University of Toronto

FINAL EXAMINATION, Dec. 21, 2001

APM 288 H1 F
Ordinary Differential Equations

Examiners: Professors B. Khesin and B.Wagneur
Duration: $2\frac{1}{2}$ hours
Total: 51 marks

1. *[3+3+4 marks]* a) Solve the differential equation

$$5x^4(1+\ln y)dx = \left(2y - \frac{x^5}{y}\right) dy.$$

Find the solution with $y(-1) = 1$.

b) Solve the initial value problem

$$\frac{dy}{dx} = 7y^{6/7}, \ y(4) = 0.$$

c) Solve the differential equation

$$y' = \frac{2x+y+1}{x+2y-4}.$$

Hint: Substitute $x = X - X_0$, $y = Y - Y_0$ for suitable constant X_0 and Y_0.

2. *[5 marks]* Suppose water is added to a tank at 12 gal/min, but leaks out at the rate of 1/4 gal/min for each gallon in the tank. What is the smallest capacity the tank can have if the process is to continue indefinitely?

3. *[6 marks]* Give the general solution of the differential equation

$$x^2y'' - xy' + y = 3x.$$

4. *[6 marks]* Find the general solution of the differential equation

$$y^{(4)} - 4y''' + 8y' - 32y = 0,$$

if one of its solutions is $y_1 = 3e^{4x}$. Sketch the roots of the characteristic equation in the complex plane.

5. *[5 marks]* Solve the differential equation

$$(2e^y - x)y' = 1.$$

6. *[6 marks]* Solve the system

$$x' = 5x - 3y$$

$$y' = 6x - 4y$$

$$z' = 3x - 3y + 2z$$

7. *[7 marks]* For the system of differential equations

$$x' = (y - 4)(x + y)$$

$$y' = 2x - y$$

a) describe the locations of all critical points, b) classify their types (i.e., specify whether they are nodes, saddles, etc) and stability, and c) try to sketch the phase portrait of the system.

8. *[6 marks]* Let $A(t)$ be a matrix-valued function, such that $A(t)$ is a nondegenerate $n \times n$ matrix for every t. Show that the determinant $Y(t) = \det A(t)$ is a function satisfying the following differential equation:

$$Y' = tr(A'(t)A^{-1}(t))Y,$$

where $tr(B)$ is the trace (i.e., the sum of the diagonal entries) of a matrix B, $'$ denotes the derivative d/dt, and A^{-1} is the inverse to A.