Name____Student Number___________________

Lecture Section:___________

UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

FINAL EXAMINATION, April 17, 2000
First Year

ECE115s-Electricity and Magnetism
Exam Type: A

Examiners: G.V. Eleftheriades
C. Bantin
F. Gaspari

Closed book.
Only the following calculators will be allowed: casio 991; sharp 520; TI 30.
Answer the questions in the spaces provided on the facing page.
All questions have equal weight.
For numerical answers specify units.

1	2	3	4	5	6	TOTAL

USEFUL FORMULAS

$$e = 1.6 \times 10^{-19}[C]$$

$$m_e = 9.11 \times 10^{-31}[kg]$$

$$\mu_o = 4\pi \times 10^{-7}\left[\frac{Tm}{A}\right] or [H/m]$$

$$\varepsilon_o = 8.85 \times 10^{-12}\left[\frac{C^2}{Nm^2}\right] or [F/m]$$

$$\int \frac{x\,dx}{(x^2 + a^2)^{3/2}} = \frac{1}{(x^2 + a^2)^{1/2}}$$

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2(x^2 + a^2)^{1/2}}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$F = \frac{1}{4\pi\varepsilon_o}\frac{|q_1||q_2|}{r^2}$$

$$E = \frac{\lambda}{2\pi\varepsilon_o r} \text{ infinite line charge}$$

$$E = \frac{1}{2\pi\varepsilon_o}\frac{p}{z^3}(dipole)$$

$$E = \frac{1}{4\pi\varepsilon_o}\frac{qz}{(z^2 + R^2)^{3/2}}(ring)$$

$$E = \frac{\sigma}{2\varepsilon_o}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)(disk)$$

$$\varepsilon_o\oint \bar{E} \cdot \overline{dA} = q_{enc}\text{(free space)}$$

$$E = \frac{\sigma}{\varepsilon_o}\text{(conducting surface)}$$

$$E = \frac{\sigma}{2\varepsilon_o}\text{(insulating surface)}$$

$$\Delta V = V_f - V_i = -\frac{W}{q} = -\int_i^f \bar{E} \cdot \overline{ds}$$

$$V = \frac{1}{4\pi\varepsilon_o}\frac{q}{r} \text{ (point charge)}$$

$$E_x = -\frac{\partial V}{\partial x}, E_y = -\frac{\partial V}{\partial y}, E_z = -\frac{\partial V}{\partial z}$$

$$q = CV \ , \ q = q_o e^{-t/RC}$$

$$C = \frac{\varepsilon_o A}{d}(plates)$$

$$C = 2\pi\varepsilon_o L/\ln(b/a)(cylinder)$$

$$C = 4\pi\varepsilon_o ab/(b-a)\text{(spherical capacitor)}$$

$$V_C = \mathcal{E}(1 - e^{-t/RC}), \text{(charging a capacitor)}$$

$$\varepsilon_o\oint K\bar{E} \cdot \overline{dA} = q_{enc}(dielectric)$$

$$I = \frac{dq}{dt}$$

$$R = \frac{V}{I}$$

$$\bar{J} = \sigma\bar{E} \ , \ \bar{J} = (ne)\bar{v} = \rho\bar{v}$$

$$P = VI$$

$$EMF = \frac{dW}{dq}$$

$$W = \frac{1}{2}Vq$$

$$\bar{\tau}_B = \bar{\mu} \times \bar{B}$$

$$F_B = q\bar{v} \times \bar{B}$$

$$\bar{F}_B = I\bar{L} \times \bar{B}$$

$$\overline{dB} = \frac{\mu_o}{4\pi}I\overline{ds} \times \bar{r}/r^3$$

$$EMF = \oint \bar{E} \cdot \overline{ds} = -\frac{d\Phi}{dt} \quad , \quad \Phi_B = \int \bar{B} \cdot \overline{dA}$$

