Convex Optimization
Homework 2

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1 LP Duality

Consider the two problems

$$\begin{aligned} &\min_x c^T x \\ \text{s.t. } &Ax = b \\ &x \geq 0 \end{aligned} \tag{P}$$

$$\begin{aligned} &\max_y b^T y \\ \text{s.t. } &A^T y \leq c \end{aligned} \tag{D}$$

1.1

The Lagrangian associated to (P)

$$L(x, \lambda, \mu) = c^T x - \lambda^T (Ax - b) - \mu^T x = (c^T - \lambda^T A - \mu^T)x + \lambda^T b$$

The dual function

$$g(\lambda, \mu) = \begin{cases} \lambda^T b & \text{if } A^T \lambda + \mu - c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem of (P) can be written as

$$\max_{\substack{\mu \geq 0, \lambda \\ A^T \lambda + \mu - c = 0}} \quad b^T \lambda$$

Which can be simplified as equivalent to (D)

$$\begin{aligned} &\min_z b^T z \\ \text{s.t. } &A^T z \leq c \end{aligned}$$

1.2

In fact (D) is equivalent to

$$\begin{aligned} \min_{y} \; & -b^T y \\ \text{s.t.} \; & A^T y \le c \end{aligned} \tag{1}$$

The Lagrangian associated

$$L(y, \lambda, \mu) = -b^T y + \mu^T (A^T y - c) = (-b^T + \mu^T A^T) y - \mu^T c$$

The dual function

$$g(\lambda, \mu) = \begin{cases} -\mu^T c & \text{if } A\mu - b = 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem of (D) can be written as

$$\max_{\substack{\mu \ge 0 \\ A\mu = b}} -c^T \mu$$

Which can be simplified as equivalent to (P)

$$\begin{aligned} \min_{z} \; & c^T z \\ \text{s.t.} \; & Az = b \\ & z \ge 0 \end{aligned}$$

1.3

Consider the following problem

$$\begin{aligned} \min_{x,y} \; & c^T x - b^T y \\ \text{s.t.} \; & Ax = b \\ & x \ge 0 \\ & A^T y \le c \end{aligned} \tag{SD}$$

The Lagrangian of this problem

$$\begin{aligned} L(x, y, \lambda, \mu) &= c^T x - b^T y - \lambda^T (Ax - b) - \mu_1^T x + \mu_2^T (A^T y - c) \\ &= (c^T - \lambda^T A - \mu_1^T) x + (-b^T + \mu_2^T A^T) y + \lambda^T b - \mu_2^T c \end{aligned}$$

The dual function

$$g(\lambda, \mu) = \begin{cases} \lambda^T b - \mu_2^T c & \text{if } A^T \lambda + \mu_1 - c = A\mu_2 - b = 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem of (SD)

$$\begin{aligned} \max_{\lambda,\mu} \; & b^T \lambda - c^T \mu \\ \text{s.t.} \; & A^T \lambda + \mu_1 = c \\ & \mu \ge 0 \\ & A\mu_2 = b \end{aligned}$$

Or simplified as

$$
\begin{aligned}
\max_{z,t}\ & b^T z - c^T t \\
\text{s.t.}\ & A^T z \leq c \\
& t \geq 0 \\
& At = b
\end{aligned}
\tag{SDD}
$$

As (SD) $\Leftrightarrow$ (SDD), we deduce than that (SD) is self-dual.

1.4

Assume that (SD) is feasible and bounded, let $(x^\star, y^\star)$ be the optimal solution.
Let $\mathcal{D}_{\mathrm{P}}$ and $\mathcal{D}_{\mathrm{D}}$ be the feasible sets of (P) and (D). We notice that the constraint of (SD) are independent between x and y. Then $\mathcal{D}_{\mathrm{P}} \times \mathcal{D}_{\mathrm{D}}$ is the feasible set of (SD). We have

$$
\begin{aligned}
\min_{x \in \mathcal{D}_{\mathrm{P}}} c^T x &= \min_{x \in \mathcal{D}_{\mathrm{P}}} \left(c^T x - b^T y^\star\right) + b^T y^\star \geq c^T x^\star - b^T y^\star + b^T y^\star = c^T x^\star \\
\max_{y \in \mathcal{D}_{\mathrm{D}}} b^T y &= \max_{y \in \mathcal{D}_{\mathrm{D}}} \left(b^T y - c^T x^\star\right) + c^T x^\star \leq b^T y^\star - c^T x^\star + c^T x^\star = b^T y^\star
\end{aligned}
$$

