

Constructor Theory of Thermodynamics

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The laws of thermodynamics, powerful for countless purposes, are not exact: both their phenomenological and their statistical-mechanical versions are valid only at ‘macroscopic scales’, which are never defined. Here I propose a new, exact and scale-independent formulation of the first and second laws of thermodynamics, using the principles and tools of the recently proposed constructor theory. Specifically, I improve upon the axiomatic formulations of thermodynamics (Carathéodory, 1909; Lieb & Yngvason, 1999) by proposing an exact and more general formulation of ‘adiabatic accessibility’. This work provides an exact distinction between work and heat; it reveals an unexpected connection between information theory and the *first* law of thermodynamics (not just the second); it resolves the clash between the irreversibility of the ‘cycle’-based second law and time-reversal symmetric dynamical laws. It also achieves the long-sought unification of the axiomatic version of the second law with Kelvin’s.

1. Introduction

An insidious gulf separates existing formulations of thermodynamics from other fundamental physical theories. Those formulations are *approximate* and *scale-dependent* – i.e., they have only a certain domain of applicability, or hold only at a certain ‘scale’, or level of ‘coarse-graining’, none of which are ever specified. So existing thermodynamics can provide powerful predictions and explanations of ‘macroscopic’ systems such as Victorian heat engines, but not

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about ‘microscopic’ ones, such as individual quantum systems. Consequently, the conventional wisdom is that thermodynamics is not a fundamental theory of physics at all.

In sharp contrast, in this paper I propose a *scale-independent* and *exact* formulation of the first and the second laws of thermodynamics – that is to say, a formulation that does not rely on approximations, such as ‘mean values on ensembles’, ‘coarse-graining procedures’, ‘thermodynamic equilibrium’, or ‘temperature’. This is a radically new approach, improving upon so-called *axiomatic thermodynamics* (Carathéodory, 1909; Lieb & Yngvason, 1999), and using the principles and tools of the recently proposed *constructor theory* (Deutsch, 2013) and the associated *constructor theory of information* (Deutsch & Marletto, 2015).

The key idea is a new, constructor-theoretic definition of *adiabatic accessibility*, based on Lieb&Yngvason’s, but providing an *exact* distinction between work and heat. This then reveals an exact link between thermodynamics and (constructor-theoretic) information theory, not only via the second law, as one would expect (Landauer, 1961; Bennett, 1987), but also via the *first*. It also unifies Carathéodory’s approach with more traditional ones (e.g. Kelvin’s).

Existing approaches to thermodynamics can be classified as follows: those, like Carathéodory’s and the statistical-mechanical approaches, that state the second law in terms of *spontaneous processes on isolated, confined systems*; and those that state it in what we shall see with hindsight are informal constructor-theoretic terms, i.e., as the *impossibility* of certain physical transformations being performed by *devices operating in a cycle*, such as Kelvin’s and Clausius’s, and Lieb&Yngvason’s. The formulation in this paper belongs to the latter tradition; hence, as I shall explain, it does not clash with time-reversal symmetric dynamical laws – which is, in part, why it can be exact (see section 2.1).

I shall now summarise why constructor theory is needed in my approach (section 1.1); the specific problems I shall address (section 1.2); and the logic of the solution (section 1.3).

1.1. The role of constructor theory

The radically new mode of explanation of constructor theory allows thermodynamics to be approached from a new direction – namely that the physical world can be described and explained exclusively via statements about which physical transformations, more precisely ‘*tasks*’ (section 2), are *possible*, which are *impossible*, and why. This is in contrast with *the prevailing conception of fundamental physics*, under which physical laws are predictors of what *must happen*, given boundary conditions in spacetime that sufficiently fix the state.

One of constructor theory’s key insights is that there is a fundamental difference between *a task being possible* and *a process being permitted* by dynamical laws. The latter means that the process occurs spontaneously (i.e., when the physical system is isolated, with no interactions with the surroundings) given certain boundary conditions. In contrast, a task is deemed ‘possible’ if the laws of physics allow for arbitrarily accurate approximations to a *constructor* for the physical transformation the task represents. A constructor (see section 2) is an object that, if presented with one of the designated inputs of the task, produces (one of) the corresponding outputs, and retains the ability to do this again. This allows it to operate ‘in a cycle’ – a concept familiar in thermodynamics, and indeed, idealised heat engines are constructors. But the concept of a constructor is extremely general – for example, computers and chemical catalysts can be regarded as approximately-realised constructors. In reality no perfect constructor ever occurs, because of errors and deterioration; but whenever a task is possible the behaviour of a constructor for that task can be approximated to arbitrarily high accuracy. Under constructor theory (despite its name!) laws are expressed referring exclusively to the possibility or impossibility of tasks, *not* to constructors.

The laws of constructor theory are *principles* – laws about laws – i.e. they underlie other physical theories (such as laws of motion of elementary particles, etc.), called subsidiary theories in this context. These principles express regularities among all subsidiary theories, including new regularities

that the prevailing conception cannot adequately capture. Thus, constructor theory is not just a framework (such as resource theory, (Coecke *et. al.*, 2014), or category theory (Abramsky & Coecke, 2009)) for reformulating existing theories: it also has new laws of its own.

Essential for the present work are the new principles of the *constructor theory of information* (Deutsch & Marletto, 2015). They express the regularities in nature that are implicitly required by theories of information (e.g. Shannon's), via *exact* statements about possible and impossible tasks, thus giving full physical meaning to the hitherto fuzzily defined notion of information. These laws will be used (see section 5) to express the exact connection between thermodynamics and information.

1.2. The problem

There are several not-quite-equivalent formulations of the laws of thermodynamics (Bailyn, 1994; Buchdahl, 1966). Here I shall summarise them by merging Carathéodory's approach with its most recent formulation, by Lieb and Yngvason (Lieb & Yngvason, *op. cit.*), on which I shall improve. This approach is notable for improving upon Kelvin's (Marsland *et. al.*, 2015), in that it defines 'heat' – a notoriously fuzzy concept – in terms of 'work'. However, as I shall explain, it too is affected by serious problems. In short, it lacks definite physical content; also, just like all other existing formulations, it relies on unphysical idealisations, which make the domain of applicability of its laws ill-defined.

The primitive notion in that approach is that of an *adiabatic enclosure*:

An adiabatic enclosure is one that only allows 'mechanical coordinates' [see below] of a physical system to be modified.

Then, the *first law of thermodynamics* states that all ways of 'doing work' on adiabatically enclosed systems (i.e. those inside an adiabatic enclosure) are equivalent (Atkins, 1998) in the sense that:

The work required to change an adiabatically enclosed system from one specified state to another specified state is the same however the work is done.

This allows one to introduce internal energy as an additive function of state; and then ‘heat’ is defined as the difference ΔQ between the work ΔU required to drive the system from the state x to the state y when adiabatically enclosed, and the work ΔW required to drive the system between the same two states when the adiabatic enclosure is removed. The classic expression of the first law is then:

$$\Delta U = \Delta W + \Delta Q$$

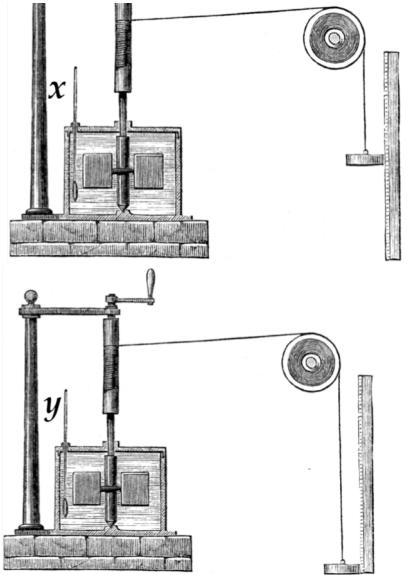
The fundamental problem here is that everything rests on the above-mentioned ‘mechanical coordinates’, but it is never specified what properties the coordinates of a system must have for them to be ‘mechanical’. Mechanical coordinates are only defined ostensively – i.e., by listing some variables of real physical systems (e.g. momentum, volume) that seem to possess the intuitive property. However, ostensive definitions have little physical content, and none at a fundamental level. Is the x -component of a quantum spin a mechanical coordinate? Existing thermodynamics gives no indication. So the ‘completely mechanical definition of heat’, at which Caratheodory’s approach aimed, is never actually achieved; nor, therefore, is his distinction between work and heat (Uffink, 2001).

The same problem affects the second law, which is rooted in the notion of adiabatic accessibility of ‘macrostates’ x and y of a physical system (Lieb&Yngvason, *op. cit.*):

The state x is adiabatically accessible from the state y if the physical transformation $\{x \rightarrow y\}$ can be brought about by a device capable of operating in a cycle [a constructor], with the sole side-effect being the displacement of a weight in a gravitational field.

The second law can then be stated as (Caratheodory, 1909):

In any neighbourhood of any point x there exists a point y such that y is not adiabatically accessible from x .



A famous example of such x and y is given by Joule's experiment (see **Figure 1**) to measure 'the mechanical equivalent of heat'. The constructor in question consists of the stirrer and the pulley (as they undergo no net change and can work in a cycle) and the 'weight' includes the string. x and y are two macrostates in the thermodynamic space – labelled by their different temperatures x and y . Under known

Figure 1: The state y , at a higher temperature, is laws of physics, if $y > x$, y is adiabatically accessible from x , at a lower adiabatically accessible from x , but not temperature, but not vice-versa. (Adapted from: *Harper's New Monthly Magazine*, No. 231, August, 1869).

The problem here is that 'adiabatically accessible' is given only an ad-hoc definition, again with little physical significance. What is the physical property of the weight that makes the transformation between x and y adiabatic, when accomplished with the weight being the sole side-effect? No answer can be found in thermodynamics. Yet answering that question is essential to defining the domain of applicability of the second law, and to its physical content. In its current form, the law may not be applicable in more general situations – such as in information-processing nanoscale devices (see, e.g., Goold *et. al.*, 2016; Brandão *et al.*, 2015), where side-effects of transformations are not clearly related to a weight or other intuitively defined mechanical coordinate.

The first and second laws proposed in this paper are not affected by the above problems. In particular, they *are scale-independent*: thus, although this goes

well beyond the scope of this paper, they should be applicable to nano-scale devices too.

Another problem with all existing formulations of thermodynamics, and which does not arise in my approach, is that they rely on the existence of ‘equilibrium states’ (or equivalent), which are in turn vaguely defined. Equilibrium is when “all the fast things have happened and the slow things have not” (Feynman, 1972) – but this condition is never strictly satisfied in reality. A more rigorous definition is that equilibrium states are those “which, once attained, remain constant in time thereafter until the external conditions are unchanged” (Brown & Uffink, 2001). Such states never exactly occur either, because of fluctuations (both in the classical (Brown *et. al.*, 2009) and quantum (Linden *et. al.*, 2009) domains). Existing formulations postulate equilibrium states via two routes. One is the so-called “*minus-first*” law – (Brown & Uffink *op. cit.*):

An isolated system in an arbitrary initial state within a finite fixed volume will spontaneously attain a unique state of equilibrium.

The other is the *zeroth law* (e.g. (Atkins, 1998), which is used to define temperature by requiring transitivity of the thermodynamic equilibrium relation between any two physical systems.

As I shall explain, using constructor theory I shall not need to postulate the existence of equilibrium states via the minus-first law (see section 2.1) and I shall introduce an equivalent of the zeroth law (see section 6) which does not rely on there being such equilibrium states, nor temperature. It will be expressed, of course, as an exact statement about certain tasks being possible. The reason why constructor-theoretic statements are exact is that they do not require the *existence* of an ideal constructor for possible tasks (see section 2.1); rather, they refer exclusively to possible/impossible tasks, i.e., to physical laws allowing/not allowing a sequence of arbitrarily accurate approximations to the ideal constructor.

