UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, APRIL 2001

Third Year- Program: Engineering Science

ECE 351S - PROBABILITY AND RANDOM PROCESSES

Exam Type: D

Examiner: *S.Pasupathy*

- A single aid sheet (8.5"x11", two-sided, handwritten) and a non- programmable calculator are the **only aids allowed.**
- Answer **all** nine [9] questions.
- The value of each question is indicated beside each question; total marks = 65.
- Start each new question on a new page.
- If you need to make any assumptions, state them clearly.
- Answers should be clear, crisp and brief; answers without logical reasoning steps showing *all* the work will **not** be given credit.
- Lengthy reproductions of text material should be avoided. Credit is for **solving** the problems.
- In this exam, pdf=probability density function; cdf=cumulative distribution function; pmf=probability mass function; iid = independent, identically distributed.

3 marks

1. If A, B and C are three events, explain carefully why the following assignment of probabilities is not possible:

   (a) $P(A) = .65$, $P(A \cap B^C) = .60$, $P(A \cap B) = .10$

   (b) $P(A) = .5, P(B) = .5, P(C) = .5, P(A \cap B) = 0,$
   and $P(A \cap B \cap C) = .2$.

4 marks

2. Twenty people at a party compare birthdays. What is the probability that ten of the people have their birthday in January?

8 marks

3. Choose two points X and Y at random and independently on a line AB with length l. Find the probability that the distance from Y to X is at least twice from A to X.

6 marks

4. X is a uniform random variable over (0,1). Find a function $g(x)$ such that the pdf of $Y = g(X)$ is

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

10 marks

5. Random X and Y have the joint pmf

$$p_{X,Y} = \begin{cases} c|x+y| & x = -2, 0, 2; y = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of the constant c?

   (b) What is $P[Y < X]$?

   (c) What is $P[Y > X]$?

   (d) What is $P[Y = X]$?

   (e) What is $P[X < 1]$?

12 marks

6. At 12 noon on a weekday, we begin recording new call attempts at a telephone switch. Let X denote the arrival time of the first call, as measured by the number of seconds after noon. Let Y denote the arrival time of the second call. In the most common model used in the telephone industry, X and Y are continuous random variables with joint pdf

$$f_{X,Y} = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

where λ is a positive constant.

Find the marginal pdf's $f_X(x)$, $E[X]$, and $f_Y(y)$ and the conditional pdf's $f_{X/Y}(x/y)$ and $f_{Y/X}(y/x)$. Sketch the pdf's.

6 marks

7. Let X be an exponential random variable with an expected value of unity(1). What is the variance of X? Let $M_n(X)$ denote the sample mean of n independent samples of X. How many samples n are needed to guarantee that the variance of $M_n(X)$ is no more than 0.01?

8 marks

8. The input to a digital filter is an iid random sequence $..., X_{-1}, X_0, X_1, ...$ with E[X_i]=0 and VAR[X_i]=1. The output is also a random sequence $..., Y_{-1}, Y_0, Y_1, ...$ The relationship between the input sequence and output sequence is expressed in the formula

$$Y_n = X_n + X_{n-1} \quad \text{for all integers } n$$

Find the expected value function $E[Y_n]$, and the autocovariance function $C_Y[m, k]$ of the output.

8 marks

9. $X(t)$ is a wide sense stationary random process with average power equal to 1. Let θ denote a random variable with uniform distribution over $[0, 2\pi]$ such that $X(t)$ and θ are independent.

   (a) What is $E[X^2(t)]$?

   (b) What is $E[\cos(2\pi f_c t + \theta)]$?

   (c) Let $Y(t) = X(t)\cos(2\pi f_c t + \theta)$. What is $E[Y(t)]$? How does $E[Y(t)]$ change with $X(t)$?

   (d) What is the average power of $Y(t)$?

