On the Determinant, Inverse, and Eigenvalues of Infinite Matrices

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A Project submitted to the **Department of Mathematics** in fulfillment of the requirements of Course No. MAT 433 (Course Title: Project in Mathematics and Relevant Fields) for the degree of
Bachelor of Science (Honors) in Mathematics.

Department of Mathematics
Shahjalal University of Science and Technology, Sylhet, Bangladesh
February 22, 2024

DECLARATION

We hereby declare that the work reported in the project titled **"On the Determinant, Inverse, and Eigenvalues of Infinite Matrices"** submitted for the partial fulfillment of the B.Sc. (Honors) degree at the Department of Mathematics, Shahjalal University of Science and Technology, Sylhet is a record of our work carried out under the supervision of Dr Mohammad Salah Uddin.

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CERTIFICATE

It is to certify that **Iftekhar Al Mahmud, Rafiur Rahman, Durjoy Das, and Md Arif Hossain** have submitted this project under my supervision for partial fulfillment of the B.Sc. (Honors) degree in Mathematics on the topic **"On the Determinant, Inverse, and Eigenvalues of Infinite Matrices"**. It is further certified that the above candidates have carried out the project work under my guidance during the academic session 2022-2023 at the Department of Mathematics, Shahjalal University of Science and Technology.

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ACKNOWLEDGEMENT

First and foremost, we would like to express our sincere gratitude to our supervisor, **Dr Mohammad Salah Uddin**, Associate Professor, Department of Mathematics, Shahjalal University of Science and Technology.
We are grateful for his assistance and support throughout the project.

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Abstract

This project systematically studies the foundational principles governing infinite matrices and their diagonally dominant properties, progressing through a structured exploration of critical topics. Initially, attention is directed toward the introduction of the matrix logarithm, where practical implications in real and complex domains are discussed alongside fundamental matrix-related definitions. Subsequently, determinant properties are thoroughly investigated, encompassing their utility in solving linear equations and methodologies such as Cramer's rule. The analysis then extends to matrix inverses, providing definitions and examples within both finite and infinite contexts. Following this, the focus shifts to eigenvalues and eigenvectors, emphasizing the role of the trace of the logarithm in determinant calculations and exploring convergence behaviors. Moreover, this project includes the algorithms and corresponding computer codes developed and implemented for approximating determinants, finding inverse matrices, and determining eigenvalue approximations using series expansions.

Keywords: infinite matrix, matrix logarithm, determinant, inverse, eigenvalue, error analysis.

CHAPTER 1

Introduction

An infinite matrix is a matrix of infinite dimensions in which both the number of rows and columns are infinite. Infinite matrices, the forerunner and a primary constituent of many branches of classical mathematics (infinite quadratic forms, integral equations, differential equations, etc.) and the modern operator theory, are revisited to demonstrate its profound influence on the development of many branches of mathematics, classical and contemporary, replete with applications.

According to Michael Bernkopf [**1**], the history of a general theory of infinite matrices begins with Henri Poincare in 1884. After Poincare, Helge Von Koch was the next to take up the study, and by 1893, he had proved most of the theorems about infinite matrices and their determinants. Later in 1906, a tremendous impulse was given to the subject when David Hilbert used infinite quadratic forms, which are equivalent to infinite matrices, to solve the integral equation

$$f(s) = \phi(s) + \lambda \int_a^b K(s,t)\phi(t)\, dt.$$

According to P. N. Shivakumar and K. C. Shivakumar [**2**], applications of matrices are found in most scientific fields. In every branch of physics, including classical mechanics, optics, electromagnetism, quantum mechanics, etc, they are used to study physical phenomena, such as the motion of rigid bodies. Computer graphics are used to manipulate 3D models and project them onto a 2-dimensional screen. In probability theory and statistics, stochastic matrices are used to describe sets of probabilities. Matrix calculus generalizes classical analytical notions such as derivatives and exponentials to higher dimensions. A significant branch of numerical analysis is devoted to developing efficient algorithms for matrix computations, a subject that is centuries old and an expanding area of research.

This project endeavors to elucidate the fundamental principles governing infinite matrices, with a particular focus on their determinant properties. Through an exploration of parallels between finite and infinite matrices, key concepts such as eigenvalues, eigenvectors, and solutions to systems of linear equations are examined in detail.

Chapter 2 introduces the matrix logarithm, emphasizing its practical implications in both real and complex domains, while also discussing various matrix-related definitions and fundamental mathematical structures such as groups, rings, and fields.

Chapter 3 delves deeply into determinant properties, addressing scenarios in both finite and infinite matrices, showcasing their practical utility in solving linear equations and exemplifying methodologies like Cramer's rule.

Chapter 4 extensively analyzes matrix inverses, providing definitions and illustrative examples predominantly within finite contexts, and extending the discussion to encompass infinite scenarios, including propositions on matrix invertibility.

Chapter 5 focuses on eigenvalues and eigenvectors, providing precise definitions and examples, highlighting the importance of employing the trace of the logarithm in determinant calculations, and exploring convergence behaviors through graphical representations of determinant errors.

Finally, Chapter 6 includes the algorithms and corresponding MATLAB source codes developed and used for approximating determinants of matrices, finding inverse matrices, and determining approximations for multiple eigenvalues using series expansion.

In this project, we abstain from delving into discussing fundamental theorems specific to infinite matrices. Instead, we offer an overview of properties inherent to finite matrices that find applicability and relevance within infinite matrix contexts.

CHAPTER 2

Preliminaries

The chapter delves into matrix definitions, rank analysis, and linear transformations, notably exploring matrix logarithms in real and complex domains, focusing on complex matrices with positive eigenvalues. It introduces a method for deriving complex matrix logarithms via Taylor series expansion contingent upon convergence. Section 2 covers convergent sequences and series alongside an analysis of norms in the real line. Section 3 discusses vector space definitions and dimensions with examples. Lastly, Section 4 explores fundamental algebraic structures like groups, rings, integral domains, and fields, supplemented with illustrative examples.

1. Groups, rings, and fields

Most of the definitions recalled in this section belong to the classical book by Joseph Gallian [**3**].

Definition 2.1. A **group** is a nonempty set G on which there is defined a binary operation satisfying the following properties:

(1) Closure: If a and b are two elements in the group G, then the product $a * b$ is also in G.

(2) Associativity: The operation is associative; that is, $(a * b) * c = a * (b * c)$ for all a, b, c in G.

(3) Identity: There is an element e (called the identity) in G such that $a * e = e * a = a$ for all a in G.

(4) Inverses: For each element a in G, there is an element b in G (called an inverse of a) such that $a * b = b * a = e$.

Example 2.1. The set of integers $\mathbb{Z}$ with the operation of addition forms a group as follows:

(1) Closure: The sum of any two integers is also an integer.

(2) Associativity: Addition is associative.

(3) Identity element: The identity element for addition is 0.

(4) Invertibility: For every integer a, there exists an inverse $(-a)$ such that $a + (-a) = 0$.

DEFINITION 2.2. An algebraic structure $(R, +, *)$ consisting of a non-empty set R with two binary compositions (to be denoted additively and multiplicatively) is called a **ring** if the following properties are satisfied:

(1) $(R, +)$ is an abelian group.

(2) Multiplication is associative, i.e., $(ab)c = a(bc)$ for all $a, b, c \in R$.

(3) Multiplication distributes addition, i.e., $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

EXAMPLE 2.2. The set of integers $\mathbb{Z}$ with addition and multiplication forms a ring because it satisfies closure, associativity, the existence of an additive identity, additive inverses, and the distributive property.

DEFINITION 2.3. A ring $(R, +, *)$ is called an **integral domain** if it is a commutative ring with unity and without zero divisors. An example of an integral domain is the set of integers $\mathbb{Z}$ with the usual operations of addition and multiplication.

(1) Closure under addition and multiplication: For any two integers a and b, their sum $a + b$ and product $a \cdot b$ are also integers.

(2) Associativity of addition and multiplication: Both addition and multiplication operations are associative in $\mathbb{Z}$.

(3) Commutativity of addition and multiplication: Both addition and multiplication operations are commutative in $\mathbb{Z}$.

(4) Existence of additive and multiplicative identity: The additive identity is 0, and the multiplicative identity is 1.

(5) No zero divisors: In $\mathbb{Z}$, if the product of two integers is zero $(a \cdot b = 0)$, then either $a = 0$ or $b = 0$. There are no non-zero divisors of zero.

Example 2.3. The set of integers $\mathbb{Z}$ with addition and multiplication satisfies the properties of an integral domain.

Definition 2.4. A ring $(R, +, *)$ is called a **field** if it is a commutative ring with unity in which every non-zero element has a multiplicative inverse. The set of complex numbers $\mathbb{C}$ with the operations of addition and multiplication forms a field.

(1) Closure under addition and multiplication: For any two complex numbers $a+bi$ and $c+di$, where a, b, c, d are real numbers, their sum and product are also complex numbers.

(2) Associativity of addition and multiplication: Both addition and multiplication operations are associative in $\mathbb{C}$.

(3) Existence of additive and multiplicative identity: The additive identity is $0+0i$, and the multiplicative identity is $1+0i$.

(4) Existence of additive and multiplicative inverses: For every nonzero complex number $a+bi$, there exists an additive inverse $-a-bi$ such that $a+bi+(-a-bi)=0$, and there exists a multiplicative inverse $\frac{1}{a+bi}$ such that $(a+bi)\cdot\frac{1}{a+bi}=1$.

(5) Distributive property: Multiplication distributes over addition, meaning $(a+bi)\cdot(c+di) = ac-bd+(ad+bc)i$ for any complex numbers $a+bi, c+di$.

Example 2.4. The set of complex numbers $\mathbb{C}$ with addition and multiplication satisfies the properties of a field.

Definition 2.5. A **Gaussian integer** is a complex number $a+ib$ such that both a and b are integers.

Example 2.5. Consider the complex number $3-2i$. Here, the real part $a=3$ and the imaginary part $b=-2$. Therefore, $3-2i$ is a Gaussian integer.

2. Sequence and series

Most of the definitions recalled in this section belongs to the classical books by Walter Rudin [**4**] and Terence Tao [**5**].

DEFINITION 2.6. . Let $\varepsilon > 0$ be a real number, and let L be a real number. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers is said to be ε-close to L iff a_n is ε-close to L for every $n \geq N$, i.e., we have $|a_n - L| \leq \varepsilon$ for every $n \geq N$. We say that a sequence $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε-close to L. We say that a sequence $(a_n)_{n=m}^{\infty}$ converges to L iff it is eventually ε-close to L for every real $\varepsilon > 0$. A **convergent sequence** is a sequence of real numbers that approaches a specific limit as the sequence progresses.

EXAMPLE 2.6. Consider the sequence (a_n), where

$$a_n = \frac{1}{n}.$$

Here, (a_n) is a sequence defined by the reciprocal of natural numbers $1, 2, 3, \ldots$. The terms of the sequence are

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad \ldots.$$

As n approaches infinity, a_n approaches zero. The limit of this sequence as n approaches infinity is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, the sequence $(\frac{1}{n})$ is a convergent sequence, and its limit is 0. In general, a sequence (a_n) is said to converge to a limit L if, for any small positive number ϵ, there exists a positive integer N such that $|a_n - L| < \epsilon$ for all $n \geq N$.

DEFINITION 2.7. Let $\sum_{n=m}^{\infty} a_n$ an be a formal infinite series. For any integer N $\geq$ m, we define the N^{th} partial sum S_N of this series to be $S_N = \sum_{n=m}^{N} a_n$; of course, S_N is a real number. If the sequence $(S_N)_{n=m}^{\infty}$ converges to Some limit L as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is **convergent series**, and converges to L; we also write $L = \sum_{n=m}^{\infty} a_n$, and say that L is the sum of the infinite series $\sum_{n=m}^{\infty} a_n$. If the partial sums S_N diverge, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is divergent, and we do not assign any real number value to that series.

EXAMPLE 2.7. We consider an example of a convergent series.The series is given by as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Mathematically, the sum is expressed as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

By p-series test, which states that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$. In this case, $p = 2$, so the series converges.

3. Vectors and vector norms

Most of the definitions recalled in this section belong to the classical book by Gilbert Strang [**6**].