Note that deriving thermodynamics from statistical mechanics (Landsberg, 1978) does not solve any of the above problems. Statistical mechanics's aim is to reconcile the second law of thermodynamics with the prevailing conception. Thus, the second law in this context requires there to be irreversibility in the spontaneous evolution of confined, isolated systems (Wallace, 2015a); such irreversibility is usually cast in terms of *entropy* being a globally monotonically increasing function (Uffink, *op. cit.*). However, deriving or even reconciling such statements with the microscopic time-reversal symmetric dynamical laws is notoriously problematic, because of the inevitability of Poincaré recurrence (Wallace, 2015b). To that end, the prevailing conception has produced a number of *models* where time-reversal asymmetric, second-law like behaviour arises from microscopic time-reversal symmetric laws. Such models, however, adopt approximations involving ensembles or coarse-graining procedures, or statistical (probabilistic) assumptions about the process of equilibration – e.g. its “probably” leading to the “most probable” configuration – defined with respect to a natural measure on phase space, not from anything real; or about some specially selected, ad hoc, initial conditions. So once more, albeit successful in all sorts of problem-situations, these approximation schemes lead to laws that are not exact and rest on ad-hoc assumptions. Factual, exact statements in the unphysical limit of an ensemble cannot possibly imply any factual, exact statement about a single system; and resorting to some ‘coarse-graining’ scheme only means that the domain of applicability of the laws is vaguely defined. Hence, in this regard, the statistical-mechanics path to foundations for thermodynamics is no less problematic than the phenomenological approach.

Now, the above problems are nowadays generally regarded as unsolvable: thermodynamic laws are expected to hold only approximately and at macroscopic scales. This capitulation in the face of foundational problems generates a number of troubling open issues. For instance, updated versions of Maxwell’s demon (e.g. Szilard’s engine (Lex & Reff, 1990)), purporting to violate the second law of thermodynamics, are hard to exorcise. In such models, the working medium of the alleged perpetual motion machine of the second kind is constituted by a *single* particle. Since the second law is only

known in scale-dependent forms, it is difficult to pin down what exactly it is that exorcises the demon, because it is controversial what exactly the second law forbids in that context (Bennett, 1987; Earman & Norton, 1999). Similar problems arise in the new field of quantum thermodynamics (Goold *op.cit.*, 2016; Brandão *op.cit.*, 2015), which investigates the implications of the laws of thermodynamics for quantum systems such as atomic-scale ‘heat-engines’. In addition, the known connection between existing information theory (classical and quantum) and thermodynamics (Bennett, *op. cit.* ; Landauer, *op. cit.*) – together with the isolated case of the entropy of an individual black hole (Beckenstein, 1972) – strongly suggest that there is indeed an exact, scale-independent formulation of thermodynamic laws.

In this paper I show that the ‘cycle’ form of the laws of thermodynamics can be stated exactly.

1.3. The logic of the solution

The key step of my construction is to recast *adiabatic accessibility* in constructor theory (section 5). To that end, I define a class of physical systems – *work media* – which would include idealised weights, springs and flywheels – by stating in exact, constructor-theoretic terms what tasks must be possible on them. I then define ‘adiabatic accessibility’ in terms of work media. This definition is very general – it can be applied to any subsidiary theory complying with the principles of constructor theory. An exact, scale-independent formulation of the first and second laws follows.

As I mentioned above, those laws are not confined to equilibrium thermodynamics only, nor do they rely on any notion of temperature. The connection with equilibrium thermodynamics (which will not be explored in this work but is an interesting future application of it) may be established via the constructor-theoretic version of the *zeroth law* (section 6). As in traditional thermodynamics, the zeroth law is an ‘afterthought’ (Atkins, 2007): it comes at the end of the construction. However, unlike in the traditional formulation, is not about *temperature*, but about the possibility of a certain class of tasks.

As a consequence, the Carathéodory's approach is unified with Kelvin's. In short, the two approaches differ as follows: Kelvin's statement of the second law sets a definite direction to the 'irreversibility' of the second law, stating that it is impossible to convert heat completely into work (without any other side-effects), but it is possible to do the reverse; Carathéodory-type statements are weaker, in that they only imply the irreversibility of some transformation, without specifying which direction is forbidden (Uffink, *op. cit.*; Marsland *et. al.*, *op. cit.*). Here the two approaches are properly unified, and heat and work given exact, constructor-theoretic characterisations.

In addition, as I shall explain in section 2.1, expressing the laws in terms of possible or impossible tasks, and not about spontaneous processes happening, has two significant consequences for thermodynamics: (i) it is no longer necessary to require that *equilibrium states* exist; (ii) the statement of the second law requiring that a certain task (say, $\{x \rightarrow y\}$) be *possible*, but the task representing the inverse transformation, $\{y \rightarrow x\}$, be *impossible*, is shown not to clash with time-reversal symmetric dynamical laws. This is because, in short, a task being possible, unlike a process happening, requires a constructor for the task; and the time-reverse of a process including a constructor for $\{x \rightarrow y\}$ need not include a constructor for $\{y \rightarrow x\}$ (section 2.1).

A notable difference is that in constructor theory the role of entropy is not as central as in classical axiomatic thermodynamics. The second law's physical content is *not* that it forbids tasks that decrease (or increase) the entropy of a system, but that it forbids certain tasks to be performed adiabatically (as defined in section 5), while requiring the inverse physical transformation to be performable adiabatically. Entropy enters the picture *after* the second law, as a quantitative classification of tasks, so that tasks that change entropy by the same amount belong to the same class; but the physical content of the second law resides elsewhere: in the definition of adiabatically accessible.

Finally, the *first* law of constructor-theoretic thermodynamics is connected to (constructor) information theory, exactly. This is a novel development: in all previous treatments it is only the second law that is so connected.

1.4. The threads

This paper has two main threads. One is that by expressing physical laws in terms of possible and impossible tasks, instead of processes occurring or not (with some probability), one can express more about physical reality. Specifically, one can accommodate *counterfactual properties* of physical systems – about what can, or cannot, be done to them. This is the key, for instance, to the new definition of adiabatic accessibility: ‘work media’, on which the definition relies, are defined by their counterfactual properties. Thus, by switching to that mode of explanation, it is possible to formulate laws about entities which appear to be inherently fuzzy in the prevailing conception of physics.

The other thread is that information-based concepts, such as distinguishability, provide physical foundations for the notions of heat and work, and for distinguishing between them. This is possible because in the constructor theory of information (section 3), the fuzzily-defined traditional notion of information (Timpson, 2013) is replaced by exact ones. In particular, nothing in the constructor-theoretic notion of information relies on any subjective, agent-based, or probabilistic/statistical statements about reality.

2. Constructor Theory

In constructor-theoretic physics the primitive notion of a ‘physical system’ is replaced by the slightly different notion of a *substrate*, which is a physical system some of whose properties can be changed by a physical transformation brought about by a constructor, which is in turn a substrate that undergoes no net change in its ability to do this. The other primitive elements are *tasks* (as defined below), and the statements about their being possible/impossible. Intuitively a task specifies the states of all the substrates allowed at the beginning and end of the transformation, except those of the constructor(s) bringing it about.

Attributes and variables. For any substrate, a subsidiary theory must provide its *states*, *attributes* and *variables*. These are physical properties of the substrate, which are represented in several interrelated ways. An *attribute* of a substrate is formally defined as a *set of all the states* in which the substrate has that property. For example, a substrate might be a die on a table. Its upturned face is a substrate that can have *six attributes* $n : n \in \{1, 2, \dots, 6\}$, each one consisting of a vast number of states representing, say, the configuration of the atoms of the die. Given these attributes, one can construct others by set-wise union or intersection. For example, the attribute ‘*odd*’ of the upturned face, denoted by *odd*, is the union of all the odd-numbered attributes: $\text{odd} = 1 \cup 3 \cup 5$. Similarly for the attribute ‘*even*’: $\text{even} = 2 \cup 4 \cup 6$. Attributes therefore generalise and make exact the notion of ‘macrostates’; the crucial difference is that attributes, unlike macrostates, are not the result of any approximation (e.g. coarse-graining) procedures. An *intrinsic* attribute is one that can be specified without referring to any other specific system. For example, ‘showing the same number’ is an intrinsic attribute of a pair of dice, but ‘showing the same number as the other one in the pair’ is not an intrinsic attribute of either of them. In quantum theory, ‘being entangled with each other’ is an intrinsic attribute of a qubit *pair*; ‘having a particular density operator’ is an intrinsic attribute of a qubit; if the rest of its quantum state describes entanglement with other systems, then the attribute is non-intrinsic. In this paper A physical *variable* is defined in a slightly unfamiliar way as any *set of disjoint attributes* of the same substrate. In quantum theory, this includes not only all observables, but many other constructs, such as any set $\{x, y\}$ where the attributes *x* and *y* each contain a single state $|x\rangle$ and $|y\rangle$ respectively, not necessarily orthogonal. Whenever a substrate is in a state in an attribute $x \in X$, where *X* is a variable, we say that *X* is *sharp* (on that system), with the *value* *x* – where the *x* are members of the set *X* of labels of the attributes in *X*². As a shorthand, “*X* is sharp in *a*” shall mean that the attribute *a* is a subset of

² I shall always define symbols explicitly in their contexts, but for added clarity I use this convention: Small Greek letters ($\gamma, \delta, \alpha, \beta, \omega, \mu, \nu, \tau, \sigma$) denote states; *small italic boldface* denotes; **CAPITAL ITALIC BOLDFACE** denotes variables; *small italic* denotes labels; **CAPITAL ITALIC** denotes sets of labels; **CAPITAL BOLDFACE** denotes physical systems; and capital letters with arrow above (e.g. \vec{C}) denote constructors.

some attribute in the variable X . In the case of the die, ‘parity’ is the variable $P = \{even, odd\}$. So, when the die’s upturned face is, say, in the attribute 6 , we say that “ P is sharp with value $even$ ”. Also, we say that P is sharp in the attribute 6 , with value $even$ – which means that $6 \subseteq even$. In quantum theory, the z -component-of-spin variable of a spin- $\frac{1}{2}$ particle is the set of two attributes: that of the z -component of the spin being $\frac{1}{2}$, and $-\frac{1}{2}$. That variable is sharp when the qubit is in a pure state with spin $\frac{1}{2}$ or $-\frac{1}{2}$ in the z -direction, and is non-sharp otherwise.

Tasks. A *task* is the *abstract specification* of a set of *physical transformations* on a substrate, which is transformed from having one attribute to having another. It is expressed as a *set of ordered pairs of input/output attributes* $x_i \rightarrow y_i$ of the substrates. I shall represent it as:

$$\mathfrak{A} = \{x_1 \rightarrow y_1, x_2 \rightarrow y_2, \dots\}.$$

The $In(\mathfrak{A}) = \{x_i\}$ are the legitimate *input attributes*, the $Out(\mathfrak{A}) = \{y_i\}$ are the *output attributes*. The *transpose* of a task \mathfrak{A} , denoted by \mathfrak{A}^\sim , is such that $In(\mathfrak{A}^\sim) = Out(\mathfrak{A})$ and $Out(\mathfrak{A}^\sim) = In(\mathfrak{A})$. A task where $In(\mathfrak{A}) = V = Out(\mathfrak{A})$ for some variable V will be referred to as ‘a task \mathfrak{A} over V ’.

A **constructor** for the task \mathfrak{A} is defined as a physical system that would cause \mathfrak{A} to occur on the substrates and *would remain unchanged in its ability to cause that again*. Schematically:

Input attribute of substrates $\xrightarrow{\text{Constructor}}$ Output attribute of substrates

where constructor and substrates jointly are isolated. This scheme draws upon two primitive notions that must be given physical meanings by the subsidiary theories, namely: the substrates with the input attribute are *presented* to the constructor, which *delivers* the substrates with the output attribute. A constructor is *capable of performing* \mathfrak{A} if, whenever presented with the substrates (where it and they are *in isolation*) with a legitimate input attribute of \mathfrak{A} (i.e., in *any* state in that attribute) it delivers them in *some* state in one of the corresponding output attributes, regardless of how it acts on the substrate when it is presented in any other attribute. For instance, a task on the die substrate is $\{even \rightarrow odd\}$; and a constructor for it is a device that

must produce *some* of the die's attributes contained in *odd* whenever presented when *any* of the states in *even*, and retain the property of doing that again. In the case of the task $\{6 \rightarrow \text{odd}\}$ it is enough that a constructor for it delivers *some* state in the attribute *odd* – by switching 6 with, say, 1.

The fundamental principle. A task \mathbf{T} is *impossible* (denoted as \mathbf{T}^x) if there is a law of physics that forbids its being carried out with arbitrary accuracy and reliability by a constructor. Otherwise, \mathbf{T} is *possible*, (denoted by \mathbf{T}^v). This means that a constructor capable of performing \mathbf{T} can be physically realised with arbitrary accuracy and reliability (short of perfection). As I said, heat engines, catalysts and computers are familiar examples of *approximations* to constructors. So, ' \mathbf{T} is possible' means that it can be brought about with arbitrary accuracy, but *it does not imply that it will happen*, since it does not imply that a constructor for it will ever be built and presented with the right substrate. Conversely, a prediction that \mathbf{T} will happen with some probability would not imply \mathbf{T} 's possibility: that 'rolling a seven' sometimes happens when shooting dice does not imply that the task 'roll a seven under the rules of that game' can be performed with arbitrarily high accuracy.

