UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
FINAL EXAMINATION, APRIL 1997
Third Year - Programs 05bme, 05ce, 05e
ECE356S - SYSTEM AND SIGNAL ANALYSIS II
Examiner - B.A. Francis

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1. (a) **[6 marks]** Find the transfer function from u to y for the linear time-invariant system modeled by the state equations

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \mathbf{x} + u. \end{aligned}$$

(b) **[2 marks]** Let $\mathbf{A}$ be a real $n \times m$ matrix and $\mathbf{b}$ a real n-dimensional vector. Define the function

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$$

mapping $\mathbb{R}^m$ to $\mathbb{R}^n$. Is $\mathbf{f}$ a linear transformation or not? Defend your answer.

(c) **[4 marks]** Consider a linear time-invariant system modeled by the equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{x}(t)$ is a real n-dimensional vector, $\mathbf{u}(t)$ is a real m-dimensional vector, and $\mathbf{A}$ and $\mathbf{B}$ are constant real matrices. The state signal depends on the initial state and the input signal linearly in the following way:

$$\mathbf{x} = G\mathbf{x}_0 + H\mathbf{u}.$$

Here G and H are linear transformations. What are their domains and co-domains? (If $T : \mathcal{X} \to \mathcal{Y}$ is a linear transformation, $\mathcal{X}$ is called its *domain* and $\mathcal{Y}$ its *co-domain*.)

(d) **[3 marks]** This problem involves a linear time-invariant system modeled by state equations, where the input $u(t)$ and output $y(t)$ are 1-dimensional. Suppose the output is denoted $y_1(t)$ when the input is a unit step and

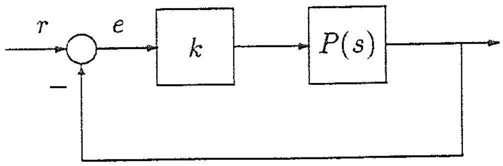
$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and suppose the output is denoted $y_2(t)$ when the input is a unit step and

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Find in terms of $y_1(t)$ and $y_2(t)$ the output when the input is a unit step and $\mathbf{x}(0) = 0$.

2. Consider the feedback control system



where $P(s) = 1/(s+1)$.

(a) [**5 marks**] Find the minimum $k > 0$ such that the steady-state absolute error $|e(t)|$ is less than or equal to 0.01 when r is the unit step.

(b) [**5 marks**] Find the minimum $k > 0$ such that the steady-state absolute error $|e(t)|$ is less than or equal to 0.01 for all inputs of the form

$$r(t) = \cos(\omega t), \quad 0 \leq \omega \leq 4.$$

3. (a) [**8 marks**] A linear time-invariant discrete-time system has input $x(k)$, output $y(k)$, and impulse response

$$g(k) = \begin{cases} 1, & k = -1, 1 \\ 2, & k = 0 \\ 0, & \text{else.} \end{cases}$$

Given

$$X(z) = \frac{z^2}{3(3-z)}, \quad \text{ROC}: \ |z| < 3,$$

find the z-transform of $y(k)$. Find and plot $y(k)$.

(b) [**7 marks**] Consider a causal linear discrete-time system with input $x(k)$ and output $y(k)$ and modeled by the equation

$$y(k) = y(k-1) + 2x(k) - x(k-2).$$

Find the matrix representation and the transfer function. Is the system BIBO stable?

4. (a) **[5 marks]** Let $N = 8$ and define $x(k)$ for $k = 0, 1, \ldots, N-1$ by

$$x(k) = \begin{cases} 1, & k = 3 \\ 0, & \text{else.} \end{cases}$$

Plot the magnitude and phase of the DFT of $x(k)$.

(b) **[5 marks]** Continue with the same N and $x(k)$. As you just saw, all elements of the DFT of $x(k)$ are non-zero. Find a different orthogonal set of basis functions with respect to which the transform of $x(k)$ has the fewest number of non-zero elements (i.e., is maximally compressed).

(c) **[5 marks]** The continuous-time signal $\sin(2\pi \times 500t)$ is sampled at 8 kHz for 2 seconds starting at $t = 5$. Graph the magnitude and phase of the DFT of the resulting data.

(d) **[5 marks]** Prove that the DFT basis functions are orthogonal.

5. **[10 marks]** This problem asks you to design a *sampling-rate converter*. Suppose $x(t)$ is a continuous-time signal bandlimited to less than 1 Hz. Suppose it has been sampled at 2 Hz, producing the discrete-time signal $y_d(k)$, but we actually had wanted it sampled at 3 Hz, producing $w_d(k)$. Specify a discrete-time system that will convert $y_d(k)$ to $w_d(k)$. Explain how your system works. Hint: Upsample $y_d(k)$ by 3, then lowpass filter, then downsample by 2. You are *not allowed* to reconstruct $x(t)$ and then sample.