Surname	Given Name(s)	Student Number

UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Final Examination, December 17, 2001
Second Year — Programs 7 and 9

ECE 203F — Discrete Mathematics

Examination Type: A
Examiners: B. J. Frey and F. R. Kschischang

Instructions

- This examination paper consists of thirteen (13) pages (including this one). Please make sure that you have a complete paper.
- Write your name and student number in the space provided at the top of each page.
- Answer all nineteen questions. The value of each question is noted on each page. A total of 100 marks is available.
- Answer each question directly on the examination paper, using the back of each page if necessary. Indicate clearly where your work can be found. Put your name and student number at the top of any additional pages, and hand them in with this paper.
- This is a *closed book* (Type A) examination. Aids are not permitted. Calculators are not permitted.
- Show all steps and present all results clearly. Please be neat.
- Time: $2\frac{1}{2}$ hours. Allocate your time carefully!

Enter the first letter of

your last name here.

EXAMINER'S REPORT

p.2		/7
p.3		/10
p.4		/8
p.5		/8
p.6		/12
p.7		/9
p.8		/7
p.9		/10
p.10		/6
p.11		/8
p.12		/7
p.13		/8
Total:		/100

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1. Fermat's Last Theorem states that the equation $x^n + y^n = z^n$ has no solutions in positive integers x, y, z, whenever n is an integer greater than 2.

3 marks (a) By defining the appropriate propositional functions (be sure to specify the domain of discourse), express Fermat's Last Theorem as a quantified statement that does not use the negation symbol.

2 marks (b) Repeat for the negation of Fermat's Last Theorem, using the same propositional functions as in part (a), ensuring that no quantifier is under a negation symbol.

1 mark (c) Note that $3^2 + 4^2 = 5^2$. Does this prove that Fermat's Last Theorem is false? (Justify your answer.)

1 mark (d) Note that $3^3 + 4^3 \neq 5^3$. Does this prove that Fermat's Last Theorem is true? (Justify your answer.)

7 marks

5 marks

2. Show by induction that for any integer $n \geq 0$,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

5 marks

3. Show by induction that any postage of eight cents or more can be achieved using only 3-cent and 5-cent stamps.

10 marks

4. The relation R is defined on the integers as follows:

$$(a, b) \in R \text{ if } a - b \text{ is divisible by } 5.$$

3 marks (a) Prove that R is an equivalence relation.

2 marks (b) Specify the different equivalence classes.

3 marks 5. If X is an n-element set and Y is an m-element set, how many functions are there from X to Y?

8 marks

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6. Let $\mathbb{R}^+$ denote the set of *positive* real numbers, and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as follows:

$$f(x) = x^2, \qquad g(x) = \frac{1}{x}.$$

1 mark (a) Is $f(x)$ injective (i.e., a *one-to-one* function)?

1 mark (b) Is $g(x)$ surjective (i.e., an *onto* function)?

1 mark (c) Which of $f(x)$ and $g(x)$ are bijective?

2 marks (d) Find the compositions $f \circ f$, $f \circ g$, $g \circ f$ and $g \circ g$.

3 marks 7. How many different strings of length 7 can be formed by permuting the letters of "TORONTO"?

8 marks

8. Consider the set of binary strings of length nine.

1 mark (a) How many nine-bit strings are there?

2 marks (b) How many nine-bit strings start 101 or end with 101?

2 marks (c) How many nine-bit strings have 4 ones?

2 marks (d) How many nine-bit strings are palindromes, i.e., read the same from either end? (An example of such a nine-bit string is 110010011.)

5 marks 9. In how many ways can six distinct Electrical Engineering students and 4 distinct Computer Engineering students wait in line if no two Computer Engineering students stand together?

12 marks

10. A muffin shop sells ten varieties of muffins by the half-dozen (i.e., six at a time). Assume that at least six muffins of each variety are available.

3 marks (a) How many different half-dozen combinations are possible?

3 marks (b) How many different half-dozen combinations are possible if at least one muffin must be blueberry?

3 marks **11.** Prove that

$$\sum_{k=0}^{n}(-2)^k\binom{n}{k} = (-1)^n.$$

9 marks

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12. A particular family has four children.

1 mark (**a**) What is the probability of all boys?

2 marks (**b**) What is the probability of exactly two girls?

2 marks (**c**) What is the probability of all boys, given that there is at least one boy?

2 marks (**d**) Which event has higher probability: two children of one sex and two of the other, or three of one sex and one of the other?

7 marks

5 marks

13. Prove that in any connected simple graph G of more than one vertex, at least two vertices have the same degree. [*Hint:* apply the Pigeonhole Principle.]

5 marks

14. A group of 35 students (20 wearing hats and the rest hatless) are lined up for movie tickets. If x is standing behind y, then x is said to be "1-back" from y. In general, if x is standing behind a person that is "i-back" from y, then x is "$i+1$-back" from y. Prove that 4-back from at least one hat-wearing student in line is another hat-wearing student.

10 marks

15. Twelve people are required to serve on a jury. The birth months of the jury members can be recorded as an ordered 12-tuple $(b_1, b_2, \ldots, b_{12})$, where b_i denotes the birth month of the ith juror.