Non-probabilistic, counterfactual properties – i.e. about what *does not happen*, but could – are the centrepiece of constructor theory's mode of explanation, on which this approach to thermodynamics depends. It is expressed by its fundamental principle:

- I. All (other) laws of physics are expressible solely in terms of statements about which tasks are possible, which are impossible, and why.

Hence principle I requires subsidiary theories to have two crucial properties: (i) They must define a topology over the set of physical processes they apply to, which gives a meaning to a sequence of approximate constructions *converging* to an exact performance of \mathbf{T} ; (ii) They must be *non-probabilistic* – since they must be expressed exclusively as statements about possible/impossible tasks. The latter point may seem to make the task of expressing the laws of thermodynamics particularly hard, but that is only an artefact of the prevailing conception (which tries to cast the second law as a model to provide predictions of what will happen to a system evolving spontaneously) that makes probabilities appear to be central to the second

law. In fact none of the laws, in the constructor-theoretic formulation, use probabilistic statements.

Principle of Locality. A pair of substrates S_1 and S_2 may be regarded as a single substrate $S_1 \oplus S_2$. Constructor theory requires all subsidiary theories to provide the following support for the concept of such a combination. First, $S_1 \oplus S_2$ is indeed a substrate. Second, if subsidiary theories designate any task as possible which has $S_1 \oplus S_2$ as input substrate, they must also provide a meaning for *presenting* S_1 and S_2 to the relevant constructor as the substrate $S_1 \oplus S_2$. Third, and most importantly, they must conform to Einstein's (1949) *principle of locality* in the form:

- II. There exists a mode of description such that the state of $S_1 \oplus S_2$ is the pair (ξ, ζ) of the states³ ξ of S_1 and ζ of S_2 , and any construction undergone by S_1 and not S_2 can change only ξ and not ζ .

This, like many of the constructor-theoretic principles I shall be using, is tacitly assumed in all formulations of thermodynamics. It is pleasant that constructor theory states them explicitly, so that their physical content and consequences are exposed.

Unitary quantum theory satisfies II, as is explicit in the Heisenberg picture (Deutsch & Hayden, 2000).

Tasks may be composed into networks to form other tasks, as follows. The *parallel composition* $\mathfrak{A} \otimes \mathfrak{B}$ of two tasks \mathfrak{A} and \mathfrak{B} is the task whose net effect on a composite system $M \oplus N$ is that of performing \mathfrak{A} on M and \mathfrak{B} on N . When $\text{Out}(\mathfrak{A}) = \text{In}(\mathfrak{B})$, the *serial composition* $\mathfrak{B}\mathfrak{A}$ is the task whose net effect is that of performing \mathfrak{A} and then \mathfrak{B} on the same substrate. Parallel and serial composition must satisfy the *composition law*

- III. The serial or parallel composition of possible tasks is a possible task,

which is a tacit assumption both in information theory and thermodynamics, and finds an elegant expression in constructor theory. Note however that in

³ In which case the same must hold for intrinsic attributes.

constructor theory tasks can be impossible and the composition of two impossible tasks may result in a possible one. In fact, that two impossible tasks give rise to a possible one is the signature of the existence of a conservation law (section 4).

2.1. Possible tasks *vs* permitted processes

In line with axiomatic thermodynamics, the second law of thermodynamics in constructor theory takes the form of what I shall call, adapting a term from (Buchdahl, 1966), a *law of impotence*: a law requiring some task to be possible, and its transpose to be impossible. Such irreversibility, unlike that required by the second law that refers to spontaneous processes, is compatible with time-reversal symmetric dynamical laws. This fact, which has been known for some time (Uffink *op. cit.*), can be expressed rigorously in constructor theory because of the fundamental difference between a task being possible and a process being permitted.

To explain how, I shall consider a physical system whose dynamics are expressible in the prevailing conception, but which also conforms to constructor theory. Let the system's state space, containing all its states σ described but the subsidiary theory be Γ . In the prevailing conception, a process is represented as a trajectory – the sequence of states the system goes through as the evolution unfolds: $P \doteq \{\sigma_t \in \Gamma : t_i < t < t_f\}$. A process is *permitted* under the theory if it is a solution of the theory's equations of motion; let W be the set of all permitted processes. Let the map R transform each state σ into its ‘time-reverse’ $R(\sigma)$. For example, R may reverse the sign of all momenta and magnetic fields. Also, define the time-reverse P^- of a process P by: $P^- \doteq \{(R\sigma)_{-t} \in \Gamma : -t_f < t < -t_i\}$, (Uffink, *op. cit.*). The subsidiary theory is called time-reversal invariant if the set W of permitted processes is closed under time reversal, i.e. if and only if: $P^- \in W \Leftrightarrow P \in W$.

Now consider a time-reversal invariant subsidiary theory and the constructor-theoretic statement that the task \mathbf{T} is possible, but \mathbf{T}^- is impossible. As we said, the second law (like any law of impotence) is expressed via a statement

of this kind. We can immediately see that those two facts are compatible with one another:

That \mathbf{T} is possible implies that the process P_ϵ corresponding to an approximation to a constructor \vec{C} performing \mathbf{T} to accuracy ϵ , is permitted for any ϵ (short of perfection). That the theory is time-reversal invariant implies that the process $P_{-\epsilon}$ too is permitted. But, crucially, $P_{-\epsilon}$ does not correspond to the task \mathbf{T}^- being performed to accuracy ϵ ; because the reversed time-evolution of the approximate constructor running in the process P_ϵ is not the dynamical evolution of an approximate constructor for \mathbf{T}^- , to accuracy ϵ . Thus, the statement that \mathbf{T} is possible and \mathbf{T}^- is not possible is compatible with time-reversal invariant dynamical laws, as promised. In constructor theory, laws are about tasks being possible or impossible, not about processes. This is why they can be exact statements, not contradicted by, e.g., the existence of fluctuations.

Another remarkable consequence of stating thermodynamics in terms of possible and impossible tasks only is that need not require ‘equilibrium states’ to exist (defined as states that physical systems evolve spontaneously to, but which never change thereafter unless the external conditions change). As I remarked in the introduction, some existing formulations depend on this impossible requirement, via the so-called ‘minus-first’ law (Brown & Uffink, op.cit.). In constructor theory, it will only be necessary to require there to be a particular class of intrinsic attributes, which I shall call *thermodynamic attributes*. These attributes, however, need not have a definite temperature and include many more than equilibrium states. As explained in section 1, thermodynamics in constructor theory is more general than standard equilibrium thermodynamics: the notion of temperature need never be invoked. Specifically, I shall require the following principle to hold:

IV. Attributes that are unchanged except when acted upon are possible.

Such attributes will be called ‘thermodynamic attributes’: in short, they are attributes of a physical system that can be stabilised to arbitrarily high accuracy; in the case of a qubit, they include quantum states that are very far

from equilibrium: for instance, its pure states. The principle requires the possibility of bringing about such attributes to any accuracy short of perfection.

For example, consider a glass of water with temperature x . Principle IV requires that the attribute x of the glass and water can be stabilised to any arbitrary accuracy, short of perfection. This is compatible with fluctuations occurring. For example, for higher accuracies, stabilisation will require inserting various insulating materials (part of the constructor) around the glass of water to keep it at temperature x to perform the task to that accuracy, while for lower accuracies just the glass by itself will suffice. So, principle IV is fundamentally different from the requirement that “equilibrium states exist”. For while the former is not contradicted by fluctuations occurring, the latter is. Classical thermodynamics relies on the latter requirement and is therefore problematic, given the occurrence of fluctuations. Constructor theoretic thermodynamics relies on the former. This is yet another reason why its laws can be exact.

As remarked in the introduction, although (perfect) constructors never occur in reality, just like equilibrium states, constructor-theoretic laws are exact because they are formulated exclusively in terms of possible/impossible tasks, not in terms of constructors. In particular, they never require perfect constructors to exist. Whenever requiring a task to be possible, one refers implicitly to a sequence of ever improving (but never perfect) approximations to the ideal constructor, with no limit on how each approximation can be improved. That there is, or there is not, a limit to how well the task can be performed (defining a impossible or possible task respectively), is an exact statement.

As a consequence of laws being stated in terms of possible/impossible tasks, the emergence of an arrow of time and the second law of thermodynamics appear as entirely distinct issues (Barbour, *et. al* 2014; Marsland *et. al.*, *op. cit.*). The former is about the spontaneous evolution of isolated physical systems, as established by, e.g., the minus-first law; the latter is about the possibility of certain tasks on finite subsystems of an isolated system. In line with the

traditions of studies in thermodynamics, I shall assume here the existence of an unambiguous 'before and after' in a physical transformation. This must be explained under constructor theory by subsidiary theories about time, in terms of constructor-theoretic interoperability laws (see section 3) concerning tasks of synchronising 'clocks' (Marletto&Vedral, in preparation) – *not* in terms of the ordering established by minus-first law, as in current thermodynamics (Marsland *et. al.*, *op. cit.*).

For present purposes, I shall restrict attention to subsidiary theories that allow for an unlimited number of substrates to be prepared in their thermodynamic states; and I shall require substrates to be *finite*. Having defined *generic substrates* as those substrates that occur in unlimited numbers (Deutsch & Marletto, 2015), I shall assume that:

- V. The task of preparing any number of instances of any substrates with any one of its thermodynamic attributes from generic substrates is possible.

This is a working assumption about cosmology. It would be enough that it hold for a subclass of physical systems only, to which we confine attention for present purposes.

The constructor-theoretic concept of *side-effect*, which will be essential in understanding the notion of 'adiabatically possible' (section 5), is then introduced as follows: If $(\mathcal{A} \otimes \mathcal{T})'$ for some task \mathcal{T} on generic resources (as defined in Deutsch & Marletto, 2015), \mathcal{A} is possible with side-effects, which is written \mathcal{A}^L , and \mathcal{T} is the side-effect.

3. Constructor theory of information

I shall now summarise the principles of the *constructor theory of information* (Deutsch & Marletto, 2015), which I shall use in sections 5 and 6 to define 'work media' and 'heat media', and to distinguish work from heat. The principles express the exact properties required of physical laws by the theories of (classical) information, computation and communication. An important point here is that nothing that follows is probabilistic or 'subjective'. Information is understood in constructor theory in terms of

objective, *counterfactual* properties of substrates ('information media') – i.e. about what tasks are possible or impossible on them.

The logic of the construction is that one *first* defines *a class of substrates* as those on which certain tasks are possible/impossible. In the constructor theory of information these capture the properties of a physical system that would make it capable of instantiating what has been informally referred to as 'information'. Then, one expresses principles about them.

First, a *computation⁴ medium* with *computation variable* V (at least two of whose attributes have labels in a set V) is defined as a substrate on which the task $\Pi(V)$ of performing *every* permutation Π defined via the labels V

$$\Pi(V) \doteq \bigcup_{x \in V} \{x \rightarrow \Pi(x)\}$$

is possible (with or without side-effects). $\Pi(V)$ defines a logically *reversible computation*.

Information media are computation media on which additional tasks are possible. Specifically, a variable X is *clonable* if for some attribute x_0 of S the computation on the composite system $S \oplus S$

$$\bigcup_{x \in X} \{(x, x_0) \rightarrow (x, x)\}, \quad (0)$$

namely *cloning* X , is possible (with or without side-effects)⁵. An *information medium* is a substrate with at least one clonable computation variable, called an *information variable* (whose attributes are called *information attributes*). For instance, a qubit is a computation medium with *any* set of two pure states, even if they are not orthogonal (Deutsch & Marletto, 2015); with a set of two *orthogonal* states it is an information medium. Apart from that definition, information media must also obey the principles (i.e. substantive laws) of constructor information theory, which I now review:

Interoperability of information. Let X_1 and X_2 be variables of substrates S_1 and S_2 respectively, and $X_1 \times X_2$ be the variable of the composite substrate

⁴ This is just a label for the physical systems with the given definition. Crucially, it entails no reliance on any a-priori notion of computation (such as Turing-computability).

