UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

DEPARTMENT OF MECHANICAL AND INDUSTRIAL ENGINEERING

FINAL EXAM

MIE230F - ENGINEERING ANALYSIS

Examiner: Cliff K.K. Lun

INSTRUCTIONS:

- Close books *Type B* exam.
- Value for each question is indicated below.
- Students can use the backs of the pages for their work. Show all steps of calculations clearly.
- Final answers involving complex numbers must be expressed in the form of $x+iy$

Date: December 19, 2001 (Wednesday) **Duration**: 2:00 p.m. - 4:30 p.m. (2.5-hour)

Student Name: ______________________ **I.D.#**: ______________

	SCORE
1. (20 points)	
2. (20 points)	
3. (15 points)	
4. (15 points)	
5. (15 points)	
6. (15 points)	
TOTAL (100 points)	

1. **Verify** the *Stokes' Theorem* for the vector field $v = z^2 i + 2xj - y^3 k$ over the upper portion (i.e. $z \geq 0$) of the surface S given by $z = 4 - (x^2 + y^2)$. Indicate the sense of circulation on the bounding curve C

2. (a) Evaluate the integral directly

$$\oint_C (3x^2 + y)dx + 4xydy$$

where C is the triangular region with vertices (0,0), (2,0) and (0,4). Curve C is traversed in a counterclockwise manner.

(b) Verify the Green's Theorem for the above integral.

3 Find all solutions of the equation $\sinh z = 1 + i$.

4. Evaluate $\oint_C \frac{z+1}{z(2z+i)(2z-5)^3}\,dz$ where $C : |z-1| = 2$ is the counterclockwise path. Identify what kind of singular points there are.

5 Evaluate $\oint_C z^4 \sin(\frac{1}{z})dz$ where C is the unit circle $|z| = 1$, counterclockwise. Identify what kind of singular points there are.

6 Evaluate the integral of real rational function $\int_0^{2\pi} \frac{\cos\theta}{1-\frac{1}{2}\cos\theta}\,d\theta$

MIE230F - Formula Sheet

Gradient : $\nabla f(r) = f_x i + f_y j + f_z k$, $\quad r = xi + yj + zk$

Diretional derivative in u $\quad f_u'(r) = \nabla f(r) \cdot u$

Tangent plane equation at r_o : $\quad \frac{\partial f(r_o)}{\partial x}(x - x_o) + \frac{\partial f(r_o)}{\partial y}(y - y_o) + \frac{\partial f(r_o)}{\partial z}(z - z_o) = 0$

Normal line equation at r_o $\quad r = r_o + \nabla f(r_o)t$

Double Integrals Mean - Value Theorem :

$$\iint_\Omega f(x,y)\lambda(x,y)dxdy = f(x_o, y_o)\iint_\Omega \lambda(x,y)dxdy$$, where $f(x_o, y_o)$ is the average value on Ω.

Polar Coordinates : $x = r\cos\theta$, $y = r\sin\theta$, $\quad \iint_\Omega f(x,y)dxdy = \iint_\Gamma f(r,\theta)rdrd\theta$

Centroids: $\bar{x}A = \iint_\Omega xdxdy$, $\quad \bar{y}A = \iint_\Omega ydxdy$, $\quad A = \iint_\Omega dxdy$

Center of Mass: $x_m M = \iint_\Omega x\lambda(x,y)dxdy$, $\quad y_m M = \iint_\Omega y\lambda(x,y)dxdy$, $\quad M = \iint_\Omega \lambda(x,y)dxdy$

Moment of Inertia: $I_x = \iint_\Omega y^2 dxdy$, $\quad I_y = \iint_\Omega x^2 dxdy$

Triple Integrals Mean - Value Theorem :

$$\iiint_T f(x,y,z)\lambda(x,y,z)dxdy = f(x_o, y_o, z_o)\iiint_T \lambda(x,y,z)dxdydz$$, $f(x_o, y_o, z_o)$ is the average value on T

Cylindrical Coord. : $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $\quad \iiint_T f(x,y,z)dxdydz = \iiint_S f(r,\theta,z)rdrd\theta dz$

Spherical Coord. : $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$,

$$\iiint_T f(x,y,z)dxdydz = \iiint_S f(\rho,\theta,\phi)\rho^2 \sin\phi d\rho d\theta d\phi$$

Jacobains : $\quad \iint_\Omega f(x,y)dxdy = \iint_\Gamma f(u,v)|J(u,v)|dudv \qquad J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

Line Integral: $\quad \int_C h(r) \cdot dr = \int_a^b h(r(u)) \cdot r'(u)du$

Straight line parametric equation from $a=(a_1,a_2)$ to $b=(b_1,b_2)$: $\quad r(u) = (1-u)a + ub$, $\quad u \in [0,1]$

Green's Theorem: $\quad \iint_\Omega \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy = \oint_C Pdx + Qdy$

Divergence (Gauss) Theorem: Flux $\quad \iiint_T (\nabla \cdot v)dxdydz = \oiint_S (v \cdot n)d\sigma$

Stokes' Theorem: $\quad$ **Circulation** $\quad \iint_S (\nabla \times v) \cdot nd\sigma = \oint_C v(r) \cdot dr$

