Surname: ______________ Given Name: ______________ Student No.: ______________

University of Toronto
Faculty of Applied Science and Engineering
Department of Electrical and Computer Engineering

Final Examination, December 9, 1999
Fourth Year — Program 5 (Option 5CE) and Program 9

ECE 418F — Data Communications

Examination Type: D
Examiner: F. R. Kschischang

Instructions:

- This examination paper consists of twelve [12] pages (including this one). Please make sure that you have a complete paper.
- Write your name and student number in the space provided at the top of each page.
- Answer each question directly on the examination paper, using the back of each page if necessary. Indicate clearly where your work can be found.
- Show all steps and present all results clearly. **State any assumptions that you make.**
- Answer **all** five [5] questions. Each question is worth 20 marks. A total of 100 marks is available.
- This is a type D examination. **The only aids permitted are class notes, the textbook by Gibson, and a calculator.**
- Time: 2 $\frac{1}{2}$ hours.

Examiner's Report

1.		/20
2.		/20
3.		/20
4.		/20
5.		/20
Total:		/100

1. Consider a memoryless[1] source that produces symbols at random from the alphabet $\{1, 2, 3, \ldots, M\}$, where each symbol is equally likely to occur. It is known that a binary Huffman code for such a source has an average codeword length of

$$L = 1 + B - 2^B/M, \tag{1}$$

where B is the smallest integer for which $2^B - M \geq 0$.

7 marks

(**a**) For each M ranging from 2 to 8, draw a Huffman tree for the source, and verify that the average codeword length is given by (1).

For the remainder of this question, suppose that the memoryless source produces symbols at random from the alphabet $\{a, b, c\}$, with $P[a] = P[b] = P[c] = 1/3$. We wish to design a code for a long sequence of symbols produced by this source.

2 marks

(**b**) What is the minimum number of bits per source symbol that any uniquely decodable code can possibly achieve for this source?

9 marks

[1] Memoryless just means that each source output is *independent* of all other source outputs.

4 marks

(c) One possible approach to design an efficient code would be to group the source output into blocks of length k, and then use a fixed-length binary code for each possible block. For example, if $k = 1$, there are 3 possible output blocks, $\{a, b, c\}$, and so the source could be represented at a cost of $L = 2$ bits/symbol. If $k = 2$, there are 9 possible output blocks $\{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$, requiring 4 bits per block, or $L = 2$ bits per symbol. For each k ranging from 3 to 10, compute the number of bits per symbol required by the shortest possible fixed length code, entering your result in the following table.

k	1	2	3	4	5	6	7	8	9	10
L	2	2								

6 marks

(d) An alternative approach would be to represent the possible blocks of length k using a Huffman code. For each k ranging from 1 to 6, compute the number of bits per symbol required by the Huffman code, entering your result in the following table. [*Hint*: use equation (1).]

k	1	2	3	4	5	6
L						

1 mark

(e) What is the smallest value of k for which the Huffman code is more efficient than the *best* fixed-length code found in part (c)?

11 marks

2. Let $s_1(t)$ and $s_2(t)$ be two real-valued signals with $\langle s_1(t), s_1(t)\rangle = \langle s_2(t), s_2(t)\rangle = E$, and $\langle s_1(t), s_2(t)\rangle = R$, where $|R| \leq E$.

4 marks **(a)** Show that $s_1(t) - s_2(t)$ and $s_1(t) + s_2(t)$ are orthogonal.

4 marks **(b)** Use the result of part (a) to find an orthonormal basis $\phi_1(t)$ and $\phi_2(t)$ for the signal space spanned by $s_1(t)$ and $s_2(t)$.

8 marks

4 marks

(c) Find the coordinates of $s_1(t)$ and $s_2(t)$ with respect to the orthonormal basis found in (b).

4 marks

(d) Find the squared Euclidean distance between $s_1(t)$ and $s_2(t)$ as a function of E and R.

4 marks

(e) For which value of R is the squared Euclidean distance between $s_1(t)$ and $s_2(t)$ maximized? Plot, for this value of R, the location of $s_1(t)$ and $s_2(t)$ with respect to the orthonormal basis found in (b).

12 marks

3. Figure 1 shows two 8-point signal constellations in the signal space spanned by orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$.

