Linear Mappings

28. 4. 2015

definition: A mapping $L : \mathbb{R}^m \to \mathbb{R}^n$ is said to be a linear mapping if for any $u, v \in \mathbb{R}^m$ and any scalar α,

$$L(\underline{u} + v) = L(u) + L(v)$$
$$L(\alpha \underline{u}) = \alpha\, L(u)$$

For any matrix $A \in \mathbb{R}^{m,n}$ we can associate with it a linear **mapping** L_A as:

$$\forall \underline{u} \in \mathbb{R}^m \qquad L_A(\underline{u}) = A\underline{u}.$$

In principle, any linear mapping is completely defined by its values on the basis vectors.

Example: Lets consider $L : \mathbb{R}^2 \to \mathbb{R}^2$ and basis in $\mathbb{R}^2$

$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Lets assume that $L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ and $L\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

what will be $L\begin{pmatrix} 1 \\ 4 \end{pmatrix}$?

Since b_1 and b_2 form basis in our space, $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ can be represented as a linear combination of basis vectors:

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

we can find α_1 and α_2 $\qquad \alpha_1 = 2$ and $\alpha_2 = 1$.

Then, $L\begin{pmatrix} 1 \\ 4 \end{pmatrix} = L\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = 2L\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \times L\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$= 2\begin{pmatrix} 7 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 14 \\ 7 \end{pmatrix}}}$$

For any linear mapping, we can associate with it a matrix.

<u>Proof</u>: Consider linear mapping $L: \mathbb{R}^m \to \mathbb{R}^n$.
Consider also standard basis $E_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, E_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Lets denote by $A_1 = L(E_1)$, $A_2 = L(E_2)$, $\dots$, $A_m = L(E_m) \in \mathbb{R}^n$

If we consider arbitrary vector $x \in \mathbb{R}^m$. then $\underline{x} = x_1 \times E_1 + \dots x_m E_m$
and also,

$$L(\underline{x}) = L(x_1 E_1 \dots + x_m E_m) = x_1 \cdot L(E_1) + \dots + x_m L(E_m)$$

$$= x_1 A_1 + \dots + x_m \times A_m = A\underline{x},$$ where A is a matrix whose columns are $A_1, A_2, \dots, A_m$.

We found matrix A associated with the linear mapping L.

<u>Example</u>: Consider linear mapping $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ - projection from $\mathbb{R}^3$ to $\mathbb{R}^2$.

We consider,

$$L(E_1) = L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A_1$$

$$L(E_2) = L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A_2$$

$$L(E_3) = L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = A_3$$

Then $\underline{\underline{A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}}$

Matrix associated with linear mapping in particular basis

From now on we will focus ~~on~~ primarily on linear mappings $L: V \to V$

Assume $b_1, \dots, b_n$ form basis in space V.
Then any vector $u \in V$ can be written as

$$\underline{u} = u_1 b_1 + \dots u_n b_n$$

We can call $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$, the coordinates of $\underline{u}$ in basis $b_1 \dots b_n$.

Consider linear mapping $L: V \to V$, How does the matrix associated with L looks for basis $b_1 \dots b_n$?

Since $b_1, \dots b_n$ is a basis of V, then we can write.

$$L(b_1) = C_{11}b_1 + C_{12}b_2 + \dots + C_{1n}b_n$$

$$\vdots$$

$$L(b_n) = C_{n1}b_1 + C_{n2}b_2 + \dots + C_{nn}b_n.$$

Now if we take arbitrary vector $\underline{u} \in V$,

$$\underline{u} = u_1 b_1 + u_2 b_2 + \dots + u_n b_n = \sum_{i=1}^{n} u_i b_i$$

then,

$$L(\underline{u}) = L\left(\sum_{i=1}^{n} u_i b_i\right) = \sum_{i=1}^{n} u_i L(b_i) = \sum_{i=1}^{n} u_i \sum_{j=1}^{n} C_{ij} b_j =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} u_i C_{ij} b_j = \sum_{j=1}^{n} b_j \sum_{i=1}^{n} u_i C_{ij}$$

$$= \sum_{i=1}^{n} C_{i1} u_i \times b_1 + \sum_{i=1}^{n} C_{i2} u_i \times b_2 + \dots + \sum_{i=1}^{n} C_{in} u_i \times b_n$$

Therefore, we got,

$$L(\underline{u}) = \begin{pmatrix} \sum_{i=1}^{n} C_{i1} u_i \\ \vdots \\ \sum_{i=1}^{n} C_{in} u_i \end{pmatrix} = C^T u$$

↑ coordinates in basis $\underline{b_1} \cdots \underline{b_n}$

On coordinate vectors our linear mapping is represented by $L(\underline{u}) = C^T \underline{u}$ for given basis $\underline{b}_1, \dots, \underline{b}_n$

Note: for a different basis we will have different coordinates of vectors as well as different associated matrix.

<u>Example</u> : Consider $\mathbb{R}^3$ and basis b_1, b_2, b_3.

Assume
$$L(b_1) = b_1 + b_2$$
$$L(b_2) = 5b_1 - b_2 + 3b_3$$
$$L(b_3) = -1b_1 + 4b_2 - 7b_3$$

The matrix associated with this linear mapping is

$$\begin{pmatrix} 1 & 5 & -1 \\ 1 & -1 & 4 \\ 0 & 3 & -7 \end{pmatrix} = C^T$$

Lets say we have a vector whose coordinates in basis b_1, b_2, b_3 are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 & -1 \\ 1 & -1 & 4 \\ 0 & 3 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \times b_1 + 1 \times b_2 + 0 \times b_3 .$$