UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, APRIL 2001

ECE527S — PHOTONICS I

Exam Type: A

Examiner — E. H. Sargent

Please **answer any 5 questions** out of 7 questions.

You have 2.5 hours — an average of half an hour per problem.

A six-page equation sheet is included following the questions. Feel free to detach it.

1. Design devices to filter multi-wavelength optical signals.
2. Design a waveguide-based co-directional coupler.
3. Design high- and low-reflectivity coatings for light-emitting devices.
4. Evaluate a chirped fibre Bragg grating for dispersion pre-compensation.
5. Determine optical gain spectra of 1-D and 3-D superlattices.
6. 2 x 2 Optical waveguide switch with bow-tie electrode using free-carrier plasma effect induced total internal reflection in SiGe alloy.
7. Total internal reflection and frustrated total internal reflection.

1. **(10 marks). Design devices to filter multi-wavelength optical signals.**

(a) **(4 marks)** Design a Fabry-Perot etalon (figure 1.1) which will pass light of frequency 200 THz but reject light of frequency 200.1 THz and 199.9 THz with greater than 20 dB rejection ratio (figure 1.2(a)). Draw a labelled diagram of your device and specify mirror reflectance and mirror separation.

(b) **(6 marks)** Suppose that it is important to pass with essentially uniform transmittance (less than 1 dB variation) all light within 0.05 THz of the 200 THz centre frequency while still rejecting light of frequency 200.1 THz and 199.9 THz with greater than 20 dB rejection ratio (figure 1.2(b) - keep in mind that the figure plots transmittance, but you may be more accustomed to looking at the *reflectance* of a multilayer stack.).

Show that this cannot be done using a Fabry-Perot device.

Design a set of two multilayer periodic structures in series which will meet this new specification. Draw a labelled diagram of your device and specify layer refractive indices, layer thicknesses, and number of layers required.

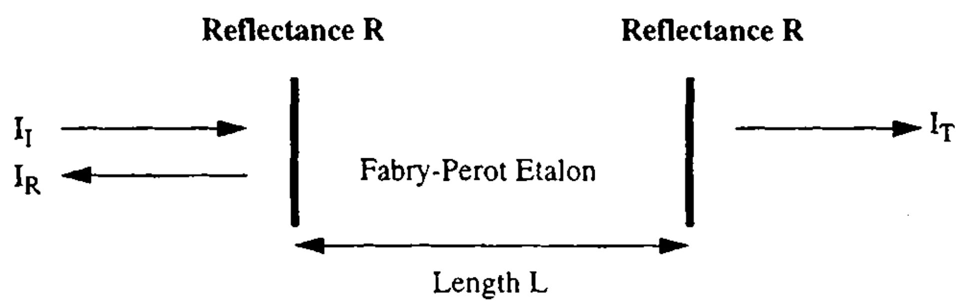


Figure 1.1. A Fabry-Perot Etalon.

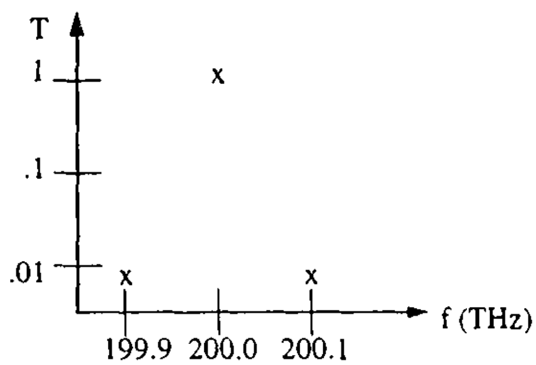


Figure 1.2(a).
Desired transmittance for question 1.(a).

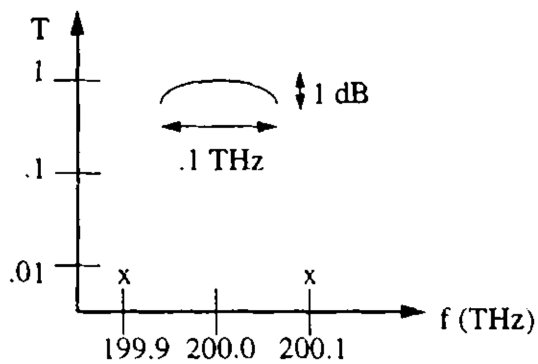


Figure 1.2(b)
Desired transmittance for question 1.(b).

2. **(10 marks). Design a waveguide-based co-directional coupler.**

(a) **(1 mark).** Design a slab waveguide (figure 2.1) which supports exactly two guided TE modes for light of wavelength 1 um. The core index is 1.55 and the cladding index 1.45.

(b) **(1 mark).** Show that these modes are orthogonal.

(c) **(1 mark).** Suppose that you found from (a) that the fundamental guided mode has effective index 1.53 and the higher confined mode has effective index 1.48. What grating period will enable resonant codirectional energy transfer between these modes? Again consider light to be of wavelength 1 um.

(d) **(5 marks).** Specify the remaining features of the device:

- longitudinal grating profile, including layer indices of refraction
- transverse grating profile
- total grating length

such that the device:

- has transferred 50% of λ=1 um power launched in the fundamental mode into the first-order mode by the time it reaches the far end of the device;
- has transferred no more than 0.5% of λ=1.1 um fundamental mode light into the first-order mode by the time it reaches the far end of the device.

(e) **(2 marks).** Depict and explain a co-directional-coupling-based electro-optic device in which:

i) the efficiency of conversion between codirectionally-propagating modes could be tuned while not changing the resonant wavelength;

ii) the efficiency of conversion between codirectionally-propagating modes could be left unchanged while tuning the resonant wavelength of mode conversion.

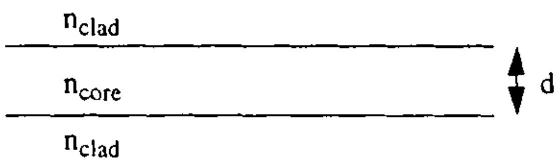


Figure 2.1. Slab waveguide.

3. **(10 marks). Design high- and low-reflectivity coatings for light-emitting devices.**

(a) **(2 marks).** Design an anti-reflection (AR) coating which will allow light from a medium of refractive index 3 (semiconductor) to be coupled with high efficiency into a medium of refractive index 1 (air). Consider light of wavelength 1 um. Specify both the thickness and index of your coating. You may use materials ranging in refractive index from 1.5 to 2.3.

(b) **(1 mark).** Estimate the bandwidth of your anti-reflection coating. You will need to choose and justify a reasonable (possibly somewhat arbitrary) definition of bandwidth in this case.