$$L = N\frac{\Phi}{i}(inductance) \ , \ i = i_f(1 - e^{-Rt/L})$$

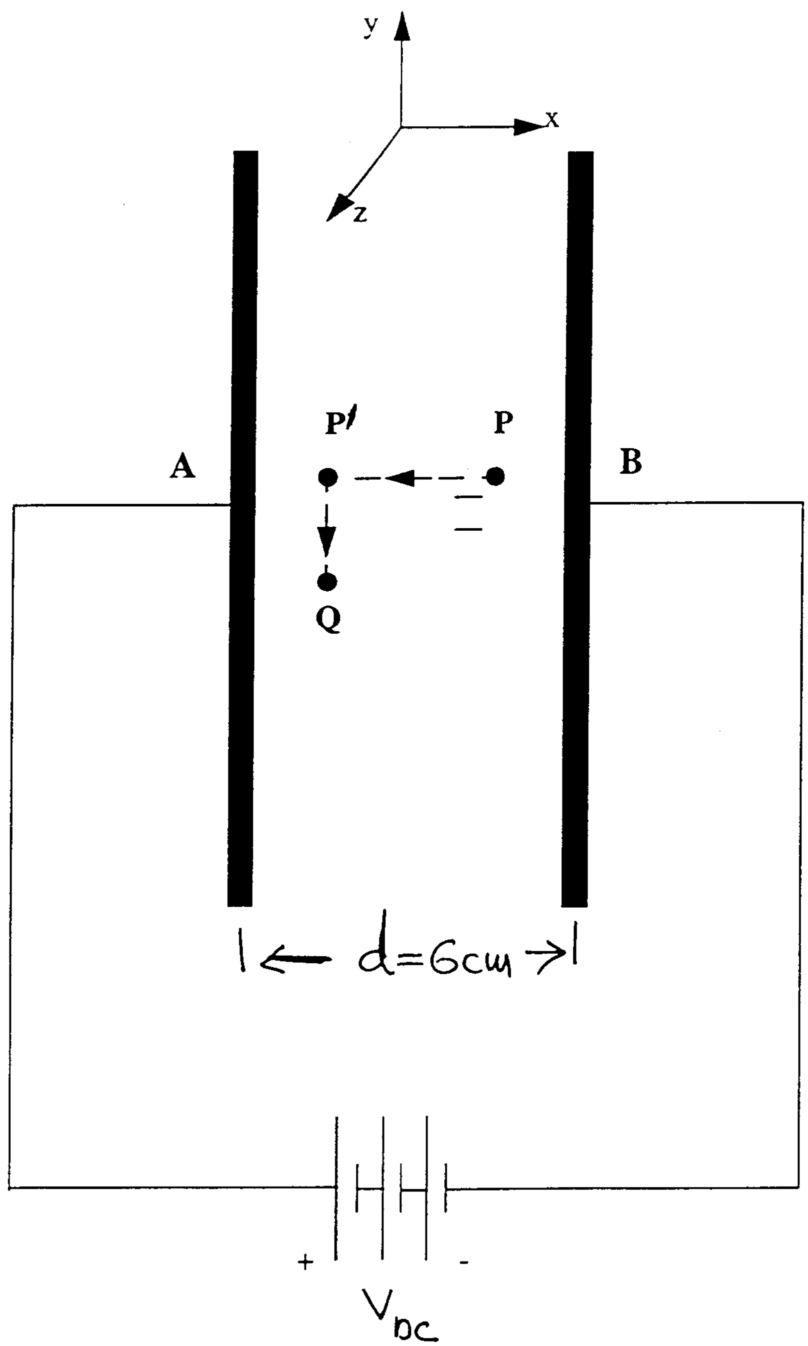
$$B = \mu_o In(soleniod)$$

$$B = \mu_o I/2\pi R(wire)$$

$$\oint \bar{B} \cdot \overline{ds} = \mu_o I_{enc}$$

Problem 1

Three 100V power supplies are assembled in series and connected between two parallel plates (A and B) as shown in the figure below. The separation between the plates is 6 cm and is much smaller than the z and y dimensions of the plates.



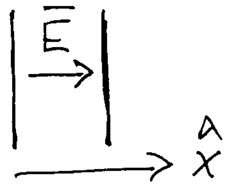
0·5 units
1.0 direction
1·0 magnitude
0·5 mark, numerical
0·5 for 3x100

2·5 a) Find the magnitude and the direction of the electric field between the plates.

0.5 $V_{DC} = 3 \times 100V = 300V$

1·0 $E = \frac{V_{DC}}{d} = \frac{300}{0.06} = 5\times10^3 \, V/m$ along the $\hat{x}$-direction

1·0



$\bar{E} = 5\times10^3 \, \hat{x} \; V/m$

2·5 b) What is the work done in moving a charge q=+10^{-6} C from point P to point Q passing through P', where PP'=4.0 cm, P'Q=2.0 cm, and P' and Q have the same x,z coordinates.

0.5 mark if positive

$$W = q\int_P^Q \bar{E}\cdot d\bar{l} = q\int_P^{P'} \bar{E}\cdot d\bar{l} + \cancel{q\int_{P'}^{Q} \bar{E}\cdot d\bar{l}} \rightarrow 0$$

$$= q\,\hat{x}E\,(PP')\cdot(-\hat{x}) = -q(5000)(0.04) = -2\times10^{-4}\,J.$$

(external work must be done).

