LATTICE POINTS AND SIMULTANEOUS CORE PARTITIONS

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ABSTRACT. We observe that for a and b relatively prime, the "abacus construction" identifies the set of simultaneous (a, b)-core partitions with lattice points in a rational simplex. Furthermore, many statistics on (a, b)-cores are piecewise polynomial functions on this simplex.

We apply these results to rational Catalan combinatorics. Using Ehrhart theory, we reprove Anderson's theorem [2] that there are $(a+b-1)!/a!b!$ simultaneous (a, b)-cores, and using Euler-Maclaurin theory we prove Armstrong's conjecture [5] that the average size of an (a, b)-core is $(a+b+1)(a-1)(b-1)/24$.

We conjecture a unimodality result for q rational Catalan numbers, and make preliminary investigations in applying these methods to the (q, t)-symmetry and specialization conjectures. We prove these conjectures for low degree terms and when $a = 3$, connecting them to the Catalan hyperplane arrangement and proving an apparently new result about permutation statistics along the way.

1. INTRODUCTION

The goal of this paper is to establish lattice point geometry as a foundation for rational Catalan combinatorics. Rational Catalan numbers, and their q and (q, t) analogs, are a natural generalization of Catalan numbers that, apart from their intrinsic combinatorial interest, appear in the study of Hecke algebras [8] and compactified Jacobians of singular curves [9, 10] .

Our point of entry to rational Catalan combinatorics is Anderson's [2] result that rational Catalan numbers count simultaneous (a, b)-core partitions. Simultaneous core partitions are our main object of study, and the first result of this paper is another proof of Anderson's theorem.

1.1. Background: Simultaneous cores and rational Catalan numbers.

A *partition of n* is a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq \lambda_k > 0$ of positive integers so that $\sum_{i=1}^{k} \lambda_i = n$. We call n the *size* of the partition and denote it by $|\lambda|$; we call k the *length* of λ and denote it by $\ell(\lambda)$.

1.1.1. *Hooks and Cores.* We frequently identify λ with its Young diagram, in English notation – that is, we draw the parts of λ as the columns of a collection of boxes.

Definition 1.1. The *arm* $a(\square)$ of a cell $\square$ is the number of cells contained in λ and above $\square$, and the *leg* $l(\square)$ of a cell is the number of cells contained in λ and to the right of $\square$.

The *hook length* $h(\square)$ of a cell is $a(\square) + l(\square) + 1$.

Example 1.2. The cell $(2, 1)$ of $\lambda = 3 + 2 + 2 + 1$ is marked s; the cells in the leg and arm of s are labeled a and l, respectively.

$$\begin{array}{|c|c|c|c|} \hline \\ \cline{1-1} & a & \\ \cline{1-3} & s & l & l \\ \hline \end{array} \qquad \begin{aligned} a(s) &= \#a = 1 \\ l(s) &= \#l = 2 \\ h(s) &= 4 \end{aligned}$$

We now introduce our main object of study.

Definition 1.3. An *a-core* is a partition that has no hook lengths of size a. An (a, b)*-core* is a partition that is simultaneous an a-core and a b-core.

Example 1.4. We have labeled each cell $\square$ of $\lambda = 3 + 2 + 2 + 1$ with its hook length $h(\square)$.

$$\begin{array}{|c|c|c|c|} \hline 1 \\ \cline{1-3} 4 & 2 & 1 \\ \hline 6 & 4 & 3 & 1 \\ \hline \end{array}$$

We see that λ is *not* an a-core for $a \in \{1, 2, 3, 4, 6\}$; but it *is* an a-core for all other a.

1.1.2. *Rational Catalan numbers.* Recall that the Catalan number $\mathbf{Cat}_n = \frac{1}{2n+1}\binom{2n+1}{n}$. Catalan numbers count hundreds of different combinatorial objects; for instance, the number of lattice paths from $(0, n)$ to $(n + 1, 0)$ that stay strictly below the line connecting these two points.

The rational Catalan numbers are defined by a natural two variable generalization of these.

Definition 1.5. For a, b relatively prime, the *rational Catalan number*, or (a, b) *Catalan number* $\mathbf{Cat}_{a,b}$ is

$$\mathbf{Cat}_{a,b} = \frac{1}{a+b}\binom{a+b}{a}$$

The rational Catalan number $\mathbf{Cat}_{a.b}$ counts the number of lattice paths from $(0, a)$ to $(b, 0)$ that stay beneath the line from $(0, a)$ to $(b, 0)$. This is consistent with the specialization $\mathbf{Cat}_{n,n+1} = \mathbf{Cat}_n$.

1.2. We prove the following two theorems:

Theorem 1.6 (Anderson [2])**.** *If a and b are relatively prime, the number of (a, b)-core partitions is $\mathbf{Cat}_{a,b}$.*

Theorem 1.7. *The average size of an (a, b)-core is $(a + b + 1)(a - 1)(b - 1)/24$.*

Remark 1.8. Theorem 1.7 was conjectured by Armstrong in 2011, and first appearing in print in [5]. Proving Armstrong's conjecture was the initial motivation for this work.

Stanley and Zanello [12] have proven the Catalan case ($a = b + 1$) of Armstrong's conjecture by different methods.

There are two main tools in the proofs of these theorems: the abacus construction, Ehrhart / Euler-Maclaurin theory. We breifly recall these ideas before giving a high-level overview of the proof.

1.2.1. *Abaci.* The main tool used to study a-cores is the "abacus construction". We review this construction in detail in Section 2. For now, we observe that there are at least two variants of the abacus construction in the literature.

The first construction, which we call the *positive abacus*, gives a bijection between a core partitions and $\mathbb{N}^{a-1}$. Anderson's original proof used the positive abacus as part of a bijection between (a, b)-cores and (a, b)-Dyck paths, which were already known to be counted by $\mathbf{Cat}_{a,b}$.

The second construction, which we call the *signed abacus*, gives a bijection between a-core partitions and points in the $a - 1$ dimensional lattice

$$\Lambda_a = \left\{ c_1, \ldots, c_a \in \mathbb{Z} \middle| \sum c_i = 0 \right\}$$

We call Λ_a the *a-charge lattice*. The second construction is used in [7], to study the t-core crank. In particular, it shown that under the signed abacus, the size of an a-core is a quadratic function on Λ_a.

Our proof uses the signed lattice, and directly proves the count, without passing through (a, b)-Dyck paths.

1.2.2. *Ehrhart / Euler-Maclaurin.* The main novelty necessary to prove Armstrong's conjecture is to apply the techniques of Ehrhart / Euler-Maclaurin theory to core partitions.

The essential idea of Ehrhart theory is that counting lattice points in polytopes is a discrete approximation of the volume of the polytope, and so should behave similarly. Euler-Maclaurin theory extends this to summing polynomials over the lattice points being a discrete analog to the integral of the function over the polytope.

Let V be an n dimensional real vector space, and $\Lambda \subset V$ an n dimensional lattice. For example, $\Lambda \subset V = \mathbb{Z}^n \subset \mathbb{R}^n$. A lattice polytope $P \subset V$ is a polytope all of whose vertices are points of Λ.

For t a positive integer, let tP denote the tth dilate of P; that is, the polytope obtained by scaling all the vertices by t. We want to count the number of lattice points in tP; that is, we want to study the function

$$L(P, T) = \#\{\Lambda \cap tP\}$$

Clearly, the volume of tP is t^n times the volume of P. Ehrhart showed that in fact, $L(P, t)$ is a degree n polynomial in t; Ehrhart theory refers to the study of these polynomials.

Similarly, if f is a polynomial of degree d on V, then we have that $\int_{tP} f$ is a polynomial of degree $d+n$. Euler-Maclaurin theory says that the discrete analog

$$L(f,P,t) = \sum_{x \in \Lambda \cap tP} f(x)$$

is also a polynomial of degree $d+n$.

1.2.3. *Initial motivation.* To explain the method used to prove Theorems **??**], we begin with the following

False Hope 1. Fix b. Then under the signed abacus construction, the set of (a,b)-cores maps to $a\Delta$, for some $(b-1)$-dimensional integral polytope Δ.

The false hope provides an obvious strategy to prove Anderson's theorem and Armstrong's conjecture. If the false hope were true, Ehrhart theory would imply that, for a relatively prime to a fixed b, $|\mathcal{C}_{a,b}|$ would be a polynomial of degree $b-1$ in a. It is clear from the definition that this polynomiality property holds for $\mathbf{Cat}_{a,b}$. Thus, proving Anderson's theorem for a fixed b reduces to showing that two polynomials are equal, which only requires checking finitely many values.

Furthermore, it is a theorem that thee size of a b core partition is a quadratic function on the lattice. Thus, if False Hope were true Euler-Maclaurin theory would give that the total size of all (a,b) cores was a polynomial of degree $b+1$, and again we could hope to exploit this polynomiality in a proof.

Note that since the degree of the polynomial $\mathbf{Cat}_{a,b}$ grow as b grows, proving Anderson and Armstrong's conjecture as a whole by this method would still requires checking infinitely many values.

1.2.4. The False Hope is not quite true, but the strategy outlined above is essentially the one we follow. One minor tweak needed to the False Hope is that as we vary a, the polytope is not only scaled, by also changed by a linear transformation. These transformations preserve the number of lattice points and the quadratic function giving the size of the partitions, and so does not pose any real difficulties.

More troubling is that the polytope $\mathcal{C}_{a,b}$ is not integral, but only rational. Recall that a polytope Δ is *rational* if there is some $k \in \mathbb{Z}$ so that $k\Delta$ is a lattice polytope.

Ehrhart and Euler/Maclaurin theory can be extended to rational polytopes at the cost of replacing polynomials by *quasipolynomial*.

Definition 1.9. A function $f : \mathbb{Z} \to \mathbb{C}$ is a quasipolynomial of degree d and period n if there exist n polynomials $p_0, \dots, p_{n-1}$ of degree d, so that for $x \in k + n\mathbb{Z}$, we have $f(x) = p_k(x)$.

Example 1.10.

Since $\mathbf{Cat}_{a,b}$ is defined only for a and b relatively prime, it fits nicely into the quasipolynomial framework. For fixed a, and for b in a fixed residue class mod

a, we have that $\mathbf{Cat}_{a,b}$ is polynomial; it just so happens that for any residue class relatively prime to a, these polynomials are identical.

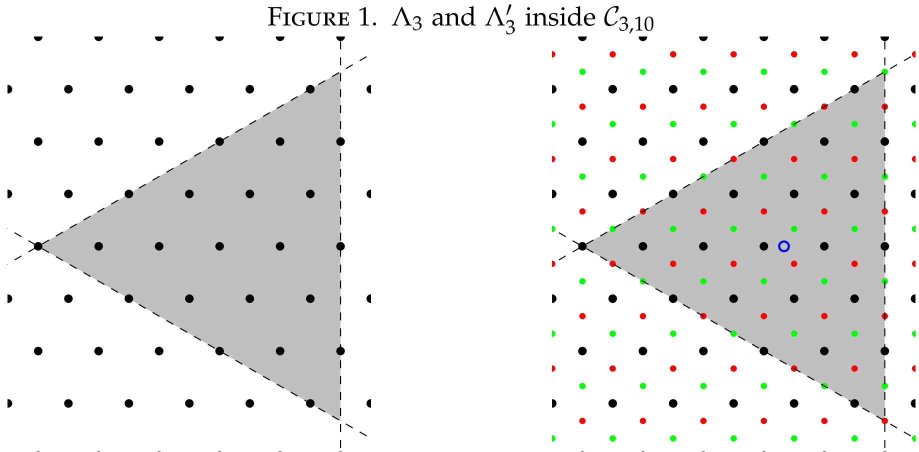
Such "accidental" equalities between the polynomials for different residues happen frequently in Ehrhart theory, but are still mysterious in general. Perhaps the most studied manifestation of this is *period collapse* (see [11] and references), where the quasipolynomial is in fact a polynomial.

