbb{R}^n\to\mathbb{R}^n$ $\frac{\partial f_i}{\partial x_k}$ conti on $S$ $\forall j.k$ and $J_f(x)\neq 0$ $\forall x\in S$ then $f$ is an open mapping

pf: $V\overset{\text{open}}{\subseteq}S$ $V=\bigcup_{x\in V}B_x$ $f(V)=\bigcup_{x\in V}f(B_x)$

Inverse function Thm: $S \subseteq \mathbb{R}^n$ open, $f: S \to \mathbb{R}^n$ $f \in C^k(S)$ $k \geq 1$ $a \in S$ s.t. $J_f(a) \neq 0$

Then $\exists$ open sets $X \cdot Y$ $a \in X \subseteq S$ $f(a) \in Y \subseteq f(S) = T$

s.t. ① $Y = f(X)$ $f$: 1-1 on $X$ ② $\exists g: Y \to X$ s.t. $g(f(x)) = x$ $\forall x \in X$ ③ $g \in C^k(Y)$

pf: $\because J_f(x) \neq 0$ $\exists B(a;r) \subseteq S$ s.t. $f$ is 1-1 on $B(a;r)$, $B = B(a;\frac{r}{2})$ $f(B)$ contains $B(f(a);p) = Y$ $X = f^{-1}(Y) \cap B$ — ①

assume $\det \begin{pmatrix} \nabla f(z_1) \\ \vdots \\ \nabla f(z_n) \end{pmatrix} \neq 0$ $z_1, \dots, z_n \in B(a)$ (by contineness) consider $f: \overline{B} \xrightarrow{\subseteq B(a)} f(\overline{B})$ closed mapping, $\exists g = f^{-1}$ — ②

$f \in C^1 \Rightarrow g \in C^1$ $g = (g_1, \dots, g_n)$ consider $g_i$, $\frac{g_i(y + tu_k) - g_i(y)}{t}$ WLOG $\left( \begin{matrix} x \overset{f}{\underset{g}{\leftrightarrow}} y \\ x' \leftrightarrow y + tu_k \end{matrix} \right)$

$\frac{f_j(x') - f_j(x)}{t} \overset{MVT}{=\!=} \nabla f_j(z_j) \cdot (x' - x) \cdot \frac{1}{t}$ LHS $= \begin{cases} 1 & j \neq k \\ 0 & j = k \end{cases}$ $(\because \frac{y + tu_k - y}{t} = u_k$ std basis for $\mathbb{R}^n)$

compute $\nabla f_j(z_j)$, $\nabla f_1(z_1) \cdot \frac{x'-x}{t} = 0 \dots$, $\nabla f_k(z_k) \cdot \frac{x'-x}{t} = 1, \dots$ $\nabla f_n(z_n) \frac{x'-x}{t} = 0$

$\because \det \begin{pmatrix} \nabla f_1(z_1) \\ \vdots \\ \nabla f_n(z_n) \end{pmatrix} \neq 0$ by Crammer's rule $\frac{g(y + tu_k) - g(y)}{t} = \frac{\det(\ )}{\det \begin{pmatrix} \nabla f_1(z_1) \\ \nabla f_n(z_n) \end{pmatrix}} \xrightarrow{t \to 0} \frac{\square}{J_f(x)}$ exsits $\because J_f(x) \neq 0$

by MI, assume $g \in C^{k-1}(Y)$, $f \in C^k \to g \in C^k$, $x = g(y)$ $\lim_{t \to 0} \frac{g^{(k-1)}(y + tu_k) - g^{(k-1)}(y)}{t} = \frac{\square}{J_f(g^{(k-1)}(y))}$

$\therefore (g^{(k-1)})' \in C^{k-1} \Rightarrow g \in C^k$ — ③

Implicit function Thm: $f_j: \mathbb{R}^{n+k} \to \mathbb{R}^n$ $j = 1, \dots, n$ $(x;t) \in \mathbb{R}^{n+k}$ $x \in \mathbb{R}^n$ $t \in \mathbb{R}^k$ $f_j \in C^k(\mathbb{R}^n)$ $f(x_0, t_0) = 0$ $j = 1, 2, \dots, n$

Assume $\det(D_i f_j(x_0; t_0)) \neq 0$ $D_i = \frac{\partial}{\partial x_i}$

Then $\exists$ open set $t_0 \in T \subseteq \mathbb{R}^k$ $\exists! g: T \to \mathbb{R}^n$ $g \in C^k(T)$ $g(t_0) = x_0$ $f_j(g(t) \cdot t) = 0$ $\forall j$

pf: consider $F(x;t) = \begin{pmatrix} f_1(x;t) \\ \vdots \\ f_n(x;t) \\ t_1 \\ \vdots \\ t_k \end{pmatrix}$ $F: \underset{(x_0, t_0)}{\mathbb{R}^{n+k}} \to \underset{\mapsto (0, t_0)}{\mathbb{R}^{n+k}}$ (view $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$) $(x_0, t_0) \in \underbrace{(f_1 = \dots = f_n = 0)}_{\text{manifold}}$

$F$ is projection from $\mathbb{R}^{n+k}$ to $\mathbb{R}^k$ in $(x_0, t_0)$'s neighborhood

$J_F = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial t_1} & \cdots & \frac{\partial f_1}{\partial t_k} \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial t_1} & \cdots & \frac{\partial f_n}{\partial t_k} \\ & 0 & & & I_k & \end{pmatrix}$ $\det J_F = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & & \frac{\partial f_n}{\partial x_n} \end{pmatrix} (x_0, t_0) \neq 0$ impliment inverse function thm

$\exists G: \underset{(0, t_0) \mapsto (x_0, t_0)}{Y^{\subseteq \mathbb{R}^{n+k}} \to X^{\subseteq \mathbb{R}^{n+k}}}$ $G(x, t) = (\underset{x'}{V(x, t)}, \underset{t'}{W(x, t)})$ $F(x', t') = (f(x', t'), t') = (x, t)$ $\therefore t' = t$

$\therefore G(x, t) = (V(x, t) \cdot t)$ $G \in C^k \Rightarrow V(x, t) \in C^k$ let $T = Y \cap (\{0\} \times \mathbb{R}^k) = \{t \in \mathbb{R}^k \mid (0, t) \in Y\} \underset{\text{open}}{\subseteq} \mathbb{R}^k$, $t_0 \in T$

let $g(t) = V(0, t) \in C^k$ $g(t_0) = V(0, t_0) = x_0$

$F(V(x, t), t) = (x, t)$ $f(V(x, t), t) = x$ let $x = 0$ $f(V(0, t), t) = 0 \Rightarrow f(g(t), t) = 0$

For $f(g(t), t) = f(h(t), t)$ $g, h: T \to \mathbb{R}^n$ $(f(g(t), t), t) = (f(h(t), t), t) \in \mathbb{R}^{n+k}$

$G(f(g(t), t), t) = (g(t), t)$ $G(f(h(t), t), t) = (h(t), t) \Rightarrow g(t) = h(t)$ $\because G$ 1-1

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$n=1$ $f:(a,b)\to\mathbb{R}$ $c\in(a,b)$ local extremum $f(c)\geq f(x)$ $\forall x\in(c-\delta,c+\delta)$
$f(c)\leq f(x)$

$c$: saddle pt $\forall\delta>0$ if $\exists$ $x_1\in(c-\delta,c+\delta)$ $f(x_1)<f(c)$
$x_2\in(c-\delta,c+\delta)$ $f(x_2)>f(c)$

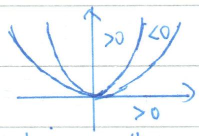
necessary cond. If $f$ diff at $c$ and $c$ is a local extremum $\Rightarrow f'(c)=0$

$f\in C^n(a,b)$ $c\in(a,b)$ If $f^{(1)}(c)=\dots=f^{(n)}(c)=0$ then ① $n$: even $f^{(n)}(c)>0$ $c$: local min, $f^{(n)}(c)<0$ $c$: local max ② $n$: odd $f^{(n)}(c)\neq0\Rightarrow c$: saddle pt

$f(x)-f(c)=\frac{f^{(n)}(\bar{x})}{n!}(x-c)^n$ $c<\bar{x}<x$ or $c>\bar{x}>x$ $n$: even $x\neq c$, $n$: odd $x\neq c$

$f:\mathbb{R}^n\to\mathbb{R}$ $f$ is diff If $f$ has a local extremum at $c$

Ex: $f(x,y)=(y-x^2)(y-2x^2)$




but any line pass $(0,0)$ would pass $>0$ area

$\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$ $\therefore$ $(0,0)$ is saddle pt.

If $f$ is diff at $c$ and $\nabla f(c)=0$ then we call $c$ is a sationary pt.

$f(x+a)-f(a)=\nabla f(a)\cdot x+\frac{1}{2}f''(z;x)$ $z\in L(a,a+x)$, $a$: stationary $\nabla f(a)=0$

$f(x+a)-f(a)=\frac{1}{2}f''(z;x)=\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial f}{\partial x_j\partial x_i}(z)x_jx_i=\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_j\partial x_i}(a)x_jx_i+\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2 f}{\partial x_j\partial x_i}(z)-\frac{\partial^2 f}{\partial x_j\partial x_i}(a)\right)x_ix_j$

$\left|\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2 f}{\partial x_j\partial x_i}(z)-\frac{\partial^2 f}{\partial x_j\partial x_i}(a)\right)x_ix_j\right|\leq\frac{1}{2}\left(\sum_{i,j=1}^{n}\left|\frac{\partial^2 f}{\partial x_j\partial x_i}(z)-\frac{\partial^2 f}{\partial x_j\partial x_i}(a)\right|\right)\|x\|^2=\frac{1}{2}E(x)\|x\|$

If $\frac{\partial^2 f}{\partial x_j\partial x_i}(x)$ is conti at $a$ $\Rightarrow\lim_{x\to a}E(x)=0$

Thm: Suppose $\frac{\partial^2 f}{\partial x_j\partial x_i}$ is conti at $a$ $\forall i,j$ $\underset{\text{interior pt}}{\nabla f(a)=0}$ let $Q(x)=\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(a)x_i\cdot x_j$ (Hessian)

then (i) $Q(x)>0$ $\forall x\neq0\Rightarrow f(a)$ is a local min (ii) $Q(x)<0$ $\forall x\neq0\Rightarrow f(a)$ is a local max

(iii) $Q(x_1)>0$ $Q(x_2)<0$ for some $x_1,x_2\neq0\Rightarrow a$ is a saddle pt

pf: $Q(x)$: homog of order 2 $0\neq\lambda\in\mathbb{R}$ $Q(\lambda x)=\lambda^2Q(x)$

$x\in S^n$: compact $Q(x)>0$ $Q(x)\geq m>0$ $\forall x\in S^n$ $\forall x\neq0\Rightarrow Q(x)\geq m\|x\|^2$ $\left(x=\|x\|\frac{x}{\|x\|},\ Q(x)=\|x\|^2Q\left(\frac{x}{\|x\|}\right)\right)$

$f(x+a)-f(a)=\frac{1}{2}Q(x)+\frac{1}{2}E(x)\|x\|^2\geq\frac{1}{2}(m\|x\|^2+E(x)\|x\|^2)=\frac{1}{2}\|x\|^2(m+E(x))>\frac{1}{2}\|x\|^2\left(\frac{m}{2}\right)$ $\left(\lim_{x\to0}E(x)=0\ \exists\ \delta>0\ 0<|x|<\delta\right)$

