MAT [illegible] S, 1998 Finals Akcoglu, Stangeby

...uate

$$\int \frac{dx}{x^2(x-1)^2}.$$

2a. Find the total length of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$, where a is a positive constant and $0 \le t \le 2\pi$.

2b. Find the volume of the region obtained by rotating the above astroid about the x-axis. (Note that it is enough to rotate the upper half of the curve, corresponding to $0 \le t \le \pi$.)

3. Find the area of the region that lies inside the polar curve $r = 1 - \sin\theta$, $0 \le \theta \le 2\pi$, and outside the circle $r = 1$.

4. A number a is given as

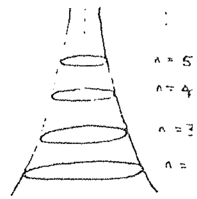
$$a = \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots}}}}.$$

(More precisely, $a = \lim_{k\to+\infty} a_k$, where $a_1 = \sqrt{3}$ and $a_{k+1} = \sqrt{3 + a_k}$ for $k \ge 1$.) Show the existence of this number and find its value.

...all tower of circular cross section is reinforced by horizontal circular discs (like large coins) one meter apart and of negligible thickness. The radius of the disc at height n is $1/(n \log n)$, $n \geq 2$. Assume that the tower has infinite height.

5a. Can the discs be made from a finite amount (area) of material? Why?

5b. If the discs are further strengthened by wires going around their circumference (like tires), will the total length of the wire be finite or not? Why?



6. For each of the following two series find all the (real) values of x for which the series is convergent. Find the sum of the series for these values of x.

$$\sum_{k=1}^{\infty} \frac{2^{kx}}{k}, \quad \sum_{k=1}^{\infty} (2^k + 3^k) x^k .$$

7. Find a polynomial $P(x)$ such that

$$\left|P(x) - \int_0^x \frac{1}{1+t^8}\,dt\right| < 10^{-4}|x|$$

for each x such that $|x| < 1/2$. Justify your result.

8. The points on a helix are given in term of the parameter t by

$$\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + \lambda t\,\mathbf{k}\,,$$

where a and λ are positive constants. Find the equation of the normal (that is, the principal normal) to the curve at the point P with parameter t, and show that this normal lies on the surface

$$\frac{y}{x} = \tan\frac{z}{\lambda}\,.$$

What is the curvature of the helix at the point P?

... the tangent plane to the surface

$$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$$

at the point (x_0, y_0, z_0). Here a is a positive constant.

9b. Show that the sum of the squares of the intercepts of this plane on the coordinate axes is constant.

10a. Find the maximum value of $f(x, y, z) = (xyz)^{1/3}$, given that $x + y + z =$ constant and $x, y, z \geq 0$. Hence show that

$$(xyz)^{1/3} \leq \frac{1}{3}(x + y + z)$$

for nonnegative x, y, z.

10b. Is the general result

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

true for nonnegative variables $x_1, x_2, \ldots, x_n$? Give reasons.