

# Bayesian Learning for Classification using a Uniform Dirichlet Prior

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Bayesian approaches to machine learning attempt to make better decisions by exploiting *prior knowledge* regarding the data-generating distribution:

### Informative

- If the prior is localized around the true data-generating model, low-risk decisions can be made even with limited training data
- Priors that assign low weighting to the true model may not be able to realize satisfactory performance

### Non-Informative

- Learners designed with approximately uniform priors will not perform as well as those made with well-selected informative priors
- Avoid high risk inherent to learners made by mismatched informative priors

- Often, priors are termed non-informative as long as they are approximately uniform over their limited support. For example, a parametric regression function might use a high covariance Gaussian vector prior to characterize a subset of probability distributions
- The uniform Dirichlet distribution is unique in that it has full support over the space of data-generating distributions and is thus truly non-informative
- Additionally, it is a *conjugate prior* for independent, identically distributed observations and leads to a closed-form model posterior distribution

## Objective and Bayesian Setup

**Observable random element:**  $x \in \mathcal{X}$

**Unobservable random element:**  $y \in \mathcal{Y}$

**Observable training data:**  $D \in \mathcal{D} = \{\mathcal{Y} \times \mathcal{X}\}^N$

Independently, identically distributed according to an unknown probability mass function (PMF)

$$\theta \in \Theta = \left\{ \theta \in \mathbb{R}_{\geq 0}^{\mathcal{Y} \times \mathcal{X}} : \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \theta(y, x) = 1 \right\},$$

such that  $P_{y,x|\theta}(y, x|\theta) = P_{D_n|\theta}(y, x|\theta) = \theta(y, x)$ .

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*Alternate Notation:*  $\theta \Leftrightarrow (\theta', \tilde{\theta})$

- Marginal model  $\theta' \equiv \sum_{y \in \mathcal{Y}} \theta(y, \cdot) = P_{x|\theta}$  over the set  $\mathcal{X}$
- Conditional models  $\tilde{\theta}(x) \equiv \theta(\cdot, x)/\theta'(x) = P_{y|x,\theta}$  over the set  $\mathcal{Y}$

# Sufficient Statistic

Training data PMF:

$$P_{D|\theta}(D|\theta) = \prod_{y \in \mathcal{Y}} \prod_{x \in \mathcal{X}} \theta(y, x)^{\bar{N}(y, x; D)}$$

is expressed via

$$\bar{N}(y, x; D) = \sum_{n=1}^N \delta[(y, x), D_n], \text{ with range}$$

$$\bar{\mathcal{N}} = \left\{ \bar{n} \in \mathbb{Z}_{\geq 0}^{\mathcal{Y} \times \mathcal{X}} : \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \bar{n}(y, x) = N \right\}$$

- Empirical count  $\bar{N}(D)$  is a *sufficient statistic* for the model  $\theta$
- $|\bar{\mathcal{N}}| = \binom{N + |\mathcal{Y}| |\mathcal{X}| - 1}{|\mathcal{Y}| |\mathcal{X}| - 1} \leq |\mathcal{D}|$   
 $\Rightarrow$  **Efficient Transform**

$$\Downarrow \quad \Downarrow$$

**Compact Representation**

**Express distributions using new random process**  $\bar{n} \equiv \bar{N}(D) \in \bar{\mathcal{N}}$

## Design Metric

User-selected function  $\mathcal{L} : \mathcal{Y} \times \mathcal{Y} \mapsto \mathbb{R}_{\geq 0}$ .  
Most commonly selected for classification  
is the 0–1 loss:

$$\mathcal{L}(h, y) = 1 - \delta[h, y]$$

- \* All misclassifications penalized equally

## Design Task

Create a classification function  $f : \bar{\mathcal{N}} \mapsto \mathcal{Y}^{\mathcal{X}}$   
that minimizes the conditional expected  
loss, or conditional “risk”,

$$\begin{aligned}\mathcal{R}_{\Theta}(f; \theta) &= E_{x, \bar{n} | \theta} \left[ E_{y | x, \theta} \left[ \mathcal{L} (f(x; \bar{n}), y) \right] \right] \\ &= 1 - E_{x, \bar{n} | \theta} \left[ \tilde{\theta}(f(x; \bar{n}); x) \right]\end{aligned}$$

