# LATTICE POINTS AND SIMULTANEOUS CORE PARTITIONS

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Abstract. We observe that for $a$ and $b$ relatively prime, the "abacus construction" identifies the set of simultaneous $(a, b)$-core partitions with lattice points in a rational simplex. Furthermore, many statistics on $(a, b)$-cores are piecewise polynomial functions on this simplex.

We apply these results to rational Catalan combinatorics. Using Ehrhart theory, we reprove Anderson's theorem [3] that there are $(a + b - 1)!/a!b!$ simultaneous $(a, b)$-cores, and using Euler-Maclaurin theory we prove Armstrong's conjecture [8] that the average size of an $(a, b)$-core is $(a + b + 1)(a - 1)(b - 1)/24$. Our methods also give new derivations of analagous formulas for the number and average size of self-conjugate $(a, b)$-cores.

We conjecture a unimodality result for $q$ rational Catalan numbers, and make preliminary investigations in applying these methods to the $(q, t)$-symmetry and specialization conjectures. We prove these conjectures for low degree terms and when $a = 3$, connecting them to the Catalan hyperplane arrangement and proving an apparently new result about permutation statistics along the way.

## 1. Introduction

This paper establishes lattice point geometry as a foundation for the study of simultaneous core partitions, and, more generally, rational Catalan combinatorics.

Rational Catalan numbers, and their $q$ and $(q, t)$ analogs, are a natural generalization of Catalan numbers that, apart from their intrinsic combinatorial interest, appear in the study of Hecke algebras [12] and compactified Jacobians of singular curves [13, 14]. It is a theorem of Anderson that simultaneous core partitions are counted by rational Catalan numbers.

Our first result is to give a new proof of Anderson's theorem by identifying simultaneous core partitions with lattice points in a rational simplex. After this identification is made, many other results follow quite naturally.

### 1.1. Background: Simultaneous cores and rational Catalan numbers.

A *partition of $n$* is a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq \lambda_k > 0$ of positive integers so that $\sum_{i=1}^{k} \lambda_i = n$. We call $n$ the *size* of the partition and denote it by $|\lambda|$; we call $k$ the *length* of $\lambda$ and denote it by $\ell(\lambda)$.

1.1.1. *Hooks and Cores.* We frequently identify $\lambda$ with its Young diagram, in English notation – that is, we draw the parts of $\lambda$ as the columns of a collection of boxes.

**Definition 1.1.** The *arm* $a(\square)$ of a cell $\square$ is the number of cells contained in $\lambda$ and above $\square$, and the *leg* $l(\square)$ of a cell is the number of cells contained in $\lambda$ and to the right of $\square$.

The *hook length* $h(\square)$ of a cell is $a(\square) + l(\square) + 1$.

**Example 1.2.** The cell $(2, 1)$ of $\lambda = 3 + 2 + 2 + 1$ is marked $s$; the cells in the leg and arm of $s$ are labeled $a$ and $l$, respectively.

| | | | |
|---|---|---|---|
| | | | |
| | $a$ | | |
| | $s$ | $l$ | $l$ |

$$a(s) = \#a = 1$$
$$l(s) = \#l = 2$$
$$h(s) = 4$$

We now introduce our main object of study.

**Definition 1.3.** An *a-core* is a partition that has no hook lengths of size $a$. An $(a, b)$*-core* is a partition that is simultaneously an $a$-core and a $b$-core.

**Example 1.4.** We have labeled each cell $\square$ of $\lambda = 3 + 2 + 2 + 1$ with its hook length $h(\square)$.

| | | | |
|---|---|---|---|
| 1 | | | |
| 4 | 2 | 1 | |
| 6 | 4 | 3 | 1 |

We see that $\lambda$ is *not* an $a$-core for $a \in \{1, 2, 3, 4, 6\}$; but it *is* an $a$-core for all other $a$.

1.1.2. *Rational Catalan numbers.* Recall that the Catalan number $\mathbf{Cat}_n = \frac{1}{2n+1}\binom{2n+1}{n}$. Catalan numbers count hundreds of different combinatorial objects; for example, the number of lattice paths from $(0, n)$ to $(n + 1, 0)$ that stay strictly below the line connecting these two points.

Rational Catalan numbers are a natural two parameter generalization of $\mathbf{Cat}_n$.

**Definition 1.5.** For $a, b$ relatively prime, the *rational Catalan number*, or $(a, b)$ *Catalan number* $\mathbf{Cat}_{a,b}$ is

$$\mathbf{Cat}_{a,b} = \frac{1}{a+b}\binom{a+b}{a}$$

The rational Catalan number $\mathbf{Cat}_{a,b}$ counts the number of lattice paths from $(0, a)$ to $(b, 0)$ that stay beneath the line from $(0, a)$ to $(b, 0)$. This is consistent with the specialization $\mathbf{Cat}_{n,n+1} = \mathbf{Cat}_n$.

1.2. Simultaneous cores and rational Catalan numbers are connected by:

**Theorem 1.6** (Anderson [3])**.** *If $a$ and $b$ are relatively prime, the number of $(a, b)$-core partitions is $\mathbf{Cat}_{a,b}$.*

Our main result is a new proof of Theorem 1.6 using the geometry of lattice points in rational polyhedra. This new viewpoint easily extends to prove other results; chief among them a proof of Armstrong's conjecture:

**Theorem 1.7.** The average size of an $(a,b)$-core is $(a+b+1)(a-1)(b-1)/24$.

**Remark 1.8.** Theorem 1.7 Armstrong conjectured Theorem 1.7 in 2011; it first appeared in print in [8].

Stanley and Zanello [16] have proven the Catalan case ($a = b+1$) of Armstrong's conjecture by different methods, and building on their work Aggarwal [2] has proven the case $a = mb+1$.

The two main tools in the proofs of the theorem are the abacus construction and Ehrhart theory. We briefly recall these ideas before giving a high-level overview of the proof.

1.2.1. *Abaci.* The main tool used to study $a$-cores is the "abacus construction". We review this construction in detail in Section 2. For now, we observe that there are at least two variants of the abacus construction in the literature.

The first construction, which we call the *positive abacus*, gives a bijection between $a$ core partitions and $\mathbb{N}^{a-1}$. Anderson's original proof used the positive abacus as part of a bijection between $(a,b)$-cores and $(a,b)$-Dyck paths, which were already known to be counted by $\mathbf{Cat}_{a,b}$.

We will only make use of the second construction, which we call the *signed abacus*. The signed abacus is a bijection between $a$-core partitions and points in the $a-1$ dimensional lattice

$$\Lambda_a = \left\{ c_1, \ldots, c_a \in \mathbb{Z} \middle| \sum c_i = 0 \right\}$$

We give a detailed review of the signed abacus in Section 2. For now we just mention one result that is key to our proof of Armstrong's conjecture:

**Theorem 1.9.** Under the signed abacus bijection, the size of an $a$-core is given by the quadratic function

$$Q(c_1, \ldots, c_a) = \frac{a}{2} \sum c_i^2 + \sum i c_i$$

We prove this as Theorem 2.10.

1.2.2. *Ehrhart / Euler-Maclaurin.* The number of lattice points in a polytope can be viewed as a discrete version of the volume of a polytope. Ehrhart theory is the study of this analogy. A gentle introduction to Ehrhart theory may be found in [5].

Let $V$ be an $n$ dimensional real vector space, and $\Lambda \subset V$ an $n$ dimensional lattice. Conretely, $\Lambda = \mathbb{Z}^n, V = \mathbb{R}^n$. A lattice polytope $P \subset V$ is a polytope all of whose vertices are points of $\Lambda$.

For $t$ a positive integer, let $tP$ denote the $t$th dilate of $P$, the polytope obtained by scaling $P$ by $t$. For $t \geq 0$, define $L(P,t)$ to the number of lattice points in $tP$:

$$L(P,t) = \#\{\Lambda \cap tP\}$$

Clearly, the volume of $tP$ is $t^n$ times the volume of $P$. Ehrhart showed that, in parallel to this fact, $L(P,t)$ is a degree $n$ polynomial in $t$. Ehrhart theory refers to the study of these polynomials.

Other than the fact that $L(P,t)$ is a polynomial of degree $n$, the one fact from Ehrhart theory we will use is Ehrhart reciprocity. If we scale a polytope by a negative number, then keeping track of orientation the volume changes by $(-t)^n$. The polynomial $L(P,t)$ will not be even or odd, and so $L(P,-t)$ cannot be $(-1)^n$ times the number of lattice points in $-P$. Ehrhart reciprocity states that instead, $L(P,-t)$ is $-(1)^n L(P^\circ,t)$, where $P^\circ$ here denotes the interior of $P$.

The results of Ehrhart theory extend to an analogy between integrating a polynomial over a region and summing it over the lattice points in a polytope. This is an extension of the familiar "sum of the first $n$ cubes" type formulas.

More specifically, if $f$ is a polynomial of degree $d$ on $V$, then we have that $\int_{tP} f$ is a polynomial of degree $d+n$. Euler-Maclaurin theory says that the discrete analog

$$L(f,P,t) = \sum_{x\in\Lambda\cap tP} f(x)$$

is also a polynomial of degree $d+n$. Ehrhart reciprocity also extends:

$$L(f,P,-t) = (-1)^n L(f,-P^\circ,t)$$

1.2.3. *Initial motivation.* To explain the method used to prove Theorems 1.6 and 1.7, we begin with the following

**False Hope 1.** Fix $a$. Under the signed abacus construction, the set of $(a,b)$-cores are exactly those lattice points in $bP$, for some integral polytope $P \subset V_a$.

If the false hope were true, Ehrhart theory would imply that, for $b$ relatively prime to a fixed $a$, $|\mathcal{C}_{a,b}|$ would be a polynomial of degree $a-1$ in $b$. It is clear from the definition that this polynomiality property holds for $\mathbf{Cat}_{a,b}$. Thus, proving Anderson's theorem for a fixed $a$ reduces to showing that two polynomials are equal, which only requires checking finitely many values.

Furthermore, it is known that the size of an $a$-core is a quadratic function $Q$ on the lattice. Thus, if False Hope were true Euler-Maclaurin theory would give that the total size of all $(a,b)$ cores was a polynomial of degree $a+1$ in $b$ and again we could hope to exploit this polynomiality in a proof.

Note that since the degree of the polynomial $\mathbf{Cat}_{a,b}$ grow as $b$ grows, proving Anderson and Armstrong's conjecture as a whole by this method would still requires checking infinitely many values.

1.2.4. The False Hope is not quite true, but the strategy outlined above is essentially the one we follow. The set of $b$ cores inside the lattice of $a$ cores *does* form a polytope (a simplex, actually), which we will call $\mathbf{SC}_a(b)$ for *Simplex of Cores*.

One minor tweak needed to the False Hope is that as we vary $b$ $\mathbf{SC}_a(b)$ is not only scaled, but also changed by a linear transformation. These transformations preserve the number of lattice points and the quadratic function $Q$ giving the size of the partitions, and so do not pose any real difficulties.

More troubling is that the polytope $\mathbf{SC}_a(b)$ is not integral, but only rational. Recall that a polytope $P$ is *rational* if there is some $k \in \mathbb{Z}$ so that $kP$ is a lattice polytope.

1.2.5. *Rational Polytopes and quasipolynomials.* Ehrhart and Euler/Maclaurin theory can be extended to rational polytopes at the cost of replacing polynomials by *quasipolynomial*.

**Definition 1.10.** A function $f : \mathbb{Z} \to \mathbb{C}$ is a quasipolynomial of degree $d$ and period $n$ if there exist $n$ polynomials $p_0, \dots, p_{n-1}$ of degree $d$, so that for $x \in k + n\mathbb{Z}$, we have $f(x) = p_k(x)$.

**Example 1.11.** Let $P$ be the polytope $x, y \geq 0, 2x + y \leq 1$. Then

$$\#\{tP \cap \mathbb{Z}^2\} = \begin{cases} \frac{t^2+4t+4}{4} & t \text{ even} \\ \frac{t^2+4t+3}{4} & t \text{ odd} \end{cases}$$

Since $\mathbf{Cat}_{a,b}$ is defined only for $a$ and $b$ relatively prime, it fits nicely into the quasipolynomial framework. For $a$ fixed, and $b$ in a fixed residue class mod $a$, $\mathbf{Cat}_{a,b}$ is a polynomial. It just so happens that residue classes relatively prime to $a$ have identical polynomials. 7 Such "accidental" equalities between the polynomials for different residue classes happen frequently in Ehrhart theory, but are mysterious in general. Perhaps the most studied manifestation of this is *period collapse* (see [15] and references), where the quasipolynomial is in fact a polynomial. In our case, symmetry considerations give an elementary explanation of the "accidental" equalities between the polynomials for different residue classes.