(c) **(2 marks).** Design a high-reflection (HR) coating which will ensure that less than 1% of optical power incident from a medium of refractive index 3 (semiconductor) is transmitted into a medium of index 1 (air). Consider light of wavelength 1 um. Specify both the thicknesses and indices of the layers which make up your coating. You may use materials ranging in refractive index from 1.5 to 2.3.

(d) **(1 mark).** Estimate the bandwidth of your high-reflection coating.

(e) **(4 marks).** Consider a device whose front mirror consists of your device from (a) and whose back mirror consists of your device from (c). In between the HR and AR coatings lies a bulk semiconductor p-n junction across which current is forward-injected in order to provide gain over the wavelength range 0.95 - 1.05 um:

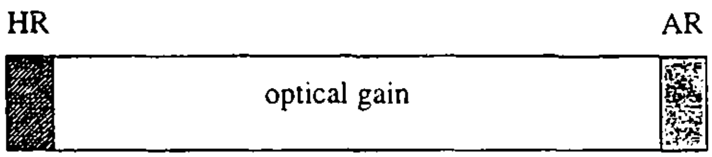


Fig. 3.1. Device with optical gain and HR and AR facet coatings.

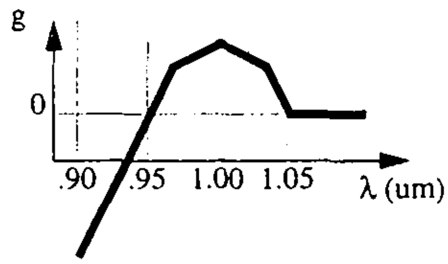


Fig. 3.2. Gain spectrum of the semiconductor material lying between the two mirrors.

Plot, as the forward current through the gain-providing region is increased, the current-dependence of carrier density in the gain-providing region; and of optical power emitted from each of the front and back facets. Plot a spectrum of the light emitted from the device at various currents both small and large. Explain the power spectrum of the device with reference to the reflectance spectra of the facet coatings and, if appropriate, with reference to fulfilment of the condition for zero net round-trip gain. Suggest a reason to use one HR and one AR mirror instead of two equal-reflectivity mirrors with the same reflectance product.

4. **(10 marks). Evaluate a chirped fibre Bragg grating for dispersion pre-compensation.** The spatial period of a grating is chirped continuously from 0.24 um to 0.26 um from position 0 to z_0 according to the relation:

$$\nu(z) = \nu(0)\left(1 - \frac{z}{z_0}\right) + \nu(z_0)\frac{z}{z_0}$$

$$\frac{1}{\Lambda(z)} = \frac{1}{0.24um}\left(1 - \frac{z}{z0}\right) + \frac{1}{0.26um}\frac{z}{z0}$$

The grating length is $z_0 = 10$ cm. The periodic variation in index along the length of the device may be expressed:

$$n(z) = 2 + 1\times10^{-4}\sin\left(2\pi\frac{z}{\Lambda(z)}\right)$$

and is depicted in Fig. 4.1:

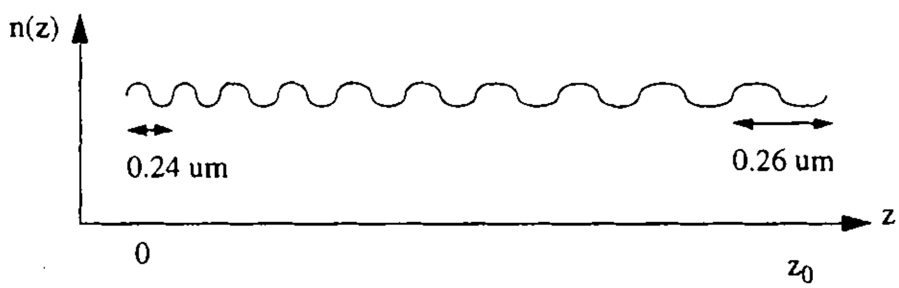


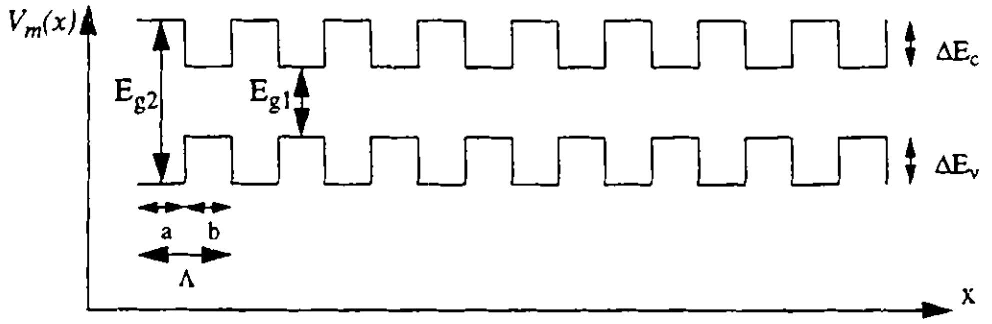
Fig. 4.1. Index profile of chirped grating.

(a) **(4 marks)** Plot the reflectance spectrum for the grating. Provide quantitative results for both the horizontal (wavelength) and vertical (reflectance) values.

(b) **(3 marks)** Plot the group delay -- the delay experienced by the peak of the light pulse -- as a function of wavelength for <u>reflected</u> light. Neglect any material dispersion -- focus purely on the relative delay incurred by different wavelengths as a result of different effective locations of reflection within the chirped grating.

(c) **(3 marks)** A 20 ps-wide transform-limited optical pulse is pre-dispersed by reflecting it off of the chirped grating. It is then launched down 1000 km of optical fibre. What dispersion parameter (sign and magnitude in ps/(nm km)) inside the fibre of length 1000 km will result in complete reversal of the effect of the pre-dispersion?

5. **(10 marks). Determine optical gain spectra of 1-D and 3-D superlattices.** A 1-D superlattice is depicted in the figure. It consists of a periodic stack of layers of alternating material composition and having different bandgaps. Their band offsets are chosen such that electrons and holes are <u>each</u> confined to the layers of bandgap E_{g1}:



The wavefunctions for electrons and holes within this mesoscopic potential may be written:

$$\psi(x) = F(x)u(x) \tag{1}$$

where the envelope function $F(x)$ describes the slowly-varying component arising from the mesoscopic potential $V_m(x)$ (e.g. conduction band edge profile due to different conduction band offsets) and the Bloch function $u(x)$ accounts for the bulk crystal behaviour of the carriers. The envelope function $F(x)$ obeys the Schrodinger-like equation:

$$\frac{\partial^2 F}{\partial x} + \frac{2m^*}{\hbar^2}[E - V_m(x)]F(x) = 0 \tag{2}$$

where m^* is the effective mass of the carrier in question and E the energy eigenvalue.

