Appendix E

More examples of Lagrange interpolation

E.1 Lagrange polynomials

We wish to find the polynomial interpolating the points

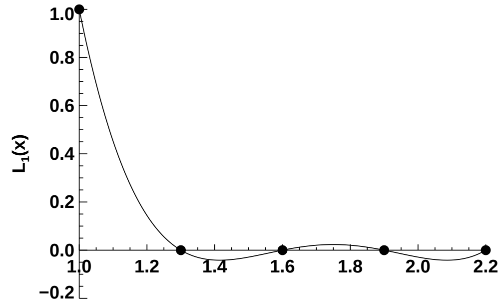
x	1	1.3	1.6	1.9	2.2
$f(x)$	0.1411	−0.6878	−0.9962	−0.5507	0.3115

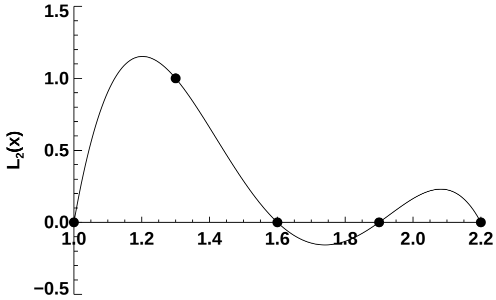
where $f(x) = \sin(3x)$, and estimate $f(1.5)$.

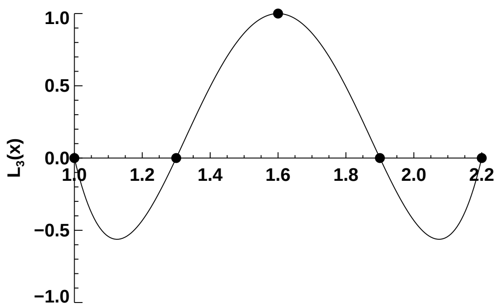
First, we find Lagrange polynomials $L_k(x)$, $k = 1 \dots 5$,

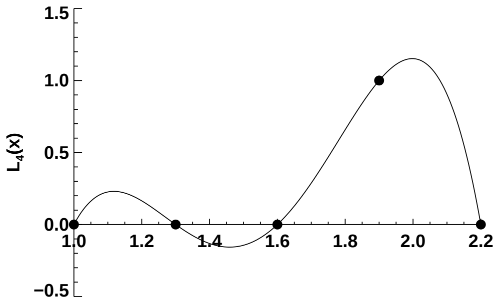
$$L_1(x) = \frac{(x-1.3)(x-1.6)(x-1.9)(x-2.2)}{(1-1.3)(1-1.6)(1-1.9)(1-2.2)}, \quad L_2(x) = \frac{(x-1)(x-1.6)(x-1.9)(x-2.2)}{(1.3-1)(1.3-1.6)(1.3-1.9)(1.3-2.2)}$$
$$L_3(x) = \frac{(x-1)(x-1.3)(x-1.9)(x-2.2)}{(1.6-1)(1.6-1.3)(1.6-1.9)(1.6-2.2)}, \quad L_4(x) = \frac{(x-1)(x-1.3)(x-1.6)(x-2.2)}{(1.9-1)(1.9-1.3)(1.9-1.6)(1.9-2.2)}$$
$$L_5(x) = \frac{(x-1)(x-1.3)(x-1.6)(x-1.9)}{(2.2-1)(2.2-1.3)(2.2-1.6)(2.2-1.9))}$$

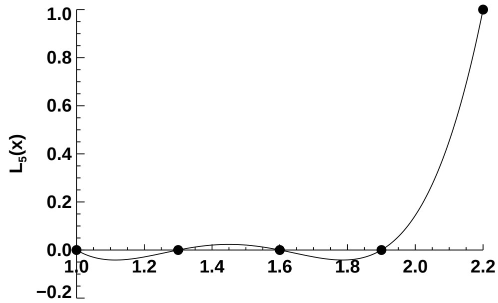
with the following graphs,





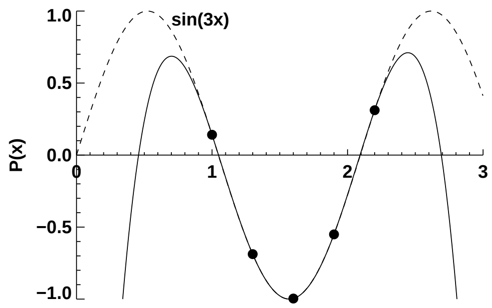






Clearly, $L_k(x_i) = \delta_{ik}$. Next, the polynomial approximation can be assembled,

$$P(x) = 0.1411 \times L_1(x) - 0.6878 \times L_2(x) - 0.9962 \times L_3(x) - 0.5507 \times L_4(x) + 0.3115 \times L_5(x).$$