DEFINITION 2.8. Let V be a **vector space** and $\{v_1, v_2, ..., v_n\}$ a finite set of vectors in V. We call $\{v_1, v_2, ..., v_n\}$ a basis for V if and only if

(1) $\{v_1, v_2, ..., v_n\}$ is linearly independent and

(2) $\{v_1, v_2, ..., v_n\}$ spans V.

In $\mathbb{R}^2$, the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis because the following hold.

(1) They span $\mathbb{R}^2$: Any vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(2) They are linearly independent: No scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$ can add up to the zero vector unless all the scalars are zero.

EXAMPLE 2.8. Consider the vectors $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ in V. Their sum and scalar multiplication are

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \text{and} \quad 2\mathbf{u} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}.$$

DEFINITION 2.9. The **dimension** of a vector space is equal to the maximum number of linearly independent vectors contained in it.

EXAMPLE 2.9. We consider the vector space $\mathbb{R}^2$, the two-dimensional real vector space. A standard basis for $\mathbb{R}^2$ is given by the set of vectors

$$\{(1, 0), (0, 1)\}.$$

These vectors are the standard unit vectors along the x-axis and y-axis. They are linearly independent and span the entire $\mathbb{R}^2$ space.

Therefore, the dimension of $\mathbb{R}^2$ is 2 because the basis has two linearly independent vectors. In general, the dimension of a vector space is the number of vectors in any basis for that space.

DEFINITION 2.10. A **linear transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$ is any function from one Euclidean space $\mathbb{R}^n$ to another $\mathbb{R}^m$ which obeys the following two axioms.
(a) Additivity : $\forall\ \mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, we have $T(\mathbf{x} + \mathbf{x}') = T(\mathbf{x}) + T(\mathbf{x}')$.
(b) Homogeneity : $\forall\ \mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, we have $T(c\mathbf{x}) = cT(\mathbf{x})$.

EXAMPLE 2.10. Consider the map $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 3x + 2$. To check its linearity, we need to check the following two properties.

(1) For any two real numbers x and y, $f(x+y)$ should be equal to $f(x)+f(y)$.

(2) For any real number x and any scalar c, $f(cx)$ should be equal to $cf(x)$.

For the given map, we have

$$\begin{aligned} f(x + y) &= 3(x + y) + 2 \\ &= 3x + 3y + 2. \end{aligned}$$

$$\begin{aligned} f(x) + f(y) &= (3x + 2) + (3y + 2) \\ &= 3x + 3y + 4. \end{aligned}$$

So,

$$f(x+y) \neq f(x)+f(y).$$

Again,

$$\begin{aligned} f(cx) &= 3(cx)+2 \\ cf(x) &= c(3x+2) \\ &= 3cx+2c. \end{aligned}$$

So,

$$f(cx) \neq cf(x).$$

Therefore, the map $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x)=3x+2$ is not linear.

DEFINITION 2.11. The **norm** or **length** of a vector $\mathbf{x}$ in $\mathbb{R}^n$ is defined by $\|\mathbf{x}\| = \{\sum_{i=1}^{n} \mathbf{x}_i^2\}^{\frac{1}{2}}$, where $\mathbf{x} = \{x_1, x_2, ..., x_n\}$. The distance between two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is the real number.

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \{\sum_{i=1}^{n}(x_i-y_i)^2\}^{\frac{1}{2}}$$

EXAMPLE 2.11. Suppose we have a vector $\mathbf{v} = [3,-4,5]$. The Euclidean norm of this vector is calculated as follows:

$$\|\mathbf{v}\|_2 = \sqrt{3^2+(-4)^2+5^2} = \sqrt{9+16+25} = \sqrt{50}$$

So, the Euclidean norm of the vector $\mathbf{v}$ is $\|\mathbf{v}\|_2 = \sqrt{50}$.

DEFINITION 2.12. A **multiplicative norm** on a commutative unital ring is a function from the non-zero elements of the commutative unital ring to the integers with the property that the norm of a nonzero product of two elements equals the product of their norms.

EXAMPLE 2.12. Imagine we have a set of complex numbers where both the real and imaginary parts are integers, called Gaussian integers. We define a norm function called the Euclidean norm, denoted by $||\cdot||$ as follows:

$$\|a+bi\| = \sqrt{(a^2+b^2)}$$

This function assigns a non-negative real number to each Gaussian integer. It satisfies the following properties.

$$\|0\| = 0$$

(i.e., the norm of the Gaussian integer zero is zero) and

$$\|a.b\| = \|a\| \cdot \|b\|$$

(i.e., the norm of the product of two Gaussian integers is the product of their norms).

Therefore, the Euclidean norm is a multiplicative norm on the ring of Gaussian integers.

4. Matrices of infinite dimension

Most of the definitions recalled in this section belong to the articles by Howard E. Haber [**7**] and Nicholas J. Higham [**8**].

DEFINITION 2.13. A **complex matrix** is a rectangular array of complex numbers arranged in m rows and n columns. The set of all m-by-n complex matrices is denoted as $\mathbb{C}^{m\times n}$ or $M_{m\times n}(\mathbb{C})$, where $\mathbb{C}$ represents the set of complex numbers.

EXAMPLE 2.13. In the following, both A and B are complex matrices.

$$A = \begin{bmatrix} 1-\mathrm{i} & 2\mathrm{i} & -2+\sqrt{3} \\ -2-\sqrt{3}\mathrm{i} & -2\sqrt{2} & -2-\sqrt{2}\mathrm{i} \end{bmatrix}$$

$$B = \begin{bmatrix} -2\sqrt{2}\mathrm{i} & 1-\sqrt{2} & -4\sqrt{2} \\ -1+\sqrt{3} & 2-\sqrt{3}\mathrm{i} & -2+\sqrt{2} \\ -3\sqrt{2} & -3+\sqrt{2} & -4+\sqrt{3}\mathrm{i} \end{bmatrix}$$

DEFINITION 2.14. A **complete matrix** is a matrix in which all elements are specified or defined. In other words, a complete matrix has every entry filled or assigned, with no missing elements. The dimensions of a complete matrix are fully defined, with the number of rows and columns specified.

EXAMPLE 2.14. In the following, C is a complete matrix.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

DEFINITION 2.15. An **incomplete matrix** is a matrix in which one or more elements are undefined or missing. The dimensions of an incomplete matrix may or may not be fully specified, but it contains at least one entry that is not provided.

EXAMPLE 2.15. In the matrix D below, the entries represented by '-' are undefined or missing. The dimensions of the matrix are still specified, that is, 3-by-3 but some of the elements are incomplete. So, D is an incomplete matrix.

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & - & 6 \\ 7 & 8 & - \end{bmatrix}$$

DEFINITION 2.16. A symmetric matrix M with real entries is **positive-definite** if, for every nonzero real column vector $\mathbf{x}$, the real number $\mathbf{x}^{\mathrm{T}}Mx$ is positive. Here, $\mathbf{x}^{\mathrm{T}}$ denotes the transpose of $\mathbf{x}$. More generally, a Hermitian matrix, which is a complex matrix equal to its conjugate transpose, is considered positive-definite if the real number x^*Mx is positive for every nonzero complex column vector x, where x^* represents the conjugate transpose of x.

EXAMPLE 2.16. The following matrix A is symmetric because $A = A^T$, and it is positive definite because for any non-zero column vector x, the quadratic form x^TAx will always be greater than zero.

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

DEFINITION 2.17. The **rank** of a matrix A is the maximum number of linearly independent rows or columns in the matrix.

EXAMPLE 2.17. Consider the matrix A as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

To find the rank of the matrix A, we can perform row operations to bring it to its echelon form or reduced row-echelon form (row-reduced form). The rank is then the number of non-zero rows in that form.

Firstly we perform the following row operations on A.

$$\mathrm{R}_2^{'} = \mathrm{R}_2 + (-4) \cdot \mathrm{R}_1$$
$$\mathrm{R}_3^{'} = \mathrm{R}_3 + (-7) \cdot \mathrm{R}_1$$

These give us

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}.$$

Now, we perform the following row operations to simplify further.

$$\mathrm{R}_3^{'} = \mathrm{R}_3 + (-2) \cdot \mathrm{R}_2.$$

This yields

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that the number of non-zero rows of A is 2. Therefore, the rank of the matrix A is 2.

Definition 2.18. Let A be a real or complex n-by-n matrix. The **matrix exponential** of A is defined via its Taylor series

$$e^A = I + \sum_{n=1}^{\infty} \frac{A^n}{n!},$$

where I is the n-by-n identity matrix. Notably, the radius of convergence for this series is infinite, indicating that the series converges for all matrices A.

Example 2.18. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

We can calculate the matrix exponential using the following Taylor series.

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \dots$$

We calculate some of the terms of the above series as follows:

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, substituting these into the series we get

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots.$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \frac{1}{24} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \cdots.$$

Simplifying, we can see that this series converges to a matrix. The matrix exponential e^A in this example is

$$e^A = \begin{bmatrix} e & 2e \\ 0 & e^{-1} \end{bmatrix}.$$

DEFINITION 2.19. The **matrix logarithm** should be an inverse function to the matrix exponential. However, because the complex logarithm is a multi-valued function, the concept of the matrix logarithm is not as straightforward as the matrix exponential. Let A be a complex n-by-n matrix with no real negative or zero eigenvalues. Then, there is a unique logarithm, denoted by $\ln A$, all of whose eigenvalues lie in the strip $-\pi < \text{Im } z < \pi$ of the complex z-plane. We refer to $\ln A$ as the principal logarithm of A, which is defined on the cut complex plane, where the cut runs from the origin along the negative real axis. If A is a real matrix (subject to the conditions just stated), then its principal logarithm is real.

For an n-by-n complex matrix A, we can define $\ln A$ via its Taylor series expansion, under the assumption that the series converges. The matrix logarithm is then defined as

$$\ln A = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(A-I)^k}{k},$$

whenever the series converges, where I is the n-by-n identity matrix. The series converges whenever $\|(A-I)\| < 1$, where $\|\cdot\|$ indicates a suitable matrix norm.

If the matrix A satisfies $(A-I)^k = 0$ for all integers $m > N$ (where N is some fixed positive integer), then $A - I$ is called nilpotent, and A is called unipotent. If A is unipotent, then the series given terminates, and $\ln A$ is well-defined independently of the value of $\|(A-I)\|$. For later use, we also note that if $\|(A-I)\| < 1$, then $I - A$ is non-singular, and $(I-A)^{-1}$ can be expressed as an infinite geometric series:

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Example 2.19. Let

$$A = \begin{bmatrix} 0.8763 & 0.7373 & 0.2400 \\ 0.3106 & 1.2792 & 0.1884 \\ 0.2902 & 0.1537 & 0.1851 \end{bmatrix}.$$

We know that

$$\log(A) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(A-I)^k}{k}.$$

If we expand this, we get

$$\log(A) = (A-I) - \frac{(A-I)^2}{2} + \frac{(A-I)^3}{3} - \frac{(A-I)^4}{4} + \cdots$$

$$= \begin{bmatrix} -0.1237 & 0.7373 & 0.2400 \\ 0.3106 & 0.2792 & 0.1884 \\ 0.2902 & 0.1537 & -0.8149 \end{bmatrix} - \begin{bmatrix} -0.2807 & 0.6616 & 0.2832 \\ 0.2591 & 0.1113 & 0.2016 \\ 0.4025 & 0.0879 & -1.1962 \end{bmatrix}$$

$$+ \begin{bmatrix} -0.2863 & 0.7484 & 0.3413 \\ 0.2871 & 0.1665 & 0.2381 \\ 0.4992 & 0.0840 & -1.4131 \end{bmatrix} - \begin{bmatrix} -0.3197 & 0.7266 & 0.3656 \\ 0.2688 & 0.1352 & 0.2476 \\ 0.5563 & 0.0563 & -1.5624 \end{bmatrix}$$

$$+ \cdots$$

$$= \begin{bmatrix} -0.3637 & 0.7554 & 0.4744 \\ 0.2523 & 0.1515 & 0.3026 \\ 0.7760 & -0.0110 & -2.0666 \end{bmatrix}.$$

DEFINITION 2.20. The **principal logarithm** of a complex number z is defined as follows:

$$\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z)$$

where

(1) ln is the natural logarithm,

(2) $|z|$ is the magnitude (or modulus) of z,

(3) i is the imaginary unit,

(4) $\operatorname{Arg}(z)$ is the principal argument of z, which is the angle θ such that $-\pi < \theta \leq \pi$, measured in radians.

