Faculty of Applied Science and Engineering

Final Exam April 20, 2001 - 14:00 - 16:30
First Year – AEELEBASC, AECPEBASC

ECE115S – Electricity and Magnetism

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Type A - Closed book.
Only Calculators approved by Registrar. (3)
Answer the questions in the spaces provided or on the facing page.
For numerical answers specify units.
DO NOT UNSTAPLE
All questions have equal weight.

1	2	3	4	5	6	TOTAL

$$g = 9.80 \left[\frac{m}{s^2}\right] \qquad c = 3.00\times 10^{8}\left[\frac{m}{s}\right]$$

$$e = 1.60\times 10^{-19}\,[C] \qquad m_e = 9.11\times 10^{-31}\,[kg]$$

$$\mu_0 = 4\pi\times 10^{-7}\left[\frac{T\cdot m}{A}\right] \qquad \epsilon_0 = 8.85\times 10^{-12}\left[\frac{C^2}{N\cdot m^2}\right]$$

$$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{1}{(x^2+a^2)^{1/2}} \qquad \int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\,(x^2+a^2)^{1/2}}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = ln\left(x+\sqrt{x^2+a^2}\right) \qquad F = \frac{1}{4\pi\epsilon_0}\frac{|q_1||q_2|}{r^2}$$

$$B = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\ \text{(on axis of ring)} \qquad E \approx \frac{1}{2\pi\epsilon_0}\frac{p}{z^3}\ \text{(axis of dipole)}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{qz}{(z^2+R^2)^{3/2}}\ (ring) \qquad E = \frac{\sigma}{2\epsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)\ \text{(axis of disk)}$$

$$\epsilon_0\oint \mathbf{E}\cdot d\mathbf{A} = q_{enc}\text{(free space)} \qquad E = \frac{\sigma}{\epsilon_0}\ \text{(conducting surface)}$$

$$E = \frac{\sigma}{2\epsilon_0}\ \text{(insulating surface)} \qquad \Delta V = V_f - V_i = -\frac{W}{q} = -\int_i^f \mathbf{E}\cdot ds$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \qquad E_x = -\frac{\partial V}{\partial x},\quad E_y = -\frac{\partial V}{\partial y},\quad E_z = -\frac{\partial V}{\partial z}$$

$$q = CV \qquad C = \frac{\epsilon_0 A}{d}\ \text{(plates)}$$

$$C = 2\pi\epsilon_0 L\,/\,ln\,(b/a)\ \text{(cylinder)} \qquad C = 4\pi\epsilon_0\, ab\,/\,(b-a)\ \text{(spherical capacitor)}$$

$$C = 4\pi\epsilon_0 R\ \text{(sphere)} \qquad \epsilon_0\oint K\mathbf{E}\cdot d\mathbf{A} = q_{enc}\text{(dielectric)}$$

$$I = \frac{dq}{dt} \qquad R = \frac{V}{I}$$

$$\mathbf{E} = \rho\mathbf{J} \qquad P = VI$$

$$EMF = \frac{dW}{dq} \qquad W_{stored} = \frac{1}{2}Vq = \frac{1}{2}CV^2$$

$$\boldsymbol{\tau}_B = \boldsymbol{\mu}\times\mathbf{B} \qquad \mathbf{F}_B = q\mathbf{v}\times\mathbf{B}$$

$$\mathbf{F}_B = I\,\mathbf{L}\times\mathbf{B} \qquad d\mathbf{B} = \frac{\mu_0}{4\pi} I ds\times\mathbf{r}\,/\,r^3$$

$$B = \mu_0 I\,\Phi\,/\,4\pi R\ \text{(arc)} \qquad B = \mu_0 I\,n\ \text{(solenoid)}$$

$$B = \mu_0 I\,/\,2\pi R\ \text{(wire)} \qquad \oint B\cdot ds = \mu_0 I_{enc}$$

$$q_{charging} = C\mathcal{E}\left(1-e^{-t/RC}\right);\ q_{discharging} = q_0 e^{-t/RC} \qquad \Phi_B = \int \mathbf{B}\cdot d\mathbf{A}$$

$$(1-x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)x^2}{2!}+\cdots\quad (x^2<1) \qquad \boldsymbol{EMF} = \oint \mathbf{E}\cdot d\mathbf{s} = -\frac{d\Phi}{dt}$$

cloth 2. All four objects are composed of different materials. You are informed that when rod A is rubbed by cloth 1 the rod will become negatively charged because of the triboelectric effect. You perform an experiment to determine the triboelectric sequence of the four materials. The results of this experiment are as follows.

