Lecture 4: Equality Constrained Optimization

Tianxi Wang

wangt@essex.ac.uk

2.1 Lagrange Multiplier Technique

(a) *Classical Programming*

- $\max\ f(x_1, x_2, ..., x_n) \quad \rightarrow \quad$ objective function

  where $x_1, x_2, ..., x_n$ are instruments/control variables

- subject to a constraint $g(x_1, x_2, ..., x_n) = b$.

- The most popular technique to solve this constrained optimization problem is to use the Lagrange multiplier technique.

(b) *Lagrangean Method*

- We introduce a new variable λ, called the Lagrange multiplier, and set up the Lagrangean.

- For example if we want to maximize $f(x_1, x_2)$ subject to $g(x_1, x_2) = b$, the Lagrange function is

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[b - g(x_1, x_2)]$$

- Necessary conditions for a local maximum/minimum requires setting all first-order conditions equal to zero.

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0 \tag{1}$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0 \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = b - g(x_1, x_2) = 0 \tag{3}$$

- The third first-order condition (equation (3)) automatically guarantees the constraint is satisfied.

- From the first-order condition we then solve for the critical values x_1^*, x_2^* and λ^*.

  Divide equations (1) and (2) to eliminate λ

  $$\frac{\partial f/\partial x_1}{\partial f/\partial x_2} = \frac{\partial g/\partial x_1}{\partial g/\partial x_2}$$

  and re-arranging equation (3)

  $$b = g(x_1, x_2)$$

  we now have two equations to find x_1^*, x_2^*.

  To compute λ^* re-arrange equation (1)

  $$\lambda = \frac{\partial f/\partial x_1}{\partial g/\partial x_1}$$

  and plug in x_1^* and x_2^*.

- Example: $f = x^{\frac{1}{2}}y^{\frac{1}{2}}$ s.t. $ax + cy = b$.

  $$L = x^{\frac{1}{2}}y^{\frac{1}{2}} + \lambda[b - ax - cy]$$

  <u>F.O.Cs</u>:

  $$L_x = \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}} - \lambda a = 0 \tag{i}$$

  $$L_y = \frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}} - \lambda c = 0 \tag{ii}$$

  $$L_\lambda = b - ax - cy = 0 \tag{iii}$$

  Dividing (i) by (ii) yields:

  $$\frac{y}{x} = \frac{a}{c} \rightarrow y = \frac{ax}{c} \tag{iv}$$

Using (iv) to substitute out y from (iii) yields x^*

$$x^* = \frac{b}{2a}$$

and substituting x^* into (iv) yields y^*

$$y^* = \frac{b}{2c}$$

To find λ^* re-arrange (i) and substitute the values obtained for x^* and y^*

$$\lambda^* = \frac{1}{2a} x^{*-\frac{1}{2}} y^{*\frac{1}{2}} = \frac{1}{2a} \left(\frac{b}{2a}\right)^{-\frac{1}{2}} \left(\frac{b}{2c}\right)^{\frac{1}{2}}$$
$$\lambda^* = \frac{1}{2a^{\frac{1}{2}} c^{\frac{1}{2}}}$$

Section 2.2: Interpretation of the Lagrange Multiplier λ

- Since x_1^*, x_2^* and λ^* are all functions of the constraint parameter b (where b is a fixed exogenous parameter), what happens to these critical values when we change b?

$$L^* = f(x_1^*, x_2^*) + \lambda^*[b - g(x_1^*, x_2^*)]$$

- Totally differentiating L^* with respect to b we find

$$\begin{aligned}\frac{dL^*}{db} = f_{x_1}\frac{dx_1^*}{db} + f_{x_2}\frac{dx_2^*}{db} + [b - g(x_1^*, x_2^*)]\frac{d\lambda^*}{db} \\ +\lambda^*\left(1 - g_{x_1}\frac{dx_1^*}{db} - g_{x_2}\frac{dx_2^*}{db}\right)\end{aligned}$$

Re-arranging:

$$\begin{aligned}\frac{dL^*}{db} = \left(f_{x_1} - \lambda^* g_{x_1}\right)\frac{dx_1^*}{db} + \left(f_{x_2} - \lambda^* g_{x_2}\right)\frac{dx_2^*}{db} \\ +[b - g(x_1^*, x_2^*)]\frac{d\lambda^*}{db} + \lambda^*\end{aligned}$$

where f_{x_1},f_{x_2},g_{x_1} and g_{x_2} are all evaluated at the optimum.