2·5 c) What is the potential difference between initial point P and final point P' and between initial point P' and final point Q?

$$\Delta V_{PP'} = V_{P'} - V_P = \int_{P'}^{P} \bar{E}\cdot d\bar{l} = (5000\hat{x})\cdot(PP')(\hat{x})$$

$$= (5000)(0.04) = 200V$$

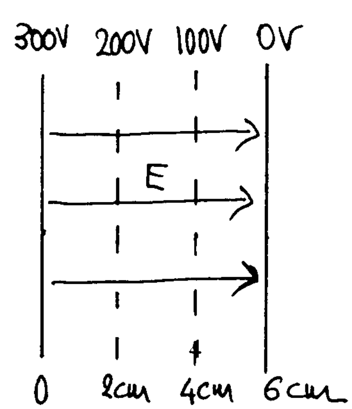
-0.5 if negative

1·5

1·0

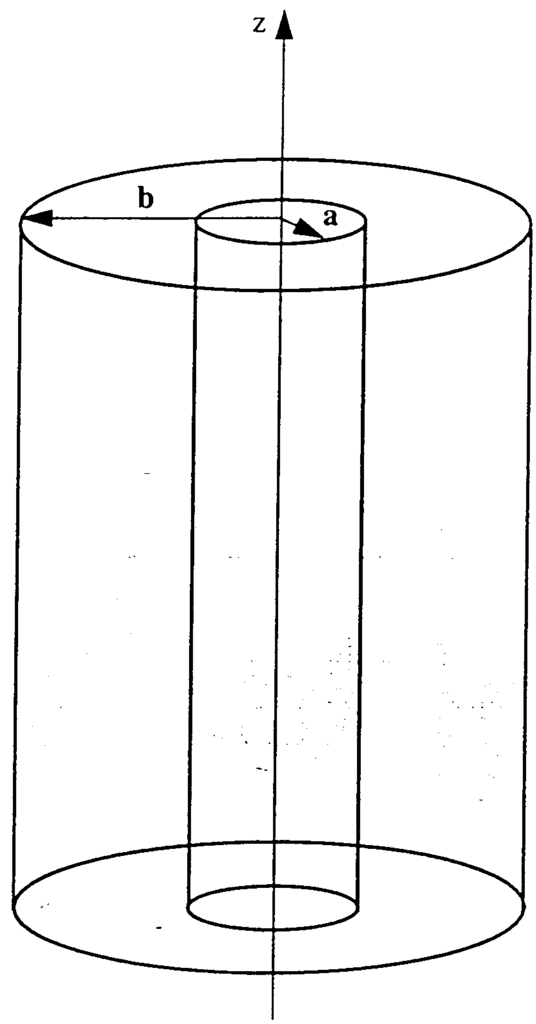
$$\Delta V_{P'Q} = 0$$

2·5 d) Sketch the equipotential surfaces in 100V intervals between the two plates.



Problem 2

Consider a cylindrical non conductive shell of uniform positive volume charge density ρ in the region a<r<b.



2.5 a) What is the electric field for r<a? Justify your answer.

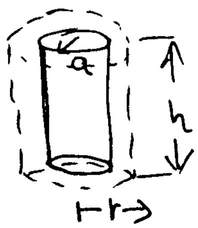
From Gauss's law and symmetry

$$\varepsilon_0 \int \bar{E} \cdot d\bar{A} = q_{enc} = 0 \quad \Rightarrow \quad \bar{E} \equiv 0.$$

2.5 b) Derive an expression for the electric field in the region a<r<b.

$$\varepsilon_0 \int \bar{E}\cdot d\bar{A} = q_{enc} \Rightarrow \varepsilon_0 E\,(2\pi r)\cancel{h} = (\pi r^2 - \pi a^2)\cancel{h}\rho$$

Hence $\bar{E} = \rho \dfrac{r^2 - a^2}{2r\varepsilon_0}\hat{r}$



2.5 c) Derive an expression for the electric field for r>b.

From above, set $r = b$ to obtain

$$\bar{E} = \rho\,\frac{b^2 - a^2}{2r\varepsilon_0}\hat{r}$$

2.5 d) What uniform line charge should be placed at r=0 (i.e., along the z-axis) to reduce the external field (r>b) to zero?

For a uniform line charge $\bar{E} = \dfrac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$

Hence $-\dfrac{\lambda}{\cancel{2}\pi\cancel{\varepsilon_0}\cancel{r}} = \rho\,\dfrac{b^2 - a^2}{\cancel{2}\cancel{\varepsilon_0}\cancel{r}} \Rightarrow$