$x=\lambda x_1$ $f(\lambda x_1+a)-f(a)=\frac{1}{2}\lambda^2Q(x_1)+\frac{1}{2}E(\lambda x_1)\lambda^2\|x_1\|^2=\frac{1}{2}\lambda^2(Q(x_1)+E(\lambda x_1)\|x_1\|^2)>\frac{1}{2}Q(x_1)>0$

$f(\lambda x_2+a)-f(a)=\frac{1}{2}\lambda^2(Q(x_2)+E(\lambda x_2)\|x_2\|^2)<\frac{1}{2}Q(x_2)$

$\mathbb{R}^n$ consider $Q(x)=\sum_{i,j=1}^{n}a_{ij}x_ix_j$ $a_{ij}\in\mathbb{R}$ Quadratic form

symmetric if $a_{ij}=a_{ji}$, positive definite (正定) if $Q(x)>0$ $\forall x\neq0$ negative definite (負定) if $Q(x)<0$ $\forall x\neq0$

symmetric quadratic form $Q(x)=(x_1,\dots,x_n)\begin{pmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{n1}&\cdots&a_{nn}\end{pmatrix}\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}=x^TMx$ $\lambda_1\cdots\lambda_n$ eigenvalue

$Q(x)=\lambda_1\tilde{x}_1^2+\dots+\lambda_n\tilde{x}_n^2$ positive definite $\Leftrightarrow\lambda_1>0\dots\lambda_n>0$ negative definite $\Leftrightarrow\lambda_1<0,\dots,\lambda_n<0$
$\Leftrightarrow$ $\Delta_0>0$ $\Delta_1<0$ $\Delta_2>0$ $\dots$

$\Delta_0=1$ $\Delta_1=\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}$ $\cdots$ $\Delta_n=M$

Q&C

Thm: $f: \mathbb{R}^2 \to \mathbb{R}$, $C^2$, $a \in \mathbb{R}^2$, $\frac{\partial^2 f}{\partial x \partial x}(a) = A$, $\frac{\partial^2 f}{\partial y \partial x}(a) = B$, $\frac{\partial^2 f}{\partial y^2}(a) = C$, $\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$ suppose $\nabla f(a) = 0$

then (1) $A > 0$, $\Delta > 0 \Rightarrow f(a)$: local min (2) $A < 0$, $\Delta > 0 \Rightarrow f(a)$: local max (3) $\Delta < 0 \Rightarrow a$: saddle pt (4) $\Delta = 0$ inconclusion

pf: (1)(2) $Ax^2 + 2Bxy + Cy^2 = A(x^2 + 2\frac{B}{A}xy + \frac{C}{A}y^2) = A(x^2 + 2\frac{B}{A}xy + (\frac{B}{A}y)^2) + Cy^2 - \frac{B^2}{A}y^2 = A(x + \frac{B}{A}y)^2 + \frac{\Delta}{A}y^2 > 0$

(3) $A \neq 0$, $\Delta < 0$ $\Downarrow$ $= (ax - by)(ax + by)$ two lines $\to$ saddle pt

(4) $AC - B^2 = -B^2 < 0$, $B \neq 0$ $2Bxy + Cy^2 \begin{cases} = 2Bxy & C = 0 \\ = (2Bx + Cy)y & C \neq 0 \end{cases}$ two lines

$f(x, y) = x^4 + y^4$ $(0, 0)$ min $f(x, y) = -x^4 - y^4$ $(0, 0)$ max $f(x, y) = x^4 - y^4$ $(0, 0)$ saddle

Lagrange's multiplier $f(x, y, z)$, $g(x, y, z) = 0$ 4/18

level set $\{(x, y, z) \mid f(x, y, z) = c\} = L_c$ want $L_c$ intersect with $g(x, y, z) = 0$ $\nabla f = \lambda \nabla g$

Ex: $y = x^3$ $y - x^3 = 0$ $f(x, y) = y$ $\nabla f = (0, 1)$, at $(0, 0)$ $\nabla f = \nabla g$ but $(0, 0)$ is not extreme

Thm (Lagrange multiplier) let $S$ be an open set in $\mathbb{R}^n$ $f \in C^1(S)$ $(g_1, \dots, g_m) \in C^1(S)$ $m < n$, $X = \{x \in S \mid g_1(x) = \dots = g_m(x) = 0\}$

Assume $(\nabla g_1, \dots, \nabla g_m)$ linear indep ($C^1$-manifold) $\left| \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_m} \end{pmatrix} (x_0) \right| \neq 0$

$\exists \delta > 0$ s.t. $f(x_0) \geq f(x)$ $\forall x \in B(x_0 ; \delta)$ or $f(x_0) \leq f(x)$ $\forall x \in B(x_0 ; \delta)$

then $\exists \lambda_1, \dots, \lambda_m$ s.t. $\nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \dots + \lambda_m \nabla g_m(x_0)$

pf: $\underbrace{\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \\ \frac{\partial f}{\partial x_{m+1}} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}}_{\nabla f(x_0)} (x_0) = \lambda_1 \underbrace{\begin{pmatrix} \frac{\partial g_1}{\partial x_1} \\ \vdots \\ \frac{\partial g_1}{\partial x_m} \\ \frac{\partial g_1}{\partial x_{m+1}} \\ \vdots \\ \frac{\partial g_1}{\partial x_n} \end{pmatrix}}_{\nabla g_1(x_0)} (x_0) + \dots + \lambda_m \underbrace{\begin{pmatrix} \frac{\partial g_m}{\partial x_1} \\ \vdots \\ \frac{\partial g_m}{\partial x_m} \\ \frac{\partial g_m}{\partial x_{m+1}} \\ \vdots \\ \frac{\partial g_m}{\partial x_n} \end{pmatrix}}_{\nabla g_m(x_0)} (x_0)$ $\uparrow$ : $\exists \lambda_1 \cdots \lambda_m$ linear comb

(lower part: $\overset{?}{=} \lambda_1 (\cdots) + \dots + \lambda_m (\cdots)$)

by implicit function thm $x_1 = x_1(x_{m+1}, \dots, x_n)$ $\cdots$ $x_m = x_m(x_{m+1}, \dots, x_n)$ $x_1 \cdots x_m \in C^1$

$f(x_1(x_{m+1}, \dots, x_n), \dots, x_m(x_{m+1}, \dots, x_n), x_{m+1}, \dots, x_n)$

$g_1(x_1(x_{m+1}, \dots, x_n), \dots, x_m(x_{m+1}, \dots, x_n), x_{m+1}, \dots, x_n) \equiv 0$

$\vdots$

$g_m(x_1(x_{m+1}, \dots, x_n), \dots, x_m(x_{m+1}, \dots, x_n), x_{m+1}, \dots, x_n) \equiv 0$

$\frac{\partial f}{\partial x_{m+1}} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial x_{m+1}} + \dots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial x_{m+1}} + \frac{\partial f}{\partial x_{m+1}} = 0$

$\lambda_1 \frac{\partial g_1}{\partial x_{m+1}} = \lambda_1 \frac{\partial g_1}{\partial x_1}\frac{\partial x_1}{\partial x_{m+1}} + \dots + \frac{\partial g_1}{\partial x_m}\frac{\partial x_m}{\partial x_{m+1}} + \frac{\partial g_1}{\partial x_{m+1}} = 0$

$\vdots$

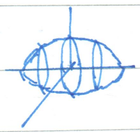
$\lambda_m \frac{\partial g_m}{\partial x_{m+1}} = \lambda_m \frac{\partial g_m}{\partial x_1}\frac{\partial x_1}{\partial x_{m+1}} + \dots + \frac{\partial g_m}{\partial x_m}\frac{\partial x_m}{\partial x_{m+1}} + \frac{\partial g_m}{\partial x_{m+1}} = 0$

$\Rightarrow \frac{\partial x_1}{\partial x_{m+1}}\left(\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \dots - \lambda_m \frac{\partial g_m}{\partial x_1}\right) = 0$

$\therefore \frac{\partial f}{\partial x_{m+1}} = \lambda_1 \frac{\partial g_1}{\partial x_{m+1}} + \dots + \lambda_m \frac{\partial g_m}{\partial x_{m+1}}$

Q&C

Ex: Quadric surface $Ax^2+By^2+Cz^2+2Dxy+2Eyz+2Fzx=0$

ellipsoid:  problem: maximize $x^2+y^2+z^2=f(x,y,z)$ constrain $q(x,y,z)-1=g(x,y,z)=0$

$$q(x_1,x_2,x_3)=\sum_{i,j=1}^{3} a_{ij}x_ix_j$$

by Lagrange's multiplier $\nabla f+\lambda\nabla g=0$ $\nabla f+\lambda\nabla q=0$

$x=(x_1,x_2,x_3)$ $x\cdot\nabla f+\lambda x\cdot\nabla q=0$ $f(x)+\lambda q(x)=0$ $\because q(x)=1$ $\therefore -f(x)=\lambda\neq 0$

$\frac{1}{\lambda}\nabla f+\nabla q=0$ $-\frac{1}{\lambda}\nabla f-\nabla q=0$ $t\nabla f-\nabla q=0$

$$\begin{cases}(a_{11}-t)x_1+a_{12}x_2+a_{13}x_3=0\\ a_{21}x_1+(a_{22}-t)x_2+a_{23}x_3=0\\ a_{31}x_1+a_{32}x_2+(a_{33}-t)x_3=0\end{cases}$$

has nonzero sols $\therefore$ $\begin{vmatrix}a_{11}-t & a_{12} & a_{13}\\ a_{21} & a_{22}-t & a_{23}\\ a_{31} & a_{32} & a_{33}-t\end{vmatrix}=0$

$t_1,t_2,t_3$

$t_i=\frac{1}{f(x_1^i,x_2^i,x_3^i)}$ semi-axes $\sqrt{\frac{1}{t_1}},\sqrt{\frac{1}{t_2}},\sqrt{\frac{1}{t_3}}$

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$\boxed{\text{Multiple Riemann Integral 重黎曼積分}}$

$\mathbb{R}^n$ $I = I_1 \times \cdots \times I_n$ $I_i$ one-dim interval

Def: compact interval $I$ $I = I_1 \times \cdots \times I_n = [a_1, b_1] \times \cdots \times [a_n, b_n]$ $P_1 \in \mathcal{P}(I_1) \cdots P_n \in \mathcal{P}(I_n)$ $P = P_1 \times \cdots \times P_n \in \mathcal{P}(I)$

$P_i$ partition $I_i$ into $m_i$ subintervals $P$: partition $I$ into $m_1 \times \cdots \times m_n$ subintervals

Def: Let $f$ be define and bounded on a compact interval $I$, let $P \in \mathcal{P}(I)$ which divides $I$ into $m$ subintervals $(I_1, \dots, I_m)$ 4/21

then $S(P, f) = \sum_{k=1}^{m} f(t_k) \mu(I_k)$ is called a Riemann sum of $f$ wrt $P$ $t_k \in I_k$

$f$ is said to be Riemann integrable on $I$ ($f \in R$ on $I$)

if $\exists A \in \mathbb{R}$ and given $\varepsilon > 0$ $\exists P_\varepsilon \in \mathcal{P}(I)$ s.t. for any $P \geq P_\varepsilon$ we have $|S(P, f) - A| < \varepsilon$ for any choice of $t_k$

Def: $f$: defined and bounded on $I$ $P \in \mathcal{P}(I)$ divided $I$ into $I_1 \cdots I_m$ let $M_k = \sup_{I_k} f$ $m_k = \inf_{I_k} f$

let $U(P, f) = \sum_{k=1}^{m} M_k \mu_k(I)$ upper Reimann sum, $L(P, f) = \sum_{k=1}^{m} m_k \mu_k(I)$ lower Riemann sum $L(P, f) \leq U(P, f)$