(a) **(1 mark).** State Floquet's theorem applied to the envelope function in the periodic mesoscopic potential. Substitute into (2) to obtain a dispersion relation (not necessary to simplify) between the electron energy E and envelope Bloch wavevector K.

(b) **(3 marks).** Plot the dispersion relation and explain its key features, including the establishment of superlattice minibands and minigaps. Instead of solving in detail, reason in analogy with the dispersion relation for electromagnetic waves inside an infinitely periodic dielectric stack studied in class.

(c) **(1.5 marks).** Plot (pictorially, not quantitatively) the function $g_{max}(E)$ for the superlattice. Compare with the case of a single quantum well. Explain.

(d) **(1.5 marks).** Repeat (c) for the case of a 3-D superlattice (a 3-D periodic array of different-bandgap materials; thus an array of coupled quantum dots). Compare with the case of a single quantum dot. Explain.

(e) **(3 marks).** Plot the gain spectrum for the 1-D superlattice for three excitation levels:
equilibrium, no quasi-Fermi level separation
quasi-Fermi-level separation = energy difference between electron and hole first miniband energies
quasi-Fermi-level separation = 3kT + energy difference between electron and hole first miniband energies

6. **(10 marks). 2 x 2 Optical waveguide switch with bow-tie electrode using free-carrier plasma effect induced total internal reflection in SiGe alloy.**

In IEEE Photonics Technology Letters (March 2001 issue), Li and Chua describe switching based on tunable total internal reflection as depicted in the figure:

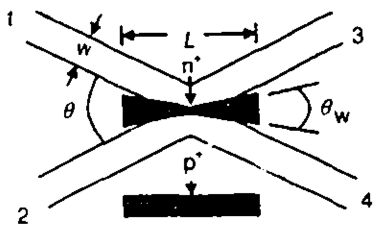


Fig. 1. Schematic diagram of the 2×2 SiGe intersectional rib optical waveguide switch with a bow-tie electrode.

The two waveguides of width w and composed of a p-type SiGe guiding layer grown on a p-Si substrate intersect an angle θ. A bow-tie-shaped electrode with angle θ_w is located at the intersection region of the waveguides. On the top surface of the intersection region, a bow-tie abrupt p-n^+ junction is made to inject the carriers under forward bias into the carrier injection region. Consider 1 um light throughout.

(a) **(3 marks).** The device uses injection of free carriers (excess electrons in the conduction band, excess holes in the valence band) to change the refractive index. This situation is described remarkably well by the simple classical oscillator model which we developed in class, evaluated in the simplifying case in which the optical frequency ω greatly exceeds the resonance frequency of the medium ω_0 (the plasma frequency in this case). If by varying forward injection the electron density in the SiGe may be changed by 10^{16} cm^{-3}, how much index change can be effected? Feel free to neglect factors of order unity such as n and m^*/m_0.

(b) **(1 mark).** You will have found in (a) that the free carrier plasma effect results in a reduction in refractive index. Explain with the help of a diagram how this can be used to effect switching of light input on port 1 into either port 3 or 4 based on total internal reflection.

(c) **(2 marks).** The authors chose $\theta = 2^\circ$ and $\theta_w = 1.5^\circ$. What index change is required in the bowtie region to achieve total internal reflection?

(d) **(4 marks).** Design a directional coupler switch to perform the same function using the same index-changing effect. Compare quantitative results for your directional coupler and the reported total internal reflection device with respect to:

length (the TIR device had length 600 um);

optical bandwidth over which uniform operation is achieved;

vulnerability to optical losses in the index-changing medium.

7. **(10 marks). Total internal reflection and frustrated total internal reflection.** Starting from Maxwell's equations, the constitutive relations, and the definition of the Poynting vector, show that:

(a) For light incident from semi-infinite medium 1 onto semi-infinite medium 2 with $n_1 > n_2$ and at an angle greater than the critical angle, the time-averaged normal component of the Poynting vector in the second medium is zero. Explain the significance of your result.

(b) For light incident from semi-infinite medium 1 onto finite layer 2 and thence onto semi-infinite medium 3 with $n_1=n_3$ and $n_1>n_2$ and at an angle greater than the critical angle for the interface between 1 and 2, the time-averaged normal component of the Poynting vector in the third medium may be non-zero. Explain the significance of your result.

ECE527S - Photonics I

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \quad \nabla \cdot \mathbf{D} = \varrho, \quad \nabla \cdot \mathbf{B} = 0.$$

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mathbf{M},$$

$$\varepsilon_0 = 8.854\times 10^{-12} F/m \qquad \mu_0 = 4\pi\times 10^{-7} H/m$$

$$\int \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int \mathbf{F} \cdot d\mathbf{l} \qquad (1.1\text{-}10)$$

$$\int \nabla \cdot \mathbf{F}\, dV = \int \mathbf{F} \cdot d\mathbf{S} \qquad (1.1\text{-}7)$$

$$B_{2n} - B_{1n} = 0 \quad D_{2n} - D_{1n} = \sigma \qquad \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0, \quad \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K},$$

$$\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \qquad \nabla^2 H - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0 \qquad \psi = A\exp(j(\omega t - \vec{k}\cdot\vec{r}))$$

$$k^2 = \mu\varepsilon\omega^2 \qquad \overrightarrow{v_{phase}} = \frac{\omega}{|\vec{k}|}\hat{k} \qquad v_{phase} = \frac{\omega}{\sqrt{\mu\varepsilon}\,\omega} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad v_{group} = \frac{\partial\omega}{\partial k}$$

$$c_0 \approx 3\times 10^8 (m/s) \qquad H_0 = \frac{kE_0}{\mu\omega} = E_0\left(\frac{\sqrt{\mu\varepsilon\omega^2}}{\mu\omega}\right) = E_0\sqrt{\frac{\varepsilon}{\mu}} \quad H_0 = \frac{E_0}{\eta} \quad \eta = \sqrt{\frac{\mu}{\varepsilon}} \quad \eta_0 = 377\Omega$$

Homogeneous dielectric interface

TE (s) wave

$$r_s = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}, \qquad t_s = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}.$$

TM (p) wave

$$r_p = \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}, \qquad t_p = \frac{2n_1\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}.$$

(if $\mathbf{H}_i$ is taken to be out of the page and $\mathbf{H}_r$ into the page.)