$$\lambda = -\pi\rho\,(b^2 - a^2)$$

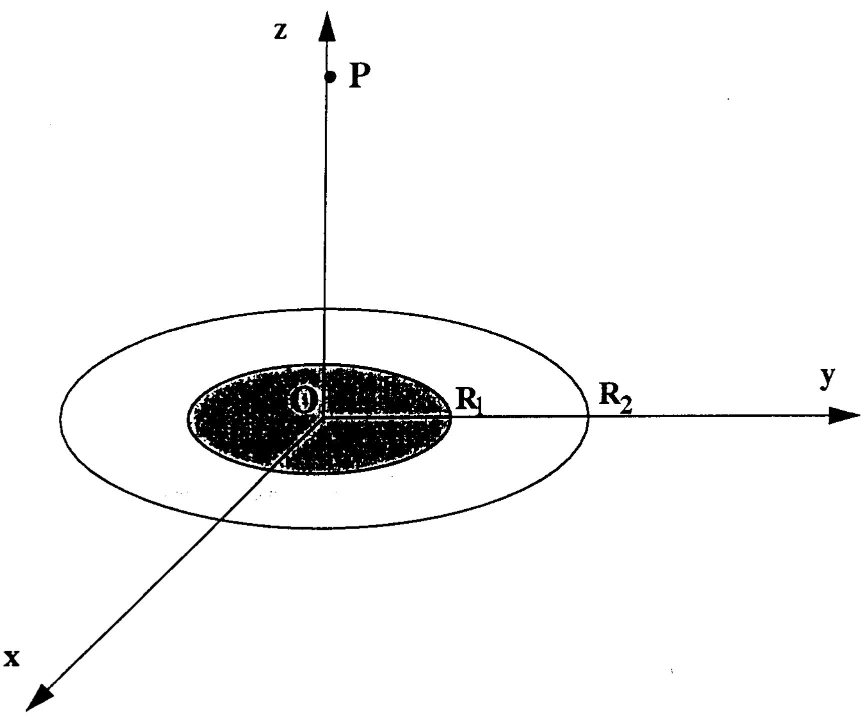
Problem 3

Consider the disk charge distribution shown below:

$\sigma_1 = 10\mu C/m^2$ over $0 \le r \le R_1$ and

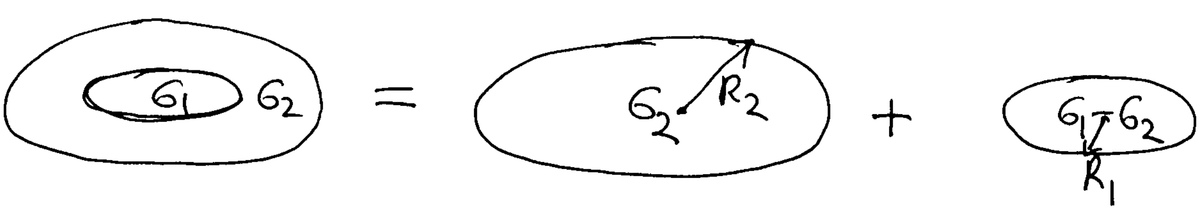
$\sigma_2 = -10\mu C/m^2$ over $R_1 < r \le R_2$,

where $R_1 = 2cm$ and $R_2 = 4cm$.



a) Calculate the electric field vector at P(0,0,5cm).

Use superposition :



Hence

$$E = \frac{\sigma_2}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R_2^2}}\right) + \frac{\sigma_1-\sigma_2}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R_1^2}}\right)$$

$$= \frac{-10\times10^{-6}}{2(8.85\times10^{-12})}\left(1 - \frac{0.05}{\sqrt{(0.05)^2+(0.04)^2}}\right) + \frac{20\times10^{-6}}{2(8.85\times10^{-12})}\left(1 - \frac{0.05}{\sqrt{(0.05)^2+(0.02)^2}}\right)$$

$$= -564971.75(0.219) + 1129943.5(0.0715)$$

$$= -4.29\times10^{4}\ \text{V/m}$$

I.e. $\bar{E} = -4.29\times10^{4}\,\hat{z}\ \text{V/m}$

3 b) Calculate the electric field vector at the origin O(0,0,0).

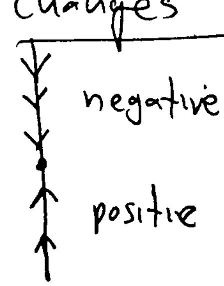
At $z=0$, $\bar{E} = \hat{z}\frac{\sigma_2}{2\varepsilon_0} + \frac{\sigma_1-\sigma_2}{2\varepsilon_0}\hat{z} = \frac{\sigma_1}{2\varepsilon_0}\hat{z}$

I.e. as if only the interior disk was participating.

$$\bar{E} = \frac{10\times10^{-6}}{2(8.85\times10^{-12})}\hat{z} = 5.65\times10^{5}\,\hat{z}\ \text{V/m}$$

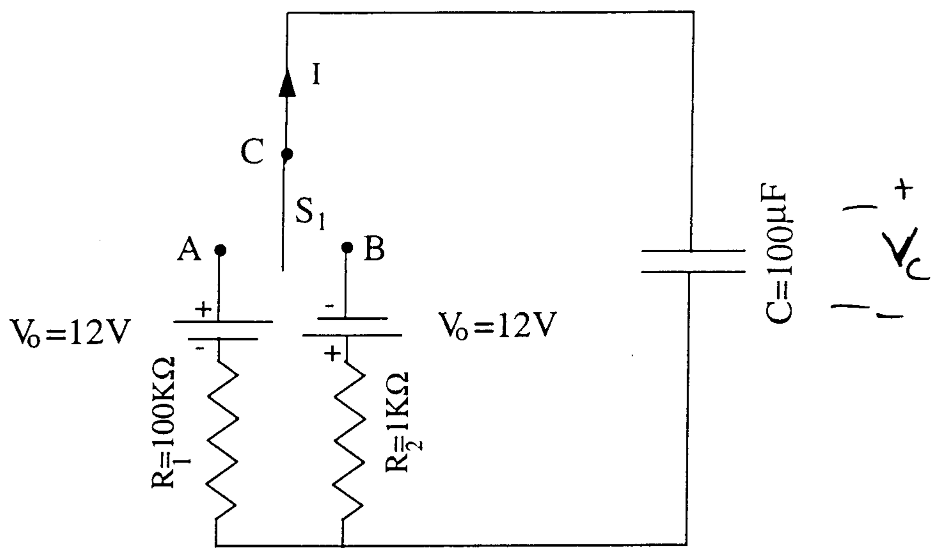
1 c) On the given diagram, sketch the electric field along the z-axis.

