UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, MONDAY DECEMBER 17, 2001

Third Year – Engineering Science (Biomedical, Computer, Electrical, Nanoengineering)
Fourth Year – Engineering Science (Manufacturing)

ECE355F - Signal Analysis and Communications

Exam Type: A

Examiner: D. Kundur

- No aids allowed (including calculators).
- Three important tables from the text book have been provided on the last pages of the exam.
- If a particular question seems unclear, please explicitly state any reasonable assumptions and proceed with the problem.
- "Sketch" questions require that all axes and significant points on the graph be labeled.
- Please answer ALL questions.

1. **(10 marks)**

   Short answers are required for the following conceptual questions:

   (a) Is the system $y(t) = 3x(t) + 4$ linear, where $x(t)$ is the input and $y(t)$ is the output? Yes or No? Explain why or why not briefly.

   (b) Is the system $y(t) = x(\sin(t))$ causal, where $x(t)$ is the input and $y(t)$ is the output? Yes or No? Explain why or why not briefly.

   (c) Briefly state what it means when we say that the impulse response of a linear time invariant (LTI) system *completely characterizes* the system.

   (d) Explain briefly what Gibb's phenomenon is in relation to the Fourier Series representation of signals.

   (e) Provide one advantage of using the Fourier Transform over the Fourier Series in general signal processing analysis.

   (f) Describe intuitively the information provided by the phase and magnitude of the Fourier Transform of a signal. *Hint: recall the lecture with example pictures of Ralph Fiennes and Meg Ryan.*

   (g) What is one possible motivation to sample an analog signal?

   (h) True or False? Given two arbitrary systems S_1 and S_2, S_1 is the inverse of S_2 if and only if S_2 is the inverse of S_1.

   (i) True or False? In single-sideband modulation, a bandwidth equal to that of the information bearing signal is used for transmission, however the signal we transmit is complex (i.e., has non-zero real and imaginary parts).

   (j) True or False? All linear constant coefficient differential and difference equations with conditions of initial rest represent LTI systems.

2. **(10 marks)**

   Determine if each of the following systems is invertible. If it is, construct the inverse system. If it is not, find two input signals to the system that have the same output.

   Please note: $x(t)$ denotes the input and $y(t)$ the output of a continuous-time system, and $x[n]$ denotes the input and $y[n]$ the output of a discrete-time system.

   (a) $y(t) = 3x(t) + 4$

   (b) $y(t) = x(3t + 4)$

   (c) $y[n] = x[3n + 4]$

   (d) $y(t) = tx(t)$

   (e) $y[n] = x[n - 1]x[n + 1]$

3. **(10 marks)**

Consider a continuous-time LTI system denoted S with impulse response $h(t) = e^{2t}u(1-t)$.

(a) Compute the output $y(t)$ of S for the input $x(t) = u(t) - 2u(t-2) + u(t-5)$ where $u(t)$ is the unit step function.

(b) Is the system:

i. linear?

ii. time-invariant?

iii. causal?

iv. memoryless?

v. bounded input bounded output (BIBO) stable?

(c) Is there a linear constant coefficient differential equation (with auxiliary conditions of initial rest) that describes S? Explain why or why not. Note: you don't have to find the differential equation, just state whether or not it exists.

4. **(10 marks)**

Let

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{cases}$$

be a periodic signal with fundamental period $T = 2$ and Fourier coefficients a_k.

(a) Let b_k be the Fourier Series coefficients of $\frac{dx(t)}{dt}$. Determine b_k for all k.

(b) Determine a_k for all k. Tables 3.1, 4.1 and 4.2 are provided at the end of this exam.

(c) If $x(t)$ is passed through the filter shown in Figure 1. What is the corresponding output $y(t)$?

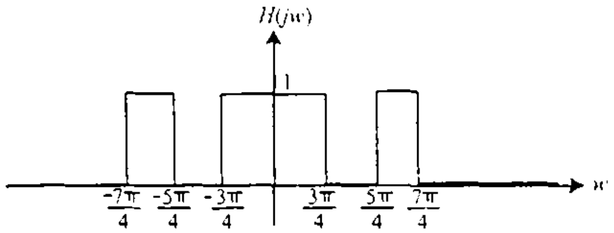

Figure 1: Frequency selective filter. Then input to the filter is $x(t)$ and the output $y(t)$.

5. **(10 marks)**

For your fourth year Engineering Science thesis, you have designed and implemented a set of amazing A/D and D/A converter chips; they are so amazing that the A/D chip performs ideal sampling (as modeled in class by multiplication of the analog signal with an ideal impulse train) and the D/A chip performs ideal bandlimited interpolation with an ideal lowpass filter. They work for signals with a Nyquist frequency of at most $2\omega_0$.

Before you show your supervisor, Prof. No-Nonsense, your brilliant chips, you check the initial specifications one last time to make sure everything is okay. You see that the specifications require that the design work for signals with a Nyquist frequency of less than or equal to $8\omega_0$!!! To pass your thesis, Prof. No-Nonsense wants you to test it out on a continuous-time signal $x(t)$ with frequency response shown in Figure 2(a).

You realize that you're in luck as the test signal is a narrow bandwidth bandpass signal. You learned some tools in your third year Signal Analysis and Communications course to *modify* the test signal to make it work with your chips. You consider adding compensation stages A and B (taken from parts of your second year design project) shown in Figure 2(b), so that the input into A is $x(t)$ and it produces $w(t)$ which can work with your A/D chip. Stage B takes $w(t)$ and produces the signal $x(t)$.