1.2.6. In Lemma 3.5 we show that the the polyhedron $\mathbf{SC}_a(b)$ is isomorphic to a rational simplex we call $\mathbf{TD}_a(b)$ (for *Trivial Determinant*) that we now describe. Let $L_k$ be the one dimensional representation of $\mathbb{Z}_a$ where $1 \in \mathbb{Z}_a$ acts as $\exp(2\pi i k/a)$. Then any $b$ dimensional representation $V$ of $\mathbb{Z}_a$ may be written as

$$V = \bigoplus_{k=0}^{a-1} L_k^{\oplus z_k}$$

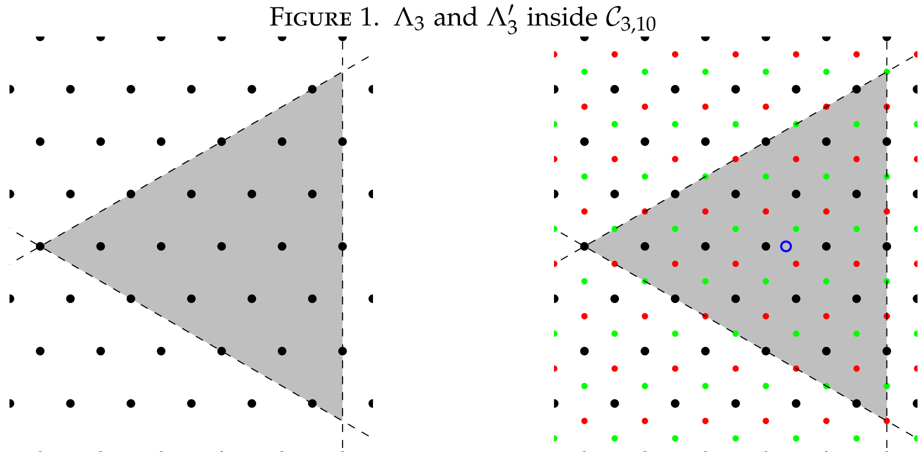
for nonnegative integers $z_i$ satisfying $\sum z_i = b$. Thus, the set of all $b$ dimensional representations of $\mathbb{Z}_a$ is isomorphic to the standard simplex $b\Delta_a$, which has $\binom{a-1+b}{b}$ lattice points.

The simplex $\mathbf{TD}_a(b)$ is obtained by considering only those representations that have trivial determinant (i.e., $\wedge^b V \cong L_0$), or equivalently restricting to the index $a$ sublattice of $z_i$ satisfying $\sum i z_i = 0 \pmod{a}$.

However, we could just as well have considered the set of representations with determinant isomorphic to $L_k$ for any $k$. Tensoring $V$ by $L_1$ corresponds to the cyclic permutation of coordinates $z_k \mapsto z_{k+1}$, and changes the determinant of $V$ by tensoring by $L_b$ (where we are using periodic indices). Thus, the dual $\mathbb{Z}_a$ acts on the set of all $b$ dimensional representations, and when $b$ is relatively prime to $a$ this action is free, and each orbit contains exactly one representation with trivial determinant. Hence, the number of points in $\mathbf{TD}_a(b)$ is exactly one $a$th of the number of points in $b\Delta_a$, namely $\binom{a-1+b}{b}/a = \mathbf{Cat}_{a,b}$.

The situation is illustrated in Figure 1. The left hand picture shows $\mathbf{TD}_3(10) \cong \mathbf{SC}_3(10)$, while the right hand picture shows the standard simplex $10\Delta_3$. The black dots are the representations with trivial determinant, while the red and green dots are those representations with determinant $L_1$ and $L_2$. Rotating about the blue circle by 120 degrees corresponds to tensoring by $L_1$ and permutes the different colored dots.

FIGURE 1. $\Lambda_3$ and $\Lambda'_3$ inside $\mathcal{C}_{3,10}$



Thus, we see that the rational Catalan number $\mathbf{Cat}_{a,b}$ counts the isomorphism classes of $b$-dimensional representations of $\mathbb{Z}_a$ with determinant 1, or equivalently, $\mathbf{Cat}_{a,b} = \dim_{\mathbb{C}} \operatorname{Sym}^b(\mathbb{C}[\mathbb{Z}_a])^{\mathbb{Z}}_a$. Furthermore, the identification of $\mathbf{SC}_a(b)$ with $\mathbf{TD}_a(b)$ proves Anderson's theorem.

1.2.7. *Self-conjugate simultaneous cores.* The lattice point technique easily adapts to treat the case of self-conjugate simultaneous cores. Ford, Mai and Sze have shown [10] that self-conjugate $(a, b)$-core partitions are counted by

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor}$$

Armstrong conjectured, and Chen, Huang and Wang recently proved [7], that the average size of self-conjugate $(a, b)$-core partitions is the same as the average size of all $(a, b)$-core partitions, namely $(a + b + 1)(a - 1)(b - 1)/24$.

In Section 3.3 we give new proofs of both of these results. A key idea is that the action of conjugation on $\mathbf{SC}_a(b)$ corresponds to the action of taking dual representations on $\mathbf{TD}_a(b)$.

1.3. $q$ **and** $(q,t)$**-analogs.** The second half of the paper, namely Sections 4 and 5, addresses $q$ and $(q,t)$ analogs of $\mathbf{Cat}_{a,b}$.

1.3.1. *q-analogs.* The $q$-rational Catalan numbers $\mathbf{Cat}_{a,b}(q)$ are defined by the obvious $q$-analog of $\mathbf{Cat}_{a,b}$:

$$\mathbf{Cat}_{a,b}(q) = \frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

It is nontrivial that the coefficients of $\mathbf{Cat}_{a,b}(q)$ are positive. The main question we pursue in Section 4 is whether our lattice point view can shed any light on this positivity question.

An obvious hope is that $\mathbf{Cat}_{a,b}(q)$ is a sum over the lattice points in $\mathbf{SC}_{a,b}$ of $q^L$, where $L$ is some linear function. This does not appear to be true. However, we conjecture that there is an index $a^{a-2}$ sublattice $\Lambda'$ of the lattice of cores, and a function $\iota$ on the cosets $\mathfrak{c}$ of $\Lambda'$, so that if $\mathbf{Cat}_{a,b}(q)$ is the sum over the lattice points in $\mathbf{SC}_{a,b}$ of $q^{L+\iota}$; this would give an expression for $\mathbf{Cat}_{a,b}(q)$ as a sum of $a^{a-2}$ terms of the form

$$q^{\iota(\mathfrak{c})}\begin{bmatrix} S(\mathfrak{c}) \\ a-1 \end{bmatrix}_{q^a}$$

which would explain the positivity of the coefficients of $\mathbf{Cat}_q(a,b)$.

Furthermore, this conjectural formula leads naturally to a unimodality conjecture about $\mathbf{Cat}_{a,b}(q)$. Recall that a sequence $a_1, dots, a_n$ is *unimodal* if there is some $k$ so that

$$a_1 \leq a_2 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k-1} \geq a_{k-2} \cdots \geq a_n$$

The coefficients of $\mathbf{Cat}_{a,b}$ are not unimodal. However, we conjecture that, if we fix $0 \leq k < a$, and look only at the coefficients of $\mathbf{Cat}_{a,b}(q)$ of the form $q^{an+k}$, the resulting sequences are unimodal.

1.3.2. The $(q,t)$-rational Catalan numbers were original defined in terms of summing certain statistics over lattice paths, but Armstrong, have given an analogous formula over simultaneous core. In particular, let $\mathbf{SC}_{a,b}$ denote the set of all $(a,b)$-cores,

$$\mathbf{Cat}_{a,b}(q,t) = \sum_{\lambda \in \mathbf{SC}_{a,b}} q^{\ell(\lambda)} t^{s\ell'(\lambda)}$$

where $\ell(\lambda)$ is the length of $\lambda$, and $s\ell'(\lambda)$ is the *co-skew length*, a statistic introduced

Our main result in the second half of the paper is that $\ell$ and $s\ell'$ are piecewise linear functions on $\Delta_{a,b}$, and thus that lattice point geometry should be useful in these studying $\mathbf{Cat}_{a,b}(q,t)$ and $\mathbf{Cat}_{a,b}(q)$. In particular, the piecewise linearity implies that for fixed $a$, and for $b$ in a fixed residue class $k \pmod a$, $\mathbf{Cat}_{a,b}(q,t)$

has a uniform formula as a rational function. We explicitly compute this rational function when $a = 3$; let $b = 3k + 1 + \delta$, where $\delta \in \{0, 1\}$; then:

$$\mathbf{Cat}_{a,b}(q,t) = \frac{t^{3k+\delta}}{(1-qt^{-1})(1-q^2t^{-1})} + \frac{q^k t^{k+1} + q^{k+\delta}t^{k+\delta} + q^{k+1}t^k}{(1-q^{-1}t^2)(1-q^2t^{-1})} + \frac{q^{3k+\delta}}{(1-tq^{-1})(1-t^2q^{-1})}$$

This provides an avenue of attack toward the symmetry and specialization conjectures about $\mathbf{Cat}_{a,b}(q,t)$. The symmetry conjecture states that $\mathbf{Cat}_{a,b}(q,t)$ is symmetric in $q$ and $t$, and and the specialization conjecture states

$$q^{(a-1)(b-1)/2}\mathbf{Cat}_{a,b}(q,1/q) = \mathbf{Cat}_{a,b}(q)$$

Using lattice point methods, we prove these conjectures for the high and low degree terms of $\mathbf{Cat}_{a,b}$, independent of $a$ and $b$.

We also derive an explicit formula for $\mathbf{Cat}_{3,b}(q,t)$ as a rational function, from which it is easy to prove both conjectures when $a$ is 3.

for $a = 3$ and any $b$.

Second, it turns out that the chambers of linearity for $\ell$ and $s\ell'$ are interesting hyperplane arrangements – the walls of $\ell$ are the walls of the $A_{n-1}$ arrangement, and the walls of $s\ell'$ are the walls of a deformation of the $A_{n-1}$ arrangement known as the Catalan arrangement. This gives a connection between $\mathbf{Cat}_{a,b}(q,t)$ and $\mathbf{Cat}_a$, which we don't believe was well known. Furthermore, it suggests that a thorough understanding of the geometry of the Catalan arrangement could lead to a full proof of the specialization and symmetry conjectures.

Finally, in order to prove the specialization and symmetry conjectures for low degree terms we are lead to define and investigate a new permutation statistic $\mathbf{siz}(\sigma)$.

**Lemma 1.12.**

$$\sum_{\sigma \in S_n} q^{\mathbf{siz}(\sigma)} t^{\mathbf{maj}(\sigma)} = \prod_{k=1}^{n} [k]_{q^{n+1-k}t}$$

**Definition 1.13.** The $A_{a-1}$ hyperplane arrangement is the set of the $\binom{a}{2}$ hyperplanes $x_i = x_j$ in the $a-1$ dimensional vector space $\sum x_i = 0$.

There are $n!$ regions of the $A_{a-i}$ region, which are indexed by partitions $\sigma$; the region indexed by $\sigma$ is where $x_{\sigma(0)} < z_{\sigma(1)} < \cdots < x_{\sigma(a)}$.

Another hyperplane arrangment that is pertinent is the Catalan arrangement, which is a deformation of the $A_a$ arrangement.

**Definition 1.14.** A hyperplane arrangement $\mathcal{A}'$ is a *deformation* of an arrangement $\mathcal{A}$ if every hyperplane in $\mathcal{A}'$ is parallel to one in $\mathcal{A}$.

**Definition 1.15.** The *Catalan arrangement* $\mathcal{C}_a$ is the union of the $3\binom{a}{2q}$ hyperplanes $x_i - x_j \in \{-1, 0, 1\}, i < j$.

The name *Catalan arrangement* comes from the fact that $\mathcal{C}_a$ has $n!C_n$ regions.

We have already seen the hyperplanes in the Catalan arrangement appearing, if $b\mathcal{C}_n$ denotes the Catalan arrangement scaled by $b$ (so $x_i - x_j \in \{-b, 0, b\}$), then the hyperplanes that define the simplex of $b$-cores are in $b\mathcal{C}_n$.

We now give an informal discussion of how $\mathbf{core}_a(\mathbf{x})$ depends on the chamber of $\mathcal{A}_a$.

**Example 1.16.** Consider the lattice path of a large random $a$-core. At the start, every segment of the path slopes down; then there is a section where one out of every $a$ segments slopes up; then another large section where two out of every $a$ slope up, then 3 out of every $a$ steps slope up, until eventually the path hits the $x$-axis, from which point every step slopes up.

In the first sections, all steps that slope up correspond to electrons on the same runner $i$ of the $a$-abacus. The $i$ that occurs is the one with $x_i$ is minimum. Similarly, in the second section, all of segments corresponding to electrons on runner $i$ slope up, but also the segments corresponding to electrons on runner $j$, where $x_j$ is the second smallest of all the $x_k$.

The ordering of the $x_i$ tell us the ordering the up-steps on the $i$th abacus happen.