Note that the electric field changes sign as we move along the z-axis



Problem 4

In the circuit below, switch S_1 alternates between positions A and B every T=5s. Initially, i.e. at t=0 the switch first touches point A. Also, capacitor C is initially completely discharged.



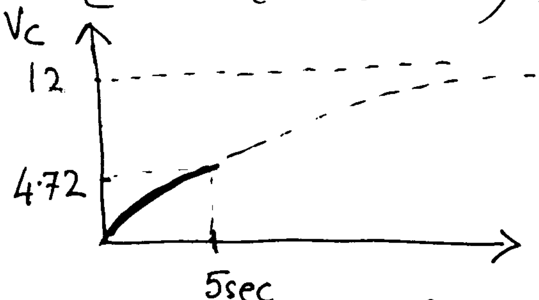
4

a) Calculate and sketch the voltage across the capacitor for $0 \le t \le 5s$.
What is the voltage at t=5s?

Let $T = 5s$, $\tau_1 = R_1 C = (100 \times 10^3)(100 \times 10^{-6}) = 10 sec$

$V_c = V_1 + V_2 e^{-t/\tau_1}$ At $t=0$, $V_c = 0 = V_1 + V_2$; At $t \to \infty$, $V_c = +12 = V_1$ $\Rightarrow V_1 = -V_2 = 12$

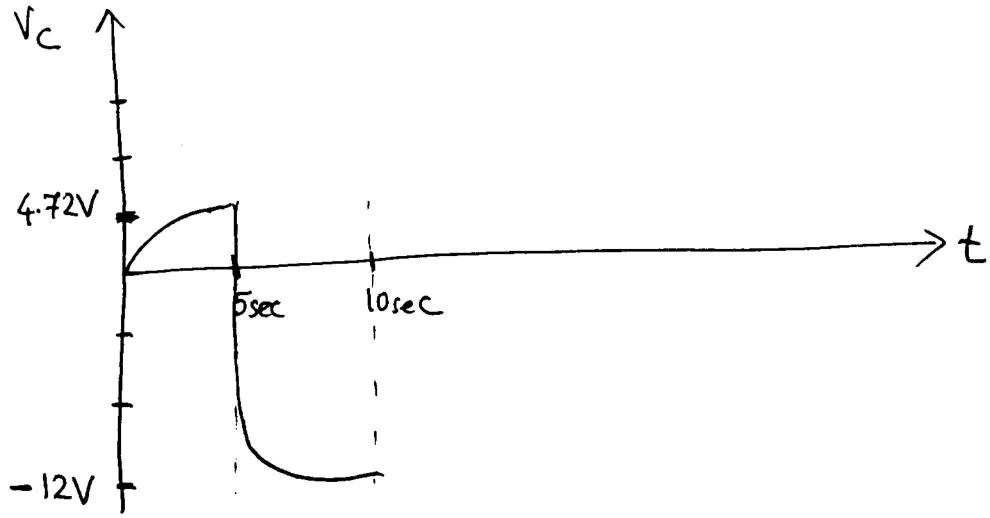
Hence $V_c = 12(1 - e^{-t/\tau_1})$. At $t=T$, $V_c = 12(1-e^{-1/2}) = \underline{4.72V}$



2

b) Calculate and sketch the voltage across the capacitor for $5s \leq t \leq 10s$.
What is the voltage at t=10s?

At $t = 5s$, the switch touches point B.
Hence $\tau_2 = R_2 C = (1\times10^3)(100\times10^{-6}) = 0.1\,sec \ll T = 5sec$
and the capacitor fast discharges to $-12V$.



2

c) Calculate and sketch the voltage across the capacitor for $10s \leq t \leq 15s$.
What is the voltage at t=15s?

