UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, APRIL 2001

Second Year — Program: Mechanical Engineering

AER234S: NUMERICAL METHODS

Examiner: Professor C. P. T. Groth

Instructions: *(a)* Exam Type A: closed book, no notes or textbook permitted.
(b) Calculator Type 2: no programmable calculators.
(c) Relative mark values for questions are indicated on exam.

15% 1. *(a)* What are the final values that are assigned to the variables for the four FORTRAN 77 arithmetic expressions listed below?

```
J=10**(1/5)        K=32.0/4.04/4.0        U=2D+3 + 2D-3        V=10*4.00**3**2/256.0
```

Here, J and K are integer variables (INTEGER*4), U is a double-precision real variable (REAL*8), and V is single-precision real variable (REAL*4).

(b) How many passes (cycles) will be made through each of the four DO loops in the FORTRAN code given below?

```
  DO 6 I=-6,6        DO J=5,12,3          DO 7777 L=13,11,13        DO L=8,4,-2
6 CONTINUE           END DO          7777 CONTINUE                  END DO
```

In the DO loops given above, I, J, and L are integer variables (INTEGER*4).

(c) The midpoint rule applied to the evaluation of the integral $I = \int_a^b f(x)dx$ yields an absolute integration error of $\epsilon = |I - M(\Delta x)| = 0.08$ using $n - 1 = 10$ subintervals. How many subintervals should you use to reduce the absolute error by a factor of four?

(d) Arrange the following quadrature methods in order of increasing accuracy: Gauss quadrature, Simpson's rule, right rectangular rule, and midpoint rule.

15% 2. Consider the 3×3 system of linear equations given by $\mathbf{Ax} = \mathbf{b}$ where the coefficient matrix, $\mathbf{A}$, and right-hand-side (RHS) vector, $\mathbf{b}$, are given by

$$\mathbf{A} = \begin{bmatrix} 10 & 1 & -1 \\ 20 & 2 & 2 \\ -1 & 1 & 10 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix}.$$

(a) Solve this system of equations and determine the solution vector, $\mathbf{x}$, using Gaussian elimination with partial pivoting. Please show all relevant steps.

(b) Using an initial guess of $\mathbf{x}^{(0)} = [0, 0, 0]^T$, determine an improved estimate for the solution vector, $\mathbf{x}$, using one (1) interactive step of the Gauss-Seidel method. Is $\mathbf{A}$ diagonally dominant? Justify your answer. Do you expect the Gauss-Seidel method to converge?

30% 3. A ball is thrown directly upwards from an initial height of 18 m with an initial velocity of 12 m/s. Under the action of the Earth's gravitational force, the ball decelerates at a constant rate of 9.8 m/s^2 in the downward direction. The trajectory of the ball is such that it initially travels upwards, reaches a maximum height of more than 25 m, and then falls and hits the ground at

$h = 0$ more than 3 s later. The acceleration, a, vertical velocity, v, and height above the ground, h, of the ball at any time, t, after release are given by the following expressions:

$$a(t) = \frac{dv}{dt} = \frac{d^2h}{dt^2} = -9.8\,, \qquad v(t) = \frac{dh}{dt} = 12 - 9.8t\,, \qquad h(t) = 18 + 12t - 4.9t^2\,.$$

(a) The ball strikes the ground sometime between $t = 3$ and $t = 4$ s after its initial release. Using two (2) iterative steps of the bisection method, calculate the appropriate root of $h(t)$ and estimate the time when the ball hits the ground. Provide an error bound for this estimate.

(b) Repeat the calculation of part (a), this time using two (2) iterative steps of the Newton-Raphson (Newton's) method to estimate the time when the ball hits the ground. Start with an initial estimate of $t = 3$ s.

(c) Of the two methods considered in parts (a) and (b), which would generally be expected to provide an estimate for the time when the ball hits the ground in fewer iterations?

(d) Estimate the velocity of the ball at $t = 2$ s using a first-order accurate, forward, finite difference formula for the first derivative, $h' = v = dh/dt$, with $\Delta t = 0.1$ s. Compare the numerical result with the exact result given by the expression for $v(t)$ above. Note that several finite difference formulas are provided at the end (on the last page) of the exam.

(e) Repeat the calculation of part (d), this time using a second-order accurate, central, finite difference formula for the first derivative, $h' = v = dh/dt$, with $\Delta t = 0.1$ s to estimate the velocity of ball at $t = 2$ s. Compare this result to the result of part (d) and explain the differences in the calculated values for the ball velocity.

(f) Using appropriate analysis, show that the truncation error of the second-order accurate, central finite difference approximation for the first derivative, $h' = dh/dt$, is $O(\Delta t^2)$.

(g) Estimate the height of the ball above the ground at $t = 2$ s by integrating the expression for $v(t)$ above between $t = 0$ and $t = 2$ s using the 3-point (2 subintervals) trapezoidal rule. Give the formula for the 3-point trapezoidal rule as part of your answer. Note that $h(t) = 18 + \int_0^t v(t)dt$. Compare the numerical result with the exact result given by the expression for $h(t)$ above and explain the accuracy of your calculation. Why is it that any n-point trapezoidal rule, for $n \geq 2$, will provide the exact result for the ball height?

