University of Toronto
Department of Electrical and Computer Engineering

FINAL EXAMINATION, APRIL 22, 1998
Third Year – Engineering Science

ECE351S – Probability and Random Processes

Exam Type: B
Examiner: Amir H. Banihashemi

(a) Check that you have all five [5] pages.

(b) Time allowed: 2.5 hours.

(c) No aids allowed (not even a calculator).

(d) Some formulas which might be useful are given in pages 4 and 5.

(e) [50] constitutes full marks. The value of each question is indicated beside the question.

(f) **Attempt all the six questions. Justify all your answers.**

(g) In this exam:
cdf = cumulative distribution function
pdf = probability density function
pmf = probability mass function
iid = independent and identically distributed

[2] 1. (a) The random variable X is uniformly distributed on the interval $(-1, 1)$. Compute $E[(X-2)^3]$.

[2] (b) An oil company strikes oil in any particular drilling with probability 0.1. If the drillings are performed independently, and each hole drilled costs 0.5 million dollars, what is the expected cost of finding oil if the company keeps drilling until it finds oil.

[3] (c) Suppose that X_1 and X_2 are random variables, each with zero mean, variance σ^2, and correlation coefficient $\rho_{X_1,X_2} = 0$. Find the correlation coefficient of random variables X_1 and $Y_1 = X_1 - X_2$.

[2] (d) Having access to a software which generates a random variable X that is uniformly distributed on the interval $(0,1)$, find a transformation $g(.)$ such that $Y = g(X)$ has an exponential distribution with parameter λ.

[4] (e) Suppose that continuous random variables $X_1, \ldots, X_n$ are iid with cdf $F(x)$. Find the joint cdf $F_{Y,Z}(y,z)$ of $Y = \max\{X_1, \ldots, X_n\}$ and $Z = \min\{X_1, \ldots, X_n$

[2] (f) The random variable θ is uniformly distributed over $[0, 2\pi)$. The random variables X and Y are defined as: $X = \sin(\theta)$ and $Y = \cos(\theta)$. Are X and Y independent? Are they uncorrelated?

[3] (g) Suppose that $Z = \sum_{i=1}^{50} X_i + \sum_{j=1}^{20} Y_j$, where the random variables X_i are iid with exponential distribution ($\lambda = 5$), and the random variables Y_j are iid with mean 0.2 and variance 0.05. If X_i's are independent of Y_i's, find an approximation for the pdf of Z using the central limit theorem.

2. Let X and Y be iid random variables with the pdf:

$$f(x) = \frac{1}{x^2},\ 1 \le x < \infty\,.$$

[2] (a) What is the probability that $X > 2Y$.

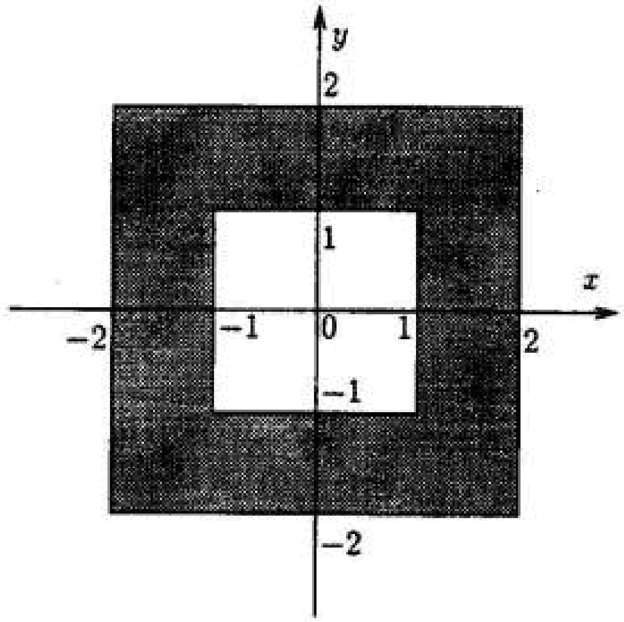
[4] (b) Find the pdf of the random variable $Z = X/Y$.