Although the general strategy outlined above can be carried forth with quasipolynomials instead of polynomials, it is more difficult as the periods of the quasipolynomials are increasing with b, and so many more values must be checked. In the end, we could not find a way to prove Theorems **??** using solely quasipolynomiality. Instead, we found a clear explanation of the "accidental" equality of the polynomials for different residue classes, which we now explain.

Although $\mathcal{C}_{a,b}$ is not a lattice polytope with respect to the lattice Λ_a of a-cores, $\mathcal{C}_{a,b}$ will be a lattice polytope with respect to a refinement of Λ_a. In particular, there is a lattice Λ'_a containing Λ_a with index $[\Lambda_a : \Lambda'_a] = a$, so that with respect to the lattice Λ'_a, the polytope $\mathcal{C}_{a,b}$ is the standard simplex $\sum_{i=1}^{a} x_i = b, 0leqx_i \in \mathbb{Z}$.

Furthermore, consider the action of $\mathbb{Z}_a$ cyclicly permutating the variables, i.e., $x_i \mapsto x_{i+1}, x_a \mapsto x_0$. When b is relatively prime to a, this action does not have any fixed points. Furthermore, in this case the $\mathbb{Z}_a$ orbit of each point in Λ'_a contains exactly one element of Λ_a. This is illustrated in Figure 1: the left hand picture shows $\Lambda_3 \subset \mathbb{C}_{3,10}$, while the right hand picture shows Λ'_3. The black dots are elements of Λ_3, while the red and green dots are the other cosets of Λ_3 inside of Λ'_3, and are obtained by rotating around the blue circle.

FIGURE 1. Λ_3 and Λ'_3 inside $\mathcal{C}_{3,10}$



From this we see that the number of points in $\mathcal{C}_{a,b} \cap \Lambda_a$ is exactly $1/a$ of the number of points in the standard simplex, which is $\binom{a+b-1}{b}$. Since

$$\frac{1}{a}\binom{a+b-1}{b} = \frac{1}{a+b}\binom{a+b}{b} = \mathbf{Cat}_{a,b}$$

this proves Anderson's theorem.

Thus, $\mathbf{Cat}_{a,b}$ is the number of degree b monomials $p_1^{e_1}\cdots p_a^{e_a}$ with $\sum e_i = b$ and $\sum ie_i = 0 \mod a$. Alternatively, let V_a be the regular representation of $\mathbb{Z}_a$. Then $\mathbf{Cat}_{a,b}$ is the dimension of the invariant part of the bth symmetric power of V_a:

$$\mathbf{Cat}_{a,b} = \dim_{\mathbb{C}}\left(\mathrm{Sym}^b(V_a)\right)^{\mathbb{Z}_a}$$

1.3. q **and** (q,t)**-analogs.** The second half of the paper, comprosing Sections **??** and

1.3.1. Section **??** looks at the q-rational Catalan numbers $\mathbf{Cat}_{a,b}(q)$, which are the obvious extension of rational Catalan numbers obtained by replacing all numbers in the definition of $\mathbf{Cat}_{a,b}(q)$ by their q analog:

$$\mathbf{Cat}_{a,b}(q) = \frac{1}{[a+b]_q}\binom{a+b}{a}.$$

It is a not obvious fact, proven in , that the coefficients of $\mathbf{Cat}_{a,b}(q)$ are positive. The main question we pursue in Section **??** is whether our lattice point view can shed any light on this positivity question.

We observed in Equation **??** that $\mathbf{Cat}_{a,n}$ is the dimension of $\mathrm{Sym}^b(\mathbb{C}[\mathbb{Z}_a])^{\mathbb{Z}}_a$; on the other hand. On the other hand, we could let $\mathbb{C}^*$ act on $\mathbb{C}[\mathbb{Z}_a]$ by $\lambda\cdot[k] = \lambda^k[k]$; then the q binomial coefficient are computing the $\mathbb{C}^*$ weight on $\mathrm{Sym}^b(\mathbb{C}[\mathbb{Z}_a])$.

It is then natural to hope that the coefficients of $\mathbf{Cat}_{a,b}(q)$ are computing the dimensions of the $\mathbb{C}^*$ eigenspaces on the invariant part of $\mathrm{Sym}^b(\mathbb{C}[\mathbb{Z}_a])$. This hope, however, is false.

However, this line of thought leads us to conjectural formula for $\mathbf{Cat}_{a,b}(q)$ as a sum of a^{a-2} terms, each of which is the product of q^b times a q^a binomial coefficients.

This conjectural formula leads naturally to a unimodality conjecture about $\mathbf{Cat}_{a,b}(q)$. Recall that a sequence $a_1, dots, a_n$ is unimodal if there is some k so that

$$a_1 \leq a_2 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k-1} \geq a_{k-2} \cdots \geq a_n$$

The coefficients of $\mathbf{Cat}_{a,b}$ are not unimodal. However, we conjecture that, if we fix $0leqk < a$, and look only at the coefficients of $\mathbf{Cat}_{a,b}(q)$ of the form q^{an+k}, the resulting sequences are unimodal.

1.3.2. The (q,t)-rational Catalan numbers were original defined in terms of summing certain statistics over lattice paths, but Armstrong, have given an analogous formula over simultaneous core. In particular, let $\mathcal{SC}_{a,b}$ denote the set of all (a,b)-cores,

$$\mathbf{Cat}_{a,b}(q,t) = \sum_{\lambda\in\mathcal{SC}_{a,b}} q^{\ell(\lambda)}t^{s\ell'(\lambda)}$$

where $\ell(\lambda)$ is the length of λ, and $s\ell'(\lambda)$ is the *co-skew length*, a statistic introduced by λ.

Our main result in the second half of the paper is that ℓ and $s\ell'$ are piecewise linear functions on $\Delta_{a,b}$, and thus that lattice point geometry should be useful in these studying $\mathbf{Cat}_{a,b}(q,t)$ and $\mathbf{Cat}_{a,b}(q)$.

This result is for a variety of reasons: it provides a progress toward proving some conjectures about $\mathbf{Cat}_{a,b}(q,t)$. First, it is conjectured that $\mathbf{Cat}_{a,b}(q,t)$ is symmetric in q and t, and that

$$q^{(a-1)(b-1)/2}\mathbf{Cat}_{a,b}(q,1/q) = \mathbf{Cat}_{a,b}(q).$$

As a corollary of Lemma **??**, we are able to prove these conjectures for the high and low degree terms of $\mathbf{Cat}_{a,b}$:

Second, it turns out that the chambers of linearity for ℓ and $s\ell'$ are interesting hyperplane arrangements – the walls of ℓ are the walls of the A_{n-1} arrangement, and the walls of $s\ell'$ are the walls of a deformation of the A_{n-1} arrangement known as the Catalan arrangement. This gives a connection between $\mathbf{Cat}_{a,b}(q,t)$ and $\mathbf{Cat}_{a}$, which we don't believe was well known. Furthermore, it suggests that a thorough understanding of the geometry of the Catalan arrangement could lead to a full proof of the specialization and symmetry conjectures.

Finally, in order to prove Corollary [**?**] we are lead to define and investigate a new permutation statistic $\mathbf{siz}(\sigma)$.

Lemma 1.11.
$$\sum_{\sigma\in S_n} q^{\mathbf{siz}(\sigma)}t^{\mathbf{maj}(\sigma)} = \prod_{k=1}^{n}[k]_{q^{n+1-k}t}$$

Definition 1.12. The A_{a-1} hyperplane arrangement is the set of the $\binom{a}{2}$ hyperplanes $x_i = x_j$ in the $a-1$ dimensional vector space $\sum x_i = 0$.

There are $n!$ regions of the A_{a-i} region, which are indexed by partitions σ; the region indexed by σ is where $x_{\sigma(0)} < z_{\sigma(1)} < \cdots < x_{\sigma(a)}$.

Another hyperplane arrangment that is pertinent is the Catalan arrangement, which is a deformation of the A_a arrangement.

Definition 1.13. A hyperplane arrangement $\mathcal{A}'$ is a *deformation* of an arrangement $\mathcal{A}$ if every hyperplane in $\mathcal{A}'$ is parallel to one in $\mathcal{A}$.

Definition 1.14. The *Catalan arrangement* $\mathcal{C}_a$ is the union of the $3\binom{a}{2}$ hyperplanes $x_i - x_j \in \{-1,0,1\}, i<j$.

The name *Catalan arrangement* comes from the fact that $\mathcal{C}_a$ has $n!C_n$ regions.

We have already seen the hyperplanes in the Catalan arrangement appearing, if $b\mathcal{C}_n$ denotes the Catalan arrangement scaled by b (so $x_i - x_j \in \{-b,0,b\}$), then the hyperplanes that define the simplex of b-cores are in $b\mathcal{C}_n$.

We now give an informal discussion of how $\mathbf{core}_a(\mathbf{x})$ depends on the chamber of $\mathcal{A}_a$.

Example 1.15. Consider the lattice path of a large random a-core. At the start, every segment of the path slopes down; then there is a section where one out of every a segments slopes up; then another large section where two out of every a slope up, then 3 out of every a steps slope up, until eventually the path hits the x-axis, from which point every step slopes up.

In the first sections, all steps that slope up correspond to electrons on the same runner i of the a-abacus. The i that occurs is the one with x_i is minimum. Similarly, in the second section, all of segments corresponding to electrons on runner i slope up, but also the segments corresponding to electrons on runner j, where x_j is the second smallest of all the x_k.

The ordering of the x_i tell us the ordering the up-steps on the ith abacus happen.

1.4. **Chamber dependence of $\mathbf{core}_a c$.** The lattice of charges Λ_a is essentially the A_{a-1} lattice. Although we have given a uniform descirption of the partition corresponding to $\mathbf{core}_a(c)$ for any charge vector c, and shown the size $|\mathbf{core}_a(c)|$ is a global polynomial in c, in many ways $\mathbf{core}_a(c)$ has a chamber dependence on the A_{a-1} hyperplane arrangement. By this we mean that if we restrict to a given chamber of this hyperplane arrangement, then $\mathbf{core}_a(c)$ behaves nicely, but if c crosses one of the walls of the hyperplane arrangement, then $\mathbf{core}_a(c)$ undergoes a qualitative change.

We illustrate this now with an informal example.

