University of Toronto

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATIONS, DECEMBER 2001

First Year – Programs 1,2,3,4,6,7,8,9

MAT 198F
Linear Algebra

SURNAME ______________________

GIVEN NAME ______________________

STUDENT NO. ______________________

INSTRUCTOR ______________________

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INSTRUCTIONS:

- Closed Book Examination.
- **Only approved non-programmable calculators are permitted.**
- Please answer all questions.
- Present your solutions in the space provided; use the back of the **preceding** pages if more space is required.

TOTAL MARKS: 100

The value for each question is shown in parentheses after the question number.

MARKER'S REPORT	
1	
2	
3	
4	
5	
6	
TOTAL	

1. [20 marks: 2 marks for each part] Indicate whether each of the following statements is true (T) or false (F), and give a brief justification for your choice.

(a) If E and F are any 3×3 elementary matrices, then $EF = FE$.

(b) If 3 is an eigenvalue of the square matrix A, then 27 is an eigenvalue of the matrix A^3.

(c) If the 6×6 matrix B is obtained from the 6×6 matrix A by replacing the third column of A with the sum of the second and fourth columns of A, then $\det B = \det A$.

(d) If λ is an eigenvalue of the $n \times n$ matrix A, then $\lambda^2 + 1$ is an eigenvalue of $A^2 + I$, where I is the $n \times n$ identity matrix.

(e) The set of all $n \times n$ symmetric matrices is a subspace of $M^{n,n}$.

(f) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is an orthogonal basis of $\mathbf{R}^m$, with respect to the usual dot product, and A is the $m \times m$ matrix with $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m$ as its columns. Then the rows of $(14A)^T$ form an orthogonal basis of $\mathbf{R}^m$.

(g) If A and B are two 2×3 matrices such that $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution spaces, then $A = B$.

(h) $(3, 2, -4)$ is the coordinate vector of $-3 + x - 4x^2$, relative to the ordered basis $\{x + 1, x - 1, 1 + x + x^2\}$ of P_2.

(i) The value of y in the solution for the system of equations

$$\begin{aligned} 2x + 4y + z &= 6 \\ x - y + 3z &= 3 \\ -x + y - 4z &= -3 \end{aligned}$$

is $y = 0$.

(j) $\dim(P_n) = n + 1$

2.(a) [5 marks] Find a real scalar "a" so that :

$$\dim\{\operatorname{span}\{ax + 2x^2, a + x, a + x^2\}\} = 2$$

2.(b) [5 marks] Given $A \in R^{n\times n}, B \in R^{n\times n}$, let the inner product $< A, B > = \operatorname{trace}(A^T B)$; in this case, what is the length of A if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix}$?

2.(c) [5 marks] Find the least squares straight line fit to the three points (0,0), (1,2), and (2,7).

3. [10 marks: 5 marks for each part] Find the following:

(a) parametric equations of the line of intersection of the two planes with equations $x + y - z = 6$ and $3x - y + 3z = 4$.

(b) $\det \begin{pmatrix} 1 & 1 & 2 & -5 \\ -1 & 0 & 1 & 0 \\ 5 & -1 & 0 & -2 \\ 0 & 1 & -1 & 4 \end{pmatrix}$

4.(a) [10 marks] Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

(a) Show that for any value of θ the matrix A is orthogonal.

(b) Show that for any vector $\mathbf{v}$ in $\mathbf{R}^2$, $\| A\mathbf{v} \| = \| \mathbf{v} \|$.

(c) Let P be any 2×2 orthogonal matrix with determinant equal to 1. Show that $P = A$, for some value of θ.

4.(b) [10 marks] For which values of k does the following system of equations

$$\begin{aligned} 2x + ky - z &= 2 \\ y + kz &= 2 \\ kx + y &= 2 \end{aligned}$$

have

(a) no solutions ?

(b) a unique solution?

(c) infinitely many solutions?

5.(a) [10 marks] Let $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

5.(b) [5 marks] Consider the differential equation $\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x$. Find a solution to this differential equation given that $x(10) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and thence find the value of $x(0)$.

6.(a) [10 marks] For this question let the inner product on $\mathbf{R}^3$ be defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 4u_2v_2 + u_3v_3.$$

Let S be the subset of vectors in $\mathbf{R}^3$ which are orthogonal to $(1, -1, 2)$, with respect to the above inner product.

(i) Show that S is a subspace of $\mathbf{R}^3$.

(ii) Find an *orthonormal* basis of S.

6.(b) [10 marks] Let $\mathbf{S}$ be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form $(a-c, a-b, c, 2a+b+c)$, where a, b, and c are in $\mathbf{R}$. Find an orthogonal basis of $\mathbf{S}$, relative to the usual dot product in $\mathbf{R}^4$.