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UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER 2001

First Year – Program 5

MAT 194F - CALCULUS I

Exam Type A

Time allowed: 2-1/2 hours

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Each question is worth a maximum of 10 marks. Please begin each solution on a new page of your examination book.

1. Determine

(a)

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx$$

(b)

$$\int \frac{1-x}{\sqrt{1-x^2}}\,dx$$

(c)

$$\int (\tan^2 x + \cos^2 x)\,dx$$

(d)

$$\frac{d}{dx}\int_{\sqrt{x}}^{\sqrt{2x}} \sqrt{1+t^4}\,dt$$

(e)

$$\frac{d}{dx}\left[(1+x^2)^{3/2}x^{(\ln x)}\right]$$

2. Determine

(a)

$$\lim_{x\to\infty}\frac{\sinh x}{2e^x}$$

(b)

$$\lim_{x\to 9}\frac{x^{3/2}-\text{[illegible]}}{x-9}$$

(c)

$$\lim_{x\to\infty}(x^2-\sqrt{x^4-3x^2+3})$$

(d)

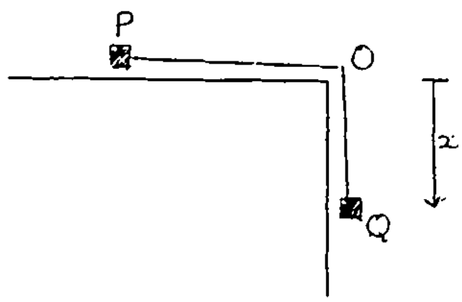
$$\lim_{x\to 0}\left(\frac{\sin x}{x^3}-\frac{1}{x^2}\right)$$

(e)

$$\lim_{x\to 0} x^{\sin x}$$

(e) Sketch the graph of the equation $y = f(x)$ showing all important features.

7. Two equal point masses P and Q are connected by a taut, massless, inextensible string of length L. They sit on a horizontal table top. At time $t = 0$, Q drops from rest over the edge and, falling vertically from point O with constant acceleration a, drags P across the tabletop.



Let x represent the vertical distance between O and Q, and A the area of triangle POQ. Consider x and A as functions of the time t.

(a) Obtain a polynomial expression for $A(t)$.

(b) When is $A(t)$ maximum? Determine its maximum value.

(c) Sketch the graph of $A(t)$, labelling all its important features.

8. Two streams of beer are flowing into a mixing vessel of volume 4000 litres. The vessel initially contains 1000 litres of beer of concentration 0.1 kg of alcohol per litre. Stream A has an alcoholic concentration of 0.06 kg of alcohol per litre. Stream B has an alcohol concentration of 0.02 kg of alcohol per litre. The flow rate of stream A is 100 L/min and that of stream B is 200 L/min. Once the volumetric capacity of the vessel is reached, the beer flows out at the same rate it flows in. At all times, the contents of the vessel are well mixed by constant stirring. Find $C(t)$, the concentration of the beer in the vessel, in kg of alcohol per litre, for $t > 0$, and sketch the graph of this function.

9. (a) Solve the differential equation

$$y'' - 3y' + 2y = 0$$

with the boundary conditions $y(1) = e$, $y'(1) = e^2$.

(b) Solve the differential equation

$$\frac{y'' + 2y}{3y' - 2xe^x} = 1$$

with boundary conditions $y(1) = y'(1) = 0$.

10. Let $g(x)$ be a function defined and differentiable on $[a, b]$. The following result is true:

If $g'(a) < k < g'(b)$, then there exists a number c for which $a < c < b$ and $g'(c) = k$.

(a) Explain why it is inappropriate to apply the Intermediate Value Theorem to the function g' in proving this result.

(b) Establish the result by applying the Extreme Value Theorem to the function $f(x) = g(x) - kx$, or otherwise.

END