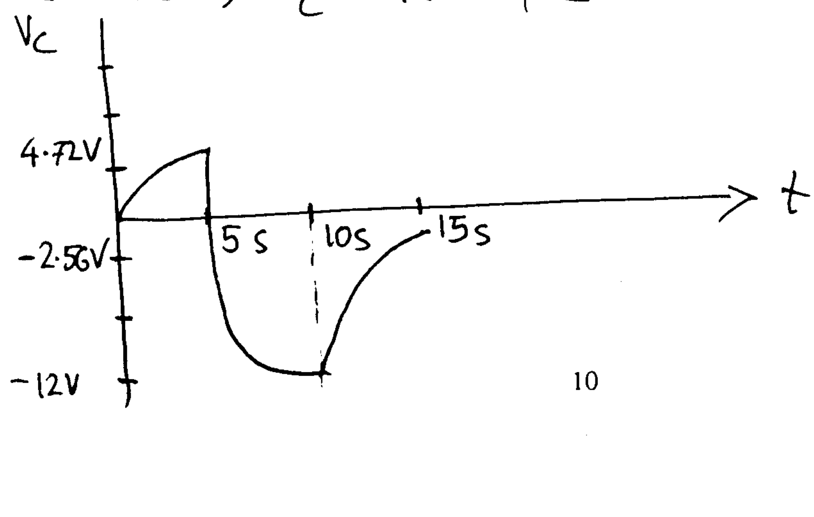
$$V_c = V_1 + V_2 e^{-\frac{t-2T}{\tau_1}}$$

At $t = 10s$, $V_c = -12 = V_1 + V_2$
As $t \to \infty$, $V_c = 12 = V_1$

$$\Rightarrow V_1 = 12V,\quad V_2 = 24V$$

I.e. $V_c = 12 - 24 e^{-\frac{t-2T}{\tau_1}}$

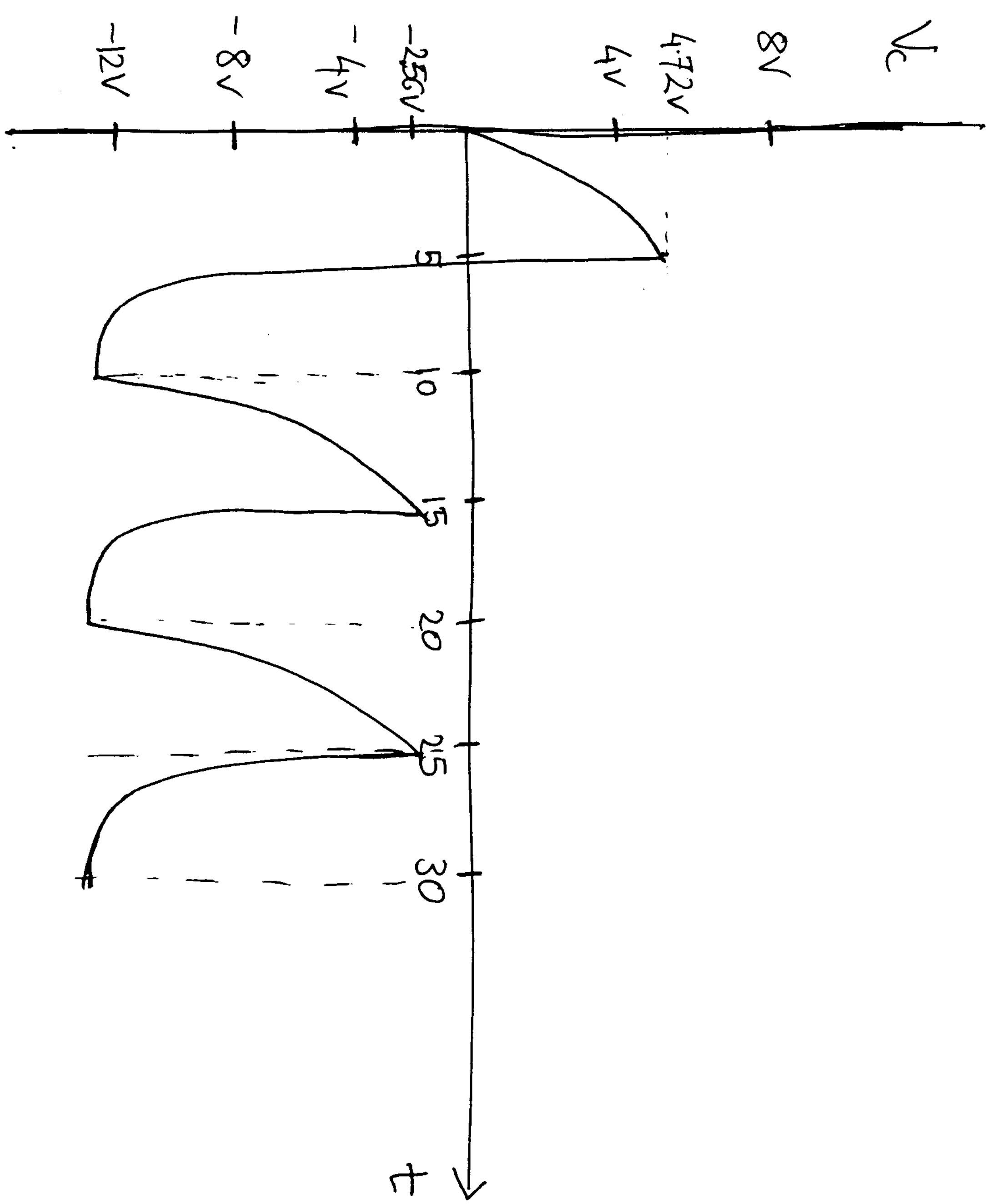
At $t = 15s$, $V_c = 12 - 24 e^{-1/2} = -2.56V$