Brewster angle $\quad \theta_B = \operatorname{atan}\left(\frac{n2}{n1}\right)$

Critical angle (TIR) $\quad \theta_c = \operatorname{asin}\left(\frac{n2}{n1}\right)$

Reflectance, Transmittance

$$R_s = \left|\frac{\mathbf{x}\cdot\mathbf{S}'_{1s}}{\mathbf{x}\cdot\mathbf{S}_{1s}}\right|, \quad R_p = \left|\frac{\mathbf{x}\cdot\mathbf{S}'_{1p}}{\mathbf{x}\cdot\mathbf{S}_{1p}}\right| \qquad R_s = |r_s|^2, \qquad T_s = \frac{n_2\cos\theta_2|t_s|^2}{n_1\cos\theta_1}$$

$$T_s = \left|\frac{\mathbf{x}\cdot\mathbf{S}_{2s}}{\mathbf{x}\cdot\mathbf{S}_{1s}}\right|, \quad T_p = \left|\frac{\mathbf{x}\cdot\mathbf{S}_{2p}}{\mathbf{x}\cdot\mathbf{S}_{1p}}\right| \qquad R_p = |r_p|^2, \qquad T_p = \frac{n_2\cos\theta_2|t_p|^2}{n_1\cos\theta_1}$$

Single homogeneous dielectric layer

$$n(x) = \begin{cases} n_1, & x < 0, \\ n_2, & 0 < x < d, \\ n_3 & d < x. \end{cases} \qquad E(x) = \begin{cases} Ae^{-ik_{1x}x} + Be^{ik_{1x}x}, & x < 0, \\ Ce^{-ik_{2x}x} + De^{ik_{2x}x}, & 0 < x < d, \\ Fe^{-ik_{3x}(x-d)}, & d < x, \end{cases}$$

$$A + B = C + D$$
$$k_{1x}(A - B) = k_{2x}(C - D),$$
$$Ce^{-ik_{2x}d} + De^{ik_{2x}d} = F,$$
$$k_{2x}(Ce^{-ik_{2x}d} - De^{ik_{2x}d}) = k_{3x}F.$$

$$F = A\frac{4k_{1x}k_{2x}e^{-ik_{2x}d}}{(k_{1x} + k_{2x})(k_{2x} + k_{3x}) + (k_{1x} - k_{2x})(k_{2x} - k_{3x})e^{-2ik_{2x}d}}$$

$$B = A\frac{(k_{1x} - k_{2x})(k_{2x} + k_{3x}) + (k_{1x} + k_{2x})(k_{2x} - k_{3x})e^{-2ik_{2x}d}}{(k_{1x} + k_{2x})(k_{2x} + k_{3x}) + (k_{1x} - k_{2x})(k_{2x} - k_{3x})e^{-2ik_{2x}d}}$$

$$C = \tfrac{1}{2}F\left(1 + \frac{k_{3x}}{k_{2x}}\right)e^{ik_{2x}d} \qquad D = \tfrac{1}{2}F\left(1 - \frac{k_{3x}}{k_{2x}}\right)e^{-ik_{2x}d},$$

$$r_{12} = \frac{k_{1x} - k_{2x}}{k_{1x} + k_{2x}} \qquad t_{12} = \frac{2k_{1x}}{k_{1x} + k_{2x}}$$

$$r_{23} = \frac{k_{2x} - k_{3x}}{k_{2x} + k_{3x}} \qquad t_{23} = \frac{2k_{2x}}{k_{2x} + k_{3x}}$$

$$r \equiv \frac{B}{A} = \frac{r_{12} + r_{23}e^{-2i\phi}}{1 + r_{12}r_{23}e^{-2i\phi}} \qquad t \equiv \frac{F}{A} = \frac{t_{12}t_{23}e^{-i\phi}}{1 + r_{12}r_{23}e^{-2i\phi}} \qquad \phi = k_{2x}d = \frac{2\pi d}{\lambda}n_2\cos\theta_2$$

Fabry-Perot Interferometer

If R is single interface reflectance $R = r_{12}^2 = r_{23}^2$ then single homogeneous layer transmittance is: $|t|^2 = \frac{(1 - R)^2}{(1 - R)^2 + 4R\sin^2\phi}$

The finesse is the ratio of the transmission peak spacing (free spectral range, or FSR) to FWHM of peaks: $F \equiv \frac{\pi\sqrt{R}}{1 - R}$

Infinite dielectric stack

$$n(x) = \begin{cases} n_2, & 0 < x < b; \\ n_1, & b < x < \Lambda; \end{cases} \qquad n(x) = n(x+\Lambda) \qquad \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \qquad \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-n} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \qquad \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

TE (s) wave

$$A = e^{ik_{1x}a}\left[\cos k_{2x}b + \frac{1}{2}i\left(\frac{k_{2x}}{k_{1x}} + \frac{k_{1x}}{k_{2x}}\right)\sin k_{2x}b\right]$$

$$B = e^{-ik_{1x}a}\left[\frac{1}{2}i\left(\frac{k_{2x}}{k_{1x}} - \frac{k_{1x}}{k_{2x}}\right)\sin k_{2x}b\right]$$

$$C = e^{ik_{1x}a}\left[-\frac{1}{2}i\left(\frac{k_{2x}}{k_{1x}} - \frac{k_{1x}}{k_{2x}}\right)\sin k_{2x}b\right]$$

$$D = e^{-ik_{1x}a}\left[\cos k_{2x}b - \frac{1}{2}i\left(\frac{k_{2x}}{k_{1x}} + \frac{k_{1x}}{k_{2x}}\right)\sin k_{2x}b\right]$$

TM (p) wave

$$A_{TM} = e^{ik_{1x}a}\left[\cos k_{2x}b + \frac{1}{2}i\left(\frac{n_2^2 k_{1x}}{n_1^2 k_{2x}} + \frac{n_1^2 k_{2x}}{n_2^2 k_{1x}}\right)\sin k_{2x}b\right]$$

$$B_{TM} = e^{-ik_{1x}a}\left[\frac{1}{2}i\left(\frac{n_2^2 k_{1x}}{n_1^2 k_{2x}} - \frac{n_1^2 k_{2x}}{n_2^2 k_{1x}}\right)\sin k_{2x}b\right]$$

$$C_{TM} = e^{ik_{1x}a}\left[-\frac{1}{2}i\left(\frac{n_2^2 k_{1x}}{n_1^2 k_{2x}} - \frac{n_1^2 k_{2x}}{n_2^2 k_{1x}}\right)\sin k_{2x}b\right]$$

$$D_{TM} = e^{-ik_{1x}a}\left[\cos k_{2x}b - \frac{1}{2}i\left(\frac{n_2^2 k_{1x}}{n_1^2 k_{2x}} + \frac{n_1^2 k_{2x}}{n_2^2 k_{1x}}\right)\sin k_{2x}b\right]$$