Example 1.16. The boundary path of a large a-core can be decomposed into $a+1$ regions, labeled with $i \in \{0, 1, \dots, a\}$. On the ith region, i out of every a steps will be left, and $a-i$ will be down; thus on the ith region the path will have slope $-(a-i)/i$.

This description of a-cores is clear from the abacus description. In the region zero, all the runners have unfilled energy states; we cross into the first region as soon as one of the runners start having filled energy states. In general, the ith region is exactly those regions where i of the runners have filled energy states.

We get a chamber structure on the space of a-cores by considering *which* of the i runners have filled or empty energy states.

1.5. **Outline.** In section **??** we introduce standard notation and bijections about a-core partitions. Section **??** applies this to simultaneous core partitions and proves our main results. Section **??** describes some conjectural applications of these ideas to the q and (q,t) generalizations of (a,b)-Catalan numbers.

1.6. **Acknowledgements.** I learned about Armstrong's conjecture over dinner after speaking in the MIT combinatorics seminar. I would like to thank Jon Novak for the invitation, Fabrizio Zanello for telling me about the conjecture, and funding bodies everywhere for supporting seminar dinners.

2. Abaci and Electrons

In this section we recall the fermionic viewpoint of partitions and the abaci model of a-cores. The main results are that a-cores are in bijection with points on the "charge lattice" Λ_a, and the size of a given a-core is given by a quadratic function on the lattice.

2.1. **The fermionic viewpoint.**

In this section, we introduce Dirac's electron sea and its relation to partitions and Frobenius notation.

2.1.1. The following is a motivating fairy tale and should not be mistaken for an attempt at accurate physics or accurate history.

According to quantum mechanics, the possible energies levels of an electron are quantized – they can only be half integers. The Dirac equation predicts that electrons can have negative energy, and hence the possible energy levels of an electron are half integers i.e., elements of $\mathbb{Z}_{1/2} = \{a + 1/2 | a, \in \mathbb{Z}\}$. However, the physical meaning of a negative energy electron is elusive.

Dirac's *electron sea* solves the problem of negative energy electrons by redefining the vacuum state **vac**. The Pauli exclusion principle states that each possible energy state can have at most one electron in it; thus, we can view any set of electrons as a subset $S \subset \mathbb{Z}_{1/2}$. Intuitively, the vacuum state **vac** should consist of empty space with no electrons at all, and hence correspond to the set $S = \emptyset \subset \mathbb{Z}_{1/2}$.

Dirac suggested redefining **vac** to be an infinite "sea" of electrons; specifically, in the vacuum state every negative energy level should be filled with an electron, and none of the positive energy states filled. Pauli's exclusion principle implies we cannot add a negative energy electron to **vac**, but we can add any positive energy electron to **vac**, and so Dirac's electron sea solves the problem of negative energy electrons.

In addition, Dirac's electron sea predicts the positron, a particle that has the same energy levels as an electron, but positive charge. Namely, a positron corresponds to a "hole" in the electron sea, that is, a negative energy level *not* filled with an electron. Removing a negative energy electron results in adding positive charge and positive energy, and hence can be interpreted as a having a positron.

2.1.2. We are thus led to the following definitions:

Definition 2.1. Let $\mathbb{Z}^{\pm}_{1/2}$ denote the set of all positive/negative half integers, respectively.

The vacuum $\mathbf{vac} \subset \mathbb{Z}_{1/2}$ is the set $\mathbb{Z}^{-}_{1/2}$.

A *state* S is a set $S \subset \mathbb{Z} + 1/2$ so that the symmetric difference $S \Delta \mathbf{vac} = (S \cap \mathbb{Z}^{+}_{1/2}) \cup (S^c \cap \mathbb{Z}^{-}_{1/2})$ is finite. States should be interpreted as a finite collection of electrons (the elements of $S \cap \mathbb{Z}^{+}_{1/2}$) and positrons (the elements of $S^c \cap \mathbb{Z}^{-}_{1/2}$).

The *charge* $c(S)$ of a state S is the number of positrons minus the number of electrons:

$$c(S) = \#S \cap Z^+_{1/2} - \#S^c \cap \mathbb{Z}^-_{1/2}$$

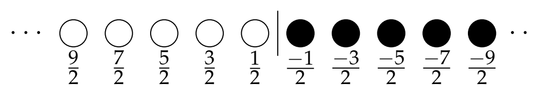
The *energy* $e(S)$ of a state S is the sum of all the energies of the positrons and the electrons:

$$e(S) = \sum_{k \in \mathbb{Z}^+_{1/2} \cap S} k + \sum_{k \in \mathbb{Z}^-_{1/2} \cap S^c} -k$$

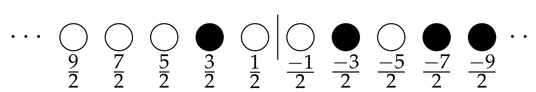
2.1.3. *Maya Diagrams.* It is convenient to represent states S as *Maya Diagrams*

The Maya diagram of S is an infinite line of circles, one for each element of $\mathbb{Z}_{1/2}$, with the positive circles extending to the left and the negative direction to the right. A black "stone" is placed on the circle corresponding to $k \in \mathbb{Z}_{1/2}$ if and only if $k \in S$, that is, if the energy level k is occupied by an electron.

Example 2.2. The Maya diagram corresponding to the vacuum vector **vac** is shown below.



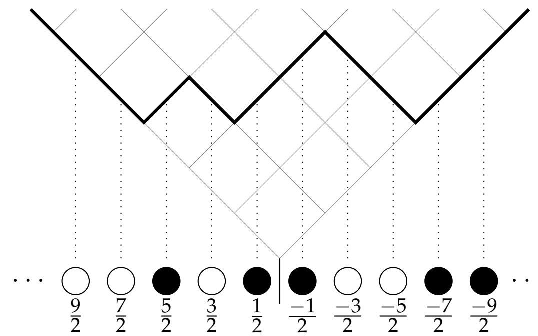
Example 2.3. The following Maya diagram illustrates the state S consisting of an electron of energy $3/2$, and two positrons, of energy $1/2$ and $5/2$.



2.2. **Paths.** There is a bijection from the set of partitions $\mathcal{P}$ to the set of charge 0 states, that sends a partition $\lambda \in \mathcal{P}_n$ of size n to a state S_λ with energy $e(S_\lambda) = n$. This bijection can be described in two ways: as recording the boundary path of λ, or recording the modified Frobenius coordinates of *lambda*.

2.2.1. We draw partitions in "Russian notation" – rotated $\pi/4$ radians counterclockwise and scaled up by a factor of $\sqrt{2}$, so that each segment of the border path of λ is centered above a half integer. For each segment of the border path, we place an electron in the corresponding energy level if that segment of the border slopes up, and we leave the energy state empty if that segment of border path slopes down.

Example 2.4. We illustrate the bijection in the case of $\lambda = 3 + 2 + 2$. The corresponding state S_λ consists of two electrons with energy $5/2$ and $1/2$, and two positrons with energy $3/2$ and $5/2$.



2.2.2. *Frobenius Coordinates.* The energies e_i of the electrons and the positrons of λ are the *modified Frobenius coordinates*,

Dissect the partition λ with the vertical line through 0. The left side of λ consists of c rows, where c is the number of electrons, and the length of the ith row will be the energy of the ith electron. Similarly, the right side consists of c pieces, with lengths the energies of the positrons.

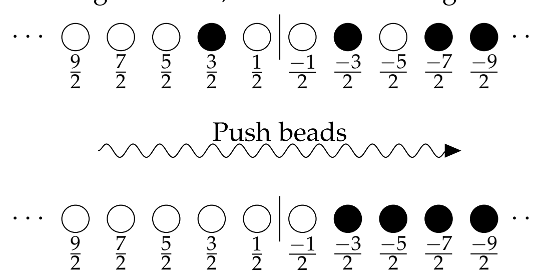
Example 2.5.

2.2.3. *Non-zero charge.* The bijection between states of charge zero and partitions may be modified to give a bijection between the states of charge c and partitions, for any $c \in \mathbb{Z}$: simply translate the the grid the partition is drawn on by c.

2.3. **Abaci.** Rather than view the Maya diagram as a series of stones in a line, we now view it as beads on the runner of an abacus. Sliding the beads to be right justified allows the charge of the state to be read off, as it is easy to see how many electrons have been added or are missing from the vacuum state.

In what follows, we mix our metaphors and talk about electrons and protons on runners of an abacus.

Example 2.6. Consider Example 2.3, where the Maya diagram consists of two positrons and an electron. Pushing the beads to be right justified, we see the first bead is one step to the right of zero, and hence the original state had charge 1.



2.3.1. *Cells and hook lengths.* The cells $\square \in \lambda$ are in bijection with the *inversions* of the boundary path; that is, by pairs of segments $(\text{step}_1, \text{step}_2)$, where step_1 occurs before step_2, but step_1 is traveling NE and step_2 is traveing SE. The bijection sends λ to the segments at the end of its arm and leg.

In the fermionic viewpoint, cells of λ are in bijection with pairs $(e, e-k), e \in \mathbb{Z}_{1/2}, k > 0$ of a filled energy level e and an empty energy level $e-k$ of lower energy; we call such a pair an *inversion*. The hook length $h(\square)$ of the corresponding cell is k.

If $(e, e-k)$ is such a pair, reducing the energy of the electron from e to $e-k$ changes λ by removing the rim hook corresponding to the cell $\square$. This rim-hook has length k.

Example 2.7. The cell $\square = (2,1)$ of $\lambda = 3+3+2$ has hook length $h(\square) = 3$, and corresponds to the electron in energy state $1/2$ and the empty energy level $-5/2$; which are three apart.

2.4. **Bijections.** Rather than place the electrons corresponding to λ on one runner, place them on a different runners, putting the energy levels $ka - i - 1/2$ on runner i.

If the hooklength $h(\square) = ka$ is divisible by a, then the two energy levels of inversion$(\square)$ lie on the same runner. Similarly, any inversion of energy states on the same runner corresponds to a cell with hook length divisible by a.

Thus, λ is an a-core if and only if the beads on each runner of the a-abacus are right justified. Although the total charge of all the runners must be zero, the charge need not be evenly divided among the runners. Let c_i be the charge on the ith runner; then we have $\sum c_i = 0$, and the c_i determine λ.

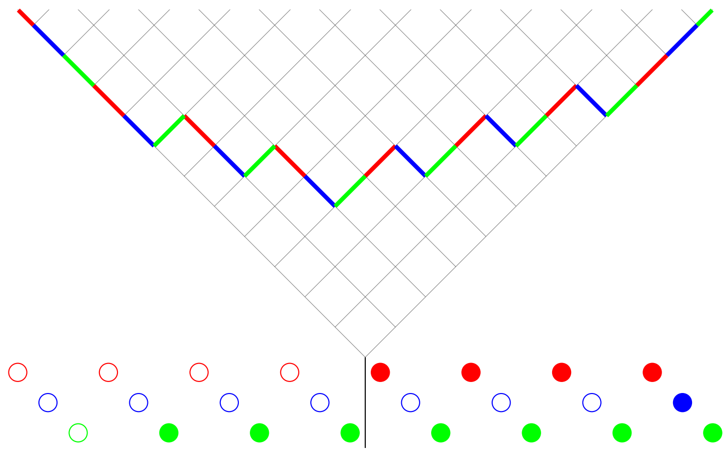
Similarly, given any $\mathbf{c} = (c_0, \ldots, c_{a-1}) \in \mathbb{Z}^a$ with $\sum c_i = 0$, there is a unique right justified abacus with charge c_i on the ith runner. The coresponding partition is an a-core which we denote $\mathbf{core}_a(\mathbf{c})$.