Thm: If $P_1 \subseteq P_2$ $U(P_2, f) \leq U(P_1, f)$ $L(P_2, f) \geq L(P_1, f)$

Def: $\overline{\int_I} f(x)\,dx = \inf_P \{U(P, f)\}$ upper integral $\underline{\int_I} f(x)\,dx = \sup_P \{L(P, f)\}$ lower integral

Def (Riemann's condition) $f$: defined and bounded on $I$, $f$ is said to satisfy Riemann's condition

if given $\varepsilon > 0$ $\exists P_\varepsilon$ s.t. for any $P \geq P_\varepsilon$ we have $0 \leq U(P, f) - L(P, f) < \varepsilon$

Thm: $\overline{\int_I} (f+g)\,dx \leq \overline{\int_I} f\,dx + \overline{\int_I} g\,dx$, $\underline{\int_I} (f+g)\,dx \geq \underline{\int_I} f\,dx + \underline{\int_I} g\,dx$

$I = I_1 \cup I_2$ nonoverlapping $\overline{\int_I} f\,dx = \overline{\int_{I_1}} f\,dx + \overline{\int_{I_2}} f\,dx$ $\underline{\int_I} f\,dx = \underline{\int_{I_1}} f\,dx + \underline{\int_{I_2}} g\,dx$

Thm: $f$: defined and bounded on $I$ then following statements are equivalent

(i) $f \in R$ on $I$ (ii) $f$ satisfies Reimann's condition (iii) $\overline{\int_I} f(x)\,dx = \underline{\int_I} f(x)\,dx$

Def (measure zero set) $E \subseteq \mathbb{R}^n$ we say $E$ has n-measure zero $E \subseteq \bigcup_{k=1}^{\infty} I_k$ $\sum_{k=1}^{\infty} \mu(I_k) < \varepsilon$

if given $\varepsilon > 0$ $\exists$ countable intervals that cover $E$ and the sum of the n-measure of the intervals is less than $\varepsilon$

Def: We say $f$ has property A a.e. (almost everywhere) means $f$ has property A on $I$ possibly except on a subset of n-measure zero

lemma: $E_k \subseteq \mathbb{R}^n$ $k = 1, 2 \cdots$ $|E_k|_n = 0$ $\Rightarrow |\bigcup_{k=1}^{\infty} E_k|_n = 0$

Thm: $E \subseteq \mathbb{R}^{n-1} \Rightarrow |E|_n = 0$

Thm (Lesbegue Criterion) $f$: defined and bounded on $I$ (compact interval)

then $f \in R$ on $I \Leftrightarrow$ the set of discontinuities of $f$ on $I$ has n-measure zero

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$f \in C([a,b]\times[c,d])$ $\int_I f(x,y)\,dxdy$ exsits ? $\int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$ $\int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$

$f$: bounded on $[a,b]\times[c,d]$ $\int_I f(x,y)\,dxdy$ exsits ?

Thm $f$ defined and bounded on $[a,b]\times[c,d]=Q$

then $\underline{\int}_Q f(x,y)\,dxdy \le \underline{\int}_c^d \left(\underline{\int}_a^b f(x,y)\,dx\right) dy \le \overline{\int}_c^d \left(\overline{\int}_a^b f(x,y)\,dx\right) dy \le \overline{\int}_Q f(x,y)\,dxdy$

If $f \in R$ on $Q \Rightarrow \int_Q f(x,y)\,dxdy = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$ (Fubini Thm)

(same result to $\int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$ )

pf: let $M=\sup_Q f$ $F(y)=\overline{\int}_a^b f(x,y)\,dx \le M(b-a)$ let $P_1=\{a=x_0<x_1<\cdots<x_n=b\}$ $P_2=\{c=y_0<\cdots<y_m=d\}$

$I_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$ $M_{ij}=\sup_{I_{ij}} f$ $m_{ij}=\inf_{I_{ij}} f$

$\overline{\int}_c^d \left(\overline{\int}_a^b f(x,y)\,dx\right) dy = \overline{\int}_c^d \left(\sum_{i=1}^n \overline{\int}_{x_{i-1}}^{x_i} f(x,y)\,dx\right) dy \le \sum_{i=1}^n \overline{\int}_c^d \left(\overline{\int}_{x_{i-1}}^{x_i} f(x,y)\,dx\right) dy = \sum_{i=1}^n\sum_{j=1}^m \overline{\int}_{y_{j-1}}^{y_j}\left(\overline{\int}_{x_{i-1}}^{x_i} f(x,y)\,dx\right) dy$

$\le \sum_{i=1}^n\sum_{j=1}^m M_{ij}\,\mu(I_{ij}) = U(P,f)$

$\underline{\int}_c^d \left(\underline{\int}_a^b f(x,y)\,dx\right) dy \ge \sum_{i=1}^n \underline{\int}_c^d \left(\underline{\int}_{x_{i-1}}^{x_i} f(x,y)\,dx\right) dy := \sum_{i=1}^n\sum_{j=1}^m \underline{\int}_{y_{j-1}}^{y_j}\left(\underline{\int}_{x_{i-1}}^{x_i} f(x,y)\,dx\right) dy$

$\ge \sum_{i=1}^n\sum_{j=1}^m m_{ij}\,\mu(I_{ij}) = L(P,f)$

$\underline{\int}_Q f(x,y)\,dxdy = \sup\{L(P,f)\}$ $\overline{\int}_Q f(x,y) = \inf\{U(P,f)\}$ done

4/25 Jordan content

$S\subseteq I$: compact interval $P\in\mathcal{P}(I)$ $\underline{J}(P,S)=\sum_{I_k\subseteq S}\mu(I_k)$ $\overline{J}(P,S)=\sum_{I_k\cap(S\cup\partial S)\neq\emptyset}\mu(I_k)$ $\overline{J}(P,S)\ge\underline{J}(P,S)$

$\inf_P\{\overline{J}(P,S)\}=\overline{c}(S)$ Jordan outer content $\sup_P\{\underline{J}(P,S)\}=\underline{c}(S)$ Jordan inner content

$S$: Jordan measurable if $\overline{c}(S)=\underline{c}(S)=c(S)$ call $c(S)$ Jordan content

$c(S)=0$ $\forall\varepsilon>0$ $\exists P$ s.t. $\overline{J}(P,S)<\varepsilon \Rightarrow$ measure zero

$\mathbb{Q}\cap[0,1]\subseteq[0,1]$ $\overline{J}(P,S)=1$ $\not\Leftarrow$ content zero

Thm: $S\subseteq I$ compact interval $\Rightarrow \overline{c}(\partial S)=\overline{c}(S)-\underline{c}(S)$

pf: $P\in\mathcal{P}(I)$ $J(P,\partial S)=\overline{J}(P,S)-\underline{J}(P,S)\ge\overline{c}(S)-\underline{c}(S)$ $\inf_P \overline{J}(P,\partial S)=\overline{c}(\partial S)\ge\overline{c}(S)-\underline{c}(S)$

given $\varepsilon>0$ $\exists P_1\in\mathcal{P}(I)$, $\exists P_2\in\mathcal{P}(I)$ s.t. $\overline{J}(P_1,S)<\overline{c}(S)+\varepsilon$ $\underline{J}(P_2,S)>\underline{c}(S)-\varepsilon$

let $P=P_1\cup P_2$ $\overline{c}(\partial S)\le\overline{J}(P,\partial S)=\overline{J}(P,S)-\underline{J}(P,S)\le\overline{J}(P_1,S)-\underline{J}(P_2,S)<\overline{c}(S)-\underline{c}(S)+2\varepsilon$

$\overline{c}(\partial S)<\overline{c}(S)-\underline{c}(S)+2\varepsilon \Rightarrow \overline{c}(\partial S)\le\overline{c}(S)-\underline{c}(S)$

$S \subseteq I$ compact interval $f: S \to \mathbb{R}$ bounded Define $g: I \to \mathbb{R}$, $x \mapsto g(x) = \begin{cases} f(x) & x \in S \\ 0 & x \in I-S \end{cases}$

Def: $f \in R$ on $S$ if $g \in R$ on $I$ write $\int_I g(x)dx = \int_S f(x)dx$

Thm: $S \subseteq I$ compact interval Assume $S$ is Jordan measurable $f: S \to \mathbb{R}$ bounded

then $f \in R$ on $S \Leftrightarrow$ the set of discontinuities of $f$ on $S$ is n-measure zero

Thm: $S$: compact Jordan measurable $\Rightarrow \int_S 1\,dx$ exsits and $\int_S 1\,dx = c(S)$

pf: $S \subseteq I$ $P \in \mathcal{P}(I)$ $\int_S 1\,dx = \inf_P U(P,f)$ $U(P,f) = \bar{J}(P,S) = \bar{c}(S) = c(S)$

Thm: $f \in R$ on $S \subseteq \mathbb{R}^n$ Jordan measurable, suppose $S = A \cup B$ $A, B$: Jordan measurable $\mathring{A} \cap \mathring{B} = \emptyset$ 5/2

then $f \in R$ on $A$, $f \in R$ on $B$ and $\int_S f\,dx = \int_A f\,dx + \int_B f\,dx$

pf: $f \in R$ on $A, B$ $\because f \in R$ on $S$ disconti of $f$ on $S$ is n-measure zero

$S = \sum f(t_k)\mu(I_k) = \underset{I_k \cap A \neq \emptyset}{S_A} + \underset{I_k \cap B \neq \emptyset}{S_B} - \underset{\substack{I_k \cap A \neq \emptyset \\ I_k \cap B \neq \emptyset}}{S_C}$

Thm: $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ conti $\phi_1(x) < \phi_2(x)$ $x \in [a,b]$ let $S = \{(x,y) \mid a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}$

$f: S \to \mathbb{R}$ bounded If $f \in R$ on $S$ then $\iint_S f(x,y)\,dx\,dy = \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$

Thm: $S$: Jordan measurable $f \in R$, $g \in R$ on $S$ suppose $f \le g$ on $S$ $\Rightarrow \int_S f\,dx \le \int_S g\,dx$

Thm: $S$: Jordan measurable $f, g \in R$ on $S$ $g \ge 0$ $m = \inf_S f$ $M = \sup_S f$ then $\exists\, m \le \lambda \le M$ s.t. $m \int_S g\,dx \le \underset{= \lambda \int_S g(x)dx}{\int_S fg\,dx} \le M \int_S g\,dx$

pf: $m \le f(x) \le M$ $m g(x) \le f(x)g(x) \le M g(x)$ $m \int_S g(x)dx \le \int_S f(x)g(x)dx \le M \int_S g(x)dx$

① $\int_S g(x)dx = 0$ $\lambda$: arbitrary ② $\int_S g(x)dx > 0$ $\lambda = \dfrac{\int_S f(x)g(x)\,dx}{\int g(x)\,dx}$

In particular, $g(x) \equiv 1$ $m\,c(S) \le \int_S f(x)dx \le M\,c(S)$

Thm: $S$: Jordan measurable $f \in R$ on $S$ $T \subseteq S$ $c(T) = 0$ $g: S \to \mathbb{R}$ bounded $g = f$ on $S - T \Rightarrow g \in R$ on $S$ and $\int_S g(x)dx = \int_S f(x)dx$

## Sequence of Functions — Rudin

Q1: Let $\{f_k\}$ be a seq of uniformly bounded functions on E. Does $\{f_k\}$ have a subseq of functions that conv ptwise on E?

Q2: Let $\{f_k\}$ be a seq of conv functions on E (ptwise). Does $\{f_k\}$ have a subseq of functions that conv uniformly on E?