1 mark (a) Show that the number of distinct ordered 12-tuples is 12^{12}.

(b) Assuming that each of the ordered 12-tuples has the same probability of occurrence, find the probability that:

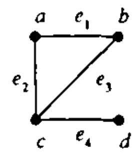
1 mark i. the jurors have birthdays all in the same month;

2 marks ii. the jurors have birthdays all in different months;

1 mark iii. at least one juror has a birthday in December;

1 mark iv. no jurors have a birthday in December.

6 marks



16. Consider the graph G shown above.

1 mark (a) Write an adjacency matrix for G.

1 mark (b) Write an incidence matrix for G.

2 marks (c) How many path of length 3 are there from vertex a to vertex c in G?

4 marks **17.** Let A be the adjacency matrix for a simple graph. Prove or disprove: the $(i, i)th$ element of A^2 is the degree of vertex v_i.

8 marks

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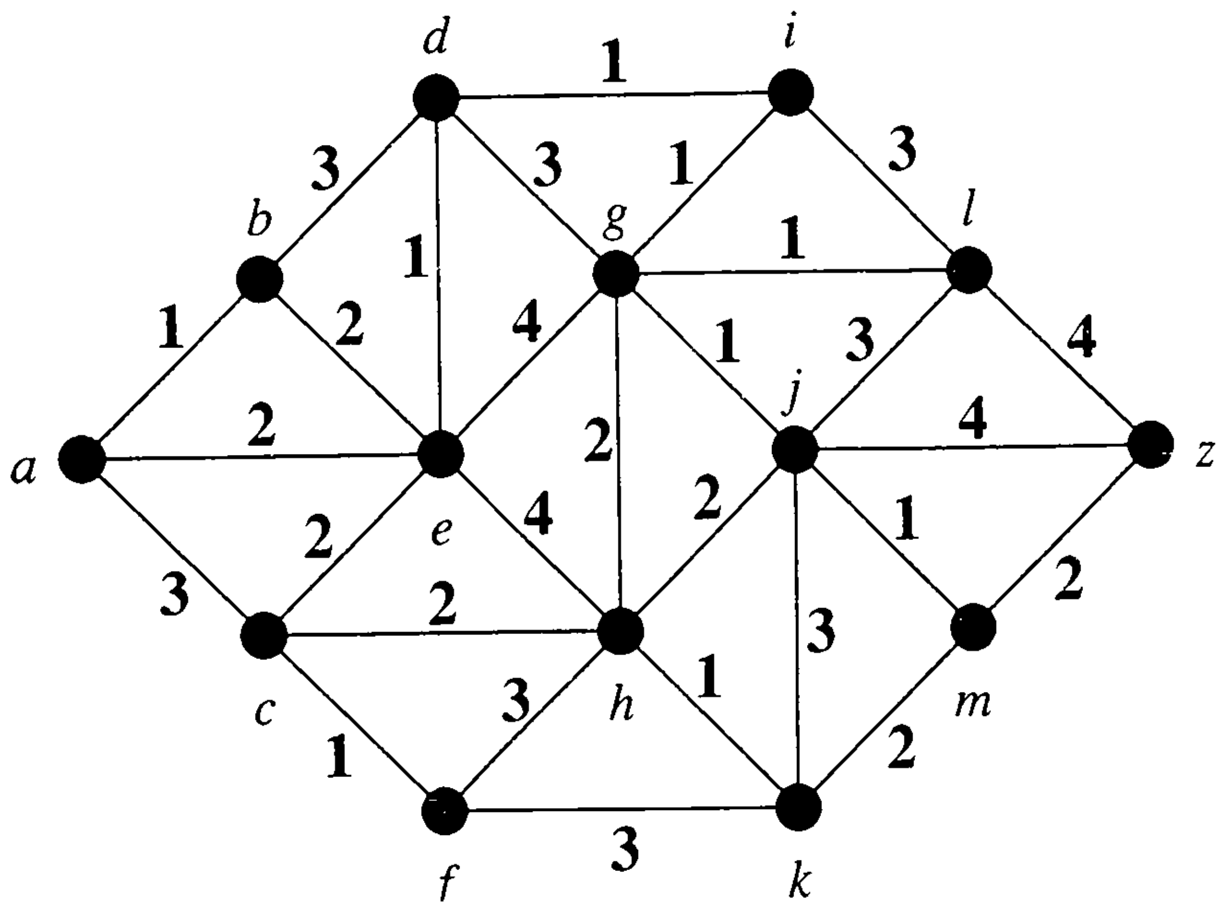
18. Consider the complete bipartite graph $K_{m,n}$.

1 mark (a) For which values of m and n does $K_{m,n}$ have an Euler circuit?

3 marks (b) For which values of m and n does $K_{m,n}$ have a Hamiltonian circuit?

3 marks (c) For which values of m and n is $K_{m,n}$ planar?

7 marks



8 marks 19. Consider the weighted graph G shown above. Use Dijkstra's algorithm to find the minimum-weight path from vertex a to vertex z in G. What is the weight of this path?

8 marks