Experiment.	Rod A is rubbed by the cloth below.	Rod B is rubbed by the cloth below	The two rods
1	1	1	Repel
2	2	2	Attract
3	1	2	Attract
4	2	1	repel

Hint: The triboelectric series is a list of materials arranged so that if a material placed low on the list is rubbed with a material that is higher on the list, the higher placed material will become positively charged and the lower placed material will become negatively charged.

From the above results deduce the order of the four objects in the triboelectric sequence. Record your answer in the table below.

Triboelectric series deduced from results in table above			
Highest (*i*)	(*ii*)	(*iii*)	Lowest (*iv*)
1	B	2	A

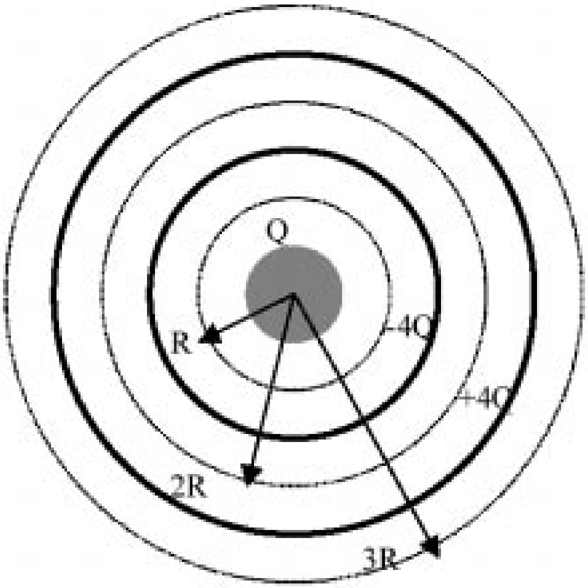
Show the steps you used to deduce your answer.
Like charges repel and unlike attract.
Given TS = 1, A where TS means *triboelectric series* established so far.
Expt. 1 B1 makes B –ve and so TS = 1,B
Expt. 3 B2 makes B +ve and so TS = B, 2 → 1,B,2
Expt. 2 A2 makes A –ve and so TS = 2,A → 1,B,2,A

These 3 results can only be explained if TS = 1, B, 2, A
OR
Expt. 1 B1 makes B –ve and so TS = 1,B
Expt.4 A2 makes A –ve and so TS = 2, A
Expt 2 B2 makes B +be and so TS = B, 2
These 3 results can only be explained if TS = 1, B, 2, A
OR
Make an arbitrary initial assumption, say (A,1,2 ,B), and then reorder according to the information given:
A1 makes A –ve. Therefore swap A and 1 so (A,1,2,B) becomes (1,A,2,B)
Expt 1 A is –ve so B must be -ve so 1 above B. Therefore (1,A,2,B) is ok
Expt 3 A is –ve so B must be +ve so 2 below B. Therefore swap 2 and B so (1,A,2,B) becomes (1,A,B,2).
Expt 2 B is +ve (from Expt 3) so A must be –ve so 2 is above A. Thus (1,A,B,2) becomes (1,B,2,A)

Question 1 Part B 5 marks

The figure shows, in cross section, a central metal ball (gray), two spherical metal shells (black) and three spherical Gaussian surfaces (dotted lines) of radii R, 2R & 3R, all with the same center. The charge on the three objects is:

- Ball + Q.
- Smaller shell -4Q.
- Larger shell +4Q.



Define the electric flux outward - 4through each Gaussian Surface as follows.

Gaussian Surface Radius	Electric flux outward through each Gaussian Surface
R	$\Phi_1 = Q/\varepsilon_0$
2R	$\Phi_2 = -3Q/\varepsilon_0$
3R	$\Phi_3 = Q/\varepsilon_0$

1) Rank these three fluxes from most negative to most positive: 2marks

Ranked list of flux Φ_1, Φ_2 & Φ_3.		
Most Negative		Most Positive
Φ_2		$\Phi_1 = \Phi_3$

Justify Your Answer:

Gauss's Law
Enclosed Charge as above
R → Q, 2R → (1-4)Q, 3R → (1-4+4)Q

2) Write an expression for the magnitude, E_1, of the electric field at the smallest Gaussian Surface i.e. the surface with radius R. What is the direction of this field. 1mark

$$|E|\ 4\Pi R^2 = \Phi_1 = Q/\varepsilon_0$$

$$|E_1| = Q/\varepsilon_0 4\Pi R^2$$

Direction outward.