1.4. **Chamber dependence of $\mathbf{core}_a c$.** The lattice of charges $\Lambda_a$ is essentially the $A_{a-1}$ lattice. Although we have given a uniform descirption of the partition corresponding to $\mathbf{core}_a(c)$ for any charge vector $c$, and shown the size $|\mathbf{core}_a(c)|$ is a global polynomial in $c$, in many ways $\mathbf{core}_a(c)$ has a chamber dependence on the $A_{a-1}$ hyperplane arrangement. By this we mean that if we restrict to a given chamber of this hyperplane arrangement, then $\mathbf{core}_a(c)$ behaves nicely, but if $c$ crosses one of the walls of the hyperplane arrangement, then $\mathbf{core}_a(c)$ undergoes a qualitative change.

We illustrate this now with an informal example.

**Example 1.17.** The boundary path of a large $a$-core can be decomposed into $a+1$ regions, labeled with $i \in \{0, 1, \dots, a\}$. On the $i$th region, $i$ out of every $a$ steps will be left, and $a - i$ will be down; thus on the $i$th region the path will have slope $-(a-i)/i$.

This description of $a$-cores is clear from the abacus description. In the region zero, all the runners have unfilled energy states; we cross into the first region as soon as one of the runners start having filled energy states. In general, the $i$th region is exactly those regions where $i$ of the runners have filled energy states.

We get a chamber structure on the space of $a$-cores by considering *which* of the $i$ runners have filled or empty energy states.

1.5. **Outline.** In section 2 we introduce standard notation and bijections about $a$-core partitions. Section 3 applies this to simultaneous core partitions and proves our main results. Sections 4 and 5 describes some conjectural applications of these ideas to the $q$ and $(q,t)$ generalizations of $(a,b)$-Catalan numbers.

### 1.6. **Further directions.**

1.6.1. *Simultaneous cores for 3 or more moduli.* Recently proven by [18].

, or to partitions that are simultaneous $a$-core for 3 or more values of $a$.

1.7. **Acknowledgements.** I learned about Armstrong's conjecture over dinner after speaking in the MIT combinatorics seminar. I would like to thank Jon Novak for the invitation, Fabrizio Zanello for telling me about the conjecture, and funding bodies everywhere for supporting seminar dinners.

## 2. Abaci and Electrons

This section is a review of the fermionic viewpoint of partitions and the abaci model of $a$-cores. It contains no new material. The main results are that $a$-cores are in bijection with points on the "charge lattice" $\Lambda_a$, and the size of a given $a$-core is given by a quadratic function on the lattice.

2.1. **Fermions.** We begin with a motivating fairy tale. It should not be mistaken for an attempt at accurate physics or accurate history.

2.1.1. *A fairy tale.* According to quantum mechanics, the possible energies levels of an electron are quantized – they can only be half integers i.e., elements of $\mathbb{Z}_{1/2} = \{a + 1/2 | a, \in \mathbb{Z}\}$. In particular, basic quantum mechanics predicts electrons with negative energy. Physically, it makes no sense to have negative energy electrons, so these negative energy electrons were a problem that needed explaining.

Dirac's *electron sea* solves the problem of negative energy electrons by redefining the vacuum state **vac**. The Pauli exclusion principle states that each possible energy state can have at most one electron in it; thus, we can view any set of electrons as a subset $S \subset \mathbb{Z}_{1/2}$. Intuitively, the vacuum state **vac** should consist of empty space with no electrons at all, and hence correspond to the set $S = \emptyset \subset \mathbb{Z}_{1/2}$.

Dirac suggested isntead to take **vac** to be an infinite "sea" of negative energy electrons. Specifically, in the vacuum state every negative energy level should be filled with an electron, but none of the positive energy states should be filled with an electron. Then by Pauli's exclusion principle we cannot add a negative energy electron to **vac**, but positive energy electrons can be added as usual. Thus, Dirac's electron sea solves the problem of negative energy electrons.

As an added benefit, Dirac's electron sea predicts the positron, a particle that has the same energy levels as an electron, but positive charge. Namely, a positron corresponds to a "hole" in the electron sea, that is, a negative energy level *not* filled with an electron. Removing a negative energy electron results in adding positive charge and positive energy, and hence can be interpreted as a having a positron.

2.1.2. We are thus led to the following definitions:

**Definition 2.1.** Let $\mathbb{Z}^{\pm}_{1/2}$ denote the set of all positive/negative half integers, respectively.

The vacuum $\mathbf{vac} \subset \mathbb{Z}_{1/2}$ is the set $\mathbb{Z}^{-}_{1/2}$.

A *state* $S$ is a set $S \subset \mathbb{Z} + 1/2$ so that the symmetric difference $S\mathbf{xorvac} = (S \cap \mathbb{Z}^{+}_{1/2}) \cup (S^c \cap \mathbb{Z}^{-}_{1/2})$ is finite. States should be interpreted as a finite collection of electrons (the elements of $S \cap \mathbb{Z}^{+}_{1/2}$) and positrons (the elements of $S^c \cap \mathbb{Z}^{-}_{1/2}$).

The *charge* $c(S)$ of a state $S$ is the number of positrons minus the number of electrons:

$$c(S) = \#S \cap Z^{+}_{1/2} - \#S^c \cap \mathbb{Z}^{-}_{1/2}$$

The *energy* $e(S)$ of a state $S$ is the sum of all the energies of the positrons and the electrons:

$$e(S) = \sum_{k \in \mathbb{Z}^{+}_{1/2} \cap S} k + \sum_{k \in \mathbb{Z}^{-}_{1/2} \cap S^c} -k$$

2.1.3. *Maya Diagrams.* It is convenient to have a graphical representation of states $S$.

The *Maya diagram* of $S$ is an infinite sequence of circles on the $x$-axis, one circle centerred at each element of $\mathbb{Z}_{1/2}$, with the positive circles extending to the left and the negative direction to the right. A black "stone" is placed on the circle corresponding to $k \in \mathbb{Z}_{1/2}$ if and only if $k \in S$.

**Example 2.2.** The Maya diagram corresponding to the vacuum vector **vac** is shown below.

$$\begin{array}{cccccc|cccccc} \cdots & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\ & \frac{9}{2} & \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-3}{2} & \frac{-5}{2} & \frac{-7}{2} & \frac{-9}{2} & \end{array}$$

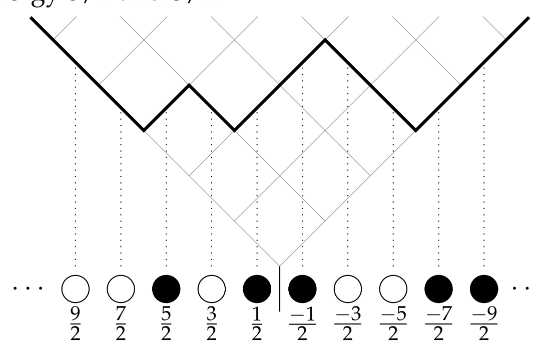
**Example 2.3.** The following Maya diagram illustrates the state $S$ consisting of an electron of energy $3/2$, and two positrons, of energy $1/2$ and $5/2$.

$$\begin{array}{cccccc|cccccc} \cdots & \bigcirc & \bigcirc & \bigcirc & \bullet & \bigcirc & \bigcirc & \bullet & \bigcirc & \bullet & \bullet & \cdots \\ & \frac{9}{2} & \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-3}{2} & \frac{-5}{2} & \frac{-7}{2} & \frac{-9}{2} & \end{array}$$

2.2. **Paths.** We now describe a bijection between the set of partitions $\mathcal{P}$ to the set of charge 0 states, that sends a partition $\lambda \in \mathcal{P}_n$ of size $n$ to a state $S_\lambda$ with energy $e(S_\lambda) = n$. This bijection can be understood in two ways: as recording the boundary path of $\lambda$, or recording the modified Frobenius coordinates of $\lambda$.

2.2.1. We draw partitions in "Russian notation" – rotated $\pi/4$ radians counterclockwise and scaled up by a factor of $\sqrt{2}$, so that each segment of the border path of $\lambda$ is centered above a half integer on the $x$-axis. We traverse the boundary path of $\Lambda$ from left to right. For each segment of the border path, we place an electron in the corresponding energy level if that segment of the border slopes up, and we leave the energy state empty if that segment of border path slopes down.

**Example 2.4.** We illustrate the bijection in the case of $\lambda = 3+2+2$. The corresponding state $S_\lambda$ consists of two electrons with energy $5/2$ and $1/2$, and two positrons with energy $3/2$ and $5/2$.




2.2.2. *Frobenius Coordinates.* The energies of the electrons and the positrons of $\lambda$ are the *modified Frobenius coordinates*,

The $y$-axis dissects the partition $\lambda$ into two pieces. The left side of $\lambda$ consists of $c$ rows, where $c$ is the number of electrons. The length of the $i$th row is the energy of the $i$th electron. The right half of $\lambda$ also consists of $c$ rows, with lengths the energies of the positrons. Note that $c$ here is the size of the *Durfee square* of $\lambda$.

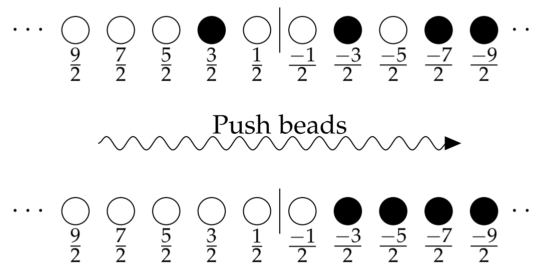
**Example 2.5.** Consider Example 2.4. If the $y$-axis was drawn in, left of the $y$-axis would be two rows, the bottom row having length 2.5 and the top row length .5 – these were precisely the energies of the electrons in $S$. Similarly, the right hand side has two rows of length 2.5 and 1.5, the energies of the positrons in $S$.

2.2.3. *Non-zero charge.* The bijection between partitions and states of charge zero may be modified to give a bijection between partitions and states of charge $c$ for any $c \in \mathbb{Z}$. Simply translate the partition to the right by $c$.

2.3. **Abaci.** Rather than view the Maya diagram as a series of stones in a line, we now view it as beads on the runner of an abacus. Sliding the beads to be right justified allows the charge of the state to be read off, as it is easy to see how many electrons have been added or are missing from the vacuum state.

In what follows, we mix our metaphors and talk about electrons and protons on runners of an abacus.

**Example 2.6.** Consider Example 2.3, where the Maya diagram consists of two positrons and an electron. Pushing the beads to be right justified, we see the first bead is one step to the right of zero, and hence the original state had charge 1.




2.3.1. *Cells and hook lengths.* The cells $\square \in \lambda$ are in bijection with the *inversions* of the boundary path; that is, by pairs of segments $(\text{step}_1, \text{step}_2)$, where $\text{step}_1$ occurs before $\text{step}_2$, but $\text{step}_1$ is traveling NE and $\text{step}_2$ is traveing SE. The bijection sends $\lambda$ to the segments at the end of its arm and leg.

In the fermionic viewpoint, cells of $\lambda$ are in bijection with pairs $(e, e-k), e \in \mathbb{Z}_{1/2}, k > 0$ of a filled energy level $e$ and an empty energy level $e-k$ of lower energy; we call such a pair an *inversion*. The hook length $h(\square)$ of the corresponding cell is $k$.

If $(e, e-k)$ is such a pair, reducing the energy of the electron from $e$ to $e-k$ changes $\lambda$ by removing the rim hook corresponding to the cell $\square$. This rim-hook has length $k$.

**Example 2.7.** The cell $\square = (2, 1)$ of $\lambda = 3 + 3 + 2$ has hook length $h(\square) = 3$, and corresponds to the electron in energy state $1/2$ and the empty energy level $-5/2$; which are three apart.

2.4. **Bijections.** Rather than place the electrons corresponding to $\lambda$ on one runner, place them on $a$ different runners, putting the energy levels $ka - i - 1/2$ on runner $i$.

If the hooklength $h(\square) = ka$ is divisible by $a$, then the two energy levels of inversion$(\square)$ lie on the same runner. Similarly, any inversion of energy states on the same runner corresponds to a cell with hook length divisible by $a$.

Thus, $\lambda$ is an $a$-core if and only if the beads on each runner of the $a$-abacus are right justified. Although the total charge of all the runners must be zero, the charge need not be evenly divided among the runners. Let $c_i$ be the charge on the $i$th runner; then we have $\sum c_i = 0$, and the $c_i$ determine $\lambda$.

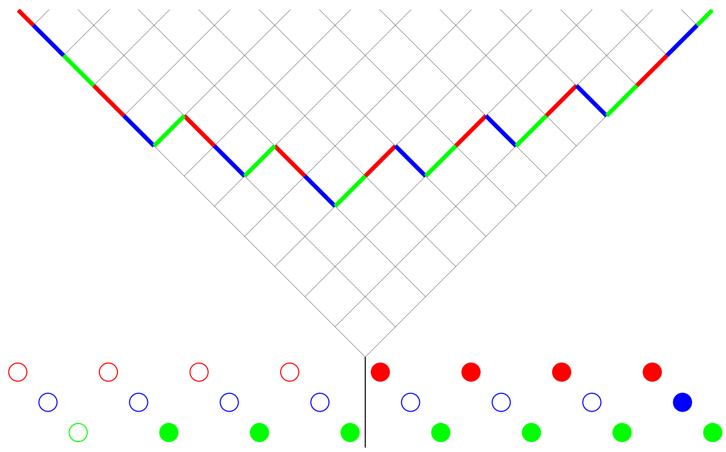
Similarly, given any $\mathbf{c} = (c_0, \ldots, c_{a-1}) \in \mathbb{Z}^a$ with $\sum c_i = 0$, there is a unique right justified abacus with charge $c_i$ on the $i$th runner. The coresponding partition is an $a$-core which we denote $\mathbf{core}_a(\mathbf{c})$.

