31.3.15

We can define $\underline{e} = \underline{b} - \underline{p}$, $\underline{e}$ should be orthogonal to all $a_1, \dots, a_m$.

$$\langle a_1, e\rangle = a_1^T \times (\underline{b} - A\hat{x}) = 0$$
$$\vdots$$
$$\langle a_m, e\rangle = a_m^T \times (\underline{b} - A\hat{x}) = 0$$

$$\Rightarrow \underbrace{\begin{pmatrix} \text{---} a_1^T \rightarrow \\ \vdots \\ \text{---} a_m^T \rightarrow \end{pmatrix}}_{A^T} (b - A\hat{x}) = 0$$

$$\therefore \quad A^T(\underline{b} - A\underline{\hat{x}}) = 0$$

$$A^T b - A^T A \hat{x} = 0$$

<u>Theorem</u>: A has linearly independant columns.

Then $\underbrace{A^T A}_{\substack{\text{square} \\ \text{symmetric} \\ \text{invertible}}}$ is invertible

Proof: later

$$\underline{\hat{x}} = (A^T A)^{-1} A^T \underline{b}$$

$$\underline{p} = A\hat{x} = \underbrace{A(A^T A)^{-1} A^T}_{P\text{ - projection matrix}} b. \quad \text{- projection vector.}$$

X———————— MIDTERM ————————X

14.4.2015

Determinant

lets consider matrix $A \in \mathbb{R}^{n,n}$ - square matrix.

Determinant of A is a number, usually written as $\det(A)$ or $|A|$.

lets consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = |A| = ad - bc$.

$$A^{-1} = \frac{1}{\underbrace{ad - bc}_{\det(A)}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ - Inverse of A.

In order for A^{-1} to exist det A should not be equal to 0.

If $\det A = 0$, then A^{-1} doesn't exist, A^{-1} is not invertible.

A is a singular matrix.

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\det A^{-1} = \frac{d}{ad-bc} \times \frac{a}{ad-bc} - \frac{-b}{ad-bc} \times \frac{-c}{ad-bc} = \frac{da - bc}{(ad-bc)^2} = \frac{1}{ad-bc}$$

$$= \frac{1}{\det A}.$$

*

$\det A^* = ab - ab = 0$

* consider $A^* = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$

lets consider now $A' = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ - switched rows.

$\det A' = cb - ad = -\det A$.

<u>The following properties are true for any $n \times n$ matrix.</u>

1. determinant of $I \in \mathbb{R}^{n,n}$ (identity matrix) is equal to 1.
2. If 2 rows matrix $A \in \mathbb{R}^{n,n}$ are exchanged, determinant changes its sign.

<u>eg.</u> Permutation matrix P - identity matrix with rows exchanged.

→ If rows are exchanged odd number of times, then $\det P = -1$.

→ If rows are exchanged even numbers of times, then $\det P = 1$.

3. The determinant is a linear function of each row, all other rows stay the same.

<u>eg1</u>: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

<u>eg2</u>: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$

<u>eg3</u> $\begin{vmatrix} 4 & 8 & 8 \\ 3 & 7 & 9 \\ 2 & 1 & 4 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ 2 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 3 & 7 & 9 \\ 2 & 1 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 8 & 8 \\ 3 & 7 & 9 \\ 2 & 1 & 4 \end{vmatrix}$

These 3 properties completely define the determinant.

④. If $A \in \mathbb{R}^{n,n}$ has, 2 equal rows, then $\det A = 0$.
↳ at-least
(i.e. 2 or more equal rows)

Proof: lets assume that rows i and j are equal

we can exchange these rows. This resulting matrix A' is in fact equal to A.

but ② → $\det A' = -\det A$

$A' = A$, $\therefore \det A' = \det A$

$\Rightarrow \det A = -\det A = 0$

⑤ If we add a multiple of one row to another row, determinant doesn't change.

Proof: $A = \begin{pmatrix} \text{row } 1 \\ \text{row } n \end{pmatrix}$

$$\begin{vmatrix} \text{row } 1 \rightarrow \\ \text{row } i \rightarrow \\ \text{row } j + 2\,\text{row } i \rightarrow \\ \text{row } n \rightarrow \end{vmatrix} \overset{③}{=} \begin{vmatrix} \text{row } 1 \rightarrow \\ \text{row } i \rightarrow \\ \text{row } j \rightarrow \\ \text{row } n \rightarrow \end{vmatrix} + 2 \underbrace{\begin{vmatrix} \text{row } 1 \\ \text{row } i \\ \text{row } i \\ \text{row } n \end{vmatrix}}_{\overset{④}{=} 0} = \det A$$

Remark: our standard row operation in gaussian elimination do not change the determinant.

