

# Chapter 1

## Vectors

### 1.1 Introduction

#### Definition

A vector is an ordered collection of  $n$  numbers

#### Definition

Let us consider vector  $\underline{u} \in \mathbb{R}^n$ . The  $i$ -th component of vector

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

is  $u_i$

#### Definition

Let us consider vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$ . Vector  $\underline{w} \in \mathbb{R}^n$  is a sum of  $\underline{u}$  and  $\underline{v}$ ,  $\underline{w} = \underline{u} + \underline{v}$ , if  $w_i = u_i + v_i$  for all  $i = 1, \dots, n$

#### Definition

1. Vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$  are equal, if  $u_i = v_i$  for all  $i = 1, \dots, n$
2. A scalar is just another name for real number
3. Let us consider a scalar  $\alpha \in \mathbb{R}$  and vector  $\underline{u} \in \mathbb{R}^n$ . A product of  $\alpha$  and  $\underline{u}$  is defined as:

$$\alpha \underline{u} = \alpha \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \vdots \\ \alpha \cdot u_n \end{pmatrix}$$

**Definition**

Let us consider scalars  $\alpha$  and  $\beta$ , and vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$ . A sum of  $\alpha \cdot \underline{u} + \beta \cdot \underline{v}$  is called a linear combination of vectors  $\underline{u}$  and  $\underline{v}$ .

**Definition**

Vector  $\underline{u} \in \mathbb{R}^n$  is called a zero vector if all  $u_i = 0$ ,  $i = 1, \dots, n$ . The zero vector is often written as  $\underline{0} \in \mathbb{R}^n$ .

**1.2 Vector Representations and Operations****1.3 Dot Product (Scalar product)****Definition**

Let us consider two vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$ . The dot (or scalar) product of vectors  $\underline{u}$  and  $\underline{v}$  is defined as

$$\langle \underline{u}, \underline{v} \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^n u_i v_i$$

**1.4 Length of a Vector****Definition**

The length of vector  $\underline{u} \in \mathbb{R}^n$ ,  $\|\underline{u}\|$ , is defined as  $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$ . Sometimes it is also called the Euclidian norm of  $\underline{u}$ .

**1.5 Unit Vectors****Definition**

A vector with length equal to 1 is called a unit vector

**1.6 Angle between Vectors****1.7 Cauchy Schwarz inequality**

## Chapter 2

# Matrices

### 2.1 Matrix Operations

#### Definition

Let us consider matrices  $A \in \mathbb{R}^{n,m}$  and  $B \in \mathbb{R}^{n,m}$  where  $n$  = rows,  $m$  = columns. Matrix  $C \in \mathbb{R}^{n,m}$  is a sum of  $A$  and  $B$ ,  $C = A + B$ , if  $C_{ij} = A_{ij} + B_{ij}$  for all  $i = 1, \dots, n, j = 1, \dots, m$

#### Definition

A product of a scalar  $\alpha$  and a matrix  $A \in \mathbb{R}^{n,m}$  is defined as  $(\alpha A)_{ij} = \alpha \cdot A_{ij}$ ,  $\forall i = 1, \dots, n; j = 1, \dots, m$ .

### 2.2 Matrix-Matrix Multiplication

#### Definition

Let us consider matrix  $A \in \mathbb{R}^{n,m}$  and  $B \in \mathbb{R}^{m,l}$ . Then  $C = A \cdot B$  is an  $n$  by  $l$  matrix,  $C \in \mathbb{R}^{n,l}$  such that

$$C_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$$

### 2.3 Linear System of Equations

### 2.4 Inverse of a Matrix

**Definition**

Let us consider a matrix  $A \in \mathbb{R}^{n,n}$  (square matrix). Matrix  $B \in \mathbb{R}^{n,n}$  is called an inverse of  $A$ , if

$$A \cdot B = I \quad \text{AND} \quad B \cdot A = I$$

(Both conditions are vital)

**2.5 Special Matrices****2.6 Elementary Transition Matrices****Definition**

We can define the elementary transition matrix  $I_{pq} \in \mathbb{R}^{n,n}$

$$(I_{pq}) = \begin{cases} 1 & i = p, q = j \\ 0 & \text{otherwise} \end{cases}$$

If we take a matrix  $A \in \mathbb{R}^{n,n}$  then when calculating  $I_{pq}$  we take row  $q$  of  $A$ , put it into row  $p$ , replace everything else with 0.

We can also define:

$$\begin{aligned} E_{pq}(l) &= I + l \cdot I_{pq}, l \in \mathbb{R} - \text{scalar} \\ E_{pq}(l) \cdot A &= (I + lI_{pq}) \cdot A = A + l \cdot I_{pq}A \end{aligned}$$

We take row  $q$  of  $A$ , multiply it by  $l$ , add it to row  $p$  of  $A$

$$E_{pq}^{-1}(l) = E_{pq}(-l)$$

## Chapter 3

# Gaussian Elimination

### Definition

Permutation matrix  $P$  is an identity matrix with rows in any order.