From you second year design project, you have continuous-time multipliers, synchronized carriers at frequencies integer multiples of $2\omega_0$ (specifically you have $\cos(2m\omega_0 t)$ for $m \in \mathbb{Z}$), and ideal lowpass and bandpass filters with variable cutoff frequencies (i.e., you can arbitrarily set the cutoff frequency of each filter).

(a) What is the Nyquist frequency of the test signal $x(t)$?

(b) Design your compensation stage A. Provide a specific block diagram and draw the spectra of $w(t)$. Keep your design as simple as possible.

(c) Design your compensation stage B. Provide a specific block diagram. Keep your design as simple as possible.

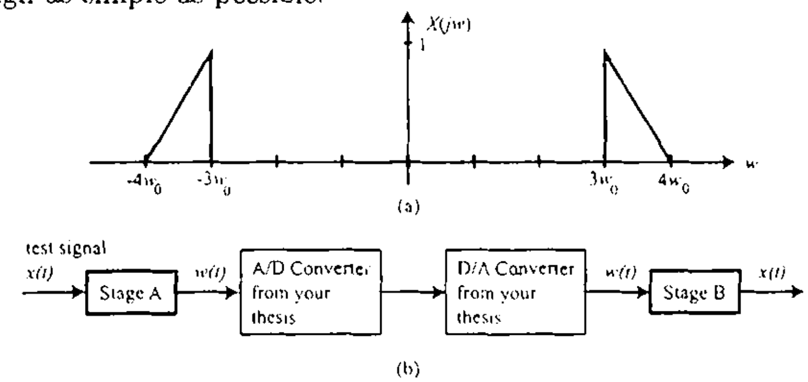

Figure 2: (a) Test signal used to prove your thesis works, (b) Compensation strategy to make your chip work for the test signal $x(t)$.

6. **(10 marks)**

Hint before your start:

To answer this question you will need to make use of the following function $u_1(t)$ defined as the derivative of the unit impulse function $\delta(t)$ with respect to time. You will not be asked to do anything too sophisticated with $u_1(t)$. Just note the following useful properties:

- $u_1(t) = \frac{d}{dt}\delta(t)$
- $\delta(t) = \int_{-\infty}^{t} u_1(\tau)d\tau$
- $x(t) * u_1(t) = u_1(t) * x(t) = \frac{dx(t)}{dt}$

The problem statement:

Consider a device D for measuring the temperature of a liquid. It is often reasonable to model such a device as an LTI system that does not respond instantaneously to temperature changes[1].

Assume that the response of the temperature measuring device to a **unit step** in temperature is

$$s(t) = (1 - e^{-t/2})u(t). \tag{1}$$

(a) Design an LTI compensatory system S_c that, when provided with the output of the measuring device D, produces an output equal to the instantaneous temperature of the liquid. We call this an *inverse* system because it "undoes" the non-ideal effects of the measuring device. Find the impulse response of S_c denoted $h_c(t)$.

(b) Suppose that there are slight inaccuracies in the temperature measurements due to small, erratic noise-like phenomena within the measuring device. We model the overall output of this more realistic system as the sum of the response to the measurement device characterized by Equation (1) and an interfering "noise" signal $n(t)$. Figure 3 depicts this where this model is contained in the dashed rectangle. Our compensatory system S_c of part (a) is applied to the output signal of the noisy measuring device.

   i. Suppose $n(t) = \sin(\omega_0 t)$. What is the contribution of $n(t)$ to the output of the inverse system S_c? That is, you need to find the component of $z(t)$ in Figure 3 which is a result of $n(t)$.

   ii. How does this output change as ω_0 is increased?

(c) Describe any performance trade-offs observed in this problem.

[1] This model is based on the response characteristics of the measuring element (e.g., mercury in a thermometer).

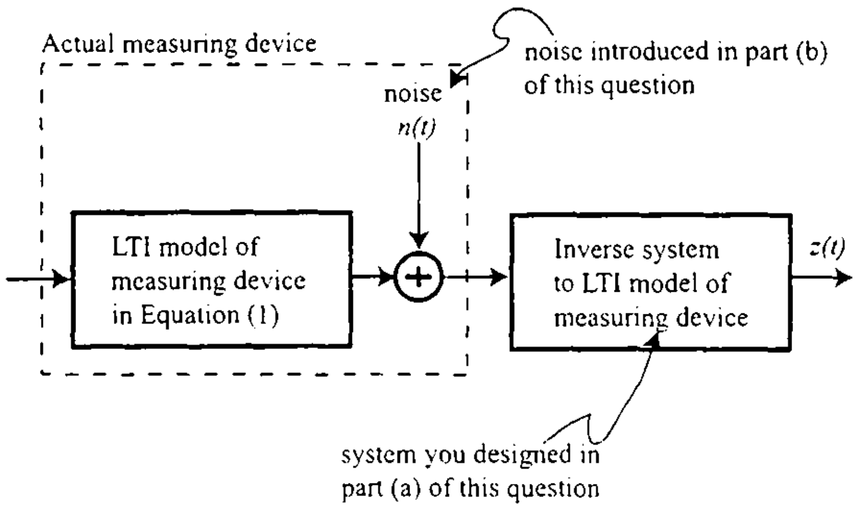

Figure 3: System addressed in Question 6(b). The inverse system derived in Question 6(a) is applied to the combined signal from the output of the measure device modeled in Equation (1) plus the noise introduced in part (b).

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	$\begin{matrix} a_k \\ b_k \end{matrix}$
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$x(t)e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	Ta_kb_k
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\begin{matrix} \mathcal{Re}\{a_k\} \\ j\mathcal{Im}\{a_k\} \end{matrix}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals		

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—