We have shown:

Lemma 2.8. There is a bijection

$$\mathbf{core}_a : \{(c_0, \ldots, c_{a-1} | c_i \in \mathbb{Z}, \sum c_i = 0\} \to \{\lambda | \lambda \text{ is in } a\text{-core}\}$$

Example 2.9. We illustrate that $\mathbf{core}_3(0,3,-3) = 7+5+3+3+2+2+1+1$.



2.5. **Size of an a-core.**

Theorem 2.10.

$$|\mathbf{core}_a(\mathbf{c})| = \frac{a}{2}\sum_{k=0}^{a-1} c_k^2 + kc_k$$

We are not sure where exactly where this theorem originates; a stronger version is used in [7] and [6], to prove certain generating functions of partitions are modular forms.

Proof. If $c_k > 0$ the kth runner has c_k positrons, with energies

$$\begin{aligned} &(k+1/2),\\ &(k+1/2)+a,\\ &(k+1/2)+2a,\\ &\quad\vdots \qquad \vdots\\ &(k+1/2)+(c_k-1)a \end{aligned}$$

and so the particles on the kth runner have total energy

$$\frac{a}{2}(c_k^2 - c_k) + (k+1/2)c_k.$$

If $c_k < 0$, the kth runner has $-c_k$ electrons, and a similar calculation shows they have a total energy of

$$\frac{a}{2}(c_k^2 + c_k) - c_k(a-k-1/2) = \frac{a}{2}(c_k^2 - c_k) + (k+1/2)c_k.$$

Since $\sum c_k = 0$, the total energy of all particles simplifies to $\frac{a}{2}\sum(c_k^2 + kc_k)$. $\square$

3. Simultaneous Cores

We now turn to studying the set of b-cores within the lattice Λ_a of a-cores.

3.1. (a,b)-cores form a simplex.

First, some notation and conventions.

Let $r_a(x)$ be the remainder when x is divided by a, and $q_a(x)$ to be the integer part of x/a, so that $x = aq_a(x) + r_a(x)$ for all x. Furthermore, we use cyclic indexing for $\mathbf{c} \in \Lambda_a$; that is, for $k \in \mathbb{Z}$, we set $c_k = c_{r_a(k)}$.

Lemma 3.1. Within the lattice of a cores, the set of b cores are the lattice points satisfying the inequalities

$$c_{i+b} - c_i \leq q_a(b+i)$$

for $i \in \{0, \ldots, a-1\}$.

Proof. Fix $\mathbf{c} \in \Lambda_a$, and consider the corresponding a-abacus.

Let $\lambda = \mathbf{core}_a(\mathbf{c})$ be an a core, and let e_i denote the energy of the highest electron the ith runner. We claim that $\mathbf{core}_a(\mathbf{c})$ is a b-core if an only if for each i, the energy state $e_i - b$ is filled.

Certainly this condition is necessary. To see that it is sufficient, suppose that λ is an a-core, and that $e_i - b$ are all filled. To see λ is a b core, we must show that for any filled energy level L, that $L - b$ is filled.

Suppose that L is on the ith runner; then $L = e_i - aw$ for some $w \geq 0$, and so $L - b = (e_i - b) - aw$. But by supposition $e_i - b$ is a filled state, and $e_i - b - aw$ is to the right of it and on the same runner, and so it must be filled since λ is an a-core.

Now, the energy state $e_i - b$ is on runner $r_a(i+b)$, and so λ is b-core if and only if $e_i - b \leq e_{i+b}$ (recall that we are using cyclic indexing).

Substituting $e_k = -ac_k - r(k) - 1/2$ and simplifying gives that our inequality is equivalent to

$$a(c_{i+b} - c_i) \leq b + i - r_a(i+b)$$

and hence to

$$c_{i+b} - c_i \leq q_a(b+i).$$

$\square$

We have a hyperplanes in an $a-1$ dimensional space; they either form a simplex or an unbounded polytope.

Remark 3.2. The same analysis sheds light on the case when a and b are not relatively prime, which has been studied in [3].

Let $d = \gcd(a,b)$; then any d-core is also an (a,b)-core, and so there are no longer finitely many (a,b)-cores.

The inequalities given for $\mathcal{SC}_a(b)$ still describe the space of (a,b)-cores when a,b are no longer relatively prime, but these inequalities no longer describe a simplex. The inequalities no longer relate all the c_i to each other; rather, they decouple into d sets of a/d of variables

$$S_0 = \{xc_0, c_d, c_{2d}, \ldots, c_{a-d}\}$$
$$S_1 = \{c_1, c_{d+1}, \ldots, c_{a-d+1}\}$$
$$\cdots$$
$$S_{d-1} = \{c_{d-1}, c_{2d-1}, \ldots, c_{a-1}\}$$

The charges c_i in a given group must be close together, but for any vector $(v_0, \ldots, v_{d-1})$ with $\sum v_i = 0$, we may shift each element of S_i by v_i and all inequalities will still be satisfied.

In particular, the shifts of the zero vector are easily seen to be the d core partitions, and we see the set of (a,b)-core partitions is finite number of translates of the lattice of d-cores within the lattice of a-cores.

3.1.1. *Coordinate shift.* In the charge coordinates $\mathbf{c}$, neither the hyperplanes defining the set of b cores nor the quadratic form Q are symmetrical about the origin. We shift coordintaes to remedy this.

Definition 3.3. Define $\mathbf{s} = (s_1, \ldots, s_a) \in V_a$ by

$$s_i = \frac{i}{a} - \frac{a-1}{2a}$$

The i/a term ensures $s_{i+1} - s_i = 1/a$; subtracting $\frac{a-1}{2a}$ ensures that $\mathbf{s} \in V_a$, i.e. $\sum s_i = 0$.

Lemma 3.4. In the shifted charge coordinates

$$x_i = c_i + s_i$$

the inequalities defining the set of b cores become

$$x_{i+b} - x_i \leq b/a$$

and the size of an a-core is given by

$$Q(\mathbf{x}) = -\frac{a^2-1}{24} + \frac{a}{2}\sum_{i=0}^{a-1} x_i^2$$

Proof. That the linear term of Q vanishes in the $\mathbf{x}$ coordinates follows immediately from the definition of $\mathbf{s}$. The constant term of Q in the $\mathbf{x}$ coordinates is $-\frac{a}{2}\sum_{i=0}^{a-1} s_i^2$, which a short computation shows is $\frac{a^2-1}{24}$.

The statement about the set of b-cores follows from the computation

$$\begin{aligned}
x_{i+b} - x_i &= c_{i+b} - c_i + s_{i+b} - s_i \\
&\leq q_a(i+b) + r_a(i+b)/a - i/a \\
&= (b+i)/a - i/a \\
&= b/a
\end{aligned}$$

□

Another change of variables makes clear the shape these inequalities cut out.

Lemma 3.5. If a and b are relatively prime, then after the change of variables

$$y_i = x_i - x_{i+b} + \frac{b}{a}$$

the simplex of (a, b)-cores becomes the scaled standard simplex

$$b\Delta_{a-1} = \{(y_0, \ldots, y_{a-1}) \Big| \sum y_i = b\}$$

and the shifted lattice $\Lambda + \mathbf{s}$ goes to the index a lattice:

$$\sum_{k=0}^{a-1} (a-k) y_{i+kb} = (i + a/2) \mod a$$

Proof. It is immediate that the y_i satisfy $\sum y_i = b$ and $y_i \geq 0$. If $x_i \in \Lambda + s$, the fractional part of y_i is the fractional part of $s_i - s_{i+b}$, which is negative the fractional part of b/a, and hence y_i is an integer.

To show which sublattice $\Lambda + s$ maps to, and to show the map is surjective onto the simplex, we invert it. Note that $x_{i+b} = x_i - z_i + b/a$.

$$\begin{aligned}
x_i &= x_i \\
x_{i+b} &= x_i + b/a - z_i \\
x_{i+2b} &= x_i + 2b/a - z_i - z_{i+b} \\
&\cdots \\
x_{i+(a-1)b} &= x_i + (a-1)b/a - z_i - z_{i+b} - \cdots - z_{i+(a-2)b}
\end{aligned}$$

Summing these equations, the left hand side is $\sum x_i = 0$. Thus, we have

$$ax_i = (a-1)z_i + (a-2)z_{i+b} + \cdots + z_{i+(a-2)b} - b(a-1)/2$$

and the map is invertible.

Further noting that if $x_i \in \Lambda + s$, then $ax_i = ac_i + i - (a-1)/2$, and so

$$\sum_{k=0}^{a-1} (a-1-k) z_{i+kb} = i + (a-1)(b-1)/2 \mod a$$

Thus, the images of $x_i \in \Lambda + s$ lie in an index a sublattice of the standard lattice; it is a further easy check that this sublattice is equivalent to one of the cosets of the sublattice described in the introduction.

$\square$

Corollary 3.6 (Anderson [2])**.** The number of simultaneous (a, b)-cores is $\mathbf{Cat}_{a,b}$.

Proof. This follows quickly from Lemma 3.5.

The scaled simplex $b\Delta_a$ has $\binom{a+b-1}{a-1}$ usual lattice points. Cyclicly permuting the variables preserves $b\Delta_a$ and the standard lattice, and when b is relatively prime to a it cyclicly permutes the a cosets of the charge lattice.

Thus the standard lattice points in $b\Delta_a$ are equidistributed among the a-cosets of the charge lattice, and hence each one contains $\frac{1}{a}\binom{a+b-1}{a-1} = \mathbf{Cat}_{a,b}$.

$\square$

3.2. **The size of simultaneous cores.** We now have all the ingredients needed to prove Armstrong's conjecture. We derive it as a consequence of:

Theorem 3.7. For fixed a, and b relatively prime to a, the average size of an (a, b)-core is a polynomial of degree 2 in b.

Proof. For fixed a, the number of a-cores is $1/a$ times the number of lattice points in $b\Delta_{a-1}$, which is a polynomial $F_a(b)$ of degree $a - 1$. In the x-coordinates $Q = \mathbf{core}_a$ is invariant under S_a, and in particular rotation, we see that the sum of the sizes of all (a, b)-cores is $1/a$ times the sum of Q over the lattice points in $b\Delta_{a-1}$. By Euler-Maclaurin theory, the number of points in $b\Delta_{a-1}$ is a polynomial $G_a(b)$ of degree $a + 1$.

Thus, the average value of an (a, b)-core is $G_a(b)/F_a(b)$, the quotient of a polynomial of degree $a + 1$ by a polynomial of degree $a - 1$. To show this is a polynomial of degree two in b, we need to show that every root of F_a is a root of G_a.

