University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATIONS, APRIL 1997

First Year – Program 5

MAT 195S
Calculus II

Examiners: Prof. S.H. Smith
Prof. P. A. Sullivan
Duration – $2\frac{1}{2}$ hours

NAME: ______________________________

STUDENT NO: ______________________________

INSTRUCTIONS:

Your answers should be written in the space provided on the question paper; if necessary, also write on the back of the preceding page.

Answer all questions.

Non-programmable calculators allowed.

Values for each question are shown in brackets [ ] in the left margin.
Total marks are 130

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TOTAL	

1. (a) Integrate $\int \frac{dx}{(x-x^2)^{\frac{1}{2}}}$

(b) Evaluate $\int_0^1 x^2 \ln(1+x^2)dx$

(c) Integrate $\int \frac{\ln(1-x)}{x^{\frac{1}{2}}}dx$

(d) For what values of the constant p does the improper integral

$\int_0^1 \frac{\ln(1-x)}{x^p}dx$ exist?

2. Sketch the graph of $r = a(1 - \cos 2\theta)$ for constant $a > 0$. Find the total area enclosed by this curve.

(a) Show that $\lim_{n\to\infty} (a^n + b^n)^{\frac{1}{n}} = b$ for $0 < a < b$

(b) Evaluate $\lim_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$

(c) Evaluate $\lim_{x \to 1} \left(\frac{2}{x^2 - 1} - \frac{3}{x^3 - 1} \right)$

(d) Generalize part (c) to show that $\lim_{x \to 1} \left(\frac{m}{x^m - 1} - \frac{n}{x^n - 1} \right) = \frac{n - m}{2}$ for positive integers m and n.

(a) For what values of x is the series $\sum_{k=1}^{\infty} \frac{x^k}{k^2 + k^{-2}}$ convergent

(b) For what values of x is the series $\sum_{k=0}^{\infty} 2^{\sqrt{k}} x^k$ convergent?

Sum the series $\sum_{k=1}^{\infty} \frac{k^2+1}{k} x^k$; state the values of x for which the result is valid.

(a) Give the Taylor series for $(1+x^4)^{-\frac{1}{2}}$, and hence calculate $\int_0^{\frac{1}{2}} \frac{dx}{(1+x^4)^{\frac{1}{2}}}$ to three decimal places.

(b) Without completing the details, explain what strategy you would develop to approximate $\int_2^{\infty} \frac{dx}{(1+x^4)^{\frac{1}{2}}}$

If $F(x)$ is a continuous function for $a \leq x \leq b$, then the mean value theorem of integral calculus states that there is some value c between a and b such that $\int_a^b f(x)dx = (b-a)f(c)$.

(a) If $p(x)$ is a positive continuous function for $a \leq x \leq b$ show that

$$\int_a^b f(x)p(x)dx = f(\xi) \int_a^b p(x)dx \text{ for some } a \leq \xi \leq b,$$

(b) Hence or otherwise, prove that if $F(x)$ and $G(x)$ are continuous functions for $a \leq x \leq b$, with continuous first derivatives, and with $G'(x)$ never zero, then $\dfrac{F(b)-F(a)}{G(b)-G(a)} = \dfrac{F'(\xi)}{G'(\xi)}$ for some $a \leq \xi \leq b$.

A particle starts at the origin at time $t = 0$ and then slides down a curve whose radius vector $\underline{r}(t)$ is given by $\underline{r} = (t^2 \sin t, t^2 \cos t, -t)$.

Describe the curve geometrically.

What is the speed of the particle at time t?

(c) At what angle does the path cut the circular cylinder $x^2 + y^2 = 1$?

8. If $f(x,\, y) = (x+y)\; F(\frac{y}{x})$, for arbitrary functions F, show that $x f_x + y f_y = f$, , and hence that

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = 0.$$

(a) Find the equation of the target plane to the ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point $(x_0,\, y_0,\, z_0)$.

(b) Hence show that the plane $lx + my + nz = p$ is a tangent plane when

$a^2 l^2 + b^2 m^2 + c^2 n^2 = p^2$.

10. Given the function

$$f(x,y) = \begin{cases} \dfrac{xy^2}{(x^2+y^2)^{3/2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that both f_x and f_y exist at the origin, but that f is not continuous there.

11. (a) Find the equation of the osculating plane at the point with parameter t of the circular helix given by $\underset{\sim}{r}(t) = (a\cos t, a\sin t, \lambda t)$; a, λ are positive constants

(b) Hence show that the lines through the origin parallel to the binomials lie on the surface of the cone $a^2(x^2+y^2) = \lambda^2 z^2$. (The binomial is the unit vector which is perpendicular to the osculating plane.)