(h) The weighting coefficients, ω_i, and the Gauss points, x_i, for the two-point (2) Gaussian quadrature rule are

$$\omega_1 = 1\,, \quad \omega_2 = 1\,, \quad x_1 = -\frac{1}{\sqrt{3}} \approx -0.577350269\,, \quad x_2 = \frac{1}{\sqrt{3}} \approx 0.577350269\,.$$

Give the two-point Gauss quadrature formula for approximating the value of the integral, I, given by

$$I = \int_{-1}^{1} f(x)dx\,.$$

What class of functions, $f(x)$, can be integrated exactly by the two-point Gauss formula?

(i) Repeat the calculation of part (g), this time using the two-point Gaussian quadrature rule to estimate the height of the ball above the ground at $t = 2$ s. Again compare the numerical result with the exact result given by the expression for $h(t)$ above and explain the accuracy of your calculation.

20% 4. Reconsider the falling ball of Problem #3. In general, the height of the ball, $h(t)$, at any time, t, after its release is given by the equation

$$h(t) = h_o + v_o t - \frac{1}{2}gt^2\,,$$

where h_o is the initial height of the ball above the ground at $t = 0$ (i.e., $h_o = h(t = 0)$), v_o is the initial upward velocity of the ball, and g is the acceleration of the ball due to gravity. All three are constants. Given the following measurements of the ball height as a function of time during its decent:

Time (s): t_i	0.5	1	1.5	2	2.5
Ball Height (m): $h_i = h(t = t_i)$	22.85	25	25	22.2	17.3

a least squares fit of this data to the expression for $h(t)$ above can be used to determine a value for the acceleration due to gravity, g. Illustrate how this may be done by answering the following questions:

(a) Assuming you are given n unspecified data points (h_i, t_i), select an appropriate error measure for use in a least squares fit of the ball height data to the expression $h(t) = h_o + v_o t - gt^2/2$ and write a general expression for the total error of the fit.

(b) Determine the set of conditions that minimize total error for the fit. Again, assume you are given n unspecified data points (h_i, t_i).

(c) Reformulate the system of equations for h_o, v_o and g found in part (b) and express the equations in the form $\mathbf{Ax} = \mathbf{b}$. This system of equations may be solved to obtain a value of the gravitational acceleration, g. You are not required to solve the equations!

(d) Referring to the tabulated data given above, is the number of data points given sufficient to estimate a value for g? Justify your answer.

20% 5. Given n data points (x_i, y_i) for $1 \leq i \leq n$, a cubic spline interpolating polynomial for the points can be defined on each of the $n - 1$ subintervals, i, $[x_i, x_{i+1}]$ by the third-order polynomial

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 ,$$

where a_i, b_i, c_i, and d_i are the coefficients of the cubic spline for the appropriate subinterval i. Write a function subprogram using the FORTRAN 77 programming language that uses this expression for $s_i(x)$ to calculate and return the value of the cubic spline interpolating polynomial at any given value x. When writing the subprogram you must address the following points:

- The function subprogram should be called SPLINE.
- Ensure that you declare all of variables used by the subprogram and perform all arithmetic using double precision real (REAL*8) variables and constants.
- The value of x should be stored in the double precision real variable X.
- Assume that you are given eleven (11) data points (x_i, y_i) (i.e., $n = 11$ and $n - 1 = 10$) and the values of x_i and y_i are stored in the FORTRAN 77 one-dimensional array variables XI and YI. Select an appropriate length and index range for these array variables.
- Assume also that the coefficients of the cubic spline a_i, b_i, c_i, and d_i have already been computed and are stored in the FORTRAN 77 array variables AI, BI, CI, and DI. An appropriate length and index range must also be selected for these variables.
- The input variables to subprogram SPLINE should include X, XI, YI, AI, BI, CI, and DI. The computed value of the spline function should be returned by the function subprogram.
- Use appropriate logic to determine the subinterval and value of the spline function at x.
- The FORTRAN function subprogram SPLINE should also write or print the input value X and the value of the cubic spline interpolating polynomial to the standard output device.

Finite-Difference Formulas for $f' = df/dx$

Central difference: $f_i' \approx \dfrac{f_{i+1} - f_{i-1}}{2\Delta x}, \quad O(\Delta x^2)$

$$f_i' \approx \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x}, \qquad O(\Delta x^4)$$

Forward difference: $f_i' \approx \dfrac{f_{i+1} - f_i}{\Delta x}, \quad O(\Delta x)$

$$f_i' \approx \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2\Delta x}, \quad O(\Delta x^2)$$

Backward difference: $f_i' \approx \dfrac{f_i - f_{i-1}}{\Delta x}, \quad O(\Delta x)$

$$f_i' \approx \frac{3f_i - 4f_{i-1} + f_{i-2}}{2\Delta x}, \quad O(\Delta x^2)$$