- Now at the optimum we have $f_{x_1} - \lambda^* g_{x_1} = 0$, $f_{x_2} - \lambda^* g_{x_2} = 0$ and $b - g(x_1^*, x_2^*) = 0$.

  Thus we obtain:

$$\frac{dL^*}{db} = \lambda^*$$

- Therefore the Lagrange multiplier tells us the effect of a change in the constraint via parameter b on the optimal value of the objective function f.

- Note: for this interpretation of λ^* the Lagrangean must be formulated where the constraint enters as $\lambda[b - g(x_1, x_2)]$ and not $\lambda[g(x_1, x_2) - b]$.

Section 2.3: Second-Order Conditions for Constrained Optimization

(a) *Sufficient conditions for a local max/min*

- Suppose we have an n variable function $f(x_1, x_2, ..., x_n)$ and one constraint $g(x_1, x_2, ..., x_n) = b$.
- Construct a Bordered Hessian matrix of the Lagrange function where the bordered elements are first-order partial derivatives of the constraint g and the remaining elements are second-order partial derivatives of the Lagrangean function L.

$$H^B = \begin{bmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}$$

- Note that the all partial derivatives in this matrix are evaluated at the critical values (x_1^*,x_2^*,...,x_n^*;λ^*)
- Check the sign of the leading principal minors.

- Sufficient condition for a local max: the bordered Hessian is negative definite.

  $| H_1^B |< 0$, $| H_2^B |> 0$, $| H_3^B |< 0$, $| H_4^B |> 0$ ...

  where

$$| H_1^B |= \begin{vmatrix} 0 & g_1 \\ g_1 & L_{11} \end{vmatrix}$$

$$| H_2^B |= \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}$$

  and so on.

- Sufficient condition for a local min: the bordered Hessian is positive definite.

  $| H_1^B |< 0$, $| H_2^B |< 0$, $| H_3^B |< 0$...

- Example: $z = xy$ s.t. $x + y = 6$

$$L = xy + \lambda[6 - x - y]$$

  <u>nec</u>:

$$\begin{aligned} L_x = y - \lambda = 0 \\ L_y = x - \lambda = 0 \\ L_\lambda = 6 - x - y = 0 \\ \Rightarrow x^* = y^* = \lambda^* = 3 \end{aligned}$$

  <u>suff</u>:

$$\begin{aligned} L_{xx} = L_{yy} = 0 \\ L_{xy} = L_{yx} = 1 \\ g_x = g_y = 1 \end{aligned}$$

Construct the Bordered Hessian matrix of the Lagrange function

$$| H^B |= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

Check the sign of leading principal minors:

$| H_1^B |= 0 - 1 < 0.$

$| H_2^B |= -1(0-1) + 1(1-0) = 2 > 0.$

Thus we have a local max at $x^* = y^* = 3$.

