Dispersal kernel integration

Integrating dispersal kernels from Bode et al. (2018), which have the general form $p_{d,k,\theta} = ke^{-(kd)^{\theta}}$.

- For $\theta = 1$, the dispersal kernel has an analytical indefinite integral, which is below.
- For $\theta = 2$ (confirm the same is true for 0.5 and 3), there is no elementary indefinite integral but there is a definite integral when integrating from $-\infty$ to ∞ and from 0 to ∞. We can confirm analytically that the sum under the integral from 0 to ∞ is 1 (see below) but for partial integration, like among sites, we will have to integrate numerically in R.
- In code from Bode et al. (2018), he integrates numerically using the function *integral* in MatLab, which says it uses global adaptive quadrature. The function *integrate* in R also uses adaptive quadrature and integrated the 4 dispersal kernals from 0 to ∞ as 1 with absolute error e-05 or less.

Gaussian integration and the Gamma function

From `https://en.wikipedia.org/wiki/Gaussian_integral`, see that:

$$\int_0^{\infty} e^{-ax^b} dx = \frac{\Gamma\frac{1}{b}}{ba^{\frac{1}{b}}} \tag{1}$$

$\theta = 1$ (eqn. 1 in Bode et al. (2018))

Equation as written in Bode et al. (2018) (1):

$$p_1(d,k) = ke^{-kd} \tag{2}$$

Re-written with $k = z$, where $z = e^k$ using the k KC is estimating, and $d = x$:

$$p_1(x, z) = ze^{-zd} \tag{3}$$

Integrating over x from a to b:

$$\begin{aligned} p_{ij} &= \int_a^b ze^{-zx}dx \\ &= z\int_a^b e^{-zx}dx \\ &= z(\frac{-1}{z})[e^{-zb} - e^{-za}] \\ &= -e^{-e^k b} + e^{e^k a} \end{aligned} \tag{4}$$

Check total sum under integral:

$$\begin{aligned} p_{0,\infty} &= \int_0^\infty ze^{-zx}dx \\ &= z\int_0^\infty e^{-zx}dx \\ &= z(\frac{1}{z}) \\ &= 1 \end{aligned} \tag{5}$$

$\theta = 2$ (eqn. 6a in Bode et al. (2018))

Equation as written in Bode et al. (2018) (6a) but with z instead of k and x instead of $d_{i,j}$:

$$p_2(x) = \frac{2z}{\Gamma(\frac{1}{2})}e^{-(zx)^2} \tag{6}$$

Integrating over the whole dispersal kernel, using Gaussian integration formula (eqn. 1) with $a = z^2$ and $b = 2$:

$$\begin{aligned} p_{0,\infty} &= \int_0^\infty \frac{2z}{\Gamma(\frac{1}{2})}e^{-(zx)^2}dx \\ &= \frac{2z}{\Gamma(\frac{1}{2})}\int_0^\infty e^{-z^2x^2}dx \\ &= \frac{2z}{\Gamma(\frac{1}{2})}\frac{\Gamma(\frac{1}{2})}{2(z^2)^{\frac{1}{2}}} \\ &= 1 \end{aligned} \tag{7}$$

$\theta = 3$ (eqn. 6b in Bode et al. (2018))

Equation as written in Bode et al. (2018) (6b) but with z instead of k and x instead of $d_{i,j}$:

$$p_3(x) = \frac{3z}{\Gamma(\frac{1}{3})} e^{-(zx)^3} \tag{8}$$

Integrating over the whole dispersal kernel, using Gaussian integration formula (eqn. 1) with $a = z^3$ and $b = 3$:

$$\begin{aligned} p_{0,\infty} &= \int_0^\infty \frac{3z}{\Gamma(\frac{1}{3})} e^{-(zx)^3} dx \\ &= \frac{3z}{\Gamma(\frac{1}{3})} \int_0^\infty e^{-z^3x^3} dx \\ &= \frac{3z}{\Gamma(\frac{1}{3})} \frac{\Gamma(\frac{1}{3})}{3(z^3)^{\frac{1}{3}}} \\ &= 1 \end{aligned} \tag{9}$$

References

Michael Bode, David H Williamson, Hugo B Harrison, Nick Outram, and Geoffrey P Jones. Estimating dispersal kernels using genetic parentage data. *Methods in Ecology and Evolution*, 9(3):490–501, 2018.