EXAMPLE 2.20. For a complex number $z = x + iy$, where x and y are real numbers, the principal logarithm can be expressed as follows:

$$\operatorname{Log}(z) = \ln\sqrt{x^2 + y^2} + i \operatorname{Arg}(z)$$

DEFINITION 2.21. A consistent matrix norm $\|\cdot\| : \mathbb{C}^{m\times n} \to \mathbb{R}$ is said to be **sub-multiplicative** if for all $A, B \in \mathbb{C}^{m\times n}$ it satisfies

$$\|AB\| \leq \|A\| \cdot \|B\|.$$

EXAMPLE 2.21. Consider the matrices A and B as follows:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

We have

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

Now, we have

$$||A|| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30} \approx 5.4772,$$

$$||B|| = \sqrt{5^2 + 6^2 + 7^2 + 8^2} = \sqrt{174} \approx 13.1909,$$

and

$$||AB|| = \sqrt{19^2 + 22^2 + 43^2 + 50^2} \approx 72.0694.$$

Here

$$\|AB\| = 72.0694 < (5.4772)(13.1909) = \|A\| \cdot \|B\|.$$

This result shows that the matrix norm considered in this case is a sub-multiplicative norm.

CHAPTER 3

Determinant of Infinite Matrices

This chapter begins by thoroughly exploring the determinants of finite matrices, delving into their properties and applications. Emphasis is placed on how determinants are utilized in solving systems of linear equations within the finite matrix framework. The subsequent section extends this analysis to infinite matrices, examining the conditions that govern their determinants. Special attention is given to the complexities introduced by matrices of infinite dimension. The chapter concludes by applying determinants to resolve systems of linear equations involving infinite matrices. Overall, the chapter aims to provide a comprehensive understanding of both finite and infinite matrix determinants and their practical implications in the solution of linear systems. Here, we follow Matysiak et al. [**9**] for most of the defintions and results.

1. Determinant of a finite matrix

For every square matrix $A = [a_{ij}]$, $1 \leq i, j \leq n$, the **determinant** is a scalar value associated to A that is a real or complex number, and it is denoted by $\det(A)$ or $|A|$. For $n = 1$, we assume $\det(A) = a_{11}$ and for $n \geq 2$, we define $\det(A)$ recursively, as follows:

For some fixed i,

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot C_{ij}$$

For some fixed j,

$$\det(A) = \sum_{i=1}^{n} a_{ij} \cdot C_{ij}$$

where C_{ij} is the cofactor of the element a_{ij}, $1 \leq i, j \leq n$. Recall that the cofactor of an element of a matrix A depends on the determinant of a lower order submatrix of A. In particular, for $n = 2$, we have

$$\det(A) = a_{11}C_{22} + a_{12}C_{21} = a_{11}a_{22} - a_{12}a_{21}.$$

Consider the matrix A as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Fix $i = 1$. Then we find $\det(A)$ as follows:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

where

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix},$$

$$\vdots$$

$$C_{1n} = (-1)^{1+n} \begin{vmatrix} a_{21} & \cdots & a_{n-1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} \end{vmatrix}.$$

2. Determinant of an infinite matrix

A square matrix of infinite dimension is an infinite matrix in which the number of rows is equal to the number of columns. Let $M_n(R)$, $n \in \mathbb{N} \cup \{\infty\}$, denote the set of all square matrices of dimension n with coefficients from any integral domain R such that all the rows and columns of a matrix are convergent series.

A primary reason for choosing n as an extended natural number, that is, $n \in \mathbb{N} \cup \{\infty\}$, is that upper and lower bounds of the extended natural numbers always exist [**2**]. Note that, ∞ is the largest element and 0 is the smallest element in extended natural numbers. Also, every subset of extended natural numbers has the supremum and infimum.

The following result gives us a way to compute the determinant of an infinite dimensional square matrix.

THEOREM 2.1. [**9**] Let A_n, where $n \in \mathbb{N} \cup \{\infty\}$, be a square matrix such that $\operatorname{tr}(\log A) = \sum_{k=1}^{\infty}(-1)^{k+1}\operatorname{tr}(\frac{(A-I)^k}{k})$ is a convergent series. Then $\det(A)$ is determined as follows:

$$\det(A) = \exp(\operatorname{tr}(\log A))$$

The following two examples illustrate the above result in the case of finite square matrices.

Here, we use the MATLAB code Source Code 6.1 (Algorithm 1) to evaluate the terms of the logarithmic series involved in the aforesaid formula.

EXAMPLE 3.1. Let

$$A = \begin{bmatrix} 0.8763 & 0.7373 & 0.2400 \\ 0.3106 & 1.2792 & 0.1884 \\ 0.2902 & 0.1537 & 0.1851 \end{bmatrix}.$$

From Example 2.19, we have

$$\log(A) = \begin{bmatrix} -0.3637 & 0.7554 & 0.4744 \\ 0.2523 & 0.1515 & 0.3026 \\ 0.7760 & -0.0110 & -2.0666 \end{bmatrix}.$$

Then

$$\operatorname{tr}(\log A) = -2.2788$$

and hence

$$\det A = \exp(\operatorname{tr}(\log A)) = 0.1024008543.$$

The approximated determinant of A after 157 iterations is 0.1024008544.

The graph in Figure 3.1 illustrates a rapid reduction in determinant error with increasing iterations, where the determinant error represents the disparity between the true and algorithmically estimated determinants of a matrix. Plotted on a logarithmic scale, each unit on the y-axis signifies a ten-fold change in determinant error. Notably, the graph signifies swift convergence of the algorithm, attaining a minimal error level around the 157th iteration.

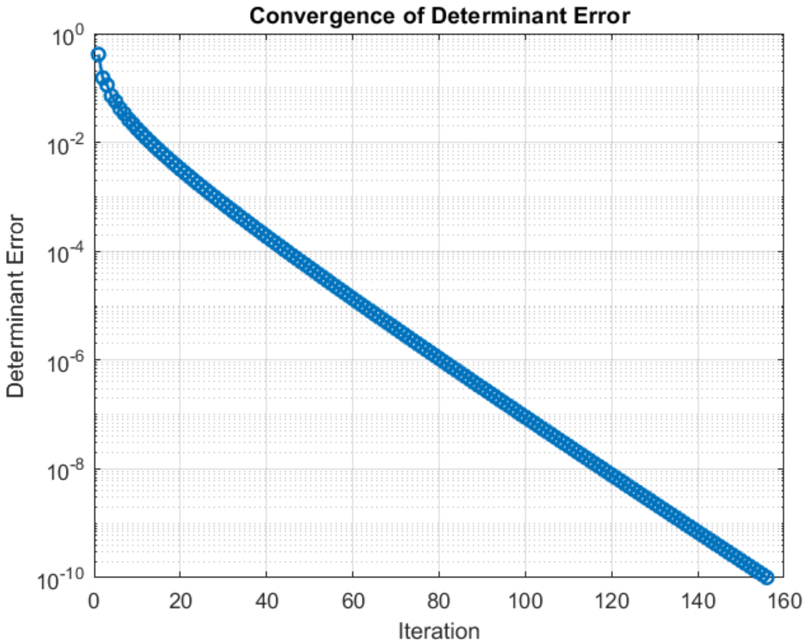

Figure 3.1. Convergence of determinant error over iterations (Case 1)

EXAMPLE 3.2. Consider the following matrix A.

$$A = \begin{bmatrix} 1.0000 & 0.0050 & 0.0050 & 0 & 0 & 0 & 0.0000 \\ 0.0100 & 0.6000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7000 & 0 & 0 & 0 & 0 \\ 0 & 0.1200 & 0 & 0.8000 & 0.1000 & 0 & 0.0010 \\ 0.0010 & 0 & 0.1250 & 0 & 0.9000 & 0 & 0.0010 \\ 0 & 0 & 0 & 0 & 0 & 0.9500 & 0 \\ 0 & 0.3000 & 0.2500 & 0 & 0 & 0 & 0.9800 \end{bmatrix}$$

We find that $\|I - A\| = 0.554479 < 1$. Thus $\log(A)$ exists. By using series expansion, we have

$$\log(A) = \begin{bmatrix} 0 & 0.0064 & 0.0059 & 0 & 0 & 0 & 0 \\ 0.0128 & -0.5109 & -0.0000 & 0 & 0 & 0 & -0 \\ 0 & 0 & -0.3567 & 0 & 0 & 0 & 0 \\ -0.0010 & 0.1724 & -0.0100 & -0.2231 & 0.1178 & 0 & 0.0011 \\ 0.0011 & -0.0002 & 0.1569 & 0 & -0.1054 & 0 & 0.0011 \\ 0 & 0 & 0 & 0 & 0 & -0.0513 & 0 \\ -0.0021 & 0.3873 & 0.3004 & 0 & 0 & 0 & -0.0202 \end{bmatrix}.$$

Then by the definition of the logarithm of a matrix, we have

$$\det(A) = \exp(\operatorname{tr}(\log(A)) = 0.2815109532.$$

We find that the exact (up to 10 decimal places) determinant of A is 0.2815109532 which equals the calculated determinant of A. Thus, the approximated determinant is correct up to 10 decimal places.

The graph in Figure 3.2 shows how the determinant error decreases as the number of iterations increases. The determinant error is the difference between the actual determinant of a matrix and the estimated determinant using an algorithm. The graph is plotted on a logarithmic scale, which means that each unit on the y-axis represents a ten-fold change in the determinant error. The graph shows that the algorithm converges quickly, reaching a very small error after about 35 iterations.

3. Determinant of matrix products

To define the determinant of the product of infinite matrices we need the idea of equinumerous sets. A set A is **equinumerous** to a set B, written $A \approx B$, if and only if there is a one-to-one function from A onto B. For example, the sets $\mathbb{N}$ and $S = \{2, 4, 6, ...\}$ are equinumerous because the function $f : \mathbb{N} \to S$ defined as $f(n) = 2n$ is both one-one and onto.

DEFINITION 3.1. Consider the matrices $A_{m\times n}$ and $B_{n\times k}$, where $m, n, k \in \mathbb{N} \cup \{\infty\}$. For the matrix product $A_{m\times n}B_{n\times k}$, we get the following conclusions.

(a) If $m = \infty$, $k = \infty$, then $AB = C_{\infty\times\infty}$.

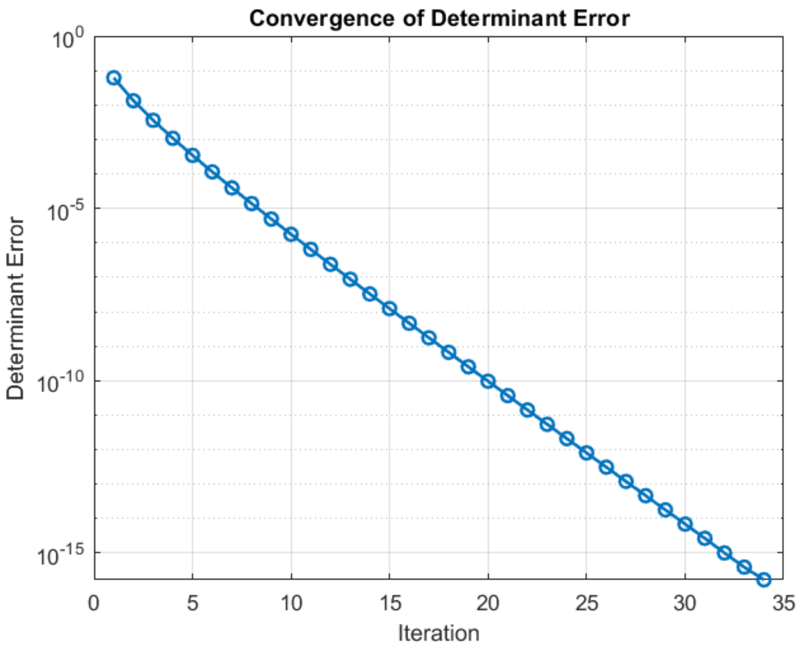

Figure 3.2. Convergence of determinant error over iterations (Case 2)

(b) If $n = \infty$, then $AB = C_{m\times k} = [c_{ij}]$, where $c_{ij} = \sum_{l=1}^{\infty} a_{il}b_{lj}$, $1 \le i \le m$, $1 \le j \le k$, are all convergent series.

(c) If A and B are square matrices of infinite dimension then $AB = C$ holds.

(d) If A and B are matrices of infinite dimension, and the number of rows in A is equinumerous to the number of columns in B, then we find the product AB only if rows of A and columns of B are convergent series.