We already know from 3.6 that the roots of F_a are $-1, -2, \ldots, -(a - 1)$. We now give another derivation of this fact, using Ehrhart reciprocity, that will easily adapt to shown these are also roots of G_a.

Ehrhart reciprocity says that $F_a(-x)$ is, up to a sign, the number of points in the *interior* of $x\Delta_{a-1}$. The interior consists of the points in $x\Delta_{a-1}$ none of whose coordinates are zero, and so the first interior point in $x\Delta_{a-1}$ is $(1, 1, \ldots, 1) \in a\Delta_{a-1}$. Thus, $F_a(b)$ vanishes at $b = -1, \ldots, -(a - 1)$, and as it has degree $a - 1$ it has no other roots.

Ehrhart reciprocity extends to Euler-Maclaurin theory, to say that up to a sign $Q_a(-x)$ is the sum of F of the interior points of $x\Delta_{a-1}$. Thus $Q_a(-x)$ also vanishes at $b = -1, \ldots, -(a - 1)$, and so P_a/Q_a is a polynomial of degree 2.

$\square$

Corollary 3.8. When (a,b) are relatively prime, the average size of an (a,b) core is $(a+b+1)(a-1)(b-1)/24$

Proof. Fix a, and let $P_a(b) = G_a(b)/F_a(b)$ be the degree two polynomial that gives the average value of the (a,b)-cores when a and b are relatively prime. As we know $P_a(b)$ is a polynomial of degree 2, we can determine it by computing a few values by hand; together with (a,b) symmetry of cores, $P_a(b)$ is also a polynomial of degree two in a, and hence we can compute the exact formula by calculating only a few values.

This is essentially the route we take, but use Ehrhart reciprocity to minimize the computation needed.

First, we find the two roots of $P_a(b)$. As the only 1 core is the empty partition, we have $F_a(1) = 1$ and $G_a(1) = 0$, and so $(b-1)$ is a factor of P_a.

Ehrhart reciprocity gives that $G_a(-a-1)$ is, up to a sign, the sum of Q over the lattice points in the interior of $(a+1)\Delta_a$, which are just the lattice points contained in Δ_a, and hence equal to $G_a(1) = 0$, and so $(b+a+1)$ is a factor of G_a.

By symmetry between a and b, we must have the average size for an (a,b) core is $A(a-1)(b-1)(a+b+1)$ for some constant A independent of a and b.

To see that $A = 1/24$, note that although there are no points of Λ_s in $\mathcal{S}_a(0) = 0$, the standard simplex $\delta_{a-1}(0)$ contains the origin as a lattice point, and so $P_a(0) = Q(0) = -(a^2-1)/24$. □

4. Toward q-analogs

In this section, we apply our lattice point and simplex point of view on simultaneous cores to the q-analog of rational Catalan numbers; the next section approaches (q,t)-analogs.

4.1. q-numbers.

Recall the standard q analogs of n, $n!$ and $\binom{n}{k}$:

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$
$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$$
$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

These three functions are polynomials with positive integer coefficients, i.e., they are elements of $\mathbb{N}[q]$.

The q rational Catalan numbers are given by the obvious formula:

Definition 4.1.

$$\mathbf{Cat}_{a,b}(q) = \frac{1}{[a+b]_q}\binom{a+b}{a}_q = \frac{(1-q^{b+1})(1-q^{b+2})\cdots(1-q^{b+a-1})}{(1-q^2)(1-q^3)\cdots(1-q^a)}$$

4.2. **Graded vector spaces.** One place q analogs occur naturally is in graded vector spaces.

Definition 4.2. If V is a graded vector space, with V_k denoting the weight k subspace of V, we define

$$\dim_q V = \sum_{k \in \mathbb{N}} q^k \dim V_k.$$

Proposition 4.3. Let p_i be a variable of weight i, then $\mathbb{C}[p_1, \ldots, p_n]$ has finite dimensional graded pieces, and

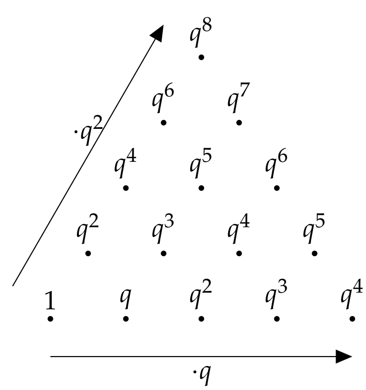
$$\dim_q \mathbb{C}[p_1, \ldots, p_n] = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

If V is a vector space with $\dim_q V = [n]_q$, then

$$\dim_q \mathrm{Sym}^b V = \binom{n+b-1}{n-1}_q$$

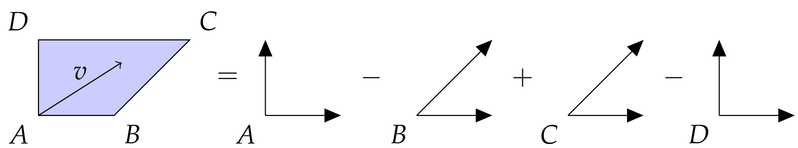
These statements can be interpretted geometrically in terms of lattice points The monomials in $\mathbb{C}[p_1, \ldots, p_n]$ correspond to the lattice points in an n dimensional unimodular cone; the monomials in $\mathrm{Sym}^b V$ correspond to lattice points in the scaled standard simplex $b\Delta_{a-1}$; the q-analogs of the statements listed above are q counting the lattice points, where the weights of the ith primitive lattice vector on the ray of the cone has weight q^i.

Example 4.4. The following diagram illustrates $\binom{b+a-1}{a-1}_q$ as q-counting standard lattice points in $b\Delta_{a-1}$ in the case $a = 3$ and $b = 4$. Letting b go to infinity corresponds to extending the arrows and the lattice points between them infinitely far to the upper right, showing that $\prod_{k=1}^{a-1} \frac{1}{1-q^k}$ q-counts the points in a standard cone.



4.3. **A q-version of cone decompositions.**

4.3.1. *Lawrence Varchenko.* Recall the decomposition of a simplicial polytope $\mathcal{P}$ in a vector space V of dimension n as a signed sum of cones based at their vertices, called the Lawrence-Varchenko decomposition:



First, pick a generic direction vector $v \in V$. At each vertex v_i, n facets of $\mathcal{P}$ meet; if we extend these facets to hyperplanes, they cut V into orthants. Let $\mathcal{C}_k$ be the orthant at v_i that contains our direction vector v. Let f_i be the number of hyperplanes that must be crossed to get from $\mathcal{C}_i$ to P.

Then:

$$\mathcal{S} = \sum_{i=0}^{k} (-1)^{f_i} \mathcal{C}_i$$

To deal correctly with the boundary of P, one must correctly include or exclude portions of the boundary of $\mathcal{C}_k$, but this subtlety won't matter to us.

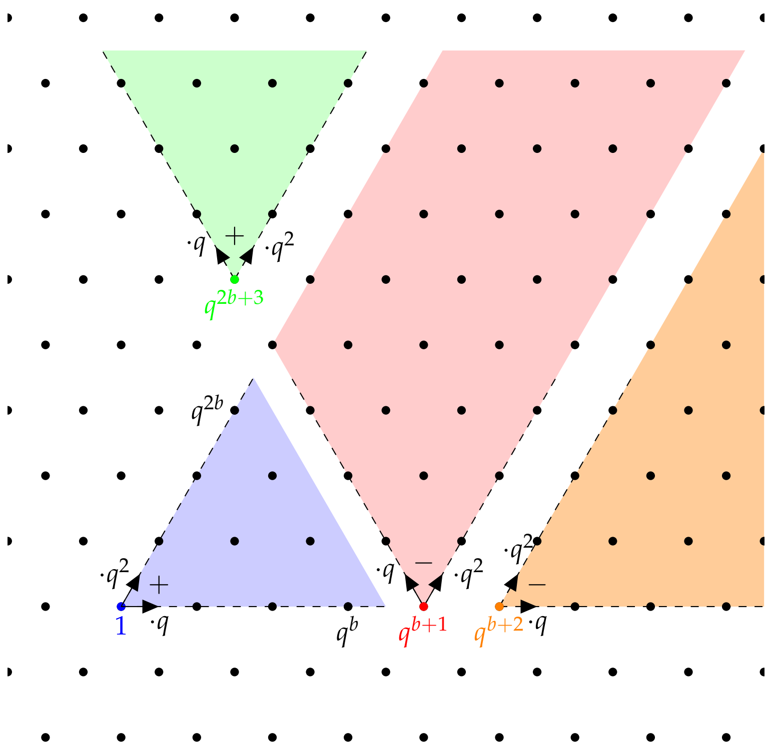
4.3.2. The algebraic structure of $\binom{b+a-1}{a-1}$ suggests a refinement of the Lawrence-Varchenko decomposition of $b\Delta_{a-1}$ for q-counting the lattice points.

Expanding the numerator of $\binom{a+b-1}{a-1}$ as $(1-q^{b+1})\cdots(1-q^{b+a-1})$ there are $\binom{a-1}{k}$ terms obtained from choosing 1 from $n-k$ factors and q^M from k factors. Each such term has sign $(-1)^k$, and the exponent of q is slightly larger than kb. We will interpret these $\binom{a-1}{k}$ terms as making up the polarized tangent cone at the kth vertex.

The polarized tangent cone at the kth vertex v_k will not carry the standard q-grading. However, it appears the cone at v_k may be subdivided into $\binom{a-1}{k}$ smaller cones that do have the standard q-grading, essentially by intersecting with the A_{a-1} hyperplane arrangement translated to v_k.

Example 4.5. We illustrate the decomposition of $b\Delta_2$ suggested by

$$\binom{b+2}{2}_q = \frac{1}{(1-q)(1-q^2)}\left(1 - q^{b+1} - q^{b+2} + q^{2b+3}\right)$$



Together with $\mathbf{Cat}_{a,b} = \dim_{\mathbb{C}}(\mathrm{Sym}^b\,\mathbb{C}[\mathbb{Z}_a])^{\mathbb{Z}_a}$, one might hope that we could give $\mathbb{C}[\mathbb{Z}_a]$ a grading so that we have

$$\mathbf{Cat}_{a,b}(q) = \dim_q(\mathrm{Sym}^b\,\mathbb{C}[\mathbb{Z}_a])^{\mathbb{Z}_a}$$

This naive hope does not appear possible. However, we now describe a conjectural weakening of it.

4.4. **Sublattices and shifting.** We begin by rewriting $\mathbf{Cat}_{a,b}(q)$. Since $[a]_{q^k} = (1-q^{ak})/(1-q^k)$, we have

$$\begin{aligned}
\mathbf{Cat}_{a,b}(q) &= \frac{(1-q^{b+1})(1-q^{b+2})\cdots(1-q^{b+a-1})}{(1-q^2)\cdots(1-q^a)} \cdot \frac{[a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}}}{[a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}}} \\
&= \frac{(1-q^{b+1})(1-q^{b+2})\cdots(1-q^{b+a-1})}{[a-1]_{q^a}}[a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}}
\end{aligned}$$

Observe that the fraction is similar to the q^a-count of the lattice points inside a simplex of size b/a, and that the product of $[a]_{q^i}$ is a q analog of a^{a-2}.

