An Algebraic Identity Leading to Wilson's Theorem Sebastián Martín Ruiz

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In most text books on number theory Wilson's theorem is proved by applying Lagrange's theorem concerning polynomial congruences [1,2,3,4]. Hardy and Wright also give a proof using quadratic residues [3]. In this note Wilson's theorem is derived as a corollary to an algebraic identity.

Theorem 1:

For all integers $n \ge 0$ and for all real numbers x

$$\sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (x-i)^{n} = n!$$

Proof:

We proceed by induction. Let

$$f_n(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n$$

It is easy to show that $f_0(x) = 1 = 0!$

Assume $f_k(x) = k! \ \forall \ x \in \mathbb{R}$

We consider

$$f'_{k+1}(x) = \frac{d}{dx} \left[\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (x-i)^{k+1} \right] =$$

$$= (k+1) \sum_{i=0}^{k+1} (-1)^i \frac{(k+1)!}{i!(k+1-i)!} (x-i)^k$$

Splitting off the i = k + 1 term and using

$$\frac{(k+1)!}{i!(k+1-i)!} = \frac{k!}{i!(k-i)!} \cdot \frac{k+1}{k+1-i} = \binom{k}{i} \left(1 + \frac{i}{k+1-i}\right)$$

we get:

$$f'_{k+1}(x) = (k+1) \left[\sum_{i=0}^{k} (-1)^{i} {k \choose i} (x-i)^{k} + (-1)^{k+1} [x - (k+1)]^{k} \right] + (k+1) \left[\sum_{k=1}^{k} (-1)^{i} \frac{k! (x-i)^{k}}{(i-1)! (k+1-i)!} \right]$$

$$= (k+1) \left[f_k(x) + \sum_{j=0}^k (-1)^{j+1} {k \choose j} (x-1-j)^k \right] \text{ where } j = i-1$$

$$= (k+1) \left[f_k(x) - f_k(x-1) \right] = (k+1) \left[k! - k! \right] = 0.$$

Thus $f_{k+1}(x)$ is constant and in particular

$$f_{k+1}(x) = f_{k+1}(k+1) =$$

$$= \sum_{i=0}^{k} (-1)^{i} \frac{(k+1)!}{i!(k-i)!} (k+1-i)^{k}$$

Since the i = k + 1 term is zero

$$f_{k+1}(x) = (k+1)\sum_{i=0}^{k} (-1)^{i} {k \choose i} (k+1-i)^{k} =$$

$$=(k+1)f_k(k+1) = (k+1)k! = (k+1)!$$

so
$$f_k(x) = k! \Rightarrow f_{k+1}(x) = (k+1)!$$

Therefore
$$f_n(x) = n! \ \forall x \in \mathbb{R} \ \forall n \in \mathbb{N}$$

Corollary 1: (Wilson's theorem)

For any prime number P, we have: $(p-1)! \equiv p-1 \pmod{p}$.

Proof:

Let n = p - 1 where P is prime. We use the formula for x = 0.

$$\sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} (-i)^{p-1} = (p-1)! \tag{1}$$

But for $p > 1 \ge 1$:

$$0 \equiv \binom{p}{i} = \binom{p-1}{i-1} + \binom{p-1}{i} \pmod{p}$$

Thus

$$\begin{pmatrix} p-1 \\ i \end{pmatrix} \equiv -\begin{pmatrix} p-1 \\ i-1 \end{pmatrix} \pmod{p} \\
\begin{pmatrix} p-1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pmod{p} \\
1 \end{pmatrix} \Rightarrow \begin{pmatrix} p-1 \\ i \end{pmatrix} \equiv (-1)^{i} \pmod{p}$$

Using this result in (1) we obtain

$$\sum_{i=0}^{p-1} (-1)^i (-1)^i (-i)^{p-1} \equiv (p-1)! \pmod{p}$$

If p > 2, p is odd and therefore p-1 is even. Thus we have the relation $(-i)^{p-1} = i^{p-1}$ which allows us to obtain

$$\sum_{i=0}^{p-1} i^{p-1} \equiv (p-1)! \pmod{p} \quad (2)$$

For on the other hand since P is not a factor of i, and using Fermat's theorem, we have

$$i^{p-1} \equiv 1 \pmod{p} \quad (3)$$

Combining (2) and (3), we can conclude:

$$\sum_{i=1}^{p-1} 1 \equiv (p-1)! \pmod{p}$$

and this last relation can be written in the form

$$(p-1)! \equiv p-1 \pmod{p}$$

Corollary 2:

For all integers $n \ge 0$ and for all real numbers x

$$\sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (x-i)^{n-j} = 0 \quad 1 \le j \le n$$

Proof:

We consider the algebraic identity of theorem 1

$$f_n(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n$$

Differentiating j times

$$f_n^{(j)}(x) = n(n-1)\cdots[n-(j-1)]\sum_{i=0}^n (-1)^i \binom{n}{i} (x-i)^{n-j} = 0$$

Then,

$$\sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (x-i)^{n-j} = 0 \quad 1 \le j \le n$$

References:

- 1: T.M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York (1976)
- 2: R. D. Carmichael, *Theory of Numbers*, John Wiley and Sons, New York (1914)
- 3: G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Fourth edition, Oxford University Press (1960)
- 4: Kenneth Ireland and Michael Rosen, *A classical introduction to modern number theory*, second edition, Springer-Verlag, New York (1990)

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The Mathematical Gazette Volume 80 ,number 489 November 1996 pp.579-582