3) Let E_2 and E_3 be the magnitude of the electric field at the Gaussian Surfaces with radius 2R and 3R respectively. Compute the following: 2 marks

Electric field magnitude, E	Ratio E/E_1.	Direction of field E
For $E = E_2$	= 3/4	inward
For $E = E_3$	= 1/9	Outward

Justify Your Answer:
$E2/E1 = (-3Q/2^2R^2)/(Q/R^2) = (-3/4)/1 = -(3/4)$
$E3/E1 = (Q/3^2R^2)/(Q/R^2) = = (1/9)/1 = 1/9$

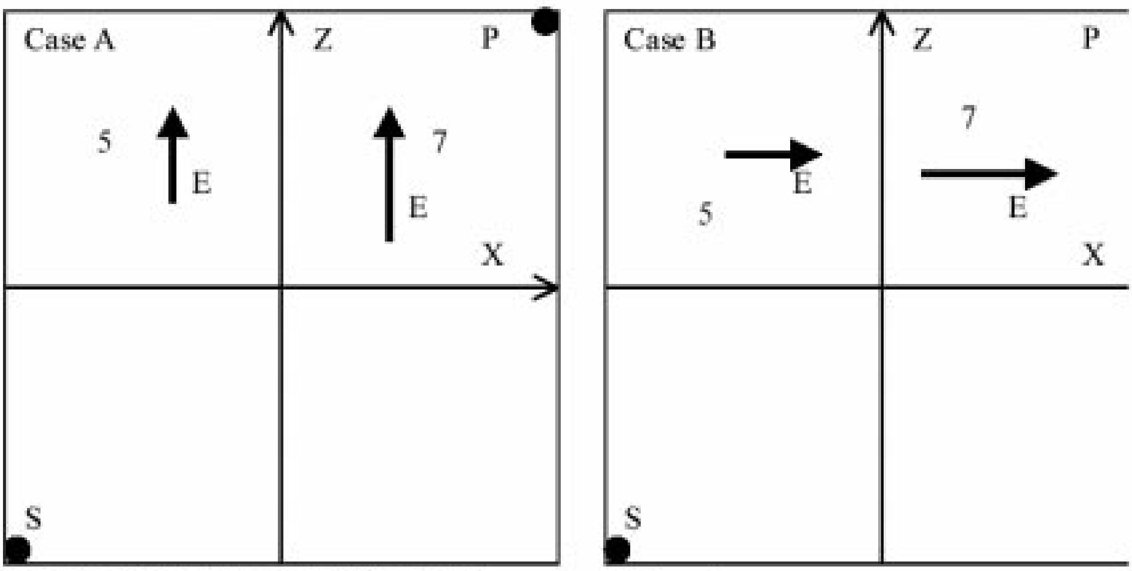
field, E, has a discontinuity at the yz plane. The field (in SI Units) in each case is:

$$\vec{E}_A = \begin{cases} 7\hat{k} & x > 0 \\ 5\hat{k} & x < 0 \end{cases} \quad \text{and} \quad \vec{E}_B = \begin{cases} 7\hat{i} & x > 0 \\ 5\hat{i} & x < 0 \end{cases}$$

In these equations *i* and *k* are unit vectors in the x and z directions respectively.
Furthermore P is the point (1, 0, 1) and S is the point (-1,0,-1).
For each case answer the following questions.

Is the electric field conservative?

1) If **so** then:
   a) What is the electric potential of point P with respect to point S?
   b) Is there any unbalanced electric charge in the region?
   c) If
      - So: where, in the region shown, is it and what is it's magnitude and sign?
      - Not: state why.
2) If **not** then:
   a) What is the value of the integral, $-\int_S^P \vec{E}.\overrightarrow{dl}$ along the path specified below,
      i) along a straight line joining points S and P?
      ii) along a straight line joining points S and (1,0,-1) and then along a straight line joining points (1,0,-1) and P.
   b) What might have caused the field?