The following result gives an important property regarding the determinant of the product of infinite matrices.

Proposition 3.1. **[9]** Let A be an m-by-n matrix, and let B be an n-by-m matrix, where $m, n \in \mathbb{N} \cup \{\infty\}$. Let $1 \le j_1, j_2, \ldots, j_m \le n$. Let $A_{j_1 j_2 \ldots j_m}$ denote the m-by-m matrix consisting of columns $j_1, j_2, \ldots, j_m$ of A. Let $B_{j_1 j_2 \ldots j_m}$ denote the m-by-m matrix consisting of rows $j_1, j_2, \ldots, j_m$ of B. Then

$$\det(AB) = \sum_{1 \le j_1 < j_2 < \ldots < j_m \le n} \det(A_{j_1 j_2 \ldots j_m}) \det(B_{j_1 j_2 \ldots j_m}).$$

Proof. First, we will show the proof in the finite version. Let $(k_1, k_2, \ldots, k_m)$ be an ordered m-tuple of integers. Let $\eta(k_1, k_2, \ldots, k_m)$ denote the sign of $(k_1, k_2, \ldots, k_m)$.

Let $(l_1, l_2, \ldots, l_m)$ be the same as $(k_1, k_2, \ldots, k_m)$ except for k_i and k_j having been transposed. Then, from Transposition is of Odd Parity

$$\eta(l_1, l_2, \ldots, l_m) = -\eta(k_1, k_2, \ldots, k_m).$$

Let $(j_1, j_2, \ldots, j_m)$ be the same as $(k_1, k_2, \ldots, k_m)$ by arranging into non-decreasing order. That is $j_1 \leq j_2 \leq \ldots \leq j_m$. Then, it follows that

$$\det(B_{k_1 \ldots k_m}) = \eta(k_1, k_2, \ldots, k_m) \det(B_{j_1 \ldots j_m}).$$

Hence

$$\begin{aligned}
\det(AB) &= \sum_{1 \leq l_1, \ldots, l_m \leq m} \eta(l_1, \ldots, l_m) \left(\sum_{k=1}^{n} a_{1k} b_{k,l_1} \right) \ldots \left(\sum_{k=1}^{n} a_{mk} b_{k,l_m} \right) \\
&= \sum_{1 \leq k_1, \ldots, k_m \leq n} \sum_{k=1}^{n} a_{1k} \ldots a_{mk} \sum_{1 \leq l_1, \ldots, l_m \leq m} \eta(l_1, \ldots, l_m) b_{k_1,l_1} \ldots b_{k_m,l_m} \\
&= \sum_{1 \leq k_1, \ldots, k_m \leq n} \sum_{k=1}^{n} a_{1k} \ldots a_{mk} \det(B_{j_1 \ldots j_m}) \\
&= \sum_{1 \leq k_1, \ldots, k_m \leq n} \sum_{k=1}^{n} a_{1k} \ldots a_{mk} \det(B_{k_1 \ldots k_m}) \\
&= \sum_{1 \leq k_1, \ldots, k_m \leq n} a_{1k_1} \ldots a_{mk_m} \eta(k_1, \ldots, k_m) \det(B_{j_1 \ldots j_m}) \\
&= \sum_{1 \leq j_1 \leq j_2 \leq \ldots \leq j_m \leq n} \det(A_{j_1 \ldots j_m}) \det(B_{j_1 \ldots j_m}).
\end{aligned}$$

If two j's are equal then

$$\det(A_{j_1 \ldots j_m}) = 0.$$

For infinite matrices, we put ∞-tuple of the form $(k_1, k_2, k_3, \ldots)$ and put $1 \leq j_1, j_2, j_3, \ldots < n = \infty$. □

The following example illustrates the result obtained above in the case of finite dimensional matrices.

Example 3.3. Consider the non-square matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}.$$

According to the given proposition, consider all possible combinations of distinct indices $j_1 < j_2$ from the set $\{1, 2, 3\}$ to form submatrices $A_{j_1j_2}$ and $B_{j_1j_2}$. The determinant of the product $A_{j_1j_2}B_{j_1j_2}$ is then calculated, and the sum of all these determinants should be equal to $\det(AB)$.

Let us consider $j_1 = 1$ and $j_2 = 2$. Then we have

$$A_{12} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B_{12} = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}.$$

Then

$$\det(A_{12}) = -3 \quad \text{and} \quad \det(B_{12}) = -2.$$

Again for $j_1 = 1$ and $j_2 = 3$, we have

$$A_{13} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \quad \text{and} \quad B_{13} = \begin{bmatrix} 7 & 8 \\ 11 & 12 \end{bmatrix}.$$

Then

$$\det(A_{13}) = -6 \quad \text{and} \quad \det(B_{13}) = -4.$$

Again for $j_1 = 2$ and $j_2 = 3$, we have

$$A_{23} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad B_{23} = \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix}.$$

Then

$$\det(A_{23}) = -3 \quad \text{and} \quad \det(B_{23}) = -2.$$

Now by summing up all the determinants, we have

$$\begin{aligned} \det(AB) &= \sum_{1 \le j_1 < j_2 \le 3} \det(A_{j_1j_2}) \det(B_{j_1j_2}) \\ &= (-3) \cdot (-2) + (-6) \cdot (-4) + (-3) \cdot (-2) = 36. \end{aligned}$$

On the other hand, we calculate AB as follows:

$$AB = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Then $\det(AB) = 36$.

So, the final result of $\det(AB)$ is 36, and it matches the sum of the determinants of the submatrices as predicted by the proposition. The steps include calculating the determinants for all valid combinations of indices and summing them up.

4. Applications of determinant

Let A be an m-by-n matrix over an arbitrary field $\mathbb{F}$ $(m, n \in \mathbb{N} \cup \{\infty\})$. There is an associated linear mapping $f : \mathbb{F}^n \to \mathbb{F}^m$ defined by $f(x) = Ax$. The rank of A is the dimension of the image f. This definition has the advantage that it can be applied to any linear map without the need for a specific matrix. Consider the system of equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots &= b_3 \\ &\vdots \end{aligned}\right\}$$

denoted as $AX = B$. By Cramer's system, we mean a system in which the number of equations is equal to the number of unknowns. Then, Cramer's theorem states that in the finite case, the system has a unique solution provided we have n equations. Hence, individual values for the unknowns are given by

$$x_i = \frac{A_{x_i}}{\det(A)},$$

for $i = 1, 2, \ldots,$ where A_{x_i} is the determinant of the matrix obtained by replacing the i^{th} column of A by the column vector B. If $\operatorname{tr} A_{x_i}$ and $\operatorname{tr} A$ are convergent series, then Cramer's formula holds for the infinite case.

On the other hand, for a system of equations $AX = B$ where A, X, B can have an infinite dimension, it implies $X = A^{-1}B$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{12} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \end{bmatrix}$$

At first, we find the determinant of the coefficient matrix of A as follows:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{12} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Then we find the determinant A_{x_1} as follows:

$$A_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} & \cdots \\ b_2 & a_{22} & a_{23} & \cdots \\ b_3 & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Then we have

$$x_1 = \frac{A_{x_1}}{\det(A)}.$$

Similarly, we find

$$A_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} & \cdots \\ a_{21} & b_2 & a_{23} & \cdots \\ a_{31} & b_3 & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}.$$

which gives

$$x_2 = \frac{A_{x_2}}{\det(A)}$$

and we find

$$A_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 & \cdots \\ a_{21} & a_{22} & b_2 & \cdots \\ a_{31} & a_{32} & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

which gives

$$x_3 = \frac{A_{x_3}}{\det(A)}.$$

Similarly for all $i = 1, 2, 3, \ldots$, we find

$$A_{x_i} = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & b_1 & \cdots \\ a_{21} & a_{22} & \cdots & \cdots & b_2 & \cdots \\ a_{31} & a_{32} & \cdots & \cdots & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}.$$

These give the value of i^{th} variable x_i for all $i = 1, 2, 3, \ldots$ as follows:

$$x_i = \frac{A_{x_i}}{\det(A)}.$$

CHAPTER 4

Inverse of Infinite Matrices

This chapter delves into inverse matrices, with Section 1 concentrating on finite matrices and defining matrix inverses, accompanied by an illustrative example. In Section 2, the discussion extends to infinite matrices, introducing a corollary on the determinant relationship in matrix products. A proposition outlines conditions for expressing the inverse of a matrix as an infinite series, contingent upon the convergence of matrix rows and columns, along with the sub-multiplicative norm of the matrix difference being less than 1. A practical example involving finite matrices is provided for clarity.Here, we follow most of the definitions from the classical book Richard E. Bellman [**10**] and the article P.N. Shivakumar [**11**].

1. Inverse of a finite matrix

Let A be a square matrix. Then A is **invertible** if there exits a matrix B such that $AB = BA = I$. Then we call the matrix B, the **inverse** of A, and write $B = A^{-1}$.

For example, consider the following matrices A and B.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

We find that

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = I$$

and

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = I.$$

Therefore, $A^{-1} = B$.

2. Inverse of an infinite matrix

DEFINITION 4.1. *A matrix $A \in M_n(K)$ (where $n \in \mathbb{N} \cup \{\infty\}$, K be a field) is* **invertible** *if and only if the map $f : K^n \to K^n$ defined by $f(x) = Ax$ is invertible, where elements of K^n are considered as column vectors.*

EXAMPLE 4.1. Let $K = \mathbb{R}$ (the field of real numbers) and consider the following matrix $A \in M_2(\mathbb{R})$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Now, let's define the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = Ax$, where x is a column vector in $\mathbb{R}^2$. The matrix-vector multiplication Ax corresponds to the standard matrix multiplication.

If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then

$$Ax = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Now, let's check if the map f is invertible. According to the theorem, the matrix A is invertible if and only if f is invertible.

To find the inverse of A, you can use the formula for the inverse of a 2x2 matrix

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

Here, $\det(A)$ is the determinant of A, which is $(2 \times 3) - (1 \times 1) = 5$. Therefore,

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

Now, f has an inverse, and you can verify that $f^{-1}(y) = A^{-1}y$. This means that f is an invertible map.

So, the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ is invertible, and the map $f(x) = Ax$ is invertible as well.

The following proposition gives us a way to compute the inverse of an infinite matrix.

PROPOSITION 2.1. [**9**] Let A be a matrix in which every row and column forms a convergent series such that $\|I - A\| < 1$, where $\|\cdot\|$ is a submultiplicative norm. Then

$$A^{-1} = I + (I - A) + (I - A)^2 + \cdots .$$

PROOF. We see that $I + (I - A) + (I - A)^2 + \cdots$ is a geometric series with $I - A$ as the ratio of the consecutive terms of the series. Since $\|I - A\| < 1$, we have the sum of the series as follows:

$$I + (I - A) + (I - A)^2 + \cdots = \frac{I}{I - (I - A)} = \frac{I}{A} = A^{-1}.$$

□

We illustrate the above result in the following two examples in the case of finite matrices with submultiplicative norms less than 1.

EXAMPLE 4.2. Let

$$A = \begin{bmatrix} 0.6015 & 0.6618 & 0.1938 \\ -0.4642 & 1.0997 & -0.6004 \\ -0.2306 & -0.1662 & 1.5319 \end{bmatrix}.$$

We see that the submultiplicative norm of A is 0.8804410. Thus, A is invertible.