⁵ The usual notion of cloning, as in the no-cloning theorem (Wootters & Zurek, 1982), is (1) with X as the set of all attributes of S .

$\mathbf{S}_1 \oplus \mathbf{S}_2$ whose attributes are labelled by the ordered pair $(x, x') \in X_1 \times X_2$, where X_1 and X_2 are the sets of labels of X_1 and X_2 respectively, and \times denotes the Cartesian product of sets. The *interoperability principle* is elegantly expressed as a constraint on the composite system of information media (and on their information variables):

VI. The combination of two information media with information variables X_1 and X_2 is an information medium with information variable $X_1 \times X_2$.

This expresses the property that information can be copied from any one information medium to any other; which makes it possible to regard information media as a class of substrates. Interoperability laws for heat and work will be introduced in sections 5 and 6.

The concept of '**distinguishable**' – which is used in the zeroth law (in section 6) – can be defined in constructor theory without circularities or ambiguities. A variable X of a substrate \mathbf{S} is *distinguishable* if

$$\left(\bigcup_{x \in X} \{x \rightarrow i_x\} \right) \not\models \quad (0)$$

where $\{i_x\}$ is an information variable (whereby $i_x \cap i_{x'} = \emptyset$ if $x \neq x'$). I write $x \perp y$ if $\{x, y\}$ is a distinguishable variable. Information variables are necessarily distinguishable, by the interoperability principle VI. Note that 'distinguishable' in this context is not the negative of 'indistinguishable', as used in statistical mechanics to refer to bosons and fermions. Rather, it means that it is possible to construct a 'single-shot' machine that is capable of discriminating between any two attributes in the variable. For instance, any two non-orthogonal states of a quantum system are not distinguishable, in this sense. I do not use the term 'indistinguishable' in this paper.

I shall use the principle (Deutsch&Marletto 2015) that:

VII. A variable whose attributes are all pairwise distinguishable is distinguishable.

This is trivially true in quantum theory, for distinguishable pairs of attributes are orthogonal pairs of quantum states – however, it must be imposed for general subsidiary theories.

4. Conservation of energy

In constructor theory, conservation laws cannot be formulated via the usual dynamical considerations. They, too, must be expressed solely as statements about possible/impossible tasks. As we shall see, the notion of a conserved quantity (and in particular energy) will refer to a particular pattern of possible/impossible tasks. To describe that pattern, I shall now introduce a powerful constructor-theoretic tool – an *equivalence relation* ‘ \simeq ’ (pronounced ‘is-like’) on the set of all tasks on a substrate, which, ultimately, I shall use to express the intuitive notion that two tasks in the same equivalence class ‘would violate the principle of the conservation of energy by the same amount’.

‘Is-like’ equivalence relation. Given any two *pairwise tasks* $\mathfrak{A} = \{x \rightarrow y\}$, $\mathfrak{B} = \{x' \rightarrow y'\}$, we say that $\mathfrak{A} \simeq \mathfrak{B}$ if and only if

$$[(\mathfrak{A}^\sim \otimes \mathfrak{B})' \wedge (\mathfrak{A} \otimes \mathfrak{B}^\sim)'].$$

This is an equivalence relation over the set of all pairwise tasks on *thermodynamic* attributes of substrates under a given subsidiary theory. So, the family of all equivalence classes generated by ‘is-like’ is a *partition* of that set. I shall assume initially that each class is labeled by a vector $\underline{\Delta} = (\Delta_i)$ of functions $\Delta_i : S_M \Rightarrow \mathfrak{R}$, where \mathfrak{R} is, for simplicity of exposition, the set of the real numbers, with the property that $\Delta_i(\mathfrak{A}) = \Delta_i(\mathfrak{B}), \forall i$ if and only if $\mathfrak{A} \simeq \mathfrak{B}$.

The physical meaning of the partition into equivalence classes is that *tasks in the same class, if impossible, are impossible for the same reason*: this is guaranteed by the fact that $(\mathfrak{A}^\sim \otimes \mathfrak{B})'$ and $(\mathfrak{A} \otimes \mathfrak{B}^\sim)'$. For example, suppose in the prevailing conception there is only one conservation law of some scalar (e.g. energy); in constructor theory, this corresponds to the tasks \mathfrak{A} and \mathfrak{B} that would both violate the conservation law by the same amount being in the same class. Both are impossible, but $(\mathfrak{A}^\sim \otimes \mathfrak{B})'$ and $(\mathfrak{A} \otimes \mathfrak{B}^\sim)'$ because the two tasks balance one another when performed in parallel, so that overall there is no violation of the conservation law. Similarly, consider a law of impotence, such as the second law, requiring tasks decreasing (or increasing)

a given function (such as entropy) to be impossible, but their transpose to be possible. Then, is-like is such that tasks requiring a change in that function by the same amount belong to the same class.

Accordingly, it is natural to regard the labels of the equivalence classes as the amount by which the conserved quantity (or the monotone such as entropy) is changed. To ensure that this identification is physically meaningful, I shall now show that under the additional constraints of locality (principle II), the functions Δ_i must be additive.

To this end, it is necessary to define another is-like relation on the set of thermodynamic attributes, based on the *serial composition* of two tasks $\mathfrak{A}^\sim \mathfrak{B}$ and $\mathfrak{A}\mathfrak{B}^\sim$ (whenever it is defined, i.e., whenever $Out(\mathfrak{A}) = Out(\mathfrak{B})$):

$$\mathfrak{A} \overset{\bullet}{\simeq} \mathfrak{B} \leftrightarrow [(\mathfrak{A}^\sim \mathfrak{B})' \wedge (\mathfrak{B}^\sim \mathfrak{A})']$$

One can easily prove that this, too, is an equivalence relation on the set of all pairwise tasks on thermodynamic attributes, that can be serially composed.

By the composition law, $\mathfrak{A} \simeq \mathfrak{B} \Leftrightarrow \mathfrak{A} \overset{\bullet}{\simeq} \mathfrak{B}$ whenever $Out(\mathfrak{A}) = Out(\mathfrak{B})$ because:

$$\mathfrak{A} \simeq \mathfrak{B} \Rightarrow [(\mathfrak{A}^\sim \otimes \mathfrak{B})(\mathfrak{B} \otimes \mathfrak{B}^\sim)]' \Rightarrow (\mathfrak{A}^\sim \mathfrak{B})'$$

Hence, the partitions into equivalence classes generated by $\mathfrak{A} \simeq \mathfrak{B}$ and by

$\mathfrak{A} \overset{\bullet}{\simeq} \mathfrak{B}$ are the same whenever they are both defined. Therefore they are both

represented by the same labelling Δ , $\mathfrak{A} \simeq \mathfrak{B}$ being an extension of $\mathfrak{A} \overset{\bullet}{\simeq} \mathfrak{B}$ to the whole set of pairwise tasks on thermodynamic attributes.

Properties of *is-like*. In order to explain how the conservation law emerges, it is necessary to show a number of interesting properties of *is-like* (which hold for the serial-composition version too).

One is that if $\mathfrak{A} \simeq \mathfrak{B}$ and \mathfrak{A}' , then \mathfrak{B}' . This is because, by the composition law (principle III),

$$\mathfrak{A} \simeq \mathfrak{B} \wedge \mathfrak{A}' \Rightarrow [(\mathfrak{A}^\sim \otimes \mathfrak{B})(\mathfrak{A} \otimes \mathfrak{I})]' \Rightarrow \mathfrak{B}'$$

Moreover, tasks in the same class as the unit task \mathbf{I} are all possible, and so are their transposes: $\mathbf{A} \simeq \mathbf{I} \Rightarrow [(\mathbf{A}^\sim \otimes \mathbf{I})' \wedge (\mathbf{A} \otimes \mathbf{I})'] \Rightarrow \mathbf{A} \simeq \mathbf{I} \wedge \mathbf{A}' \wedge \mathbf{A}^{\sim'}$.

In addition, if two tasks are in the same class, their transposes must be in the same class, too: $\mathbf{A} \simeq \mathbf{B} \Rightarrow \mathbf{A}^\sim \simeq \mathbf{B}^\sim$. Thus, one can set $\Delta_i(\mathbf{A}) = \alpha \Delta_i(\mathbf{A}) + k, \forall i$, for some real numbers $\alpha \neq 1$ and non-zero k (representing a rescaling and a uniform translation). Since $\Delta_i((\mathbf{A}^\sim)^\sim) = \Delta_i(\mathbf{A}) = \alpha^2 \Delta_i(\mathbf{A}) + (\alpha+1)k$, it must be $\alpha = -1$, whereby I choose $k=0$. This in turn implies that $\Delta_i(\mathbf{I}) = 0 \Leftrightarrow \Delta_i(\mathbf{A}) = -\Delta_i(\mathbf{A}^\sim)$.

Let me temporarily drop the index i . Consider *any two substrates* $\mathbf{M}_1, \mathbf{M}_2$, and the pairwise tasks $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in S_{M_1} \cup S_{M_2}$, such that $\mathbf{A} \simeq \mathbf{B} \wedge \mathbf{C} \simeq \mathbf{D}$. It follows that $\mathbf{A} \otimes \mathbf{C} \simeq \mathbf{B} \otimes \mathbf{D}$. Thus, by locality, I require that $\Delta(\mathbf{A} \otimes \mathbf{B}) = \Phi(\Delta(\mathbf{A}), \Delta(\mathbf{B}))$, for some function Φ with the property that:

$$\begin{aligned}\Delta(\mathbf{A} \otimes \mathbf{I}) &= \Phi(\Delta(\mathbf{A}), 0) = \Delta(\mathbf{A}) = \Phi(0, \Delta(\mathbf{A})) \\ \Delta(\mathbf{A} \otimes \mathbf{A}^\sim) &= \Phi(\Delta(\mathbf{A}), -\Delta(\mathbf{A})) = 0 = \Phi(0, 0) = \Delta(\mathbf{I} \otimes \mathbf{I}) \\ \Phi(\Phi((\Delta(\mathbf{A}), \Delta(\mathbf{B})), \Delta(\mathbf{C})) &= \Phi(\Phi((\Delta(\mathbf{C}), \Delta(\mathbf{A})), \Delta(\mathbf{B}))) \\ &= \Phi(\Phi((\Delta(\mathbf{B}), \Delta(\mathbf{C})), \Delta(\mathbf{A}))) + \text{cyclic permutations}\end{aligned}$$

where the last two lines follows from the associativity of parallel-composing tasks with one another. In order to satisfy these constraints I shall pick the solution $\Delta(\mathbf{A} \otimes \mathbf{B}) = \Phi(\Delta(\mathbf{A}), \Delta(\mathbf{B})) = \Delta(\mathbf{A}) + \Delta(\mathbf{B})$ – which requires additivity of the corresponding components of the vector of labels $\underline{\Delta}$.

Now, since, for each component, $\Delta(\mathbf{A} \otimes \mathbf{B}) = \Delta((\mathbf{A} \otimes \mathbf{B})(\mathbf{B} \otimes \mathbf{B}^\sim)) = \Delta(\mathbf{A}\mathbf{B})$, we have the same additivity property with respect to serial composition: $\Delta(\mathbf{A}\mathbf{B}) = \Delta(\mathbf{A}) + \Delta(\mathbf{B})$. Again by locality, I can assume that there exists a function $F : \Delta(\mathbf{T}) = \Delta(F(y); F(x))$. Since for any two serially-composable tasks $\mathbf{A} = \{x \rightarrow y\}, \mathbf{B} = \{y \rightarrow z\}$ it must be:

$$\Delta(\mathbf{A}\mathbf{B}) = \Delta(F(y), F(x)) + \Delta(F(z), F(y)), \forall y.$$

Hence $\Delta(\mathfrak{A}) = F(\mathbf{y}) - F(\mathbf{x})$. The additivity of $\Delta(\mathfrak{A})$ implies the additivity of the function F , too: $F((\mathbf{x}, \mathbf{y})) = F(\mathbf{x}) + F(\mathbf{y})$.

Now, it is easy to prove that there can only be *three kinds* of equivalence classes. One kind consists of only one class, containing the unit task and other possible tasks only, and it is labelled by $\Delta_i = 0, \forall i$. Given a representative element \mathfrak{A} in that class, its transpose \mathfrak{A}^\sim is in that class too, and it is possible. Next is the kind of class with the property that each task \mathfrak{A} in the class is impossible, and its transpose, in the class labelled by $-\Delta_i(\mathfrak{A}), \forall i$, is also impossible. A familiar example where these classes are non-empty is when there are tasks that violate a given conservation law by the same amount.