Ex: $f_k(x) = \sin(kx)$ $x \in [0, 2\pi]$ $k \in \mathbb{N}$ $|f_k(x)| \leq 1$ $[0, 2\pi]$ compact

Lebesgue dominated convergence theorem (LDCT) $f_n(x)$ conti on E, $|f_n(x)| \leq \phi(x)$ $\int_E \phi(x)dx < \infty$ $\lim_{n\to\infty} f_n(x) = f(x)$ (pointwise)

then $\lim_{n\to\infty} \int_E f_n(x)dx = \int_E \lim_{n\to\infty} f_n(x)\,dx = \int_E f(x)\,dx$

pf of Ex: suppose $\{f_k(x)\}$ has a subseq of functions $\{f_{k_j}(x)\}$ that conv ptwise on $[0, 2\pi]$

$\lim_{j\to\infty} (\sin k_{j+1}x - \sin k_j x) = 0$ $\lim_{j\to\infty} (\sin k_{j+1}x - \sin k_j x)^2 = 0$ $|\sin k_{j+1}x - \sin k_j x|^2 \leq 4$ $\int_0^{2\pi} 4\,dx = 8\pi$

$\lim_{j\to\infty} \int_0^{2\pi} (\sin k_{j+1}x - \sin k_j x)^2 dx \overset{LDCT}{=} \int_0^{2\pi} \lim_{j\to\infty} (\sin k_{j+1}x - \sin k_j x)^2 dx = \int_0^{2\pi} 0\,dx = 0$

$\lim_{j\to\infty} \int_0^{2\pi} (\sin^2 k_{j+1}x + \sin^2 k_j x - 2\sin k_j x \sin k_{j+1}x)\,dx = \lim_{j\to\infty} 2\pi = 2\pi$ $\cancel{=}$

$\int_0^{2\pi} \sin^2\theta\,d\theta = \int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta = \frac{\theta}{2} - \sin 2\theta \Big|_0^{2\pi} = \pi$

$\int_0^{2\pi} \sin\alpha x \sin\beta x\,dx = \int_0^{2\pi} \frac{1}{2}(\cos(\alpha-\beta)x - \cos(\alpha+\beta)x)\,dx = 0$

Ex: $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$ $x \in [0,1]$ $f_n(0) = \frac{0}{1} = 0$ $\forall n$ $\lim_{n\to\infty} f_n(x) = 0$ $\forall x \in [0,1]$ $\because f_n(\frac{1}{n}) = 1$

Def: Let $\mathcal{F}$ be a collection of functions on $(E, d)$. We say that $\mathcal{F}$ is equicontinuous on $(E, d)$

if given $\varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ if $d(x,y) < \delta$ $\forall f \in \mathcal{F}$

Thm (Arzelà–Ascoli) Suppose that $\mathcal{F}$ is a pointwise bounded equicontinuous collection of complex functions on a metric space X

and that X contains a countable dense subset E then every seq $\{f_n\}$ in $\mathcal{F}$ has a subseq

that converges uniformly on every compact subsets of X

Thm: Let $(K, d)$ be a compact metric space, then K is separable

pf: $K = \bigcup_{x\in K} B(x; \frac{1}{k}) = \bigcup_{j=1}^{m_k} B(x_j^{(k)}, \frac{1}{k})$ let $E = \bigcup_{k=1}^{\infty} \{x_1^{(k)}, \dots, x_{m_k}^{(k)}\}$ countable

Thm: Let $\{f_k\}$ be a pointwise bounded seq on a countable set E then $\{f_k\}$ has a subseq $\{f_{n_k}\}$ that convs on every pt of E

pf: write $E = \{x_1, \dots, x_\ell, \dots\}$. $\{f_k(x_1)\}$ bounded $f_{11}, \dots, f_{1n}, \dots$ $\{f_{1k}(x_2)\}$ bounded $f_{21}, \dots, f_{2n}, \dots$

$\{f_{2k}(x_3)\}$ bounded $f_{31}, \dots, f_{3n}, \dots$ ... ... $f_{m1}, f_{m2}, \dots, f_{mn}, \dots$

pick $f_{11}, f_{22}, \dots, f_{mm}, \dots$ it conv on every pt of E

Thm: $\{f_n\}$ is a seq of conti functions on a compact metric space $K$

If $\{f_n\}$ converges uniformly on $K$ then $\{f_n\}$ are equicontinuous on $K$

pf: given $\varepsilon>0$ $\exists\, n_0\in\mathbb{N}$ s.t. $|f_{n+k}(x)-f_n(x)|<\varepsilon$ $\forall x\in K$ $k\in\mathbb{N}$ $n\geq n_0$

$\exists\,\delta>0$ s.t. $|f_{n_0}(x)-f_{n_0}(y)|<\varepsilon$ if $d(x,y)<\delta$

For $m>n_0$ $|f_m(x)-f_m(y)|\leq|f_m(x)-f_{n_0}(x)|+|f_{n_0}(x)-f_{n_0}(y)|+|f_{n_0}(y)-f_m(y)|<3\varepsilon$

let $\delta_j>0$ be the corresponding $\delta$ for $f_j$ $1\leq j\leq n_0-1$ choose $\delta_0=\min\{\delta,\delta_1,\cdots,\delta_{n_0-1}\}$

Thm (Arzela-Ascoli) Let $\mathcal{F}$ be a collection of pointwise bounded equicontinuous complex functions on a metric space $X$ that contains a countable dense subset, then every seq of functions from $\mathcal{F}$ contains a subseq that converges uniformly on each compact subset of $X$

pf: let $\{f_n(x)\}\subseteq\mathcal{F}$. let $E=\{x_1,x_2,\cdots\}$ be a countable dense subset of $X$

$\therefore$ By Cantor diagonal process we can extract a subseq $\{f_{n_k}\}$ of $\{f_n\}$ s.t. $\{f_{n_k}\}$ converge at every point in $E$

let $K\subseteq X$ be a compact subset. By equicontinuity of $\mathcal{F}$ given $\varepsilon>0$ $\exists\,\delta>0$ s.t. $\forall f\in F$ $|f(x)-f(y)|<\varepsilon$ if $d(x,y)<\delta$

$K=\bigcup_{y\in K}B(y;\frac{\delta}{2})=\bigcup_{j=1}^{m}B(y_j;\frac{\delta}{2})$ Since $E$ is dense in $X$ hence $\exists\, x_j\in E$ s.t. $x_j\in B(y_j;\frac{\delta}{2})$ $1\leq j\leq m$

$\because \{f_{n_k}(x_j)\}$ converges given $\varepsilon>0$ $\exists\, k_0\in\mathbb{N}$ s.t. if $k,k'\geq k_0$ $k\in\mathbb{N}$ $1\leq j\leq m$ $|f_{n_k}(x_j)-f_{n_{k'}}(x_j)|<\varepsilon$

If $p\in K$ $p\in B(y_j;\frac{\delta}{2})$ for some $j$ $\therefore d(p,x_j)<\delta$

$|f_{n_k}(p)-f_{n_{k'}}(p)|\leq|f_{n_k}(p)-f_{n_k}(x_j)|+|f_{n_k}(x_j)-f_{n_{k'}}(x_j)|+|f_{n_{k'}}(x_j)-f_{n_{k'}}(p)|<3\varepsilon$

Thm (Weierstrass approximation thm) Let $f(x)$ be a real-valued continous function on $[a,b]$

then given $\varepsilon>0$ $\exists$ a polynomial $p(x)$ s.t. $|f(x)-p(x)|<\varepsilon$ $\forall x\in[a,b]$

pf: consider $t=\frac{x-a}{b-a}$ may assume $[a,b]=[0,1]$ may assume $f(0)=f(1)=0$ $g(x)=f(x)-(f(0)+x(f(1)-f(0)))$ $g(0)=g(1)=0$

For each $n\in\mathbb{N}$ Define $Q_n(x)=c_n(1-x^2)^n$ on $[-1,1]$ s.t. $\int_{-1}^{1}Q_n(x)dx=1$

$1=c_n\int_{-1}^{1}(1-x^2)^n dx>c_n\cdot 2\int_0^{\frac{1}{\sqrt{n}}}(1-nx^2)dx=2c_n(x-\frac{n}{3}x^3)\Big|_0^{\frac{1}{\sqrt{n}}}=2c_n(\frac{1}{\sqrt{n}}-\frac{1}{3}\frac{1}{\sqrt{n}})=\frac{4}{3}\frac{1}{\sqrt{n}}\cdot c_n>\frac{c_n}{\sqrt{n}}\Rightarrow c_n<\sqrt{n}$

$\Big(\underset{h(0)=0}{h(x)=(1-x^2)^n-(1-nx^2)}\nearrow\ \because h'(x)=n(1-x^2)(-2x)+2nx=2nx(1-(1-x^2)^n)>0\Big)$ $x\geq\delta$ $Q_n(x)=c_n(1-x^2)^n\leq c_n(1-\delta^2)^n\to 0$

define $0\leq x\leq 1$ $P_n(x)=\int_{-1}^{1}f(x+t)Q_n(t)dt\overset{s=x+t}{=}\int_{-1+x}^{1+x}f(s)Q_n(s-x)ds=\int_0^1 f(s)Q_n(s-x)ds$ polynomial in $x$

Estimate $|P_n(x)-f(x)|=\left|\int_{-1}^{1}f(x+t)Q_n(t)dt-\int_{-1}^{1}f(x)Q_n(t)dt\right|=\left|\int_{-1}^{1}(f(x+t)-f(x))Q_n(t)dt\right|\leq\int_{-1}^{1}|f(x+t)-f(x)|Q_n(t)dt$

(given $\varepsilon>0$ $\exists\,\delta>0$ s.t. $|f(\alpha)-f(\beta)|<\varepsilon$ if $|\alpha-\beta|<\delta$) $=\int_{-1}^{-\delta}+\int_{-\delta}^{\delta}+\int_{\delta}^{1}=\mathrm{I}+\mathrm{II}+\mathrm{III}\leq 2\varepsilon$

① $\mathrm{II}\leq\varepsilon\int_{-\delta}^{\delta}Q_n(t)dt\leq\varepsilon\int_{-1}^{1}Q_n(t)dt=\varepsilon$ ② $\mathrm{I},\mathrm{III}$ $|f(x)|\leq M$ $|f(x+t)|\leq M$ $\mathrm{I},\mathrm{III}\leq 2M\left(\int_{-1}^{-\delta}Q_n(t)dt+\int_{\delta}^{1}Q_n(t)dt\right)\leq 2M\cdot 2c_n(1-\delta^2)^n\leq 4M\sqrt{n}(1-\delta^2)^n\to 0$

Cor: For $f(x)=|x|$ and $a>0$, $\exists$ a seq of poly $p_n(x)$ s.t. $p_n(0)$ and $p_n(x)$ conv to $f(x)$ uniformly on $[-a,a]$

on $[-a,a]$ $\exists\, q_n(x)\to|x|$ uniformly $q_n(0)=0$ let $p_n(x)=q_n(x)-q_n(0)$

STONE: K: compact set, $\mathcal{A}$: functions space on K, $\mathcal{A}$ is called an algebra

if (i) $f+g\in\mathcal{A}$ if $f,g\in\mathcal{A}$ (ii) $cf\in\mathcal{A}$ if $f\in\mathcal{A}$ $c\in\mathbb{C}$ or $\mathbb{R}$ (iii) $fg\in\mathcal{A}$ if $f,g\in\mathcal{A}$

uniform cloure $\mathcal{B}=\{g \mid g$ is the uniform limit of $f_n$ $f_n\in\mathcal{A}\}$

