EE1330: Digital Signal Processing, Spring 2017

Indian Institute of Technology Bhilai

Assignment Solutions

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(1) Theory

Fourier Series

(1)(a) $f(t) = t \quad -\pi < t < \pi \Rightarrow T = 2\pi$

$$a_n = \frac{1}{\pi}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n t}{2\pi}\right) dt$$

$f(t)$ is odd $\Rightarrow a_n = 0$

$$b_n = 1/\pi \int_{-T/2}^{T/2} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left[\frac{-t\cos nt}{n} + \frac{\sin(nt)}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-2\pi\cos n\pi}{n}\right] = -\frac{2}{n}(-1)^n$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = 0$$

$$f(t) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin(nt)$$

$$f(t) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \dots\right]$$

(1)(b) $f(t) = |t| \quad -\pi < t < \pi$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi n t}{2\pi}\right) dt$$

Since, $f(t)$ is even, $b_n = 0$

$$a_n = \frac{2}{2\pi}\int_{-T/2}^{T/2} |t|\cos(nt)\,dt$$

$$= \frac{1}{\pi} 2\int_{0}^{\pi} t\cos(nt)\,dt$$

$$a_n = \frac{2}{\pi}\left[\frac{\sin(nt)}{n}\cdot t + \frac{\cos(nt)}{n^2}\right]_0^{\pi}$$

(1.)

$$a_n = \frac{2}{\pi}\left[\frac{\cos(n\pi)-1}{n^2}\right] = \frac{2}{\pi}\frac{((-1)^n-1)}{n^2}$$

for even n, $a_n = 0$ $\qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|t|$

for odd n $\quad a_n = \frac{-4}{\pi n^2}$

$a_0 = \pi$

$$f(t) = \frac{a_0}{2} + \sum_{k=0}^{\infty} \frac{-4}{\pi(2k+1)^2}\cos(2k+1)t$$

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \cdots\right]$$

(1)(c) $\quad f(t) = \sin^2 t \qquad -\pi < t < \pi$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \sin^2 t \;\sin\left(\frac{2\pi n t}{2\pi}\right) dt$$

since $\sin^2 t \sin(nt)$ is odd $\Rightarrow b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 t \cos(nt)\,dt$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin^2 t \cos(nt)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2t}{2}\right)\cos(nt)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{\cos(nt)}{2}dt - \frac{1}{\pi}\int_0^{\pi}\frac{[\cos((n+2)t) + \cos((n-2)t)]}{2}dt$$

for $n \neq 2$ $\quad a_n = \frac{2}{\pi}(0) - \frac{1}{2\pi}(0) = 0$

$n = 2 \quad a_n = 0 - \frac{1}{2\pi}\int_0^{\pi}(0+1)\,dt$

$a_2 = -1/2 \qquad a_0 = \frac{2}{\pi}\int_0^{\pi}\sin^2(t)\,dt = \frac{2}{2} = 1$

$$f(t) = \frac{1}{2} - \frac{\cos 2t}{2} = \frac{1-\cos 2t}{2}$$

(2)

(1) (d) $f(t) = \cos 5t \qquad -\pi < t < \pi$

Since $f(t)$ is even $f(t)\sin(t)$ will be odd hence $b_n = 0$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \cos(5t)\cos\left(\frac{2\pi n t}{2\pi}\right) dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos((n+5)t) + \cos((n-5)t)\right] dt$$

for $n \neq 5 \qquad a_n = 0$

for $n = 5 \qquad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\, dt \Rightarrow a_5 = 1$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(5t)\,dt = 0$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$f(t) = a_5 \cos 5t = \cos(5t)$$

(1) (e) $f(t) = t^2 \qquad -\pi < t < \pi$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} t^2 \sin\left(\frac{2\pi n t}{2\pi}\right) dt$$