We use the MATLAB code Source Code 6.2 (Algorithm 2) to compute the terms $(I - A)^n$ of the following series $A^{-1} = I + (I - A) + (I - A)^2 + \cdots$ up to 42 terms terms. Some of the terms are given as follows:

$$I - A = \begin{bmatrix} 0.3985 & -0.6618 & -0.1938 \\ 0.4642 & -0.0997 & 0.6004 \\ 0.2306 & 0.1662 & -0.5319 \end{bmatrix}$$

$$(I - A)^2 = \begin{bmatrix} -0.1931 & -0.2300 & -0.3715 \\ 0.2772 & -0.1975 & -0.4692 \\ 0.0464 & -0.2576 & 0.3380 \end{bmatrix}$$

$$(I-A)^3 = \begin{bmatrix} -0.2693 & 0.0889 & 0.0969 \\ -0.0894 & -0.2417 & 0.0773 \\ -0.0231 & 0.0512 & -0.3434 \end{bmatrix}$$

$$(I-A)^4 = \begin{bmatrix} -0.0437 & 0.1855 & 0.0540 \\ -0.1300 & 0.0961 & -0.1689 \\ -0.0647 & -0.0469 & 0.2179 \end{bmatrix}$$

$$(I-A)^5 = \begin{bmatrix} 0.0811 & 0.0194 & 0.0911 \\ -0.0461 & 0.0484 & 0.1727 \\ 0.0027 & 0.0837 & -0.1315 \end{bmatrix}$$

Finally, we have

$$A^{-1} = \begin{bmatrix} 1.0034 & -0.6622 & -0.3864 \\ 0.5379 & 0.6116 & 0.1717 \\ 0.2094 & -0.0333 & 0.6132 \end{bmatrix}.$$

The graphical representation in Figure 4.1 illustrates the diminishing error in the computation of the inverse of a finite matrix with an increasing number of iterations. Specifically, the analysis focuses on 42 iterations, utilizing a logarithmic scale to quantify the error. The graphical depiction strongly implies that the iterative process for determining the inverse matrix exhibits rapid and precise convergence, attesting to its efficiency and accuracy.

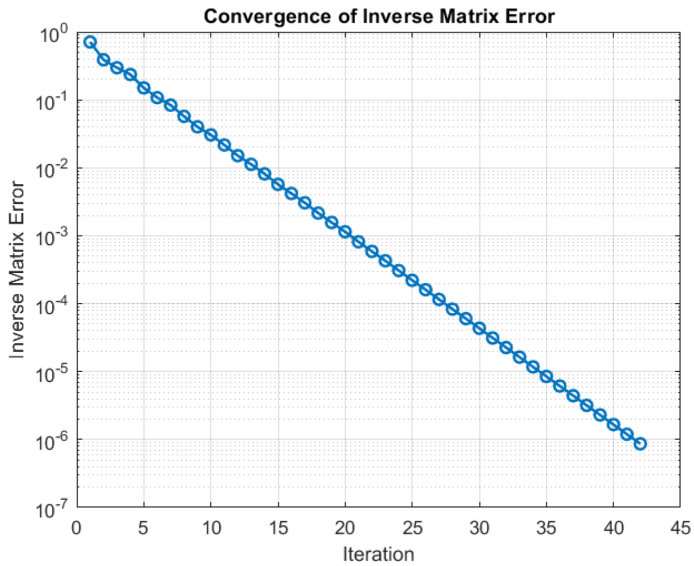

Figure 4.1. Error plot during iterative inverse computation (Case 1)

EXAMPLE 4.3. Let

$$B = \begin{bmatrix} 1.2000 & -0.0200 & -0.1200 & -0.0600 & -0.0800 & -0.0030 & -0.0050 \\ -0.0800 & 0.9997 & -0.0030 & -0.0400 & -0.0200 & -0.0070 & -0.0060 \\ -0.0100 & -0.0100 & 0.9997 & -0.0060 & -0.0070 & -0.0040 & -0.0006 \\ -0.0300 & -0.0100 & -0.0200 & 0.9994 & -0.0200 & -0.0500 & -0.0700 \\ -0.0300 & -0.2141 & -0.0007 & -0.0942 & 0.9999 & -0.0009 & -0.0001 \\ -0.0800 & -0.0120 & -0.0004 & -0.0009 & -0.9890 & 0.9994 & -0.0005 \\ -0.0300 & -0.0010 & -0.0050 & -0.0020 & 0 & -0.3000 & 0.9998 \end{bmatrix}.$$

We see that the submultiplicative norm of B is 0.995135. Therefore, B is invertible.

We use the MATLAB code Source Code 6.2 (Algorithm 2) to compute the terms $(I-B)^n$ of the series $B^{-1} = I + (I-B) + (I-B)^2 + \cdots$ up to 29 terms. We find

$$(I-B) = \begin{bmatrix} -0.2000 & 0.0200 & 0.1200 & 0.0600 & 0.0800 & 0.0030 & 0.0050 \\ 0.0800 & 0.0003 & 0.0030 & 0.0400 & 0.0200 & 0.0070 & 0.0060 \\ 0.0100 & 0.0100 & 0.0003 & 0.0060 & 0.0070 & 0.0040 & 0.0006 \\ 0.0300 & 0.0100 & 0.0200 & 0.0006 & 0.0200 & 0.0500 & 0.0700 \\ 0.0300 & 0.2141 & 0.0007 & 0.0942 & 0.0001 & 0.0009 & 0.0001 \\ 0.0800 & 0.0120 & 0.0004 & 0.0009 & 0.9890 & 0.0006 & 0.0005 \\ 0.0300 & 0.0010 & 0.0050 & 0.0020 & 0 & 0.3000 & 0.0002 \end{bmatrix}$$

$$(I-B)^2 = \begin{bmatrix} -0.0134 & 0.0064 & 0.0104 & 0.0068 & 0.0142 & 0.0041 & 0.0032 \\ 0.0005 & 0.0018 & 0.0014 & 0.0017 & 0.0051 & 0.0006 & 0.0005 \\ 0.0036 & 0.0116 & 0.0004 & 0.0053 & 0.0215 & 0.0004 & 0.0004 \\ -0.0013 & 0.0026 & 0.0019 & 0.0019 & 0.0077 & 0.0030 & 0.0008 \\ 0.0175 & 0.0030 & 0.0044 & 0.0102 & 0.0089 & 0.0067 & 0.0083 \\ 0.0058 & 0.0645 & 0.0025 & 0.0295 & 0.0021 & 0.0006 & 0.0003 \\ 0.0182 & 0.0043 & 0.0038 & 0.0021 & 0.2992 & 0.0005 & 0.0004 \end{bmatrix}$$

$$(I-B)^3 = \begin{bmatrix} -0.0084 & -0.0015 & 0.0057 & 0.0023 & 0.0084 & 0.0010 & 0.0001 \\ 0.0017 & 0.0030 & -0.0014 & 0.0009 & 0.0033 & 0.0014 & 0.0005 \\ 0.0036 & 0.0116 & 0.0004 & 0.0053 & 0.0215 & 0.0004 & 0.0004 \\ -0.0013 & 0.0026 & 0.0019 & 0.0019 & 0.0077 & 0.0030 & 0.0008 \\ 0.0175 & 0.0030 & 0.0044 & 0.0102 & 0.0089 & 0.0067 & 0.0083 \\ 0.0058 & 0.0645 & 0.0025 & 0.0295 & 0.0021 & 0.0006 & 0.0003 \\ 0.0050 & 0.0009 & 0.0015 & 0.0032 & 0.0030 & 0.0020 & 0.0025 \end{bmatrix}$$

$$(I-B)^4 = \begin{bmatrix} 0.0020 & 0.0017 & -0.0010 & 0.0003 & 0.0004 & 0.0001 & 0.0001 \\ -0.0004 & 0.0008 & 0.0006 & 0.0007 & 0.0018 & 0.0002 & 0.0001 \\ 0.0000 & 0.0002 & 0.0001 & 0.0001 & 0.0003 & 0.0001 & 0.0000 \\ 0.0011 & 0.0047 & 0.0006 & 0.0027 & 0.0011 & 0.0005 & 0.0005 \\ 0.0010 & 0.0017 & -0.0001 & 0.0008 & 0.0030 & 0.0004 & 0.0001 \\ -0.0019 & 0.0025 & 0.0024 & 0.0021 & 0.0083 & 0.0031 & 0.0008 \\ 0.0050 & 0.0009 & 0.0015 & 0.0032 & 0.0030 & 0.0020 & 0.0025 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 0.8408 & 0.0354 & 0.1023 & 0.0600 & 0.0787 & 0.0089 & 0.0087 \\ 0.0708 & 1.0110 & 0.0126 & 0.0485 & 0.0396 & 0.0128 & 0.0098 \\ 0.0101 & 0.0132 & 1.0017 & 0.0084 & 0.0132 & 0.0050 & 0.0013 \\ 0.0367 & 0.0326 & 0.0252 & 1.0135 & 0.0959 & 0.0727 & 0.0714 \\ 0.0440 & 0.2208 & 0.0089 & 0.1078 & 1.0209 & 0.0108 & 0.0092 \\ 0.1117 & 0.2336 & 0.0175 & 0.1130 & 1.0173 & 1.0124 & 0.0105 \\ 0.0589 & 0.0723 & 0.0134 & 0.0378 & 0.3079 & 0.3042 & 1.0038 \end{bmatrix}$$

The graph in Figure 4.2 visually represents the decreasing error in computing the inverse of a finite matrix as the number of iterations increases, with a specific focus on 29 iterations. Employing a logarithmic scale to measure the error, the illustration strongly suggests that the iterative process for determining the inverse matrix converges swiftly and accurately, highlighting its efficiency and precision.

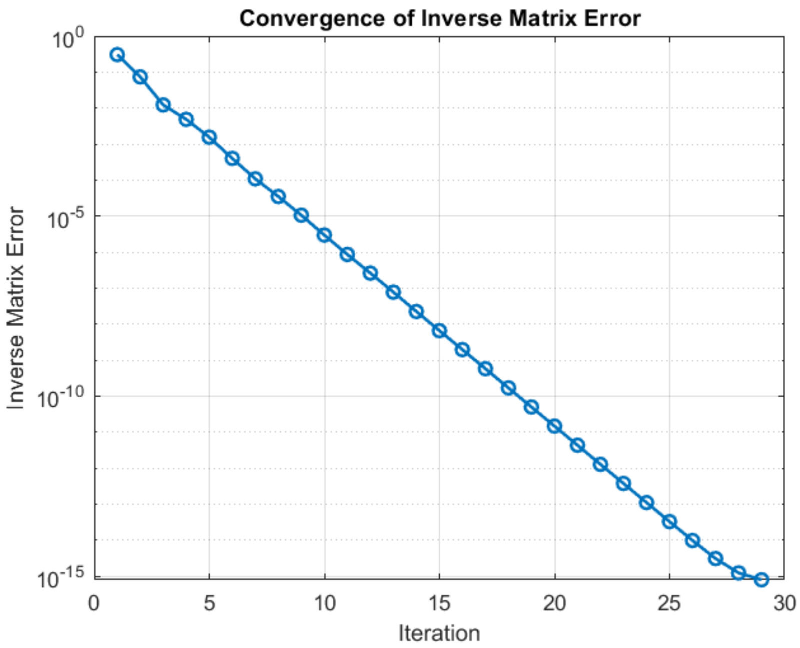

Figure 4.2. Error plot during iterative inverse computation (Case 2)

We observe that

$$BB^{-1} = \begin{bmatrix} 1.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & 1.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 1.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 1.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & -0.0000 & 1.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 1.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 1.0000 \end{bmatrix}.$$

3. Applications of matrix inverse

We can represent any infinite system of linear equations by $BX = C$, where B is the coefficient matrix, X is the column vector of variables, and C is the right-hand side vector of the constants. If B is an infinite square matrix and has its inverse then we can apply the matrix inverse method to solve an infinite system of linear equations represented by $BX = C$. This method is illustrated in the following example in the case of a finite system of linear equations such that the coefficient matrix B satisfies the condition of sub-multiplicative norm.

