University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
First Year - Program 5

FINAL EXAMINATIONS, APRIL 1996

MAT 195S
Calculus II

Examiners: Prof. P. A. Sullivan
Prof. S. H. Smith

Duration - $2\frac{1}{2}$ hours

1. (a) [3] Integrate $\int x \arctan x \, dx$.

(b) [5] Evaluate $\int_0^{\frac{1}{4}\pi} \tan^3 x \, dx$.

(c) [5] Integrate $\int \frac{dx}{x(x^2-1)^{\frac{1}{2}}}$.

(d) [4] Does the improper integral $\int_1^{\infty} \frac{dx}{x(x^3-1)^{\frac{1}{2}}}$ exist?

(e) [ ] Show that $\int_0^x \left\{ \int_0^u f(t)\,dt \right\} du = \int_0^x f(u)(x-u)\,du$ for continuous functions $f(x)$.

(f) [6] Show that if $f(x)$ is twice differentiable then

$$f(x) = f(a) + f'(a)\cdot(x-a) + \int_a^x (x-t)f''(t)\,dt.$$

2. (a) [7] Evaluate $\lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)$

(b) [ ] $\lim_{x\to 0}\left(\cot^2 x - \frac{1}{x^2}\right)$

(c) [7] $\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}$ when $\lim_{n\to\infty} a_n = a$.

3. (a) Does the series $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}} \sin\left(\frac{\pi}{k}\right)$ converge?

(b) [6] For what values of x does the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ converge?

3. (c) [7] For what values of x does the series $\sum_{k=1}^{\infty} \frac{2^{kx}}{k}$ converge? Sum the series for such values.

4. (a) Find the Taylor series for $f(x) = \int_0^x \sin(u^2)\,du$, and calculate $f(1)$ to three decimal places.

(b) Find the first three non-zero terms in the Taylor series expansion for $(1+\cos x)^{-1}$ when x is small.

4. (c) Find the first four terms in the power series solution $f(x) = \sum_{k=0}^{\infty} a_n x^n$ to the differential equation $f' = f^2$, given that $f(0) = 1$. Can you identify the series?

5. (a) Give a geometrical description of the curve $\underline{r}(t) = (t\cos t, t\sin t, t^2)$.

(b) At what angle does this curve cut the sphere $|\underline{r}| = 1$?

6. (a) Check whether the following limits exist

(i) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y^2}$, (ii) $\lim_{(x,y)\to(1,1)} \frac{x-y^4}{x^3-y^4}$;

6. (b) The first mean value theorem for a differentiable function $f(x,y)$ can be given as $f(x+h,y) - f(x,y) = hf_x(x+\theta h, y)$ for some quantity θ with $0<\theta<1$; there is a corresponding expression involving f_y. Use these to show that if f_{xy} and f_{yx} are continuous then $f_{xy} = f_{yx}$.

7. (a) If $u(r,t) = \frac{1}{r}F(r-ct)$ where c is a constant, show that $u_{rr} + \frac{2}{r}u_r = \frac{1}{c^2}u_{tt}$.

(b) When $x = e^u \cos v$, $y = e^u \sin v$, show that $f_u^2 + f_v^2 = (x^2+y^2)(f_x^2+f_y^2)$.

8. The temperature distribution in a plate is given by $T(x,y) = 10 + 3xy$. Find the path a heat-seeking particle (which always moves in the direction of greatest increase in temperature) would follow if it starts at the point (a,b).

9. Find the maximum value of the product $uvwxyz$, for positive values of the variables, given that the sum $u+v+w+x+y+z = 6$.