(b) *A Further Look at the Bordered Hessian*

- Border Hessian used in constrained optimization problems

$$| H^B |= \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}$$

- Bordered Hessian used to test quasiconcavity/quasiconvexity

$$| B |= \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

- Two key differences:
  (1) Bordered elements in $\mid B \mid$ are first-order partial derivatives of the function f rather than g.
  (2) Remaining elements in $\mid B \mid$ are second-order partial derivatives of f rather than the Lagrange function L.
- However in the special case of a linear constraint $g(x_1, x_2, ..., x_n) = a_1x_1 + ..., a_nx_n = m$
  (1) The second-order partial derivatives of Lagrange function L reduces to the second-order partial derivatives of f:
  $L_{ij} = f_{ij}$
  (2) The border in $\mid B \mid$ is simply that of $\mid H^B \mid$ multiplied by a positive scalar λ.
  (3) Therefore: $\mid B \mid = \lambda^2 \mid H^B \mid$
- Under this special case the leading principle minors $\mid B_i \mid$, $\mid H_i^B \mid$ must share the same sign. i.e. if $\mid B \mid$ satisfies the sufficient condition for strict quasiconcavity then $\mid H^B \mid$ must satisfy the second-order sufficient condition for constrained maximization.

Section 2.4: Economic Applications

(a) *Utility Maximization and Consumer Demand*

- $U(x, y)$ s.t. $P_x x + P_y y = M$
- Standard consumer problem. Consumer must maximize utility subject that she spends all her income M on purchasing two goods x, y, where the prices of both goods are market determined and hence exogenous and we will assume that the marginal-utility functions are continuous and positive i.e. $U_x, U_y > 0$.

$$L = U(x, y) + \lambda[M - P_x x - P_y y]$$
$$L_x = U_x - \lambda P_x = 0$$
$$L_y = U_y - \lambda P_y = 0$$
$$L_\lambda = M - P_x x - P_y y = 0$$

- First-order equations imply that the consumer equalizes the ratio of marginal utility to the price for each good:

$$\frac{U_x}{P_x} = \frac{U_y}{P_y} = \lambda \qquad \text{(i)}$$

- Here the Lagrange multiplier can be interpreted as the marginal utility of money when utility is maximized:

$$\lambda^* = \frac{\partial U^*}{\partial M}$$

- An alternative interpretation of (i) is:

$$-\frac{U_x}{U_y} = -\frac{P_x}{P_y}$$

i.e. Marginal rate of substitution (the slope of the indifference curve) = price ratio (the slope of the budget constraint).

- Recalling that an indifference curve is the locus of combinations of x and y that yield a constant level of utility U:

$$\begin{aligned} dU = U_x dx + U_y dy = 0 \\ U_x + U_y \frac{dy}{dx} = 0 \\ \Rightarrow \frac{dy}{dx} = -\frac{U_x}{U_y} \end{aligned}$$

Since $U_x, U_y > 0$ the slope must be negative.

- Re-arranging the budget constraint

$$y = \frac{M}{P_y} - \frac{P_x}{P_y} x$$

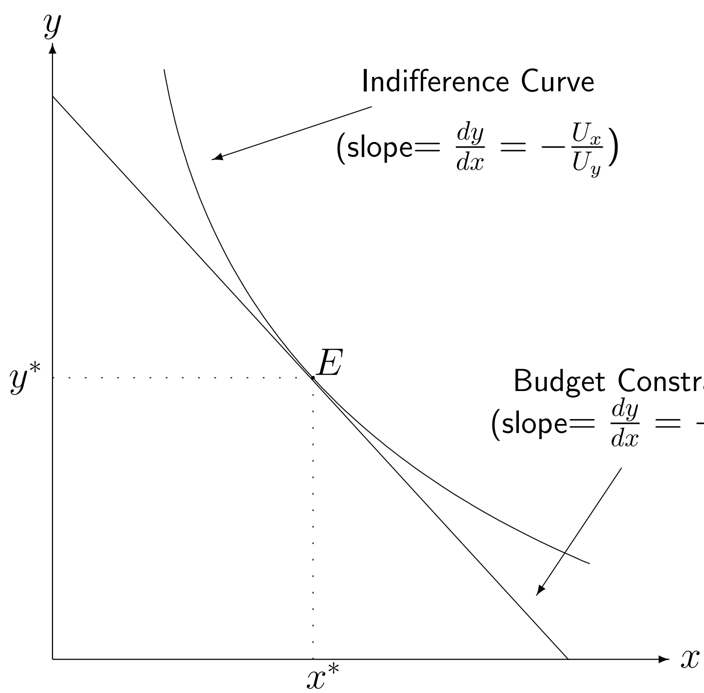

Figure 1: Utility Maximization

- Second-order conditions

$$| H^B | = \begin{vmatrix} 0 & P_x & P_y \\ P_x & U_{xx} & U_{xy} \\ P_y & U_{xy} & U_{yy} \end{vmatrix}$$