3. Random variables X and Y are uniformly distributed in the region shown in the following figure.

[3] (a) Compute the probabilities $P[X + Y \ge 2]$ and $P[X + Y > 2 \mid |X| \le 1]$.

[3] (b) Find the marginal pdf and the marginal cdf of X.

[2] (c) Find the conditional pdf $f_Y(y|x)$ and compute $P[Y < 1.5|X = 0.5]$.



[2] (d) Are the two random variables X and Y uncorrelated? Are they independent?

4. Let X and Y be independent standard Gaussian random variables ($N(0,1)$).

[3] (a) Prove that $U = X + Y$ and $W = (X - Y)^2$ are independent.

[2] (b) Find the pdf of W.

[4] 5. A waitress is responsible for n customers sitting along a lunch counter. If the customers are equally spaced, each d feet apart, what is the average distance that she walks from customer to customer assuming that at any given time each customer (including the one just being served) is equally likely to ask for service. (Hint: Let the customers be numbered from 1 to n. Let X be the number of customer that she is taking order from at present. Let Y be the distance that she walks from the current customer to her next customer. Use $E[Y] = E[E[Y|X]]$.)

6. Let A and B be iid random variables with the following moments:

$$E[A] = E[B] = 0\,, \quad \text{and} \quad E[A^2] = E[B^2] = \sigma^2\,.$$

The random process $X(t)$ is defined by $X(t) = A\cos(2\pi t) + B\sin(2\pi t)$.

[3] (a) Prove that $X(t)$ is wide-sense stationary.

[2] (b) Is $X(t)$ strict-sense stationary?

[2] (c) Is $X(t)$ mean-ergodic?

- Conditional probability: $P(B|A) = \frac{P(B\cap A)}{P(A)}$, $P(A) \neq 0$.
- Conditional pdf: $f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{d}{dy}F_Y(y|x)$, $f_X(x) \neq 0$.
- Marginal pdf: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$.
- Mean: $E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$, $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy$. $E[Y] = E[E[Y|X]]$, where $E[Y|X] = g(X)$ is a function of X and $g(x) = \int_{-\infty}^{\infty} y f_Y(y|x)dy$.
- Variance: $VAR[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$.
- Correlation coefficient: $\rho_{X,Y} = \frac{COV(X,Y)}{\sigma_X \sigma_Y}$, where $COV(X,Y) = E[(X-\bar{x})(Y-\bar{y})] = E[XY] - E[X]E[Y]$. X and Y are uncorrelated if $\rho_{X,Y} = 0$.
- Random variables X and Y are independent iff $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, $\forall x, y$. In this case, $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ for any functions $g(\cdot)$ and $h(\cdot)$. Also, for $Z = X+Y$, $f_Z(z) = f_X(z) * f_Y(z)$.
- Total probability theorem: $P(A) = \sum_{i=1}^{n} P(A\cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$, where B_i's are disjoint events which partition the sample space. Other forms: $P(A) = \int_{-\infty}^{\infty} P(A|X=x)f_X(x)dx$, $f_Y(y) = \int_{-\infty}^{\infty} f_Y(y|x)f_X(x)dx$.
- Binomial distribution: $p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}, k = 0,1,\ldots,n$, $E[X] = np$, $VAR[X] = np(1-p)$.
- Geometric distribution: $p_X(k) = p(1-p)^{k-1}, k = 1,2,3,\ldots$, $E[X] = \frac{1}{p}$, $VAR[X] = \frac{1-p}{p^2}$.
- Poisson distribution with parameter α: $p_X(k) = \frac{\alpha^k e^{-\alpha}}{k!}, k = 0,1,2,\ldots$ and $\alpha > 0$.
- Exponential distribution with parameter λ: $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0; \\ 0 & \text{otherwise,} \end{cases}$

  where $\lambda > 0$, $E[X] = \frac{1}{\lambda}$, $VAR[X] = \frac{1}{\lambda^2}$.
- Gaussian (or Normal) distribution $N(m,\sigma^2)$: $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}e^{-(x-m)^2/(2\sigma^2)}$.
- Q function: $Q(z) = \int_z^{\infty} \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$.
- Pdf of two jointly Gaussian random variables:
  $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-m_X)^2}{\sigma_X^2} - \frac{2\rho(x-m_X)(y-m_Y)}{\sigma_X\sigma_Y} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right]\right\}$,
  where $X \sim N(m_X,\sigma_X^2)$ and $Y \sim N(m_Y,\sigma_Y^2)$, and $\rho = \rho_{X,Y}$. We also have: $aX + bY \sim N(am_X + bm_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y)$.
- Pdf of $Y = g(X)$ in terms of the pdf of X, for g a differentiable real function:

$$f_Y(y) = \sum_{k=1}^{n}\left[\frac{f_X(x)}{|g'(x)|}\right]_{x=x_k},$$

  where $g'(x) = \frac{d}{dx}g(x)$, and $x_1, x_2, \ldots, x_n$ are the solutions of the equation $g(x) = y$.

- Pdf of linear transformations: Let $\mathbf{Z} = A\mathbf{X}$. Then $f_{\mathbf{Z}}(\mathbf{z}) = \frac{f_{\mathbf{X}}(A^{-1}\mathbf{z})}{|\det(A)|}$.

- Pdf of general transformations: Let the set of equations $\mathbf{z} = \mathbf{g}(\mathbf{x})$ have a unique solution given by $\mathbf{x} = (x_1, \ldots, x_n) = (h_1(\mathbf{z}), \ldots, h_n(\mathbf{z})) = \mathbf{h}(\mathbf{z})$. Then $f_{\mathbf{Z}}(\mathbf{z}) = \frac{f_{\mathbf{X}}(\mathbf{h}(\mathbf{z}))}{|J(\mathbf{x})|}$, or equivalently $f_{\mathbf{Z}}(\mathbf{z}) = f_{\mathbf{X}}(\mathbf{h}(\mathbf{z}))|J(\mathbf{z})|$, where $J(\mathbf{x})$ and $J(\mathbf{z})$ are the Jacobians of the transformation and its inverse, respectively, i.e.,

$$J(\mathbf{x}) = \det \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \cdot & \cdots & \cdot \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix} \quad \text{and} \quad J(\mathbf{z}) = \det \begin{pmatrix} \frac{\partial h_1}{\partial z_1} & \cdots & \frac{\partial h_1}{\partial z_n} \\ \cdot & \cdots & \cdot \\ \frac{\partial h_n}{\partial z_1} & \cdots & \frac{\partial h_n}{\partial z_n} \end{pmatrix}.$$

- Central limit theorem: Let $X_1, X_2, \ldots$ be an infinite sequence of iid random variables with mean μ and variance σ^2, and let $Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}$. Then the cdf of Z_n approaches that of a Gaussian random variable with zero mean and unit variance.

- For a random process $X(t)$:

  1. Mean: $m_X(t) = E[X(t)]$.
  2. Autocorrelation function: $R_X(t, t-\tau) = E[X(t)X(t-\tau)]$.
  3. Autocovariance function: $C_X(t, t-\tau) = E[(X(t) - m_X(t))(X(t-\tau) - m_X(t-\tau))] = R_X(t.t-\tau) - m_X(t)m_X(t-\tau)$.

- Strict-sense stationary random process: $F_{X(t_1)\cdots X(t_k)}(x_1, \ldots, x_k) = F_{X(t_1+\tau)\cdots X(t_k+\tau)}(x_1, \ldots$ for all time shifts τ, all k, and all sampling times $t_1, t_2, \ldots, t_k$.

- Wide-sense stationary random process: $E[X(t)] = m$ and $E[X(t)X(t-\tau)] = R_X(\tau)$.

- Ergodic theorem: Let $X(t)$ be a wide-sense stationary process with $m_X(t) = m$, then $\langle X(t)\rangle = m$ in the mean square sense, if and only if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|u|}{2T}\right) C_X(u)du = 0,$$

  where $C_X(u)$ is the autocovariance function of $X(t)$, and $\langle X(t)\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)dt$ is the time average of $X(t)$. Such an $X(t)$ is called mean-ergodic.

- Series: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, $|x| < 1$.

- Trigonometric identities: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$. $\sin(2a) = 2\sin(a)\cos(a)$, $\cos(2a) = 2\cos^2(a) - 1$.