EXAMPLE 4.4. Consider the system of linear equations that takes the following matrix form.

$$\begin{bmatrix} 1.2 & -0.02 & -0.12 & -0.06 & -0.08 & -0.003 & -0.005 \\ -0.08 & 0.9997 & -0.003 & -0.040 & -0.02 & -0.007 & -0.006 \\ -0.01 & -0.01 & 0.9997 & -0.006 & -0.007 & -0.004 & -0.0006 \\ -0.03 & -0.01 & -0.02 & 0.9994 & -0.02 & -0.05 & -0.07 \\ -0.03 & -0.2141 & -0.0007 & -0.0942 & 0.9999 & -0.0009 & -0.0001 \\ -0.08 & -0.012 & -0.0004 & -0.0009 & -0.989 & 0.9994 & -0.0005 \\ -0.03 & -0.001 & -0.0050 & -0.0020 & 0 & -0.3000 & 0.9998 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \end{bmatrix}$$

Now we solve this system of linear equations with the help of the inverse matrix as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1.2 & -0.02 & -0.12 & -0.06 & -0.08 & -0.003 & -0.005 \\ -0.08 & 0.9997 & -0.003 & -0.040 & -0.02 & -0.007 & -0.006 \\ -0.01 & -0.01 & 0.9997 & -0.006 & -0.007 & -0.004 & -0.0006 \\ -0.03 & -0.01 & -0.02 & 0.9994 & -0.02 & -0.05 & -0.07 \\ -0.03 & -0.2141 & -0.0007 & -0.0942 & 0.9999 & -0.0009 & -0.0001 \\ -0.08 & -0.012 & -0.0004 & -0.0009 & -0.989 & 0.9994 & -0.0005 \\ -0.03 & -0.001 & -0.0050 & -0.0020 & 0 & -0.3000 & 0.9998 \end{bmatrix}^{-1} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \end{bmatrix}$$

From Example 4.3, we can demonstrate that

$$B^{-1} = \begin{bmatrix} 0.8408 & 0.0354 & 0.1023 & 0.0600 & 0.0787 & 0.0089 & 0.0087 \\ 0.0708 & 1.0110 & 0.0126 & 0.0485 & 0.0396 & 0.0128 & 0.0098 \\ 0.0101 & 0.0132 & 1.0017 & 0.0084 & 0.0132 & 0.0050 & 0.0013 \\ 0.0367 & 0.0326 & 0.0252 & 1.0135 & 0.0959 & 0.0727 & 0.0714 \\ 0.0440 & 0.2208 & 0.0089 & 0.1078 & 1.0209 & 0.0108 & 0.0092 \\ 0.1117 & 0.2336 & 0.0175 & 0.1130 & 1.0173 & 1.0124 & 0.0105 \\ 0.0589 & 0.0723 & 0.0134 & 0.0378 & 0.3079 & 0.3042 & 1.0038 \end{bmatrix}.$$

Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0.8408 & 0.0354 & 0.1023 & 0.06 & 0.0787 & 0.0089 & 0.0087 \\ 0.0708 & 1.011 & 0.0126 & 0.0485 & 0.0396 & 0.0128 & 0.0098 \\ 0.0101 & 0.0132 & 1.0017 & 0.0084 & 0.0132 & 0.005 & 0.0013 \\ 0.0367 & 0.0326 & 0.0252 & 1.0135 & 0.0959 & 0.0727 & 0.0714 \\ 0.044 & 0.2208 & 0.0089 & 0.1078 & 1.0209 & 0.0108 & 0.0092 \\ 0.1117 & 0.2336 & 0.0175 & 0.113 & 1.0173 & 1.0124 & 0.0105 \\ 0.0589 & 0.0723 & 0.0134 & 0.0378 & 0.3079 & 0.3042 & 1.0038 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \end{bmatrix}.$$

By matrix multiplication, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0.1966 \\ 0.2668 \\ 0.3180 \\ 0.5647 \\ 0.6177 \\ 1.2318 \\ 1.0786 \end{bmatrix}.$$

So we find the unknown variables $x_1 = 0.1966, x_2 = 0.2668, x_3 = 0.3180, x_4 = 0.5647, x_5 = 0.6177, x_6 = 1.2318$, and $x_7 = 1.0786$.

CHAPTER 5

Eigenvalues of Infinite Matrices

This chapter centers its attention on eigenvalues and eigenvectors within the context of infinite matrices. Section 1 serves as a comprehensive introduction to these fundamental concepts, particularly emphasizing methodologies, practical applications, and their significance in the realm of finite matrices. The subsequent transition to Section 2 involves an in-depth analysis of eigenvalues and eigenvectors within the domain of infinite matrices. This analytical exploration encompasses the resolution of characteristic equations, employing determinant and identity matrices, and addresses computational intricacies, such as exponentiation and logarithmic transformations. The systematic elucidation of each eigenvalue and its corresponding eigenvector contribute valuable insights into the inherent properties characterizing infinite matrices.

1. Eigenvalues of a finite matrix

Let A be a square matrix of order n and $X \in \mathbb{C}^n$ be a nonzero vector for which

$$AX = \lambda X$$

for some scalar $\lambda \in \mathbb{C}$. Then λ is called an **eigenvalue** of the matrix A, and X is called an **eigenvector** of A associated to λ. The set of all eigenvalues of a matrix A is denoted by $\sigma(A)$ and is referred to as the spectrum of A.

EXAMPLE 5.1. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}.$$

The eigenvalues are those λ for which $\det(A-\lambda I)=0$. Now

$$\begin{aligned}
\det(A-\lambda I) &= \det\left(\begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\
&= \begin{vmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{vmatrix} \\
&= (2-\lambda)(-1-\lambda) - 10 \\
&= \lambda^2 - \lambda - 12.
\end{aligned}$$

The eigenvalues of A are the solutions of the quadratic equation $\lambda^2-\lambda-12 = 0$, namely $\lambda_1 = -3$ and $\lambda_2 = 4$.

As we have discussed, if $\det(A-\lambda I) = 0$, then the equation $(A-\lambda I)x = b$ has either no solutions or infinitely many. When we take $b = 0$, however, it is clear by the existence of the solution $x = 0$ that there are infinitely many solutions (i.e., we may rule out the "no solution" case). If we continue using the matrix A from the example above, we can expect nonzero solutions x (infinitely many of them, in fact) of the equation $Ax = \lambda x$ precisely when $\lambda = -3$ or $\lambda = 4$. Let us proceed to characterize such solutions.

First, we work with $\lambda = -3$. The equation $Ax = \lambda x$ becomes $Ax = -3x$. Writing $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and using the matrix A from above, we have

$$\begin{aligned}
Ax &= \begin{bmatrix} 2x_1 + 2x_2 \\ 5x_1 - x_2 \end{bmatrix}, \\
-3x &= \begin{bmatrix} -3x_1 \\ -3x_2 \end{bmatrix}.
\end{aligned}$$

Setting these equal, we get

$$\begin{aligned}
\begin{bmatrix} 2x_1 + 2x_2 \\ 5x_1 - x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 \\ -3x_2 \end{bmatrix} &\Rightarrow 2x_1 + 2x_2 = -3x_1 \quad \text{and} \quad 5x_1 - x_2 = -3x_2 \\
&\Rightarrow 5x_1 = -2x_2 \\
&\Rightarrow x_1 = -\frac{2}{5}x_2.
\end{aligned}$$

This means that, while there are infinitely many nonzero solutions (solution vectors) of the equation $Ax = -3x$, they all satisfy the condition that the first entry x_1 is $-2/5$ times the second entry x_2. Thus all solutions of this equation can be characterized by

$$\begin{bmatrix} 2t \\ -5t \end{bmatrix} = t \begin{bmatrix} 2 \\ -5 \end{bmatrix},$$

where t is any real number. The nonzero vectors x that satisfy $Ax = -3x$ are called eigenvectors associated with the eigenvalue $\lambda = -3$. One such eigenvector is $u_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, and all other eigenvectors corresponding to the eigenvalue (-3) are simply scalar multiples of u_1 i.e. u_1 spans this set of eigenvectors.

Similarly, we can find eigenvectors associated with the eigenvalue $\lambda = 4$ by solving $Ax = 4x$ as follows:

$$\begin{bmatrix} 2x_1 + 2x_2 \\ 5x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix}$$

$$\Rightarrow 2x_1 + 2x_2 = 4x_1 \text{ and } 5x_1 - x_2 = 4x_2$$

$$\Rightarrow x_1 = x_2$$

Hence, the set of eigenvectors associated with $\lambda = 4$ is spanned by $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. Eigenvalues of an infinite matrix

Let A be any infinite square matrix. The $\mathrm{tr}(\log(A-\lambda I))$, where λ is a scalar, is a logarithmic series. We note that $\mathrm{tr}(\log(A-\lambda I)) = \sum_{k=1}^{\infty}(-1)^{k+1}\,\mathrm{tr}(\frac{(A-\lambda I-I)^k}{k})$.

Definition 5.1. Let A be an infinite square matrix such that $\mathrm{tr}(\log(A-\lambda I))$ is a convergent series for some scalars λ. Then λ is called an **eigenvalue** of the infinite matrix A if and only if

$$\exp(\mathrm{tr}(\log(A - \lambda I))) = 0.$$

We see that the above definition agrees the definition of the eigenvalues of a finite matrix A. Because, if λ is an eigenvalue of A, by using the definition of the determinant of an infinite square matrix, we have

$$\det(A - \lambda I) = \exp(\mathrm{tr}(\log(A - \lambda I))) = 0.$$

As in the definition of the eigenvalues of an infinite square matrix, to find all the eigenvalues of an infinite matrix A, we must solve the equation $\exp(\mathrm{tr}(\log(A-$

$\lambda I))) = 0$ for all possible values of λ. In the following example, we use this definition in the case of the eigenvalues of a finite matrix. Here, we use the MATLAB code Source Code 6.3 (Algorithm 3).

EXAMPLE 5.2. Let

$$A = \begin{bmatrix} 1.2000 & -0.0200 & -0.1200 & -0.0600 & -0.0800 & -0.0030 & -0.0050 \\ -0.0800 & 0.9997 & -0.0030 & -0.0400 & -0.0200 & -0.0070 & -0.0060 \\ -0.0100 & -0.0100 & 0.9997 & -0.0060 & -0.0070 & -0.0040 & -0.0006 \\ -0.0300 & -0.0100 & -0.0200 & 0.9994 & -0.0200 & -0.0500 & -0.0700 \\ -0.0300 & -0.2141 & -0.0007 & -0.0942 & 0.9999 & -0.0009 & -0.0001 \\ -0.0800 & -0.0120 & -0.0004 & -0.0009 & -0.9890 & 0.9994 & -0.0005 \\ -0.0300 & -0.0010 & -0.0050 & -0.0020 & 0 & -0.3000 & 0.9998 \end{bmatrix}.$$

We find that $\|I - A\| = 0.995135$ which is less than 1. This ensures that we can compute the eigenvalues of A. We find the eigenvalues of A as follows:

$$\lambda_1 = 0.705572, \quad \lambda_2 = 1.007391,$$

$$\lambda_3 = 1.007391, \quad \lambda_4 = 1.200285,$$

$$\lambda_5 = 1.139452, \quad \lambda_6 = 1.139452, \text{ and}$$

$$\lambda_7 = 0.998357.$$

For $\lambda_1 = 0.705572$, we have

$$\det(A - \lambda_1 I) = 0.$$

Then

$$\exp(\operatorname{trace}(\log(A - \lambda_1 I))) = 0.000002.$$

Thus, the final determinant error for eigenvalue $\lambda_1 = 0.0000021186$ after 1000 iterations.

For $\lambda_2 = 1.007391$, we have

$$\det(A - \lambda_2 I) = 0.$$

Then

$$\exp(\operatorname{trace}(\log(A - \lambda_2 I))) = 0.000000.$$

Thus, the final determinant error for eigenvalue $\lambda_2 = 0.0000000000$ after 32 iterations.

For $\lambda_3 = 1.007391$, we have

$$\det(A - \lambda_3 I) = 0.$$

Then

$$\exp(\operatorname{trace}(\log(A - \lambda_3 I))) = 0.000000.$$

Thus, the final determinant error for eigenvalue $\lambda_3 = 0.0000021186$ after 32 iterations.

For $\lambda_4 = 1.200285$, we have

$$\det(A - \lambda_4 I) = 0.$$

Then

$$\exp(\operatorname{trace}(\log(A - \lambda_4 I))) = 0.000000.$$

Thus, the final determinant error for eigenvalue $\lambda_4 = 0.0000000000$ after 5 iterations.

For $\lambda_5 = 1.139452$, we have

$$\det(A - \lambda_1 I) = 0.$$

Then

$$\exp(\operatorname{trace}(\log(A - \lambda_5 I))) = 0.00000.$$

Thus, the final determinant error for eigenvalue $\lambda_5 = 0.0000000001$ after 6 iterations.

For $\lambda_6 = 1.139452$, we have

$$\det(A - \lambda_6 I) = 0.$$

Then

$$\exp(\operatorname{trace}(\log(A - \lambda_6 I))) = 0.000000.$$

Thus, the final determinant error for eigenvalue $\lambda_6 = 0.0000000001$ after 6 iterations.

For $\lambda_7 = 0.998357$, we have

$$\det(A - \lambda_7 I) = 0.$$

Then

$$\exp(\text{trace}(\log(A - \lambda_7 I))) = 0.0000000000.$$

Thus, the final determinant error for eigenvalue $\lambda_7 = 0.0000000000$ after 12 iterations.

