UNIVERSITY OF TORONTO
DEPARTMENT OF ELECTRICAL & COMPUTER ENGINEERING
FINAL EXAMINATION - DECEMBER 1994
ECE355F - SYSTEM AND SIGNAL ANALYSIS I
Third Year - Programs 5ce, 5e; 5p
EXAMINER - W.M. Wonham

Each of the three main questions is worth 1/3 of the total mark.
Please answer each of the main questions in a separate examination booklet.

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1. Let $f(t) = e^{-\alpha|t|}$, $-\infty < t < \infty$, where $\alpha > 0$. Define

$$g(t) := \sum_{k=-\infty}^{\infty} f(t+2k), \qquad -\infty < t < \infty$$

1.1 Calculate $g(0)$.

1.2 Show that g is periodic.

1.3 Calculate the complex exponential Fourier series (CEFS) of g.

1.4 As usual, define the Nth partial sum of CEFS as

$$g_N := \sum_{n=-N}^{N} \cdots$$

and the corresponding approximation error as $e_N(t) := g(t) - g_N(t)$. Estimate the maximum error magnitude

$$\max(e_N) := \max\{|e_N(t)| \ : \ -\infty < t < \infty\}$$

as a function of N and α, when N is large (i.e. $N\pi >> \alpha$).

1.5 Estimate a value of N sufficient to guarantee that $\max(e_N)/g(0) < 0.001$. Sketch the dependence of your estimate on α.

1.6 Calculate the average power of g at zero frequency (i.e. 'd.c. power' of g).

1.7 Estimate a value of N sufficient to guarantee that the ratio

$$\frac{\text{power of } e_N}{\text{d.c. power of } g}$$

is less than 0.001. Sketch the dependence of your estimate on α.

2. Let $f(t)$, $-\infty < t < \infty$, be a real-valued signal with L_1 and L_2 norms both finite. Consider a linear time-invariant filter (called $\mathbb{H}$, say) with impulse response h, given by

$$h(u) = f(-u+\tau), \qquad -\infty < u < \infty$$

where τ is a fixed real number.

2.1 When the input to $\mathbb{H}$ is f, let the output signal be $g(t)$, $-\infty < t < \infty$. Calculate the Fourier transform $\hat{g}(\omega)$ in terms of the Fourier transform $\hat{f}(\omega)$.

2.2 With g as in 2.1, use the Fourier integral representation of g to calculate the specific output value $g(\tau)$, in terms of the energy of f.

2.3 With g as in 2.1, use 2.2 to show that $g(t)$ is maximized with respect to t ($-\infty < t < \infty$) when $t = \tau$.

2.4 Under what condition on f is the filter $\mathbb{H}$ causal?

2.5 Specifically let

$$f(t) = \begin{cases} \sin t, & 0 \le t \le \pi \\ 0, & \text{otherwise} \end{cases}$$

and let $\tau = 2\pi$. Carefully define h and sketch the graphs of f and h.

2.6 With f as in 2.5, write detailed expressions for $g(t)$, in each of the four ranges

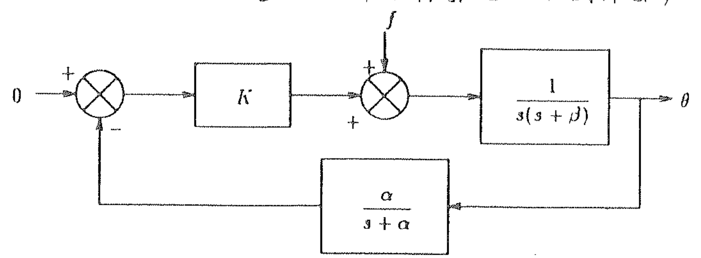
$$t \le \pi, \quad \pi \le t \le 2\pi, \quad 2\pi \le t \le 3\pi, \quad t \ge 3\pi$$

Use the picture to write down, but don't evaluate, the relevant integrals.

2.7 From the result of 2.6, just evaluate $g(\pi)$, $g(2\pi)$ and $g(3\pi)$. Sketch the graph of $g(t)$ for $0 \le t \le 4\pi$.

2.8 Consider the general case when the input to $\mathbb{H}$ is a real-valued signal $r(t)$, $-\infty < t < \infty$, with energy $\|r\|^2 = 1$. Denote the output signal by s. Use Schwarz to show that $s(\tau)$ is a positive maximum with respect to all such r when $r(t) = f(t)/\|f\|$ for all t, $-\infty < t < \infty$.

3.



The system with block diagram displayed represents a regulator whose purpose is to hold the output angle variable θ approximately constant at the 'setpoint' value of 0, in the face of 'disturbances' represented by the input signal f. The parameters α, β, K are positive real numbers.

3.1 Find the transfer function $\hat{h}(s) := \hat{\theta}(s)/\hat{f}(s)$ (on the assumption that all internal initial values of the system are 0).

3.2 Find the range of K such that the system is BIBO stable.

3.3 Write $\hat{k}(s)$ for the limiting form of $\hat{h}(s)$ as the angle sensor (represented by the block in the feedback return path) becomes 'perfect'. What can be said about the stability of $\hat{k}(s)$?

3.4 Suppose f satisfies the differential equation and initial conditions

$$d^2 f(t)/dt^2 = 0, \quad t > 0; \qquad f(0) = f_o, \quad f'(0) = f_1$$

Find the Laplace transform $\hat{f}(s)$.

3.5 With f as in 3.4, what conditions must hold on the parameters K, α, β, f_o and f_1 to guarantee that $\lim \theta(t)$ $(t \to \infty)$ exists and is finite? In that case, evaluate the limit.

3.6 With $f \equiv 0$, find the critical frequency at which the system may oscillate spontaneously when it is just on the boundary between stability and instability.

3.7 For what range of K does the system have a steady-state frequency response? For such K, find the approximate amplitude and phase of the frequency response for very high frequencies ω.

3.8 Assume $\alpha = 2$, $\beta = 1$, $K = 30$, and that $f \equiv 0$. Verify carefully that $\theta(t)$ may oscillate with an amplitude that grows like e^t as $t \to \infty$.