4.4.1. This algebraic expression is suggestive of the simplex of (a, b)-cores. The lattice of a-cores is index a within the standard lattice. The sublattice $\Lambda_T = (a\mathbb{Z})^{a-1}$, has index a^{a-1} inside the standard lattice, and hence a^{a-2} within the lattice of a-cores.

The intersection of each coset $\mathfrak{c}$ of Λ_T with the simplex of (a, b)-cores is a $k\Delta_{a-1}$, where k is slightly smaller than b/a, and depends on b and $\mathfrak{c}$.

It appears that $\mathbf{Cat}_{a,b}(q)$ is q^a counting the lattice points in each coset $\mathfrak{c}$, but then shifting the result by a factor of $q^{\iota(\mathfrak{c})}$ for some $\iota(\mathfrak{c})$.

Algebraically, this suggests

Conjecture 4.6. There is an *age* function ι on the cosets $\mathfrak{c} \in \Lambda/\Lambda_T$, so that

$$\sum_{\mathfrak{c}\in\Lambda/\Lambda_T} q^{\iota(\mathfrak{c})} = [a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}}$$

and

$$\mathbf{Cat}_{a,b}(q) = \sum_{\mathfrak{c}\in\Lambda/\Lambda_T} q^{\iota(\mathfrak{c}} \binom{b/a - s(\mathfrak{c}, b) + a - 1}{a-1}_{q^a}$$

where the q^a binomial coefficient q^a-counts the points in $\mathfrak{c} \cap \mathcal{SC}_{a-1}(b)$.

Remark 4.7. We could not find an obvious candidate for an explicit form of ι in general.

Remark 4.8. Conjecture 4.6 was motivated in part by Chen-Ruan cohomology [4, 1], which has found applications to the Ehrhart theory of rational polytopes [13]. Chen-Ruan cohomology $H^*_{CR}(\mathcal{X})$ is a cohomology theory for an orbifold (or Deligne-Mumford stack) $\mathcal{X}$. As a vector space, $H^*_{CR}(\mathcal{X})$ is the usual cohomology of a disconnected space $\mathcal{IX}$. One component C_0 of $\mathcal{IX}$ is isomorphic to $\mathcal{X}$. The other components $C_\alpha, \alpha \neq 0$ are called *twisted sectors* and are (covers of) fixed point loci in $\mathcal{X}$. The pertinent feature for us is that the grading of the cohomology of the twisted sectors are *shifted* by rational numbers, $\iota(\alpha)$, that is

$$H^k_{CR}(\mathcal{X}) = \bigoplus_\alpha H^{k-\iota(\alpha)}(C_\alpha)$$

The function ι is known as the "degree shifting number" or "age".

Orbifolds could potentially be connected to our story through toric geometry, and the well known correspondence between lattice polytopes and polarized toric varieties. When the polytope is only rational, in general the toric variety is an orbifold. The simplex of (a, b)-cores in Λ_a corresponds orbifold $[\mathbb{P}^a/\mathbb{Z}_a]$. More specifically, there is an torus equivariant orbifold line bundle $\mathcal{L}$ over $\mathbb{P}^a/\mathbb{Z}_a$, so that the lattice points in $\mathcal{SC}(a, b)$ correspond to the torus equivariant sections of $\mathcal{L}^b$.

In the fan point of view, the cosets of the lattice correspond exactly to group elements of isotropy groups, and hence to twisted sectors.

This discussion is rather vague, and at this point, there is no concrete connection between $\mathbf{Cat}_{a,b}(q)$ and the geometry of the orbifold $\mathbb{P}^a/\mathbb{Z}_a$ it would be very interesting to find one.

Note that if Conjecture 4.6 holds, it would give another proof, presumably more combinatorial, that $\mathbf{Cat}_{a,b}(q)$ are all positive. Furthermore, with some control on $\iota(\mathfrak{c})$ and $s(\mathfrak{c}, n)$, Conjecture 4.6 suggests:

Conjecture 4.9. For every residue class $r, 0 \leq r < a$, the coefficients of q^{ak+r} in $\mathbf{Cat}_{a,b}(q)$ are unimodal.

4.4.2. *Examples.*

Example 4.10. By expanding both sides, it is straightforward to check the identities

$$\mathbf{Cat}_{3,3k+1}(q) = \binom{k+2}{2}_{q^3} + q^2\binom{k+1}{2}_{q^3} + q^4\binom{k+1}{2}_{q^3}$$
$$\mathbf{Cat}_{3,3k+2}(q) = \binom{k+2}{2}_{q^3} + q^2\binom{k+2}{2}_{q^3} + q^4\binom{k+1}{2}_{q^3}$$

Example 4.11. When $a = 4$ and $b = 4k+1$,

$$\begin{aligned}\mathbf{Cat}_{4,4k+1}(q) = &\binom{k+3}{3}_{q^4} &&+q^4\binom{k+2}{3}_{q^4} &&+q^8\binom{k+2}{3}_{q^4} &&+q^{12}\binom{k+1}{3}_{q^4}\\ &+q^5\binom{k+2}{3}_{q^4} &&+q^9\binom{k+2}{3}_{q^4} &&+q^9\binom{k+1}{3}_{q^4} &&+q^{13}\binom{k}{3}_{q^4}\\ &+q^2\binom{k+2}{3}_{q^4} &&+q^6\binom{k+2}{3}_{q^4} &&+q^6\binom{k+1}{3}_{q^4} &&+q^{10}\binom{k+1}{3}_{q^4}\\ &+q^3\binom{k+2}{3}_{q^4} &&+q^7\binom{k+2}{3}_{q^4} &&+q^{11}\binom{k+1}{3}_{q^4} &&+q^{15}\binom{k+1}{3}_{q^4}\end{aligned}$$

Here, the terms have been grouped so that the coefficients on each line have the same residue mod 4, making it easy to verify the unimodality conjecture.

5. Toward (q, t)-analogs

We now turn toward applying the lattice-point viewpoint toward (q, t)-analog $\mathbf{Cat}_{a,b}(q, t)$, original defined in terms of lattice points, and translated to simultaneous cores in [5].

The (q, t)-rational catalan numbers count simultaneous cores with respect to two statistics, the length and (co)-skew-length. Our main result here is that these statistics are piecewise linear functions on the simplex of (a, b)-cores, and the domains of linearity are essentially chambers of the Catalan arrangement. This suggests that $C_{a,b}(q, t)$ should be expressible as the sum of C_a closed form functions of (q, t), and that a thorough understanding of the geometry of the

Catalan arrangement and its interaction with the lattice of cores could result in a proof of (q,t)-symmetry.

We examine this for $a=3$, and use this to prove $q-t$-symmetry for general (a,b) and low q-degree.

5.1. **Simultaneous cores and (q,t)-rational Catalan numbers.**

Definition 5.1. Let $a<b$ be relatively prime, and λ an (a,b)-core. The *b-boundary* of λ consists of all cells $\square\in\lambda$ with $h(\square)<b$.

We can group the parts of λ into a classes by taking $\lambda_i - i \mod a$; (note, at least one class is empty since λ is an a-core). The *a-parts of* λ consist of the maximal λ_i among each of the i residue classes.

The *skew length of* $\lambda, s\ell(\lambda)$ is the number of cells of λ that are in an a-row of λ and in the b-boundary of λ. The *co-skew-length* $s\ell'(\lambda)$ is $(a-1)(b-1)/2-s\ell(\lambda)$.

Definition 5.2. Let $a<b$ coprime. The (q,t)*-rational Catalan number* is

$$\mathbf{Cat}_{a,b}(q,t)=\sum_{\lambda} q^{\ell(\lambda)}t^{s\ell'(\lambda)}$$

Conjecture 5.3 (Specialization)**.**

$$\sum_{\lambda} q^{\ell(\lambda)+s\ell(\lambda)} = q^{(a-1)(b-1)/2}\mathbf{Cat}_{a,b}(q,1/q)=\mathbf{Cat}_{a,b}(q)$$

Conjecture 5.4 (Symmetry)**.**

$$\mathbf{Cat}_{a,b}(q,t)=\mathbf{Cat}_{a,b}(t,q)$$

5.1.1. *Results.* Our main result is that the statistics ℓ and $s\ell$ in the definition of $\mathbf{Cat}_{a,n}(q,t)$ are piecewise polynomial on the b/a-dilation of the Catalan arrangement (Definition 1.13). More precisely:

Proposition 5.5.

$$\ell(\mathbf{x})=-\frac{a-1}{2}+a\max x_i$$

Proposition 5.6. Let $\lfloor x\rfloor_0=\max(0,\lfloor x\rfloor)$. Then

$$s\ell(\mathbf{x})=\sum_{i,j=0}^{a}\lfloor x_i-x_j\rfloor_0-\lfloor x_i-x_j-b/a\rfloor_0$$

From Propositions **??**, it is immediate that ℓ and $s\ell$ are invariant under the S_a action permuting the coordinates.

5.1.2. As a basic check, we now illustrate that Propositions 5.5 and 5.6 give the correct results for the smallest and large $(a-b)$-cores; we will use these results later.

Example 5.7 (The empty partition)**.** The empty partition corresponds to the vector s; recall $s_i = i/a - (a-1)/(2a)$. The largest of the s_i is $s_{a-1} = (a-1)/(2a)$, and so $\ell(s) = a(a-1)/(2a) - (a-1)/2 = 0$.

Since $s_i - s_{i-1} = 1/a$, we have $s_i - s_j < 1$, and so $\lfloor s_i - s_j \rfloor_0 = 0$. Verifying that $s\ell(s) = 0$.

Example 5.8 (The largest $(a-b)$-core)**.** The largest (a,b)-core λ^M is the one vertex of $\mathcal{SC}_a(b)$ that is in $\Lambda + s$. Its coordinates are some permutation of $bs = (bs_0, bs_1, \cdots bs_{a-1})$, since $s\ell$ is S_a invariant we may assume it is bs.

It is immediate that:

$$\ell(\lambda^M) = -\frac{a-1}{2} + ab\frac{a-1}{2a} = \frac{(a-1)(b-1)}{2}$$

Verifying $s\ell(\lambda^M) = (a-1)(b-1)/2$ is more complicated. We have

$$s\ell(\lambda^M) = \sum_{i<j} \left\lfloor \frac{bj}{a} - \frac{bi}{a} \right\rfloor - \left\lfloor \frac{bj}{a} - \frac{bi}{a} - \frac{b}{a} \right\rfloor$$

The summand depends only on the difference $k = j - i$, and is equal to $\lfloor kb/a \rfloor - \lfloor (k-1)b/a \rfloor$.

