UNIVERSITY OF TORONTO
MAT290F - ADVANCED ENGINEERING MATHEMATICS
FINAL EXAM, December 21, 2001

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

Instructions
NO AIDS, NO CALCULATOR
Write your solutions clearly in the spaces provided below the problem statements.
Use the back sides of the pages for rough work.

Problem	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
Total	/80

1. (a) State the conditions for a function $f(t)$ to have a Laplace transform $F(s)$.

   (b) Using the residue theorem, find $f(t)$ if

$$F(s) = \frac{1}{s^2(s-2)}.$$

2. (a) Suppose $f(t)$ is a periodic function, of period T. Derive the following expression for its Laplace transform:

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

(b) Find $F(s)$ for the periodic function with $T = 10$ and

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 10. \end{cases}$$

3. (a) State in logic format the definition that z is a boundary point of a set S.

   (b) Using this definition, prove that $z = 1 + 2i$ is a boundary point of the set of all complex numbers $x + iy$, where x ranges over all real numbers such that $-1 < x < 1$, and y ranges over all positive integers.

   (c) Consider the function

$$f(z) = \begin{cases} 1 & \text{if } z \text{ is on the unit circle } |z| = 1 \\ z & \text{if } z \text{ is not on the unit circle.} \end{cases}$$

   At what points z is $f(z)$ continuous?

4. (a) Find all roots of the equation $z^4 = -2i$. Show their locations in the complex plane.

   (b) Determine all values of $(2 - 2i)^{2/3}$. Show their locations.

5. For each of the following functions, determine all its singularities, classify their types, and calculate the residues at the singularities where they're defined:

   (a) $f(z) = z\mathrm{Re}(z)$

   (b) $f(z) = z^2 \sin \dfrac{1}{z}$

6. (a) State Cauchy's integral formula for higher derivatives and state precisely the conditions under which it holds.

   (b) Evaluate the integral $\int_{\Gamma} \frac{z \sin(3z)}{(z+4)^3}\, dz$, where Γ is the positively-oriented circle $|z - 2i| = 9$.

   (c) Does $f(z) = \bar{z}$ have an antiderivative? Justify your answer.

7. Let $f(z) = \dfrac{1}{(z-2i)(z+4)}$.

(a) Write the Laurent series of $f(z)$ about the point $z = 0$, convergent at $z = 1$.

(b) Write the Laurent series of $f(z)$ about the point $z = 2i$, convergent at $z = 5$.

8. Evaluate the integrals:

(a) $\displaystyle\int_\Gamma \frac{z^3}{z^2-2z+5}\,dz$, Γ is the positively-oriented circle $|z|=1$.

(b) $\displaystyle\int_\Gamma e^{2z}\,dz$, Γ is the counterclockwise semicircle of radius 1 from $z=0$ to $z=2i$.

(c) $\displaystyle\int_\Gamma \frac{(z-i)^2}{\sin^2 z}\,dz$, Γ is the positively-oriented circle $|z|=1$.