University of Toronto
Department of Electrical and Computer Engineering
ECE203S - Discrete Mathematics
Final Examination - April 17, 2001
Second Year - Programs 7 and 9

Examination Type: Closed Book, Calculators are not permitted.
Examiners: B.L. Bardakjian, G. Baumgartner, R.S. Minhas & E. Sousa

Number of Pages = 22
Number of Questions = 11
ANSWER ALL Please be neat, and use ink.

SURNAME:

FIRST NAME:

STUDENT NUMBER:

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*EXAMINER'S REPORT*

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## Question #1

Are the following statements *True* or *False*?

1. $\sum_{i=0}^{2}\sum_{j=-2}^{2}(ij)^2 = 50.$

2. If $f : Z \rightarrow Z$ is defined by $f(x) = \left\lfloor \frac{3x}{2} \right\rfloor$, then $f$ is one-to-one.

3. If $f : A \rightarrow B$ and $|A| > |B|$ then $f$ is onto.

4. If $f$ and $g$ are functions that only produce positive values and $f(x)$ is $O(g(x))$, then $f(x)$ is not $\Omega(g(x) + x^2)$.

5. A tree is an undirected graph with no simple circuits.

6. Every graph containing a Hamilton path contains a Hamilton circuit.

7. If all the vertices of a simple connected graph have even degree then the graph contains an Euler circuit.

8. Every 2-colorable graph is bipartite.

9. $C_4$ and $W_4$ are homeomorphic.

10. Every simple connected planar graph has a unique adjacency matrix representation.

**[10 Marks]**

**Question #2**

Consider the following predicates where *variable* $x$ represents a student and *variable* $y$ represents a course:

U(y): y is an upper-level course.
M(y): y is a math course.
F(x): x is a freshman.
A(x): x is a part-time student.
B(x): x is a full-time student.
T(x,y): student x is taking course y.

- Write each of the following statements using the predicates and any needed quantifiers:
  - (a) Every freshman is a full-time student.
  - (b) No math course is upper-level.
  - (c) Every student is taking at least one course.
  - (d) There is a part-time student who is not taking any math course.
  - (e) Every part-time freshman is taking some upper-level course.
- Write each of the following in good English without using variables in your answers:
  - (f) $\sim \exists y\, T(Joe, y)$.
  - (g) $\exists x\, A(x) \wedge \sim F(x)$.
  - (h) $\exists x \forall y\, T(x, y)$.
  - (i) $\forall x \exists y\, (B(x) \wedge F(x)) \rightarrow (M(y) \wedge T(x, y))$.

[9 Marks]

## Question #3

Suppose $f : A \to B$ is a function, and $S$ and $T$ are subsets of $A$.

(a) Prove or disprove that
$$f\left(S \cap T\right) \subseteq f\left(S\right) \cap f\left(T\right).$$

(b) Prove or disprove that
$$f\left(S \cap T\right) = f\left(S\right) \cap f\left(T\right).$$

**[9 Marks]**

## Question #4

Prove the following **from the definition of big-O**:

(a) $2x^4 + 5x^3 - 4x^2 + 3$ is $O(x^4)$. **[3 Marks]**

(b) $x^4$ is $O(2x^4 + 5x^3 - 4x^2 + 3)$. **[3 Marks]**

(c) $\sum_{j=1}^{n} j^3$ is $O(n^4)$. **[3 Marks]**

## Question #5

Is it possible to construct a connected simple planar graph of 7 vertices with the following properties? If so then draw the graph.

(a) All the vertices have degree 3. **[3 Marks]**

(b) The degrees of the vertices form a strictly increasing sequence. **[3 Marks]**

(c) The number of regions is 16. **[3 Marks]**

## Question #6

(a) A student prepares for an exam by studying a list of 10 problems. He can solve six of them. For the exam, the instructor selects five questions at random from that list of ten. What is the probability that the student can solve:
i) all five problems on the exam,
ii) at least four problems on the exam. **[5 Marks]**

(b) A six-sided die is rolled 20 times. What is the probability of getting two 1s, three 2s, two 3s, three 4s and ten 6s if:
i) the die is fair,
ii) the various probabilities for a single roll of die are $p(1) = 0.1$, $p(2) = 0.25$. $p(3) = 0.25$, $p(4) = 0.1$, $p(5) = 0.1$. and $p(6) = 0.2$. **[4 Marks]**

## Question #7

Consider the complete graph with 6 vertices, $K_6$.

(a) What is the smallest number of edges that have to be removed from this graph in order to obtain a planar subgraph? Draw the corresponding subgraph of $K_6$. **[3 Marks]**

(b) What is the smallest number of edges that have to be removed in order to obtain a subgraph that is a tree? **[1 Marks]**

(c) How many different such tree subgraphs are there (i.e. trees obtained by removing the smallest possible number of edges)? **[1 Marks]**

(d) Does $K_6$ contain an Euler circuit? If not, what is the smallest number of edges that have to be removed in order to obtain a sub-graph that has an Euler circuit? **[2 Marks]**

(e) What is the maximum number of edges that can be removed where (depending on the set of removed edges) the resulting sub-graph contains a Hamilton circuit? **[2 Marks]**

## Question #8

A cut edge in a simple connected graph is an edge such that when removed the resulting subgraph is disconnected. Show that an edge $e$ in such a graph is a cut edge if and only if $e$ is not part of any simple circuit in the graph. [**9 Marks**]

## Question #9

Consider two distinct vertices $v_1$ and $v_2$ in the graph $K_5$. Find the number of paths that begin in $v_1$ and end in $v_2$ and have length less or equal to 4? [**9 Marks**]

## Question #10

(a) Using a combinatoric argument, find the number of distinct words that can be formed using the letters of TORONTO. [**2 Marks**]

(b) What is the coefficient of $x^{45}$ in the expansion of $(x - 1/x)^{125}$? [**2 Marks**]

(c) A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen? (Hint: let the chosen books be represented by $|$ and the rest of the books by $*$.) [**5 Marks**]

## Question #11

Consider the given weighted graph.

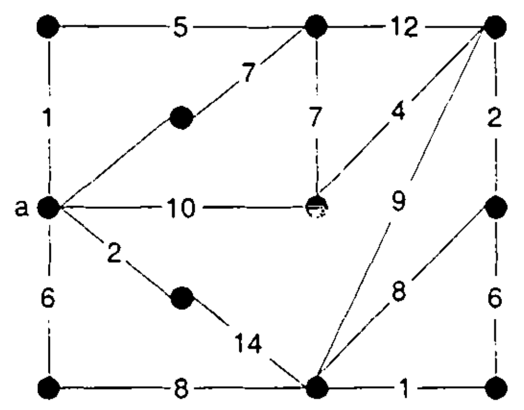


Figure 1: Undirected Weighted Graph

(a) Use Prim's algorithm to list (in the order they are added to the tree) the edges of a minimum spanning tree. What is the weight of this spanning tree? **[3 Marks]**

(b) Use Dijkstra's algorithm to compute the shortest path from vertex $a$ to all other vertices. **[6 Marks]**