Since $t^2 \sin(nt)$ is odd $b_n = 0$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} t^2 \cos(nt)\, dt$$

$$= \frac{2}{\pi}\left[t^2\frac{\sin(nt)}{n} + \frac{\cos(nt)}{n^2}\cdot 2t - \frac{2\sin(nt)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{2\pi\cos(n\pi)}{n^2}\right) = \frac{4(-1)^n}{n^2}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\, dt = \frac{2}{3\pi}\pi^3 = \frac{2\pi^2}{3}$$

$$f(t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos(nt)}{n^2}$$

(3)

(1)(f) $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ \sin t & 0 < t < \pi \end{cases}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos\left(\frac{2\pi n t}{2\pi}\right) dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin t \cos(nt)\, dt$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} \left[\sin((n+1)t) + \sin((1-n)t)\right] dt$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} \left[\sin((n+1)t) - \sin((n-1)t)\right] dt$$

$$= \frac{1}{2\pi}\left[-\frac{\cos((n+1)t)}{(n+1)} + \frac{\cos((n-1)t)}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1-\cos((n+1)\pi)}{(n+1)} + \frac{\cos((n-1)\pi)-1}{(n-1)}\right]$$

for odd n $\cos((n+1)\pi) = \cos((n-1)\pi) = +1$

& $n \neq 1 \Rightarrow a_n = 0$

for even n

$$a_n = \frac{1}{2\pi}\left[\frac{2}{n+1} - \frac{2}{n-1}\right] = \frac{1}{2\pi}\left[\frac{-4}{n^2-1}\right] = \frac{-2}{\pi(n^2-1)}$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos t\, dt = \frac{1}{\pi}\int_{0}^{\pi} \sin t \cos t\, dt$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} \sin 2t = 0$$

$$a_0 = \frac{1}{\pi}\int_{0}^{\pi} \sin t\, dt = \frac{2}{\pi}$$

(4)

(1) (g.) $f(t) = |\cos t| \quad -\pi < t < \pi$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos t| \sin(nt)\, dt$$

since $|\cos t| \sin(nt)$ is odd $b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos t| \cos(nt)\, dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos t| \cos nt\, dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \cos t \cos nt\, dt - \frac{2}{\pi}\int_{0}^{\pi} \cos t \cos nt\, dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2} [\cos((n+1)t) + \cos((n-1)t)]\, dt - \frac{2}{\pi}\int_{0}^{\pi} \cos t \cos(nt)\, dt$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi/2} [\cos((n+1)t) + \cos((n-1)t)]\, dt - \frac{1}{\pi}\int_{\pi/2}^{\pi} [\cos((n+1)t) + \cos((n-1)t)\, dt$$

$$= \frac{1}{\pi}\left[\frac{\sin((n+1)t)}{(n+1)} + \frac{\sin((n-1)t)}{(n-1)}\right]_0^{\pi/2} - \frac{1}{\pi}\left[\frac{\sin((n+1)t)}{(n+1)} + \frac{\sin((n-1)t)}{(n-1)}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{(n+1)} + \frac{\sin(n-1)\pi/2}{(n-1)}\right] + \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{(n+1)} + \frac{\sin(n-1)\pi/2}{n-1}\right)$$

for odd $n \neq 1 \quad a_n = 0$

for even $n \quad a_n = \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{(n+1)} + \frac{\sin(n-1)\pi/2}{(n-1)}\right]$

(5.)

for $n=4k$

$$a_n = \frac{2}{\pi}\left[\frac{\sin(2\pi+\pi/2)}{4k+1} + \frac{\sin(2\pi-\pi/2)}{4k-1}\right]$$

$$a_n = \frac{2}{\pi}\left(\frac{1}{4k+1} - \frac{1}{4k-1}\right) = \frac{2\times(-2)}{\pi(16k^2-1)} = \frac{-4}{\pi(16k^2-1)}$$

for $n=4k+2$

$$a_n = \frac{2}{\pi}\left(\frac{\sin(n+1)\pi/2}{(n+1)} + \frac{\sin(n-1)\pi/2}{n-1}\right)$$

$$= \frac{2}{\pi}\left(\frac{\sin(2\pi+3\pi/2)}{(4k+3)} + \frac{\sin(2\pi+\pi/2)}{(4k+1)}\right)$$

$$= \frac{2}{\pi}\left(\frac{-4k-1+4k+3}{(4k+3)(4k+1)}\right) = \frac{2}{\pi}\left(\frac{2}{(4k+3)(4k+1)}\right)$$

for $n=1$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos t| \cos t \, dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \cos^2 t \, dt - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos^2 t \, dt = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos t| \, dt = 4/\pi$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

$$f(t) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos(4k+2)t}{(4k+3)(4k+1)} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(4kt)}{(16k^2-1)}$$