# Clairvoyant Hypothesis and Risk

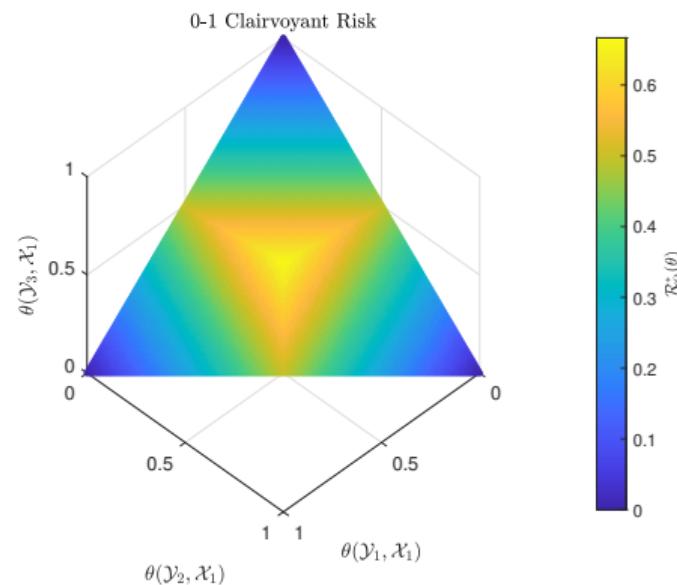
The “clairvoyant” classifier  $f_\theta : \Theta \mapsto \mathcal{Y}^\mathcal{X}$  is maximum *a posteriori* (MAP):

$$f_\Theta(x; \theta) = \arg \min_{h \in \mathcal{Y}} E_{y|x, \theta} [1 - \delta[h, y]]$$

$$\equiv \arg \max_{y \in \mathcal{Y}} \tilde{\theta}(y; x)$$

↓      ↓

$$\mathcal{R}_\Theta^*(\theta) \equiv 1 - \sum_{x \in \mathcal{X}} \theta'(x) \max_{y \in \mathcal{Y}} \tilde{\theta}(y; x)$$



**Clairvoyant Risk = Lower Bound for Conditional Risk**

↓    ↓    **Model Unknown. Select Prior  $p_\theta$**     ↓    ↓

$$\mathcal{R}(f) = E_\theta [ \mathcal{R}_\Theta(f; \theta) ] = 1 - E_{x, \bar{n}} [ P_{y|x, \bar{n}} (f(x; \bar{n})) ]$$

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**Optimal classifier:** Bayesian MAP

$$f^*(x; \bar{n}) = \arg \max_{y \in \mathcal{Y}} P_{y|x, \bar{n}}(y | x, \bar{n})$$

**Minimum Bayes Probability of Error:**

$$\mathcal{R}(f^*) = 1 - E_{x, \bar{n}} \left[ \max_{y \in \mathcal{Y}} P_{y|x, \bar{n}} (y | x, \bar{n}) \right]$$

## Predictive Distributions

Bayesian PMF is the conditional expectation of the true PMF  $P_{y|x, \theta} \equiv \tilde{\theta}(x)$ :

$$P_{y|x, \bar{n}} = E_{\theta|x, \bar{n}} [ P_{y|x, \theta} ] \equiv \mu_{\tilde{\theta}(x)|x, \bar{n}}$$

# Distributions: Prior to Predictive

# Dirichlet Prior PDF

The probability density function (PDF) for the model random process  $\theta \in \Theta$  is Dirichlet with parameters  $\alpha(\cdot, \cdot) = 1$ :

$$\begin{aligned} p_{\theta}(\theta) &= \text{Dir}(\theta; \alpha) \Big|_{\alpha(\cdot, \cdot)=1} \\ &= \left[ \beta(\alpha)^{-1} \prod_{y \in \mathcal{Y}} \prod_{x \in \mathcal{X}} \theta(y, x)^{\alpha(y, x)-1} \right]_{\alpha(\cdot, \cdot)=1} \\ &= (|\mathcal{Y}| |\mathcal{X}| - 1)! \end{aligned}$$

$$\begin{aligned} &\Rightarrow p_{\tilde{\theta}|\theta'}(\tilde{\theta}|\theta') = p_{\tilde{\theta}}(\tilde{\theta}) \\ &\Rightarrow = \prod_{x \in \mathcal{X}} \text{Dir}(\tilde{\theta}(x); \alpha(\cdot, x)) \Big|_{\alpha(\cdot, x)=1} \\ &\quad = [(|\mathcal{Y}| - 1)!]^{|\mathcal{X}|} \end{aligned}$$