-2

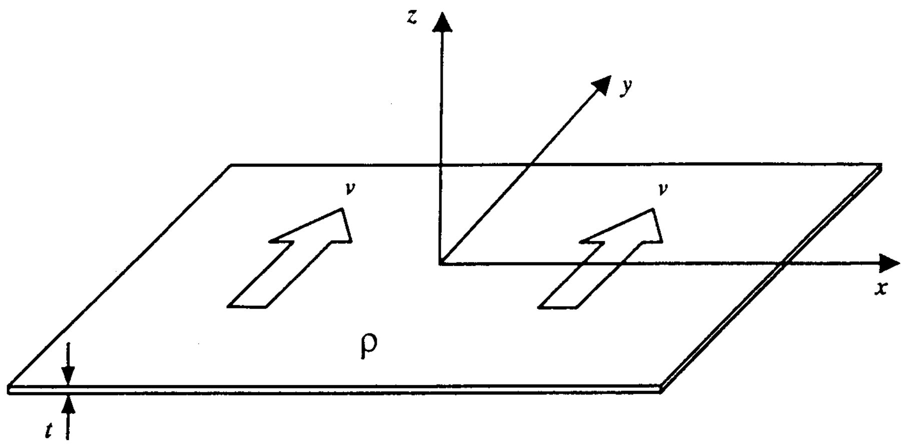
d) Sketch the voltage across the capacitor for $0 \le t \le 30s$. Clearly label all important quantities.

The voltage will alternate between -12V and -2.56V

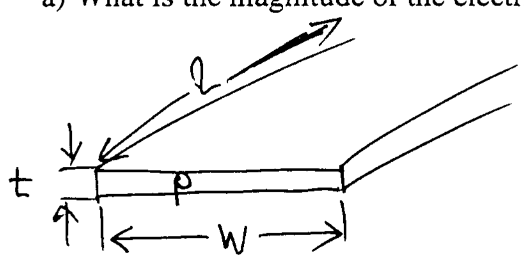


Problem 5

Consider an infinite sheet of positive charge carriers, with thickness t= 1 mm, lying in the x-y plane. The volume charge density within the sheet is $\rho = 10^{-6} C/m^3$.



3 a) What is the magnitude of the electric field at a point above the sheet?



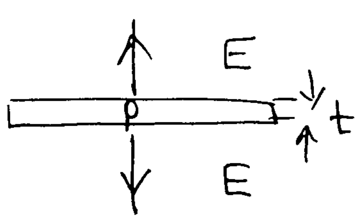
$$\rho = \frac{q}{wtl} = \left(\frac{q}{wl}\right)\frac{1}{t} = \frac{\sigma}{t} \quad \text{i.e}$$

$$\sigma = \rho t = (10^{-6})(1\times10^{-3}) = 0.001\times10^{-6}\ C/m^2$$

For a sheet of charge, $E = \frac{\sigma}{2\varepsilon_0} = 56.5\ V/m$

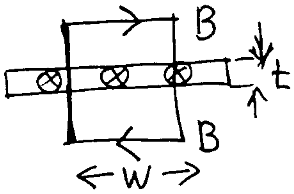
$$E = \frac{\rho t}{2\varepsilon_0}$$

1 b) Show the direction of the electric field on the diagram.



Now assume that the charge density is everywhere moving with a uniform speed v=100,000 m/s in the positive y-direction.

3 c) What is the magnetic field, B, at a point above the sheet?



From Ampere's law $\oint \bar{B} \cdot d\bar{s} = \mu_0 i_{enc} \Rightarrow$

$B(2W) = \mu_0 i_{enc}$

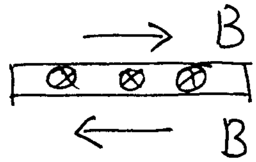
what is i_{enc}? $i_{enc} = J A = \rho v w t$

Hence, $B(2\cancel{w}) = \mu_0 \rho v \cancel{w} t$ i.e.

$$B = \frac{\mu_0 \rho v t}{2} = \frac{(4\pi \times 10^{-7})(10^{-6})(1\times10^{5})(1\times10^{-3})}{2} = 6.283\times10^{-11}\ T$$

1 d) Show the direction of the magnetic field on the diagram.