We have shown:

**Lemma 2.8.** There is a bijection

$$\mathbf{core}_a : \{(c_0, \dots, c_{a-1} | c_i \in \mathbb{Z}, \sum c_i = 0\} \to \{\lambda | \lambda \text{ is in } a\text{-core}\}$$

**Example 2.9.** We illustrate that $\mathbf{core}_3(0, 3, -3) = 7+5+3+3+2+2+1+1$.



2.5. **Size of an $a$-core.**

**Theorem 2.10.**

$$|\mathbf{core}_a(\mathbf{c})| = \frac{a}{2} \sum_{k=0}^{a-1} c_k^2 + kc_k$$

We are not sure where exactly where this theorem originates; a stronger version is used in [11] and [9], to prove certain generating functions of partitions are modular forms.

*Proof.* If $c_k > 0$ the $k$th runner has $c_k$ positrons, with energies

$$\begin{aligned}
&(k+1/2),\\
&(k+1/2)+a,\\
&(k+1/2)+2a,\\
&\quad\vdots \qquad \vdots\\
&(k+1/2)+(c_k-1)a
\end{aligned}$$

and so the particles on the $k$th runner have total energy

$$\frac{a}{2}(c_k^2 - c_k) + (k + 1/2)c_k.$$

If $c_k < 0$, the $k$th runner has $-c_k$ electrons, and a similar calculation shows they have a total energy of

$$\frac{a}{2}(c_k^2 + c_k) - c_k(a - k - 1/2) = \frac{a}{2}(c_k^2 - c_k) + (k + 1/2)c_k.$$

Since $\sum c_k = 0$, the total energy of all particles simplifies to $\frac{a}{2}\sum(c_k^2 + kc_k)$. $\square$

## 3. Simultaneous Cores

We now turn to studying the set of $b$-cores within the lattice $\Lambda_a$ of $a$-cores.

### 3.1. $(a, b)$-cores form a simplex.

First, some notation and conventions.

Let $r_a(x)$ be the remainder when $x$ is divided by $a$, and $q_a(x)$ to be the integer part of $x/a$, so that $x = aq_a(x) + r_a(x)$ for all $x$. Furthermore, we use cyclic indexing for $\mathbf{c} \in \Lambda_a$; that is, for $k \in \mathbb{Z}$, we set $c_k = c_{r_a(k)}$.

**Lemma 3.1.** Within the lattice of $a$ cores, the set of $b$ cores are the lattice points satisfying the inequalities

$$c_{i+b} - c_i \leq q_a(b + i)$$

for $i \in \{0, \ldots, a - 1\}$.

*Proof.* Fix $\mathbf{c} \in \Lambda_a$, and consider the corresponding $a$-abacus.

Let $\lambda = \mathbf{core}_a(\mathbf{c})$ be an $a$ core, and let $e_i$ denote the energy of the highest electron the $i$th runner. We claim that $\mathbf{core}_a(\mathbf{c})$ is a $b$-core if an only if for each $i$, the energy state $e_i - b$ is filled.

Certainly this condition is necessary. To see that it is sufficient, suppose that $\lambda$ is an $a$-core, and that $e_i - b$ are all filled. To see $\lambda$ is a $b$ core, we must show that for any filled energy level $L$, that $L - b$ is filled.

Suppose that $L$ is on the $i$th runner; then $L = e_i - aw$ for some $w \geq 0$, and so $L - b = (e_i - b) - aw$. But by supposition $e_i - b$ is a filled state, and $e_i - b - aw$ is to the right of it and on the same runner, and so it must be filled since $\lambda$ is an $a$-core.

Now, the energy state $e_i - b$ is on runner $r_a(i + b)$, and so $\lambda$ is $b$-core if and only if $e_i - b \leq e_{i+b}$ (recall that we are using cyclic indexing).

Substituting $e_k = -ac_k - r(k) - 1/2$ and simplifying gives that our inequality is equivalent to

$$a(c_{i+b} - c_i) \leq b + i - r_a(i + b)$$

and hence to

$$c_{i+b} - c_i \leq q_a(b + i).$$

$\square$

We have $a$ hyperplanes in an $a-1$ dimensional space; they either form a simplex or an unbounded polytope.

**Remark 3.2.** The same analysis sheds light on the case when $a$ and $b$ are not relatively prime, which has been studied in [4].

Let $d = \gcd(a, b)$; then any $d$-core is also an $(a, b)$-core, and so there are no longer finitely many $(a, b)$-cores.

The inequalities given for $\mathbf{SC}_a(b)$ still describe the space of $(a, b)$-cores when $a, b$ are no longer relatively prime, but these inequalities no longer describe a simplex. The inequalities no longer relate all the $c_i$ to each other; rather, they decouple into $d$ sets of $a/d$ of variables

$$\begin{aligned} S_0 &= \{xc_0, c_d, c_{2d}, \ldots, c_{a-d}\} \\ S_1 &= \{c_1, c_{d+1}, \ldots, c_{a-d+1}\} \\ &\cdots \\ S_{d-1} &= \{c_{d-1}, c_{2d-1}, \ldots, c_{a-1}\} \end{aligned}$$

The charges $c_i$ in a given group must be close together, but for any vector $(v_0, \ldots, v_{d-1})$ with $\sum v_i = 0$, we may shift each element of $S_i$ by $v_i$ and all inequalities will still be satisfied.

In particular, the shifts of the zero vector are easily seen to be the $d$ core partitions, and we see the set of $(a, b)$-core partitions is finite number of translates of the lattice of $d$-cores within the lattice of $a$-cores.

3.1.1. *Coordinate shift.* In the charge coordinates $\mathbf{c}$, neither the hyperplanes defining the set of $b$ cores nor the quadratic form $Q$ are symmetrical about the origin. We shift coordintaes to remedy this.

**Definition 3.3.** Define $\mathbf{s} = (s_1, \ldots, s_a) \in V_a$ by

$$s_i = \frac{i}{a} - \frac{a-1}{2a}$$

The $i/a$ term ensures $s_{i+1} - s_i = 1/a$; subtracting $\frac{a-1}{2a}$ ensures that $\mathbf{s} \in V_a$, i.e. $\sum s_i = 0$.

**Lemma 3.4.** In the shifted charge coordinates

$$x_i = c_i + s_i$$

the inequalities defining the set of $b$ cores become

$$x_{i+b} - x_i \leq b/a$$

and the size of an $a$-core is given by

$$Q(\mathbf{x}) = -\frac{a^2-1}{24} + \frac{a}{2}\sum_{i=0}^{a-1} x_i^2$$

*Proof.* That the linear term of $Q$ vanishes in the $\mathbf{x}$ coordinates follows immediately from the definition of $\mathbf{s}$. The constant term of $Q$ in the $\mathbf{x}$ coordinates is $-\frac{a}{2}\sum_{i=0}^{a-1} s_i^2$, which a short computation shows is $-\frac{a^2-1}{24}$.

The statement about the set of $b$-cores follows from the computation

$$\begin{aligned} x_{i+b} - x_i &= c_{i+b} - c_i + s_{i+b} - s_i \\ &\le q_a(i+b) + r_a(i+b)/a - i/a \\ &= (b+i)/a - i/a \\ &= b/a \end{aligned}$$

□

Although we will often use the $x$ coordinates, to show that the simplex of $\mathbf{SC}_a(b)$ is isomorphic to the simplex $\mathbf{TD}_a(b)$ of trivial determinant representations, another change of variables is needed:

**Lemma 3.5.** Let $a$ and $b$ be relatively prime, and let

$$k = -\frac{b+1}{2} \pmod a$$

Then the change of variables

$$z_i = x_{ib+k} - x_{(i+1)b+k} + b/a$$

gives an isomorphism between the rational simplices $\mathbf{SC}_a(b)$ and $\mathbf{TD}_a(b)$.

*Proof.* It is immediate that the $z_i$ satisfy $\sum z_i = b$ and $z_i \ge 0$. The integrality of the $z_i$ follows from the fact that the fractional part of $x_i - x_j$ is $(i-j)/a$. We must show $\sum i z_i = 0 \mod a$.

One computes:

$$\sum_{i=0}^{a-1} i z_i = \sum_{i=0}^{a-1} x_i + \frac{b}{a}\sum_{i=0}^{a-1} i - a x_k$$

Since the fractional part of $x_k$ is $s_k = k/a - (a-1)/2a$, plugging in the definition of $k$ gives that $a x_k = -b/2 \pmod a$. Since $\sum x_i = 0$ and $\sum i = (a-1)a/2$, we see $\sum i z_i = 0 \pmod a$.

A further computation shows this change of variables is invertible. □

**Corollary 3.6** (Anderson [3])**.** The number of simultaneous $(a,b)$-cores is $\mathbf{Cat}_{a,b}$.

*Proof.* This follows quickly from Lemma 3.5.

The scaled simplex $b\Delta_a$ has $\binom{a+b-1}{a-1}$ usual lattice points. Cyclicly permuting the variables preserves $b\Delta_a$ and the standard lattice, and when $b$ is relatively prime to $a$ it cyclicly permutes the $a$ cosets of the charge lattice.

Thus the standard lattice points in $b\Delta_a$ are equidistributed among the $a$-cosets of the charge lattice, and hence each one contains $\frac{1}{a}\binom{a+b-1}{a-1} = \mathbf{Cat}_{a,b}$.

□

3.2. **The size of simultaneous cores.** We now have all the ingredients needed to prove Armstrong's conjecture. We derive it as a consequence of:

**Theorem 3.7.** For fixed $a$, and $b$ relatively prime to $a$, the average size of an $(a,b)$-core is a polynomial of degree 2 in $b$.

*Proof.* For fixed $a$, the number of $a$-cores is $1/a$ times the number of lattice points in $b\Delta_{a-1}$, which is a polynomial $F_a(b)$ of degree $a-1$. In the $x$-coordinates $Q = \mathbf{core}_a$ is invariant under $S_a$, and in particular rotation, we see that the sum of the sizes of all $(a,b)$-cores is $1/a$ times the sum of $Q$ over the lattice points in $b\Delta_{a-1}$. By Euler-Maclaurin theory, the number of points in $b\Delta_{a-1}$ is a polynomial $G_a(b)$ of degree $a+1$.

Thus, the average value of an $(a,b)$-core is $G_a(b)/F_a(b)$, the quotient of a polynomial of degree $a+1$ by a polynomial of degree $a-1$. To show this is a polynomial of degree two in $b$, we need to show that every root of $F_a$ is a root of $G_a$.

We already know from 3.6 that the roots of $F_a$ are $-1, -2, \dots, -(a-1)$. We now give another derivation of this fact, using Ehrhart reciprocity, that will easily adapt to shown these are also roots of $G_a$.

Ehrhart reciprocity says that $F_a(-x)$ is, up to a sign, the number of points in the *interior* of $x\Delta_{a-1}$. The interior consists of the points in $x\Delta_{a-1}$ none of whose coordinates are zero, and so the first interior point in $x\Delta_{a-1}$ is $(1,1,\dots,1) \in a\Delta_{a-1}$. Thus, $F_a(b)$ vanishes at $b = -1, \dots, -(a-1)$, and as it has degree $a-1$ it has no other roots.

Ehrhart reciprocity extends to Euler-Maclaurin theory, to say that up to a sign $Q_a(-x)$ is the sum of $F$ of the interior points of $x\Delta_{a-1}$. Thus $Q_a(-x)$ also vanishes at $b = -1, \dots, -(a-1)$, and so $P_a/Q_a$ is a polynomial of degree 2.

□

**Corollary 3.8.** When $(a,b)$ are relatively prime, the average size of an $(a,b)$ core is $(a+b+1)(a-1)(b-1)/24$

*Proof.* Fix $a$, and let $P_a(b) = G_a(b)/F_a(b)$ be the degree two polynomial that gives the average value of the $(a,b)$-cores when $a$ and $b$ are relatively prime. As we know $P_a(b)$ is a polynomial of degree 2, we can determine it by computing only three values.

First, we find the two roots of $P_a(b)$. As the only 1 core is the empty partition, we have $F_a(1) = 1$ and $G_a(1) = 0$, and so $P_a(1) = 0$.

Ehrhart reciprocity gives that $G_a(-a-b)$ is, up to a sign, the sum of $Q$ over the lattice points in the interior of $(a+b)\Delta_a$, which are just the lattice points contained in $b\Delta_a$, and hence equal to $G_a(b)$. In particular, $P_a(-a-1) = 0$.