<u>JUSTIFY YOUR ANSWERS</u> Marking scheme for each case

- State CONSERVATIVE or NOT CONSERVATIVE. 1 mark
- Justify by evaluating E.dl on two paths or a loop. 3 marks
- Possible sources. 1 mark

<u>CASE A</u>

2) NOT CONSERVATIVE

a)

$-\{[\sqrt{2} \bullet 5\cos(45)] + [\sqrt{2} \bullet 7\cos(45)]\} = -12V$

$-\{2 \bullet 5\cos(90) + 2 \bullet 7\cos(0) = -14V$

b)

Changing magnetic fields

CASE B

1) CONSERVATIVE

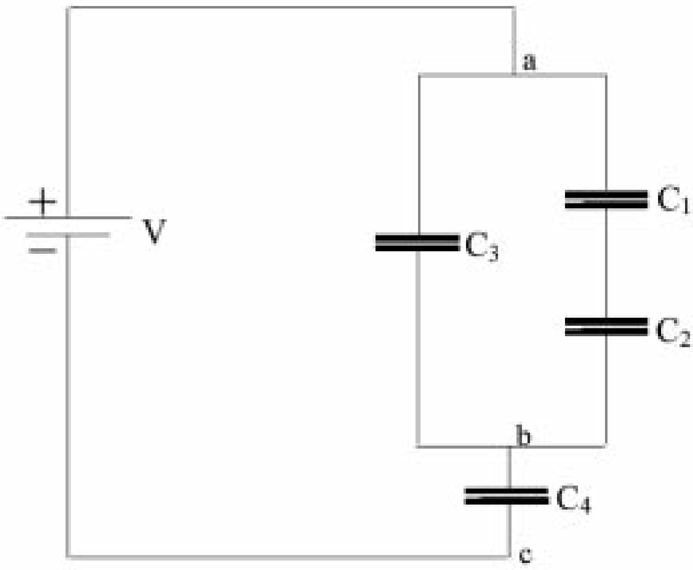
a)

$-\{[\sqrt{2} \bullet 5\cos(45)] + [\sqrt{2} \bullet 7\cos(45)]\} = -12V$

$-\{1 \bullet 5\cos(0) + 1 \bullet 7\cos(0) + 2 \bullet 7\cos(90) = -12V$

A=100 cm^2 and a separation d=2 μ (1μ=10^{-6} m). Capacitors C_1, and C_2 are filled with a slab of paraffin (dielectric constant k=2.4), capacitor C_3 is filled with neoprene (k=6.7) and capacitor C_4 is filled with paper (k=3.7). The battery provides a 100V potential across its terminals. Use two significant digits.

8/8



1) Calculate the equivalent capacitance between points a and c. **[2/2]**

C_1=C_2=($k\varepsilon_0$A/d)=2.4x(8.85x10^{-12})x(0.01)/2x10^{-6}=**1.062x10^{-7} F = 0.11 μF**

C_3=0.29 μF , C_4=0.164 μF **(0.5)**

C_{12}=[C_1C_2/(C_1+C_2)]=**0.053 μF** (equivalent C of capacitors 1 and 2) **(0.5)**

C_{123}=(C_{12}+C_3)=**0.35 μF** (equivalent C of capacitors 1, 2 and 3) **(0.5)**

C_{eq}=[$C_{123}C_4$/(C_{123}+C_4)]= 0.11 μF (equivalent C between a and c) **(0.5)**

2) Calculate the potential difference between points b and c, i.e., V_b-V_c. **[2/2]**

Q=C_{eq}V= 0.11 μF x 100 V = 1.11 x 10^{-5} C (Total charge) **(1)**

C_{123} and C_4 are in series ➔ they have the same charge Q

V_{bc}=Q/C_4= 67.68 V (Potential difference between b and c) **(1)**

3) Calculate the charge Q_1 and Q_3 on capacitors C_1 and C_3 when they are fully charged. **[2/2]**

$Q1= 1.71 \times 10^{-6}$ C **V_{ab}=100-67.68=32.32V => V_{C1}=1/2 V_{ab}** **$Q1=C_1 \times V_{C1}$**	$Q3=9.5 \times 10^{-6}$ C **V_{ab}=100-67.68=32.32V** **$Q3=C_3 \times V_{ab}$**