There are $(a-1)$ pairs (i,j) with $i - j = 1$, and in general $a - k$ pairs with $i - j = k$, and so we have

$$\begin{aligned}
s\ell(\lambda^M) &= \sum_{k=1}^{a-1} (a-k) \left\lfloor \frac{b}{a} k \right\rfloor - (a-k) \left\lfloor \frac{b}{a}(k-1) \right\rfloor \\
&= \sum_{k=1}^{a-1} \left\lfloor \frac{b}{a} k \right\rfloor \\
&= \sum_{k=1}^{a-1} \frac{b}{a} k - \sum_{k=1}^{a-1} \left\langle \frac{b}{a} k \right\rangle \\
&= \frac{b}{a} \frac{(a-1)a}{2} - \frac{1}{a} \frac{(a-1)a}{2} \\
&= \frac{(a-1)(b-2)}{2}
\end{aligned}$$

where the second line follows from telescoping the sum, the third line applies $\lfloor x \rfloor = x - \langle x \rangle$, and the third line applies $\sum i = n(n+1)/2$ and the fact that, since a and b are relatively prime, kb will take on every residue class mod a exactly once as k ranges from 1 to a.

5.2. **Length and Skew Length are piecewise linear.** In this section we prove Propositions 5.5 and Lemma 5.6.

5.2.1. *Proof of Proposition 5.5 - length is piecewise linear.*

Proof. We first translate $\ell(\lambda^S)$ into fermionic language. Let e be the lowest energy state of S that is not occupied by an electron. Then $\ell(\lambda^S)$ is the number of electrons with energy greater than e.

Recall that the highest energy occupied state on the ith runner is $-ac_i - i - 1/2$, and so the lowest unoccupied state is a higher, and hence $e = \min_i -ac_i - i - 1/2 + a$.

Let m be the runner of the a-abacus that has the lowest unoccupied energy state. For $i \neq m$, there are roughly $c_m - c_i \geq 0$ electrons on the ith runner that have energy great than e. The exact number depends on which of i and m is bigger: if $i < m$, there are exactly $c_m - c_i$ such electrons, while if $i > m$, there are only $c_m - c_i - 1$ such electrons.

There are $a - 1 - m$ runners with $i > m$, and hence we have

$$\begin{aligned}\ell(\mathbf{core}_a(\mathbf{c})) &= -(a-1-m) + \sum_{i \neq m} c_m - c_i \\ &= -(a-1-m) + ac_m\end{aligned}$$

where the second line follows by adding $\sum c_i = 0$ to the expression.

Since $x_i = c_i + i/a - (a-1)/(2a)$, it follows that

$$\ell(\mathbf{core}_a(x)) = -(a-1)/2 + a \max x_i m$$

$\square$

5.2.2. *Proof of Proposition 5.6 - skew length is piecewise linear.*

Definition 5.9. For λ and (a, b)-core, let $s\ell^T_{i,j}(\lambda)$ be the number of cells in the ith a-part with unoccupied state on the jth runner.

Furthermore let $s\ell^S_{ij}(\lambda)$ be the number of such cells with hook length less than b, and $s\ell^B_{ij}(\lambda)$ be the number of such cells with hook length greater than b.

Here, T, S and B are short for *total, small* and *big*.

From Definition 5.9 it is clear that

$$s\ell(\lambda) = \sum_{i \neq j} s\ell^S_{ij}(\lambda)$$

$$s\ell^S_{ij}(\lambda) = s\ell^T_{ij}(\lambda) - s\ell^B_{ij}(\lambda)$$

and so Proposition follows from

Lemma 5.10. Let $\lambda = \mathbf{core}_a(\mathbf{x})$ be an (a, b)-core. Then:

$$\begin{aligned}s\ell^T_{ij}(\lambda) &= \lfloor x_i - x_j \rfloor_0 \\ s\ell^B_{ij}(\lambda) &= \lfloor x_i - x_j - b/a \rfloor_0\end{aligned}$$

Proof. Recalling that cells are in bijection with pairs (e, f), with e, f energy levels, e filled and f empty, we see that $s\ell^T_{ij}$ counts pairs (e, f) with e the highest energy level on the ith runner, f any empty state on the jth runner. Thus, $s\ell^T_{ij}(\lambda)$ is the number of unoccupied states on the jth runner with energy less than e.

Recalling that the highest energy electron on the ith runner has energy $e_i = -ac_i - i - 1/2$, and that the energy of each state to the left increases by a, we have

$$\begin{aligned} s\ell^T_{ij}(\lambda) &= q_a\left(-ac_i - i - 1/2 - (-ac_j - j + -/2)\right) \\ &= q_a(-a(x_i - x_j)) \\ &= \lfloor x_j - x_i \rfloor_0 \end{aligned}$$

For $s\ell^B_{ij}(\lambda)$, we want hooklengths of size at least b, so begin by reducing the energy of the first electron on the ith runner by b. We now want to count ways of moving the resulting electron onto the jth runner, and so by our calculation of $s\ell^T_{ij}(\lambda)$ we immediately have

$$s\ell^B_{ij}(\lambda) = \lfloor x_j - x_i - b/a \rfloor_0$$

$\square$

5.3. **Low degree (q, t)-symmetry.** In this section we show, for all (a, b), that (q, t)-symmetry holds when the degree of one of the monomials are small.

More precisely, we show

Corollary 5.11 (Low degree (q, t)-symmetry)**.**

$$[t^k]\mathbf{Cat}_{a,b}(q, t) = [t^k]\mathbf{Cat}_{a,b}(t, q)$$

for

5.3.1. Let $\mathcal{D}$ denote the dominant chamber

$$\mathcal{D} = \{x \in V_a | x_1 \leq x_1 \leq x_2 \leq \cdots \leq x_a\}$$

Then $\mathcal{D}$ is a fundamental domain for the action of S_a on V_a, and we will use $\mathfrak{c}$ to denote the polyhedron $\mathcal{D} \cap \mathcal{SC}_a(b)$. We will consider the image of $\Lambda_S \cap \mathcal{SC}_a(b)$ under the S_a action as a subset of $\mathfrak{c}$; Since the points of Λ_S have distinct coordinates, each point has a unique representative in $\mathcal{D}$.

Definition 5.12. For $1 \leq i \leq a - 1$, let

$$v_i = \left(\underbrace{\frac{i}{a} - 1, \ldots \frac{i}{a} - 1}_{i \text{ times}}, \underbrace{\frac{i}{a}, \cdots \frac{i}{a}}_{a-i \text{ times}} \right)$$

One can see that the v_i generate the lattice Λ_R, and that locally near x_0, the $\mathfrak{c} = x_0 + \sum t_i v_i$, with $t_i \geq 0$, while near x_∞ we have $\mathfrak{c} = x_\infty - \sum t_i v_i$.

Because ℓ is a linear function on $\mathcal{D}$ it is immediate from the definitions of ℓ and v_i that, for any point $x \in \mathfrak{c}$ we have

$$\ell(x+v_i) = \ell(x) + i$$

Because the difference of two entries of v_i is 0 or 1, we see that $s\ell$ is a piecewise linear function when restricted to elements of the lattice Λ_R.

The dependence of $s\ell$ on v_i depends on which chamber of the Catalan arrangement we are in. Near x_0, we have $s\ell(x) = \sum_{i<j} x_j - x_i$, and so

$$s\ell(x+v_i) = s\ell(x) + i(a-i)$$

However, near x_∞, we have that $x_j - x_i > b$ if $j \neq i+1$, so $s\ell(x) = \sum_i x_{i+1} - x_i = x_a - x_1$, and

$$s\ell(x+v_i) = s\ell(x) + 1$$

This discussion is summarized as follows:

Lemma 5.13. Let $f \in \{\ell, s\ell\}$. For x near x_0, let $\Delta_i f = f(x+v_i) - f(x)$, and near x_∞ let $\Delta_i f = f(x-v_i) - f(x)$. Then:

	$\Delta_i \ell$	$\Delta_i s\ell'$
0	i	$-i(a-i)$
∞	$-i$	1

5.3.2. *Contribution near x_∞.* Near x_∞, the only orbifold lattice points in **core** are the ones near the a vertices of $\mathcal{SC}_a(b)$, and hence here the orbifold lattice Λ_O is just the rotated lattice Λ_R.

Since x_∞ has

$$t^{(a-1)(b-1)/2} \prod_{k=1}^{a-1} \frac{1}{(1-qt^{-k})}$$

5.3.3. *Contribution near 0.* Consider the contribution of points in $\{x_0 + \Lambda_R\} \cap \mathfrak{c}$ near x_0; from Lemma 5.13 and the values of $\ell, s\ell'$ on x_0, we have

$$\sum_{p\in} q^{s\ell'(p)} t^{\ell(p)} = q^{(a-1)(b-1)/2} \prod_{k=1}^{a-1} \frac{1}{(1-t^k q^{-k(a-k)})}$$

.

However, we have not taken account of the orbifold cosets of Λ_R. There are $a!$ chambers of $\mathcal{D}$, and the image of Λ_C under a of them come together to make Λ_R, and so there will be $(a-1)!$ orbifold cosets of Λ_R near 0.

Since $\mathfrak{c}$ is integral at 0 with respect to Λ_R, each orbifold coset γ of Λ_R will have a unique minimal representative x_γ, so that the points in $\gamma \cap \mathfrak{c}$ will be $x_\gamma + (\Lambda_R \cap \mathfrak{c})$, and the contribution near 0 of the points in γ will be the same as that of the points in $\Lambda_R + x_0$

$$q^{s\ell'(x_\gamma)} t^{\ell(x_\gamma)}$$

Thus, low degree symmetry will follow from

Proposition 5.14.

$$\sum_{\gamma \in \mathcal{OC}} q^{s\ell'(x_\gamma)} t^{\ell(x_\gamma)} = \prod_{k=1}^{a-1} [k]_{q^{-(a-k)}t}$$

5.3.4. *Proof of Proposition 5.14.* We factor the proof of Proposition 5.14 into two lemmas; the first establishes a bijection between the orbifold cosets γ and permutations in S_{a-1}, and identifies permutation statistics that correspond to ℓ and $s\ell$; the second shows that these permutation statistics have the proper distribution. Before stating these lemmas, we introduce these permutation statistics, one of which is, to our knowledge, new, and may be of independent interest.

5.3.5. *Permutation Statitistics.* The permutation statistics we need will all be defined in terms of descents and inversions; the following summarizes the standard definitions we will need here:

Definition 5.15. For $\sigma \in S_n$, let

$$\mathbf{DES}(\sigma) = \Big\{ i \in [1, n-1] \Big| \sigma(i) > \sigma(i+1) \Big\}$$

We use $\mathbf{des}(\sigma)$ to denote $|\mathbf{DES}(\sigma)|$, and

$$\mathbf{maj}(\sigma) = \sum_{i \in \mathbf{INV}(\sigma)} i$$

.

Recall that

$$\mathbf{inv}(\sigma) = \big| \big\{ (i,j) \big| 1 \leq i < j \leq n, \sigma(i) > \sigma(j) \big\} \big|$$

Our new statistic is the *size* of σ, written $\mathbf{siz}(\sigma)$:

Definition 5.16.

$$\mathbf{siz}(\sigma) = \left(\sum_{i \in \mathbf{DES}(\sigma)} (n+1-i)i \right) - \mathbf{inv}(\sigma)$$

Our motivation for the definition of **siz** are the following two lemmas, which together immediately prove Proposition 5.14

Lemma 5.17. There is a labeling of the orbifold cosets by partitions $\sigma \in S_{a-1}$, so that if v_σ be the minimum vector in the coset labeled by σ, then:

$$\ell(v_\sigma) = \mathbf{maj}(\sigma)$$
$$s\ell(v_\sigma) = \mathbf{siz}(\sigma)$$

Lemma 5.18.

$$\sum_{\sigma \in S_n} q^{\mathbf{siz}(\sigma)} t^{\mathbf{maj}(\sigma)} = \prod_{k=1}^{n} [k]_{q^{n+1-k}t}$$

Remark 5.19. The name *size* was chosen in reference to the size of a partition: by Lemma 5.18, for fixed k and ℓ, as n grows large the number of permutations $\sigma \in S_n$ with $\mathbf{maj}(\sigma) = \ell$ and $\mathbf{siz}(\sigma) = k$ stabilizes to the number of partitions with length ℓ and size k.