Finally, we compute $P_a(0)$. It is clear that $\mathcal{S}_a(0) = \{0\}$. Although this is not a point of $\Lambda_a$, it is in $\Lambda'_a$, and so $P_a(0) = Q(0) = -(a^2-1)/24$. □

3.3. **Self-conjugate $(a,b)$-cores.** In Lemma 3.9, we show that the bijection between $(a,b)$-cores and elements of $\mathrm{Sym}^b(\mathbb{C}[\mathbb{Z}_a])^{\mathbb{Z}_a}$, or dimension $b$ representations of $\mathbb{Z}_a$ with determinant 1, conjugating a partition corresponds to the inversion sending $g$ to $g^{-1}$, or sending a representation $V$ to its dual $V^*$. In the lattice point of view, this is just a linear map, and hence the self-dual $(a,b)$-cores correspond to the lattice points in the fixed point locus of $T$.

We show in Lemma 3.10, that the lattice points in $\cap T$ are naturally the lattice points in the $\lfloor a/2 \rfloor$ dimensional simplex $\lfloor b/2 \rfloor \Delta_{\lfloor a/2 \rfloor + 1}$, hence rederiving the count of simultaneous $(a,b)$-core partitions.

Once we have done this, an analogous application of Euler-Maclaurin theory reproves the statement about the average value.

Let $T : V_a \to V_a$ be the linear map given by

$$T(c_i) = -c_{a-i}$$

It is easy to check that when translated to core partitions, $T$ corresponds to taking the conjugate, that is:

$$\mathbf{core}_a(c)^T = \mathbf{core}_a(T(c))$$

Thus the set of self-conjugate $(a,b)$-cores is the $T$ fixed locus of $\mathbf{SC}_a(b)$.

Since $T(s) = s$, the same formula holds for the shifted coordines $x$.

**Lemma 3.9.** Under the isomorphism between $\mathbf{SC}_a(b)$ and $\mathbf{TD}_a(b)$ established in Lemma 3.5, taking the conjugate partition corresponds to taking the dual $\mathbb{Z}_a$ representation.

*Proof.* We want to show $T(z_i) = z_{-i}$. We compute:

$$T(z_i) = T(x_{ib+k} - x_{(i+1)b+k}) = -x_{-ib-k-1} + x_{-ib-b-k-1} = x_{-ib+k-(b+1+2k)} - x_{(-i+1)b+k-(b+1+2k)}$$

And so we need $b + 1 + 2k = 0 \pmod{a}$, but this is exactly the definition of $k$ in Lemma 3.5. □

**Lemma 3.10.** The number of $b$-dimensional, self-conjugate $\mathbb{Z}_a$ representations with determinant 1 is given by

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor}$$

*Proof.* Let $a = 2k$ or $2k+1$. We give a bijection between the representations in question and $k$-tuples of non-negative integers $(z_1, \ldots, z_k)$ with $2\sum z_i \leq b$. The set of such $z_i$ are the lattice points in a $k = \lfloor a/2 \rfloor$ dimensional simplex, which are clearly counted by the given binomial coefficient.

First, suppose that $a = 2k+1$. Then the only irreducible self-conjugate representation is the identity, and so $C$ has a $k$ dimensional fixed point set consisting of points of the form $(u_0, u_1, \ldots, u_k, u_k, \ldots, u_1)$. Thus, we see that $\sum_{i=1}^{k} 2u_k \leq b$, and value of $u_0$ is fixed by $\sum_{i=0}^{a-1} u_i = b$.

When $a = 2k$, there are two irreducible self-conjugate representations, the identity and the sign representation induced by the surjection $\mathbb{Z}_a \to \mathbb{Z}_2$. Again, $C$ has a $k$ dimensional fixed point set, this time consisting of points of the form $(u_0, u_1, \ldots u_{k-1}, w_k, u_{k-1}, \ldots, u_1)$. Now for such a preresentation, having trivial determinant is equivalent to $w_k$ being even, say $w_k = 2u_k$. Then again we have $\sum_{i=1}^{k} 2u_k \leq b$, with $u_0$ being determined by $\sum_{i=-}^{a-1} u_i = b$. □

**Proposition 3.11.** Let $a$ and $b$ be relatively prime. Then the average size of a self-conjugate $(a,b)$-core is $(a-1)(b-1)(a+b+1)/24$.

*Proof.* Since $a$ and $b$ are relatively prime, at most one is even, so we my assume $a$ is odd.

The proof is essentially the same as that for all $(a,b)$-cores. One complication is that it seems we must treat odd and even values of $b$ separately. In each each case, an argument identical to Lemma 3.7 gives that the average size is a polynomial of degree 2 in $b$. A priori, we may have different polynomials for $b$ odd and $b$ even; however, the symmetry $(a,b) \mapsto (a, -a-b)$ coming from Ehrhart reciprocity still holds and interchanges odd and even values of $b$, and so if we can compute three values of either polynomial (that don't get identified by this symmetry), we will identify both polynomials.

All 1 and 2 cores are self conjugate, and this gives two values; the arguments made in Corollary 3.8 for the vanishing of the polynomial at $b = 1$ holds for self-conjugate partitions as well, giving a third value.

□

We see that

## 4. Toward $q$-analogs

In this section, we apply our lattice point and simplex point of view on simultaneous cores to the $q$-analog of rational Catalan numbers; the next section approaches $(q,t)$-analogs.

4.1. **$q$-numbers.** Recall the standard $q$ analogs of $n$, $n!$ and $\binom{n}{k}$:

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$
$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

These three functions are polynomials with positive integer coefficients, i.e., they are elements of $\mathbb{N}[q]$.

The $q$ rational Catalan numbers are given by the obvious formula:

**Definition 4.1.**

$$\mathbf{Cat}_{a,b}(q) = \frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q = \frac{(1-q^{b+1})(1-q^{b+2})\cdots(1-q^{b+a-1})}{(1-q^2)(1-q^3)\cdots(1-q^a)}$$

4.2. **Graded vector spaces.** One place $q$ analogs occur naturally is in graded vector spaces.

**Definition 4.2.** If $V$ is a graded vector space, with $V_k$ denoting the weight $k$ subspace of $V$, we define

$$\dim_q V = \sum_{k \in \mathbb{N}} q^k \dim V_k.$$

**Proposition 4.3.** Let $p_i$ be a variable of weight $i$, then $\mathbb{C}[p_1, \ldots, p_n]$ has finite dimensional graded pieces, and

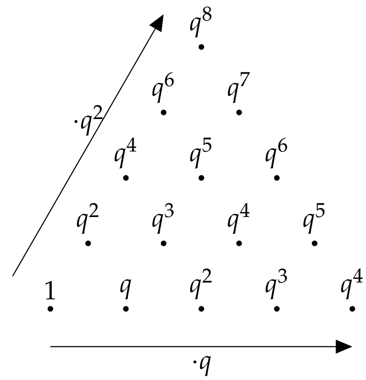
$$\dim_q \mathbb{C}[p_1, \ldots, p_n] = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

If $V$ is a vector space with $\dim_q V = [n]_q$, then

$$\dim_q \mathrm{Sym}^b V = \begin{bmatrix} n+b-1 \\ n-1 \end{bmatrix}_q$$

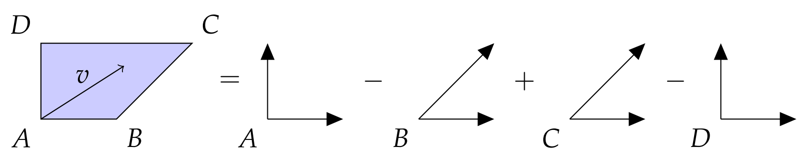
These statements can be interpretted geometrically in terms of lattice points The monomials in $\mathbb{C}[p_1, \ldots, p_n]$ correspond to the lattice points in an $n$ dimensional unimodular cone; the monomials in $\mathrm{Sym}^b V$ correspond to lattice points in the scaled standard simplex $b\Delta_{a-1}$; the $q$-analogs of the statements listed above are $q$ counting the lattice points, where the weights of the $i$th primitive lattice vector on the ray of the cone has weight $q^i$.

**Example 4.4.** The following diagram illustrates $\left[\begin{smallmatrix} b+a-1 \\ a-1 \end{smallmatrix}\right]_q$ as $q$-counting standard lattice points in $b\Delta_{a-1}$ in the case $a = 3$ and $b = 4$. Letting $b$ go to infinity corresponds to extending the arrows and the lattice points between them infinitely far to the upper right, showing that $\prod_{k=1}^{a-1} \frac{1}{1-q^k}$ $q$-counts the points in a standard cone.




4.3. **A $q$-version of cone decompositions.**

4.3.1. *Lawrence Varchenko.* Recall the decomposition of a simplicial polytope $\mathcal{P}$ in a vector space $V$ of dimension $n$ as a signed sum of cones based at their vertices, called the Lawrence-Varchenko decomposition:




First, pick a generic direction vector $v \in V$. At each vertex $v_i$, $n$ facets of $\mathcal{P}$ meet; if we extend these facets to hyperplanes, they cut $V$ into orthants. Let $\mathcal{C}_k$ be the orthant at $v_i$ that contains our direction vector $v$. Let $f_i$ be the number of hyperplanes that must be crossed to get from $\mathcal{C}_i$ to $P$.

Then:

$$\mathcal{S} = \sum_{i=0}^{k} (-1)^{f_i} \mathcal{C}_i$$

To deal correctly with the boundary of $P$, one must correctly include or exclude portions of the boundary of $\mathcal{C}_k$, but this subtlety won't matter to us.

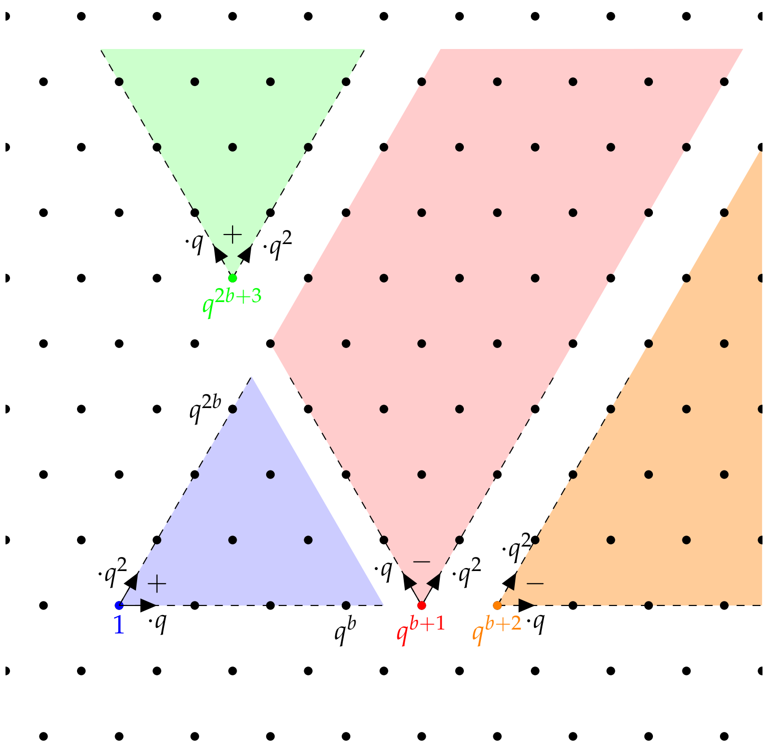
4.3.2. The algebraic structure of $\binom{b+a-1}{a-1}$ suggests a refinement of the Lawrence-Varchenko decomposition of $b\Delta_{a-1}$ for $q$-counting the lattice points.

Expanding the numerator of $\binom{a+b-1}{a-1}$ as $(1-q^{b+1})\cdots(1-q^{b+a-1})$ there are $\binom{a-1}{k}$ terms obtained from choosing 1 from $n-k$ factors and $q^M$ from $k$ factors. Each such term has sign $(-1)^k$, and the exponent of $q$ is slightly larger than $kb$. We will interpret these $\binom{a-1}{k}$ terms as making up the polarized tangent cone at the $k$th vertex.

The polarized tangent cone at the $k$th vertex $v_k$ will not carry the standard $q$-grading. However, it appears the cone at $v_k$ may be subdivided into $\binom{a-1}{k}$ smaller cones that do have the standard $q$-grading, essentially by intersecting with the $A_{a-1}$ hyperplane arrangement translated to $v_k$.

**Example 4.5.** We illustrate the decomposition of $b\Delta_2$ suggested by

$$\begin{bmatrix} b+2 \\ 2 \end{bmatrix}_q = \frac{1}{(1-q)(1-q^2)} \left(1 - q^{b+1} - q^{b+2} + q^{2b+3}\right)$$




Together with $\mathbf{Cat}_{a,b} = \dim_{\mathbb{C}}(\mathrm{Sym}^b\, \mathbb{C}[\mathbb{Z}_a])^{\mathbb{Z}_a}$, one might hope that we could give $\mathbb{C}[\mathbb{Z}_a]$ a grading so that we have

$$\mathbf{Cat}_{a,b}(q) = \dim_q(\mathrm{Sym}^b\, \mathbb{C}[\mathbb{Z}_a])^{\mathbb{Z}_a}$$

This naive hope does not appear possible. However, we now describe a conjectural weakening of it.