$$| H_1^B | = -P_x^2 < 0$$
$$| H_2^B | = -P_x^2 U_{yy} - P_y^2 U_{xx} + 2 P_x P_y U_{xy} > 0$$

Recalling from F.O.C's that $P_x = \frac{U_x}{\lambda}$ and $P_y = \frac{U_y}{\lambda}$. Therefore:

$$| H_2^B | = \frac{-U_x^2 U_{yy} - U_y^2 U_{xx} + 2 U_x U_y U_{xy}}{\lambda^2} > 0$$
$$| H_2^B | = U_x^2 U_{yy} + U_y^2 U_{xx} - 2 U_x U_y U_{xy} < 0 \quad for\ a\ local\ max$$

- But this is just the condition that the utility function be strictly quasiconcave [For Homework check this!].
- Therefore the tangency point E is a local max when the indifference curve is strictly convex to the origin i.e. when the utility function is strictly quasiconcave.
- Note that strict quasiconcavity of the utility function also ensures that the local maximum is unique and globally optimal.

Theorem 1 *In a constrained maximization problem*
$\max f(x)$ *s.t.* $g(x) = 0$
where f *and* g *are increasing functions of* x*, if:*
(a) f *is strictly quasiconcave and* g *is quasiconvex, or*
(b) f *is quasiconcave and* g *is strictly quasiconvex,*
then a locally optimal solution is unique and also globally optimal.

(b) *Cost Minimization under Cobb-Douglas Production Technology*

- $\min rk + wl$ s.t. $y = k^\alpha l^\beta$ where $\alpha, \beta > 0$.

$$L = rk + wl + \lambda[y - k^\alpha l^\beta]$$

- First-order conditions:

$$\begin{aligned} L_k &= r - \lambda \alpha k^{\alpha-1} l^\beta = 0 \\ L_l &= w - \lambda \beta k^\alpha l^{\beta-1} = 0 \\ L_\lambda &= y - k^\alpha l^\beta = 0 \end{aligned}$$

Solve for k^* and l^* to obtain the demand functions for capital and labor. Here $\lambda^* =$ marginal cost.

- Second-order conditions:

$$\begin{aligned} L_{kk} &= -\lambda \alpha(\alpha - 1) k^{\alpha-2} l^\beta \\ L_{ll} &= -\lambda \beta(\beta - 1) k^\alpha l^{\beta-2} \\ L_{kl} = L_{lk} &= -\lambda \alpha \beta k^{\alpha-1} l^{\beta-1} \\ g_k &= \alpha k^{\alpha-1} l^\beta \\ g_l &= \beta k^\alpha l^{\beta-1} \end{aligned}$$

Homework: Derive the Bordered Hessian and show that the conditions for a local minimum are satisfied. i.e. $\mid H_1^B \mid < 0$ and $\mid H_2^B \mid < 0$, since α,$\beta > 0$.

- Recall from problem set 1 Q3(b) where we showed that a Cobb-Douglas production function is strictly quasiconcave for any $\alpha, \beta > 0$. Since the objective function is linear and hence quasiconvex, the standard cost minimization problem of the firm under Cobb-Douglas technology generates a local minimum which is unique and is also a global minimum.

Theorem 2 *In a constrained minimization problem*

$\min f(x)$ *s.t.* $g(x) = 0$

(a) if f *is strictly quasiconvex and* g *is quasiconcave, or*

(b) if f *is quasiconvex and* g *is strictly quasiconcave,*

then a local minimum is unique and a global minimum.