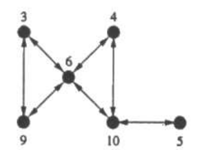

Figure 1: The graph of the relation R on the set $A := \{3,4,5,6,9,10\}$. **Note:** in this graph an edge with two arrows, e.g., $a \longleftrightarrow b$, actually represents *two* edges: (a,b) **and** (b,a).

1. The graph $G_R = (V,E)$ of the relation R on the set $A := \{3,4,5,6,9,10\}$ is shown in Fig. 1.

3 marks (a) Let x and y be variables taking on values in the set A. For each of the following predicates, explain whether or not the predicate describes the relation R.

i. $P(x,y)$: For $x \neq y$, y is an integer multiple of x.

ii. $P(x,y)$: For $x \neq y$, x and y have a prime as their greatest common divisor.

iii. $P(x,y)$: For $x \neq y$, the greatest common divisor of x and y is not 1.

1 mark (b) Write the zero-one relation matrix M_R of R.

4 marks (c) Is R reflexive? symmetric? transitive? an equivalence relation?

2 marks (d) Find the transitive closure of R.

2 marks (e) Is R a function? Why?

2 marks (f) Find a subgraph $G_F = (V, E_f)$ of the graph $G_R = (V,E)$ where $E_f \subseteq E$, such that its corresponding relation $F \subseteq R$ is a *bijective function* on A.

Consider now the underlying undirected graph of R, in which each double edge $a \longleftrightarrow b$ is replaced with a single undirected edge $\{a,b\}$. Assume the graph is weighted, where the weight of an edge $\{a,b\}$ is the integer product ab.

2 marks (g) Find the shortest paths from the vertex of smallest degree to all other vertices.

2 marks (h) Suppose that this weighted undirected graph is a road map of the small tropical island of *Freedonia*, where the vertices of G_R represent the cities of *Freedonia* and the edges of G_R representing the highways connecting the cities. Can a tourist start a tour from the most remote city (the vertex having the smallest degree) and cover each highway of the road map **exactly once**? Explain why or why not? If such a tour exists, identify it.

2 marks (i) Suppose that a hurricane hit *Freedonia* (one of the drawbacks of living in the tropics) and a resulting flood destroyed all the highways of the island. The government of *Freedonia* wants to re-build a portion of the old highway system to connect the cities of the island, but so that the total cost is minimum, where the cost of rebuilding a road is equal to its weight in the undirected graph. What is the graph of the new road map? What is its total cost?

2 marks 2. (a) The word *unless* means *if not*. Thus, for propositions p and q, the phrase "p *unless* q" is equivalent to p *if not* q. Write the truth table for p *unless* q.

4 marks (b) Let x and y be integers, and consider the predicate $Q(x,y) : x + 2y = xy$. Evaluate

$$\sim \forall x \exists y \sim Q(x,y).$$

4 marks (c) Show that $\sum_{i=1}^{n} i^3$ is $O(n^4)$.

5 marks 3. (a) State and prove the Handshaking Theorem of graph theory.

5 marks (b) At a party, a certain amount of handshaking goes on. Assuming that nobody shakes his or her own hand, and nobody shakes another person's hand more than once, prove that at least two people must have shaken the same number of hands.

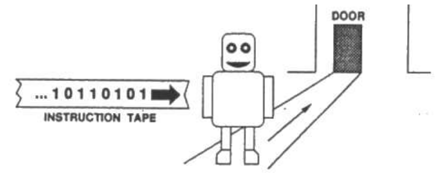

Figure 2: A robot moving towards the door according to instructions printed on the instruction tape.

4. The robot shown in Fig. 2 is controlled by an instruction tape, on which a string of bits of length n is recorded. In one unit of time, the robot reads a bit from the tape (after which the tape is advanced to the next bit position). If the bit is a one, the robot takes one step towards the door, and if the bit is a zero, the robot remains where it is. This process is then repeated during the next unit of time until all bits have been read. Assume that the tape contains a completely random string of bits, each bit being one or zero with equal probability.

2 marks (a) How many possible instruction tapes are there of length n?

2 marks (b) How many different instruction tapes of length n result in a movement of k steps towards the door?

3 marks (c) After the robot has read in n bits, for each k in the range $0 \leq k \leq n$, find the probability that it has moved k steps towards the door.

3 marks (d) Suppose $n = 12$, and that the robot will pass through the door if it takes 8 or more steps. What is the probability that the robot will pass through the door?

5. Let A be a set with $n \geq 1$ elements. In this problem it might be useful to represent each relation on A as a zero-one matrix.

3 marks (a) How many relations are there on the set A?

3 marks (b) How many relations are there on the set A that are reflexive?

4 marks (c) How many relations are there on the set A that are symmetric?

6. Twelve people are required to serve on a jury. Assuming that birthdays are uniformly distributed over months, find the probability that

3 marks (a) the jurors have birthdays all in the same month;

4 marks (b) the jurors have birthdays all in different months;

3 marks (c) at least one juror has a birthday in April.

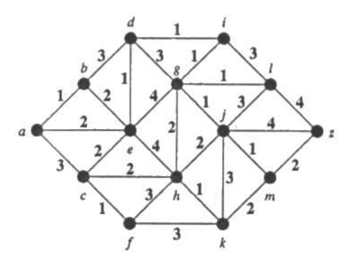

Figure 3: A weighted graph.

7. Consider the weighted graph G shown in Fig. 3.

4 marks (a) Use Dijkstra's algorithm to find the minimum-weight path from vertex a to vertex z in G. What is the weight of this path?

4 marks (b) Using any method, find a spanning tree for G of minimum total weight.

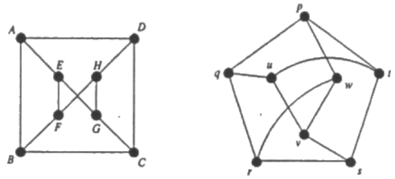

Figure 4: Two graphs.

8. Consider the two graphs shown in Fig. 4.

2 marks (a) Are the two graphs isomorphic? If not, explain why not. If so, relabel the graph on the right so as to show the isomorphism.

2 marks (b) Is the graph on the left planar? If not, explain why not. If so, redraw the graph to show that it is planar.

2 marks (c) Determine the chromatic number of the graph on the right, and exhibit a colouring that achieves this chromatic number.

2 marks (d) Does the graph on the left have an Euler path? If not, explain why not. If so, list the edges of the path.

2 marks (e) Does the graph on the right have a Hamilton path? If not, explain why not. If so, list the edges of the path.

3 marks 9. (a) How many different strings of length 11 can be obtained by rearranging the letters in the word "ENGINEERING"?

3 marks (b) Let $X = \{\mathrm{E, N, G, I, R}\}$. Eleven letters are drawn one at time from X with replacement. What is the probability that the letters drawn, *in the order in which they are drawn*, spell the word "ENGINEERING"?

4 marks (c) Eleven letters are drawn from X with replacement. What is the probability that the letters drawn can somehow be re-arranged to spell the word "ENGINEERING"?