4.4. **Sublattices and shifting.** We begin by rewriting $\mathbf{Cat}_{a,b}(q)$. Since $[a]_{q^k} = (1-q^{ak})/(1-q^k)$, we have

$$\begin{aligned}
\mathbf{Cat}_{a,b}(q) &= \frac{(1-q^{b+1})(1-q^{b+2})\cdots(1-q^{b+a-1})}{(1-q^2)\cdots(1-q^a)} \cdot \frac{[a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}}}{[a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}}} \\
&= \frac{(1-q^{b+1})(1-q^{b+2})\cdots(1-q^{b+a-1})}{[a-1]_{q^a}} [a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}}
\end{aligned}$$

Observe that the fraction is similar to the $q^a$-count of the lattice points inside a simplex of size $b/a$, and that the product of $[a]_{q^i}$ is a $q$ analog of $a^{a-2}$.

4.4.1. This algebraic expression is suggestive of the simplex of $(a,b)$-cores. The lattice of $a$-cores is index $a$ within the standard lattice. The sublattice $\Lambda_T = (a\mathbb{Z})^{a-1}$, has index $a^{a-1}$ inside the standard lattice, and hence $a^{a-2}$ within the lattice of $a$-cores.

The intersection of each coset $\mathfrak{c}$ of $\Lambda_T$ with the simplex of $(a,b)$-cores is a $k\Delta_{a-1}$, where $k$ is slightly smaller than $b/a$, and depends on $b$ and $\mathfrak{c}$.

It appears that $\mathbf{Cat}_{a,b}(q)$ is $q^a$ counting the lattice points in each coset $\mathfrak{c}$, but then shifting the result by a factor of $q^{\iota(\mathfrak{c})}$ for some $\iota(\mathfrak{c})$.

Algebraically, this suggests

**Conjecture 4.6.** There is an *age* function $\iota$ on the cosets $\mathfrak{c} \in \Lambda/\Lambda_T$, so that

$$\sum_{\mathfrak{c}\in\Lambda/\Lambda_T} q^{\iota(\mathfrak{c})} = [a]_{q^2}[a]_{q^3}\cdots[a]_{q^{a-1}}$$

and

$$\mathbf{Cat}_{a,b}(q) = \sum_{\mathfrak{c}\in\Lambda/\Lambda_T} q^{\iota(\mathfrak{c})}\begin{bmatrix} b/a - s(\mathfrak{c},b) + a - 1 \\ a-1 \end{bmatrix}_{q^a}$$

where the $q^a$ binomial coefficient $q^a$-counts the points in $\mathfrak{c} \cap \mathcal{SC}_{a-1}(b)$.

**Remark 4.7.** We could not find an obvious candidate for an explicit form of $\iota$ in general.

**Remark 4.8.** Conjecture 4.6 was motivated in part by Chen-Ruan cohomology [6, 1], which has found applications to the Ehrhart theory of rational polytopes [17]. Chen-Ruan cohomology $H^*_{CR}(\mathcal{X})$ is a cohomology theory for an orbifold (or Deligne-Mumford stack) $\mathcal{X}$. As a vector space, $H^*_{CR}(\mathcal{X})$ is the usual cohomology of a disconnected space $\mathcal{IX}$. One component $C_0$ of $\mathcal{IX}$ is isomorphic to $\mathcal{X}$. The other components $C_\alpha, \alpha \neq 0$ are called *twisted sectors* and are (covers of) fixed point loci in $\mathcal{X}$. The pertinent feature for us is that the grading of the cohomology of the twisted sectors are *shifted* by rational numbers, $\iota(\alpha)$, that is

$$H^k_{CR}(\mathcal{X}) = \bigoplus_\alpha H^{k-\iota(\alpha)}(C_\alpha)$$

The function $\iota$ is known as the "degree shifting number" or "age".

Orbifolds could potentially be connected to our story through toric geometry, and the well known correspondence between lattice polytopes and polarized toric varieties. When the polytope is only rational, in general the toric variety is an orbifold. The simplex of $(a,b)$-cores in $\Lambda_a$ corresponds orbifold $[\mathbb{P}^a/\mathbb{Z}_a]$. More specifically, there is an torus equivariant orbifold line bundle $\mathcal{L}$ over $\mathbb{P}^a/\mathbb{Z}_a$, so that the lattice points in $\mathbf{SC}(a,b)$ correspond to the torus equivariant sections of $\mathcal{L}^b$.

In the fan point of view, the cosets of the lattice correspond exactly to group elements of isotropy groups, and hence to twisted sectors.

This discussion is rather vague, and at this point, there is no concrete connection between $\mathbf{Cat}_{a,b}(q)$ and the geometry of the orbifold $\mathbb{P}^a/\mathbb{Z}_a$ it would be very interesting to find one.

Note that if Conjecture 4.6 holds, it would give another proof, presumably more combinatorial, that $\mathbf{Cat}_{a,b}(q)$ are all positive. Furthermore, with some control on $\iota(\mathfrak{c})$ and $s(\mathfrak{c}, n)$, Conjecture 4.6 suggests:

**Conjecture 4.9.** For every residue class $r, 0 \leq r < a$, the coefficients of $q^{ak+r}$ in $\mathbf{Cat}_{a,b}(q)$ are unimodal.

4.4.2. *Examples.*

**Example 4.10.** By expanding both sides, it is straightforward to check the identities

$$\begin{aligned}\mathbf{Cat}_{3,3k+1}(q) &= \begin{bmatrix} k+2 \\ 2 \end{bmatrix}_{q^3} + q^2 \begin{bmatrix} k+1 \\ 2 \end{bmatrix}_{q^3} + q^4 \begin{bmatrix} k+1 \\ 2 \end{bmatrix}_{q^3} \\ \mathbf{Cat}_{3,3k+2}(q) &= \begin{bmatrix} k+2 \\ 2 \end{bmatrix}_{q^3} + q^2 \begin{bmatrix} k+2 \\ 2 \end{bmatrix}_{q^3} + q^4 \begin{bmatrix} k+1 \\ 2 \end{bmatrix}_{q^3}\end{aligned}$$

**Example 4.11.** When $a = 4$ and $b = 4k+1$,

$$\begin{aligned}\mathbf{Cat}_{4,4k+1}(q) = &\begin{bmatrix} k+3 \\ 3 \end{bmatrix}_{q^4} && +q^4 \begin{bmatrix} k+2 \\ 3 \end{bmatrix}_{q^4} && +q^8 \begin{bmatrix} k+2 \\ 3 \end{bmatrix}_{q^4} && +q^{12} \begin{bmatrix} k+1 \\ 3 \end{bmatrix}_{q^4} \\ &+q^5 \begin{bmatrix} k+2 \\ 3 \end{bmatrix}_{q^4} && +q^9 \begin{bmatrix} k+2 \\ 3 \end{bmatrix}_{q^4} && +q^9 \begin{bmatrix} k+1 \\ 3 \end{bmatrix}_{q^4} && +q^{13} \begin{bmatrix} k+1 \\ 3 \end{bmatrix}_{q^4} \\ &+q^2 \begin{bmatrix} k+2 \\ 3 \end{bmatrix}_{q^4} && +q^6 \begin{bmatrix} k+2 \\ 3 \end{bmatrix}_{q^4} && +q^6 \begin{bmatrix} k+1 \\ 3 \end{bmatrix}_{q^4} && +q^{10} \begin{bmatrix} k+1 \\ 3 \end{bmatrix}_{q^4} \\ &+q^3 \begin{bmatrix} k+2 \\ 3 \end{bmatrix}_{q^4} && +q^7 \begin{bmatrix} k+2 \\ 3 \end{bmatrix}_{q^4} && +q^{11} \begin{bmatrix} k+1 \\ 3 \end{bmatrix}_{q^4} && +q^{15} \begin{bmatrix} k+1 \\ 3 \end{bmatrix}_{q^4}\end{aligned}$$

Here, the terms have been grouped so that the coefficients on each line have the same residue mod 4, making it easy to verify the unimodality conjecture.

## 5. Toward $(q,t)$-analogs

We now turn toward applying the lattice-point viewpoint toward $(q,t)$-analog $\mathbf{Cat}_{a,b}(q,t)$, original defined in terms of lattice points, and translated to simultaneous cores in [8].

The $(q,t)$-rational catalan numbers count simultaneous cores with respect to two statistics, the length and (co)-skew-length. Our main result here is that these statistics are piecewise linear functions on the simplex of $(a,b)$-cores, and the domains of linearity are essentially chambers of the Catalan arrangement.

This suggests that $C_{a,b}(q,t)$ should be expressible as the sum of $C_a$ closed form functions of $(q,t)$, and that a thorough understanding of the geometry of the Catalan arrangement and its interaction with the lattice of cores could result in a proof of $(q,t)$-symmetry.

We examine this for $a=3$, and use this to prove $q-t$-symmetry for general $(a,b)$ and low $q$-degree.

5.1. **Simultaneous cores and $(q,t)$-rational Catalan numbers.** We first introduce the skew length statistic needed to define $(q,t)$-rational Catalan numbers.

**Definition 5.1.** Let $a<b$ be relatively prime, and $\lambda$ an $(a,b)$-core. The *b-boundary* of $\lambda$ consists of all cells $\square \in \lambda$ with $h(\square)<b$.

We can group the parts of $\lambda$ into $a$ classes by taking $\lambda_i - i \mod a$; (note, at least one class is empty since $\lambda$ is an $a$-core). The *a-parts of* $\lambda$ consist of the maximal $\lambda_i$ among each of the $i$ residue classes.

The *skew length of* $\lambda, s\ell(\lambda)$ is the number of cells of $\lambda$ that are in an $a$-row of $\lambda$ and in the $b$-boundary of $\lambda$. The *co-skew-length* $s\ell'(\lambda)$ is $(a-1)(b-1)/2 - s\ell(\lambda)$.

**Definition 5.2.** Let $a<b$ coprime. The $(q,t)$*-rational Catalan number* is

$$\mathbf{Cat}_{a,b}(q,t) = \sum_{\lambda} q^{\ell(\lambda)} t^{s\ell'(\lambda)}$$

We will focus on understanding the following two conjectures:

**Conjecture 5.3** (Specialization)**.**

$$\sum_{\lambda} q^{\ell(\lambda)+s\ell(\lambda)} = q^{(a-1)(b-1)/2}\mathbf{Cat}_{a,b}(q,1/q) = \mathbf{Cat}_{a,b}(q)$$

**Conjecture 5.4** (Symmetry)**.**

$$\mathbf{Cat}_{a,b}(q,t) = \mathbf{Cat}_{a,b}(t,q)$$

5.1.1. *Results.* Our main result is that the statistics $\ell$ and $s\ell$ in the definition of $\mathbf{Cat}_{a,n}(q,t)$ are piecewise linear functions on a dilation of the Catalan arrangement (Definition 1.15). More precisely, in the $\mathbf{x}$ coordinates on $a$-cores, $\ell$ and $s\ell$ have the following formula

**Proposition 5.5.**

$$\ell(\mathbf{x}) = -\frac{a-1}{2} + a \max x_i$$

**Proposition 5.6.** Let $\lfloor x \rfloor_0 = \max(0, \lfloor x \rfloor)$. Then

$$s\ell(\mathbf{x}) = \sum_{i,j=0}^{a} \lfloor x_i - x_j \rfloor_0 - \lfloor x_i - x_j - b/a \rfloor_0$$

From Proposition 5.5, it is clear that length is linear on each chamber of the braid arrangement.

Note that $s\ell$ is not piecewise linear on $V_a$, but if we restrict to $\Lambda + s$, then $x_i$ can only change by an integer, so $s\ell$ will be piecewise linear on chambers of the arrangement $x_i - x_j \in \{0, \pm b/a\}$, a scaling of the Catalan arrangement.

From Propositions 5.5 and 5.6, it is immediate that $\ell$ and $s\ell$ are invariant under the $S_a$ action permuting the coordinates.

5.1.2. As a basic check, we now illustrate that Propositions 5.5 and 5.6 give the correct results for the smallest and large $(a-b)$-cores; we will use these results later.

**Example 5.7** (The empty partition)**.** The empty partition corresponds to the vector $s$; recall $s_i = i/a - (a-1)/(2a)$. The largest of the $s_i$ is $s_{a-1} = (a-1)/(2a)$, and so $\ell(s) = a(a-1)/(2a) - (a-1)/2 = 0$.

Since $s_i - s_{i-1} = 1/a$, we have $s_i - s_j < 1$, and so $\lfloor s_i - s_j \rfloor_0 = 0$. Verifying that $s\ell(s) = 0$.