(6)

(1) (h) $f(t) = \begin{cases} 1 & -\pi/2 \le t < \pi/2 \\ -1 & (-\pi \le t < -\pi/2) \cup (\pi/2 \le t < \pi) \end{cases}$

$$b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

Since $f(t)\sin(nt)$ is odd $\Rightarrow b_n = 0$

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt$$

$$= -\frac{1}{\pi} \int_{-\pi}^{-\pi/2} \cos(nt)\,dt + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(nt)\,dt - \frac{1}{\pi} \int_{\pi/2}^{\pi} \cos(nt)\,dt$$

$$= -\frac{1}{\pi}\left[\frac{\sin(nt)}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left(\frac{\sin(nt)}{n}\right)_{-\pi/2}^{\pi/2} - \frac{1}{\pi}\left(\frac{\sin(nt)}{n}\right)_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\sin(n\pi/2)}{n}\right) + \frac{1}{\pi}\left[\frac{2\sin(n\pi/2)}{n}\right] + \frac{1}{\pi}\left(\frac{\sin(n\pi/2)}{n}\right)$$

$$= \frac{4}{\pi} \frac{\sin(n\pi/2)}{n} \Rightarrow$$ for even n $a_n = 0$

for odd n $a_n = \frac{4}{\pi} \frac{\sin(n\pi/2)}{n}$

$$a_0 = \int_{-\pi/2}^{\pi/2} f(t)\, dt = 0$$

$$f(t) = \sum_{k=0}^{\infty} \frac{4}{\pi} \cos\left(\frac{(2k+1)\pi}{2}\right) \cos((2k+1)t)$$

$$= \frac{4}{\pi}\left(\cos t - \frac{\cos 3x}{3} + \frac{\cos 5x}{3} \dots\right)$$

(7)

(1)(i) $f(t) = 1 \quad , -\pi < t < \pi$

$$a_0 = \frac{2}{T}\int_{-\pi}^{\pi} 1\,dt = \frac{2}{2\pi} \times 2\pi = 2$$

$$a_n = \frac{2\times 2}{2\pi}\int_0^{\pi} \cos(nt)\,dt = \frac{2}{\pi}\,\frac{\sin(nt)}{n}\Big|_0^{\pi} = \frac{2}{\pi n}(\sin(n\pi))$$

$$= 0 \;, \; n \neq 0$$

$b_n = 0 \quad (\because f(t) \text{ is even})$

$$f(t) = \frac{2}{2} = 1$$

(1)(j) $f(t) = \begin{cases} 0 \;, & -\pi < t < 0 \\ t \;, & 0 < t < \pi \end{cases}$

$$a_0 = \frac{2}{T}\int_0^{\pi} t\,dt = \frac{2}{2\pi}\,\frac{t^2}{2}\Big|_0^{\pi} = \frac{2\pi^2}{4\pi} = \frac{2\pi}{4} = \pi/2$$

$$a_n = \frac{2}{T}\int_0^{T} f(t)\cos\left(\frac{2\pi n t}{T}\right)dt = \frac{2}{2\pi}\left[\int_0^{\pi} t\cos\left(\frac{2\pi n t}{2\pi}\right)dt\right]$$

$$= \frac{1}{\pi}\left[\frac{t\sin(nt)}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nt)\,dt\right]$$

$$= \frac{1}{\pi}\left[\pi\sin(n\pi) - \frac{1}{n}\left(-\frac{\cos(nt)}{n}\right)\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\sin(n\pi) + \frac{1}{n}(\cos n\pi - 1)\right]$$

$$a_n = \begin{cases} 0 \;, & n \text{ is even} \\ -2/n\pi \;, & n \text{ is odd} \end{cases}$$

$$b_n = \frac{2}{2\pi}\int_0^{\pi} t\sin\left(\frac{2\pi n t}{2\pi}\right)dt = \frac{1}{\pi}\left[\frac{t(-\cos(nt))}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nt)\,dt\right]$$

