UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION APRIL 17, 2001

Third Year – Program: Electrical Engineering

ECE302S – PROBABILITY AND APPLICATIONS

Exam Type: C

Examiner – D. Kundur

- A single aid sheet and a non-programmable calculator (with no communication capabilities) are the only aids allowed.

- Answer **all six** questions.

- Start each new question on a separate page.

- If you need to make any assumptions, state them clearly in your answer.

- **Show all steps and present all results clearly.**

- Time: $2\frac{1}{2}$ hours.

1. **(10 marks)**
   Prove the following starting from first principles (e.g., axioms of probability, corollaries, definition of the cdf, definition of expectation, ...)

   (a) $P[a < X \leq b] = F_X(b) - F_X(a)$ for $a < b$, where $F_X(x)$ is the cdf of X.

   (b) $P[X \geq c] = 1 - F_X(c)$ if X is a continuous random variable.

   (c) $P[X \geq c] \leq \frac{E[X]}{c}$ for X nonnegative (i.e., $f_X(x) = 0$ for $x < 0$, where $f_X(x)$ is the pdf of X), and for $c > 0$.

2. **(10 marks)**
   A web site earns revenue through on-line sales. Suppose that customer orders are all independent uniformly distributed random variables with mean of \$20 and a variance of \$5. Estimate the probability that the first five hundred customers spend altogether between \$9,000 and \$11,000. Simplify your answer analytically.

3. **(10 marks)**
   Let X_1 and X_2 be two independent noise processes in an electronic circuit.

   (a) If X_1 is a uniformly distributed random variable with a mean of 4 and a variance of 3, and X_2 is an exponential random variable with a mean of 2 and a variance of 4, compute the joint pdf of both noise processes.

   (b) At the output of the circuit the noise processes sum together to produce the overall "output" noise. Determine the pdf of this output noise $X_1 + X_2$. Sketch this pdf.

4. **(10 marks)**

   (a) A popular Internet chat service charges users on a per half-hour basis. Specifically, the charge is $1 for each half-hour or less. Suppose that the amount of time users use the service is an exponentially distributed random variable with $\lambda = 1$. Find the probability that a user pays D dollars.

   (b) The Engineering Computing Facility requires users to enter passwords for access to their e-mail. If the probability of any user making a mistake when entering their password is of $e^{-\frac{i}{2}}$, find the probability of someone having to enter their password k times before gaining entry.

   (c) For parts (a) and (b), comment on the results and probability laws applied for each situation.

5. **(10 marks)**
   Consider the following joint pdf function

   $$f_{X,Y}(x, y) = \begin{cases} c\sqrt{x^2 + y^2} & \text{for } x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute

       i. the constant c.
       ii. the mean of X, and the mean of Y.
       iii. the covariance of X and Y.
       iv. $E[\sqrt{X^2 + Y^2}]$.

   (b) As discussed in class, comment on the "linear predictability" of X given Y. Repeat for Y given X. Are X and Y independent?

   (c) Consider

   $$\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

   Find and simplify the joint pdf of V, W.

   (d) Find the pdf of $Z = X + Y$.

6. **(5 marks)**

   An individual accessing a web site is processed by server k with probability p_k, $k = 1, 2, \ldots, n$. The time taken by server k to process the customer's request for information is an exponential random variable with mean μ_k.

   (a) Find the pdf of T, the time taken to process the request.

   (b) Find $E[T]$ and $VAR[T]$.