Conditional PDF of  $\tilde{\theta}$  given  $\theta'$  is

**True predictive distribution  $\tilde{\theta}$  is independent of the marginal model and inherits a uniform PDF**

- Independence of  $\tilde{\theta}$  from  $\theta'$  implies conditional independence of  $\tilde{\theta}$  from  $x$  given  $\bar{n}$
- As the data PMF  $P_{\bar{n}|\theta}$  has exponential form, the Dirichlet PDF is a conjugate prior

$$\begin{aligned} p_{\tilde{\theta}|x,\bar{n}}(\tilde{\theta}|x, \bar{n}) &= p_{\tilde{\theta}|\bar{n}}(\tilde{\theta}|\bar{n}) \\ &= \prod_{x' \in \mathcal{X}} \text{Dir}\left(\tilde{\theta}(x'); \alpha(\cdot, x')\right) \Big|_{\alpha(\cdot, x') = 1 + \bar{n}(\cdot, x')} \end{aligned}$$

**Posterior for model  $\tilde{\theta}(x)$  is Dirichlet and dependent solely on the sufficient statistic elements  $\bar{n}(\cdot, x)$**

# Model Posterior PDF

## Asymptotic Trend

- Covariance of Dirichlet  $p_{\tilde{\theta}(x)|\bar{n}(\cdot,x)}$  decreases monotonically with concentration

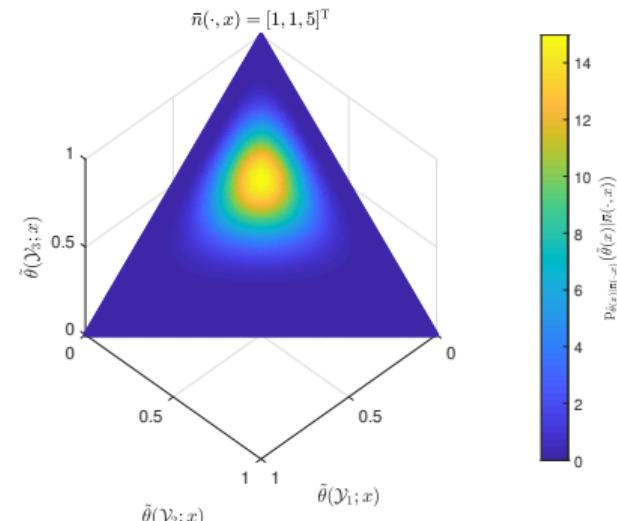
$$\alpha'(x) \equiv \sum_{y \in \mathcal{Y}} \alpha(y, x)$$

- Localized around  $\mu_{\tilde{\theta}(x)|\bar{n}(\cdot,x)} = \alpha(\cdot, x)/\alpha'(x)$

$$\Downarrow \quad \Downarrow$$

As  $n'(x) \equiv \sum_{y \in \mathcal{Y}} \bar{n}(y, x) \rightarrow \infty$ , the posteriors converge to

$$p_{\tilde{\theta}(x)|\bar{n}(\cdot,x)} \left( \tilde{\theta}(x) | \bar{n}(\cdot, x) \right) \rightarrow \delta \left( \tilde{\theta}(x) - \frac{\bar{n}(\cdot, x)}{n'(x)} \right)$$



Asymptotically consistent estimation of  $\tilde{\theta}$  due to full support of prior

# Bayesian Predictive PMF

*The Bayesian predictive PMF is a convex combination of two conditional distributions:*

$$\begin{aligned} P_{y|x,\bar{n}}(\cdot|x, \bar{n}) &= \mu_{\tilde{\theta}(x)|\bar{n}(\cdot,x)}(\bar{n}(\cdot, x)) \\ &\equiv \left( \frac{|\mathcal{Y}|}{n'(x) + |\mathcal{Y}|} \right) \frac{1}{|\mathcal{Y}|} + \left( \frac{n'(x)}{n'(x) + |\mathcal{Y}|} \right) \frac{\bar{n}(\cdot, x)}{n'(x)} \end{aligned}$$

**Prior Mean** Uniform PMF  $\mu_{\tilde{\theta}(x)} = |\mathcal{Y}|^{-1}$  dependent only on the Dirichlet parameterization