(8)

$$b_n = \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \frac{1}{n^2}\sin(nt)\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{1}{n^2}\sin(n\pi)\right]$$

$$b_n = \begin{cases} -\frac{1}{n}, & n \text{ is even} \\ 1/n, & n \text{ is odd} \end{cases}$$

Fourier Series;

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nt}{T}\right) + b_n\sin\left(\frac{2\pi nt}{T}\right)\right]$$

$$= \frac{\pi}{4} + \left[\left(-\frac{2}{\pi}\cos t + \sin t\right) + \left(-\frac{1}{2}\sin(2t)\right) + \cdots\right]$$

(9.)

Fourier Transform

Fourier Transform.

2(a) $f(t) = \begin{cases} 1 & -0.5 \le t \le 0.5 \\ 0 & \text{elsewhere.} \end{cases}$

$$X(j\Omega) = \int_{-\infty}^{\infty} f(t) e^{-j\Omega t}\, dt = \int_{-0.5}^{0.5} e^{-j\Omega t}\, dt = -\frac{e^{-j\Omega t}}{j\Omega}\Bigg|_{-1/2}^{1/2}$$

$$= \frac{e^{j\Omega/2} - e^{-j\Omega/2}}{j\Omega} = \frac{2}{\Omega}\sin\left(\frac{\Omega}{2}\right)$$

2(b) $f(t) = e^{j\Omega_0 t}$

Let its fourier transform be $X(j\Omega)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t}\, d\Omega$$

$$e^{j\Omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t}\, d\Omega$$

by comparing $X(j\Omega) = 2\pi\,\delta(\omega - \omega_0)$ where δ is dirac delta function.

2(c) $f(t) = e^{-a|t|}$

$$X(j\Omega) = \int_{-\infty}^{\infty} f(t) e^{-j\Omega t}\, dt$$

$$X(j\Omega) = \int_{-\infty}^{0} e^{at} e^{-j\Omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{t(a-j\Omega)}\, dt + \int_{0}^{\infty} e^{-t(a+j\Omega)}\, dt$$

$$= \left.\frac{e^{t(a-j\Omega)}}{(a-j\Omega)}\right]_{-\infty}^{0} + \left.\frac{e^{-t(a+j\Omega)}}{-(a+j\Omega)}\right]_{0}^{\infty}$$

$$X(j\Omega) = \frac{1}{a-j\Omega} - 0 + 0 + \frac{1}{a+j\Omega} = \frac{2a}{a^2+\Omega^2}$$

(10)

Q(d) $f(t) = \cos \Omega_0 t$ ~~By~~

By Euler's formula $\cos(\Omega_0 t) = \dfrac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$

let fourier transform of $f(t)$ be $X(j\Omega)$

by fourier transform's inverse equation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$\frac{e^{j\Omega t} + e^{-j\Omega t}}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$e^{j\Omega t} + e^{-j\Omega t} = \frac{1}{\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

By comparison and shifting property

$$X(j\Omega) = \pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right]$$

where δ is dirac delta function

Q(e) $f(t) = \sin \Omega_0 t$

By Euler's formula $f(t) = \sin \Omega_0 t = \dfrac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}$

let fourier transform of $f(t)$ be $X(j\Omega)$

by inverse fourier transform

$$\frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$e^{j\Omega_0 t} - e^{-j\Omega_0 t} = \frac{j}{\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

By comparison and shifting property

$$X(j\Omega) = -j\pi\left[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)\right]$$

(11)