### 3.1 Matrix Transposition

### Definition

Let us consider matrix  $A \in \mathbb{R}^{m,n}$ . Matrix  $B \in \mathbb{R}^{n,m}$  is called the transpose of  $A$  if  $(B)_{ij} = (A)_{ji}, i = 1 \dots n, j = 1 \dots m$

### Definition

Matrix  $A$  is called symmetric if  $A^t = A$ . Matrix  $A$  should be a square matrix,  $A \in \mathbb{R}^{n,n}$

$$\text{e.g. } A = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = A$$

$$\text{e.g. } A = I \in \mathbb{R}^{n,n} \rightarrow I^T = I$$



## Chapter 4

# Vector Spaces

### Definition

A vector space  $V$  is a set of objects, such that any two objects can be added together, any object can be multiplied by a scalar.

If two objects belong to the vector space, then their sum also belongs to the vector space.

If an object belongs to  $V$ , then the product of any scalar with this object belongs to  $V$  and the following properties are satisfied:

1.  $\forall \underline{u}, \underline{v}, \underline{w} \in V; (\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
2.  $\forall \underline{u}, \underline{v} \in V; \underline{u} + \underline{v} = \underline{v} + \underline{u}$
3. There exists unique elements  $\underline{0} \in V$ , such that  $\forall \underline{u} \in V; \underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$
4. For any  $\underline{u} \in V, \exists!(-\underline{u}) \in V$ , such that  $\underline{u} + (-\underline{u}) = \underline{0}$
5.  $\forall \underline{u}, \underline{v} \in V; \forall \alpha \in \mathbb{R}; \alpha(\underline{u} + \underline{v}) = \alpha\underline{u} + \alpha\underline{v}$
6.  $\forall \underline{u} \in V; \forall \alpha, \beta \in \mathbb{R}; (\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}$
7.  $\forall \underline{u} \in V; \forall \alpha, \beta \in \mathbb{R}; (\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$
8.  $\forall \underline{u} \in V; 1 \cdot \underline{u} = \underline{u}$  (1 is a scalar here)

### 4.1 Subspace of the Vector Space

### Definition

A subspace  $W$  of the vector space  $V$ , is a set of vectors in  $V$ , such that:

1. If  $\underline{u}, \underline{v} \in W$  then  $\underline{u} + \underline{v} \in W$
2. If  $\alpha \in \mathbb{R}, \underline{u} \in W$  then  $\alpha\underline{u} \in W$

**Definition**

Let us consider a set of vectors  $\{\underline{u}_1, \dots, \underline{u}_n\}$ . The span of vectors  $\{\underline{u}_1, \dots, \underline{u}_n\}$  is defined as

$$\mathcal{S} = \text{span}\{\underline{u}_1, \dots, \underline{u}_n\} = \{\alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n \mid \forall \alpha_1 \dots \alpha_n \in \mathbb{R}\}$$

**4.2 Linear Independence****Definition**

Let us consider vector space  $V$  and  $\underline{v}_1, \dots, \underline{v}_n \in V$ .  $\underline{v}_1, \dots, \underline{v}_n$  are linearly dependent if there exists scalars  $\alpha_1, \dots, \alpha_n$  not all equal to zero, such that  $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$

If no such scalars exist, the vectors  $\underline{v}_1, \dots, \underline{v}_n$  are linearly independent.

**Definition**

Vectors  $\underline{v}_1, \dots, \underline{v}_n \in V$  are linearly independent if the following is true:

$$\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0} \Rightarrow \text{all } \alpha_i = 0, i = 1, \dots, n$$

**Definition**

If vector space  $v$  is generated by  $\{\underline{v}_1, \dots, \underline{v}_n\}$  (in other words,  $V = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$ ) and  $\underline{v}_1, \dots, \underline{v}_n$  are linearly independent, then  $\{\underline{v}_1, \dots, \underline{v}_n\}$  is called basis of  $V$

**Definition**

Let us consider vector space  $V$  and vectors  $\underline{v}_1, \dots, \underline{v}_n$  that form a basis of  $V$ . If vector  $\underline{x} \in V$  can be written as  $\underline{x} = x_1 \underline{v}_1 + \dots + x_n \underline{v}_n$  then  $(x_1, \dots, x_n)$  are called the coordinates of  $\underline{x}$  with respect to basis  $\{\underline{v}_1, \dots, \underline{v}_n\}$

**4.3 Rank of Matrix****Definition**

The row rank of matrix  $A$  is a maximum number of linearly independent rows of matrix  $A$ .

**Definition**

The column rank of matrix  $A$  is a maximum number of linearly independent columns of matrix  $A$ .