Finally, there is the kind of class with the property that each task in it is possible, but its transpose is impossible. Such a class reflects the presence of a “law of impotence”, such as the second law of thermodynamics: tasks in the same class change entropy by the same amount. See the figure.

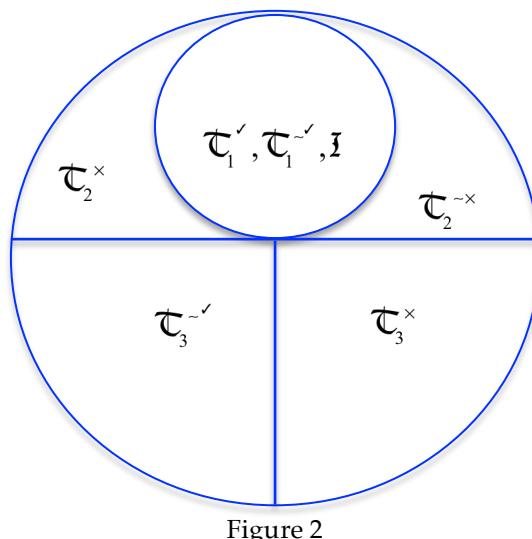


Figure 2

However, it is up to the subsidiary theory to say which one of those sectors in figure 2 is populated. For example, in the absence of a law of impotence, the first and second sector only would be populated. In thermodynamics it is customary to treat the simplified case that there is only one conservation law (namely that of energy), and one law of impotence (the second law). I shall do the same. Then there will be only two additive functions $U(\mathbf{x}), \Sigma(\mathbf{x})$ such that

$F(x) = (U(x), \Sigma(x))$ and $\Delta(\mathbf{T}) = (\Delta U(\mathbf{T}), \Delta \Sigma(\mathbf{T}))$. Thus in the first kind of class, containing the identity, $\Delta U = 0 = \Delta \Sigma$. In my notation, U will be the function involved in the conservation law. At this stage U could be any conserved quantity, not necessarily energy. It is energy only if it satisfies the first and second law of thermodynamics, as explained in sections 5 and 6. (The second law, requiring that the classes with non-zero $\Delta \Sigma$ be non-empty, corresponds to requiring that there be what in classical thermodynamics are called ‘thermodynamic coordinates’, in addition to ‘mechanical coordinates’). But in view of the above-mentioned simplification, I can call U ‘energy’. The principle of **conservation of energy** can be then stated as follows:

VIII. The task of changing the U of any substrate is impossible.

This conservation law requires the second kind of class (figure 2) to exist, containing tasks $\mathbf{T} = \{x \rightarrow y\}$ where $\Delta U(\mathbf{T}) \neq 0$, because, by VIII, $U(x) \neq U(y) \Rightarrow \{x \rightarrow y\}^x, \{y \rightarrow x\}^x$. For such tasks \mathbf{T} , $\Delta \Sigma(\mathbf{T})$ may or may not be zero. In the general case, the function $\Delta \Sigma$ will describe the remaining kind of class, generated by a law of impotence – i.e., those classes with tasks \mathbf{T} that are possible, but have an impossible transpose (the third kind in figure 2). For tasks in those classes it must be $\Delta U = 0$, because \mathbf{T} is possible, and $\Delta \Sigma \neq 0$. Nothing in the principles so far requires the subsidiary theories to permit such tasks, but as we shall see, the second law of thermodynamics will precisely require them to do so; $\Delta \Sigma$ will then be connected to what we have been calling ‘entropy’ (see section 6).

Another interesting consequence of the difference between a task being *impossible* and a process being not permitted, is that a conservation law stated in these terms implies *that U of a substrate must be bounded both above and below* – a fact that must be otherwise imposed separately in the prevailing conception (Deutsch, 2013). For, suppose that $U(y) - U(x) = \delta$ were not bounded above, and that there are no other reason why $\{x \rightarrow y\}$ is impossible. Then the task $(\mathfrak{A}_\delta \otimes \{x \rightarrow y\})'$ would be possible, for any task \mathfrak{A}_δ

: $\Delta U(\mathfrak{A}_\delta) = \delta$, $\Delta \Sigma(\mathfrak{A}_\delta) = 0$ because $\Delta(\mathfrak{A}_\delta \otimes \{x \rightarrow y\}) = 0$. The first substrate, were U not bounded, would still have the ability to perform the task $\{x \rightarrow y\}$, any number of times: it would therefore qualify as a constructor for the task $\{x \rightarrow y\}$, which would therefore be possible – contradicting the principle of conservation of energy. A similar contradiction follows from assuming that it is not bounded below. Thus, U must be bounded above and below for *any substrate*.

Energy being bounded above is an unfamiliar requirement. But in this context it is a consequence of the fact that the conservation law is about the energy of a *substrate* – i.e., a physical system that can be presented to a constructor. For there exists an energy value beyond which any physical system changes so drastically, for instance, by turning into a black hole or a vast plasma cloud – that no single (even idealised) constructor could accept it as input for arbitrarily large energies.

5. The first law of thermodynamics

I shall now recast the notion of *adiabatic accessibility* (Lieb & Yngvason, *op. cit.*) in constructor theory, in exact terms. To this end I introduce an *exactly defined* class of physical substrates, which I shall call *work media*, that generalise the ‘weight in a gravitational field’ that appeared in the original definition of adiabatic accessibility (see section 1.2). The first law will then be stated as the requirement that such media be *interoperable*, following Carathéodory’s original formulation – i.e., that all ways of ‘doing work’ are interchangeable with one another.

Work Media. A *work medium* \mathbf{M} with *work variable* $W = \{w_1, w_2, \dots, w_N\}$ is a substrate whose thermodynamic attributes $\{w_1, w_2, \dots, w_N\}$ have the property that:

1. $\{w_i \rightarrow w_j\}^x$ for all i, j .
2. For any pair of adjacent attributes $\{w_n, w_{n+1}\} \subseteq W$:

- a. $\{w_n \rightarrow w_{n+1}\} \simeq \{w'_n \rightarrow w'_{n+1}\}$ for all adjacent pairs of attributes
 $\{w'_n, w'_{n+1}\} \subseteq W$;
- b. For some attributes $x, x' \in W$ and, crucially, $x_0 \in \{w_n, w_{n+1}\}$:
 $\{(w_n, x_0) \rightarrow (w_{n+1}, x), (w_{n+1}, x_0) \rightarrow (w_n, x')\}^\vee$
- c. If $w_n = (a_n, b_n), w_{n+1} = (a_{n+1}, b_{n+1})$ for some thermodynamic attributes $\{a_n, b_n, a_{n+1}, b_{n+1}\}$, then the variables $\{a_n, a_{n+1}\}$, $\{b_n, b_{n+1}\}$ separately satisfy all the above conditions.

Condition 1 requires that $\Delta U(\{\mathbf{T}_{i,j}\}) \neq 0, \forall i, j$: for, by the conservation law VIII (and under the simplifying assumption) a necessary condition for both a task and its transpose to be impossible is that they would change the U of their substrate.

Condition 2a implies that adjacent attributes in a work variable are “equally spaced”: $\Delta U(\{\mathbf{T}_{i,j}\}) = \Delta$ for all adjacent i, j .

Whenever $\Delta U(\mathbf{T}) = \Delta U(\mathbf{T}')$ for some pairwise tasks \mathbf{T} and \mathbf{T}' over some two-fold variables V and V' , I shall say that V is *U-commensurable* with V' , and that \mathbf{T} is *U-commensurable with \mathbf{T}'* .

Condition 2b is the key counterfactual property that singles out substrates such as weights. It requires that it is possible to perform a *swap* of any two adjacent attributes $\{w_n, w_{n+1}\}$ of \mathbf{M} with a side-effect on a replica of the same substrate, initialised so that W is sharp with the same value – i.e., the substrate holds either a sharp w_n or a sharp w_{n+1} . I shall call ‘*work attributes*’ the attributes in the variable $W' \subset W$ with the property that for any two adjacent attribute $w_n, w_{n+1} \in W'$, property 2b is satisfied with both $x_0 = w_n$ and $x_0 = w_{n+1}$.

The classical-thermodynamics notion of ‘mechanical coordinates’ (section 1) can now be identified with the set of labels for each attribute in a work variable, thus becoming physically meaningful and exactly defined. A comparison with existing thermodynamics will explicate how:

Clearly, a quantum system with any number greater than 3 of equally spaced energy levels does satisfy the definition of work media: its work attributes are all except the ones labeled by extremal energy values. For instance, a weight in a gravitational field, with three distinct ‘heights’, is a work medium.⁶ Classical thermodynamics offers also a key example of substrates that do *not* satisfy that condition for work media. Consider a quantum system with a variable containing two different *thermal states*, $\{\rho_1, \rho_2\}$ – i.e., with two different temperatures 1 and 2. The reason why this is not a work medium is that:

$$\left\{(\rho_1, \rho) \rightarrow (\rho_2, \rho_x), (\rho_2, \rho) \rightarrow (\rho_1, \rho_y)\right\}^*, \forall \rho \in \{\rho_1, \rho_2\}$$

(where ρ_x, ρ_y are allowed to be any two other quantum (pure or mixed) states with different mean⁷ energies from the initial ones, as required by conservation of energy). Whatever the state ρ may be between ρ_1, ρ_2 , the *swap* with that constrained side-effect, plus the conservation of energy, would require a thermal state with a given temperature, (ρ, ρ) , to be changed to one where there is a temperature difference, such as (ρ, ρ_x) – which is forbidden by the second law in classical thermodynamics. This is a crucial difference between a system with ‘mechanical coordinates’ only (e.g. a weight in a

⁶ Work media necessarily have finite, or countably infinite, work variables. Continuous spectra can be approximated to arbitrary accuracy by composing a number of such physical systems.

⁷ A quantum thermal state with a given mean energy corresponds, in constructor theory, to a thermodynamic attribute (i.e., one that can be stabilised to arbitrary accuracy – *not* the result of some spontaneous thermalisation process of an isolated system.)

gravitational field), and one having thermal states (e.g. a reservoir with a range of possible temperatures). This difference can be stated exactly *only* by expressing the counterfactual properties of that system.

The only stationary quantum states of a single quantum system (i.e., diagonal in the energy basis) that qualify as work attributes of a work variable are pure eigenstates of the unperturbed Hamiltonian of the system (see appendix). This shows that the above definition of work media is consistent with existing thermodynamics, and that it is a good candidate to provide foundations for the new definition of adiabatic accessibility.

Condition 2c (mirroring the requirement that there are no other conservation laws) rules out from the class of work variables of composite systems where the change in U is not the only one taking place. For, in the presence of a law of impotence, a task and its transpose could both be impossible, but a change in the other label of the equivalence classes, Σ , might take place too. For instance, the variable of the composite system $\{(U_+, \Sigma_+), (U_0, \Sigma_0), (U_-, \Sigma_-)\}$, where U can be thought of as being internal energy and Σ can be thought of as being classical thermodynamics' entropy, would satisfy conditions 1-2.b. However, it does not satisfy conditions 1-2c, for $\{U_+, U_0, U_-\}$ satisfies properties 1-2b while $\{\Sigma_+, \Sigma_0, \Sigma_-\}$ does not satisfy property 2a.

It follows immediately that the minimal work variable in the presence of a single conservation law is one that has *three* thermodynamic equally spaced attributes, labelled by different values of U . Note that a perfect work medium need not exist in reality: it is enough that they be approximated arbitrarily well. Such a medium might be made, for instance, as a composite system of several systems with energy spectra that are *not* equally spaced.

Work variables are information variables. Condition 2b – requiring what I shall call the ‘swap’ property – provides an unexpected, illuminating connection between thermodynamics and information theory: any sub-variable $\{w_n, w_{n+1}\}$ of a work variable is *distinguishable*, in the exact, constructor-theoretic sense of section 3. To see how, recall that (Deutsch &

Marletto, 2015) any two disjoint intrinsic attributes x and x' are *ensemble distinguishable*, which means the following.