Upper normal: $\quad N = \nabla F(x,y,z)$, $\quad F(x,y,z) = z - f(x,y)$, $\quad for\ S \quad z = f(x,y)$

Upper unit normal & surface element:

$n = N/|N|$, $\quad d\sigma = |N|dxdy$

Complex variable : $\quad z = x + iy$

$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, $\quad |z| = r$, $\quad r^2 = x^2 + y^2$, $\quad \theta = \arg(z) = \tan^{-1}(\frac{y}{x}) \pm 2n\pi$, $\quad n = 0,1,$

principal value $\quad Arg(z) \qquad -\pi < Arg(z) \le \pi$

powers : **De Moivre Formula -** $\quad z^n = r^n(\cos n\theta + i\sin n\theta)$

roots : $\quad \sqrt[n]{z} = \sqrt[n]{r}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}) \qquad k = 0,1,\ldots,n-1$

Cauchy - Riemann criteria for analytic function : $u_x = v_y$, $\quad u_y = -v_x$

Trigonometric and hyperbolic functions :

$e^{iz} = \cos z + i\sin z \qquad \cos z = \dfrac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \qquad \cos^2 z + \sin^2 z = 1,$

$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2, \quad \sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \mp \sin z_1 \cos z_2$

$\cosh z = \dfrac{e^{z} + e^{-z}}{2} \qquad \sinh z = \dfrac{e^{z} - e^{-z}}{2} \qquad \cosh^2 iz - \sinh^2 iz = 1$

$\cosh iz = \cos z; \quad \sinh iz = i\sin z; \quad \cos iz = \cosh z; \quad \sin iz = i\sinh z;$

Logarithm : $\ln z = \ln r + i\theta \qquad r = |z| > 0, \qquad \theta = \arg z = Arg\, z \pm 2n\pi, \qquad n = 0,1,2 \ldots$

General Complex Power : $z^{c} = e^{c\ln z}, \qquad c$ – **complex power**

Cauchy Integral Theorem for an analytic $f(z)$ inside a simply connected domain : $\oint_c f(z)dz = 0$

For multiply connected domain enclosing N singular points : $\oint_c f(z)dz = \sum_{n=1}^{N} \oint_{c_n} f(z)dz$

Cauchy's Integral Formular : $\oint_c \dfrac{f(z)}{z - z_o} dz = \begin{cases} 2\pi i f(z_o) & z_o \textit{ inside } C \\ 0 & z_o \textit{ outside } C \end{cases}$

Derivative of all orders : $\oint_c \dfrac{f(z)}{(z - z_o)^{n+1}} dz = \begin{cases} \dfrac{2\pi i}{n!} f^{(n)}(z_o) & z_o \textit{ inside } C \\ 0 & z_o \textit{ outside } C \end{cases} \qquad n = 0,1,2,3 \ldots$

Some common series :

$\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^{n} \qquad |z| < 1; \qquad e^{z} = \sum_{n=0}^{\infty} \dfrac{z^{n}}{n!} \qquad |z| < \infty$

$\sin z = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad |z| < \infty; \qquad \cos z = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n}}{(2n)!} \qquad |z| < \infty$

$\sinh z = \sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+1)!} \qquad |z| < \infty; \qquad \cosh z = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(2n)!} \qquad |z| < \infty$

Laurent series:

$f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n + \sum_{n=1}^{\infty} \dfrac{b_n}{(z - z_o)^n} \qquad R_2 < |z - z_o| < R_1$

the residue of f(z) at $z = z_o$ is: $b_1 = \operatorname*{Res}_{z=z_o} f(z)$

Residue integration : $\oint_c f(z)dz = 2\pi i b_1$

Methods of finding b_1

1. **For simple poles :** $b_1 = \operatorname*{Res}_{z=z_o} f(z) = \lim_{z \to z_o} (z - z_o) f(z)$

2. **For simple poles :** $let\ f(z) = \dfrac{p(z)}{q(z)}, \quad then\ b_1 = \operatorname*{Res}_{z=z_o} f(z) = \dfrac{p(z)}{q'(z)}$

3. **For poles of order m > 1 :** $b_1 = \operatorname*{Res}_{z=z_o} f(z) = \dfrac{1}{(m-1)!} \left[\dfrac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right]_{z=z_o}$

4. **For isolated essential singularity :** **use direct Laurent series expansion.**

Residue Theorem for contour C enclosing k singular points : $\oint_c f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$

Evaluation of real integrals with cosθ and sinθ :

$let\ z = e^{i\theta}, \quad r = 1$

$\cos\theta = \dfrac{1}{2}(e^{i\theta} + e^{-i\theta}) = \dfrac{1}{2}(z + \dfrac{1}{z}), \qquad \sin\theta = \dfrac{1}{2}(e^{i\theta} - e^{-i\theta}) = \dfrac{1}{2}(z - \dfrac{1}{z})$

Evaluation of improper real integral of the type : $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{all\ poles} \operatorname{Res} f(z)$

where the degree of the denominator of *f*(x) is at least two units higher than the degree of the numerator.