Floquet's theorem: solutions in a medium periodic in x are of form $E_K(x, z) = E_K(x)e^{-i\beta z}e^{-iKx}$ where $E_K(x+\Lambda) = E_K(x)$

Imposing Floquet's theorem results in an eigenvalue equation: $\cos(K\Lambda) = \frac{1}{2}(A + D)$

For normal incidence, the dispersion relation between ω and K reduces to $\cos K\Lambda = \cos k_1 a \cos k_2 b - \frac{1}{2}\left(\frac{n_2}{n_1} + \frac{n_1}{n_2}\right)\sin k_1 a \sin k_2 b$

For the case of a quarter-wave stack, i.e. $k_1 a = k_2 b = \frac{1}{2}\pi$ we found the imaginary component of KΛ at the centre of the bandgap to be $(K_i\Lambda)_{max} = 2\frac{|n_2 - n_1|}{n_2 + n_1} \simeq \frac{\Delta n}{n}$ and estimated the width of the forbidden region to be $\Delta\omega_{gap} = \omega_0 \frac{4}{\pi}\sin^{-1}\frac{|(n_2 - n_1)|}{n_2 + n_1} \simeq \omega_0 \frac{2}{\pi}\frac{\Delta n}{n}$

Simple classical oscillator model of interaction of photons and atoms

Induced dipole moment and definition of atomic polarizability $\mathbf{p} = \alpha\mathbf{E}$

Polarization of a medium with N atoms per unit volume $\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E} = \varepsilon_0\chi\mathbf{E}$

Dielectric constant related to electric susceptibility $\varepsilon = \varepsilon_0(1 + \chi) = \varepsilon_0\left(1 + \frac{N\alpha}{\varepsilon_0}\right)$

Index of refraction $n^2 = 1 + \chi = 1 + \frac{N\alpha}{\varepsilon_0}$

Classical, phenomenological equation of motion of electrons fastened elastically to atoms: $m\frac{d^2}{dt^2}X + m\gamma\frac{d}{dt}X + m\omega_0^2 X = -eE$

Resulting induced dipole moment and atomic polarizability: $p = -eX = \frac{e^2}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}E \qquad \alpha = \frac{e^2}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}$

Refractive index, and approximation for small second term: $n^2 = 1 + \frac{Ne^2}{\varepsilon_0 m(\omega_0^2 - \omega^2 + i\gamma\omega)} \qquad n \cong 1 + \frac{Ne^2}{2\varepsilon_0 m(\omega_0^2 - \omega^2 + i\gamma\omega)}$

Real and imaginary parts of complex refractive index $n \to n - ik = 1 + \frac{Ne^2(\omega_0^2 - \omega^2)}{2\varepsilon_0 m[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]} - i\frac{Ne^2\gamma\omega}{2\varepsilon_0 m[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]}$

Group velocity

A pulse with a known initial wavevector distribution A(k) may be written in the time, space domain as $\psi(z, t) = \int_{-\infty}^{\infty} A(k)e^{i(\omega t - kz)}dk$

If its k and ω distribution is narrow, the dispersion relation may be written to first order as $\omega(k) = \omega_0 + \left(\frac{d\omega}{dk}\right)_0(k - k_0) + \cdots$

so that its time, space distribution is written approximately as $\psi(z, t) \simeq e^{i(\omega_0 t - k_0 z)}\int_{-\infty}^{\infty} A(k)e^{i((d\omega/dk)_0 t - z)(k - k_0)}dk$

or $\psi(z, t) = e^{i(\omega_0 t - k_0 z)}E[z - (d\omega/dk)_0 t] \qquad v_g = \left(\frac{d\omega}{dk}\right)_0 \qquad v_g = \frac{c}{n + \omega(dn/d\omega)} \qquad v_p = \frac{c}{n(\omega)} \qquad k = n(\omega)\frac{\omega}{c}$

Dispersion

Casual/approximate/heuristic definition and estimate of pulse dispersion $\Delta t \sim \frac{\partial T}{\partial\omega}\Delta\omega \sim D_\omega L(\Delta\omega) \sim \frac{\partial T}{\partial\lambda}\Delta\lambda \sim D_\lambda L(\Delta\lambda)$

More rigorous calculation for a Gaussian-distributed pulse. If $\omega(k) \sim \omega_0 + \omega'(k - k_0) + \frac{1}{2}\omega''(k - k_0)^2$ then if the initial wavevector distribution has Gaussian profile $A(k) = A(k_0)\exp(-(k - k_0)^2/(2\sigma_k^2))$ The time-space distribution is found to be

$\psi(z, t) = \exp(j(\omega_0 t - k_0 z))A(k_0)\sqrt{\frac{\pi}{1/(2\sigma_k^2) + j\omega''/2}}\exp(-(z - \omega' t)^2/(2\sigma_z^2))$ where $\sigma_z^2 = \frac{1}{\sigma_k^2} + j\omega'' t$. Focus on real part: $\sigma_{z(Re)}^2 = \sigma_{z0}^2 + (\omega'' t/\sigma_{z0})^2$

Finite (N-layer) dielectric stack

$$r_N = \left(\frac{b_0}{a_0}\right)_{b_N=0} \quad \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^N \begin{pmatrix} a_N \\ b_N \end{pmatrix} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}^N = \begin{pmatrix} AU_{N-1} - U_{N-2} & BU_{N-1} \\ CU_{N-1} & DU_{N-1} - U_{N-2} \end{pmatrix} \quad U_N = \frac{\sin(N+1)K\Lambda}{\sin K\Lambda}$$

$$r_N = \frac{CU_{N-1}}{AU_{N-1} - U_{N-2}} \quad |r_N|^2 = \frac{|C|^2}{|C|^2 + \left(\frac{\sin K\Lambda}{\sin NK\Lambda}\right)^2} \quad |r_N|^2 = \frac{|C|^2}{|C|^2 + \left(\frac{\sinh K_i\Lambda}{\sinh NK_i\Lambda}\right)^2} \quad \cos(K\Lambda) = \frac{1}{2}(A+D)$$

$$K\Lambda = m\pi + iK_i\Lambda$$

Symmetric Slab Waveguide

$$n(x) = \begin{cases} n_2, & |x| < d/2, \\ n_1, & \text{otherwise,} \end{cases} \qquad E_m(x) = \begin{cases} A\sin hx + B\cos hx, & |x| < \frac{1}{2}d, \\ C\exp(-qx), & \frac{1}{2}d < x, \\ D\exp(qx), & x < -\frac{1}{2}d, \end{cases}$$

