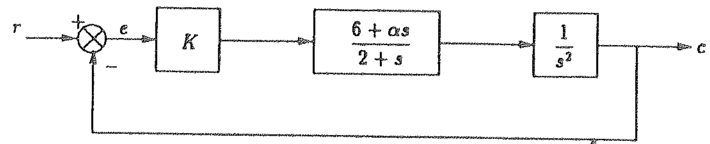
UNIVERSITY OF TORONTO
DEPARTMENT OF ELECTRICAL & COMPUTER ENGINEERING
FINAL EXAMINATION - DECEMBER 1993
ECE355F - SYSTEM AND SIGNAL ANALYSIS I
Third Year - Programs 5ce, 5e; 5p
EXAMINER - W.M. Wonham

Instruction: *Please answer each of the three main questions in a separate examination booklet.*

Marking scheme: *Each of the three main questions is worth 1/3 of the total mark. Each of the Part *.9 'challenge' questions is worth 10% of its main question.*

1.



In the tracking system TS shown, c is the controlled output, r the reference input, and e the tracking error. The parameters α and K are real numbers.

1.1 Calculate the input-to-error transfer function $\hat{e}(s)/\hat{r}(s)$.

1.2 Determine the ranges of α and K for which TS is stable.

1.3 Assume TS is stable. With $r \equiv 0$ and K fixed, imagine the value of α is slowly changed until TS just breaks into spontaneous oscillation. Calculate this critical frequency of oscillation as a function of K and α.

1.4 Let $\alpha = 3$ and $K = 1$. If TS is at rest prior to $t = 0$, and $r(t) = \cos(\sqrt{3}t)$ $(t \geq 0)$, describe $e(t)$ for large t. Express your answer as a finite sum

$$e(t) = Ae_1(t) + Be_2(t) + \ldots$$

where $A, B, \ldots$ are constants (*don't* calculate!) and $e_1, e_2, \ldots$ are functions that you *do* specify explicitly.

1.5 Let $\alpha = 12$ and $K = 1/6$. If $r \equiv 0$ and TS is assigned arbitrary nonzero initial conditions, describe $e(t)$ for all $t > 0$, in the same style as in 1.4.

1.6 Suppose TS is stable. If $r(t) = (1-t)^2$ $(t \geq 0)$ calculate, as a function of α and K, the approximate tracking error for very large t.

1.7 For what values of α and K does TS have a steady-state frequency response $e(t)$ to an input of form $r(t) = A\cos(\omega t) + B\sin(\omega t)$ $(t \geq 0)$? What is the amplitude of this response as a function of ω and the parameters A, B, α, K?

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1.8 In the block diagram, replace K by the element $Ke^{-\beta s}$, where β is a small positive parameter. What is the physical meaning of this change to the model? Intuitively, what effect would you expect this change to have on the stability ranges of K and α?

1.9 If you have time, calculate the approximate (first-order) perturbation of the stability range of α, for very small $\beta > 0$, and the corresponding perturbation in the value of the critical frequency of oscillation.

2. In this question, numerical evaluations should be done to just 2 significant figures. Consider the signal

$$f(t) = \begin{cases} e^{-\alpha(t-1)}, & t \geq 1 \\ 0, & t < 1 \end{cases}$$

where α is a positive parameter.

2.1 As a function of α, calculate the total energy of f.

2.2 As a function of α, calculate the value of T such that 99% of the total energy of f is captured by the time interval $0 \leq t \leq T$.

2.3 Calculate the Fourier transform $\hat{f}(\omega)$ $(-\infty < \omega < \infty)$ of f.

2.4 State the relationship between the L_2-norm of $\hat{f}$ and that of f.

2.5 As a function of α, calculate the value of Ω such that 99% of the energy of f is captured by the frequency range $-\Omega \leq \omega \leq \Omega$.

From now on, adopt T as the practical time duration, and Ω as the practical frequency bandwidth, of f.

2.6 If you sample f at the Nyquist rate, starting at $t = 0$, how many nonzero samples will you need to store f?

2.7 For a strictly band-limited signal h, the Nyquist-Shannon theory provides a formula for the energy of h in terms of the sample values of h. State this formula.

2.8 Let $\alpha = 1$. Assume the first nonzero sample of f is located at $t = 1$. Calculate the (supposed) energy of f as given by the formula of 2.7, using only samples from the interval $1 \leq t \leq T$. Explain any discrepancy between your result, and the true value of the energy of f.

2.9 If you have time, calculate an approximate value of the sampli ate that would reduce the discrepancy in 2.8 to about 1% of the energy of f.

3. In this question, I denotes a fixed but arbitrary subinterval of the real line. $L_2(I)$ denotes the family of all complex-valued signals defined and finitely square-integrable on I.

3.1 Define what is meant by the norm and scalar product for signals in $L_2(I)$. If the interval I has finite length, define the mean square value and the root mean square (r.m.s.) value of a signal in terms of its norm.

3.2 Define what is meant by mean-square convergence of a sequence of signals $\{f_k \mid k = 0, 1, 2, ...\}$ in $L_2(I)$, to a limit $f \in L_2(I)$.

3.3 Let

$$\Phi := \{\phi_n \mid n = 0, 1, 2, ...\}$$

denote an infinite system of signals $\phi_n \in L_2(I)$. Define what is meant by the statement that Φ is an orthonormal (ON) system on I.

3.4 Let $f \in L_2(I)$. Define what is meant by the Fourier series of f with respect to Φ, and give a formula for the Fourier coefficients of f.

3.5 State the "minimizing property" of the Fourier coefficients of f.

3.6 Let

$$f_N := \sum_{n=0}^{N} c_n \phi_n$$

be the Nth partial sum of the Fourier series of f with respect to Φ. State the 'pythagorean' relationship that holds among f, f_N, and $e_N := f - f_N$, and describe f_N in geometric terms. Illustrate with a sketch.

3.7 State Bessel's inequality for a signal $f \in$ in $L_2(I)$. What conclusion can be drawn about the behavior of the Fourier coefficients of f for large n?

3.8 Prove that the Fourier series of f converges to f in mean square, if and only if Bessel's inequality for f becomes equality.

3.9 If you have time, answer the following. Let $f \in L_2(I)$, let f_N be as in 3.6, and assume the Fourier series of f converges to f in mean square. Let $\hat{f}_N$ be the so-called Fejér partial sum

$$\hat{f}_N := \sum_{n=0}^{N} (1 - n/N) c_n \phi_n$$

and let $e_N := f - f_N$, $\hat{e}_N := f - \hat{f}_N$. Show carefully that

$$\|\hat{e}_N\|^2 = \|e_N\|^2 + \sum_{n=0}^{N} (n|c_n|/N)^2$$

If $|c_n| \leq 1/n$ for all $n > 1000$, what can be inferred about the behavior of $\hat{e}_N$ as $N \to \infty$? Justify your answer with care.