$\mathcal{A}$ is called uniformly closed if $\mathcal{A}$ contains all its uniform limit functions

Thm: $\mathcal{B}$ is uniformly closed

pf: if $f,g\in\mathcal{B}$ $\therefore \exists f_n\in\mathcal{A}, g_n\in\mathcal{A}$ s.t. $\begin{matrix} f_n\to f \\ g_n\to g\end{matrix}$ uniformly $\mathcal{A}=\{$bounded function on K$\}$

$f_ng_n\to fg$ $\quad |f_ng_n-fg-f_ng+f_ng|\le |f_n||g_n-g|+|f-f_n||g_n|$

$cf_n\to cf$ $\quad |cf-cf_n|\le|c||f-f_n|$

$f_n+g_n\to f+g$ $\quad |f_n+g_n-f-g|\le|f_n-f|+|g_n-g|$ $\quad\to \mathcal{B}$ is an algebra

$h_n\in\mathcal{B}$ $h_n\to h$ given $\varepsilon>0$ $\exists h_n$ s.t. $|h_n-h|<\varepsilon$ $\forall x\in K$ $\exists f_n\in\mathcal{A}$ s.t. $|f_n-h_n|<\varepsilon \Rightarrow |h-f_n|<2\varepsilon$ $\forall x$ $\therefore h\in\mathcal{B}$

Def: $\mathcal{A}$ is an algebra on K. We say $\mathcal{A}$ separates points of K if $\forall x,y\in K$ $x\neq y$ $\exists f\in\mathcal{A}$ s.t. $f(x)\neq f(y)$

and we say $\mathcal{A}$ vanishes at no point of K if $\forall p\in K$ $\exists$ a $f\in\mathcal{A}$ s.t. $f(p)\neq0$

Thm: $\mathcal{A}$ is a function algebra on K which separates points of K and vanishes at no point of K

then given two points $x_1\neq x_2$ of K and two const $c_1,c_2$ $\exists f\in\mathcal{A}$ s.t. $f(x_1)=c_1$ $f(x_2)=c_2$

pf: Since $\mathcal{A}$ separates point of K $\therefore\exists f\in\mathcal{A}$ s.t. $f(x_1)\neq f(x_2)$ ~~consider $g(x)=c_1\frac{f(x)-f(x_2)}{f(x_1)-f(x_2)}+\frac{f(x)-f(x_1)}{f(x_2)-f(x_1)}c_2$~~ ($f(x_1),f(x_2)$ may not in $\mathcal{A}$)

$\exists\,\alpha\in\mathcal{A},\beta\in\mathcal{A}$ s.t. $\alpha(x_1)\neq0$ $\beta(x_2)\neq0$ $\quad g(x)=c_1\frac{\alpha(x)(f(x)-f(x_2))}{\alpha(x_1)(f(x_1)-f(x_2))}+\frac{(f(x)-f(x_1))\beta(x)}{(f(x_2)-f(x_1))\beta(x_2)}c_2$

Thm (STONE) $\mathcal{A}$ is a real algebra on a compact set K suppose $\mathcal{A}$ separates points of K and $\mathcal{A}$ vanishes no point of K

then every real continuous function on K can be approximated uniformly by functions of $\mathcal{A}$, i.e, $\mathcal{B}=\{$real conti func on K$\}$

pf: (I) $f\in\mathcal{B}$ let $M=\sup_K|f|$ $\therefore$ given $\varepsilon>0$ $\exists\sum_{i=1}^m c_iy^i$ s.t. $\left|\sum_{i=1}^m c_iy^i-|y|\right|<\varepsilon$ on $[-M,M]$ $\Rightarrow\left|\sum_{i=1}^m c_if(x)^i-|f(x)|\right|<\varepsilon$ $\forall x\in K$

(II) $f,g\in\mathcal{B}\Rightarrow\max(f,g),\min(f,g)\in\mathcal{B}\Rightarrow\max\{f_1,\cdots,f_n\}\in\mathcal{B}$ $\min\{f_1,\cdots,f_n\}\in\mathcal{B}$ $\because \max\{f,g\}=\frac{f+g}{2}+\underbrace{\frac{|f-g|}{2}}_{\in\mathcal{B}}$ $\min\{f,g\}=\frac{f+g}{2}-\frac{|f-g|}{2}\in\mathcal{B}$

(III) f: real conti on K, $x\in K$ claim given $\varepsilon>0$ $\exists h_x\in\mathcal{B}$ s.t. $h_x(x)=f(x)$ $h_x(y)>f(y)-\varepsilon$ $\forall y\in K$

By thm, $\exists\, g_y\in\mathcal{B}$ $y\in K$ s.t. $g_y(x)=f(x)$ $g_y(y)=f(y)$ $\therefore g_y(y')>f(y')-\varepsilon$ if $y'\in B(y;\delta_y)$

$\therefore K\subseteq\bigcup_{y\in K}B(y;\delta_y)$ By compactness of K $\therefore K\subseteq\bigcup_{j=1}^{l}B(y_j;\delta_{y_j})$ let $h_x=\max\{g_{y_1},\cdots,g_{y_l}\}\in\mathcal{B}$ $h_x(x)=f(x)$

$h_x(y)\ge g_{y_j}(y)>f(y)-\varepsilon$ $\quad y\in B(y_j;\delta_{y_j})$

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(IV) given $\varepsilon>0$ $\exists\, g\in\mathcal{B}$ $f(x)-\varepsilon<g(x)<f(x)+\varepsilon$ $\forall x\in K$

$x\in K$ $\exists\, h_x$ s.t. $h_x(x)=f(x)$ $h_x(y)>f(y)-\varepsilon$ $\exists\,\delta_x>0$ s.t. $h_x(x')<f(x')+\varepsilon$ $x'\in B(x;\delta_x)$

$\therefore K\subseteq\bigcup_{x\in K}B(x;\delta_x)$ $\therefore K\subseteq\bigcup_{p=1}^{m}B(x_p;\delta_{x_p})$ $g(x)=\min\{h_{x_1},\cdots h_{x_m}\}\in\mathcal{B}$ $x\in K$ $f(x)-\varepsilon<g(x)\leq h_{x_p}(x)<f(x)+\varepsilon$

Def: A complex algebra is called self-adjoint if $\bar{f}\in\mathcal{A}$ for $f\in\mathcal{A}$

Thm: Let $\mathcal{A}$ be a self-adjoint complex algebra on $K$. $\mathcal{A}$ separates points of $K$ and vanishes at no points of $K$

then $\mathcal{B}=\{$complex conti func on $K\}$

pf: Let $A_R=\{$real conti func $\in\mathcal{A}\}$ $f=u+iv\in\mathcal{B}$ $\bar{f}=u-iv\in\mathcal{B}$ self-adjoint $\frac{f+\bar{f}}{2}=u\in A_R$

$x\neq y$ $f\in\mathcal{A}$ $f(x)=1$ $f(y)=0$ $u(x)=1$ $u(y)=0$ $x\in K$ $f(x)\neq0$ $e^{i\theta}f(x)>0$

$g=\tilde{u}+i\tilde{v}$ $\tilde{u}\in\mathcal{B}$ $\tilde{v}\in\mathcal{B}$ $\Rightarrow\tilde{u}+i\tilde{v}\in\mathcal{B}$

The Lebesgue Theory - Rudin

$X=\mathbb{R}^n$ $\Sigma=\{\text{some subset of } \mathbb{R}^n\}$ $\Sigma$ is said to be an algebra (ring) if $A\cdot B\in\Sigma$ then $A\cup B\in\Sigma$ $A-B\in\Sigma$

Algebra $\Sigma$ is said to be a $\sigma$-algebra if $A_k\in\Sigma, k\in\mathbb{N}$ then $\bigcup_{k=1}^{\infty} A_k\in\Sigma$

$\mathbb{R}^n-A=A^c\in\Sigma$, $A-A=\phi\in\Sigma$, $A_k\in\Sigma, k\in\mathbb{N}$ $A_k^c\in\Sigma$ $\bigcup_{k=1}^{\infty} A_k^c=\bigcap_{k=1}^{\infty} A_k\in\Sigma$

Set function: $\Sigma$: $\sigma$-algebra $\phi:\Sigma\to \begin{matrix}\mathbb{R}\cup\{\infty\}\\ \mathbb{R}\cup\{-\infty\}\end{matrix}$

A set function $\phi$ is said to be additive if $A\cdot B\in\Sigma$ $A\cap B=\phi$ then $\phi(A\cup B)=\phi(A)+\phi(B)$

" countably additive if $A_k\in\Sigma$ $A_j\cap A_k=\phi$ $j\neq k$ then $\phi(\bigcup_{k=1}^{\infty}A_k)=\sum_{k=1}^{\infty}\phi(A_k)$ $\Rightarrow \phi(\phi)=0$

$A\cdot B\in\Sigma$ additive $\phi(A\cup B)+\phi(A\cap B)=\phi(A)+\phi(B-A)+\phi(A\cap B)=\phi(A)+\phi(B)$

Thm: If $\phi\geq 0$ $A\subseteq B\Rightarrow\phi(B)\geq\phi(A)$, pf: $B=A\cup(B-A)$ $\phi(B)=\phi(A)+\underbrace{\phi(B-A)}_{\geq 0}$

Thm: $\phi$ is a countably additive set func on $\Sigma$ then if $A_k\in\Sigma$ $A_k\subseteq A_{k+1}$ $\forall k$ then $\phi(\bigcup_{k=1}^{\infty}A_k)=\lim_{k\to\infty}\phi(A_k)$

pf: $\bigcup_{k=1}^{\infty}A_k=A_1\cup(A_2-A_1)\cup\cdots\cup(A_k-A_{k-1})\cup\cdots$ $\phi(\bigcup_{k=1}^{\infty}A_k)=\phi(A_1)+\sum_{j=2}^{\infty}\phi(A_j-A_{j-1})$

$A_k=A_1\cup(A_2-A_1)\cup\cdots\cup(A_k-A_{k-1})$ $\phi(A_k)=\phi(A_1)+\sum_{j=2}^{k}\phi(A_j-A_{j-1})$

Intervals in $\mathbb{R}^n$, $I=\{(x_1,\dots,x_n)\mid a_k \lesseqgtr x_k \lesseqgtr b_k;\ a_k\cdot b_k\in\mathbb{R}\ k=1,2,\dots,n\}$ $\mathcal{E}=\{\text{elementary sets}\}$ elementary set: $\bigcup_{k=1}^{l} I_k$

Define the measure $m(I)=\prod_{k=1}^{n}(b_k-a_k)$ $I_1,\dots,I_l$: intervals $I_i\cap I_j=\phi$ if $i\neq j$ $m(I=I_1\cup\cdots\cup I_l)=\sum_{j=1}^{l}m(I_j)$

Fact ① $\mathcal{E}$ is an algebra but not a $\sigma$-algebra ② $A\in\mathcal{E}$ $A=\bigcup_{k=1}^{l}I_k$ $I_i$: disjoint intervals ③ well-define ④ $m$ is additive on $\mathcal{E}$

Def: $\phi$: nonnegative additive set func on $\mathcal{E}$ we say $\phi$ is regular

If $A\in\mathcal{E}$ given $\varepsilon>0$ $\exists$ $F\in\mathcal{E}$ $F$: closed $G\in\mathcal{E}$ $G$: open $F\subseteq A\subseteq G$ s.t. $\phi(G)-\varepsilon\leq\phi(A)\leq\phi(F)+\varepsilon$ ∴ $m$ is regular

Lebesgue-Stieltjes additive set func on $\mathbb{R}^1$, $f$ is an increasing nonnegative func on $\mathbb{R}^1$