**Conditional Empirical PMF**  $\bar{n}(\cdot, x)/n'(x)$  dependent only on training data

**As the number of training data  $n'(x)$  increases relative to the number of classes  $|\mathcal{Y}|$ , the predictive distribution tends toward the empirical PMF**

# Bayesian Classifier and Error Trends

**Optimal Hypothesis:** the *conditional majority decision*

$$f^*(x; \bar{n}) = \arg \max_{y \in \mathcal{Y}} P_{y|x, \bar{n}}(y|x, \bar{n}) = \arg \max_{y \in \mathcal{Y}} \bar{n}(y, x)$$

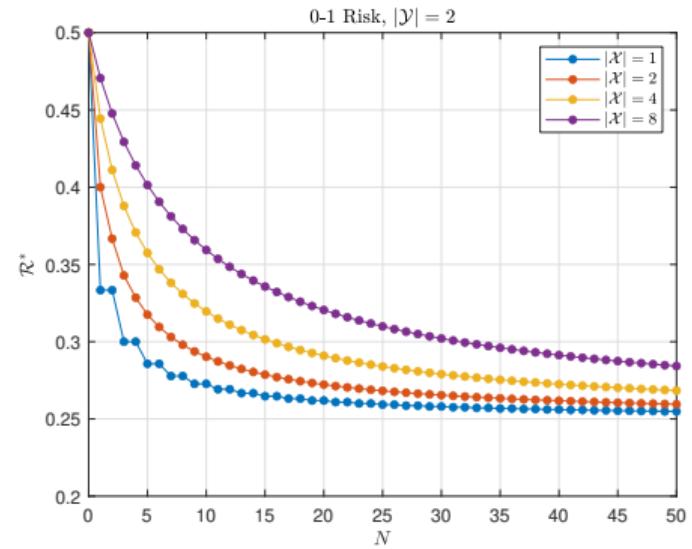
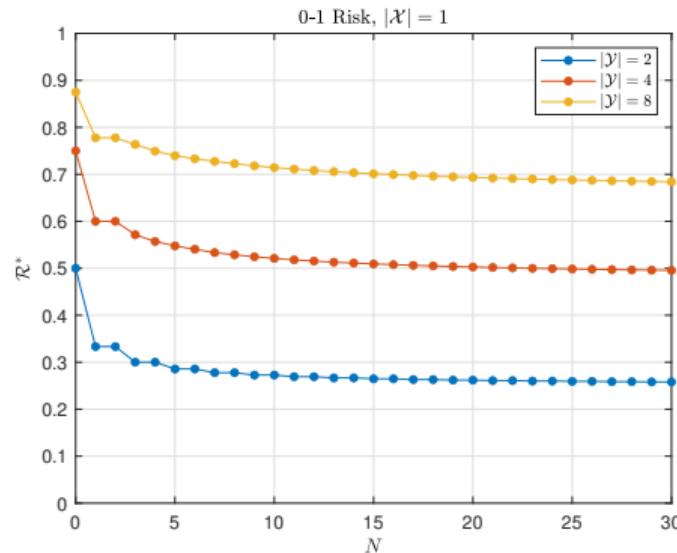
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**Minimum Expected Probability of Error:**

$$\begin{aligned} \mathcal{R}^* &= 1 - E_{x,D} \left[ \max_{y \in \mathcal{Y}} P_{y|x,D}(y|x, D) \right] = 1 - \sum_{x \in \mathcal{X}} \frac{E_{\bar{n}} \left[ \max_{y \in \mathcal{Y}} \bar{n}(y, x) \right] + 1}{|\mathcal{Y}| |\mathcal{X}| + N} \\ &= 1 - \frac{\sum_{m=1}^{|\mathcal{Y}|} \binom{|\mathcal{Y}|}{m} (-1)^{m-1} \sum_{n=0}^{\lfloor \frac{N}{m} \rfloor} \prod_{l=1}^{|\mathcal{Y}|} |\mathcal{X}|^{-1} \left(1 - \frac{mn}{N+l}\right)}{|\mathcal{Y}| + N/|\mathcal{X}|} \end{aligned}$$

