University of Toronto

Faculty of Applied Science and Engineering

Department of Electrical and Computer Engineering

FINAL EXAMINATION, APRIL 1999

Third Year - Programs 5ce, 5e

ECE 351S - PROBABILITY AND RANDOM PROCESSES

Examiner: *S.Pasupathy*

- A single aid sheet (8.5"x11", two-sided, handwritten) and a non- programmable calculator are the **only aids allowed.**
- Answer **all** six [6] questions.
- The value of each question is indicated beside each question; total marks = 55.
- Start each new question on a new page.
- If you need to make any assumptions, state them clearly.
- Answers should be clear, crisp and brief; answers without logical reasoning steps showing *all* the work will **not** be given credit.
- Lengthy reproductions of text material should be avoided. Credit is for **solving** the problems.
- In this exam, pdf=probability density function; cdf=cumulative distribution function; pmf=probability mass function.

1. (a) Consider a well-shuffled deck of cards consisting of 52 cards. of which four are queens and four are kings.

   i. Find the probability of obtaining a king in the first draw.

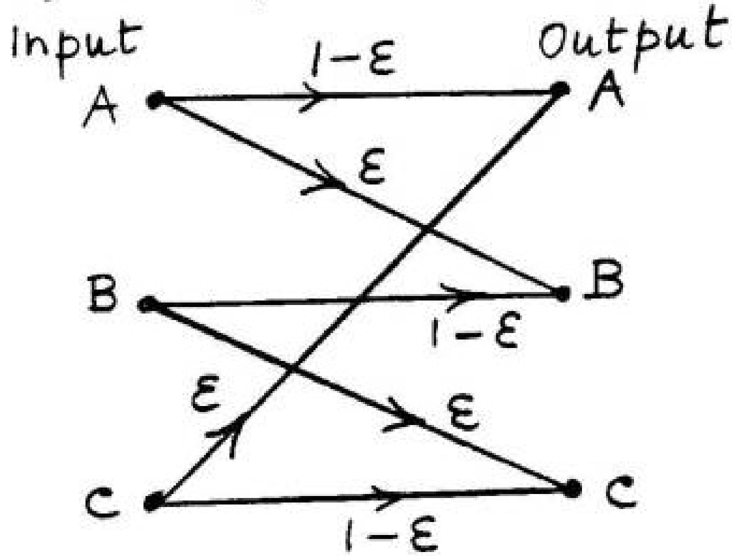
   ii. Draw a card from the deck and look at it. What is the probability of obtaining a king in the second draw? Does the answer change if you had not observed the first draw?

   iii. Suppose we draw 5 cards from the deck. What is the probability that the 5 cards include 3 kings? What is the probability that the 5 cards include 2 queens? What is the probability that the 5 cards include 3 kings and/or 2 queens ?[**Note:** There is no need to simplify the answers in this part iii) of the question]

(b) A ternary communication channel is shown below. Suppose that the input symbols A, B and C occur with probability 1/2, 1/4 and 1/4 respectively.

   i. Find the probabilities of the output symbols.

   ii. Suppose that B is observed as an output. What is the probability that the input was A? B? C?



The voltage V at the output of a microphone is a uniform random variable with limits -1 volt and 1 volt. The microphone voltage is processed by a hard limiter with cutoff points -0.5 volt and 0.5 volt. The **magnitude** of the limiter output L is a random variable such that

$$L = \begin{cases} |V| & |V| \leq 0.5 \\ 0.5 & \text{otherwise} \end{cases}$$

(a) What is the cdf $F_V(v)$? Sketch the pdf and cdf of V.

(b) What is $P[L = 0.5]$?

(c) What is the cdf of L ? the pdf of L? Sketch them both.

(d) What is $E[L]$?

3. The random variable X is uniformly distributed between -1 and 1. Let $Y = X^4$.

(a) Are the random variables X and Y orthogonal? uncorrelated? independent? Justify your answers.

(b) What is the joint pdf of the two random variables X and Y?

4. The random variables X and Y have joint pdf

$$f_{X,Y}(x,y) = c\ \sin(x+y) \quad 0 \le x \le (\pi/2), \quad 0 \le y \le (\pi/2).$$

(a) Find the value of the constant c.

(b) Find the marginal pdf's of X and Y.

(c) Find the mean and variance of X and Y.

Hint: You may need some of these formulae:

$$\sin(x \pm \frac{\pi}{2}) = \pm \cos x$$

$$\cos(x \pm \frac{\pi}{2}) = \mp \sin x$$

$$\int x \sin ax\ dx = (\sin ax - ax\ \cos ax)/a^2$$

$$\int x \cos ax\ dx = (\cos ax + ax\ \sin ax)/a^2$$

$$\int x^2\ \sin ax\ dx = (2ax\ \sin ax + 2\ \cos ax - a^2x^2\ \cos ax)/a^3$$

$$\int x^2\ \cos ax\ dx = (2ax\ \cos ax - 2\ \sin ax + a^2x^2 \sin ax)/a^3$$

5. A binary communication system transmits independent, identically distributed binary random variables , X_n, in successive time intervals. X_n assumes the value of 1 with probability p and 0 with probability $(1-p)$.

   (a) Let $S_n = \Sigma_{i=1}^{n} X_i$. What is $E[S_n]$? What is $VAR[S_n]$? What is $P[S_n = 10]$?

   (b) Let $Y_n = X_n \oplus Y_{n-1}$. Assume $Y_0 = 0$. $\oplus$ denotes modulo-2 addition. Assuming Y_n is stationary, what is $P[Y_n = 0]$? What is the significance of this result?

   (c) Let $Z_n = X_n \oplus X_{n-1}$. What is the pmf of Z_n?

6. Let W be an exponential random variable with pdf

$$f_W(w) = \begin{cases} e^{-w} & w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   with $E[W] = VAR[W] = 1$.

   (a) Find the cdf $F_{X(t)}(x)$ and the pdf $f_{X(t)}(x)$ of the time-delayed ramp process $X(t) = t - W$.

   (b) For any time $t \geq 0$, find the mean function $m_X(t)$ and the autocovariance function $C_X(t, t+\tau)$ of $X(t)$?