$\mu((a,b))=f(b-)-f(a+)$ $\mu((a,b])=f(b+)-f(a+)$ $\mu([a,b])=f(b+)-f(a-)$ $\mu([a,b))=f(b-)-f(a-)$

Def: $\mathcal{E}$ algebra $\mu$: nonnegative, additive, regular set func on $\mathcal{E}$, $E\subseteq\mathbb{R}^n$

define $\mu^*(E)=\inf_{A_n}\sum_{n}\mu(A_n)$ $A_n$: open elementary set s.t. $E\subseteq\bigcup_{n=1}^{\infty}A_n$ $\mu^*$ is called outer measure of $E$

① $\mu^*(E)\geq 0$ ② $E_1\subseteq E_2$ $\mu^*(E_2)\geq\mu^*(E_1)$

Thm ① $E\in\mathcal{E} \Rightarrow \mu^*(E)=\mu(E)$ ② $\mu^*(\bigcup_{k=1}^{\infty} E_k) \le \sum_{k=1}^{\infty} \mu^*(E)$ sub additivity

pf ① $E\in\mathcal{E}$ given $\varepsilon>0$ $\exists$ open elementary $G$ s.t. $E\subseteq G$ and $\mu(G)\le\mu(E)+\varepsilon$

$\because$ $\mu^*(E)\le\mu(G)$ and $\varepsilon$ is arbitrary $\therefore$ $\mu^*(E)\le\mu(E)$

given $\varepsilon>0$ $\exists\{A_n\}$ open $E\subseteq\bigcup_{n=1}^{\infty}A_n$ s.t. $\sum\mu(A_n)\le\mu^*(E)+\varepsilon$

$\exists F\in\mathcal{E}$ $F$: closed $F\subseteq E$ s.t. $\mu(E)\le\mu(F)+\varepsilon\le\mu(A_1\cup\cdots\cup A_N)+\varepsilon\le\sum_{n=1}^{N}\mu(A_N)+\varepsilon\le\mu^*(E)+2\varepsilon\Rightarrow\mu(E)\le\mu^*(E)$

② given $\varepsilon>0$ For each $k$ $\exists\{A_n^{(k)}\}$ open elementary s.t. $E_k\subseteq\bigcup_{n=1}^{\infty}A_n^{(k)}$ $\sum_{k=1}^{\infty}\mu(A_n^{(k)})\le\mu^*(E_k)+\frac{\varepsilon}{2^k}$

$\bigcup_{k=1}^{\infty}E_k\subseteq\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}A_n^{(k)}$ $\mu^*(\bigcup_{k=1}^{\infty}E_k)\le\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu(A_n^{(k)})\le\sum_{k=1}^{\infty}\mu^*(E_k)+\sum_{k=1}^{\infty}\frac{\varepsilon}{2^k}\le\sum_{k=1}^{\infty}\mu^*(E_k)+\varepsilon$

$A\cdot B$: two sets, define symmetric difference of $A$ and $B$ by $S(A\cdot B)=(A-B)\cup(B-A)$ $S(A\cdot A)=\emptyset$ define $d(A\cdot B)=\mu^*(S(A\cdot B))$

Fact ① $S(A\cdot B)=S(A^c\cdot B^c)$ $A-B=A\cap B^c=B^c\cap(A^c)^c=B^c-A^c$ $S(A\cdot B)=(A-B)\cup(B-A)=(B^c-A^c)\cup(A^c-B^c)=S(A^c\cdot B^c)$

② $S(A_1\cup A_2\cdot B_1\cup B_2)\subseteq S(A_1\cdot B_1)\cup S(A_2\cdot B_2)$

pf: $(A_1\cup A_2)-(B_1\cup B_2)=(A_1-B_1\cup B_2)\cup(A_2-B_1\cup B_2)\subseteq(A_1-B_1)\cup(A_2-B_2)$, $(B_1\cup B_2)-(A_1\cup A_2)\subseteq(B_1-A_1)\cup(B_2-A_2)$

③ $S(A_1\cap A_2\cdot B_1\cap B_2)\subseteq S(A_1\cdot B_1)\cup S(A_2\cdot B_2)$

pf: $S(A_1\cap A_2\cdot B_1\cap B_2)=S((A_1\cap A_2)^c\cdot(B_1\cap B_2)^c)=S(A_1^c\cup A_2^c\cdot B_1^c\cup B_2^c)\subseteq S(A_1^c\cdot B_1^c)\cup S(A_2^c\cdot B_2^c)=S(A_1\cdot B_1)\cup S(A_2\cdot B_2)$

④ $S(A_1-A_2\cdot B_1-B_2)\subseteq S(A_1\cdot B_1)\cup S(A_2\cdot B_2)$

pf: $A_1-A_2=A_1\cap A_2^c$ $S(A_1-A_2\cdot B_1-B_2)=S(A_1\cap A_2^c\cdot B_1\cap B_2^c)\subseteq S(A_1\cdot B_1)\cup S(A_2^c\cdot B_2^c)=S(A_1\cdot B_1)\cup S(A_2\cdot B_2)$

⑤ $S(A\cdot B)\subseteq S(A\cdot C)\cup S(C\cdot B)$ pf: $A-B\subseteq(A-C)\cup(C-B)$ $B-A\subseteq(B-C)\cup(C-A)$

Fact ① $d(A\cdot A)=0$ ② $d(A\cdot B)=d(B\cdot A)$ ③ $d(A\cdot C)\le d(A\cdot B)+d(B\cdot C)$ (has no $d(A\cdot B)=0$ iff $A=B$)

Ex: $A$ = countable set $B=\emptyset$ $S(A\cdot B)=(A-B)\cup(B-A)=A$ $d(A\cdot B)=\mu^*(S(A\cdot B))=\mu^*(A)=0$

Define a relation $\sim$ among all subsets $A\sim B\Leftrightarrow d(A\cdot B)=0$ $A\sim A$ $A\sim B\Rightarrow B\sim A$ $A\sim B$, $B\sim C\Rightarrow A\sim C$ (equivalent relation)

let $\mathfrak{M}_F(\mu)=\{A\mid\exists A_n\in\mathcal{E}$ s.t. $d(A_n\cdot A)=0\}$ $\mathfrak{M}(\mu)=\{$countable union of elements in $\mathfrak{M}_F(\mu)\}$

Fact ⑥ $|\mu^*(A)-\mu^*(B)|\le d(A\cdot B)$ if one of $\mu^*(A)$ and $\mu^*(B)$ is finite

pf: WLOG $\mu^*(B)<\infty$ $0\le\mu^*(B)\le\mu^*(A)$ $A=S(A\cdot\emptyset)\subseteq S(A\cdot B)\cup\underbrace{S(B\cdot\emptyset)}_{B}$

$\mu^*(A)\le\mu^*(S(A\cdot B))+\mu^*(B)$ $\mu^*(A)-\mu^*(B)\le d(A\cdot B)$

If $A\in\mathfrak{M}_F(\mu)$ then $\mu^*(A)<\infty$ by def $\exists A_n\in\mathcal{E}$ s.t. $\lim_{n\to\infty}d(A_n\cdot A)=0$ $|\mu^*(A)-\mu^*(A_n)|\le d(A_n\cdot A)\to0$

Thm: $m(\mu)$ is a $\sigma$-algebra and $\mu^*$ is countably additive on $m(\mu)$ (if $E \in m(\mu)$ use $\mu(E) = \mu^*(E)$)

pf: claim: $m_F(\mu)$ is an algebra. $\mu^*$ is additive on $m_F(\mu)$

(i) $A, B \in m_F(\mu)$ $\exists A_n, B_n \in \mathcal{E}$ s.t. $A_n \to A$, $B_n \to B$ $\Rightarrow$ $A_n \cup B_n \to A \cup B$, $A_n \cap B_n \to A \cap B$, $A_n - B_n \to A - B$

(ii) claim $\mu^*(A) < \infty$ : $|\mu^*(A_n) - \mu^*(A)| \leq d(A_n, A)$, $\mu(A_n \cup B_n) + \mu(A_n \cap B_n) = \mu(A_n) + \mu(B_n)$ $\xrightarrow{n\to\infty}$ $\mu^*(A \cup B) + \underbrace{\mu^*(A \cap B)}_{\text{if } = \emptyset} = \mu^*(A) + \mu^*(B)$

$\mu^*$ is countably additive on $m(\mu)$ $A \in m(\mu)$ $\therefore A = \bigcup_{n=1}^{\infty} A_n'$ $A_n' \in m_F(\mu)$ let $A_1 = A_1'$

$n \geq 2$ $A_n = (A_1' \cup \cdots \cup A_n') - (A_1' \cup \cdots \cup A_{n-1}')$ $\therefore A = \bigcup_{n=1}^{\infty} A_n$ disjoint. $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ subadditive

$\mu^*(A) \geq \mu^*(\bigcup_{n=1}^{m} A_n) = \sum_{n=1}^{m} A_n$ $m \to \infty$ $\therefore \mu^*(A) \geq \sum_{n=1}^{\infty} \mu^*(A_n)$ $\therefore \mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n)$

If $\mu^*(A) < \infty$ $d(A, \bigcup_{n=1}^{m} A_n) = \mu^*(\bigcup_{n=m+1}^{\infty} A_n) \leq \sum_{n=m+1}^{\infty} \mu^*(A_n) \to 0$

$\bigcup_{n=1}^{m} A_n \underset{\varepsilon}{\overset{\in m_F(\mu)}{\rightarrow}} A^{\in m(\mu)}$ $\mu^*(A) < \infty$ $\therefore A \in m_F(\mu)$, $A \in m(\mu)$, $\mu^*(A) < \infty \Leftrightarrow A \in m_F(\mu)$

$A \in m(\mu)$, $A_k \in m(\mu)$ pairwise disjoint s.t. $A = \bigcup_{k=1}^{\infty} A_k$ $\mu^*(A) = \sum_{k=1}^{\infty} \mu^*(A_k)$

(i) If $\mu^*(A_k) = \infty$ for some $k$ done (ii) If $\mu^*(A_k) < \infty$ $\forall k$ done

pf: $m(\mu)$ is a $\sigma$-algebra $A, B \in m(\mu) \Rightarrow A - B \in m(\mu)$, $\underbrace{A_k \in m(\mu)\ \forall k \Rightarrow \bigcup_{k=1}^{\infty} A_k \in m(\mu)}_{\text{trivial}}$

write $A = \bigcup_{n=1}^{\infty} A_n$ $A_n \in m_F(\mu)$ $B = \bigcup_{n=1}^{\infty} B_n$ $B_n \in m_F(\mu)$, $A - B = \bigcup_{n=1}^{\infty} A_n - B = \bigcup_{n=1}^{\infty} (A_n - B)$ goal: $A_n - B \in m_F(\mu)$

$A_n - B = A_n - (A_n \cap B)$ $A_n \cap B = A_n \cap (\bigcup_{k=1}^{\infty} B_k) = \bigcup_{k=1}^{\infty} (A_n \cap B_k)$ $A_n \cap B_k \in m_F(\mu)$ $\therefore A_n \cap B \in m(\mu)$

$\mu^*(A_n \cap B) \leq \mu^*(A_n) < \infty$ $\therefore A_n \cap B \in m_F(\mu)$ QED

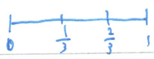
Cor1: open sets $\in m(\mu)$, closed sets $\in m(\mu)$

Cor2: $\mu$ is regular on $m(\mu)$ $A \in m(\mu)$ given $\varepsilon > 0$ $\exists F$ closed $G$ open $F \subseteq A \subseteq G$ s.t. $\mu(G - A) < \varepsilon$ $\mu(A - F) < \varepsilon$