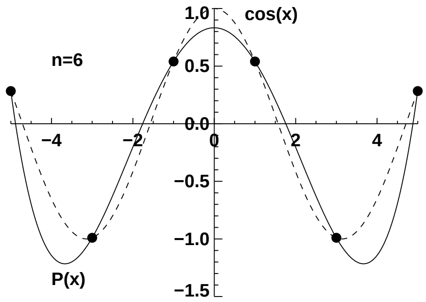


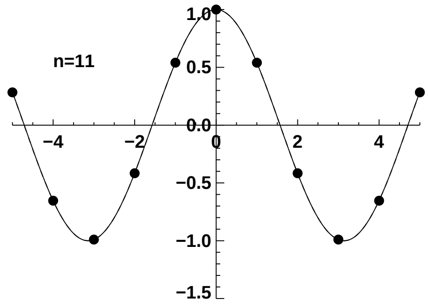
The interpolating polynomial approximates accurately the function $f(x) = \sin(3x)$ in the interval $[1, 2.2]$, with five points only.

So, $P(1.5) \approx -0.9773$ is an approximate to $f(1.5) = \sin(4.5) \approx -0.9775$ accurate within $E \approx 2 \times 10^{-4}$.

E.2 Convergence of "Lagrange" interpolation

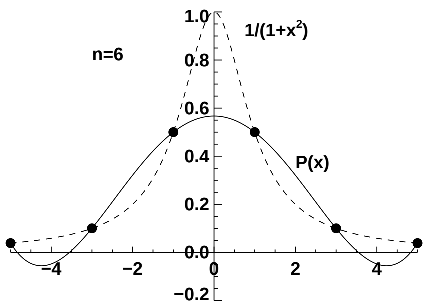
First, consider, $P(x)$, the polynomial interpolating $f(x) = \cos(x)$ through a set of equidistant points in the interval $[-5, 5]$.

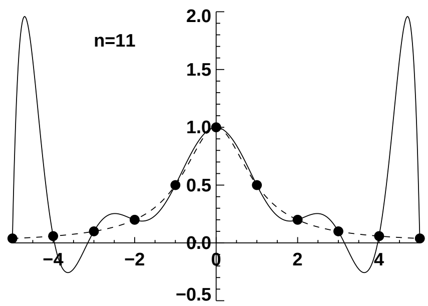




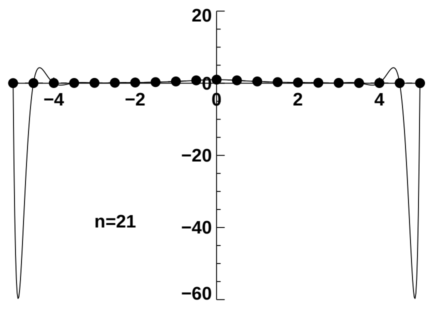
Clearly, increasing the number of equidistant points from $n = 6$ (left panel) to $n = 11$ (right panel) significantly improves the approximation of $f(x)$ by the polynomial P. In the right panel, the 10^{th} order interpolating polynomial (solid line) matches perfectly with the function $\cos(x)$.

However, Lagrange interpolation is not always accurate. For instance, consider the polynomial interpolating the Lorentz function, $f(x) = 1/(1 + x^2)$, through a set of equidistant points in the interval $[-5, 5]$.





Increasing the number of equidistant points from $n = 6$ (left panel) to $n = 11$ (right panel) improves the polynomial interpolation in the central part of $f(x)$, but large oscillations are present in the flat region.



If the number of equidistant interpolation points is increased further, these oscillations get even larger. The interpolating polynomial of degree $n - 1$, $P(x)$, does not converge to the function $1/(1 + x^2)$ as $n \to \infty$.