Dept. of Electrical and Computer Engineering

ECE354S

Spring , 2001

Final

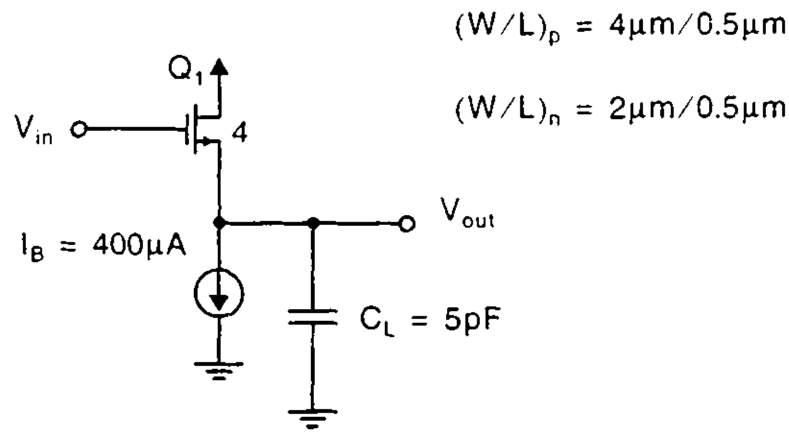
- For MOS transistors in the active region, assume $L = 0.5\mu m$, $V_{tn} = 0.7V$, $V_{tp} = -0.7V$, $\mu_n C_{ox} = 200\mu A/V^2$, $\mu_p C_{ox} = 50\mu A/V^2$, $g_s = 0.15g_m$, $k_{rn} = 8\times10^6 V/m$, and $k_{rp} = 5\times10^6 V/m$, $C_j = 1.1\times10^{-3}\ pF/(\mu m)^2$, $C_{jSW} = 2.0\times10^{-4}\ pF/\mu m$, and $C_{ox} = 5\times10^{-3}\ pF/(\mu m)^2$.
- For npn bipolar transistors, assume $\beta = 100$, $V_A = 80\ V$, $I_S = 1\times10^{-15}$, $\tau_b = 6\ ps$, $\tau_s = 1.5\ ns$, $r_b = 220\ \Omega$, and $(kT)/q = 0.028V$.
- For pnp bipolar transistors, assume $\beta = 20$, $V_A = 50\ V$, $I_S = 1\times10^{-14}$, $\tau_b = 60\ ps$, $\tau_s = 5\ ns$, $r_b = 550\ \Omega$, and $(kT)/q = 0.028V$.

- Answer all questions in the space provided on the test pages.
- Underline your answers.
- **State any assumptions you use in answering questions.**
- All questions are equally weighted.