4) Calculate the energy stored in each capacitor. **[2/2]**

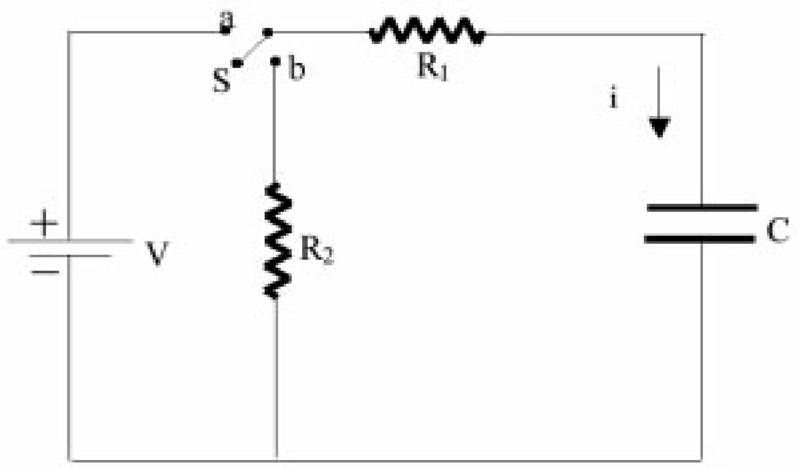
$E_{stored} = ½ CV^2 = ½ Q^2/C$ (use any formula)

$C1= 1.38 \times 10^{-5}$ J	$C2= 1.38 \times 10^{-5}$ J
$C3= 1.53 \times 10^{-4}$ J	$C4= 3.76 \times 10^{-4}$ J

Question 4

Consider the circuit below. The capacitor has a capacitance $C = 10\ \mu F$ and is initially uncharged. The resistors have the following values: R_1=47 kΩ and R_2=150 kΩ. The battery provides a 100V potential across its terminals. Use two significant digits.

10/10



1) We close the switch S at position **a**. Calculate the value of the charge, Q, on the capacitor after 100 ms.

[2/2]

$Q(t)=CV(1-e^{-t/RC})$, $RC=(47\ k\Omega)\times(10\ \mu F)$ =**0.47 s** (1)

$\mathbf{Q(0.1) = (10\ \mu F)x100x(1-e^{-(0.1/0.47)})} = 1.92 \times 10^{-4}$ C (1)

2) After 10 s we place the switch at position **b**. Can you assume that the capacitor is fully charged by now? (Justify your answer).

[1/1]

YES

t~20(RC)=> we can assume the capacitor is fully charged since e^{-20}~$2x10^{-9}$~0

3) Write down the discharging equation (i.e. with the explicit numerical values) for the current i.

[2/2]

$Q=Q_0e^{-t/RC}$

Note that now, in the discharging case, with switch at b, $\mathbf{R = R_1+R_2 = 197\ k\Omega}$ (1)

RC = 1.97 s

→ $\mathbf{I = dQ/dt = -(Q_0/RC)e^{-t/RC}} = -(10^{-3}/1.97)e^{-t/1.97}$ (1)

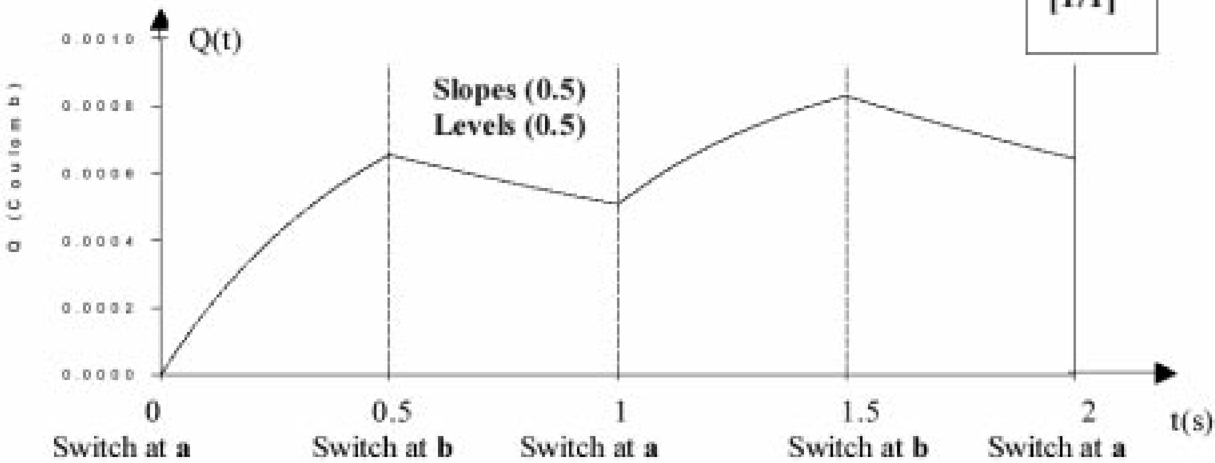
The energy dissipated is equal to the energy stored in the capacitor before discharge. **(1)**

$U = \frac{1}{2} Q^2/C = \frac{1}{2} (10^{-3})^2/10^{-5} = 0.05J$ **(1)**

Note: can use also $U = \frac{1}{2} V^2C$ or $U = i^2R_{eq}$ (where i must be integrated from 0 to inf)

5) Qualitatively sketch in the graph below the charging and discharging characteristics for 2 periods, assuming an interval of 0.5 s between every switching, and assuming that at t=0 the capacitor has no excess charge on its plates.