Let $\mathbf{S}^{(n)}$ denote a physical system $\overbrace{\mathbf{S} \oplus \mathbf{S} \oplus \dots \mathbf{S}}^{n \text{ instances}}$ consisting of n instances of \mathbf{S} , and $x^{(n)}$ the attribute $\overbrace{(x, x, \dots, x)}^{n \text{ terms}}$ of $\mathbf{S}^{(n)}$. Denote by $\mathbf{S}^{(\infty)}$ an unlimited supply of instances of \mathbf{S} . This is of course a theoretical construct, which does not occur in reality. That x and x' are ensemble-distinguishable means that the attributes $x^{(\infty)}$ and $x'^{(\infty)}$ of $\mathbf{S}^{(\infty)}$ are distinguishable. In quantum theory, this corresponds to the fact that any two different quantum states are asymptotically distinguishable – a property at the heart of so-called quantum tomography. Now, let $z \in \{w_n, w_{n+1}\} \subseteq W$, where W is a work variable of some work medium. By property 2b it is possible to apply the *swap* operation to the work medium any number of times, and the output would be a composite work medium $\mathbf{M} \oplus \mathbf{M} \oplus \mathbf{M} \dots$ with the attribute $(w_{n+1}, x, x', x, \dots)$ if $z = w_n$; and with the attribute (w_n, x', x, x', \dots) if $z = w_{n+1}$. Thus, preparing the attributes $x^{(\infty)}$ and $x'^{(\infty)}$ would be a possible task, because of the assumption V about unbounded number of thermodynamic attributes being preparable from generic resources. Those attributes can be constructed to arbitrarily high accuracy, short of perfection, using a finite number of substrates for each accuracy. The attributes x and x' are intrinsic thermodynamic attributes, so that $x^{(\infty)} \perp x'^{(\infty)}$. Thus, by preparing $z^{(\infty)}$ from $\{w_n, w_{n+1}\}$ one could distinguish w_n from w_{n+1} . From this ‘pairwise’ distinguishability of its attributes it follows, via the principle VII, that a work variable is a distinguishable variable. Thus, it is an information variable (Deutsch and Marletto, 2015). Hence all *work media are information media* – with their work variable being an information variable.

In general, the fact that the swap on a pairwise computation variable such as $\{w_n, w_{n+1}\}$ is possible need not imply that the variable is distinguishable. For instance, any two non-orthogonal quantum states can be swapped (Deutsch & Marletto, 2015). It is the presence of a conservation law that requires an

ancilla to perform a computation on attributes labeled by different values of the conserved quantity. A record of the state being swapped is left in the ancilla, whereby it is possible to distinguish the attributes in the work variable. This is the origin of the connection between a conservation law and information theory.

Interoperability of work media. Despite property 2a, two U -commensurable work variables belonging to *different* work media are not required, by their definition alone, to be such that the pairwise tasks over them are like one another. The *first law of thermodynamics* requires them to be, by requiring there to be a unique class of work media – as follows:

I. Given any two work media M_1, M_2 with commensurable work variables $W^{(1)}, W^{(2)}$:

i. For any adjacent pair $\{w, w'\} \subseteq W^{(1)}$

$$\left\{ (w, x_0) \rightarrow (w', x), (w', x_0) \rightarrow (w, x') \right\}^{\vee} \quad (1)$$

for some attributes $x, x' \in W^{(2)}$ and $x_0 \in \{w, w'\} \subseteq W^{(1)}$; and likewise when the labels 1 and 2 are interchanged.

ii. The composite substrate $M_1 \oplus M_2$ is a work medium, with variable $W \subseteq W^{(1)} \times W^{(2)}$, where W is obtained by a relabeling of $W^{(1)} \times W^{(2)}$ where attributes with the same U are assigned the same label.

Property (i) implies immediately that pairwise tasks defined on any two commensurable work variables are like one another, and that the task of transforming any two attributes in $W \subseteq W^{(1)} \times W^{(2)}$ having the same energy U into one another is possible. From now on, given a work medium M_1 , any substrate M_2 that satisfies equation (1) in property (i) above I shall call a *work-like ancilla* for M_1 . Property (ii) is an *interoperability law for work media* – it

requires the composite system of any two work media to be a work medium, which captures the intuition that a single battery can be substituted for two batteries of appropriate capacity. Together, they capture the content of the traditional formulation of the first law, that all ways of doing work are equivalent to one another (see the introduction). Two different work media with commensurable variables, such as a weight and a spring, are *interchangeable* with one another in this sense.

Quasi-work media. Work media are very special kinds of substrates, which I shall use in order to generalise the notion of adiabatic accessibility. However, they do not exhaust all physical systems that in classical thermodynamics we would characterise as “having mechanical coordinates only”. For instance, in the latter category we would include a quantum system with exactly two energy levels, but the latter would not qualify as a work medium (because it does not allow for the swap property 2.b). Thus, it is useful to introduce a class of closely related substrates, *quasi-work media*, any pair of whose attributes can be swapped via a side-effect on a work variable; but which need not be usable as work-like ancillas – i.e., they need not be usable as a side-effect for the swap (condition 2b) that defines work media. As we shall see, swapping their attributes in that way *only* changes their U (and not the possible other labels related to a law of impotence) – so, they too can be characterised as having only mechanical coordinates.

A *quasi-work medium* is a substrate with thermodynamic attributes $\{x, y\}$ (called *quasi-heat attributes*) such that $\{x \rightarrow y\} \simeq \{w \rightarrow w'\}$ for some work variable $\{w, w'\}$. All work media are quasi-work media with any pair of attributes in their work variables. Conversely, two-level system qualifies as a quasi-work medium, but not, as we said, as a work-medium. We shall see another example of a quasi-work medium in section 6. (See also table 1 for an informal summary of these notions).

Adiabatic possibility. We are now ready to give the generalisation of adiabatic accessibility, in constructor-theoretic terms. The task $\mathbb{T} = \{x \rightarrow y\}$ is

adiabatically possible, which I write as $\{x \rightarrow y\}^{\leq}$ if it is possible with a side-effect task *over work variables only*:

$$\left\{ \{x \rightarrow y\} \otimes \{w_1 \rightarrow w_2\} \right\}^{\leq}.$$

Vice-versa, it is *adiabatically impossible*, which I denote by $\{x \rightarrow y\}^{\geq}$, if

$$\left\{ \{x \rightarrow y\} \otimes \{w_1 \rightarrow w_2\} \right\}^{\geq}$$

for all work variables $\{w_1, w_2\}$. (Hence, whenever a task is possible it is also adiabatically possible, with a trivial side-effect (consisting of the unit task on some work attribute)).

This definition replaces Lieb & Yngvason's. It is more general and it is exact. It now has physical content, for work media are defined via statements about possible/impossible tasks only.

Because of interoperability of work media, it is immediate that \mathcal{T}^{\leq} and $(\mathcal{T})^{\leq}$ for any task $\mathcal{T} = \{w_1 \rightarrow w_2\}$ whose input/output attributes belong to a work variable, or to a quasi-work variable (but not necessarily for a work-like ancilla). The second law will require the existence of tasks for which \mathcal{T}^{\leq} , but \mathcal{T}^{\geq} – and hence, a new class of substrates.

6. The second law of thermodynamics

The notion of *adiabatic possibility*, which has generalised and made rigorous the notion of adiabatic accessibility, can now be used to state the second law of thermodynamics, in an exact and scale-independent way. To that end, I shall first introduce the notion of 'heat', via the powerful constructor-theoretic method of defining a class of substrates with certain counterfactual properties, obeying another interoperability law. Once more, this definition is, unlike that of classical thermodynamics, based on exact statements about possible/impossible tasks. To that end, I shall first introduce an auxiliary class of substrates – that of quasi-heat media.

Quasi-heat media. A *quasi-heat medium* is a substrate with a variable Q whose thermodynamics attributes have the property that $\forall \{h, h'\} \subseteq Q, \{h \rightarrow h'\}^{\leq}$, but $\{h' \rightarrow h\}^{\neq}$. Q is called a *quasi-heat variable*.

From the interoperability of work media, it follows immediately that a quasi-heat medium is not a work medium. If it were, then it would be possible to swap its heat attributes adiabatically, by the first law of thermodynamics (property (i)), contrary to assumption. For the same reason, it is impossible for a quasi-heat medium to be a quasi-work medium.

The second law will require the subsidiary theory to permit a particular kind of quasi-heat media, as we shall see. To introduce them it is helpful to define, given a variable H , the *symmetric variable* $H' \subseteq H \times H$ of the composite substrate $\mathbf{M} \oplus \mathbf{M}$ with the property that its attributes are invariant under the swapping the two substrates. So, for instance, if $H = \{h, h'\}$, then $H' = \{(h, h), (h', h')\}$.

Heat Media. A *heat medium* with *heat variable* H is a quasi-heat medium \mathbf{M} whose quasi-heat variable H has the following additional properties:

1. Each attribute $h \in H$ is *not distinguishable* from any other thermodynamic attribute.
2. $\mathbf{M} \oplus \mathbf{M}$ is a quasi-heat medium with the symmetric variable $H' \subseteq H \times H$
3. For any pair of attributes $\{h, h'\} \subset H$ there exists $h_F \in H$ such that $\mathbf{M} \oplus \mathbf{M}$ is a *quasi-work medium* with variable $\{(h', h), (h_F, h_F)\}$ (i.e., $\{(h', h) \rightarrow (h_F, h_F)\} \simeq \mathfrak{W}$ for some work variable \mathfrak{W}).
4. There is no work medium \mathbf{W} such that \mathbf{M} is a work-like ancilla for \mathbf{W} .

An example of attributes that would satisfy **condition 1** are thermal states: since they have support on the set of all eigenstates of energy of a quantum state, they are not perfectly distinguishable by a ‘single-shot’ measurement from any other quantum state, and thus are not distinguishable according to the constructor-theoretic definition⁸ (section 3). **Condition 2** is a generalisation of the standard property that it is impossible to ‘extract work’ from thermal states, no matter how many replicas of a substrates are available (this is also called *complete passivity*, (Skrzypczyk, 2015)). **Condition 3**, conversely, requires there to be ways of ‘extracting work’ from certain other states of a composite substrates involving thermal states. For instance, $(\mathbf{h}, \mathbf{h}')$ could be the attribute of a composite substrate of two finite heat reservoirs, at different temperatures h, h' , with the same heat capacity, in which case the attribute (h_F, h_F) is the attribute of those two reservoirs having the same temperature $h_F = \sqrt{hh'}$. It is well known that it is possible to transform the substrates made of the two reservoirs from the former attribute to the latter and vice-versa, adiabatically, by running a “reversible” heat engine (reversible in the sense of classical thermodynamics: one that has no net change in entropy), or a refrigerator, between the two reservoirs. The adiabatic side-effect would be, respectively, a weight being raised (to absorb the work done by the heat engine in depleting the temperature difference in $(\mathbf{h}, \mathbf{h}')$); and a weight being lowered to power the refrigerator which can transform (h_F, h_F) to (h, h') . Note however that constructor theory gives us an interesting new insight: because of the requirement that $\mathbf{h} \not\perp \mathbf{h}'$, the task

$$\left\{ (\mathbf{h}, \mathbf{h}', \mathbf{w}_0) \rightarrow (h_F, h_F, \mathbf{w}), (h_F, h_F, \mathbf{w}_0) \rightarrow (h, h', \mathbf{w}') \right\}$$

must be impossible for any work attributes $\mathbf{w}_0, \mathbf{w}, \mathbf{w}'$. If it were possible, then $\mathbf{h} \perp \mathbf{h}'$, (because work variables are distinguishable) – contrary to assumption. Thus, although there can be a constructor that performs each

⁸ For instance, a thermometer cannot discriminate to arbitrarily high accuracy two quantum states with different Boltzmann distributions (i.e. with two different temperatures), for they are not orthogonal states. In constructor theory, this fact is shown not to be accidental, but essential to the nature of heat.

task separately, as we said, it is impossible to have a constructor that would perform their union.

The meaning of **condition 4** becomes clear when one examines the substrates it rules out. Let us consider a heat medium \mathbf{M} , with the attributes $\{h_+, h_0, h_-\} \subset H$, that violates it – i.e., it is a work-like ancilla. Then, for some work medium \mathbf{M} with some work variable $W \supseteq \{w_+, w_0, w_-\}$ (such that attributes with the same suffix have the same energy U):

$$\left\{ \left(w_+, h_0 \right) \rightarrow \left(w_0, h_+ \right), \left(w_0, h_0 \right) \rightarrow \left(w_+, h_- \right) \right\}$$

which would imply that $\left\{\left(w_+, h_0\right) \rightarrow \left(w_0, h_+\right)\right\}'$, $\left\{\left(w_0, h_0\right) \rightarrow \left(w_+, h_-\right)\right\}'$. Thus, condition 4 rules out, from the possible assignments of adiabatically possible/impossible tasks on any 3-fold heat variable that are compatible with the definition of a heat medium, that which would require those two tasks to be possible (see figure 3, case (b)).

