Chambers of linearity It is clear from the definitions that $_{i,j}$ and $^{b}_{i,j}$, and hence and ^{b}, are piecewise linear functions of c. We now examine chamber structure and dependence of these functions.

Recall that the chambers of the A_a arrangement are labeled by permutations $\sigma \in S_a$ that determine the ordering of the c_i. More explicitly, c is in the interior of chamber σ if and only if $c_{\sigma(0)} < c_{\sigma(1)} < \cdots < c_{\sigma_{a-1}}$.

Recall that for $\sigma \in S_a$, the number of inversions of σ is

$$(\sigma) = \#\{(i,j) | 0 \leq i < j \leq (a-1), \sigma(i) > \sigma(j)\}$$

.

Then for c in the interior of chamber σ, we have align* (c)= $\sum_{i,j=0}^{a-1} \max(c_j - c_i - \delta_{j<i}, 0)$

Where the second equality follows because i is greater than i elements in $\{0, 1, \ldots, a-1\}$ and less than $(a-1-i)$ elements, and thus c_{σ_i} appears with a positive sign i times in $\sum c_{\sigma(j)} - c_{\sigma(i)}$ times and a negative sign $(a-1-i)$ times.

lemma In terms of the x-coordinates, we would expect this to be closely related to $\sum_{i<j} |x_i - x_j|$.

If $x \in C_\sigma$, we compute align* $\sum_{i<j} |x_i - x_j| = \sum_{i<j} x_{\sigma(j)} - x_{\sigma(i)}$

The second term depends only on the partition σ.

definition[SU] Let

$$(\sigma) = \{(i,j) | i < j, \sigma(i) > \sigma(j)\}$$

be the set of inversions of σ.

Define the inversion sum of (σ) to be

$$(\sigma) = \sum_{(i,j) \in (\sigma)} \sigma(j) - \sigma(i) = \sum_{(i,j) \in (\sigma)} (j - i)$$