UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, APRIL 1999

Second Year -- Engineering Science

PHY281S -- Quantum Mechanics

Exam Type: C

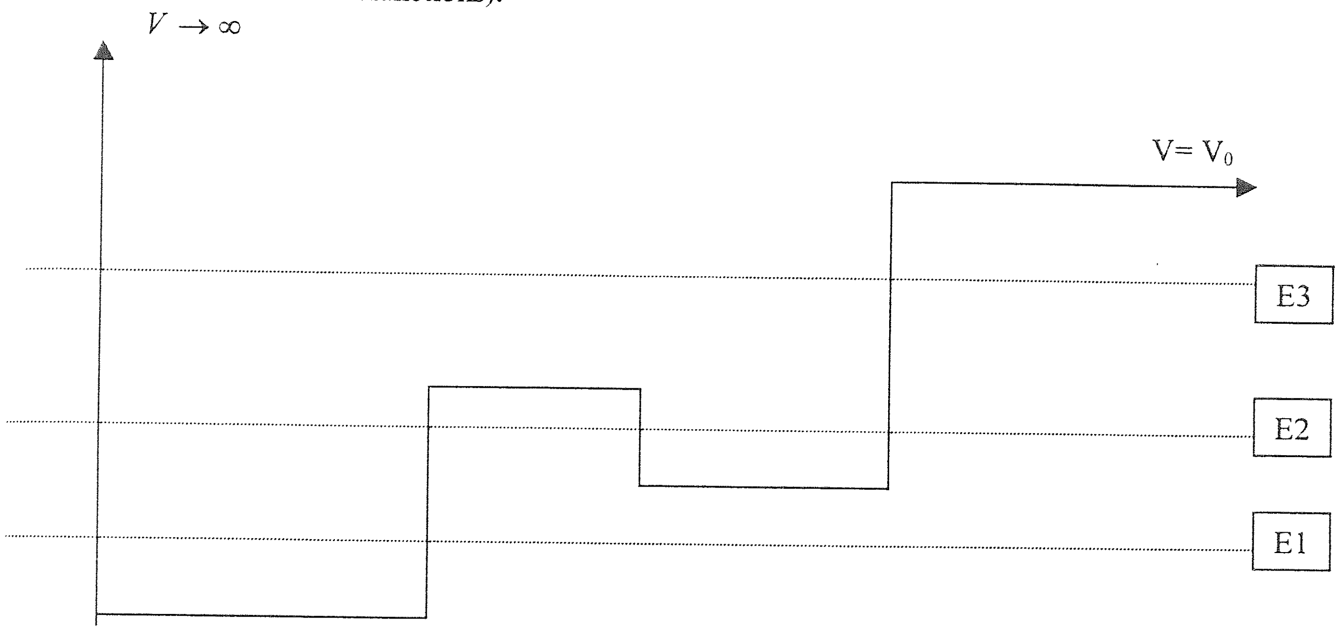
Examiner: W. Trischuk

You are only required to answer all four (4) questions. All questions are of equal value. Please note that question three is split across pages 3 and 4. For all questions you will benefit from reading the whole question before starting to answer parts of it.

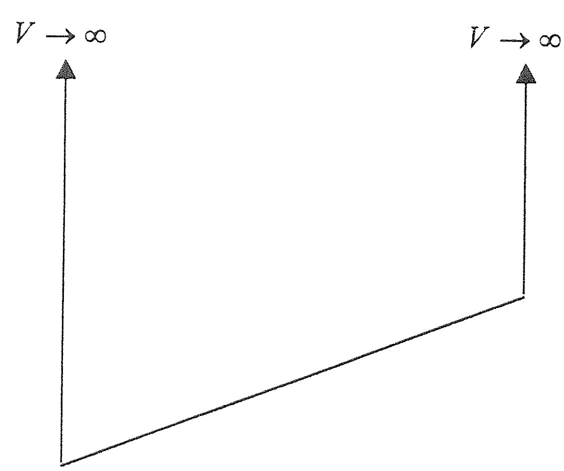
The questions start on the next page

1. a) Explain how the form of the Schrodinger equation governs the curvature of stationary wavefunction solutions. In particular how does the curvature depend on the total energy and potential energy of a particle?

   b) Suppose that the potential drawn below has solutions for the three energy levels drawn (E1, E2 and E3). Sketch the wavefunctions for each of these energy levels for each region of the potential (please indicate how the changes in potential relate to your sketched wavefunctions).



   c) Indicate the qualitative features of the wavefunction corresponding to the lowest three energy levels in the potential drawn below. Since you are not given quantitative information about the height of the sloping region at the bottom of the well you should indicate on an energy level diagram (or in some other way) the energy levels you have assumed in sketching your proposed solutions.



2. In order to achieve a time dependent probability distribution we need to consider super-positions of two or more stationary states.

a) Why must we use such super-positions?

