UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
FINAL EXAMINATION - DECEMBER 2001
ECE557F - SYSTEMS CONTROL
IV-AEESCBASCA, AEESCBASCB, AEESCBASCC, AEESCBASCE
EXAMINER - R.H. Kwong

Students may use one 8.5"×11" aid sheet in preparing their answers. The aid sheet can be either the type supplied by the Faculty Office, or it can be any other sheet with the above specified dimensions.

(10 points)

1 (a). Two pendulums having the same mass m with lengths l_1 and l_2, respectively, are mounted on a cart with mass M. An external force u is applied to the cart. Denote the angular displacements from the vertical for pendulums 1 and 2 by θ_1 and θ_2, respectively. It can be shown that the equations governing θ_1 and θ_2 are given by

$$\begin{aligned} Ml_1\ddot{\theta}_1 &= Mg\theta_1 + mg\theta_1 + mg\theta_2 - u \\ Ml_2\ddot{\theta}_2 &= Mg\theta_2 + mg\theta_1 + mg\theta_2 - u \end{aligned}$$

Let $x_1 = \theta_1$, $x_2 = \theta_2$, $x_3 = \dot{\theta}_1$, and $x_4 = \dot{\theta}_2$ be the state variables for the system.

(i) Write down the state equation describing the 2-pendulum system in the form

$$\dot{x} = Ax + Bu$$

and identify A and B.

(ii) Determine the simplest necessary and sufficient conditions for the 2-pendulum system to be controllable.

(10 points)

1 (b). The equations of a hot air balloon is given by

$$\begin{bmatrix} \dot{T} \\ \dot{v} \\ \dot{h} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & 0 & 0 \\ \sigma & -\beta & 0 & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ v \\ h \\ w \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

where T is the temperature, v the vertical velocity, h the altitude, w the wind disturbance, and u the input heat, with $\alpha > 0$, $\beta > 0$. Suppose we measure only the altitude h, i.e.,

$$y = [0\ 0\ 1\ 0] \begin{bmatrix} T \\ v \\ h \\ w \end{bmatrix}$$

Determine the simplest necessary and sufficient conditions for this system to be observable.

(20 points)

2. Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} x$$

Design a minimal order observer to estimate the state x, with the observer pole placed at -2. Your observer equation should be of the form

$$\dot{z} = Fz + Gu + Hy$$
$$\hat{x} = M \begin{bmatrix} y \\ z \end{bmatrix}$$

Identify F, G, H, and M.

(20 points)

3. Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} \alpha & 1 \end{bmatrix} x$$

where α is a constant. We would like to design a reference feedforward and state feedback control law of the form

$$u = N y_d - Kx$$

so that the closed loop system has poles located at -1 and -3, and that the output $y(t) \to y_d$ as $t \to \infty$, where y_d is a constant reference set point.

(i) Determine the gains K and N so that the above design specifications are satisfied, and determine the closed loop transfer function from y_d to y.

(ii) What happens to the control system if we let $\alpha \to 0$? Given an explanation of the resulting behaviour.

(20 points)

4. You may know from classical control theory that integral control eliminates constant disturbances as well as providing set point tracking. We investigate this property in the state space context in this problem.

Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where w is a constant unknown disturbance. The control objective is to design a controller so that the closed loop system is stable and that $y(t) \to y_d$ as $t \to \infty$ for all constant reference set points y_d, regardless of the value of w. We design the control system as follows.

(i) Augment the system dynamics with the integrator

$$\dot{\xi} = y - y_d$$

Determine the feedback gains K and K_I such that with

$$u = -Kx - K_I\xi$$

the closed loop system poles are located at $-1, -1, -2$.

(ii) Determine the transfer functions from y_d to y, and from w to y. Using these transfer functions, show explicitly that indeed $y(t) \to y_d$ as $t \to \infty$, and that the effects of w on y are asymptotically eliminated (i.e., w gives zero asymptotic contribution to y).

(Hint: You may use the following result. If

$$\begin{bmatrix} \beta_1(s) & \beta_2(s) & \beta_3(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ a_1 & s + a_2 & a_3 \\ -1 & 0 & s \end{bmatrix}^{-1}$$

then

$$\begin{aligned} \beta_1(s) &= \frac{s(s + a_2)}{s^3 + a_2 s^2 + a_1 s + a_3} \\ \beta_2(s) &= \frac{s}{s^3 + a_2 s^2 + a_1 s + a_3} \\ \beta_3(s) &= \frac{-a_3}{s^3 + a_2 s^2 + a_1 s + a_3}) \end{aligned}$$

(20 points)

5. Consider the linear system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

The optimal control cost criterion is given by

$$J = \int_0^\infty [y^2(t) + ru^2(t)]dt$$

where $r > 0$. Show that there exists a unique solution $P \geq 0$ to the associated algebraic Riccati equation and determine P. Hence find the optimal feedback law $u = -Kx$ which minimizes the quadratic cost criterion and stabilizes the closed loop system. Determine also the characteric polynomial of the closed loop system. Your answers should be expressed in terms of r.