- \* Efficient formula derived using Inclusion-Exclusion principle

# Probability of Error Trends with Class/Data Set Sizes

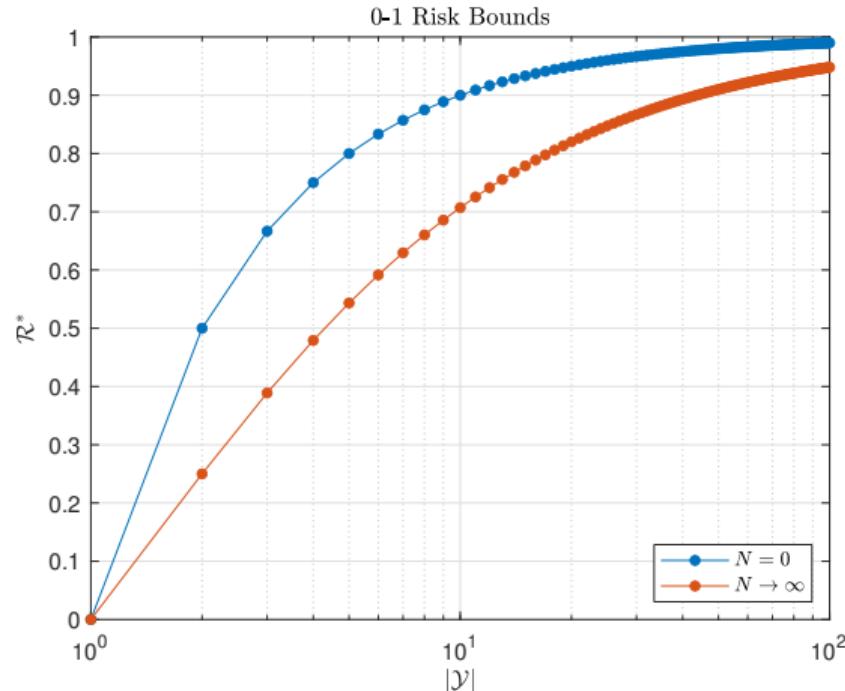


- Larger class sets  $\mathcal{Y}$  raise the lower bound on probability of error
- Larger observation set  $\mathcal{X}$  → more data  $N$  required to achieve the same level of performance

# Probability of Error Trends with Training Data Volume

- For binary classification, infinite training data only reduces the expected probability of error from 0.5 to 0.25
- As  $|\mathcal{Y}|$  increases, the probability of error tends to unity and any improvement due to training data becomes negligible

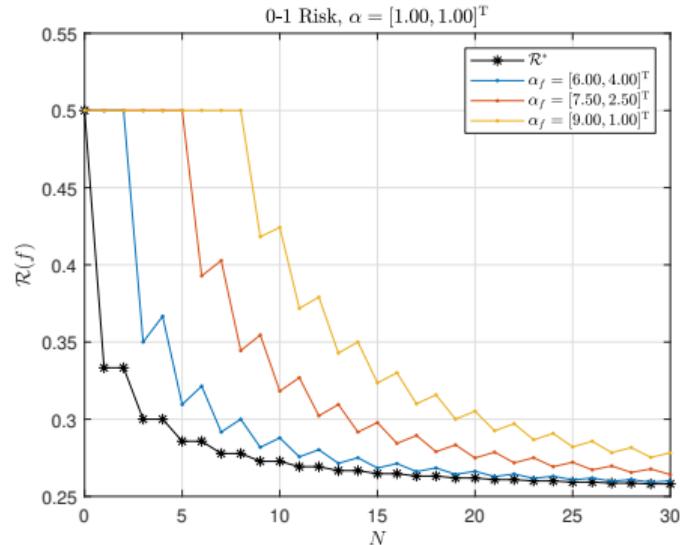
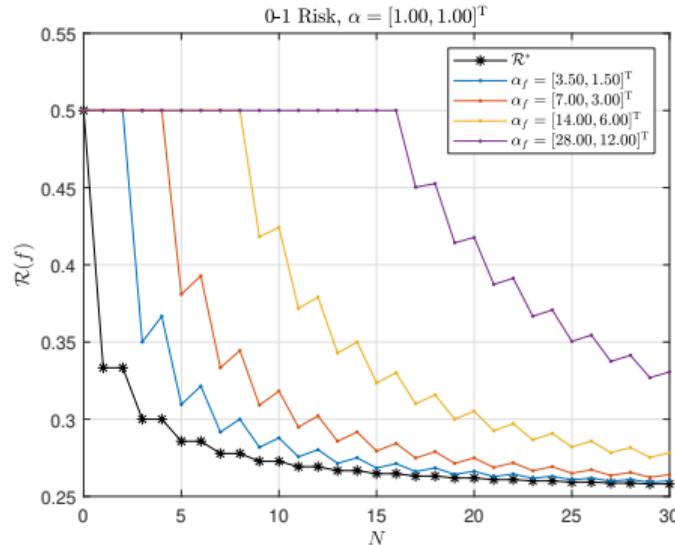
$N$	$\mathcal{R}^*$
0	$1 -  \mathcal{Y} ^{-1}$
$\rightarrow \infty$	$1 -  \mathcal{Y} ^{-1} \sum_{m=1}^{ \mathcal{Y} } m^{-1}$