In the rest of this problem we will consider a super-position of two states from the infinite square well potential (with walls at $x=0$ and $x=L$ as usual) of the form:

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-i\omega_1 t} + \frac{1}{\sqrt{2}}\psi_j(x)e^{-i\omega_j t}$$

where $\psi_1(x)$ is the ground state wavefunction, $\psi_j(x)$ is either the second or third states and the ω are the oscillation frequencies associated with the infinite square well energy levels. You should do each of the following computations for combinations of the ground state and second state as well as the ground state and third states:

i) Compute the probability $|\Psi(x,t)|^2$ in terms of the $\psi_n(x)$. You may assume that the spatial wavefunctions are purely real.

ii) What is the period, T, of the oscillation in $|\Psi(x,t)|^2$ in terms of ω_1?

iii) Sketch the probability distribution for each of the two components that make up the time depenedent state (at $t=0$ and $t=T/2$). Then sketch the combined probability distribution $|\Psi(x,t)|^2$ at $t=0$ and $t=T/2$. There is no prize for art here but feel free to elaborate on your drawings with written annotations (ie. describing the function you are sketching).

iv) Calculate the probability current $J(x=L/2, t)$. Explain why this is consistent with the sketches drawn in part iii). In particular, note the difference between the two choices of time dependent wavefunction. Explain the difference between these two combinations that leads to the qualitatively different behaviours.

3. a) Write down the three dimensional time independent Schrodinger equation. You may do this in either rectangular coordinates or any other coordinate system you chose.

b) If a bound state in three dimensions has zero angular momentum what can be said about the spatial dependence of its wavefunction?

(This question is continued at the top of the next page)

(question 3 continued from previous page)

3 c) In the rest of this problem you will consider the case of a particle confined in a cubic box having zero potential energy inside the box and infinite potential outside. The region of zero potential can be taken to be:

$$0 < x < L$$
$$0 < y < L$$
$$0 < z < L$$

Suppose there are 15 identical particles of mass *m* inside this well. For whatever reason no more than two particles can have the same wave function (ie. can have the same energy). Thus pairs must occupy successively higher energy levels.

i) What is the minimum total energy for such a system?

ii) In this minimum total energy state, at what point(s) in space would the highest energy particle most likely be found? (*NB:* Knowing no more than its energy, the highest energy particle might be in any of the multiple of wavefunctions open to it – and each of these with equal probability.)

4. A wave-packet is localised in physical space with the following wavevector distribution in configuration space:

$$B(k) = \frac{1}{\sqrt{k_2 - k_1}} \exp(ik\beta)$$

for $k_1 < k < k_2$ and zero otherwise. To answer the following questions it may help you to note that in configuration space *B(k)* plays the same role that $\psi(x)$ plays in physical space.

a) Calculate the wavefunction $\Psi(x,t=0)$. There is no need to normalise Ψ, we are only interested in the shape here.
b) What is the behaviour of the wave-packet's probability density as $|x| \to \infty$.
c) Sketch $|\Psi(x, t=0)|^2$ as a function of x.
d) Sketch the probability distribution in configuration space.
e) Using this probability distribution and the knowledge that $p = hk/2\pi$ compute the expectation value of the momentum for this wave-packet.
f) Given the spread of Ψ, in physical space (from part c) and the spread of wavevectors in configuration space (from part e) show that the Heisenberg Uncertainty principle holds.
g) Briefly explain the physical significance of the Heisenberg principle .

Some Useful Constants:

Atomic Mass Unit:	1.661×10^{-27}	kg
	10^{-24}	g
	931.5	MeV/c^2
Electron Mass:	9.109×10^{-31}	kg
	10^{-28}	g
	0.511	MeV/c^2
Elementary Charge:	1.602×10^{-19}	C
	4.803×10^{-10}	esu
Planck's Constant:	6.626×10^{-34}	J - s
	10^{-27}	erg - s
	4.136×10^{-15}	eV - s
Speed of Light:	2.998×10^{8}	m/s
	10^{10}	cm/s

Conversion Factors

1 eV = 1.602×10^{-19} J = 1.602×10^{-12} erg

1 amu = 1.492×10^{-10} J = 1.492×10^{-3} erg
= 931.5 MeV/c^2

1 Angstrom = 10^{-10} m = 10^{-8} cm

Useful Integrals

$$\int e^{ax}\sin(bx)dx = e^{ax}\frac{(a\sin(bx) - b\cos(bx))}{a^2+b^2}$$

$$\int e^{ax}\cos(bx)dx = e^{ax}\frac{(a\cos(bx) + b\sin(bx))}{a^2+b^2}$$

$$\int e^{ax}\sin^2(bx)dx = e^{ax}\sin(bx)\frac{(a\sin(bx) - 2b\cos(bx))}{a^2+4b^2} + 2b^2\int e^{ax}dx$$

$$\int e^{ax}\cos^2(bx)dx = e^{ax}\cos(bx)\frac{(a\cos(bx) + 2b\sin(bx))}{a^2+4b^2} + 2b^2\int e^{ax}dx$$