[1/1]



6) Calculate the value of:

a. Q(1)

[2/2]

Charging up to 0.5 s => $Q(0.5) = 10^{-3} \times (1-e^{-0.5/0.47}) = 6.55 \times 10^{-4}$ C = Q_0'

Discharging => $Q(1) = Q_0' e^{-t/RC} = 6.55 \times 10^{-4} e^{-0.5/1.97} = 5.08 \times 10^{-4}$ C **(1)**

b. Q(2)

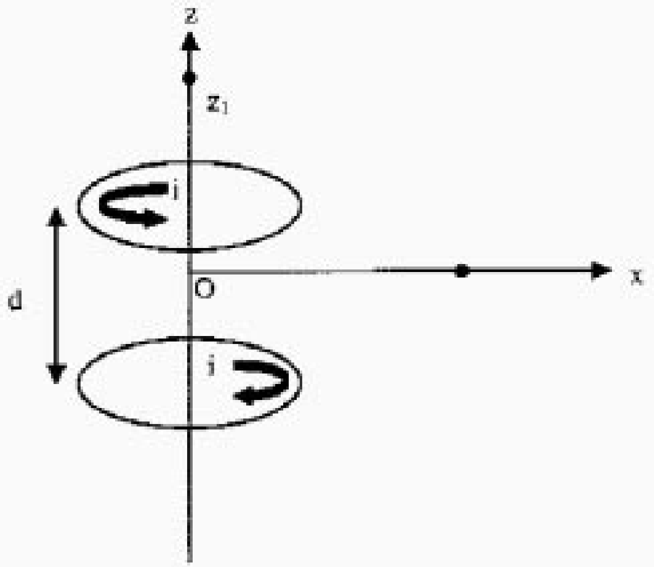
Use $V(t) = V_{final} + (V_{in}-V_{final})e^{-t/RC}$, Q=VC

⇨ $Q(1.5) = 10^{-5} \times [100 + (50.80 - 100)e^{-0.5/0.47}] =$ **8.3×10^{-4} C**

⇨ $Q(2) = 8.3 \times 10^{-4} e^{-0.5/1.97} = 6.44 \times 10^{-4}$ C **(1)**

Question 5 (10 marks)

Consider two identical current loops of radius R, with their centres lying symmetrically about the origin along the z-axis. The loops carry currents of equal magnitude but opposite directions. The centres of the loops are located at $(0,0,d/2)$ and $(0,0,-d/2)$.



1) Will the loops (pick one)

Attract	Repel	Experience no force

Justify your answer.

Think of the loops as magnetic dipoles, i.e. Poles of the same polarity repel.

N S ↑
S N ↓

2) Calculate the magnitude of the magnetic field vector $\bar{B}$. Leave your answer in symbolic form. Justify your answer.

a) At the centre of the top loop, i.e. at $(0,0,d/2)$.

Use superposition, i.e. $\bar{B}_{total} = \bar{B}_{top} + \bar{B}_{bottom} =$

$$\frac{\mu_0 i R^2}{2R^3}\hat{z} - \frac{\mu_0 i R^2}{2(R^2+d^2)^{3/2}}\hat{z} = \left(\frac{\mu_0 i}{2R} - \frac{\mu_0 i R^2}{2(R^2+d^2)^{3/2}}\right)\hat{z}$$

b) Is $\bar{B}$ pointing up or down at $(0,0,d/2)$?

$\bar{B}(0,0,d/2)$ is pointing up since $B_{top} > B_{bottom}$ at $z = d/2$

c) At the centre of the bottom loop, i.e. at $(0,0,-d/2)$.

By anti-symmetry $\bar{B}_{total}(0,0,-d/2) = -\bar{B}_{total}(0,0,d/2)$

d) Is $\bar{B}$ pointing up or down at $(0,0,-d/2)$?