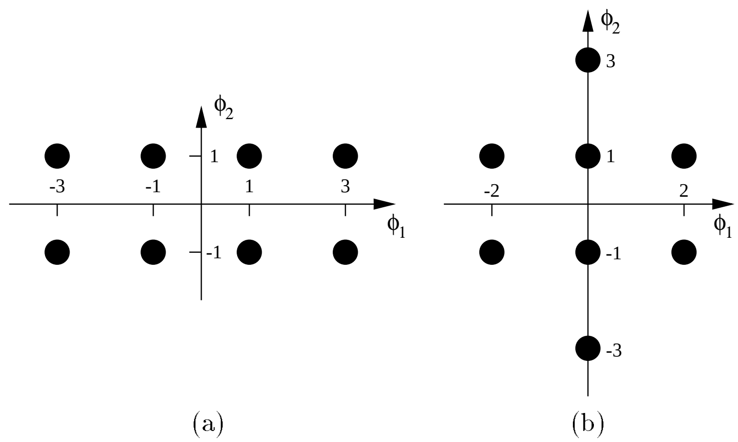


Figure 1: *Two different 8-point QAM constellations.*

4 marks (**a**) Compute the constellation figure of merit for the constellation of Fig. 1(a).

4 marks (**b**) Compute the constellation figure of merit for the constellation of Fig. 1(b).

2 marks (**c**) What is the asymptotic gain (in dB) that the better of the two constellations provides with respect to the other constellation?

10 marks

10 marks **(d)** Find an exact expression for the probability of symbol error when the constellation of Fig. 1(a) is used for transmission over an ideal additive white Gaussian noise channel with two-sided power spectral density $N_0/2$, assuming ideal maximum-likelihood detection. Express your answer in terms of the $\mathrm{erfc}(x)$ function.

10 marks

4. A binary $\{\pm 1\}$ signal is sent via two different channels, denoted 'A' and 'B', to the same destination, where the signals are combined as shown in Fig 2. In the absence of noise, the signal samples received through the two channels are $\{\pm A_A\}$ and $\{\pm A_B\}$ respectively. The noise samples n_A and n_B are independent, zero mean, Gaussian random variables with variances σ_A^2 and σ_B^2 respectively. In the combining system (called a diversity system), one of the paths has an amplifier with adjustable voltage gain K. The two paths are added and passed through a threshold detector.

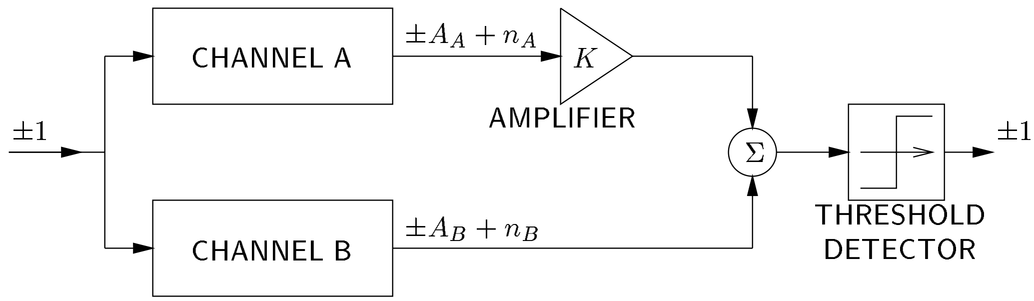


Figure 2: *A diversity combining system.*

12 marks

(a) For a fixed K, find the probability of error at the output of the detector. [*Hint*: the sum of two independent Gaussian random variables is a Gaussian random variable.]

You will find more space on the next page. . . .

12 marks

8 marks **(b)** Find the value of K that minimizes the probability of error. [*Hint*: to minimize $\mathrm{erfc}(x)$, try to maximize x.]

8 marks

5. Consider the binary code C with generator matrix

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

2 marks (**a**) What are the length and dimension of C?

4 marks (**b**) How many codewords does C have? List them all.

1 mark (**c**) What is $d_{\min}(C)$?

1 mark (**d**) Write the weight enumerator $W_C(x)$.

2 marks (**e**) Suppose C is used for error detection on a binary symmetric channel with cross-over probability $p = 10^{-4}$. Determine the probability $P[u]$ of undetected error.

10 marks

1 mark (**f**) Can C be used for error correction? If so, what is the largest number of errors C is guaranteed to be able to correct?

2 marks (**g**) Find a parity-check matrix for C.

2 marks (**h**) Compute the syndrome corresponding to the received word $r = 11001101$, and determine the position(s) of any error(s). What is the final decoded codeword?

A four-state convolutional code has the trellis diagram shown in Fig. 3. A convolutional codeword is transmitted over a binary symmetric channel, and the vector

$$r = (01, 11, 00, 01, 11, 10, 11, 01)$$

is received.