Therefore, $x^\star, y^\star$ are optimal solution of (P) and (D).
Reciprocally, if $x^\star, y^\star$ are optimal solution of (P) and (D), $(x^\star, y^\star)$ is also the optimal solution of (SD) because

$$
\min_{(x,y) \in \mathcal{D}_{\mathrm{P}} \times \mathcal{D}_{\mathrm{D}}} \left(c^T x - b^T y\right) \geq \min_{x \in \mathcal{D}_{\mathrm{P}}} c^T x - \max_{y \in \mathcal{D}_{\mathrm{D}}} b^T y = c^T x^\star - b^T y^\star
$$

Since (P) and (D) are feasible, strong duality holds for these two problems. We have proven one is dual of another, thus

$$
c^T x^\star = b^T y^\star
$$

2 Regularized Least Square

$$
\min_x \|Ax - b\|_2^2 + \|x\|_1 \tag{RLS}
$$

2.1

The conjugate of $\|x\|_1$

$$
f^\star(y) = \sup_x (y^T x - \|x\|_1) = \sum_{i=1}^{d} \sup_{x_i} (y_i x_i - |x_i|) = \begin{cases} 0 & \text{if } -1 \leq y \leq 1 \\ +\infty & \text{otherwise} \end{cases}
$$

2.2

Rewrite the problem as

$$
\begin{aligned}
\min_{x,y}\ & \|y\|_2^2 + \|x\|_1 \\
\text{s.t.}\ & y = Ax - b
\end{aligned}
$$

The Lagrangian

$$L(x,y,\lambda) = \|y\|_2^2 + \|x\|_1 + \lambda^T(Ax - b - y) = \lambda^T Ax + \|x\|_1 + \|y\|_2^2 - \lambda^T y + \lambda^T b$$

The dual function

$$\begin{aligned} g(\lambda) &= \inf_x \left(\lambda^T Ax + \|x\|_1\right) + \inf_y \left(\|y\|_2^2 - \lambda^T y\right) + \lambda^T b \\ &= -f^\star(-A^T\lambda) - \frac{\|\lambda\|_2^2}{4} + \lambda^T b \end{aligned}$$

The dual problem

$$\begin{aligned} &\max_z -\frac{\|z\|_2^2}{4} + b^T z \\ \text{s.t } &-1 \le A^T z \le 1 \end{aligned} \tag{SDD}$$

3 Data seperation

$$\min_\omega \frac{1}{n}\sum_{i=1}^n \max\{0, 1 - y_i\omega^T x_i\} + \frac{\tau}{2}\|\omega\|_2^2 \tag{Sep1}$$

$$\begin{aligned} &\min_{\omega,z} \frac{1}{n\tau}\mathbf{1}^T z + \frac{1}{2}\|\omega\|_2^2 \\ \text{s.t. } & z \ge 0 \\ & z_i \ge 1 - y_i\omega^T x_i, \forall i \end{aligned} \tag{Sep2}$$

3.1

We can rewrite

$$\begin{aligned} \text{(Sep1)} &\Leftrightarrow \min_{\substack{\omega,z \\ z_i = \max\{0, 1-y_i\omega^T x_i\}, \forall i}} \left(\mathbf{1}^T z + \frac{\tau}{2}\|\omega\|_2^2\right) \\ &\Leftrightarrow \min_{\substack{\omega,z \\ z_i \ge 1-y_i\omega^T x_i, \forall i \\ z \ge 0}} \left(\mathbf{1}^T z + \frac{\tau}{2}\|\omega\|_2^2\right) \Leftrightarrow \text{(Sep2)} \end{aligned}$$

3.2

The Lagrangian

$$\begin{aligned} L(\omega, z, \lambda, \pi) &= \frac{1}{n\tau}\mathbf{1}^T z + \frac{1}{2}\|\omega\|_2^2 + \sum_{i=1}^n \lambda_i(1 - y_i\omega^T x_i - z_i) - \pi^T z \\ &= \left(\frac{1}{n\tau}\mathbf{1} - \lambda - \pi z\right)^T z + \left(\frac{1}{2}\|\omega\|_2^2 - \lambda^T\Phi\omega\right) + \mathbf{1}^T\lambda \end{aligned}$$

With $\Phi = \mathrm{diag}(y_1, ..., y_n)(x_1^T, ..., x_n^T)^T$
The dual function

$$g(\lambda, \pi) = \begin{cases} \mathbf{1}^T \lambda - \frac{1}{2} \left\| \Phi^T \lambda \right\|_2^2 & \text{if } \pi + \lambda = \frac{1}{n\tau} \mathbf{1} \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem is simplified as

$$\begin{aligned} &\max \mathbf{1}^T \lambda - \frac{1}{2} \left\| \Phi^T \lambda \right\|_2^2 \\ \text{s.t. } &0 \leq \lambda \leq \frac{1}{n\tau} \end{aligned}$$

4 Robust LP

Considering first the LP problem for a certain x and compute its dual problem

$$\begin{aligned} &\max_a x^T a \\ \text{s.t. } &C^T a \leq d \end{aligned} \tag{1}$$