From c. $\vec{B}(0,0,-d/2)$ is pointing down.

e) At the origin, $z = 0$.

By anti-symmetry $\bar{B} \equiv 0$ over the entire x-y plane.

.5 f) Along the x-axis.

$$\bar{B} \equiv 0$$

3) Assume that z_1 is much larger than R, i.e. z_1 >>R, so that each loop behaves like a magnetic dipole. Write down a symbolic expression for $\bar{B}$ (z_1)

For $z_1 >> R$ each loop behaves like a magnetic dipole, I.e.

$B(z) = \frac{\mu_0 i R^2}{2(R^2+z^2)^{3/2}} \simeq \frac{\mu_0 i R^2}{2z^3}$. Now use superposition:

$\bar{B}(z_1) = \bar{B}_{top} + \bar{B}_{bottom} \cong \frac{\mu_0 i R^2}{2(z_1-d/2)^3}\hat{z} - \frac{\mu_0 i R^2}{2(z_1+d/2)^3}\hat{z}$.

1.5

4) On top of this assumption, suppose that d is small compared to z_1, i.e. $d << z_1$. Use the binomial theorem to put the magnetic field at the point $(0,0, z_1)$ in the form $B \cong k/z_1^n$. Determine the numerical value of the exponent n and give an expression for the constant k.

From (3) and the binomial theorem (keeping only the linear te

$\bar{B}(z_1) = \frac{\mu_0 i R^2}{2z_1^3}\left[\left(1-\frac{d}{2z_1}\right)^{-3} - \left(1+\frac{d}{2z_1}\right)^{-3}\right]\hat{z} \cong$

$\frac{\mu_0 i R^2}{2z_1^3}\left[\cancel{1} + \frac{3d}{2z_1} - \cancel{1} + \frac{3d}{2z_1}\right] = \hat{z}\frac{\mu_0 i R^2}{2z_1^3}\frac{3d}{z_1} = \frac{3\mu_0 i R^2 d}{2z_1^4}$

I.e. $k = \frac{3\mu_0 i R^2 d}{2}$ and $n=4$. The decay of the field with distance is very drastic.

1

5) If the direction of the current in the bottom loop is reversed, then the magnitude of B will (pick one):

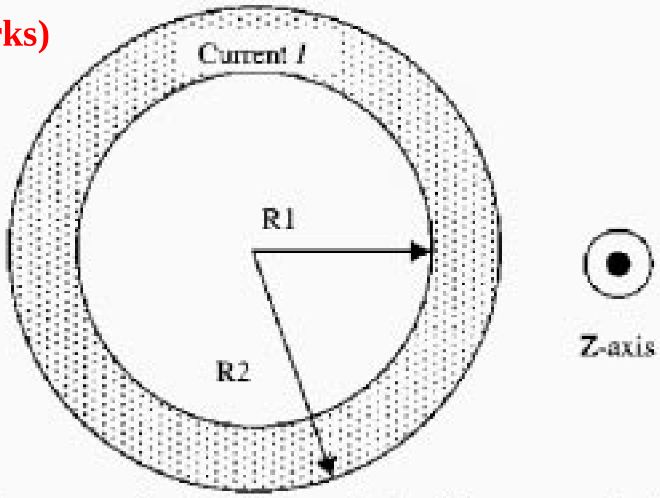
increase at z_1 but decrease at the origin 0	increase at z_1 and at the origin 0 ✓
decrease at z_1 but increase at the origin 0	decrease at z_1 and at the origin 0
none of the above	

Justify your answer

If one of the two currents is reversed then $\bar{B}$ will be enhanced <u>everywhere</u>. This becomes a 2-loop solenoid.

1) Consider an infinitely long cylindrical conducting shell extending between $R1 < r < R2$. The cylinder carries a constant current I parallel to its axis (Z-axis) which is uniformly distributed over its cross-sectional area (shaded area in figure).

Question 6 (10 marks)



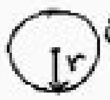
1 a) Derive an expression for the magnitude of the current density J in the region $R1 < r < R2$.

$$J = \frac{i}{A} = \frac{i}{\pi R_2^2 - \pi R_1^2}$$

1 b) Derive an expression for the magnitude of the magnetic field $\overline{B}$ in the interior of the shell, i.e. for r<R1.

i) Magnitude

From Ampere's Law $\oint_c \overline{B} \cdot d\overline{l} = \mu_0 i_{enc} = 0$



Amperian loop = Concentric circle with radius $r < R_1$

Hence due to cylindrical symmetry $\underline{\overline{B} \equiv 0}$. (B(

0.5 ii) What is the direction of $\overline{B}$?