$$h = \left[\left(\frac{n_2\omega}{c}\right)^2 - \beta^2\right]^{1/2} \qquad q = \left[\beta^2 - \left(\frac{n_1\omega}{c}\right)^2\right]^{1/2}$$

$$\begin{aligned} A\sin(\tfrac{1}{2}hd) + B\cos(\tfrac{1}{2}hd) &= C\exp(-\tfrac{1}{2}qd) \\ hA\cos(\tfrac{1}{2}hd) - hB\sin(\tfrac{1}{2}hd) &= -qC\exp(-\tfrac{1}{2}qd) \\ -A\sin(\tfrac{1}{2}hd) + B\cos(\tfrac{1}{2}hd) &= D\exp(-\tfrac{1}{2}qd) \\ hA\cos(\tfrac{1}{2}hd) + hB\sin(\tfrac{1}{2}hd) &= qD\exp(-\tfrac{1}{2}qd) \end{aligned}$$

$h\tan(\tfrac{1}{2}hd) = q$ even modes

$h\cot(\tfrac{1}{2}hd) = -q$ odd modes

Normalized coordinate system: $u = \frac{1}{2}hd \quad v = \frac{1}{2}qd$

$$u^2 + v^2 = (n_2^2 - n_1^2)\left(\frac{\pi}{\lambda}d\right)^2 = V^2$$

$u\tan u = v$ even modes

$u\cot u = -v$ odd modes

$$n_{eff} = \frac{\beta}{(\omega/c_0)}$$

Coupled Mode Theory

$$\mathbf{E} = \sum A_m(z)\mathbf{E}_m(x,y)\exp[i(\omega t - \beta_m z)] \qquad \frac{d}{dz}A_k(z) = -\frac{i\omega}{4}\frac{\beta_k}{|\beta_k|}\sum_m \langle k|\Delta\varepsilon|m\rangle A_m(z)e^{i(\beta_k-\beta_m)z}$$

$$\iint \mathbf{E}_l \cdot \mathbf{E}_m^* \, da = \frac{2\omega\mu}{|\beta_m|}\delta_{lm} \qquad \varepsilon(x,y,z) = \varepsilon_0(x,y) + \Delta\varepsilon(x,y,z) \qquad \langle k|\Delta\varepsilon|m\rangle = \int \mathbf{E}_k^* \cdot \Delta\varepsilon\, \mathbf{E}_m \, dx\, dy$$

Periodic Perturbation

$$\Delta\varepsilon = \varepsilon_1 e^{-iKz} + \varepsilon_1^* e^{iKz} \quad \text{period } K = \frac{2\pi}{\Lambda} \quad \frac{d}{dz}A_1 = -i\frac{\beta_1}{|\beta_1|}\kappa_{12}A_2e^{i\Delta\beta z} \quad \frac{d}{dz}A_2 = -i\frac{\beta_2}{|\beta_2|}\kappa_{12}^*A_1e^{-i\Delta\beta z}$$

$$\Delta\beta = \beta_1 - \beta_2 - \frac{2\pi}{\Lambda} \qquad \kappa_{12} = \frac{\omega}{4}\iint \mathbf{E}_1^* \cdot \varepsilon_1 \mathbf{E}_2 \, dx\, dy$$

If the periodic perturbation is uniform across the transverse plane, then $\kappa \approx \frac{\Delta}{n}\beta$ if index change Δ is small compared to the average index.

Contradirectional Coupling

$$\frac{d}{dz}A_1 = -i\kappa A_2 e^{i\Delta\beta z} \qquad \text{where } \Delta\beta = \beta_1 - \beta_2 - m\frac{2\pi}{\Lambda}$$

$$\frac{d}{dz}A_2 = i\kappa^* A_1 e^{-i\Delta\beta z}$$

$$\frac{d}{dz}\left(|A_1|^2 - |A_2|^2\right) = 0 \qquad s^2 = \kappa^*\kappa - \left(\frac{\Delta\beta}{2}\right)^2$$

$A_1 = A(0)$ at $z = 0$ and $A_2 = A_2(L)$ at $z = L$.

$$A_1(z) = e^{i(\Delta\beta/2)z}\left\{\frac{s\cosh s(L-z) + i(\Delta\beta/2)\sinh s(L-z)}{s\cosh sL + i(\Delta\beta/2)\sinh sL}A_1(0) + \frac{-i\kappa e^{i(\Delta\beta/2)L}\sinh sz}{s\cosh sL + i(\Delta\beta/2)\sinh sL}A_2(L)\right\}$$

$$A_2(z) = e^{-i(\Delta\beta/2)z}\left\{\frac{-i\kappa^*\sinh s(L-z)}{s\cosh sL + i(\Delta\beta/2)\sinh sL}A_1(0) + e^{i(\Delta\beta/2)L}\frac{s\cosh sz + i(\Delta\beta/2)\sinh sz}{s\cosh sL + i(\Delta\beta/2)\sinh sL}A_2(L)\right\}$$

Codirectional Coupling

$$\frac{d}{dz}A_1 = -i\kappa_{12}A_2e^{i\Delta\beta z}$$

$$\text{where } \Delta\beta = \beta_1 - \beta_2 - l\left(\frac{2\pi}{\Lambda}\right)$$

$$\frac{d}{dz}A_2 = -i\kappa_{12}^*A_1e^{-i\Delta\beta z}$$

$$\frac{d}{dz}\left(|A_1|^2 + |A_2|^2\right) = 0$$

$$A_1(z) = e^{i\frac{1}{2}\Delta\beta z}\left\{\left[\cos sz - i\frac{\Delta\beta}{2s}\sin sz\right]A_1(0) - \left[i\frac{\kappa_{12}}{s}\sin sz\right]A_2(0)\right\}$$

$$A_2(z) = e^{-i\frac{1}{2}\Delta\beta z}\left\{-i\frac{\kappa_{12}^*}{s}\sin sz\, A_1(0) + \left[\cos sz + i\frac{\Delta\beta}{2s}\sin sz\right]A_2(0)\right\}$$

$$s^2 = |\kappa_{12}|^2 + \left(\tfrac{1}{2}\Delta\beta\right)^2$$

The Directional Coupler

$$\frac{dA}{dz} = -i\kappa_{ab}Be^{i2\delta z} \qquad \frac{dB}{dz} = -i\kappa_{ba}Ae^{-i2\delta z}$$