**Example 5.8** (The largest $(a-b)$-core)**.** The largest $(a,b)$-core $\lambda^M$ is the one vertex of $\mathbf{SC}_a(b)$ that is in $\Lambda + s$. Its coordinates are some permutation of $bs = (bs_0, bs_1, \cdots bs_{a-1})$, since $s\ell$ is $S_a$ invariant we may assume it is $bs$.

It is immediate that:

$$\ell(\lambda^M) = -\frac{a-1}{2} + ab\frac{a-1}{2a} = \frac{(a-1)(b-1)}{2}$$

Verifying $s\ell(\lambda^M) = (a-1)(b-1)/2$ is more complicated. We have

$$s\ell(\lambda^M) = \sum_{i<j} \left\lfloor \frac{bj}{a} - \frac{bi}{a} \right\rfloor - \left\lfloor \frac{bj}{a} - \frac{bi}{a} - \frac{b}{a} \right\rfloor$$

The summand depends only on the difference $k = j - i$, and is equal to $\lfloor kb/a \rfloor - \lfloor (k-1)b/a \rfloor$.

There are $(a-1)$ pairs $(i,j)$ with $i - j = 1$, and in general $a - k$ pairs with $i - j = k$, and so we have

$$\begin{aligned}
s\ell(\lambda^M) &= \sum_{k=1}^{a-1} (a-k) \left\lfloor \frac{b}{a} k \right\rfloor - (a-k) \left\lfloor \frac{b}{a}(k-1) \right\rfloor \\
&= \sum_{k=1}^{a-1} \left\lfloor \frac{b}{a} k \right\rfloor \\
&= \sum_{k=1}^{a-1} \frac{b}{a} k - \sum_{k=1}^{a-1} \left\langle \frac{b}{a} k \right\rangle \\
&= \frac{b}{a} \frac{(a-1)a}{2} - \frac{1}{a} \frac{(a-1)a}{2} \\
&= \frac{(a-1)(b-2)}{2}
\end{aligned}$$

where the second line follows from reindexing the second sum, the third line applies $\lfloor x \rfloor = x - \langle x \rangle$, and the fourth line applies $\sum i = n(n+1)/2$ and the fact that, since $a$ and $b$ are relatively prime, $kb$ will take on every residue class mod $a$ exactly once as $k$ ranges from 1 to $a$.

5.2. **Length and Skew Length are piecewise linear.** In this section we prove Propositions 5.5 and5.6.

5.2.1. *Proof of Proposition 5.5 - length is piecewise linear.*

*Proof.* We first translate $\ell(\lambda^S)$ into fermionic language. Let $e$ be the lowest energy state of $S$ that is not occupied by an electron. Then $\ell(\lambda^S)$ is the number of electrons with energy greater than $e$.

Recall that the highest energy occupied state on the $i$th runner is $-ac_i - i - 1/2$, and so the lowest unoccupied state is $a$ higher, and hence $e = \min_i -ac_i - i - 1/2 + a$.

Let $m$ be the runner of the $a$-abacus that has the lowest unoccupied energy state. For $i \neq m$, there are roughly $c_m - c_i \geq 0$ electrons on the $i$th runner that have energy great than $e$. The exact number depends on which of $i$ and $m$ is bigger: if $i < m$, there are exactly $c_m - c_i$ such electrons, while if $i > m$, there are only $c_m - c_i - 1$ such electrons.

There are $a - 1 - m$ runners with $i > m$, and hence we have

$$\begin{aligned} \ell(\mathbf{core}_a(\mathbf{c})) &= -(a-1-m) + \sum_{i \neq m} c_m - c_i \\ &= -(a-1-m) + ac_m \end{aligned}$$

where the second line follows by adding $\sum c_i = 0$ to the expression.

Since $x_i = c_i + i/a - (a-1)/(2a)$, it follows that

$$\ell(\mathbf{core}_a(x)) = -(a-1)/2 + a \max x_i$$

□

5.2.2. *Proof of Proposition 5.6 - skew length is piecewise linear.*

**Definition 5.9.** For $\lambda$ and $(a,b)$-core, let $s\ell^T_{i,j}(\lambda)$ be the number of cells in the $i$th $a$-part with unoccupied state on the $j$th runner.

Furthermore let $s\ell^S_{ij}(\lambda)$ be the number of such cells with hook length less than $b$, and $s\ell^B_{ij}(\lambda)$ be the number of such cells with hook length greater than $b$.

Here, $T, S$ and $B$ are short for *total, small* and *big*.

From Definition 5.9 it is clear that

$$s\ell(\lambda) = \sum_{i \neq j} s\ell^S_{ij}(\lambda)$$

$$s\ell^S_{ij}(\lambda) = s\ell^T_{ij}(\lambda) - s\ell^B_{ij}(\lambda)$$

and so Proposition follows from

**Lemma 5.10.** Let $\lambda = \mathbf{core}_a(\mathbf{x})$ be an $(a,b)$-core. Then:

$$s\ell^T_{ij}(\lambda) = \lfloor x_i - x_j \rfloor_0$$
$$s\ell^B_{ij}(\lambda) = \lfloor x_i - x_j - b/a \rfloor_0$$

*Proof.* Recalling that cells are in bijection with pairs $(e,f)$, with $e,f$ energy levels, $e$ filled and $f$ empty, we see that $s\ell^T_{ij}$ counts pairs $(e,f)$ with $e$ the highest energy level on the $i$th runner, $f$ any empty state on the $j$th runner. Thus, $s\ell^T_{ij}(\lambda)$ is the number of unoccupied states on the $j$th runner with energy less than $e$.

Recalling that the highest energy electron on the $i$th runner has energy $e_i = -ac_i - i - 1/2$, and that the energy of each state to the left increases by $a$, we have

$$\begin{aligned} s\ell^T_{ij}(\lambda) &= q_a\left(-ac_i - i - 1/2 - (-ac_j - j + -/2)\right) \\ &= q_a(-a(x_i - x_j)) \\ &= \lfloor x_j - x_i \rfloor_0 \end{aligned}$$

For $s\ell^B_{ij}(\lambda)$, we want hooklengths of size at least $b$, so begin by reducing the energy of the first electron on the $i$th runner by $b$. We now want to count ways of moving the resulting electron onto the $j$th runner, and so by our calculation of $s\ell^T_{ij}(\lambda)$ we immediately have

$$s\ell^B_{ij}(\lambda) = \lfloor x_j - x_i - b/a \rfloor_0$$

□

5.2.3. *Rationality.* It is an immediate corollary of the piecewise linearity of the length and the skew length that, for fixed $a$, and a fixed residue class $k$ mod $a$, that for $b$ in the residue class $k$, we have

$\mathbf{Cat}_{a,b}(q,t)$ is a rational function of $q$ and $t$, and the denominator can be written so the exponents depend linearly on $b$ and $\delta$.

**Proposition 5.11.** Let $b = 3k + 1 + \delta$, where $\delta \in \{0,1\}$. Then:

$$\mathbf{Cat}_{3,b}(q,t) = \frac{t^{3k+\delta}}{(1-qt^{-1})(1-qt^{-2})} + \frac{q^k t^k (q + t + q^\delta t^\delta)}{(1-q^{-1}t^2)(1-q^2t^{-1})} + \frac{q^{3k+\delta}}{(1-tq^{-1})(1-tq^{-2})}$$

*Proof.* □

5.3. **Low degree $(q,t)$-symmetry.** In this section we show, for all $(a,b)$, that $(q,t)$-symmetry holds when the degree of one of the monomials are small.

More precisely, we show

**Corollary 5.12** (Low degree $(q,t)$-symmetry)**.**

$$[t^k]\mathbf{Cat}_{a,b}(q,t) = [t^k]\mathbf{Cat}_{a,b}(t,q)$$

for

5.3.1. Because $\ell$ and $s\ell$ are symmetric under the $S_a$ action on the $\mathbf{x}$-coordinates, we will find it convenient to quotient out by that action.

Let $\mathcal{D}$ denote the dominant chamber

$$\mathcal{D} = \{x \in V_a | x_1 \leq x_1 \leq x_2 \leq \cdots \leq x_a\}$$

Then $\mathcal{D}$ is a fundamental domain for the action of $S_a$ on $V_a$, and we will use $\mathfrak{c}$ to denote the polyhedron $\mathcal{D} \cap \mathbf{SC}_a(b)$.

We will consider the image of $\Lambda_S \cap \mathbf{SC}_a(b)$ under the $S_a$ action as a subset of $\mathfrak{c}$;

Since the points of $\Lambda_S$ have distinct coordinates, each point has a unique representative in $\mathcal{D}$.

**Definition 5.13.** For $1 \leq i \leq a-1$, let

$$v_i = \left( \underbrace{\frac{i}{a} - 1, \dots \frac{i}{a} - 1}_{i \text{ times}}, \underbrace{\frac{i}{a}, \cdots \frac{i}{a}}_{a-i \text{ times}} \right)$$

One can see that the $v_i$ generate the lattice $\Lambda_R$, and that locally near $x_0$, the $\mathfrak{c} = x_0 + \sum t_i v_i$, with $t_i \geq 0$, while near $x_\infty$ we have $\mathfrak{c} = x_\infty - \sum t_i v_i$ .

Because $\ell$ is a linear function on $\mathcal{D}$ it is immediate from the definitions of $\ell$ and $v_i$ that, for any point $x \in \mathfrak{c}$ we have

$$\ell(x + v_i) = \ell(x) + i$$

Because the difference of two entries of $v_i$ is 0 or 1, we see that $s\ell$ is a piecewise linear function when restricted to elements of the lattice $\Lambda_R$.

The dependence of $s\ell$ on $v_i$ depends on which chamber of the Catalan arrangement we are in. Near $x_0$, we have $s\ell(x) = \sum_{i<j} x_j - x_i$, and so

$$s\ell(x + v_i) = s\ell(x) + i(a - i)$$

However, near $x_\infty$, we have that $x_j - x_i > b$ if $j \neq i+1$, so $s\ell(x) = \sum_i x_{i+1} - x_i = x_a - x_1$, and

$$s\ell(x + v_i) = s\ell(x) + 1$$

This discussion is summarized as follows:

**Lemma 5.14.** Let $f \in \{\ell, s\ell\}$. For $x$ near $x_0$, let $\Delta_i f = f(x + v_i) - f(x)$, and near $x_\infty$ let $\Delta_i f = f(x - v_i) - f(x)$. Then:

| | $\Delta_i \ell$ | $\Delta_i s\ell'$ |
|---|---|---|
| 0 | $i$ | $-i(a-i)$ |
| $\infty$ | $-i$ | 1 |

5.3.2. *Contribution near $x_\infty$.* Near $x_\infty$, the only orbifold lattice points in **core** are the ones near the $a$ vertices of $\mathbf{SC}_a(b)$, and hence here the orbifold lattice $\Lambda_O$ is just the rotated lattice $\Lambda_R$.

Since $x_\infty$ has

$$t^{(a-1)(b-1)/2}\prod_{k=1}^{a-1}\frac{1}{(1-qt^{-k})}$$

5.3.3. *Contribution near 0.* Consider the contribution of points in $\{x_0+\Lambda_R\}\cap\mathfrak{c}$ near $x_0$; from Lemma 5.14 and the values of $\ell, s\ell'$ on $x_0$, we have

$$\sum_{p\in}q^{s\ell'(p)}t^{\ell(p)} = q^{(a-1)(b-1)/2}\prod_{k=1}^{a-1}\frac{1}{(1-t^kq^{-k(a-k)})}$$

.

However, we have not taken account of the orbifold cosets of $\Lambda_R$. There are $a!$ chambers of $\mathcal{D}$, and the image of $\Lambda_C$ under $a$ of them come together to make $\Lambda_R$, and so there will be $(a-1)!$ orbifold cosets of $\Lambda_R$ near 0.

Since $\mathfrak{c}$ is integral at 0 with respect to $\Lambda_R$, each orbifold coset $\gamma$ of $\Lambda_R$ will have a unique minimal representative $x_\gamma$, so that the points in $\gamma\cap\mathfrak{c}$ will be $x_\gamma+(\Lambda_R\cap\mathfrak{c})$, and the contribution near 0 of the points in $\gamma$ will be the same as that of the points in $\Lambda_R+x_0$

$$q^{s\ell'(x_\gamma)}t^{\ell(x_\gamma)}$$

Thus, low degree symmetry will follow from

**Proposition 5.15.**

$$\sum_{\gamma\in\mathcal{OC}}q^{s\ell'(x_\gamma)}t^{\ell(x_\gamma)} = \prod_{k=1}^{a-1}[k]_{q^{-(a-k)}t}$$

5.3.4. *Proof of Proposition 5.15.* We factor the proof of Proposition 5.15 into two lemmas; the first establishes a bijection between the orbifold cosets $\gamma$ and permutations in $S_{a-1}$, and identifies permutation statistics that correspond to $\ell$ and $s\ell$; the second shows that these permutation statistics have the proper distribution. Before stating these lemmas, we introduce these permutation statistics, one of which is, to our knowledge, new, and may be of independent interest.