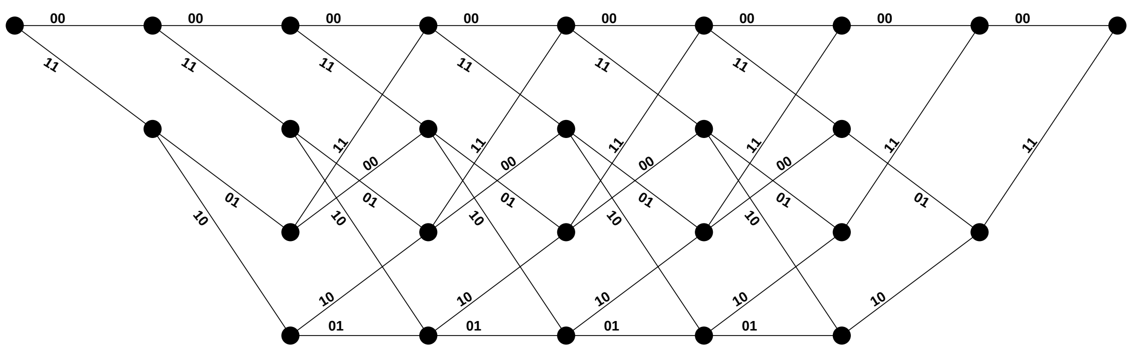


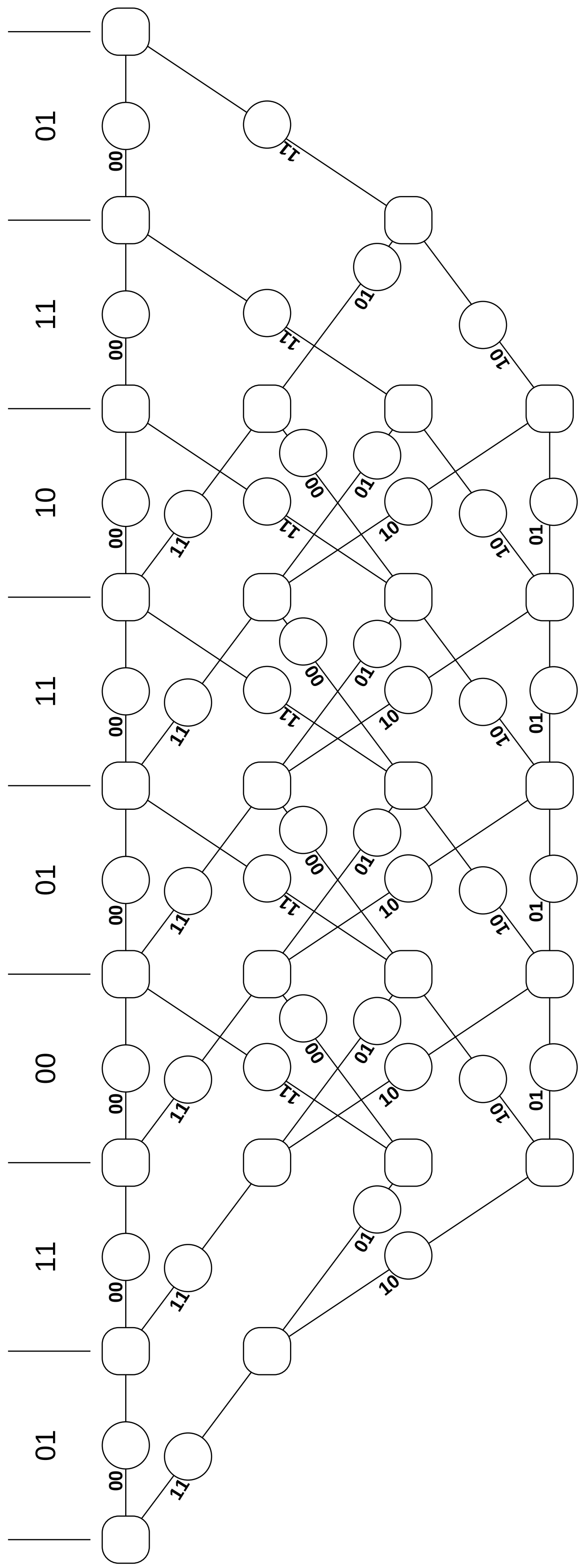
Figure 3: *Trellis diagram for a convolutional code. The label on each edge shows the corresponding output symbols.*

4 marks (**i**) Using the Viterbi algorithm Worksheet found on the next page, use the Viterbi algorithm to find the convolutional codeword that is closest to r in Hamming distance.

1 mark (**j**) What is the Hamming distance between r and the nearest convolutional codeword?

10 marks

VITERBI ALGORITHM WORKSHEET



Use the circles to record branch metrics, and the rounded boxes to record state metrics.