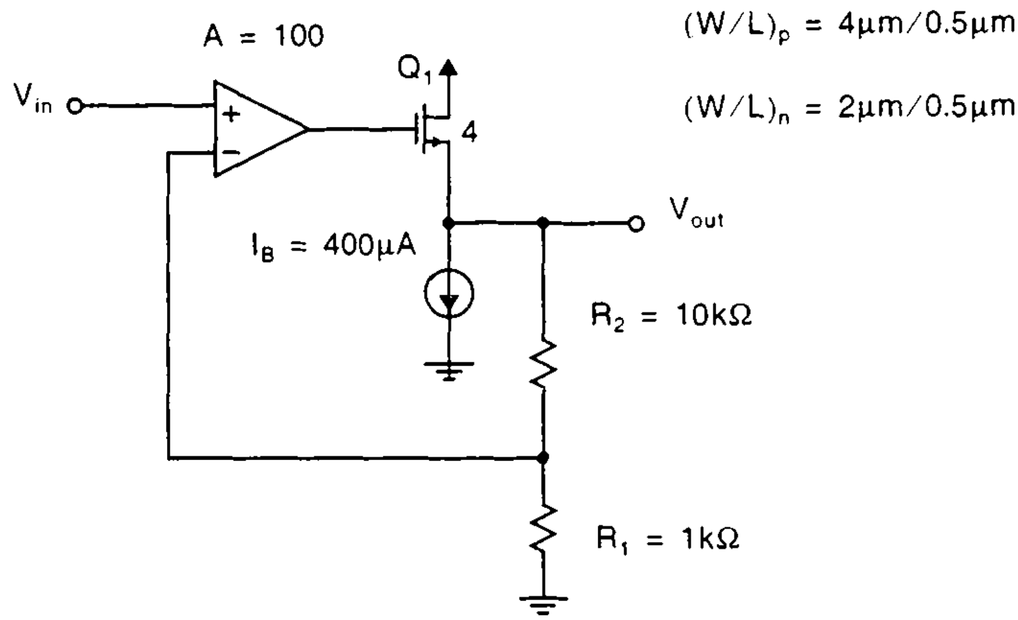
Name: ______________________

Student #:______________________

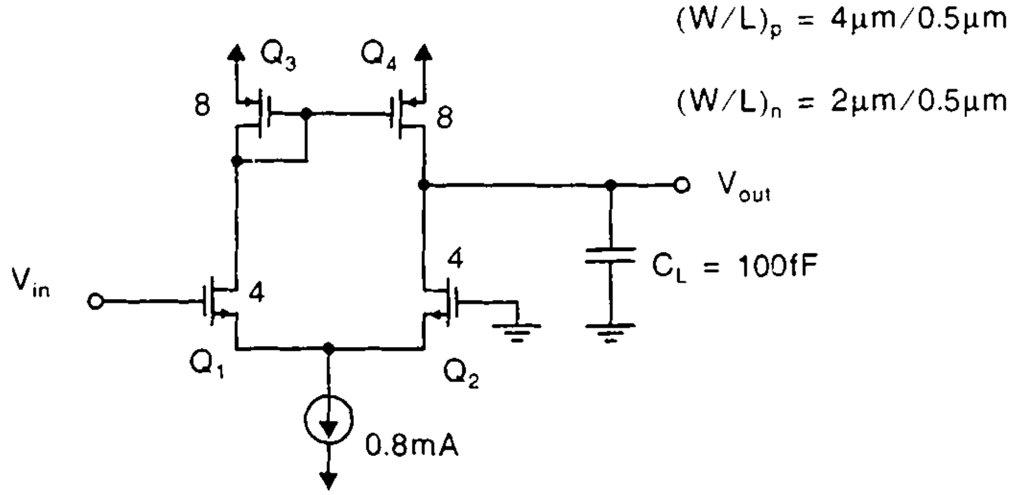
Question	Mark
1	
2	
3	
4	
Total	



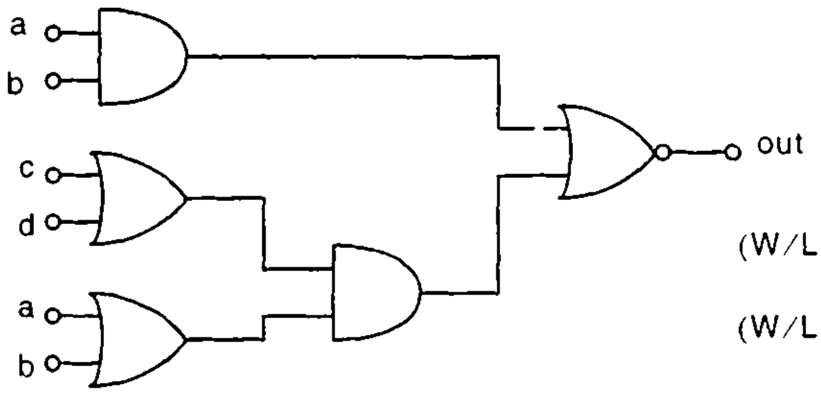
1. Assume the input bias voltage is such that Q_1 is in the active region and that Q_1 has an M-factor of 4, as shown. Assume the transistor output impedance (r_{ds}) can be ignored and that all internal parasitic capacitances except C_{gs} can be ignored. Also assume the current source is ideal. Do not ignore the body-effect. Using a nodal analysis of the small-signal model, derive a symbolic equation for the transfer function. Based on this transfer function, give symbolic equations and numerical values for the poles and zeros of the transfer function. What is the magnitude gain and phase shift at 100MHz?



2. Assume the transistor Q_1 is biased at 200μA, the gain of the amplifier is 100 and transistor Q_1 has an M-factor of 4 as shown. Taking the finite amplifier gain and non-zero output impedance into account, what is the open loop gain of the circuit? What is the closed-loop gain? Using a feedback-analysis approach, give symbolic equations and numerical answers for both the open-loop and closed-loop output impedance of the amplifier shown. State specifically any approximations used.



3. Give a symbolic equation and a numerical estimate for the parasitic capacitance and resistance seen by that capacitance for each node of the amplifier shown. Ignore all junction capacitances. State specifically any assumptions made. Based on these estimates and using open-circuit time-constant analysis techniques, give an estimate for the -3dB frequency of the amplifier.



$(W/L)_p = 4\mu m/0.5\mu m$

$(W/L)_n = 2\mu m/0.5\mu m$

4. Give a transistor-level CMOS gate that realizes the logic function shown. Simplify your solution as much as possible. Assuming all transistors have the sizes shown with $M = 1$, that the gate load capacitance is 100fF, and that $V_{DD} = 3.3V$, estimate the worst-case 70% rise and fall times of the gate.