From RHL and symmetry



2 e) If the velocity of the charge sheet is the speed of light, c, which is defined as $c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$ (m/s),

1 for value
1 for units

then what is the numerical value of the ratio $\frac{E}{B/\mu_o}$, for any ρ and t, and what are the units of this number?

$$\frac{E}{B/\mu_0} = \frac{\cancel{\rho t}}{\cancel{2}\varepsilon_0}\,\frac{\cancel{2}}{\cancel{\rho} c \cancel{t}} = \frac{1}{c\,\varepsilon_0} = \frac{\sqrt{\mu_0 \varepsilon_0}}{\varepsilon_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377$$

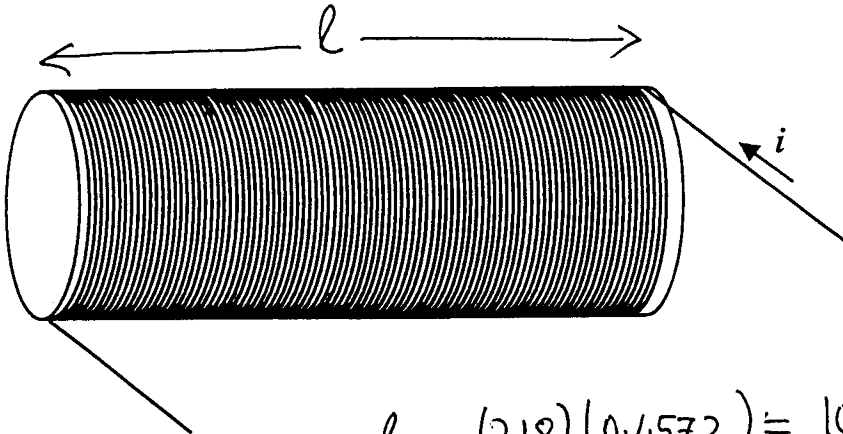
Units : $\frac{E}{B/\mu_0} = \frac{V}{\cancel{m}}\,\frac{\cancel{m}}{A} = \frac{V}{A} = \Omega$

$\oint B \cdot ds = \mu_0 i$ i.e. units for $B/\mu_0 = A/m$

Hence $\frac{E}{B/\mu_0} = 377\ \Omega$ (intrinsic impedance of free space)

Problem 6

A 3 cm diameter solenoid consists of 218 turns of #26 insulated magnet wire closely packed (i.e. adjacent turns are touching). This wire has an outside diameter of 0.4572mm, and a per-length resistance of 0.105 Ω/m.



$$\ell = (218)(0.4572) = 10\text{cm}$$

2 a) Find the inductance of the solenoid, assuming it is "ideal" (i.e. neglecting end effects).

$$L = N\frac{\Phi}{i} = N\frac{BA}{i} = N\frac{(\mu_0 \cancel{i} n)A}{\cancel{i}} = N\mu_0\frac{N}{\ell}A$$

I.e. $$L = N^2\left(\frac{\mu_0 A}{\ell}\right) = (218)^2\frac{(4\pi\times10^{-7})(\pi(0.015)^2)}{100\times10^{-3}} = 422\,\mu H$$

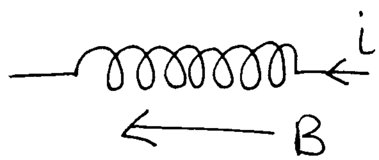
2 b) Find the magnetic field magnitude in the centre of the solenoid if i=250mA.

$$B = \mu_0 i n = \mu_0 i\frac{N}{\ell} = (4\pi\times10^{-7})(250\times10^{-3})\frac{218}{10\times10^{-2}}$$

$$= 6.85\times10^{-4}\ T$$

2 c) Show on the diagram the direction of the magnetic field for the current shown.

From RHL



2 d) If the current i is increased at a steady rate from 250 mA to 1250 mA in 50 ms, what is the magnitude of the induced *EMF* around a "virtual" loop, 1 cm in diameter, located in the centre of the solenoid normal to the cylindrical axis?

$$\frac{di}{dt} = \frac{1250-250}{50} = 20\ A/s \quad . \quad \text{Let } A' = \pi(0.5\,\text{cm})^2$$

$$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{d}{dt}(BA') = -\mu_0 \frac{N}{\ell} A' \frac{di}{dt} \quad \text{I.e.}$$

$$|\mathcal{E}| = (4\pi\times10^{-7})\left(\frac{218}{10\times10^{-2}}\right)\pi(0.5\times10^{-2})^2(20) = 4.3\ \mu V$$

2 e) If a 10 volt battery is connected across the loop (instead of the current i shown), how long will it be before the new resulting current increases to 50% of its final value?

$$R = 2\pi(0.015)(218)(0.105) = 2.168$$

$$\text{Time constant } \tau = L/R = \frac{422\times10^{-6}}{2.16} = 195\ \mu s$$

$$i = i_f\left(1-e^{-tR/L}\right), \text{ If } i = \tfrac{1}{2} i_f \Rightarrow$$

$$\frac{1}{2} = 1-e^{-tR/L}, \quad \text{I.e } \tfrac{1}{2} = e^{-t/\tau} \Rightarrow \frac{t}{\tau} = \ln 2$$

$$\text{Hence } t = \tau\ln 2 = 195 \ln 2 = 135\ \mu s$$