# Math 367      Exam 2 Solutions      Summer '99

1. A marksman hits 85% of his shots. What is the expected value for the number of shots necessary to get 20 hits?

   SOLUTION: This is a Pascal distribution with $p = .85$ and $r = 20$ (i.e., we want the expected number of Bernoulli trials with $p = .85$ needed to get 20 successes). Thus, the expected value is

   $$\frac{r}{p} = \frac{20}{.85} \approx 23.5$$

2. An urn contains 10 balls each of the following colors: Red, Blue, Green, White, Yellow, Orange.
   a) Draw 5 times without replacement from the urn. Find the
   (i) expected value
   (ii) variance
   for the number of red balls drawn.

   SOLUTION: Write $X = X_1 + X_2 + X_3 + X_4 + X_5$ where

   $$X_i = \begin{cases} 1 & \text{if the } i\text{-th ball is red} \\ 0 & \text{otherwise.} \end{cases}$$

   Then $E(X_i) = P(i\text{-th ball red}) = \frac{10}{60} = \frac{1}{6}$, and, if $i \neq j$,

   $$\begin{aligned} E(X_i X_j) &= P(i\text{-th and } j\text{-th red}) \\ &= P(j\text{-th red} \mid i\text{-th red})P(i\text{-th red}) \\ &= \frac{9}{59} \cdot \frac{10}{60} = \frac{3}{118} \end{aligned}$$

   Thus, $E(X) = E(X_1) + \cdots + E(X_5) = 5 \cdot \frac{1}{6} = \frac{5}{6}$, and

   $$E(X^2) = E(X_1) + \cdots + E(X_5) + \sum_{i \neq j} E(X_i X_j)$$

   $$= 5 \cdot \frac{1}{6} + 5 \cdot 4 \cdot \frac{3}{118},$$

   so

   $$\text{Var}(X) = \frac{5}{6} + 20 \cdot \frac{3}{118} - \left(\frac{5}{6}\right)^2$$

b) Draw 5 times without replacement from the urn. Find the
  (i) expected value
  (ii) variance
for the number of colors drawn.

SOLUTION: Let $Y$ be the number of colors *not* drawn. Then $X = 6 - Y$, and $Y = Y_1 + \cdots + Y_6$, where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th color is not drawn} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E(Y_i) = P(i\text{-th color not drawn}) = \frac{\binom{50}{5}}{\binom{60}{5}}$$

and for $i \neq j$,

$$E(Y_iY_j) = P(i\text{-th and } j\text{-th colors not drawn}) = \frac{\binom{40}{5}}{\binom{60}{5}}.$$

Thus,

$$E(Y) = E(Y_1) + \cdots + E(Y_6) = 6 \cdot \frac{\binom{50}{5}}{\binom{60}{5}}, \quad \text{and}$$

$$E(Y^2) = E(Y_1) + \cdots + E(Y_6) + \sum_{i \neq j} E(X_iX_j)$$

$$= 6 \cdot \frac{\binom{50}{5}}{\binom{60}{5}} + 6 \cdot 5 \cdot \frac{\binom{40}{5}}{\binom{60}{5}}.$$

Thus,

$$\text{Var}(Y) = 6 \cdot \frac{\binom{50}{5}}{\binom{60}{5}} + 6 \cdot 5 \cdot \frac{\binom{40}{5}}{\binom{60}{5}} - \left( 6 \cdot \frac{\binom{50}{5}}{\binom{60}{5}} \right)^2.$$

Since $X = 6 - Y$, we have

$$E(X) = 6 - E(Y) = 6 \left( 1 - \frac{\binom{50}{5}}{\binom{60}{5}} \right)$$

and

$$\text{Var}(X) = \text{Var}(Y).$$

c) Draw 5 times with replacement from the urn. Find the
   (i) expected value
   (ii) variance
   for the number of colors drawn.

SOLUTION: Again let $Y$ be the number of colors *not* drawn. Then $X = 6 - Y$, and $Y = Y_1 + \cdots + Y_6$, where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th color is not drawn} \\ 0 & \text{otherwise.} \end{cases}$$

This time, $E(Y_i) = P(i\text{-th color not drawn}) = (50/60)^5$, while $E(Y_iY_j) = P(i\text{-th and } j\text{-th colors not drawn}) = (40/60)^5$. So

$$E(Y) = E(Y_1) + \cdots + E(Y_6) = 6 \cdot \left(\frac{5}{6}\right)^5$$

and

$$E(Y^2) = E(Y_1) + \cdots + E(Y_6) + \sum_{i \neq j} E(X_iX_j)$$

$$= 6 \cdot \left(\frac{5}{6}\right)^5 + 6 \cdot 5 \cdot \left(\frac{2}{3}\right)^5,$$

so

$$\mathrm{Var}(Y) = 6 \cdot \left(\frac{5}{6}\right)^5 + 6 \cdot 5 \cdot \left(\frac{2}{3}\right)^5 - \left(6 \cdot \left(\frac{5}{6}\right)^5\right)^2.$$

Since $X = 6 - Y$,

$$E(X) = 6 - E(Y)$$

$$= 6\left(1 - \left(\frac{5}{6}\right)^5\right),$$

and

$$\mathrm{Var}(X) = \mathrm{Var}(Y).$$

3. Flip 3 coins. Then draw letters from a word with replacement as follows: If you got no heads, draw from ZERO. If you got one head, draw from ONE. If you got two heads, draw from TWO. If you got three heads, draw from THREE. In whichever case, draw from the word until you get a vowel.

   What is the expected number of draws?

   SOLUTION: This is a conditional expected value problem. The coin toss determines the word from which the letters are drawn. After choosing the word, the expected number of draws needed to get a vowel is $\frac{1}{p}$, where $p$ is the probability of drawing a vowel in a single draw. Thus,

$$E(X \mid 0H) = \frac{1}{(1/2)} = 2,$$

$$E(X \mid 1H) = \frac{1}{(2/3)} = \frac{3}{2},$$

$$E(X \mid 2H) = \frac{1}{(1/3)} = 3,$$

$$E(X \mid 3H) = \frac{1}{(2/5)} = \frac{5}{2}.$$

Thus,

$$E(X) = E(X \mid 0H)P(0H) + E(X \mid 1H)P(1H) + E(X \mid 2H)P(2H) + E(X \mid 3H)P(3H)$$

$$= 2 \cdot \frac{1}{8} + \frac{3}{2} \cdot \frac{3}{8} + 3 \cdot \frac{3}{8} + \frac{5}{2} \cdot \frac{1}{8} = \frac{3}{2}.$$

4. A 500 page book contains 300 misprints. Use the Poisson distribution to answer the following:

   a) What is the probability there is one misprint on page 29?

   SOLUTION: Here, the expected value is $m = 300/500 = .6$ misprints per page, so the Poisson distribution satisfies

$$P(X=k) = \frac{(.6)^k}{k!} e^{-.6},$$

   so the probability of one misprint is $P(X=1) = .6e^{-.6}$.

   b) What is the probability there are at least two misprints on page 29?

   SOLUTION

$$P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$$

$$= 1 - [e^{-.6} + .6e^{-.6}].$$