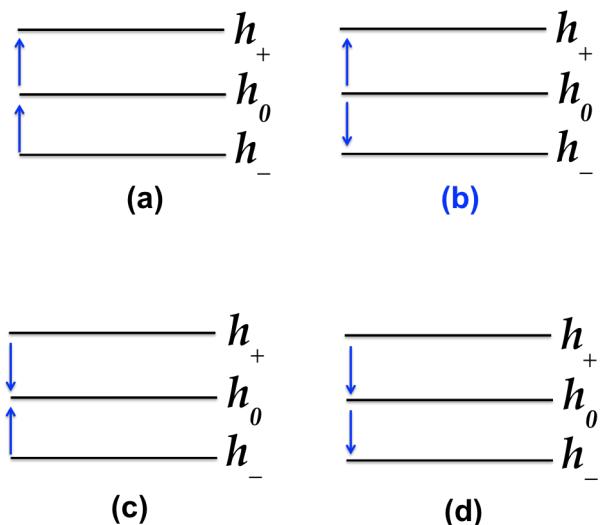


Figure 3 Possible assignments of adiabatically possible tasks on a heat medium whose attributes (+,0,-) are labelled by decreasing values of energy. (The arrows represent adiabatically possible tasks, their transpose (not represented) is adiabatically impossible by definition of heat media.)

This property of heat media will be important in deriving the Kelvin statement of the second law in this constructor-theoretic framework (section 7). Note that in classical thermodynamics that assignment, and also that corresponding to case (c) in figure 3, is ruled out by an *ancillary law* (Buchdahl, 1966; Marsland, et. al., op. cit., Lieb &

Yngvason, *op. cit.*) requiring that all adiabatically possible tasks whose transpose is adiabatically impossible either always increase (case (a)) or always decrease (case (d)) the energy. In constructor theory, that law is not necessary.

Interoperability of heat. Heat media will be required to be a class of interchangeable substrates – again, by an interoperability law. The *interoperability law for heat media* requires that:

- IX. The composite system of any two heat media with variables H_1, H_2 whose attributes have the same is-like labels, is a heat medium with heat variable $H' \subseteq H_1 \times H_2$.

The interoperability principle for heat requires therefore that the composite substrate of *any* two heat media, whatever their physical details, still satisfies properties 1–4 above. Just like other interoperability laws, it is a new physical principle. It is tacitly assumed in classical thermodynamics, but has this elegant expression in constructor-theoretic terms. Note that in the prevailing conception's approach to thermodynamics one usually demonstrates, given a particular subsidiary theory, that there exist models of physical systems displaying properties such as those required by the conditions 1–4. Here, instead, the logic is to express those properties exactly, and then illustrate the principles that substrates that classify as heat, work, quasi-work and quasi-heat media must obey.

From now on, whenever a subsidiary theory requires a pattern of possible/impossible tasks on thermodynamic attributes conforming to there being a heat medium with heat variable H , I shall say that 'A heat medium/variable H is *mandated*'. Likewise for the other categories introduced so far: work, quasi-work and quasi-heat media. See table 1 for a summary of the intuitive meaning of each of those categories.

Work media are objects like *weights*: they must have at least three distinguishable attributes with different energy, characterised by a particular ‘swap’ property.

Quasi-work media include work media, but they need not have that swap property – e.g. a quantum system with only two energy states. The task of changing any of their attributes into any other is adiabatically possible.

A **work-like ancilla** is a substrate with at least three thermodynamic attributes that can be used as a side-effect to swap work attributes of another system; but the task of changing any one of its attributes into any other may be adiabatically impossible.

A **quasi-heat medium** is a substrate with at least a pair of attributes such that the task of changing one into another is adiabatically impossible, but has an adiabatically possible transpose.

Heat media are quasi-heat media with at least three thermodynamic attributes that are not distinguishable from any other attribute and have certain additional properties (e.g. a quantum system with three different temperature states).

Table 1: Informal description of substrates appearing in constructor-theoretic thermodynamics. (Terms in **boldface italic** denote media with direct analogues in classical thermodynamics.)

The second law of thermodynamics. We are now ready to state the second law of thermodynamics, in exact terms, using the notion of heat media, as follows:

- II. Consider any two attributes x and y in any two work variables of the same medium. That substrate is also a heat medium with a heat variable H such that it contains a pair of heat attributes with the same energies as x and y .

This is the constructor-theoretic generalisation of Carathéodory’s second law.⁹ It requires that whenever a subsidiary theory mandates a work medium (whose attributes, recall, can be swapped only by changing their ‘mechanical coordinates’, with side-effects on work media only) – then given *any* pair of attributes in its work variables (including pairs with the same energy) it must

⁹ Note that the statement is constructor-theoretic because it requires the subsidiary theory to *mandate* a certain heat variable – i.e., that certain tasks be possible and impossible.

also mandate a pair of heat attributes in a heat variable with the same energies. This implies that for any pairwise task on a pair of attributes belonging to work variables (which, by definition, must be such that either both it and its transpose are possible, or they are both impossible) there is a corresponding *U-commensurable* task on a pair of heat attributes (which, by definition of heat media, must be adiabatically possible, and its transpose adiabatically impossible; or vice-versa). This is reminiscent of Caratheodory's notion of there being adiabatically inaccessible points in any 'neighbourhood' of any point in the thermodynamic space. However, unlike the former, it does require the set of thermodynamic attributes to be a continuum.

Note that no notion of entropy nor temperature has been mentioned so far; in particular, as already mentioned, this is *not* equilibrium thermodynamics. To the end of introducing an equivalent of *entropy*, let me now define another equivalence relation.

Adiabatic is-like. We say that $\mathfrak{A} \equiv \mathfrak{B}$ (read: \mathfrak{A} 'is adiabatically-like' \mathfrak{B}) if and only if

$$[(\mathfrak{A}^\sim \otimes \mathfrak{B})^{\leq} \wedge (\mathfrak{A} \otimes \mathfrak{B}^\sim)^{\leq}].$$

The same steps as in section 4 show that this is an equivalence relation on the set of all pairwise tasks on thermodynamic attributes. Under the simplifying assumption of a *single conservation law* and a *single law of impotence* (see section 5), the case where both \mathfrak{T} and its transpose are adiabatically impossible does not occur.¹⁰ Consequently, instead of a vector of labels it is enough for present purposes that there is a single real-valued, additive function $\Delta_A(\mathfrak{T})$

labeling the different classes, with the property that $\Delta_A(\mathfrak{T}) = 0 \Leftrightarrow \mathfrak{T}^{\leq} \wedge (\mathfrak{T}^\sim)^{\leq}$.

Let $\mathfrak{T} = \{x \rightarrow y\}$. Once more, by locality, I shall assume that $\Delta_A(\mathfrak{T}) = \Delta S(\mathfrak{T}) = S(y) - S(x)$ where S is an additive, real-valued function whose domain is the set of thermodynamic attributes of all substrates.

¹⁰ This simplifying assumption implies that if a task \mathfrak{T} is adiabatically impossible, then its transpose is adiabatically possible. This property is essentially the *comparability axiom* of Lieb & Yngvason, but I shall not require it in this treatment.

Entropy. Now, one can show that the function S has the following properties, which allow one to identify it as the constructor-theoretic generalisation of traditional entropy:

$$(i) \Delta S(\mathbf{T}) = 0 \Leftrightarrow \mathbf{T}^{\leq}, (\mathbf{T}^{-})^{\leq} \Leftrightarrow \mathbf{T} \cong \mathbf{I}$$

Therefore, any task $\mathbf{T} = \{w_1 \rightarrow w_2\}$ whose input/output attributes are work attributes belongs to the class labeled by $\Delta S(\mathbf{T}) = 0$.

In addition, one can show that the function Σ introduced in the is-like relation (section 4) is related to S , as follows:

$$(ii) \Delta \Sigma(\mathbf{T}) \neq 0 \Rightarrow \Delta S(\mathbf{T}) \neq 0.$$

This is because:

$$\begin{aligned} \mathbf{T}^{\vee} &\Rightarrow \mathbf{T}^{\leq} \wedge \Delta U(\mathbf{T}) = 0 = \Delta U(\mathbf{T}^{-}) \\ \mathbf{T}^{-x} &\Rightarrow \mathbf{T}^{-\leq} \vee \mathbf{T}^{-x}. \end{aligned}$$

The first line follows from the fact that whenever a task is possible, it is also adiabatically possible (the side-effect task being the unit task). The second line follows from the assumption that there is only one law of impotence. However, the option $\mathbf{T}^{-\leq}$ is not viable. For that would imply that either $(\mathbf{T}^{-})^{\vee}$ (which contradicts the premises) or that $(\mathbf{T}^{-} \otimes \{w_1 \rightarrow w_2\})^{\vee}$ for some work variable $\{w_1, w_2\}$. However, this would require that $\Delta U(\mathbf{T}^{-}) \neq 0$, by additivity of U , contrary to the assumptions. This proves (ii).

$$(iii) \forall \mathbf{T}: \mathbf{T}^{\leq} \wedge (\mathbf{T}^{-})^x, \Delta S(\mathbf{T}) \neq 0. \text{ Because, by property (ii):}$$

$$\begin{aligned} (\mathbf{T} \otimes \mathbf{w})^{\vee} &\Rightarrow \Delta U(\mathbf{T} \otimes \mathbf{w}) = 0 = \Delta U((\mathbf{T} \otimes \mathbf{w})^{-}) \\ (\mathbf{T}^{-} \otimes \mathbf{w})^x \wedge \Delta U((\mathbf{T} \otimes \mathbf{w})^{-}) &= 0, \forall \mathbf{w} \Rightarrow \Delta \Sigma(\mathbf{T} \otimes \mathbf{w}) \neq 0 \Rightarrow \Delta S(\mathbf{T} \otimes \mathbf{w}) \neq 0 \end{aligned}$$

Since $\Delta S(\mathbf{w}) = 0$, by additivity: $\Delta S(\mathbf{T}) \neq 0$.

Thus, by property (iii), whenever a task \mathbf{T} is adiabatically possible, but its transpose is not, $\Delta S(\mathbf{T}) \neq 0$; we can therefore identify it as the constructor-theoretic generalization of entropy. Just as in the case of Caratheodory's and Lieb&Yngvason, assuming additional requirements, e.g. about the continuity of the space of thermodynamic points, allows one to show that the function

has the property that whenever a task \mathbf{T} is adiabatically possible, but its transpose is not, then $\Delta S(\mathbf{T}) > 0$. However, unlike in those cases, in constructor theory the physical content of the second law resides in the notion of adiabatic possibility, *not* in the properties of the entropy function. Thus, for present purposes, I shall not assume any of those additional requirements. Because of property (ii), the partition into equivalence classes generated by S refines that generated by Σ . Therefore, one can uniquely identify a class generated by both the is-like and the adiabatic is-like equivalence relation by the labels $\underline{\Delta}(\mathbf{T}) = (\Delta U(\mathbf{T}), \Delta S(\mathbf{T}))$ where U is the function defined via the is-like relation and S is that defined by the adiabatic-is-like relation. The combination of the first and second law imply that, if there are possible and impossible tasks at all, then *all* kinds (see figure 2) of the is-like equivalence relation are present.

The Zeroth Law of Thermodynamics. As I said, the *zeroth law of thermodynamics* was an ‘afterthought’ in classical thermodynamics (Atkins, 2007): it was proposed, historically, after all the others, in order to introduce the notion of temperature. In constructor theory it is, too. However, its meaning and implications are somewhat different – in particular, they do not require the existence of the temperature in the traditional sense; nor is it about equilibration. In my formulation the zeroth law is:

- X. Given any thermodynamic attribute x , $\{x \rightarrow h\}'$ for any heat attribute h having the same energy as x .

That is to say, it is possible to convert any thermodynamic attribute into a heat attribute – i.e., one that cannot be distinguished from any other attribute. An example of such task might be that of converting some amount of purely mechanical energy completely into heat by ‘rubbing’. However, note that this law differs both in form and content from existing ancillary laws in classical thermodynamics (Buchdahl, 1966): it is *not* about spontaneous processes occurring (i.e., equilibration), but about the possibility of a task; in addition, it does not require there to be a definite sign for the change in energy

accompanying an adiabatically possible task, with an adiabatically impossible transpose.