5.4. **Proof of Lemma 5.17.**

5.4.1. *Bijection between S_{a-1} and orbifold cosets.* First, we determine a bijection between orbifold cosets and S_{a-1}.

Let $w \in \Lambda_O \cap \mathfrak{c}$, and define σ^w by

$$\frac{\sigma_i^w}{a} = \langle w_i - w_a \rangle$$

As $w \in S_a\Lambda_C$, we see σ^w is a permutation in S_{a-1}.

Since the entires of the v_i all have the same entries modulo 1, we see that$\sigma^{w+v_i} = \sigma^w$; that is, σ^w is constant on the orbifold cosets.

It is not hard to see that this map is surjective, and hence a bijection between orbifold cosets and S_{a-1}.

5.4.2. *Smallest vector in each coset.* We now describe the minimal element x^σ in the orbifold coset corresponding to σ.

Being the minimal vector x^σ in a coset means that $x^\sigma - v_i \notin \mathcal{D}$ for all i, which is equivalent to

$$x_i^\sigma + 1 > x_{i+1}^\sigma, \quad 1 \leq i \leq a-1$$

To find x^σ we will first define a vector w^σ satisfying

$$w_i^\sigma < w_{i+1}^\sigma < w_i^\sigma + 1$$

$$\langle w_i - w_a \rangle = \frac{\sigma_i}{a}$$

but does not satisfy $\sum w_i^\sigma = 0$, we will then subtract the approproiate multiple of $(1/a, \ldots, 1/a)$ to get v^σ.

We need $w_{i+1}^\sigma > w_i^\sigma$ and $\langle w_{i+1}^\sigma - w_i^\sigma \rangle = \langle \sigma_{i+1}/a - \sigma_i/a \rangle$, and so we set

$$w_{i+1}^\sigma = w_i^\sigma + \frac{\sigma_{i+1} - \sigma_i}{a} + \mathbf{des}_i(\sigma)$$

where we have conventionally set $w_0^\sigma = \sigma_0 = 0, \sigma_a = a$.

Then

$$x_i^\sigma = w_i^\sigma - \frac{1}{a}\sum_{j=1}^{a} w_j^\sigma$$

is the minimal vector in the orbifold coset labeled by σ.

5.4.3. *Simplification.* To find $\ell(x^\sigma)$ and $s\ell(x^\sigma)$, we will want to simplify our expression for x_i^σ. The following definition will help.

Definition 5.20. For $i < j$, define $\mathbf{des}_{ij}$ to be the number of descents between i and j. That is:

$$\mathbf{des}_{ij}(\sigma) = \left|\left\langle k \in \mathbf{DES}(\sigma)\middle| i \le k < j\right\}\right| = \sum_{k=i}^{j-1} \mathbf{des}_k(\sigma)$$

With this definition,

$$w_j = \frac{\sigma_j}{a} + \mathbf{des}_{1,j}(\sigma)$$

and so

$$\begin{aligned}\sum_{j=1}^{a} w_j &= \frac{1}{a}\sum_{i=1}^{a}\sigma_i + \sum_{i=1}^{a}\mathbf{des}_{1,i} \\ &= \frac{a+1}{2} + \sum_{i=1}^{a-2}(a-i)\mathbf{des}_i(\sigma)\end{aligned}$$

Thus,

$$x_j^\sigma = \frac{\sigma_j}{a} + \mathbf{des}_{1j}(\sigma) - \frac{a+1}{2a} - \frac{1}{a}\sum_{i=1}^{a-2}(a-i)\mathbf{des}_i(\sigma)$$

5.4.4. *Length of x^σ.* We compute (recalling the convention $\sigma_a = a$):

$$\begin{aligned}\ell(x^\sigma) &= a x_a^\sigma - \frac{a-1}{2} \\ &= a + a\sum_{i=1}^{a-2}\mathbf{des}_i(\sigma) - \frac{a+1}{2} - \sum_{i=1}^{a-2}(a-i)\mathbf{des}_i(\sigma) - \frac{a-1}{2} \\ &= \sum_{i=1}^{a-2} i\,\mathbf{des}_i(\sigma) \\ &= \mathbf{maj}(\sigma)\end{aligned}$$

5.4.5. *Skew length of x^σ.* We have

$$\begin{aligned}s\ell(x^\sigma) &= \sum_{1\le i<j\le a}\left\langle v_j^\sigma - v_i^\sigma\right\rangle \\ &= \sum_{1\le i<j\le a}\left\langle \frac{\sigma_j-\sigma_i}{a} + \mathbf{des}_{ij}(\sigma)\right\rangle \\ &= \sum_{1\le i<j\le a}\mathbf{des}_{ij}(\sigma) - \delta(\sigma_j < \sigma_i)\end{aligned}$$

Observe

$$\sum_{1\le i<j\le a}\delta(\sigma_j<\sigma_i) = \mathbf{inv}(\sigma).$$

and

$$\sum_{1\le i<j\le a} \mathbf{des}_{ij}(\sigma) = \sum_{k=1}^{a-2} k(a-k)\mathbf{des}_k(\sigma)$$

since for $\mathbf{des}_k$ to appear in $\mathbf{des}_{ij}$ we need $1 \le i \le k$ and $j < k \le a$, and so $\mathbf{des}_k$ appears in $k(a-k)$ different $\mathbf{des}_{ij}$.

Thus, we have shown

$$s\ell(x^\sigma) = \sum_{k=1}^{a-2} k(a-k)\mathbf{des}_k(\sigma) - \mathbf{inv}(\sigma) = \mathbf{siz}(\sigma)$$

□

5.5. **Proof of Lemma 5.18.** Before we prove Lemma 5.18, we introduce a family of codes for permuations that we call *factorization codes*; we will use a specific factorization code (the left-decreasing factorization code.

Definition 5.21. A *valid sequence of length n* is a sequence of integers $a_i, 1 \le i \le n$ such that $0 \le a_i < i$. Let $\mathbf{VS}_n$ denote the set of valid sequences; clearly $|\mathbf{VS}| = n!$.

A *permutation code* is a bijection $\phi : \mathbf{VS}_n \to S_n$.

In section 5.5.1 we introduce a family of permutation codes we call *factorization codes*; in particular, this family includes the *left-decreasing factorization code* **LD**.

Lemma 5.18 then reduces to showing:

Lemma 5.22. For a valid sequence $a \in \mathbf{VS}_n$, we have:

$$\mathbf{maj}(LD(a)) = \sum a_i$$
$$\mathbf{siz}(LD(a)) = \sum (n+1-i)a_i$$

5.5.1. *Factorization codes.* Factorization codes rest on the following simple observation. Let $C_k \in S_k$ be any k-cycle. Then $\{C_k^i\}, 0 \le i < k$ form a family of representatives for the (left or right) cosets of $S_{k-1} \subset S_k$.

Definition 5.23. A *family C of k-cycles* is a sequence $C_k, k \in \mathbb{N}$, with $C_k \in S_K$ a k-cycle.

The *right factorization* code associated to a family of k-cycles C_k is the sequence of maps $R_n^C : \mathbf{VS}_n \to S_n$ defined by

$$R_n(a) = \alpha_k = C_2^{a_2} C_3^{a_3} \cdots C_n^{a_n}$$

Similarly, the *left factorization* code associated to a family of k-cycles C_k is the the sequence of maps $L_n^C : \mathbf{VS}_n \to S_n$ defined by

$$L_n^C(a) = C_n^{a_n} C_{n-1}^{a_{n-1}} \cdots C_2^{a_2}$$

That the left and right factorization codes are in fact permutation codes follows easily from the observation using induction on n.

There are two "obvious" families of k-cycles: *increasing* cycles $C_k^+ = (1, 2, 3, \ldots, k)$, and the *decreasing* cycles $C_k^- = (k, k-1, k-2, \ldots, 1)$.

Thus, the left-decreasing factorization code L_n^- is the bijection that sends $0 \leq a_i < i$ to to

$$L_n^-(a) = (C_n^-)^{a_n}(C_{n-1}^-)^{a_{n-1}} \ldots (C_2^-)^{a_2}$$

5.5.2. *Multiplication by C_k^-.* We now inductively prove Lemma 5.22 giving **maj** and **siz** of a permutation in terms of its left decreasing factorization code.

Clearly Lemma 5.22 holds on the identity permutation, where all $a_i = 0$. Thus we must show that in such a factorization, multiplying by C_k^- raises **maj** by one and **siz** by $(n+1-k)$.

To do this, we must determine what multiplication by C_k^- does to the set **DES** of descents. When multiplying by C_k^-, we have not yet permuted the elements $(k+1), (k+2), \ldots, n$, and so $\mathbf{DES} \subset \{1, \ldots, k-1\}$. As C_k decreases $2, \ldots, j$ by 1, any comparisons involving two of these elements will remain unchanged; hence, the only descents multiplying by C_k^- could change are those involving 1, which it will change to k.

Suppose that in the one-line notation of σ the 1 is in position j; then $j-1$ will be a descent (unless $j = 1$), and j will not be a descent. After we have multiplied by c_k, the 1 will change to a k, and so $j-1$ will not be a descent, and j will be.

Thus, multiplying by C_j will either increase a descent by one, or create a new descent at 1. In either case, the major index will increase by one.

We now investigate the effect of multipication by C_k on **siz**, supposing that 1 is in position j. We first determine the change in the first term in **siz** (the sum over descents), and then determine the change this makes to the second term **inv**.

A descent at $j-1$ contributes

$$(n+1-(j-1))(j-1) = nj - j^2 + 3j - 2$$

to **siz**; a descent at j contributes

$$(n+1-j)j = nj - j^2 + j$$

and thus multiplying by C_k^- when 1 is in position $j < k$ will increase the first term of **siz** by $2 - 2j$.

We now turn to the inversions. It is clear that the only inversions that will change are those that were comparing 1. Before we multiply by C_k^-, 1 is in position j, and the $j-1$ pairs $(i, j), 1 \leq i \leq j-1$ will be inversions, and none of the $k-j$ pairs $(j, \ell), j+1 \leq \ell \leq k$ will be inversions. After we multiply by C_k^-, position j will be k; none of the pairs (i, j) will be inversions, and all of the pairs (j, ℓ) will be. Thus, **inv** increases by $k - 2j + 1$.

Thus, multiplying by C_k^- when 1 is in position $j < k$ will change **siz** by

$$n - 2j + 2 - (k - 2j + 1) = n - k + 1$$

as desired.

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