$$2\delta = (\beta_a + \kappa_{aa}) - (\beta_b + \kappa_{bb})$$

$$A_m(z) = A_0e^{i\delta z}\left\{\cos\left[(\kappa^2+\delta^2)^{1/2}z\right] - i\frac{\delta}{(\kappa^2+\delta^2)^{1/2}}\sin\left[(\kappa^2+\delta^2)^{1/2}z\right]\right\}$$

$$B_m(z) = -iA_0e^{-i\delta z}\frac{\kappa}{(\kappa^2+\delta^2)^{1/2}}\sin\left[(\kappa^2+\delta^2)^{1/2}z\right]$$

$$\kappa_{ab} = \tfrac{1}{4}\omega\varepsilon_0\int \mathcal{E}_a^* \cdot \Delta n_a^2(x,y)\mathcal{E}_b \, dx\, dy$$

$$\kappa_{ba} = \tfrac{1}{4}\omega\varepsilon_0\int \mathcal{E}_b^* \cdot \Delta n_b^2(x,y)\mathcal{E}_a \, dx\, dy$$

$$\kappa_{aa} = \tfrac{1}{4}\omega\varepsilon_0\int \mathcal{E}_a^* \cdot \Delta n_b^2(x,y)\mathcal{E}_a \, dx\, dy$$

$$\kappa_{bb} = \tfrac{1}{4}\omega\varepsilon_0\int \mathcal{E}_b^* \cdot \Delta n_a^2(x,y)\mathcal{E}_b \, dx\, dy$$

Laser Rate Equation Model

Carrier rate equation

$$\frac{dN}{dt} = \frac{\eta_i I}{qV} - \frac{N}{\tau} - v_g g N_p$$

Photon rate equation

$$\frac{dN_p}{dt} = \Gamma v_g g N_p - \frac{N_p}{\tau_p}$$

Threshold condition

$$r_1 r_2 \exp(-j2\beta_{Re} L)\exp((\Gamma g_{th} - \alpha_i)L) = 1$$

hence $\Gamma g_{th} = \alpha_i + \frac{1}{L}\ln\left(\frac{1}{R}\right)$ where $R = r_1 r_2$

Output power related to photon density

$$P_{out} = N_p h\nu V_p v_g \alpha_m$$

Output power related to injected and threshold current

$$P_{out} = \eta_i (I - I_{th}) \frac{h\nu}{q} \frac{\alpha_m}{\alpha_m + \alpha_i}$$

Differential quantum efficiency

$$\eta_d = \frac{\alpha_m}{\alpha_m + \alpha_i}\eta_i$$

Small-signal dynamic model

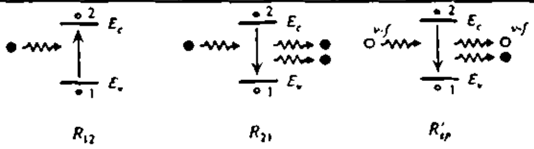
$$I = I_0 + I_1 \exp(j\omega t)$$

$$N = N_0 + N_1 \exp(j\omega t)$$

$$N_p = N_{p0} + N_{p1} \exp(j\omega t)$$

$$\frac{P_1}{I_1} = \eta_d \frac{h\nu}{q} \Big/ \left(1 - \left(\frac{\omega}{\omega_R}\right)^2 + j\frac{\omega}{\omega_R}\left(\frac{1}{\tau\omega_R} + \tau_p \omega_R\right)\right) \quad \text{where} \quad \omega_R^2 = \frac{v_g a N_{p0}}{\tau_p}, \quad \omega_R^2 = \frac{\Gamma v_g a \eta_i}{h\nu V \eta_d} P_0$$

Optical gain inside a semiconductor



Bernard-Duraffourg condition: $E_{Fc} - E_{Fv} \equiv \Delta E_F > E_{21}$

Fermi's Golden Rule: given a time-harmonic perturbation

$H = H_0 + [H'(\mathbf{r})e^{j\omega t} + h.c.]$, the maximum transition rate given all initial states filled and all final states empty is:

$$R_r = \frac{2\pi}{\hbar}|H'_{21}|^2 \rho_f(E_{21})|_{E_{21}=\hbar\omega} \quad \text{where} \quad H'_{21} \equiv \langle\psi_2|H'(\mathbf{r})|\psi_1\rangle = \int_V \psi_2^* H'(\mathbf{r})\psi_1 \, d^3\mathbf{r}$$

The net transition rate, accounting for occupation probabilities, is given by $R_{net} = R_{21} - R_{12} = R_r(f_2 - f_1)$

where $f_1 = \dfrac{1}{e^{(E_1 - E_{Fv})/kT} + 1}$ and $f_2 = \dfrac{1}{e^{(E_2 - E_{Fc})/kT} + 1}$ are the Fermi-Dirac functions describing occupation of states by electrons.

If we account for the influence of the optical field via the vector potential A, related to the electric field by $\mathcal{E} = -\partial \mathbf{A}/\partial t$, then for a time-harmonic perturbation $\mathbf{A}(\mathbf{r}, t) = \hat{\mathbf{e}}\,\mathrm{Re}\{\mathcal{A}(\mathbf{r})e^{j\omega t}\} = \hat{\mathbf{e}}\frac{1}{2}[\mathcal{A}(\mathbf{r})e^{j\omega t} + \mathcal{A}^*(\mathbf{r})e^{-j\omega t}]$, we may write $H'(\mathbf{r}) = \frac{q}{2m_0}\mathcal{A}(\mathbf{r})\hat{\mathbf{e}}\cdot\mathbf{p}$.

If wavefunctions which describe the electronic states inside a perfect bulk crystal satisfy the equation

$$H_0\psi = \left[\frac{\mathbf{p}^2}{2m_0} + V(\mathbf{r})\right]\psi = E\psi,$$

where V(r) is the periodic electronic potential presented by the crystal, then, according to the Floquet theorem, the wavefunctions take the form $\psi = e^{i\mathbf{k}\cdot\mathbf{r}}u(\mathbf{k}, \mathbf{r})$. The terms u are known as Bloch functions: they take on the periodicity of crystal potential.

