University of Toronto

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATIONS, DECEMBER 2001

Third Year - Program 5

MAT 389H1F - Complex Analysis

Exam Type: A

Examiner: Professor T. Bloom

Duration: $2\frac{1}{2}$ hours

[5] 1.(a) Find all complex numbers z such that $e^{2z} + e^z + 1 = 0$.

[5] (b) Find the coefficient of $(z-i)^8$ in the Taylor series of $\dfrac{\sin(z-i)}{z}$ centered at $z = i$.

2. Find the order of the pole and the residue at 0 for

[5] (a) $\dfrac{1+e^z}{z^2} + \dfrac{1}{z}$

[5] (b) $\dfrac{z^2}{(\sin 4z)^3}$

[5] 3.(a) What is the radius of convergence of the Taylor series of $\dfrac{e^z}{(z-1)(z+1)}$ centered at $z = 2i$.

[5] (b) Find the largest domain on which the series $\dfrac{e^{nz^2}}{n^2+1}$ converges and represents an analytic function.
Justify your conclusion in (a) and (b) above.

[10] 4.(a) How many zeros does the polynomial $P(z) = z^4 + 2z^3 + 3z^2 + z + 2$ have in the right half plane (i.e. $\{z|\mathrm{Re}(z) > 0\}$).

[6] (b) Use Rouche's theorem to show that there are two solutions to $3z^2 = e^z$ inside the circle $|z| = 1$.

[14] 5. Use the method of residues to evaluate $\displaystyle\int_0^{\pi} \frac{d\theta}{a + \sin^2\theta}$ for $a > 0$.

[7] 6.(a) What is the image of $\{(x,y)|x>0, y>0, xy>1\}$ under the mapping $z \to z^2$.

[7] (b) What is the image of the sector

$$-\frac{\pi}{4} < \mathrm{Arg}(z) < \frac{\pi}{4}$$

under the mapping $z \to \dfrac{z}{z-1}$.

[7] 7.(a) What is the value of

$$\int_{|z|=2} \left(z e^{\frac{1}{z^2}} - z^2 e^{\frac{1}{z}} \right) dz$$

where the circle $|z| = 2$ is directed counterclockwise.

[7] (b) Let C_1 be the circle $|z| = 8$ and C_2 the circle $|z| = 1$ both directed counterclockwise. What is the value of the integral $\displaystyle\int_{C_1} f(z)dz - \int_{C_2} f(z)dz$ where

$$f(z) = \frac{1}{(4z^2+1)(6z^4+5)(z^2-12z+20)}.$$

8. Let $u(x,y) = e^x(x\cos y - y\sin y)$.

[4] (a) Show that u is harmonic.

[4] (b) Find a harmonic conjugate for u.

[4] (c) Find $f(z)$ analytic such that $\mathrm{Im}\left(f(z)\right) = v$.

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[8] **BONUS**

Show that $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos x}{e^x + e^{-x}}\, dx = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}$.

HINT: Integrate an appropriate function around the rectangle with vertices $\pm R$, $\pm R + \pi i$.