5.3.5. *Permutation Statitistics.* The permutation statistics we need will all be defined in terms of descents and inversions; the following summarizes the standard definitions we will need here:

**Definition 5.16.** For $\sigma\in S_n$, let

$$\mathbf{DES}(\sigma)=\Big\{i\in[1,n-1]\Big|\sigma(i)>\sigma(i+1)\Big\}$$

We use $\mathbf{des}(\sigma)$ to denote $|\mathbf{DES}(\sigma)|$, and

$$\mathbf{maj}(\sigma) = \sum_{i \in \mathbf{INV}(\sigma)} i$$

.

Recall that

$$\mathbf{inv}(\sigma) = \left|\left\{(i,j) \middle| 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\right\}\right|$$

Our new statistic is the *size* of $\sigma$, written $\mathbf{siz}(\sigma)$:

**Definition 5.17.**

$$\mathbf{siz}(\sigma) = \left(\sum_{i \in \mathbf{DES}(\sigma)} (n+1-i)i\right) - \mathbf{inv}(\sigma)$$

Our motivation for the definition of **siz** are the following two lemmas, which together immediately prove Proposition 5.15

**Lemma 5.18.** There is a labeling of the orbifold cosets by partitions $\sigma \in S_{a-1}$, so that if $v_\sigma$ be the minimum vector in the coset labeled by $\sigma$, then:

$$\ell(v_\sigma) = \mathbf{maj}(\sigma)$$
$$s\ell(v_\sigma) = \mathbf{siz}(\sigma)$$

**Lemma 5.19.**

$$\sum_{\sigma \in S_n} q^{\mathbf{siz}(\sigma)} t^{\mathbf{maj}(\sigma)} = \prod_{k=1}^{n} [k]_{q^{n+1-k}t}$$

**Remark 5.20.** The name *size* was chosen in reference to the size of a partition: by Lemma 5.19, for fixed $k$ and $\ell$, as $n$ grows large the number of permutations $\sigma \in S_n$ with $\mathbf{maj}(\sigma) = \ell$ and $\mathbf{siz}(\sigma) = k$ stabilizes to the number of partitions with length $\ell$ and size $k$.

5.4. **Proof of Lemma 5.18.**

5.4.1. *Bijection between $S_{a-1}$ and orbifold cosets.* First, we determine a bijection between orbifold cosets and $S_{a-1}$.

Let $w \in \Lambda_O \cap \mathfrak{c}$, and define $\sigma^w$ by

$$\frac{\sigma_i^w}{a} = \langle w_i - w_a \rangle$$

As $w \in S_a \Lambda_C$, we see $\sigma^w$ is a permutation in $S_{a-1}$.

Since the entires of the $v_i$ all have the same entries modulo 1, we see that$\sigma^{w+v_i} = \sigma^w$; that is, $\sigma^w$ is constant on the orbifold cosets.

It is not hard to see that this map is surjective, and hence a bijection between orbifold cosets and $S_{a-1}$.

5.4.2. *Smallest vector in each coset.* We now describe the minimal element $x^\sigma$ in the orbifold coset corresponding to $\sigma$.

Being the minimal vector $x^\sigma$ in a coset means that $x^\sigma - v_i \notin \mathcal{D}$ for all $i$, which is equivalent to

$$x_i^\sigma + 1 > x_{i+1}^\sigma, \quad 1 \le i \le a-1$$

To find $x^\sigma$ we will first define a vector $w^\sigma$ satisfying

$$w_i^\sigma < w_{i+1}^\sigma < w_i^\sigma + 1$$

$$\langle w_i - w_a \rangle = \frac{\sigma_i}{a}$$

but does not satisfy $\sum w_i^\sigma = 0$, we will then subtract the approproiate multiple of $(1/a, \ldots, 1/a)$ to get $v^\sigma$.

We need $w_{i+1}^\sigma > w_i^\sigma$ and $\langle w_{i+1}^\sigma - w_i^\sigma \rangle = \langle \sigma_{i+1}/a - \sigma_i/a \rangle$, and so we set

$$w_{i+1}^\sigma = w_i^\sigma + \frac{\sigma_{i+1} - \sigma_i}{a} + \mathbf{des}_i(\sigma)$$

where we have conventionally set $w_0^\sigma = \sigma_0 = 0, \sigma_a = a$.

Then

$$x_i^\sigma = w_i^\sigma - \frac{1}{a} \sum_{j=1}^{a} w_j^\sigma$$

is the minimal vector in the orbifold coset labeled by $\sigma$.

5.4.3. *Simplification.* To find $\ell(x^\sigma)$ and $s\ell(x^\sigma)$, we will want to simplify our expression for $x_i^\sigma$. The following definition will help.

**Definition 5.21.** For $i < j$, define $\mathbf{des}_{ij}$ to be the number of descents between $i$ and $j$. That is:

$$\mathbf{des}_{ij}(\sigma) = \left| \left\langle k \in \mathbf{DES}(\sigma) \middle| i \le k < j \right\} \right| = \sum_{k=i}^{j-1} \mathbf{des}_k(\sigma)$$

With this definition,

$$w_j = \frac{\sigma_j}{a} + \mathbf{des}_{1,j}(\sigma)$$

and so

$$\begin{aligned}\sum_{j=1}^{a} w_j &= \frac{1}{a} \sum_{i=1}^{a} \sigma_i + \sum_{i=1}^{a} \mathbf{des}_{1,i} \\ &= \frac{a+1}{2} + \sum_{i=1}^{a-2} (a-i) \mathbf{des}_i(\sigma)\end{aligned}$$

Thus,

$$x_j^\sigma = \frac{\sigma_j}{a} + \mathbf{des}_{1j}(\sigma) - \frac{a+1}{2a} - \frac{1}{a} \sum_{i=1}^{a-2} (a-i) \mathbf{des}_i(\sigma)$$

5.4.4. *Length of $x^\sigma$.* We compute (recalling the convention $\sigma_a = a$):

$$\begin{aligned}
\ell(x^\sigma) &= ax_a^\sigma - \frac{a-1}{2} \\
&= a + a\sum_{i=1}^{a-2} \mathbf{des}_i(\sigma) - \frac{a+1}{2} - \sum_{i=1}^{a-2}(a-i)\mathbf{des}_i(\sigma) - \frac{a-1}{2} \\
&= \sum_{i=1}^{a-2} i\mathbf{des}_i(\sigma) \\
&= \mathbf{maj}(\sigma)
\end{aligned}$$

5.4.5. *Skew length of $x^\sigma$.* We have

$$\begin{aligned}
s\ell(x^\sigma) &= \sum_{1\leq i<j\leq a} \left\langle v_j^\sigma - v_i^\sigma \right\rangle \\
&= \sum_{1\leq i<j\leq a} \left\langle \frac{\sigma_j - \sigma_i}{a} + \mathbf{des}_{ij}(\sigma) \right\rangle \\
&= \sum_{1\leq i<j\leq a} \mathbf{des}_{ij}(\sigma) - \delta(\sigma_j < \sigma_i)
\end{aligned}$$

Observe

$$\sum_{1\leq i<j\leq a} \delta(\sigma_j < \sigma_i) = \mathbf{inv}(\sigma).$$

and

$$\sum_{1\leq i<j\leq a} \mathbf{des}_{ij}(\sigma) = \sum_{k=1}^{a-2} k(a-k)\mathbf{des}_k(\sigma)$$

since for $\mathbf{des}_k$ to appear in $\mathbf{des}_{ij}$ we need $1 \leq i \leq k$ and $j < k \leq a$, and so $\mathbf{des}_k$ appears in $k(a-k)$ different $\mathbf{des}_{ij}$.

Thus, we have shown

$$s\ell(x^\sigma) = \sum_{k=1}^{a-2} k(a-k)\mathbf{des}_k(\sigma) - \mathbf{inv}(\sigma) = \mathbf{siz}(\sigma)$$

□

5.5. **Proof of Lemma 5.19.** Before we prove Lemma 5.19, we introduce a family of codes for permuations that we call *factorization codes*; our proof will use a specific factorization code we call the left-decreasing factorization code.

**Definition 5.22.** A *valid sequence of length $n$* is a sequence of integers $a_i, 1 \leq i \leq n$ such that $0 \leq a_i < i$. Let $\mathbf{VS}_n$ denote the set of valid sequences; clearly $|\mathbf{VS}| = n!$.

A *permutation code* is a bijection $\phi : \mathbf{VS}_n \to S_n$.

In section 5.5.1 we introduce a family of permutation codes we call *factorization codes*; in particular, this family includes the *left-decreasing factorization code* $\mathbf{LD}$.

Lemma 5.19 then reduces to showing:

**Lemma 5.23.** For a valid sequence $a \in \mathbf{VS}_n$, we have:

$$\mathbf{maj}(LD(a)) = \sum a_i$$
$$\mathbf{siz}(LD(a)) = \sum (n+1-i)a_i$$

5.5.1. *Factorization codes.* Factorization codes rest on the following simple observation. Let $C_k \in S_k$ be any $k$-cycle. Then $\{C_k^i\}, 0 \leq i < k$ form a family of representatives for the (left or right) cosets of $S_{k-1} \subset S_k$.

**Definition 5.24.** A *family $C$ of $k$-cycles* is a sequence $C_k, k \in \mathbb{N}$, with $C_k \in S_K$ a $k$-cycle.

The *right factorization* code associated to a family of $k$-cycles $C_k$ is the sequence of maps $R_n^C : \mathbf{VS}_n \to S_n$ defined by

$$R_n(a) = \alpha_k = C_2^{a_2} C_3^{a_3} \cdots C_n^{a_n}$$

Similarly, the *left factorization* code associated to a family of $k$-cycles $C_k$ is the the sequence of maps $L_n^C : \mathbf{VS}_n \to S_n$ defined by

$$L_n^C(a) = C_n^{a_n} C_{n-1}^{a_{n-1}} \cdots C_2^{a_2}$$

That the left and right factorization codes are in fact permutation codes follows easily from the observation using induction on $n$.

There are two "obvious" families of $k$-cycles: *increasing* cycles $C_k^+ = (1,2,3,\ldots,k)$, and the *decreasing* cycles $C_k^- = (k, k-1, k-2, \ldots, 1)$.

Thus, the left-decreasing factorization code $L_n^-$ is the bijection that sends $0 \leq a_i < i$ to to

$$L_n^-(a) = (C_n^-)^{a_n}(C_{n-1}^-)^{a_{n-1}} \ldots (C_2^-)^{a_2}$$

5.5.2. *Multiplication by $C_k^-$.* We now inductively prove Lemma 5.23 giving **maj** and **siz** of a permutation in terms of its left decreasing factorization code.

Clearly Lemma 5.23 holds on the identity permutation, where all $a_i = 0$. Thus we must show that in such a factorization, multiplying by $C_k^-$ raises **maj** by one and **siz** by $(n+1-k)$.

To do this, we must determine what multiplication by $C_k^-$ does to the set **DES** of descents. When multiplying by $C_k^-$, we have not yet permuted the elements $(k+1), (k+2), \ldots, n$, and so $\mathbf{DES} \subset \{1, \ldots, k-1\}$. As $C_k$ decreases $2, \ldots, j$ by 1, any comparisons involving two of these elements will remain unchanged; hence, the only descents multiplying by $C_k^-$ could change are those involving 1, which it will change to $k$.

Suppose that in the one-line notation of $\sigma$ the 1 is in position $j$; then $j-1$ will be a descent (unless $j = 1$), and $j$ will not be a descent. After we have multiplied by $c_k$, the 1 will change to a $k$, and so $j-1$ will not be a descent, and $j$ will be.

Thus, multiplying by $C_j$ will either increase a descent by one, or create a new descent at 1. In either case, the major index will increase by one.

We now investigate the effect of multipication by $C_k$ on **siz**, supposing that 1 is in position $j$. We first determine the change in the first term in **siz** (the sum over descents), and then determine the change this makes to the second term **inv**.

A descent at $j-1$ contributes

$$(n+1-(j-1))(j-1) = nj - j^2 + 3j - 2$$

to **siz**; a descent at $j$ contributes

$$(n+1-j)j = nj - j^2 + j$$

and thus multiplying by $C_k^-$ when 1 is in position $j < k$ will increase the first term of **siz** by $2-2j$.

We now turn to the inversions. It is clear that the only inversions that will change are those that were comparing 1. Before we multiply by $C_k^-$, 1 is in position $j$, and the $j-1$ pairs $(i,j), 1 \leq i \leq j-1$ will be inversions, and none of the $k-j$ pairs $(j,\ell), j+1 \leq \ell \leq k$ will be inversions. After we multiply by $C_k^-$, position $j$ will be $k$; none of the pairs $(i,j)$ will be inversions, and all of the pairs $(j,\ell)$ will be. Thus, **inv** increases by $k-2j+1$.

Thus, multiplying by $C_k^-$ when 1 is in position $j < k$ will change **siz** by

$$n - 2j + 2 - (k - 2j + 1) = n - k + 1$$

as desired.

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