Q(f) $F(j\Omega) = \int_{-\infty}^{\infty} f(t) e^{-j\Omega t} dt$

let $\tau = t - t_0$ $\quad d\tau = dt$

$F(j\Omega) = \int_{-\infty - t_0}^{\infty - t_0} f(\tau - t_0) e^{-j\Omega(\tau - t_0)}$

$F(j\Omega) = \int_{-\infty}^{\infty} f(t - t_0) e^{-j\Omega t} e^{j\Omega t_0} dt$

$e^{-j\Omega t_0} \cdot F(j\Omega) = \int_{-\infty}^{\infty} f(t - t_0) e^{-j\Omega t} dt$

Clearly R.H.S is analysis equation of $f(t - t_0)$

Hence fourier transform of $f(t - t_0) = e^{-j\Omega t_0} \cdot F(j\Omega)$

Q(h.) $f(t) = \delta(t)$

let fourier transform be $X(j\Omega)$

$X(j\Omega) = \int_{-\infty}^{\infty} f(t) e^{-j\Omega t} dt$

$= \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt$

$= e^{-j\Omega(0)} = 1$

$X(j\Omega) = 1$

(12)
(13)

(5.) $x_c(t) = \sin(20\pi t) + \cos(40\pi t)$

$x[n] = \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right)$

sampling period – T

Solution: Let $p(t)$ be the impulse train having period T which is multiplied to $x_c(t)$ to get sampled signal $x[n]$

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

(a) $x_c(t)p(t) = \left(\sin(20\pi t) + \cos(40\pi t)\right) \sum_{k=-\infty}^{\infty} \delta(t-kT)$.

$p(t) = 1$ only at the integral multiple of T

$\therefore x_c(t) \cdot p(t) = x_c[nT] \cdot p[nT] = x[nT] \cdot 1$

$x_c(t)\,p(t) = \sin(20\pi nT) + \cos(40\pi nT)$ — (1)

Comparing eqn (1) with $x[n]$ given in question,

$\sin(20\pi nT) + \cos(40\pi nT) = \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right)$

$\Rightarrow 20\pi nT = \left(\frac{\pi}{5} + 2\pi k\right)n$ & $40\pi nT = \left(\frac{2\pi}{5} + 2\pi k\right)n$

$\Rightarrow \boxed{T = \frac{1}{100} + \frac{k}{10}}$ $k = 0, 1, 2, 3, \ldots$

$\therefore T = \frac{1}{100}, \frac{11}{100}, \ldots$

(b) Choice of T is not unique

$\because$ signal is periodic as we can see above.

19.

Q(g) $f(t) = e^{-t^2/2\sigma^2}$

Let Fourier Transform of $f(t)$ be $X(j\Omega)$

$$X(j\Omega) = \int_{-\infty}^{\infty} e^{-t^2/2\sigma^2} \cdot e^{-j\Omega t} \, dt$$

$$= \int_{-\infty}^{\infty} e^{-\left[\frac{t^2+2\sigma^2 j\Omega t}{2\sigma^2}\right]} dt$$

$$= \int_{-\infty}^{\infty} e^{-\left[\frac{t^2+2\sigma^2 j\Omega t + \sigma^4 j^2 \Omega^2}{2\sigma^2}\right]} \cdot e^{\left(\frac{\sigma^4 j^2 \Omega^2}{2\sigma^2}\right)} dt$$

$$= e^{-\frac{1}{2}\sigma^2\Omega^2} \int_{-\infty}^{\infty} e^{-\left(\frac{t-\sigma^2 j\Omega}{\sqrt{2\sigma^2}}\right)^2} dt$$

Let $\frac{t-\sigma^2 j\Omega}{\sqrt{2\sigma^2}} = k \Rightarrow dt = \sqrt{2\sigma^2}\, dk$

$$X(j\Omega) = e^{-1/2\,\sigma^2\Omega^2}\left(\int_{-\infty}^{\infty} e^{-k^2} dk\right) \cdot \sqrt{2\sigma^2}$$

$$X(j\Omega) = \sigma\sqrt{2}\, e^{-1/2\,\sigma^2\Omega^2}\left(\int_{-\infty}^{\infty} e^{-k^2} dk\right)$$

By Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$X(j\Omega) = \sigma\sqrt{2}\left[e^{-1/2(\sigma^2\Omega^2)}\right]\sqrt{\pi}$$

$$X(j\Omega) = \sigma\sqrt{2\pi}\; e^{-1/2(\sigma^2\Omega^2)}$$

(13)

?(i) $f(t) = \begin{cases} 1-t & 0 \le t < 1 \\ 1+t & -1 \le t < 0 \\ 0 & \text{elsewhere} \end{cases}$

Let $X(j\Omega)$ by fourier transform of $f(t)$

$$X(j\Omega) = \int_{-\infty}^{\infty} f(t) e^{-j\Omega t}\, dt = \int_{-1}^{0} (1+t) e^{-j\Omega t}\, dt + \int_{0}^{1} (1-t) e^{-j\Omega t}\, dt$$

$$= \int_{-1}^{1} e^{-j\Omega t}\, dt + \int_{-1}^{0} t e^{-j\Omega t}\, dt + \int_{0}^{1} (-t) e^{-j\Omega t}\, dt$$

$$= \left.\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} + \left[\frac{t e^{-j\Omega t}}{-j\Omega} + \frac{e^{-j\Omega t}}{-(j\Omega)^2}\right]_{-1}^{0} - \left[\frac{t e^{-j\Omega t}}{-j\Omega} + \frac{e^{-j\Omega t}}{-(j\Omega)^2}\right]_{0}^{1}$$

$$= \frac{e^{j\Omega} - e^{-j\Omega}}{j\Omega} + \left[\frac{1}{\Omega^2} - \frac{e^{j\Omega}}{j\Omega} - \frac{e^{j\Omega}}{\Omega^2}\right] - \left[\frac{e^{-j\Omega}}{-j\Omega} + \frac{e^{-j\Omega}}{\Omega^2} - \frac{1}{\Omega^2}\right]$$

$$= \frac{e^{j\Omega} - e^{-j\Omega}}{j\Omega} + \frac{1}{\Omega^2} - \frac{e^{j\Omega}}{j\Omega} - \frac{e^{-j\Omega}}{j\Omega} + \frac{e^{-j\Omega}}{j\Omega} - \frac{e^{-j\Omega}}{\Omega^2} + \frac{1}{\Omega^2}$$

$$= \frac{2}{\Omega^2} + \frac{-e^{-j\Omega} - e^{j\Omega}}{\Omega^2} = \frac{4}{\Omega^2} \sin^2\left(\frac{\Omega}{2}\right)$$

(14)

(7)(j) $f(t) = \begin{cases} 0, & t<0 \\ e^{-at}, & t \geq 0 \end{cases}$

Let $X(j\Omega)$ be fourier transform of $f(t)$

$$X(j\Omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\Omega t}\, dt$$

$$X(j\Omega) = \int_{-\infty}^{0} 0\, dt + \int_{0}^{\infty} e^{-at} e^{-j\Omega t}\, dt$$

$$X(j\Omega) = \int_{0}^{\infty} e^{-t(a+j\Omega)}\, dt$$

$$= \left[\frac{-e^{-t(a+j\Omega)}}{(a+j\Omega)} \right]_{0}^{\infty}$$

$$X(j\Omega) = \frac{1}{a+j\Omega}$$

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(3.) Show that $x(t) \cdot y(t) \longleftrightarrow \frac{1}{2\pi} X(j\Omega) \cdot Y(j\Omega)$

Let fourier transform of $x(t), y(t)$ be $X(j\Omega), Y(j\Omega)$ resp.