Cor3: $\varepsilon = \frac{1}{k}$ $F_k$ closed $F = \cup F_k \subseteq A$ $\mu(A - F) \leq \mu(A - F_k) < \frac{1}{k} \to 0$ $A = (A - F) \cup F$

$\Sigma_1 \cap \Sigma_2$ $\sigma$-algebra $\bigcap_{\alpha \in \Lambda} \Sigma_\alpha$ $\sigma$-algebra

$\mathcal{B} \subseteq m(\mu)$ Borel $\sigma$-algebra: smallest $\sigma$-algebra that contains open sets $A \in \mathcal{B}$ $A$: Borel set

Cantor Set 

$\lim_{n\to\infty} (\frac{2}{3})^n = 0$ perfect measure space $(X, \Sigma, \mu)$

☆ measurable space $X \in \Sigma$

① $(\mathbb{R}^n, \Sigma, m)$ ② $(\mathbb{N}, 2^{\mathbb{N}}, \#)$ counting measure ③ probability measure

measurable functions $f: \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ extended number system

f is said to be measurable if $\{x \mid f(x) > a\} \in m \quad \forall a \in \mathbb{R}$

Thm: (1) f is measurable $\Leftrightarrow$ (2) $\{x \mid f(x) \le a\} \in m \Leftrightarrow$ (3) $\{x \mid f(x) < a\} \in m \Leftrightarrow$ (4) $\{x \mid f(x) \ge a\}$ $\quad a \in \mathbb{R}$

pf: $\{x \mid f(x) \ge a\} = \bigcap_k \{x \mid f(x) > a - \frac{1}{k}\}$ $\quad \{x \mid f(x) < a\} = \mathbb{R}^n - \{x \mid f(x) \ge a\}$ $\quad \{x \mid f(x) \le a\} = \bigcap_k \{x \mid f(x) < a + \frac{1}{k}\}$ $\quad \{x \mid f(x) > a\} = \mathbb{R}^n - \{x \mid f(x) \le a\}$

Thm: f is measurable $\Rightarrow$ $|f|$: measurable

pf: $a > 0$ $\quad \{x \mid |f(x)| < a\} = \{x \mid f(x) < a\} \cap \{x \mid f(x) > -a\}$

Thm: $\{f_n\}$ measurable then (i) $\sup_n \{f_n\}$ (ii) $\inf_n \{f_n\}$ (iii) $\inf_k \sup_{n \ge k} \{f_n\}$ (iv) $\sup_k \inf_{n \ge k} \{f_n\}$ are measurable 5/26

pf: $\{x \mid \sup_n \{f_n\} > a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > a\}$, $\inf_n \{f_n\} = -\sup_n \{-f_n\}$

$\sup_{n \ge k} \{f_n\} = g_k \ge \sup_{n \ge k+1} \{f_n\} = g_{k+1}$ $\quad \inf_k \sup_{n \ge k} \{f_n\} = \lim_{k \to \infty} \sup_{n \ge k} \{f_n\}$ $\quad \sup_k \inf_{n \ge k} = \lim_{k \to \infty} \inf_{n \ge k} \{f_n\}$

$f_n(x)$ conv at $x$ $\quad \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \sup_{k \ge n} \{f_n\} = \lim_{n \to \infty} \inf_{k \ge n} \{f_n\}$

Cor: f, g measurable $\Rightarrow$ $\max\{f, g\}$, $\min\{f, g\}$ measurable, $f^+ = \max\{f, 0\} \ge 0$ $\quad f^- = -\min\{f, 0\} = \max\{-f, 0\} \ge 0$ $\quad f = f^+ - f^-$

Thm: f, g: measurable, finite, F conti on $\mathbb{R}^2$ $\Rightarrow$ $F(f, g)$ is measurable

pf: $\{x \mid F(f(x), g(x)) > a\} = \{x \mid (f(x), g(x)) \in \underbrace{\{(u,v) \mid F(u,v) > a\}}_{G \,=\, F^{-1}((t>a)) \text{ open}}\}$ $\quad G = \bigcup_{k=1}^{\infty} I_k = \bigcup_{k=1}^{\infty} (a_k, b_k) \times (c_k, d_k)$

$= \bigcup_{k=1}^{\infty} \{(f(x), g(x)) \in I_k\} = \bigcup_{k=1}^{\infty} (\{x \mid a_k < f(x) < b_k\} \cap \{x \mid c_k < g(x) < d_k\})$

$(X, \Sigma, \mu)$ $\quad E \subseteq X$ Define the characteristic function $\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c = X - E \end{cases}$

Thm: $\chi_E$: measurable $\Leftrightarrow E \in \Sigma$

pf: ($\Rightarrow$) $\{x \mid \chi_E > \frac{1}{2}\}$ ($\Leftarrow$) $\{\chi_E > a\}$ $\quad a \ge 1 = \emptyset$ $\quad 0 \le a < 1 = E$ $\quad a < 0 = X$

Simple functions $f(x) = \sum_{j=1}^{m} c_j \chi_{E_j}$ $\quad c_j \in \mathbb{R}$ $\quad E_j \cap E_k = \emptyset$ if $j \ne k$ $\quad s(x)$: measurable $\Leftrightarrow E_j \in \Sigma$ $\quad j = 1, 2, \ldots, m$

Thm: $f \ge 0$ let $E_j^{(k)} = \{x \mid \frac{j-1}{2^k} \le f(x) < \frac{j}{2^k}\}$ $\quad 1 \le j \le k \cdot 2^k$ $\quad F_k = \{x \mid f(x) \ge k\}$ define $s_k(x) = \sum_{j=1}^{k \cdot 2^k} \frac{j-1}{2^k} \chi_{E_j^{(k)}}(x) + k \chi_{F_k}$

then $s(x) \nearrow f(x)$ (uniform if f is bounded)

Thm: If f is measurable $\Rightarrow$ $s_k(x)$ is measurable

pf: $f = f^+ - f^-$ $\quad s_k(x) \nearrow f^+(x)$ $\quad t_k(x) \nearrow f^-(x)$ $\quad s_k(x) - t_k(x) \nearrow f^+(x) - f^-(x) = f(x)$

Integration. let $s(x)=\sum_{j=1}^{m} c_j \chi_{E_j}$ be a measurable simple function. For $E\in\Sigma$ define $I_E(s)=\int_E s(x)dx=\sum_{j=1}^{m} c_j\mu(E\cap E_j)=\int_E s\,d\mu$

$f\geq 0$ measurable, define $E\in\Sigma$ $I_E(f)=\sup_{0\leq s\leq f} I_E(s)$. the supremum is taken among all nonnegative simple functions that is $\leq f$

In general $f=f^+-f^-$ $f^+\geq 0$ $f^-\geq 0$ define $\int_E f\,d\mu=\int_E f^+d\mu-\int_E f^-d\mu$ $\int_E f\,d\mu$ exists if at least one of $\int_E f^+d\mu$, $\int_E f^-d\mu$ is finite

If $\int_E f\,d\mu$ is finite then we say $f$ is integrable on $E$ and write $f\in L^1(\mu)$ ($f$, $p>0$ $(\int_E|f|^p d\mu)^{\frac{1}{p}}<\infty$ $f\in L^p(\mu)$)

Thm: $f\in L^1(\mu)\Leftrightarrow |f|\in L^1(\mu)$

pf: $\int_E f^+d\mu<\infty$ $\int_E f^-d\mu<\infty$ $\Leftrightarrow$ $\int_E |f|\,d\mu=\int_E f^+d\mu+\int_E f^-d\mu$

(i) $f$: measurable bounded on $E$, $\mu(E)<\infty\Rightarrow f\in L^1(\mu)$ on $E$.

(ii) $a\leq f(x)\leq b$ $x\in E$ $\mu(E)<\infty$ $\Rightarrow a\mu(E)\leq\int_E f\,d\mu\leq b\mu(E)$

6/2 (c) $f$, $g\in L(\mu)$ on $E$ $f(x)\leq g(x)$ $\forall x\in E\Rightarrow\int_E f(x)d\mu\leq\int_E g(x)d\mu$

$\because f(x)\leq g(x)$ $\quad f^+(x)\leq g^+(x)$ $f^-(x)\geq g^-(x)$ $\quad\int_E f^+\leq\int_E g^+$ $\int_E f^-\geq\int_E g^-$

(d) $f\in L(\mu)$ on $E$ $c\in\mathbb{R}\Rightarrow cf\in L(\mu)$ on $E$ $\int_E cf\,d\mu=c\int_E f\,d\mu$

$f=f^+-f^-$ $c\geq 0$, $(cf)^+=c\cdot f^+$ $(cf)^-=c\cdot f^-$ $\quad c\leq 0$ $(cf)^+=-c\cdot f^+$ $(cf)^-=-c\cdot f^-$

(e) $\mu(E)=0$ $f$: measurable $\Rightarrow\int_E f\,d\mu=0$

(f) $f\in L(\mu)$ on $E$ $A\subseteq E$ $A\in\mathfrak{m}$, $f\in L(\mu)$ on $A$

Thm: $f$: measurable nonnegative, fixed $A\in\mathfrak{m}$ define $\phi(A)=\int_A f\,d\mu$ then $\phi$ is countably additive set func on $\mathfrak{m}$, $f\in L(\mu)$ on $X$ same result

pf: $f=\chi_E$ $\phi(A)=\int_A f\,d\mu=\int_A\chi_E\,d\mu=\mu(A\cap E)$ $\quad A=\bigcup_{k=1}^{\infty}A_k$ $A_k\in\mathfrak{m}$ $A_j\cap A_k=\emptyset$ if $j\neq k$

$\mu(A)=\mu(\bigcup_{k=1}^{\infty}A_k)=\sum_{k=1}^{\infty}\mu(A_k)$ $\quad A\cap E=\bigcup_{k=1}^{\infty}(A_k\cap E)$ $\quad\mu(A\cap E)=\sum_{k=1}^{\infty}\mu(A_k\cap E)$ ,

simple func $s(x)=\sum_{j=1}^{\infty}c_j\chi_{E_j}$ $f\geq 0$ measurable simple func $0\leq s(x)\leq f(x)$ $\int_A s(x)d\mu=\sum_{k=1}^{\infty}\int_{A_k}s(x)d\mu\leq\sum_{k=1}^{\infty}\int_{A_k}f(x)d\mu=\sum_{k=1}^{\infty}\phi(A_k)$

take sup on LHS $\phi(A)=\int_A f(x)d\mu\leq\sum_{k=1}^{\infty}\phi(A_k)$

may assume $\phi(A_k)<\infty$ $\forall k$ goal: $\phi(A_1\cup A_2)\geq\phi(A_1)+\phi(A_2)$

given $\varepsilon>0$ $\exists$ simple func $s(x)$ $\int_{A_1}s(x)d\mu\geq\int_{A_1}f(x)d\mu-\varepsilon$ $\quad\int_{A_2}s(x)d\mu\geq\int_{A_2}f(x)d\mu-\varepsilon$

$\phi(A_1\cup A_2)\geq\int_{A_1\cup A_2}s(x)d\mu=\int_{A_1}s(x)d\mu+\int_{A_2}s(x)d\mu\geq\int_{A_1}f(x)d\mu+\int_{A_2}f(x)d\mu-2\varepsilon=\phi(A_1)+\phi(A_2)-2\varepsilon$ $\varepsilon\to 0$ done

