UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
FINAL EXAMINATION, APRIL 1998
Third Year - Programs 05bme, 05ce, 05e
ECE356S - SYSTEM AND SIGNAL ANALYSIS II
Examiner - B.A. Francis

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1. The fermentation process of sugar into grain alcohol by yeast can be modelled approximately by the nonlinear state equations

$$\begin{aligned} \dot{x}_1 &= -x_1 + (1 - x_1)u \\ \dot{x}_2 &= x_1 - x_2 u, \end{aligned}$$

where x_1 is the sugar concentration, x_2 the alcohol concentration, and u the feedrate.

(a) [5 marks] Linearize the system about the equilibrium point where $x_1 = 1/4$ and find the matrices $\mathbf{A}$ and $\mathbf{B}$ in the linearized equation

$$\dot{\Delta \mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta u.$$

(b) [5 marks] Find the eigenvectors and eigenvalues of $\mathbf{A}$, and from these find the transition matrix.

(c) [5 marks] Suppose the step input $\Delta u(t) = 1/2$ is applied at $t = 0$. Does $\Delta\mathbf{x}(t)$ come to a final value, and if so, what is it?

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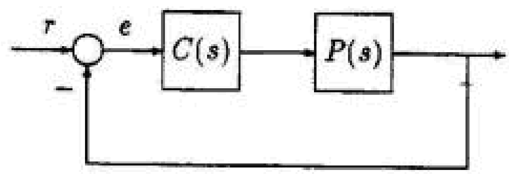
2. Consider the feedback control system



where

$$P(s) = \frac{1}{2s+1}, \quad C(s) = K_1 + \frac{K_2}{s}.$$

(a) [5 marks] Display the region in the (K_1, K_2)-plane for the feedback system to be stable.

(b) [5 marks] With $K_2 = 0$, find the minimum $K_1 > 0$ such that the steady-state absolute error $|e(t)|$ is less than or equal to 0.01 for all inputs of the form

$$r(t) = \cos(\omega t), \quad 0 \le \omega \le 4.$$

(c) [5 marks] Find suitable K_1, K_2 such that the steady-state absolute error $|e(t)|$ is less than or equal to 0.05 when $r(t)$ is the ramp of slope 1.

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3. Consider the LTI system with transfer function

$$G(z) = \frac{z-2}{2z-1}, \quad \text{ROC: } |z| > 1/2.$$

For this system

(a) [5 marks] Find and sketch the impulse response function.

(b) [5 marks] Find a real number B so that for every bounded input $x(k)$, the output $y(k)$ satisfies

$$\|y\|_\infty \le B\|x\|_\infty.$$

(c) [5 marks] Find and sketch the impulse response function of its BIBO stable inverse. Is this inverse system causal? [Two LTI discrete-time systems with impulse response functions $g(k)$ and $h(k)$, respectively, are *inverses* if the convolution of $g(k)$ and $h(k)$ equals the unit impulse $\delta(k)$.]

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4. (a) [5 marks] Find the DFT of

$$x(k) = \sin\left(2\pi\frac{3}{8}k\right) - \sin\left(2\pi\frac{1}{4}k\right), \quad k = 0, 1, \ldots, 7.$$

(b) [5 marks] Suppose we have two signals $x(k)$ ($k = 0, 1, 2, 3$) and $y(k)$ ($k = 0, 1, 2, 3$) and we interleave them like this

$$x(0), y(0), x(1), y(1), x(2), y(2), x(3), y(3),$$

forming a signal $w(k)$ ($k = 0, 1, \ldots, 7$). Find the DFT of $w(k)$ in terms of the DFTs of $x(k)$ and $y(k)$.

(c) [5 marks] Consider a signal $x(k)$ ($k = 0, 1, \ldots, N-1$) and its DFT $X(n)$. Suppose we want a formula for

$$G(f) = \sum_{k=0}^{N-1} x(k)e^{-j2\pi fk}$$

that is valid for every real f. Notice that $G(0) = X(0)$, $G(1/N) = X(1)$, etc. Find a formula for $G(f)$ in terms of $X(0), \ldots, X(N-1)$. [Thus knowing $G(f)$ only for $f = 0, 1/N, \ldots, (N-1)/N$ allows us to find $G(f)$ for all other values of f.]

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5. Construct a signal $x(t)$ that is zero for negative time and a square wave for positive time as follows:

$$x(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t < 1,\ 2 \le t < 3,\ 4 \le t < 5,\ \ldots \\ -1, & 1 \le t < 2,\ 3 \le t < 4,\ 5 \le t < 6,\ \ldots \end{cases}$$

(a) [5 marks] Sample $x(t)$ at 2 Hz to get $x_d(k)$, that is, $x_d(k) = x(k/2)$. Find the z transform of $x_d(k)$, its region of convergence, and its poles and zeros?

(b) [5 marks] Find the Fourier transform of $x_d(k)$ and sketch its magnitude Bode plot.

(c) [5 marks] Pass $x_d(k)$ through the filter with impulse response function

$$g(k) = \begin{cases} 1, & k = 0 \\ -1, & k = 1 \\ 0, & \text{else} \end{cases}$$

to get $y_d(k)$. Find and sketch $y_d(k)$.

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6. (a) [5 marks] Consider the signal

$$x(k) = \begin{cases} 0, & k < 0 \\ kr^k, & k \ge 0, \end{cases}$$

where r is real and $|r| < 1$. The Fourier transform of this signal is defined as a series. Would Gibbs phenomenon be present in the convergence of this series? [Hint: Is $x(k)$ in ℓ_1 and/or ℓ_2?]

(b) [10 marks] A signal $x(t)$ is bandlimited to frequencies less than β rad/s. It is sampled at ω_s rad/s, where $2\beta < \omega_s$, producing $x_d(k)$. A new signal $y_d(k)$ is formed by repeating each value of $x_d(k)$ twice. This can be expressed as upsampling by the factor 2, followed by filtering with the transfer function $1 + z^{-1}$. Lastly, a continuous-time signal $w(t)$ is formed via

$$w(t) = \sum_{k=-\infty}^{\infty} y_d(k) \frac{\sin \omega_N (t - k\frac{T}{2})}{\omega_N (t - \frac{T}{2})},$$

where $\omega_N = \omega_s/2$ and $T = 2\pi/\omega_s$. Find the relationship between the Fourier transforms of $w(t)$ and $x(t)$.