The graphical representations in Figure 5.1 to Figure 5.7 capture the convergence behavior of determinant errors associated with 7 eigenvalues across a series of iterative steps. This graph provides valuable insights into the iterative refinement process, illustrating how the determinant error diminishes over successive iterations, thus indicating the improving accuracy and convergence of the eigenvalue computations for the given matrix.

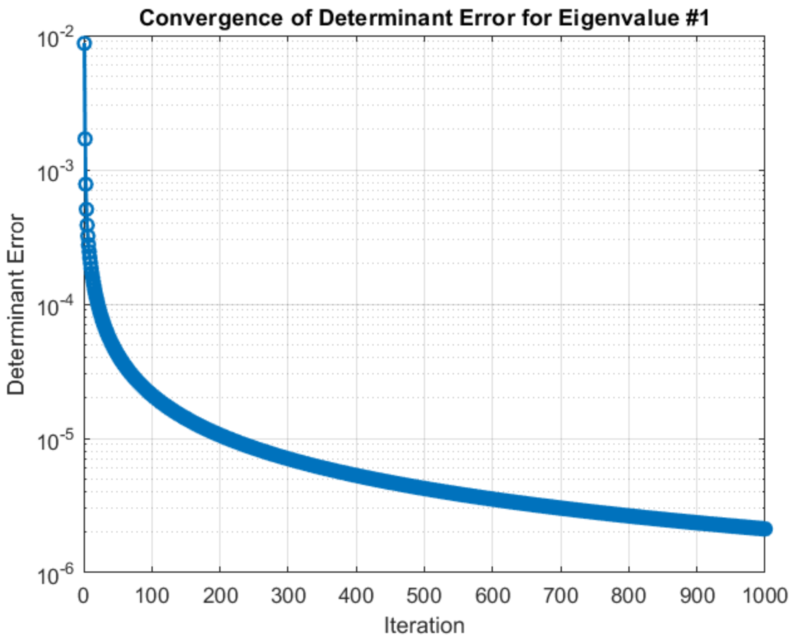

Figure 5.1. Error analysis of determinant for eigenvalue 1

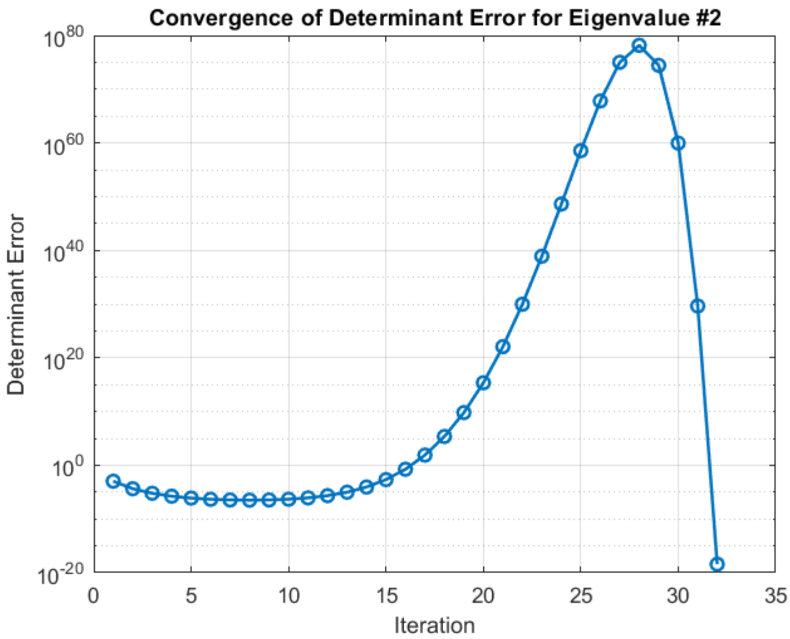

Figure 5.2. Error analysis of determinant for eigenvalue 2

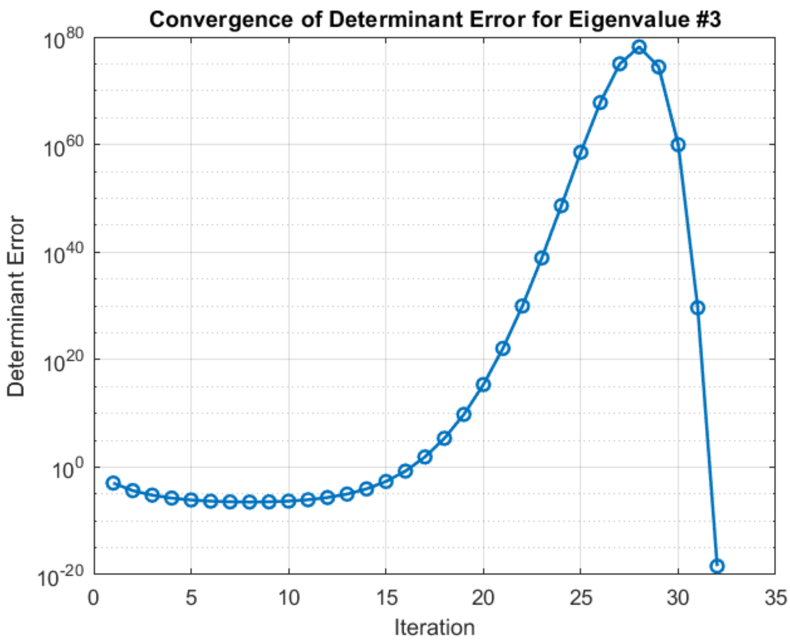

Figure 5.3. Error analysis of determinant for eigenvalue 3

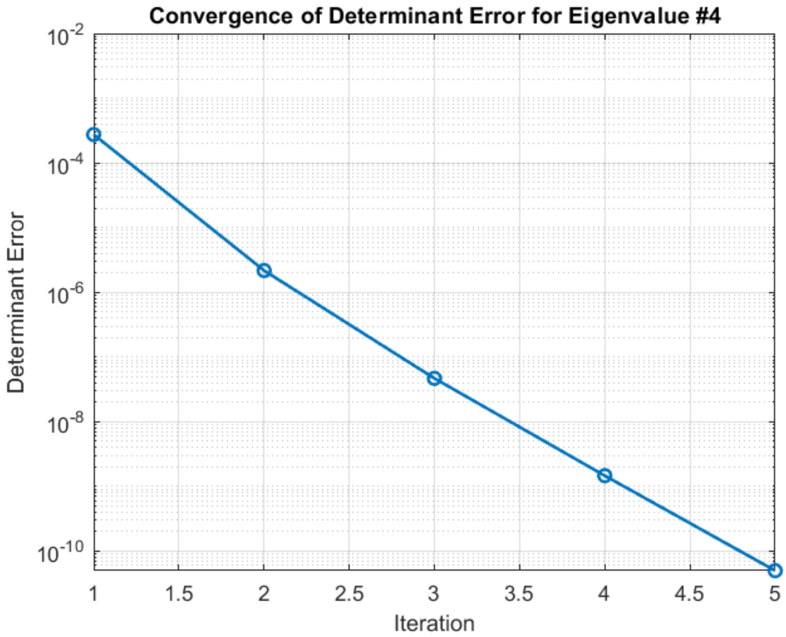

Figure 5.4. Error analysis of determinant for eigenvalue 4

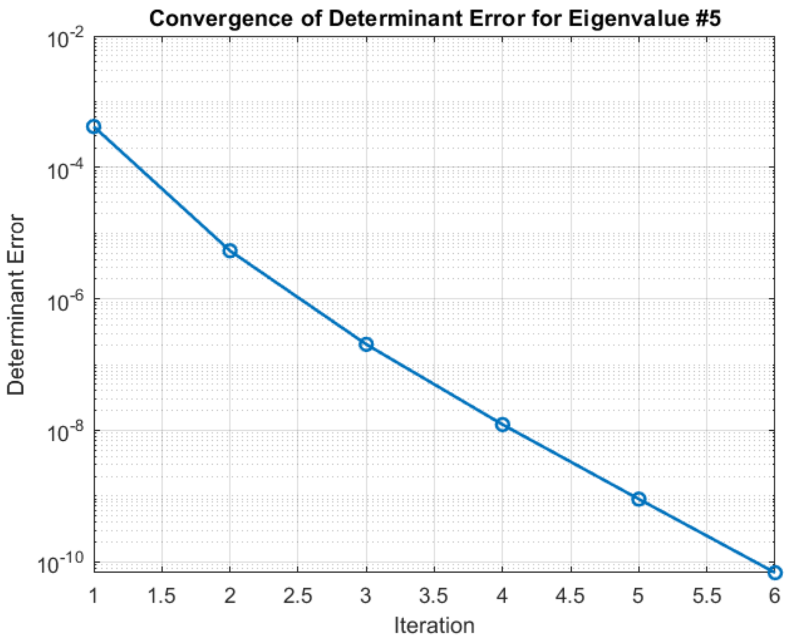

Figure 5.5. Error analysis of determinant for eigenvalue 5

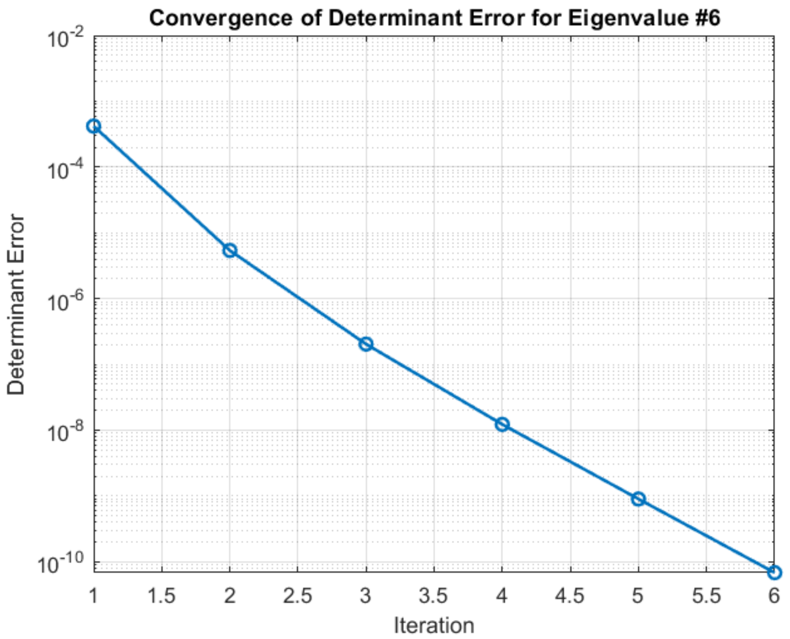

Figure 5.6. Error analysis of determinant for eigenvalue 6

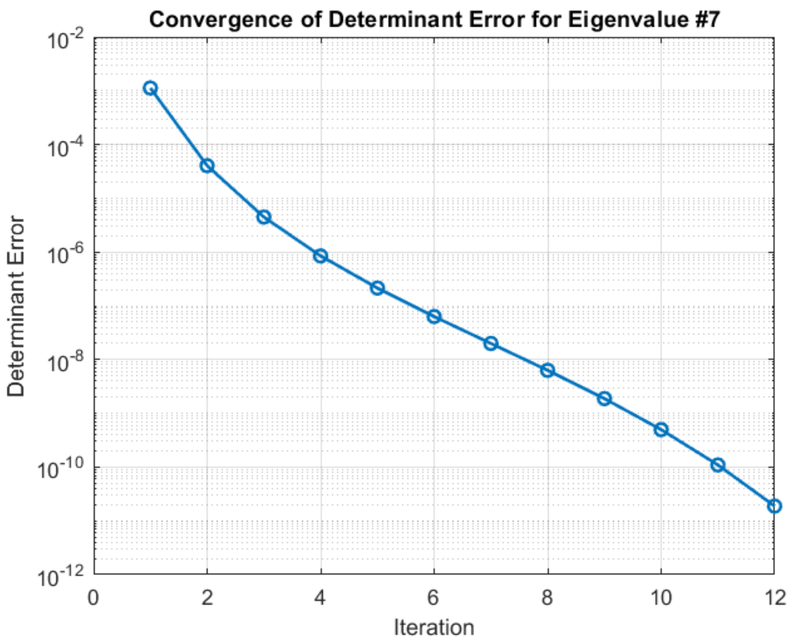

Figure 5.7. Error analysis of determinant for eigenvalue 7

3. Applications of eigenvalues

With the help of eigenvalues and eigenvectors, we can diagonalize a matrix and find its logarithm.