The only exception is exchange of rows.

⑥ If A has row of zeroes then $\det A = 0$.

Proof: Add any other row to zero row and get matrix with 2 equal rows

From ④ $\Rightarrow \det A = 0$.

⑦ lets consider A an upper or lower triangular matrix.

$$A = \begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \quad a_{nn} \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_{11} & & \\ & a_2 \ddots & 0 \\ * & & \\ & & a_{nn} \end{pmatrix}$$

Then $\det A = a_{11} \times a_{22} \times \cdots a_n$.

Proof: lets first assume that all a_{ii} are <u>not</u> equal to zeroes, $i = 1 \dots n$.

Then by adding rows, we can bring the matrix to the diagonal form.

Then we will set matrix

$$\begin{vmatrix} a_{11} & & \\ & a_{22} & 0 \\ & & \ddots \\ 0 & & a_{nn} \end{vmatrix} \overset{③}{=} a_{11} \times \begin{vmatrix} 1 & & \\ & a_{22} & 0 \\ & & \ddots \\ & 0 & a_{nn} \end{vmatrix} \overset{③}{=} a_{11} \times a_{22} \times \begin{vmatrix} 1 & & \\ & 1 & 0 \\ & & a_{33} \ddots \\ & 0 & a_{nn} \end{vmatrix}$$

$$\overset{③}{=} a_{11} \times a_{22} \times \cdots a_{nn} \times \underbrace{\begin{vmatrix} 1 & & \\ & 1 & 0 \\ & & \ddots \\ & 0 & 1 \end{vmatrix}}_{\overset{①}{=} 1} = \underline{\underline{a_{11} \times a_{22} \times a_{nn}}}$$

If a_{ii} is equal to zero now. Then using other diagonal element which are not equal to zero, we can eliminate all non-zero elements from row i, using gaussian elimination steps At the end, we will get a matrix with row of zeroes, for which determinant is equal to 0 by property ⑥ and therefore $\det A = 0 = a_{11} \times a_{22} \times \cdots a_{nn}$.

~~Remark~~:

~~When we use gaussian elimination, we bring the matrix to upper triangular form. At the end, we have pivot elements on the diagonal. If all pivot elements are non-zero elements, the determinant is $\neq 0$, since it is a product of pivot elements.~~

<u>Remark</u>: When we use gaussian elimination, we bring the matrix to upper triangular form. At the end we have pivot elements on the diagonal. If all pivot elements are non-zero elements, the determinant is not equal to zero. Since it is a product of pivot elements.
If some element on diagonal are zero, then the matrix determinant is equal to zero, the matrix does not have inverse, the matrix is singular.

(8) If A is singular then $\det A = 0$
If A is non-singular then $\det A \neq 0$.

Proof: we use gaussian elimination if all pivot elemets are non-zero (non-singular matrix) then $\det A \neq 0$ (7)
Otherwise $\det A = 0$.

(9) $\det(A \times B) = \det A \times \det B$.
Proof: Yes we can!

$$\det(A \times A^{-1}) = \det(I) = 1$$
$$\| $$
$$\det A \times \det A^{-1}$$
$$\Rightarrow \det A^{-1} = \frac{1}{\det A}.$$

(10) $\det A^T = \det A$
Proof: we'll skip it

<u>Remark</u>: Whatever we said about rows is also true for columns.

How do we compute determinant?

① We use gaussian elimination to bring the matrix to upper triangular form and then the determinant is the product of diagonal elements. Wherever we have to exchange the rows, the determinant changes its sign.

16.4.15

② Using Cofactors.

Lets consider matrix
$$A = \begin{array}{cl} \begin{pmatrix} a_{11} \cdots & a_{1j} & \cdots a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} \cdots & a_{ij} & \cdots a_{in} \\ \vdots & \vdots & \vdots \\ a_{n1} \cdots & a_{nj} & \cdots a_{nn} \end{pmatrix} & \leftarrow \text{row } i \\ \quad\quad\quad\quad \uparrow \text{ column } j & \end{array} \qquad A \in \mathbb{R}^{n,n}.$$

We can construct M_{ij} by throwing out row i and column j of A.

$M_{ij} \in \mathbb{R}^{n-1,\,n-1}$ - minor matrix.