$\phi(A)=\phi(\bigcup_{k=1}^{\infty}A_k)\geq\phi(\bigcup_{k=1}^{m}A_k)\geq\sum_{k=1}^{m}\phi(A_k)$ as $m\to\infty$ $\therefore\phi(A)\geq\sum_{k=1}^{\infty}\phi(A_k)$

Almost everywhere (a.e.) $f$ satisfies $P$ a.e. means $f$ satisfies $P$ except possibly on a measurable zero set

Ex: $f=g$ a.e $\int_E f\,d\mu = \int_E g\,d\mu$ pf: $F=\{x \mid f(x)\neq g(x)\}$ $\int_E f d\mu = \int_F f d\mu + \int_{E-F} f d\mu = \int_{E-F} g d\mu + \int_F g d\mu = \int_E g d\mu$

Thm: $f\in L(\mu)$ on $E \Rightarrow |f|\in L(\mu)$ on $E$ $\quad |\int_E f d\mu| \leq \int_E |f| d\mu$

pf: $f=f^+-f^-$ $\quad |f|=f^++f^-$ $\quad f\leq |f|$ $\quad -f\leq |f|$

Thm: $|f|\leq g$ $\quad g\in L(\mu)$ on $E \Rightarrow f\in L(\mu)$ on $E$ $\quad f^+\leq g$ $\quad f^-\leq g$

pf: $f^+\leq g$ $\quad f^-\leq g$

$\mathbb{Q}\cap[0,1]=\{x_1,x_2,\cdots\}$ let $f_k(x)=\begin{cases}1 & x=x_1,x_2\cdots x_k\\ 0 & \text{otherwise}\end{cases}$ $\quad \lim_{k\to\infty} f_k(x)=\begin{cases}1 & x\in\mathbb{Q}\cap[0,1]\\ 0 & \text{otherwise}\end{cases} \notin R$, $\lim_{k\to\infty}\int_a^b f_k(x)dx \neq \int_a^b \lim_{k\to\infty} f_k(x)dx$

Thm: (Lebesgue monotonic convergence theorem) $E\in\mathcal{M}$, $f_n(x)$ measurable on $E$ s.t. $0\leq f_1(x)\leq f_2(x)\leq\cdots\leq f_n(x)\leq\cdots$, $\lim_{n\to\infty} f_n(x)=f(x)$ a.e.

then $\lim_{n\to\infty}\int_E f_n(x)d\mu = \int_E f(x)d\mu = \int_E \lim_{n\to\infty} f_n(x)d\mu$

pf: $\int_E f_n(x)d\mu \nearrow$ $\therefore \lim_{n\to\infty}\int_E f_n(x)d\mu = \alpha$ ($\alpha$ might equal to $\infty$) $\because 0\leq f_n(x)\leq f(x)$ $\quad \int_E f_n(x)d\mu \leq \int_E f(x)d\mu$ $\therefore \alpha\leq\int_E f(x)d\mu$

let $S(x)$ be a simple func, $0\leq S(x)\leq f(x)$, $0<c<1$ let $E_n(x)=\{x \mid f_n(x)\geq cS(x)\}$ $\quad E_n\subseteq E_{n+1}$ $\quad \bigcup_{n=1}^{\infty} E_n = E$

$\int_E f_n(x)d\mu \geq \int_{E_n} f_n(x)d\mu \geq c\int_{E_n} S(x)d\mu$ let $n\to\infty$ $\int_E f_n(x)d\mu\to\alpha$ $\quad c\int_{E_n} S(x)d\mu \to c\int_E S(x)d\mu$ $\quad \alpha\geq c\int_E S(x)d\mu$

$\therefore \alpha\geq \sup_{S(x)}\int_E S(x)d\mu = \int_E f(x)d\mu$ $\quad\Rightarrow\quad \lim_{n\to\infty}\int_E f_n(x)d\mu = \int_E f(x)d\mu$

Thm: $f_1,f_2\in\mathcal{L}(\mu) \Rightarrow f_1+f_2=f\in\mathcal{L}(\mu)$ $\quad \int_E f d\mu = \int_E f_1 d\mu + \int_E f_2 d\mu$ 6/6

pf: (1) $f_1\geq 0$ $f_2\geq 0$ $\exists\, 0\leq S_k\leq f_1$ $0\leq t_k\leq f_2$ $S_k, t_k$: measurable s.t. $S_k\to f_1$ $t_k\to f_2$ $\therefore S_k+t_k\nearrow f_1+f_2=f$

$\int_E S_k d\mu + \int_E t_k d\mu = \int_E (S_k+t_k)d\mu \Rightarrow$ by LMCT $\lim_{k\to\infty}\int_E (S_k+t_k)d\mu = \int_E f d\mu$

(2) $f_1\geq 0$ $f_2<0$ $\quad f=f_1+f_2$ let $A=\{x\in E \mid f(x)\geq 0\}$ $f\geq 0$ $f_1\geq 0$ $f_2<0$ $\quad f+(-f_2)=f_1$ $\quad \int_A f d\mu + \int_A -f_2 d\mu = \int_A f_1 d\mu$ —①

$B=\{x\in E \mid f(x)<0\}$ $f<0$ $f_1\geq 0$ $f_2<0$ $\quad -f_2=f_1-f$ $\quad \int_A -f_2 d\mu = \int_A f_1 d\mu + \int_A -f d\mu$ —②

① + ② $\Rightarrow \int_E f d\mu = \int_E f_1 d\mu + \int_E f_2 d\mu$

(3) $f_1<0$ $f_2\geq 0$ the same as (2) , (4) $f_1<0$, $f_2<0$ the same as (1)

Fatou's lemma: $0\leq f_k(x)$ measurable on $E$ then $\int_E \liminf_k f_k(x)d\mu \leq \liminf_k \int_E f_k(x)d\mu$

pf: define $g_k(x)=\inf\{f_k,f_{k+1},\cdots\}$ $\quad g_k(x)\leq g_{k+1}(x)$ $\quad g_k(x)\geq 0\ \forall k$ $\quad \lim_{k\to\infty} g_k = \liminf_k f_k$

$\lim_{k\to\infty}\int_E g_k(x)d\mu = \int_E \lim_{k\to\infty} g_k(x)d\mu$ (LMCT) $\because g_k(x)\leq f_k(x)$ $\therefore \int_E g_k(x)d\mu \leq \int_E f_k(x)d\mu$

$\therefore \int_E \liminf_k f_k(x)d\mu = \int_E \lim_{k\to\infty} g_k(x)d\mu = \lim_{k\to\infty}\int_E g_k(x)d\mu = \liminf_k \int_E g_k(x)d\mu \leq \liminf_k f_k(x)d\mu$

Lebesgue dominated convergence Theorem (LDCT). $f_k(x)$ measurable on $E$ s.t. $\exists\, g(x) \in \mathcal{L}^1(\mu)$ on $E$ and $|f_k(x)| \le g(x)$ $\forall k$ a.e.

If $\lim\limits_{k\to\infty} f_k(x) = f(x)$ then $\lim\limits_{k\to\infty} \int_E f_k(x)\,d\mu = \int_E f(x)\,d\mu \ \left(= \int_E \lim\limits_{k\to\infty} f_k(x)\,d\mu\right)$

pf: $|g| \in \mathcal{L}(\mu)$ $\quad -|g(x)| \le f_k(x) \le |g(x)|$ $\quad 0 \le |g(x)| + f_k(x) \Rightarrow \int_E \underline{\lim}\,(|g(x)| + f_k(x))\,d\mu \le \underline{\lim} \int_E (|g(x)| + f_k(x))\,d\mu$

$0 \le |g(x)| - f_k(x) \Rightarrow \int_E \underline{\lim}\,(|g(x)| - f_k(x))\,d\mu \le \underline{\lim} \int_E (|g(x)| - f_k(x))\,d\mu$

$\int_E \underline{\lim}\,(|g(x)| + f_k(x))\,d\mu = \int_E (|g(x)| + f(x))\,d\mu = \int_E |g(x)|\,d\mu + \int_E f(x)\,d\mu$

$\int_E \underline{\lim}\,(|g(x)| - f_k(x))\,d\mu = \int_E (|g(x)| - f(x))\,d\mu = \int_E |g(x)|\,d\mu - \int_E f(x)\,d\mu$

$\underline{\lim} \int_E (|g(x)| + f_k(x))\,d\mu = \int_E |g(x)|\,d\mu + \underline{\lim} \int_E f_k(x)\,d\mu$

$\underline{\lim} \int_E (|g(x)| - f_k(x))\,d\mu = \int_E |g(x)|\,d\mu - \overline{\lim} \int_E f_k(x)\,d\mu$

$\Rightarrow \int_E f(x)\,d\mu \le \underline{\lim} \int_E f_k(x)\,d\mu$ and $-\int_E f(x)\,d\mu \le -\overline{\lim} \int_E f_k(x)\,d\mu$

$\Rightarrow \overline{\lim} \int_E f_k(x)\,d\mu \le \int_E f(x)\,d\mu \le \underline{\lim} \int_E f_k(x)\,d\mu \Rightarrow \int_E f(x)\,d\mu = \overline{\lim} \int_E f_k(x)\,d\mu = \underline{\lim} \int_E f_k(x)\,d\mu = \lim\limits_{k\to\infty} \int_E f_k(x)\,d\mu$

Thm: $f: [a,b] \to \mathbb{R}$ bdd suppose $f \in \mathcal{R}$ (denoted by $\mathcal{R}\int_a^b f(x)\,dx$) $\Rightarrow \underbrace{\int_a^b}_{\text{Lebesgue measure}} f(x)\,dx$ exists (Lebesgue) and $\int_a^b f(x)\,dx = \mathcal{R}\int_a^b f(x)\,dx$

pf: $f \in \mathcal{R}$, $\mathcal{R}\overline{\int_a^b} f(x)\,dx = \mathcal{R}\underline{\int_a^b} f(x)\,dx$ $\therefore \exists\, P_k \in \mathcal{P}[a,b]$ $P_k \subseteq P_{k+1}$ $\forall k$ s.t. $U(P_k, f) \searrow$, $L(P_k, f) \nearrow$

$\cdots \le L(P_k, f) \le \cdots \le \mathcal{R}\int_a^b f(x)\,dx \le \cdots \le U(P_k, f) \le \cdots$ $\quad P_k = \{a = x_0 < x_1 < \cdots < x_n = b\}$

Define $u_k(x) = \begin{cases} f(a) & x = a \\ \sup\limits_{[x_{j-1}, x_j]} f & x \in (x_{j-1}, x_j] \end{cases}$ $\quad l_k(x) = \begin{cases} f(a) & x = a \\ \inf\limits_{[x_{j-1}, x_j]} f & x \in (x_{j-1}, x_j] \end{cases}$ $\quad u_k, l_k$: simple measurable. $\int_a^b u_k(x)\,dx = U(P_k, f)$ $\int_a^b l_k(x)\,dx = L(P_k, f)$

$\cdots \le l_k(x) \le \cdots \le f(x) \le \cdots \le u_k(x) \le \cdots$ $\quad l_k(x) \nearrow L(x) \le f(x) \le U(x) \swarrow u_k(x)$ $\quad \therefore L(x), U(x)$ measurable

$\lim \int_a^b l_k(x)\,dx = \int_a^b L(x)\,dx = \int_a^b U(x)\,dx = \lim \int_a^b u_k(x)\,dx$ $\quad \therefore L(x) = U(x)$ a.e. $\quad \therefore f(x) = U(x) = L(x)$ a.e. $\therefore f(x)$ measurable

$\therefore \int_a^b f(x)\,dx = \int_a^b U(x)\,dx = \int_a^b L(x)\,dx = \mathcal{R}\int_a^b f(x)\,dx$