EXAMPLE 5.3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

To find eigenvalues of A, firslty solve $\det(A - \lambda I) = 0$ for λ. This gives

$$\det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0.$$

The three eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

We now calculate the eigenvectors associated to the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

For $\lambda_1 = 1$, we have

$$(A - I)\mathbf{v}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{v}_1 = \mathbf{0}.$$

This gives

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Similarly, for $\lambda_2 = 2$ and $\lambda_3 = 3$, we have $\mathbf{v}_2$ and $\mathbf{v}_3$ as follows:

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Now we construct the matrices P and D as follows:

$$P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now we calculate P^{-1} as follows:

$$P^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, we have the diagonal matrix $A = PDP^{-1}$ as follows:

$$PDP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we find $\log A$ as follows:

$$\log A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \log 1 & 0 & 0 \\ 0 & \log 2 & 0 \\ 0 & 0 & \log 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \log 2 & -2\log 2 \\ 0 & 0 & \log 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \log 2 & -2\log 2 + \log 3 \\ 0 & \log 2 & -2\log 2 + 2\log 3 \\ 0 & 0 & \log 3 \end{bmatrix}.$$

CHAPTER 6

Algorithms and Source Codes

Algorithm 1: Approximation determinant of a matrix by using logarithm Series Expansion

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Data: Square matrix A of size n-by-n
Result: Approximated determinant using logarithm series expansion.
Initialization:
Set parameters: `number_of_terms`, `tolerance`;
Attempt Matrix Generation:
Generate a random matrix A with elements from a standard normal distribution;
Matrix condition check:
Ensure that A satisfies the condition $\|I - A\| < 1$;
Retry with a new random matrix if conditions are not met;
Initialize Plotting:
Initialize variables for plotting the convergence of determinant error;
Create a figure for visualization;
Determinant approximation:
Calculate the logarithm using the series expansion for $\log(A)$;
Track the determinant approximation after each iteration;
Plot the determinant error after each iteration;
Visualization:
Display the original determinant, the approximated determinant, and the final determinant error;
End

Source Code 6.1.

```
function logDetApproximationWithPlot()
    % logDetApproximationWithPlot: Approximates determinant using
      series expansion
    % and plots the convergence of determinant error.
    % Parameters
    number_of_terms = 100;
    tolerance = 1e-10;
    % Matrix generation
```

```
matrixSize = randi([4, 5]);
attemptCount = 0;
% Generate a matrix meeting specific conditions
while true
    A = diag(randn(matrixSize, 1));
    if norm(eye(matrixSize) - A) > 1
        attemptCount = attemptCount + 1;
        disp(['Attempt ', num2str(attemptCount), ': Conditions
            not met. Trying again...']);
    else
        disp(['Matrix found after ', num2str(attemptCount), '
           attempts.']);
        disp('Matrix A:');
        disp(A);
        break;
    end
end
disp (" Generated Square Matrix :") ;
disp ( A ) ;
p = norm(eye(matrixSize) - A);
fprintf('Norm of (I-A): %.6f\n', p);
% Initialize variables for plotting
det_A = det(A);
det_exp_trace_log_A_series = zeros(1, number_of_terms);
det_error = zeros(1, number_of_terms);
% Initialize the figure for plotting
figure;
% Calculate the logarithm using the series expansion
log_A_series = zeros(size(A));
% Main loop for series expansion and determinant approximation
for k = 1:number_of_terms
    term = ((-1)^(k+1)) * ((A - eye(size(A)))^k) / k;
    log_A_series = log_A_series + term;
    % Compare det(A) and det(exp(trace(log_A_series)))
    det_exp_trace_log_A_series(k) = det(exp(trace(log_A_series
       )));
    % Calculate the error
    det_error(k) = abs(det_A - det_exp_trace_log_A_series(k));
    % Check for convergence
    if det_error(k) < tolerance
        break
    end
    % Plot the error after each iteration
    semilogy(1:k, det_error(1:k), '-o', 'LineWidth', 1.5);
    xlabel('Iteration');
```

```
        ylabel('Determinant Error');
        title('Convergence of Determinant Error');
        grid on;
        drawnow;
        % Pause for a short duration to allow for visualization
        pause(0.00005);
    end
    % Display results and final error
    fprintf("Original matrix determinant (up to 10 decimal places)
       : %.10f\n", det_A);
    fprintf("Approximated determinant after %d iterations: %.10f\n
       ", k, det(exp(trace(log_A_series))));
    fprintf("Final Determinant Error: %.10f\n", det_error(k));
    disp(log_A_series);
end
```

Algorithm 2: Finding Inverse Matrices by Series Expansion 2.1

Data: Square matrix A of size n-by-n

Result: Computed inverse matrix by using inverse geometric series 2.1

Initialization:

Set parameters: `tolerance`, `maxIterations`;

Attempt Matrix Generation:

Generate a random square matrix A with elements from a standard normal distribution;

Matrix Condition Check:

Ensure that $\|I - A\| < 1$;

Retry with a new random matrix if conditions are not met;

Calculate norm:

Compute the norm $\|I - A\|$;

Inverse matrix approximation:

For each iteration up to `maxIterations` (to avoid computational complexity). Calculate the inverse matrix using the inverse geometric series ;

Plot the inverse matrix error after each iteration ;

Visualization:

Display the original inverse matrix, the approximated inverse matrix, and the final determinant error;

End

Source Code 6.2.

```
function findInverseMatrices()
while true
```

```
% Generate a random square matrix size
matrixSize = randi([5, 7]); % Matrix size within the
    specified range
m = diag(randn(matrixSize, 1));
% Count the number of attempts
totalAttempts = totalAttempts + 1;
disp("Generated Square Matrix:");
disp(m);
% Get matrix size
o = size(m);
id = eye(o);
% Initialize identity and accumulation matrices
s = id;
% Calculate the norm
p = norm(id - m);
fprintf("The norm is %f \n", p);
 % Check if the norm is less than 1
if p < 1
    tol = 1e-10;
    n = 100000; % Maximum number of iterations to avoid
        computational complexity
    % Initialize variables for plotting
    inverse_error = zeros(1, n);
     % Create a figure for the error plot
    figure;
    % Iterative series expansion
    for i = 1:n
        a = (id - m)^i;
        s = s + a;
        % Calculate the error
        inverse_error(i) = norm(m * s - id);
         % Plot the error after each iteration
        semilogy(1:i, inverse_error(1:i), '-o', 'LineWidth
            ', 1.5);
        xlabel('Iteration');
        ylabel('Inverse Matrix Error');
        title('Convergence of Inverse Matrix Error');
        grid on;
        drawnow;
         % Pause for a short duration to allow for
            visualization
        pause(0.000000001);
         % Break the loop if the error is below tolerance
        if inverse_error(i) < tol
            break
```

```
                end
            end
            % Display the exact inverse matrix
            disp("Computed Inverse Matrix:");
            disp(inv(m));
            % Display the number of iterations needed for
                convergence
            fprintf("Convergence achieved up to 6 decimal places
                in %d iterations.\n", i);
            % Display the inverse matrix computed by inverse
                geometric series
            disp("Matrix obtained by expanding the series:");
            disp(s);
            % Display the total number of attempts
            fprintf("Total Attempts: %d\n", totalAttempts);
            % Display matrix size
            disp("Matrix Size:");
            disp(matrixSize);
            % Exit the loop once a suitable matrix is found
            break;
        else
            disp("Norm must be less than 1. Trying again..");
        end
    end
end
```

Algorithm 3: Determinant Approximation for Multiple Eigenvalues logarithm Series Expansion 5.1

Data: Square matrix A of size n-by-n
Result: Approximated Determinant for Multiple Eigenvalues
Initialization:
Set parameters: `maxIterations`, `tolerance`;
Attempt matrix generation:
Generate a random square matrix A with elements from a standard normal distribution;
Matrix Condition Check:
Ensure that $\|I - A\| < 1$;
Retry with a new random matrix if conditions are not met;
Calculate norm: Compute the norm $\|I - A\|$;
Calculate Eigenvalues:
Compute the eigenvalues of A;
Determinant approximation:
For each iteration up to `maxIterations`(to avoid computational complexity);
Track the determinant approximation after each iteration;
Plot the determinant error after each iteration;
Visualization
Display the original determinant by using by $(\det(A - \lambda I) = 0$, the approximated determinant $(\exp(\mathrm{tr}(\log((A - \lambda I)))) = 0)$;
Plot the determinant error after each iteration ;
End

Source Code 6.3.

```
function DetLogApproxWithEigenvals()
number_of_terms = 1000;
tolerance = 1e-4;  % Set the desired threshold for convergence
if nargin < 1
    matrixSize = randi([5,7]);
    A = randn(matrixSize);
    % Counter for attempts to generate matrix with positive
       real eigenvalues
    attemptCount = 1;
    % Keep generating matrices until all eigenvalues are
       positive real numbers
    while any(imag(eig(A)) ~= 0) || any(eig(A) < 0) || norm(
       eye(matrixSize) - A) > 1
        A = randn(matrixSize);
        attemptCount = attemptCount + 1;
```

```
    end
    fprintf('Generated matrix with positive real eigenvalues
       and norm(identity - A) < 1 after %d attempts.\n',
       attemptCount);
end
% Display the original matrix A
fprintf('Original Matrix A:\n');
disp(A);
eigenvalues_A = eig(A);
p = norm(eye(matrixSize) - A);
fprintf('Eigenvalues of A:\n');
disp(eigenvalues_A);
fprintf('Norm of (I-A): %.6f\n', p);
% Initialize variables for plotting
figure;
for eigenIdx = 1:length(eigenvalues_A)
    % Calculate the logarithm using the series expansion
    log_A_series = zeros(size(A));
    eigen_A = eigenvalues_A(eigenIdx);
    fprintf('Eigenvalue #%d: %.6f\n', eigenIdx, eigen_A);
    % Initialize variables for plotting
    det_exp_trace_log_A_series = zeros(1, number_of_terms);
    det_error = zeros(1, number_of_terms);
    for k = 1:number_of_terms
        term = ((-1)^(k+1)) * ((A - eigen_A * eye(size(A))-eye
           (size(A)))^k) / k;
        log_A_series = log_A_series + term;
        % Compare det(A) and det(exp(trace(log_A_series)))
        det_exp_trace_log_A_series(k) = (exp(trace(
           log_A_series)));
        % Calculate the error
        det_error(k) = abs(det(A - eigen_A * eye(size(A))) -
           det_exp_trace_log_A_series(k));
        if (det_error(k) <= tolerance)
            break
        end
        % Plot the error after each iteration
        subplot(length(eigenvalues_A), 1, eigenIdx);
        semilogy(1:k, det_error(1:k), '-o', 'LineWidth', 1.5);
        xlabel('Iteration');
        ylabel('Determinant Error');
        title(sprintf('Convergence of Determinant Error for
           Eigenvalue #%d', eigenIdx));
        grid on;
        drawnow;
```

```
            pause(0.005);
        end
        fprintf("Determinant of (A - eigenvalue(A) * identity): %
           .10f\n", det(A - eigen_A * eye(size(A))));
        fprintf("exp(trace(log(A - eigenvalue(A) * identity))): %
           .10f\n", det(exp(trace(log_A_series - eigen_A * eye(
           size(A))))));
        fprintf("Final Determinant Error for Eigenvalue #%d: %.10f
            after %d iteration \n", eigenIdx, det_error(end),k);
    end
end
```

Conclusion

In this project, we comprehensively explore some basic operations related to the infinite matrices. This research focuses on the comprehensive analysis of determinants, matrix inverses, and eigenvalues of infinite square matrices with a particular emphasis on finite square matrices. The investigation systematically explores finite matrix determinants and extends to their applications in solving linear systems, addressing the complexities introduced by infinite matrices. Subsequent chapters delve into matrix inverses, expressing them as infinite series, and conclude with an exploration of eigenvalues and eigenvectors, establishing connections between finite and infinite matrix domains.

The study also highlights the challenge of computing determinants for infinite matrices, employing Taylor series expansion and sub-multiplicative norm, and emphasizes the importance of an axiomatic approach rooted in functional analysis. This conceptual framework provides a robust foundation for theoretical development and holds promise for significant advancements in mathematical theory, as exemplified by historical instances like David Hilbert's use of infinite quadratic forms and contemporary programs such as the Langlands program. Approaching the study of infinite matrices through matrix properties and analysis enhances comprehension of their mathematical significance and potential applications.

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