Cofactor C_{ij} is defined as $C_{ij} = (-1)^{i+j} \det M_{ij}$

We can compute determinant of A using

→ expansion by row i = $\det A = a_{i1} \times C_{i1} + a_{i2} \times C_{i2} + \dots + a_{in} C_{in}$

→ expansion by column j = $\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}$

$$M_{ij} = \begin{pmatrix} a_{11} & a_{1j-1} & a_{1j+1} & a_{1n} \\ a_{i-1\,1} & a_{i-1\,j-1} & a_{i-1\,j+1} & a_{i-1\,n} \\ a_{i+1\,1} & a_{i+1\,j-1} & a_{i+1\,j+1} & a_{i+1\,n} \\ a_{n1} & a_{nj-1} & a_{nj+1} & a_{nn} \end{pmatrix} \in \mathbb{R}^{n-1,\,n-1}$$

det(A) - only for square matrices. det(A) = 0 - no unique solution.

Example: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2,2}$

we will use expansion by cofactors using row 1.

$$\det(A) = a_{11} C_{11} + a_{12} C_{12}$$

$$= a_{11} \cdot (-1)^{1+1} \det(a_{22}) + a_{12} \cdot (-1)^{1+2} \cdot \det(a_{21})$$

$$= a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

lets use again expansion by row 1.

$$\det(A) = a_{11} \cdot C_{ij} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} \cdot (-1)^{1+1} \cdot \left|\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}\right| + a_{12} \cdot (-1)^{1+2} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13} \cdot (-1)^{1+3} \cdot \left|\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}\right|$$

$$= a_{11} (a_{22} a_{33} - a_{32} a_{23}) - a_{12} (a_{21} a_{33} - a_{31} a_{23}) + a_{13} (a_{21} a_{32} - a_{31} a_{22})$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{31} a_{23} + a_{13} a_{21} a_{32} - a_{11} a_{32} a_{23} - a_{12} a_{21} a_{33} - a_{13} a_{31} a_{22}$$

$A \in \mathbb{R}^{3,3}$

$$\begin{array}{ccc:cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

det(A) = diagonal left → right minus diagonal right → left.

This trick with diagonals doesn't work for matrices with size greater than 3.

eg
$$\begin{cases} 3x + 2y + 4z = 1 \\ 2x - y + z = 0 \\ x + 2y + 3z = 1 \end{cases}, \quad A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} = \frac{-1}{5} \qquad y = \frac{\begin{vmatrix} 3 & 1 & 4 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} = 0$$

$$z = \frac{\begin{vmatrix} 3 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} = \frac{2}{5}$$

Inverse of A

$AX = XA = I \quad \rightarrow \quad X$ - inverse of A

$$\begin{pmatrix} a_{11} & \cdots & a_{13} \\ \vdots & & \vdots \\ a_{31} & \cdots & a_{33} \end{pmatrix} \begin{pmatrix} x_{1} & \cdots & \cdots \\ \vdots & & \vdots \\ \vdots & \cdots & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{13} \\ \vdots & & \vdots \\ \cdots & \cdots & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_{11} = \frac{\overbrace{\begin{vmatrix} 1 & a_{12} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{vmatrix}}^{= C_{11}}}{\det A} = \frac{C_{11}}{\det A}$$

$$x_{21} = \frac{C_{12}}{\det A}, \quad x_{31} = \frac{C_{13}}{\det A}$$

In general, element ij of A^{-1} can be computed as

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$

Most of the time, we use gaussian elimination to compute determinant.

We use cofactor formula mostly when A has many zeroes.

Cramer's Rule

let's consider $A\underline{x}=\underline{b}$, $A \in \mathbb{R}^{3,3}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix} = B_1$$

matrix A with 1st column replaced by $\underline{b}$

$$\det(A \times B) = \det(A) \times \det(B)$$

$$\det(A) \times \underbrace{\det\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix}}_{= x_1} = \det B_1$$

$$\det A \cdot x_1 = \det B_1$$

$$x_1 = \frac{\det(B_1)}{\det(A)}$$

Similarly, $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix} = B_2$$

obtained from A by replacing 2nd column with $\underline{b}$

$$\det A \cdot x_2 = \det B_2$$

$$x_2 = \frac{\det B_2}{\det A}$$

In general, $x_i = \dfrac{\det B_i}{\det A}$, $i = 1 \dots n$

where B_i is A with column i replaced by $\underline{b}$.,

$\det A \neq 0$.