UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Final Examination

First Year — Program 5

MAT185F — Linear Algebra

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Student Name:		
	Last Name	First Names

Student Number:

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. There are 11 pages and 6 questions in this examination paper.

For Examiners Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	10	
C		
3	20	
D		
4	20	
E		
5	20	
6	20	
Total	100	

A. Definitions

Fill in the blanks.

1(a). The *inverse* of a matrix $\mathbf{A}$ is

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/2

1(b). The *homogeneous system* associated with a matrix $\mathbf{A}$ is

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/2

1(c). The *distributive properties* of a vector space are

____

____

/2

1(d). A *linear combination* of $\{v_1, \ldots, v_n\} \subset \mathcal{V}$ is

____

____

/2

1(e). The (i, j)-*cofactor* of a square matrix $\mathbf{A}$ is

____

____

/2

B. Theorems

2(a). State two (2) equivalent conditions for a basis of a vector space $\mathcal{V}$ given that $\dim \mathcal{V} = n$.

/4

2(b). State the *Fundamental Theorem*.

/3

2(c). State the necessary conditions on the dimension of an eigenspace $\mathcal{E}$ corresponding to an eigenvalue of multiplicity m.

/3

C. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. The value of each question is 2 marks.*

3(a). The image space of an $m \times n$ matrix is identical to its row space. ☐

3(b). If $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in {}^m\mathbb{R}^n$ has a solution for at least one $\mathbf{b} \in {}^m\mathbb{R}$, then it has a solution for every $\mathbf{b} \in {}^m\mathbb{R}$ ☐

3(c). There is only one subspace of ${}^{10}\mathbb{R}$ of dimension 10. ☐

3(d). For $\mathbf{A} \in {}^m\mathbb{R}^n$, the columns of $\tilde{\mathbf{A}}$ containing a leading "1" form a basis for the column space of $\mathbf{A}$. ☐

3(e). For any $\mathbf{A} \in {}^3\mathbb{R}^7$, the solution (null) space of $\mathbf{A}^T$ has the same dimension as the solution (null) space of $\mathbf{A}$. ☐

3(f). If $\mathbf{A}$ is invertible, then rank $\mathbf{A} =$ rank $\mathbf{A}^{-1}$. ☐

3(g). For any square matrix $\mathbf{A}$, $\det \mathbf{A}^T = \det \tilde{\mathbf{A}}$. ☐

3(h). For any square matrix $\mathbf{A}$, if the rows of adj $\mathbf{A}$ are dependent, then the columns of adj $\mathbf{A}$ are also dependent. ☐

3(i). If λ is an eigenvalue of $\mathbf{A} \in {}^n\mathbb{R}^n$, then $(-1)^n\lambda$ is an eigenvalue of $-\mathbf{A}$. ☐

3(j). The eigenvectors of $\mathbf{A}$ are also the eigenvectors of $\mathbf{A}^T$. ☐

D. Multiple Choice

For the following questions, select the most complete answer (A, B, C, D or E) and indicate it in the corresponding box. The value of each question is 4 marks. Full marks will be awarded only to the most complete answer but 2 marks will be given to partially correct answers where they apply.

4(a). Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Which of the following statements is true?

I. The $(2, 3)$ element of $\mathbf{A}^{-1}$ is 0

II. $\det \mathbf{A} = 7$

III. The unique solution $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^T$ of $\mathbf{Ax} = [1 \;\; 0 \;\; 0]^T$ has $x_1 = \frac{1}{2}$

IV. The third row of $\mathbf{Y}$ such that

$$\mathbf{YA}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

is $[-2 \;\; 0 \;\; 4]$.

A. I and II
B. III and IV
C. I and IV
D. II and III
E. IV

4(b). Let

$$W_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad W_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ -k \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$$

For what values of k is $W_1 = W_2$?

A. $k = 0$
B. $k = -1$
C. $k = 1$ and $k = -1$
D. All $k \in \mathbb{R}$
E. There is no such k

4(c). Let $P = \{p \in \mathbb{P}_4 \mid p(1) = p(0) \text{ and } p(-1) = 0\} \subseteq \mathbb{P}_4$. Which of the following is a basis for P?

I. $\{1, x, x^2, x^3, x^4\}$
II. $\{x^3 - x, x^4 - x^2\}$
III. $\{x^3 - x, x^2 - x^4, -x^4 + x^3 + 2\}$
IV. $\{x - x^3, x - x^4, -x^4 + x^3 + 3\}$
V. $\{-x^4 + x^3 + 2, x^4 - x^2, x^3 - x^2 + 2\}$

A. III
B. III and IV
C. IV and V
D. III, IV and V
E. All

4(d). If $\mathbf{A} \in {}^9\mathbb{R}^9$ is diagonalizable and $\det(\lambda \mathbf{1} - \mathbf{A}) = \lambda(\lambda-2)^3(\lambda-3)^5$, which of the following statements is true?

I. $\operatorname{rank}(3\mathbf{1} - \mathbf{A}) = 4$

II. $\mathbf{A}$ is invertible

III. $\operatorname{tr} \mathbf{A} = 5$

IV. ${}^9\mathbb{R}$ has a basis containing 3 vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ such that $\mathbf{A}\mathbf{x}_i = 2\mathbf{x}_i$ for $i = 1, 2, 3$

A. I and II

B. II and III

C. III and IV

D. I, II and IV

E. All

4(e). Consider the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{x}(0) = \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Which of the following solutions is correct?

I.
$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

II.
$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

III. $\mathbf{x}(t) = \mathbf{P}\Lambda\mathbf{P}^{-1}\mathbf{x}_0$ where

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

IV. $\mathbf{A}$ is not diagonalizable and therefore a solution does not exist.

A. I

B. II

C. I and III

D. II and III

E. IV

5. Let $\{e_1, \ldots, e_n\}$ be a basis for a vector space $\mathcal{V}$ and let

$$\boldsymbol{f}_j = \sum_{i=1}^{n} p_{ij} e_i, \quad j = 1 \ldots n$$

Show that $\mathcal{V} = \operatorname{span}\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\}$ if and only if $\mathbf{P} = [p_{ij}]$ is invertible.

...cont'd

5. . . . cont'd

/20

6(a). Let $\mathbf{A} \in {}^n\mathbb{R}^n$. Show that if λ is an eigenvalue of $\mathbf{A}$ then λ^k is an eigenvalue of $\mathbf{A}^k$ for any integer $k \geq 1$.

/5

6(b). Let $\mathbf{A} \in {}^n\mathbb{R}^n$. Show that if $\lambda = 0$ is an eigenvalue of $\mathbf{A}^k$ for any integer $k \geq 1$ then $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.

/5

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6(c). Let $\mathbf{A} \in {}^n\mathbb{R}^n$ where $\mathbf{A}^k = \mathbf{O}$ $(k > 1)$ but $\mathbf{A} \neq \mathbf{O}$. Show that $\lambda = 0$ is the only eigenvalue of $\mathbf{A}$.

/5

6(d). Let $\mathbf{A} \in {}^n\mathbb{R}^n$ where $\mathbf{A}^k = \mathbf{O}$ $(k > 1)$ but $\mathbf{A} \neq \mathbf{O}$. Show that $\mathbf{A}$ is not diagonalizable.

/5