In the envelope function approximation, the wavefunctions inside a crystal which possesses some additional energetic structure (e.g. a heterostructure) may be written in the form

$$\psi = \int A(k)e^{i\mathbf{k}\cdot\mathbf{r}}u(\mathbf{k}, \mathbf{r})\, d^3\mathbf{k} \approx u(0, \mathbf{r})\int A(k)e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{k} \equiv F(\mathbf{r})u(\mathbf{r})$$

Optical gain inside a semiconductor (cont'd)

For example, the envelope functions inside a bulk crystal take the form of a plane wave $F(\mathbf{r}) = e^{-j\mathbf{k}\cdot\mathbf{r}}/\sqrt{V}$ while the envelope functions inside of a quantum well take the form $F(\mathbf{r}) = F(z)\cdot e^{-j\mathbf{k}\cdot\mathbf{r}}/\sqrt{A}$

These considerations, taken together with some approximations and some knowledge of symmetry properties of states in the conduction and valence bands of a semiconductor, permit us to obtain a simplified explicit expression for the transition matrix element H'_{21} defined above:

$$|H'_{21}|^2 = \left(\frac{q\mathcal{A}_0}{2m_0}\right)^2 |M_T|^2, \qquad \text{where} \qquad |M_T|^2 \equiv |\langle u_c|\hat{\mathbf{e}}\cdot\mathbf{p}|u_v\rangle|^2 |\langle F_2|F_1\rangle|^2$$

Note that inside a bulk material, the inner product of initial and final state envelope functions is evaluated to give $\langle F_2|F_1\rangle = \frac{1}{V}\int_V e^{j\mathbf{k}_2\cdot\mathbf{r}}e^{-j\mathbf{k}_1\cdot\mathbf{r}}\, d^3\mathbf{r} = \delta_{\mathbf{k}_1,\mathbf{k}_2}$ which indicates that momentum is conserved between initial and final states.

(The preceding expression is an approximate one in which photon momentum is neglected. The approximation is jusfied in view of the fact that photon momentum is typically much less than electron momentum.)

In the case of a quantum well, the envelope function inner product evaluates to: $\langle F_2|F_1\rangle = \frac{1}{A}\int_V F_2^*(z)e^{j\mathbf{k}_2\cdot\mathbf{r}_\parallel}F_1(z)e^{-j\mathbf{k}_1\cdot\mathbf{r}_\parallel}\, d^3\mathbf{r}$

which implies conservation of the continuously-varying momentum parallel to the well, and conservation of quantized momentum (hence quantum-confined state number n_c, n_v) perpendicular to the well.

$$= \int_z F_2^*(z)F_1(z)\, dz. \qquad (\text{with } \mathbf{k}_2 = \mathbf{k}_1)$$

To obtain an expression for the reduced density of states ρ_r in the expression for the transition rate, we developed the expression:

$$\frac{1}{\rho_r(E_{21})} = \frac{1}{\rho(k)}\frac{dE_{21}(k)}{dk} = \frac{1}{\rho(k)}\left[\frac{dE_2(k)}{dk} - \frac{dE_1(k)}{dk}\right]$$

which enables us to take k-space densities of states for the individual initial and final states, and obtain a result (known as the reduced density of states) for transition pairs. The result is particularly simple in the case of parabolic bands:

$$E_{21} = E_g + \frac{\hbar^2k^2}{2m_c} + \frac{\hbar^2k^2}{2m_v} = E_g + \frac{\hbar^2k^2}{2m_r}, \qquad \text{where} \qquad \frac{1}{m_r} \equiv \frac{1}{m_c} + \frac{1}{m_v}$$

Dimension	$\rho(k)$	$\rho(E)$
3	$\frac{k^2}{\pi^2}$	$\frac{\sqrt{E}}{2\pi^2}\left[\frac{2m}{\hbar^2}\right]^{3/2}$
2	$\frac{k}{\pi d_z}$	$\frac{m}{\pi\hbar^2 d_z}$
1	$\frac{2}{\pi d_x d_y}$	$\frac{\rho(k)}{\sqrt{E}}\left[\frac{2m}{\hbar^2}\right]^{1/2}$

Thus, in the parabolic bands approximation, we arrive at a simple mapping for the reduced density of states given knowledge of the individual band densities of states:

$$\rho_r(E_{21}) \leftrightarrow \rho_c(E_2),\ \rho_v(E_1)$$
$$E_{21} - E_g \leftrightarrow E_2 - E_c,\ E_v - E_1$$
$$m_r \leftrightarrow m_c,\ m_v$$

We may summarize our results into a final expression for gain:

$$\boxed{g_{21} = g_{max}(E_{21})\cdot(f_2 - f_1) \qquad g_{max}(E_{21}) = \frac{\pi q^2\hbar}{\pi\varepsilon_0 c m_0^2}\frac{1}{h\nu_{21}}|M_T(E_{21})|^2\rho_r(E_{21})}$$

In lower-dimensional structures such as quantum wells, wires, etc., we must include summation over discretized states:

$$g_{21} = \sum_{n_c}\sum_{n_v} g_{21}^{sub}(n_c, n_v)$$

Temperature effects in semiconductor lasers

We approximated the temperature rise associated with localized heating due to power dissipation P_{diss} as: $\Delta T = P_{diss} Z_T$
We noted that for a linear strip heat source of length l and width w on a substrate (thickness h) which is somewhat wider than the substrate thickness w_s, quasi-two-dimensional heat flow results and the thermal impedance is well approximated using the expression $Z_T \approx \frac{\ln(4h/w)}{\pi \xi l}$ where ξ is the thermal conductivity of the substrate. A typical thermal conductivity for GaAs and AlAs is 0.45 W/cm°C, and for an intermediate AlGaAs alloy is 0.11 W/cm°C.
We also noted that the dependence of threshold current on active region temperature was often described using a phenomenological expression of the form $I_{th}(T) = I_{th(Tref)} \exp\left(-\frac{T - T_{ref}}{T_0}\right)$ where T_0 typically lies in the range 50 - 100 K.

Carrier drift and diffusion

If the recombination time in a region of semiconductor is a constant, so that $R = \frac{N}{\tau}$, the carrier density profile may then be written $N(x) = N(x_0)\exp(-(x - x_0)/L)$ where L is known as the diffusion length and is given by $L^2 = D\tau$
D, the diffusivity, is related to mobility via the Einstein condition: $D/\mu = kT/q$

Nonlinear Optics

We derived a general scalar wave equation which does not assume a polarization linearly related to field: $\nabla^2 E - \frac{1}{c_0^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$
We argued that the right-hand side term could be regarded as a source term of an EM wave.
We introduced a standard form the the expansion of P about E=0: $P = \varepsilon_0 \chi E + 2dE^2 + 4\chi^{(3)}E^3 + \ldots$
We obtained our subsequent results by substituting in a time-dependent field of form: $E(t) = E_0\cos(\omega t)$
In a Kerr medium, for example, we found that we could approximate the refractive index as:

$$n(I) = n + n_2 I \text{ where } n_2 = \frac{3\eta_o}{n^2\epsilon_o}\chi^{(3)}$$