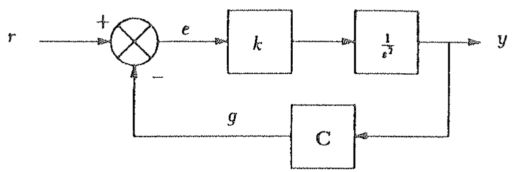
Marking scheme: Each of the four main questions is worth 1/4 of the total mark.

ECE 355 '13 FINAL

1.



In the feedback system shown, k is a constant gain, and the compensator block labelled C in the return path has transfer function $\hat{c}(s) = (s+\alpha)/(s+\beta)$. The parameters α, β are positive constants.

1.1 What is the differential equation relating g to y?

1.2 Determine the impulse response function $c(u)$ of C. Provide a formula for the compensator output signal $g(t)$ ($t \geq 0$) in terms of its initial value $g(0) = \gamma$ and the input $y(t')$ for $0 \leq t' \leq t$.

1.3 Let $\mathbb{H}$ denote the feedback system with input signal taken to be r and output e. Calculate the transfer function $\hat{h}(s)$ of $\mathbb{H}$. For what range of values of α, β, k is $\mathbb{H}$ BIBO stable?

1.4 Find the (critical) frequency of spontaneous oscillation of the system $\mathbb{H}$ when the parameters are adjusted so that $\mathbb{H}$ is on the boundary between stability and instability. If the input signal r is a sinusoid at the critical frequency, what is the general form of the output $e(t)$ for large t?

1.5 Let $r(t) = t^2$, $t \geq 0$. If $\mathbb{H}$ is stable, find $\lim e(t)$ ($t \to \infty$).

2. Consider a filter with differential equation

$$T\frac{dg(t)}{dt} + g(t) = f(t), \qquad t \geq 0$$

and initial condition $g(0) = 0$.

2.1 If the input signal is $f(t) = \cos(\omega t)$ ($t \geq 0$) find the output $g(t)$ for *all* $t \geq 0$. Check your answer when $\omega = 0$.

2.2 Suppose the input signal is $f(t) = |\sin(t)|$, $t \geq 0$. Let the Fourier series (corresponding to base interval $[0, \pi]$) be

$$f(t) = a_o/2 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)]$$

Find $a_o/2$, ω_o and b_n.

2.3 Using the results from 2.1 and 2.2, find the 'steady-state' filter output, say $g_\infty(t)$ (in terms of T and the a_n for $n \geq 1$), accurate for sufficiently large t. What would be a reasonable numerical assessment of 'sufficiently large' here?

2.4 Write expressions for the 'd.c.' power and the average 'a.c.' power corresponding to the solution $g_\infty(t)$ in 2.3. Given the time constant $T = 3\pi$, and $a_n = -4/(\pi(4n^2 - 1))$, estimate the summation limit required to compute the a.c. power numerically, with error no greater than 10^{-8} in magnitude.

3. Let $f(t)$ ($-\infty < t < \infty$) be the signal whose Fourier transform is

$$\hat{f}(\omega) = \begin{cases} 1, & |\omega| \leq 1 \\ 0, & |\omega| > 1 \end{cases}$$

3.1 Calculate f and sketch its graph.

3.2 Calculate the total energy of f.

3.3 Suppose f is sampled at times $n\tau$ ($n = ..., -1, 0, 1, ...$), and call the sampled signal g. Setting $\Omega := \pi/\tau$, provide a formula for the Fourier transform $\hat{g}$, in terms of $\hat{f}$. Sketch the graph of $\hat{g}$ when (i) $\Omega = 1.1$, and (ii) $\Omega = 0.9$.

3.4 With g as in 3.3, an attempt is made to recover f from g by passing g through an ideal low-pass filter with gain k and passband $-B \leq \omega \leq B$. Denote the filter output by $h(t)$ ($-\infty < t < \infty$), with Fourier transform $\hat{h}$. For a general value Ω, write (i) a formula for $\hat{h}$ in terms of $\hat{g}$, and (ii) a formula for $h(t)$ in terms of the samples $f(n\tau)$.

3.5 For what range of Ω will there exist suitable k and B as in 3.4 such that the signal h is identical with f? In particular evaluate your formula for $h(t)$ when $B = \Omega = 1$ and $k = \pi$.

3.6 In the case $B = \Omega = 0.9$, and $k = \pi/\Omega$, calculate the energy of h. How does your result differ from the energy of f? Discuss briefly.

4. Consider a data-processing system $\mathbb{S}$ with input signal f and output signal g, where

$$g(t) = \alpha e^{t-1} f(t-1) + \beta[f(t+1)]^2 + \gamma df(t)/dt + \int_{-\infty}^{\infty} k(t,v) f(v) dv$$

for $-\infty < t < \infty$. Here α, β, γ are real parameters, and the weighting function k is an ordinary real-valued function of t and v. The input signal class $\mathcal{F}$ consists of all real-valued signals f defined on $-\infty < t < \infty$ for which the above derivative and integral exist, and the output signal class $\mathcal{G}$ is chosen large enough to contain all the signals $\mathbb{S}f$. Assume that $\mathcal{F}$ and $\mathcal{G}$ are linear vector spaces.

4.1 Provide the 'best' condition(s) you can find, on the constants α, β, γ and the function k, which guarantee that $\mathbb{S}$ is linear. Taking a simple instance of $\mathbb{S}$, show that linearity can fail if your condition fails.

4.2 Repeat 4.1, for the property of time invariance of $\mathbb{S}$.

4.3 Repeat 4.1, for the causality of $\mathbb{S}$.

4.4 Repeat 4.1, for the BIBO stability of $\mathbb{S}$. *Hint:* Can you invent a signal f for which $\|f\|_\infty < \infty$, but whose derivative $df(t)/dt$ is an unbounded function of t?