According to 1.2 its dual is

$$\begin{aligned} &\min_z d^T z \\ \text{s.t. } &Cz = x \\ &z \geq 0 \end{aligned} \tag{2}$$

We know that $\mathcal{P} = \{a | C^T a \leq d\}$ is non empty, hence (1) is feasible, thus strong duality holds.
Therefore the initial problem of robust LP

$$\begin{aligned} &\min_x c^T x \\ \text{s.t. } &\max_{a \in \mathcal{P}} x^T a \leq b \end{aligned} \tag{3}$$

is in fact equivalent to

$$\begin{aligned} &\min_x c^T x \\ \text{s.t. } &\min_{z \in \mathcal{Q}_x} d^T z \leq b \end{aligned} \tag{4}$$

Where $\mathcal{Q}_x = \{z | Cz = x, z \geq 0\}$
Notice that

$$\min_{z \in \mathcal{Q}_x} d^T z \leq b \Leftrightarrow \exists z \in \mathcal{Q}_x : d^T z \leq b \Leftrightarrow \exists z \geq 0, Cz = x, d^T z \leq b$$

Then (4) is equivalent to

$$\begin{aligned} &\min_{x,z} c^T x \\ \text{s.t } &Cz = x \\ &z \geq 0 \\ &d^T z \leq b \end{aligned} \tag{5}$$

We conclude the proof that (3) $\Leftrightarrow$ (5).

5 Boolean LP

Consider the boolean LP problem, which is in general hard to solve

$$\begin{aligned} &\min_x c^T x \\ \text{s.t } & Ax \leq b \\ & x \in \{0,1\}^n \end{aligned} \tag{BLP}$$

Here we study two relaxations of this problem.
The LP relaxation

$$\begin{aligned} &\min_x c^T x \\ \text{s.t } & Ax \leq b \\ & x \in [0,1]^n \end{aligned} \tag{LP}$$

The Lagrangian relaxation (LR)

$$\begin{aligned} &\min_x c^T x \\ \text{s.t } & Ax \leq b \\ & x_i(1-x_i) = 0, \forall i = 1, ..., n \end{aligned} \tag{LR}$$

5.1

The Lagrangian associated to (LR)

$$\begin{aligned} L(x, \lambda, \mu) &= c^T x + \lambda^T (Ax - b) + \sum_{i=1}^n \mu_i x_i (x_i - 1) \\ &= -\lambda^T b + \sum_{i=1}^n \left(\left(c_i + (A^T \lambda)_i - \mu_i \right) x_i + \mu_i x_i^2 \right) \end{aligned}$$

The dual function

$$g(\lambda, \mu) = \begin{cases} -\lambda^T b - \sum_{i=1}^n \frac{1}{\mu_i} (c_i + A_i^T \lambda - \mu_i)^2 & \text{if } \mu \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem of (LR)

$$\begin{aligned} &\max_{\lambda, \mu} -b^T \lambda - \sum_{i=1}^n \frac{1}{4\mu_i} (c_i + A_i^T \lambda - \mu_i)^2 \\ \text{s.t. } & \lambda, \mu \geq 0 \end{aligned}$$

Which is, by maximizing on μ_i, equivalent to

$$\begin{aligned} &\max_{\lambda} -b^T \lambda + \sum_{i=1}^n \min\{0, c_i + A_i^T \lambda\} \\ \text{s.t. } & \lambda \geq 0 \end{aligned}$$

Or by posing $\nu_i = -\min\{0, c_i + A_i^T\}$

$$\begin{aligned} &\max_{\lambda} -b^T\lambda - \mathbf{1}^T\nu \\ \text{s.t. } &\lambda, \nu \geq 0 \\ &c_i + A_i^T\lambda + \nu_i \geq 0, \forall i = 1, ..., n \end{aligned} \tag{LRD}$$

5.2

The Lagrangian associated to (LP)

$$\begin{aligned} L(x, \lambda, \mu, \nu) &= c^T x + \lambda^T(Ax - b) - \mu^T x + \nu^T(x - \mathbf{1}) \\ &= (c + A^T\lambda - \mu + \nu)^T x - \lambda^T b - \nu^T \mathbf{1} \end{aligned}$$

The dual function

$$g(\lambda, \mu, \nu) = \begin{cases} -\lambda^T b - \nu^T \mathbf{1} & \text{if } c + A^T\lambda - \mu + \nu = 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem of (LP)

$$\begin{aligned} &\max_{\lambda, \nu} -b^T\lambda - \mathbf{1}^T\nu \\ \text{s.t. } &A^T\lambda + c + \nu \geq 0 \\ &\lambda, \nu \geq 0 \end{aligned} \tag{LPD}$$

We see that the lower bounds for (BLP) obtained from (LRD) and (LPD) are the same.