Let $z(t) = x(t) \cdot y(t)$.

$$Z(j\Omega) = \frac{1}{2\pi} X(j\Omega) Y(j\Omega)$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\omega) Y(j\phi) e^{jt(\omega+\phi)} \, d\omega \, d\phi$$

Let $\varphi = \omega + \phi$

$$x(t) \cdot y(t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{X(j(\varphi-\phi)) Y(j\phi) \, d\phi}{e^{-jt(\omega+\phi)}} \right] d\varphi$$

$$= \frac{2\pi}{4\pi^2} \int_{-\infty}^{\infty} e^{j\varphi t} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j(\varphi-\phi)) Y(j\phi) \, d\phi \right] d\varphi$$

$$x(t) \cdot y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\varphi t} \cdot Z(j\varphi) \, d\varphi$$

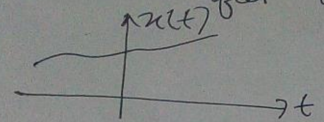
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(j\varphi) e^{j\varphi t} \, d\varphi$$

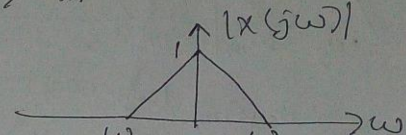
$$x(t) \cdot y(t) = z(t)$$

16.

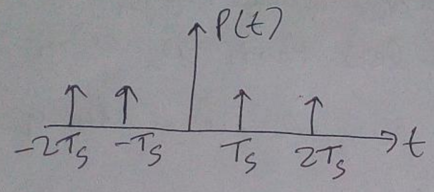
(4.) Let $x(t)$ be an analog signal that we want to sample at frequency $\frac{2\pi}{T_s} = \omega_s$

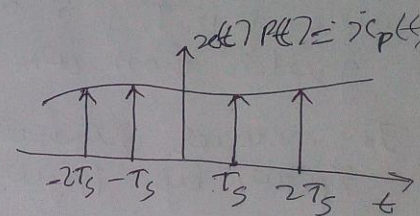
Assumption: $x(t)$ is band-limited i.e. $|X(j\omega)|$ vanishes after $\omega \geq \omega_m < \infty$





To sample this signal we use impulse train, $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$





$\therefore$ Sampled signal is, $x_p(t) = x(t)\,p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)$

Changing to frequency domain from time domain,

$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\omega) * P(j\omega)\right)$$ (Using multiplication Prop of F.T)

$$P(j\omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$\therefore\ X_p(j\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(j(\omega - n\omega_s))$$

Now two cases are possible on the basis of ω_m & ω_s

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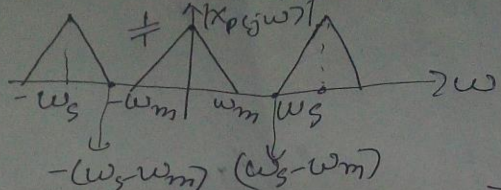
(a) when $\omega_s - \omega_m \geq \omega_m$
(b) when $\omega_s - \omega_m < \omega_m$

Case (a) when $\omega_s - \omega_m \geq \omega_m$



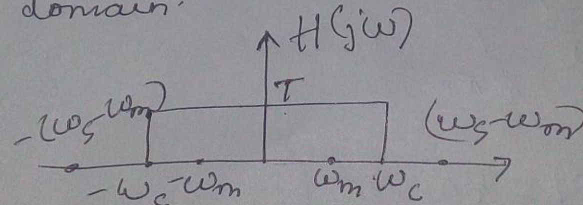
As we see that when $\omega_s - \omega_m \geq \omega_m$ $X_p(j\omega)$ is shifted versions of $X(j\omega)$. We can easily recover the original signal $X(j\omega)$ from $X_p(j\omega)$ using a low-pass filter with gain T & a cutoff frequency, ω_c, greater than ω_m & less than $\omega_s - \omega_m$.

To recover original signal we multiply $X_p(j\omega)$ by $H(j\omega)$,

$$X'(j\omega) = X_p(j\omega) \cdot H(j\omega) = X(j\omega)$$

when $\omega_s - \omega_m \geq \omega_m$

$H(j\omega)$ is the lowpass filter function in fourier domain.



ω_m is called the Nyquist frequency & $2\omega_m$ is called the Nyquist rate

$$\boxed{\omega_s \geq 2\omega_m}$$

Case (b) $\omega_s - \omega_m < \omega_m$

Due to overlap, it is not possible to recover the original signal.