**Definition**

Two subspaces  $U$  and  $W$  of vector space  $V$  are orthogonal, if  $\forall \underline{u} \in U$  and  $\forall \underline{w} \in W$ , we have  $\langle \underline{u}, \underline{w} \rangle = 0$

**Definition**

Orthogonal complement of subspace  $M$  of vector space  $V$  contains every vector orthogonal to  $M$ . This subspace is usually denoted by  $M^\perp$



## Chapter 5

# Orthogonality

### 5.1 Linear Mapping

#### Definition

Let us consider two vector spaces  $V$  and  $W$ . A function (or mapping)  $L : V \rightarrow W$  is called a *linear mapping* if the following two conditions are satisfied:

1. For any  $\underline{v} \in V$  and  $\underline{v}' \in V$ ,  $L(\underline{v} + \underline{v}') = L(\underline{v}) + L(\underline{v}')$
2. For any  $\underline{v} \in V$  and any scalar  $\alpha$ ,  $L(\alpha \underline{v}) = \alpha \cdot L(\underline{v})$

### 5.2 The 4 Fundamental Subspaces of a Matrix

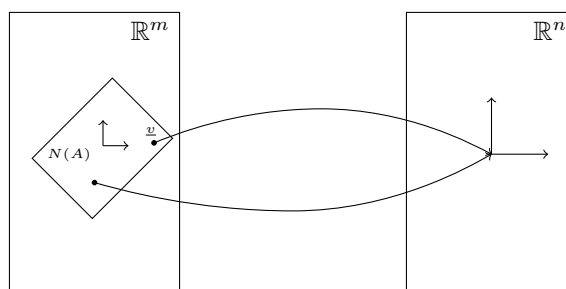
#### Definition

The span (or all possible linear combinations) of columns of the matrix  $A \in \mathbb{R}^{n,m}$  is called a *column space* of  $A$ , denoted by  $C(A)$ , where  $C(A) \subset \mathbb{R}^n$ .

#### Definition

Let us consider the matrix  $A \in \mathbb{R}^{n,m}$ ,  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . The null space of  $A$  is defined as

$$N(A) = \{\underline{v} \in \mathbb{R}^m \mid A\underline{v} = \underline{0}\}, N(A) \subset \mathbb{R}^m$$



**Definition**

The row space of a matrix  $A \in \mathbb{R}^{n,m}$  is the span of the rows of  $A$ . Clearly,  $R(A) = C(A^T)$  and  $R(A) \subset \mathbb{R}^m$ .

**Definition**

The left nullspace of  $A$  is defined as  $N(A^T)$ .  $N(A^T) \subset \mathbb{R}^n$ .

**5.3 Orthogonal Basis and Gram-Schmidt process****Definition**

Vectors  $\underline{q}_1, \dots, \underline{q}_m$  are *orthogonal* if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = 0 \quad \text{if } i \neq j$$

In other words, two vectors  $\underline{q}_i$  and  $\underline{q}_j$  are orthogonal if their dot product is zero, so two orthogonal vectors must be perpendicular.

**Definition**

Vectors  $\underline{q}_1, \dots, \underline{q}_m$  are *orthonormal* if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

In other words, two vectors  $\underline{q}_i$  and  $\underline{q}_j$  are orthonormal if they are orthogonal and they both have length of 1 (they are *unit vectors*).

**Definition**

A square matrix is called orthogonal (if its columns are orthonormal vectors) if  $Q^T Q = I$ . In this case, since it is a square matrix,  $Q Q^T = I$ .

## Chapter 6

# Determinant

6.1 Compute the Determinant

6.2 Cramer's Rule

6.3 Inverse of a Matrix



## Chapter 7

# Linear Mappings

### Definition

A mapping  $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is said to be a *linear mapping* (or linear transformation or linear map) if for any  $\underline{u}, \underline{v} \in \mathbb{R}^m$  and any scalar  $\alpha$

$$\begin{aligned}L(\underline{u} + \underline{v}) &= L(\underline{u}) + L(\underline{v}) \\L(\alpha \underline{u}) &= \alpha L(\underline{u})\end{aligned}$$

For any matrix  $A \in \mathbb{R}^{n,m}$  we can associate with it a linear mapping  $L_A$  as

$$L_A(\underline{u}) = A\underline{u} \quad \forall \underline{u} \in \mathbb{R}^m$$

where  $L_A$  is a linear mapping from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ .

In principle, any linear mapping is completely defined by its values on the basis vectors.

## 7.1 Matrix Associated with Linear Mapping in a Particular Basis

## 7.2 Change of Basis (Matrix)

### Definition

Assume that  $N \in \mathbb{R}^{n,n}$ ,  $N^{-1}$  exists.  $A' = N^{-1}AN$  is called similarity transformation.

### Definition

Matrices  $A'$  and  $A$  are called similar matrices, if  $\exists N$  such that

$$A' = N^{-1}AN$$





## Chapter 8

# Eigenvalues and Eigenvectors

### Definition

A vector  $\underline{v} \in V$ ,  $\underline{v} \neq 0$  is called an eigenvector of  $A$ , if there exists scalar  $\lambda$ , such that  $A\underline{v} = \lambda\underline{v}$ . This scalar  $\lambda$  is called an eigenvalue, corresponding to eigenvector  $\underline{v}$ .

Sometimes, eigenvectors are called characteristic vectors, and eigenvalues are called characteristic values.

## 8.1 Characteristic Polynomial

### Definition

Consider  $A \in \mathbb{R}^{n,n}$ . The characteristic polynomial is defined as

$$p_a(t) = \det(tI - A)$$



## Chapter 9

# Change of Basis

**Definition**

A set of all eigenvalues of matrix  $A \in \mathbb{R}^{n,n}$  is called spectrum of  $A$