$\overline{B} \equiv 0$, $r < R_1$

1 c) Derive an expression for the magnitude of the magnetic field $\overline{B}$ in the exterior of the shell, i.e. for r>R2.

i) Magnitude

$\oint \overline{B} \cdot d\overline{l} = \mu_0 i$

C = concentric circle with $r > R_2$

Hence $B(2\pi r) = \mu_0 i$ i.e.

$B = \frac{\mu_0 i}{2\pi r}$, $r > R_2$

0.5 ii) What is the direction of $\overline{B}$?

From RHL, $\overline{B}$ (is azimuthal) is directed counter- clockwise.

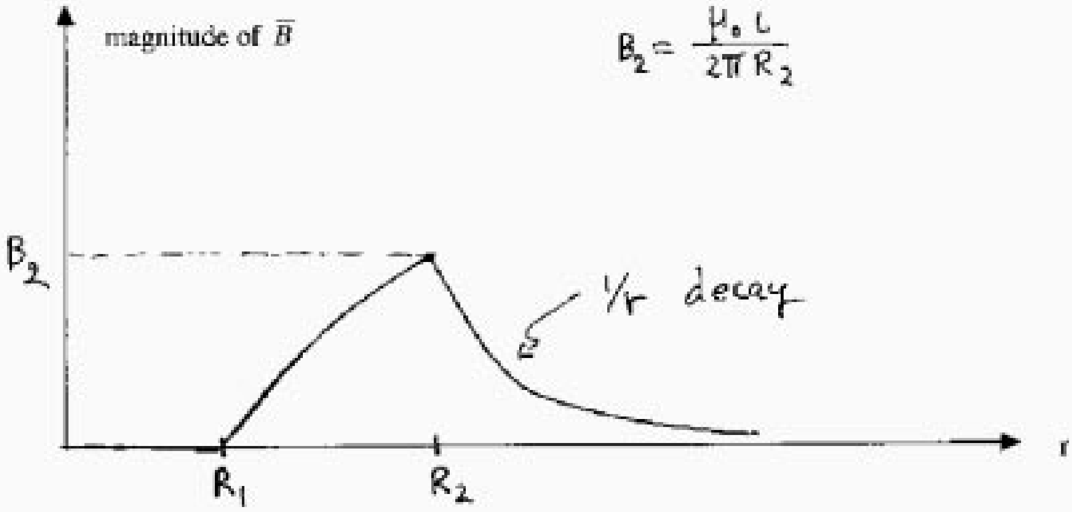
i) Magnitude $\int_c \overline{B}\cdot d\overline{l} = \mu_0 i$ i.e. $B(2\pi r) = \mu_0 J A$
$= \mu_0 \frac{i}{A}(\pi r^2 - \pi R_1^2)$

i.e. $B = \frac{\mu_0}{2\pi r}\frac{i}{(\pi R_2^2 - \pi R_1^2)}(\pi r^2 - \pi R_1^2)$

ii) What is the direction of $\overline{B}$?

i.e. $B = \frac{\mu_0 i}{2\pi r}\,\frac{r^2 - R_1^2}{R_2^2 - R_1^2}$ $R_1 < r < R_2$ and points counter-cl

1 c) Sketch the magnitude of $\overline{B}$ as a function of r, for r>0.



1 2) Now suppose that a straight wire is placed at the center of the cylindrical shell (i.e. at r=0). The wire carries the same current I as the shell but along the opposite direction (i.e. along the negative Z-axis).

1 a) Derive an expression for the magnitude of the magnetic field $\overline{B}$ in the interior of the shell, i.e. for r<R1.

i) Magnitude

In this case $\oint \overline{B}\cdot d\overline{l} = \mu_0 i \Rightarrow B(2\pi r) = \mu_0 i$ i.e.

$$B = \frac{\mu_0 i}{2\pi r}$$

0.5 ii) What is the direction of $\overline{B}$?

Azimuthal and clockwise

1.5 b) Derive an expression for the magnitude of the magnetic field $\overline{B}$ in the exterior of the shell, i.e. for r>R2.

i) Magnitude

$B(2\pi r) = \mu_0(i - i) = 0 \Rightarrow B = 0$

0.5 ii) What is the direction of $\overline{B}$?

$\overline{B} \equiv 0$