7. Kelvin's statement of the second law

I shall now prove that the constructor-theoretic version of Kelvin's statement, that "*heat cannot be converted entirely into work*", follows from the laws of thermodynamics expressed above, in the form:

The task $\{h \rightarrow w\}$ is impossible for every work attribute w and heat attribute h .

For, suppose that task were possible for some such attributes: $\{h_0 \rightarrow w_0\}^\vee$ with $U(h_0) = U(w_0)$. Consider a 3-fold work variable $\{w_+, w_0, w_-\}$ including w_0 (which must exist by definition of work attribute, section 5). By the second law there must be a heat variable H with subvariable $\{h_+, h_0, h_-\} \subseteq H$ with $U(h_+) = U(w_+)$ and $U(h_-) = U(w_-)$.

Now, recall, there are three possible cases of assigning adiabatic possibility/impossibility to any three-fold sub-variable $\{h_+, h_0, h_-\}$ of a heat variable (see figure 3). In all such cases a contradiction is reached. Suppose first that $\{h_0 \rightarrow h_-\}^\times \wedge \{h_+ \rightarrow h_0\}^\times$ (case (a) in figure 3). Then, by the interoperability of work (second line) and by the zeroth law (third line):

$$\begin{aligned} & \{(h_0, w_0) \rightarrow (w_0, w_0)\}^\vee \\ & \{(w_0, w_0) \rightarrow (w_-, w_+)\}^\vee \\ & \{(w_-, w_+) \rightarrow (h_-, w_+)\}^\vee \end{aligned}$$

By definition of adiabatically possible:

$$\{(h_0, w_0) \rightarrow (h_-, w_+)\}^\vee \Rightarrow \{h_0 \rightarrow h_-\}^\vee$$

which violates the definition of heat medium, for this means that two of its heat attributes are adiabatically accessible from one another. The same line of

argument leads to a contradiction when $\{h_o \rightarrow h_+\}^x \wedge \{h_- \rightarrow h_o\}^x$ (case (d)) and when $\{h_o \rightarrow h_+\}^x \wedge \{h_o \rightarrow h_-\}^x$ (case (c)). Therefore the laws of constructor-theoretic thermodynamics require that heat cannot be completely converted into *work*.

This constitutes the promised unification of Caratheodory's and Kelvin's approaches. Note that the principles and definitions used to achieve this result do not require a definite sign of the change in U for adiabatically possible tasks with an adiabatically impossible transpose. Thus, as I remarked, the recovery of Kelvin's statement rests on different physical laws from the ancillary laws invoked by, e.g., (Buchdahl, 1966; Lieb & Yngvason, 1999).

Only at this stage can one introduce, without circularities, the notions of 'doing work on', and 'transferring heat to', a substrate.

In any given construction to perform the task $\mathcal{T} = \{x \rightarrow y\}$ on a substrate M such that $\Delta S(\mathcal{T}) \neq 0, \Delta U(\mathcal{T}) \neq 0$, allowing for side-effects, '*the work done on a substrate*' is the change ΔW in the energy U of the *work-media* required as side-effects in the construction; and the '*heat absorbed by the substrate*' is the change ΔQ in the energy U of the *heat media* required as side-effects. Whether or not such quantities are always positive (as under the known laws of physics), negative, or lack a definite sign, is for the subsidiary theory to decide. Constructor theory allows all such possibilities.

By focusing on side-effects being instantiated in heat media or work media – i.e. two different kinds of "agents of transfer" of energy, the constructor-theoretic notion is faithful to the traditional one, while also improving on it. For example, consider (Atkins, 2004):

"Energy has been transferred from source to object *through the agency of heat*: heat is the *agent of transfer*, not the entity transferred."

In constructor theory, the crucial difference is that such different 'agents of transfer' (work media and heat media) are exactly defined, and distinct from

one another. This comes from the interoperability of work and heat – properties that cannot even be stated in the prevailing conception, but have elegant expressions in constructor theory. By the conservation of energy applied to the whole system, including the substrate in question *and* the side-effects of the construction task, one recovers the traditional formula:

$$\Delta U = \Delta W + \Delta Q$$

In constructor theory, this formula is the culmination of the construction of thermodynamics, rather than its foundation; but it has now no circularity and its quantities are exactly defined, and have physical meaning, as promised.

8. Conclusions

I have proposed a novel formulation of the first, second and zeroth laws of thermodynamics, under constructor theory. There are three main results: an exact, scale-independent, non-probabilistic formulation of those laws, via the definition of *adiabatic possibility* in terms of possible computations on a substrate; an exact connection between the *first law of thermodynamics* and information theory; and an exact distinction between work and heat, which leads to the unification of Kelvin's and Carathéodory's statements of the second law. This approach improves on Carathéodory's. The latter, too, imposes restrictions on subsidiary theories via principles, or axioms, that aim to make contact with the physical world; however, those principles are not exact, and rely on ad-hoc definitions, having little physical content. In addition, their emphasis is on defining the entropy function of state. In the constructor-theoretic approach the principles are exact, more general, and their physical content resides in the interoperability laws and in the notions of heat and work media, rather than in the existence of an entropy function.

This theory applies to single systems, and to objects that are out of equilibrium (for which thermodynamic attributes are possible). Although beyond the scope of this paper, equilibrium thermodynamics can be recovered within this picture; and connections with the recently emerged field of quantum thermodynamics can be expected to arise. These are among

the promising new avenues opened up by this approach. *But this is another story, and shall be told another time.*

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Appendix

The condition for a *quantum system* \mathbf{M} to be a work medium with a work variable including the attributes $\{\rho_a, \rho_b\}$ is that there are quantum states ρ_x , $\rho_{x'}$ satisfying the swap property (condition 2b in section 5) – i.e., such that the swap task on the variable $\{\rho_a, \rho_b\}$ is possible with side-effects only on a second instance of \mathbf{M} prepared in one of those quantum states:

$$\begin{aligned}\rho_a \otimes \rho_b &\rightarrow \rho_b \otimes \rho_{x'} \\ \rho_b \otimes \rho_b &\rightarrow \rho_a \otimes \rho_x\end{aligned}$$

Let us assume that the density operators are diagonal in the same basis as the Hamiltonian of the system \mathbf{M} , so that the states are not changed unless when acted upon by the constructor, and that each have a different expectation value for energy. Explicitly:

$$\begin{aligned}\rho_a &= \sum_{i=1}^N a_i |E_i\rangle\langle E_i| \\ \rho_b &= \sum_{i=1}^N b_i |E_i\rangle\langle E_i| \\ \rho_x &= \sum_{i=1}^N x_i |E_i\rangle\langle E_i|\end{aligned}$$

$$\text{where } \sum_{i=1}^N b_i = 1 = \sum_{i=1}^N x_i = \sum_{i=1}^N a_i \text{ and } 0 \leq a_i \leq 1, 0 \leq b_i \leq 1, 0 \leq x_i \leq 1.$$

Under unitary quantum theory, when energy is conserved, that the task is possible imposes very strict constraints on what the ‘work states’ ρ_a, ρ_b can possibly be: in short, they can only be pure states.

The second of the above equations is the one I focus on. Let us define the von Neumann entropy $S(\bar{x}) \doteq -\sum_{i=1}^N x_i \ln(x_i)$ and also the expectation value of

energy, $E(\bar{x}) \doteq \sum_{i=1}^N x_i E_i$ for any 'vector' of probabilities $\bar{x} = (x_1, x_2, \dots, x_N)$. That

equation generates a system of two equations obtained by imposing the energy conservation and the conservation of entropy (due to unitarity of quantum theory):

$$\begin{cases} S(\bar{x}) = 2S(\bar{b}) - S(\bar{a}) \\ E(\bar{x}) = 2E(\bar{b}) - E(\bar{a}) \end{cases} \Leftrightarrow \begin{cases} S(\bar{x}) = 2S(\bar{b}) - S(\bar{a}) \\ (\bar{x} - \bar{p}_0) \cdot \bar{E} = 0 \end{cases}$$

where I have introduced the notation: $\bar{E} \doteq (E_1, E_2, \dots, E_N)$ and $\bar{p}_0 \doteq 2\bar{b} - \bar{a}$.

If the states are pure, the entropy equation is an identity and therefore solution exists provided that energy is conserved.

Let us now suppose instead that $0 < a_i < 1$, $0 < b_i < 1$, $0 < x_i < 1$ (i.e., I suppose that the states are strictly mixed). I show that no pair of mixed states is allowed as a solution, unless in the trivial case where $\rho_a = \rho_b$.

The proof has two steps.

First, one shows that *the solution, if it exists, must be unique*.

For let us suppose that the set Σ of solutions contains more than one element. Since the set Σ is the intersection of two convex functions (the entropy function and the energy hyperplane appearing in the system above), Σ must be a convex set. Therefore given any two elements \bar{x}, \bar{x}' in Σ , the point $\lambda\bar{x} + (1-\lambda)\bar{x}', \lambda \in [0,1]$ is a solution too: $\lambda\bar{x} + (1-\lambda)\bar{x}' \in \Sigma$. Since it is a solution of the system, it satisfies in particular the entropy equation, i.e.:

$$S(\lambda\bar{x} + (1-\lambda)\bar{x}') = 2S(\bar{b}) - S(\bar{a}) = \lambda S(\bar{x}) + (1-\lambda)S(\bar{x}')$$

(where I have used that $S(\bar{x}) = 2S(\bar{b}) - S(\bar{a}) = S(\bar{x}')$ by construction).

This is a contradiction with the property that the entropy is a strictly concave function, i.e. that

$$S(\lambda\bar{x} + (1-\lambda)\bar{x}') < \lambda S(\bar{x}) + (1-\lambda)S(\bar{x}')$$

Hence the solution must be unique.

This implies that the solution \bar{x}_0 (if it exists) is the point at which the energy hyperplane (defined by the energy equation of the system above) touches the entropy hypersurface.

The second part of the proof shows that if \bar{x}_0 exists, then $\bar{a} = \bar{b}$.

Suppose \bar{x}_0 is the (unique) solution. Consider the tangent hyperplane of $S(\bar{x})$ at \bar{x}_0 : $\bar{z} = (\bar{x} - \bar{x}_0) \cdot \nabla S|_{x_0}$. Since \bar{x}_0 is the solution of the system, this must be *the same* hyperplane as that defined by the energy equation. In other words, the solution \bar{x}_0 of the system has the property that, for any \bar{x} :

$$(\bar{x} - \bar{x}_0) \cdot \nabla S|_{x_0} = (\bar{x} - \bar{p}_0) \cdot \bar{E}$$

where $(\bar{x} - \bar{x}_0) \cdot \nabla S|_{x_0} = -\sum_{i=1}^N (x_i - x_{0i})(\ln(x_{0i}) + 1)$ and, as I said, $\bar{p}_0 \doteq 2\bar{b} - \bar{a}$.

In particular, the equation must be satisfied for $\bar{x} = \bar{p}_0$.

Substituting $\bar{x} = \bar{p}_0$, one has:

$$0 = \sum_{i=1}^N (p_{0i} - x_{0i})(\ln(x_{0i}) + 1) = \sum_{i=1}^N (p_{0i} - x_{0i}) \ln(x_{0i})$$

because $\sum_{i=1}^N (p_{0i} - x_{0i}) = 0$ (for the summands add up to one separately).

The choice $\bar{x}_0 = \bar{p}_0 = 2\bar{b} - \bar{a}$ satisfies the equation. Moreover, since the solution of the system is unique, if there is one \bar{x}_0 that satisfies the above equation, it must be *the* solution.

On the other hand, the solution $\bar{x}_0 = \bar{p}_0$ of the system must also satisfy the entropy equation $S(\bar{x}_0) = 2S(\bar{b}) - S(\bar{a})$, whereby

$$S(2\bar{b} - \bar{a}) = 2S(\bar{b}) - S(\bar{a})$$

Since S is non-linear, this holds only if $\bar{b} = \bar{a}$. Therefore, this shows that if a solution exists, then $p_a = p_b$, as promised. Note that the presence of entanglement would make the conditions even more restrictive and thus this conclusion would not change.

Therefore work variables of a quantum system must consist of *pure eigenstates* of energy. From the separate argument mentioned in section 5 (using ensemble-distinguishability, which holds in quantum theory) it also follows that work states in the same work variable must be orthogonal – i.e., that a work variable is an information variable.