# Comparison to Informative Classifiers

## Bayes Risk

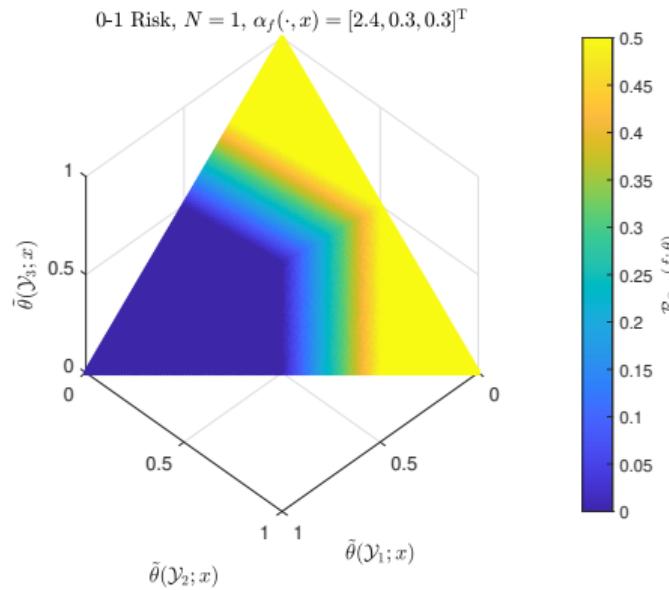
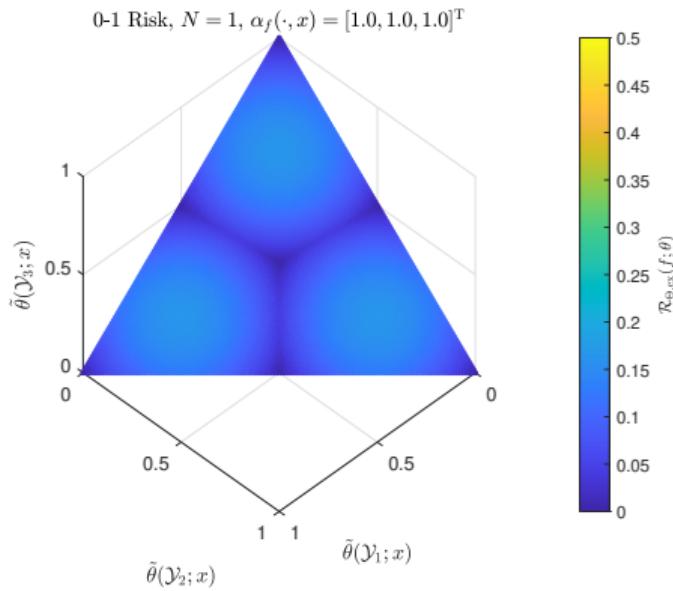
Classifiers derived from informative Dirichlet priors with  $\alpha'_f(x) \gg |\mathcal{Y}| |\mathcal{X}|$  prioritize the prior mean  $\mu_{\tilde{\theta}} = \alpha_f(\cdot, x)/\alpha'_f(x)$  for prediction



Slower adaptation  $\Rightarrow$  Higher Bayes probability of error

# Comparison to Informative Classifiers

## Excess Conditional Risk



## Trade-Off

**Uniform prior provides a robust classifier for all unknown models  $\theta$ , limiting the space of high error models. However, fewer models achieve the clairvoyant risk.**

# Conclusions

- The majority decision classifier designed with a uniform Dirichlet prior minimizes the possibility of maximal error for applications where the data distribution can not be adequately modeled
- Full support of Dirichlet priors guarantees minimal risk in the limit of training data volume
- Efficient closed-form for Bayesian probability of error provides a lower-bound for general classifiers
- Low localization Dirichlet priors result in the same classifier - these priors are sensible for recognition applications where humans perform with low-error

