University of Toronto
Faculty of Applied Science and Engineering
FINAL EXAMINATIONS -- APRIL 1999

SECOND YEAR -- ENGINEERING SCIENCE
Program 5
AER 202S -- FLUID MECHANICS

Examiner: P.A. Sullivan

Instructions:
(1) Closed book examination, no aids permitted.
(2) The questions are NOT assigned equal marks. The marks for each part are indicated at the beginning of each part.
(3) Attempt as many questions as you can. Parts of questions may be answered
(4) Marks are given for careful reasoning according to the basic principles, with algebraic errors being penalized lightly.
(5) The questions themselves contain formulae useful in other questions.
(6) Bold face quantities represent vectors.

(1a) [10 MARKS] Given two functions $f(x,y)$ and $g(x,y)$ which are continuous over a region R of the (x,y) plane bounded by a sectionally continuous convex closed curve C, show that

$$\oint_C [f(x,y)dx + g(x,y)dy] = \int_R [\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}]dR$$

(1b) [3 MARKS] Describe how this result can be extended to a region bounded by a nonconvex curve.

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(2) [6 MARKS] Given a function $f(x_1, x_2, x_3)$ which is continuous over a region R of space, for a volume $V \in R$ enclosed by a surface, it can be shown that

$$\int_V \frac{\partial f}{\partial x_j} dV = \int_S f n_j dS \quad \text{for} \quad j = 1, 2, 3.$$

where n_j is the j-th component of the unit outward normal $\boldsymbol{n}$ on S. State the two vector field theorems that follow from this expression.

(3a) [8 MARKS] Given that, if

$$F(x) = \int_a^b f(x,y)dy \quad \text{then} \quad \frac{dF}{dx} = \int_a^b \frac{\partial f}{\partial x} dy$$

for a and b constant, show that

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(x,y)dy = \int_{g_1(x)}^{g_2(x)} \frac{\partial f}{\partial x} dy + f(x,g_2(x))\frac{dg_2}{dx} - f(x,g_1(x))\frac{dg_1}{dx}$$

(3b) [8 MARKS] Given the function $F(t)$ defined by

$$F(t) = \int_0^{a(t)} \int_0^{b(t)} f(x,y)\,dy\,dx$$

find dF/dt.

(4a) [7 MARKS] Recall that, for a curve C in the (x,y)-plane enclosing a region R, if it is described parametrically by $x = x(t)$ and $y = y(t)$, the enclosed area A is given by the line integral

$$A = -\frac{1}{2}\oint_C [y(t)\frac{dx}{dt} - x(t)\frac{dy}{dt}]dt$$

Use this result to show that, for a transformation of co-ordinates $x = x(u,v)$ and $y = y(u,v)$,

$$A = \frac{1}{2}\oint_{C^*} [x\frac{\partial y}{\partial u} - y\frac{\partial x}{\partial u}]du + [x\frac{\partial y}{\partial v} - y\frac{\partial x}{\partial v}]dv$$

where C^* is the curve bounding the image of R in the (u,v) plane, namely R^*.

(4b) [9 MARKS] Hence show that

$$A = \iint_{R^*} \frac{\partial(x,y)}{\partial(u,v)} du\,dv$$

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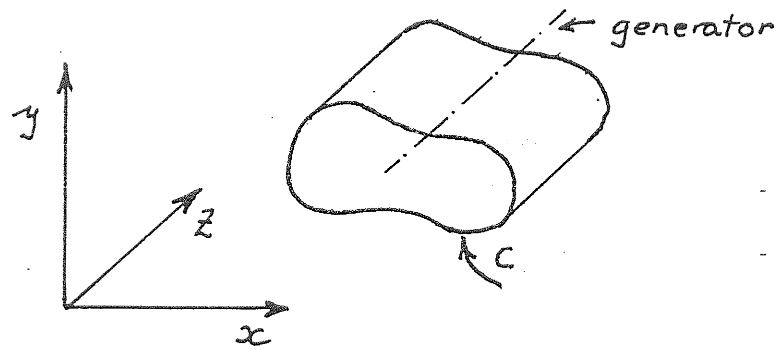
(5) [14 MARKS] For the triangular region R having vertices at $(x,y) = (0,0), (1,0), (0,1)$, evaluate

$$\int_R e^{g(x,y)} dR \quad \text{where} \quad g(x,y) = \frac{y - x}{y + x}$$

Hint: Use the transformation $x + y = u$ and $x - y = v$.

(6a) [6 MARKS] A right cylinder having its generator parallel to the $\boldsymbol{i}_z$ axis intersects the (x,y) plane forming the curve C shown in the diagram below.



It is immersed in a fluid for which the pressure field has the form

$$p(x,y,z) = a + bx + cy$$

If the curve C is described parametrically in terms of the distance s along it, that is $x = x(s)$ and $y = y(s)$, by formulating the surface integral, show that the pressure force $\boldsymbol{F}_p$ per unit length in the $\boldsymbol{i}_z$ direction acting on the cylinder can be reduced to the line integral

$$\boldsymbol{F}_p = - \oint_C [p(x,y)\frac{dy}{ds}\boldsymbol{i}_x - p(x,y)\frac{dx}{ds}\boldsymbol{i}_y]ds$$

(6b) [8 MARKS] If A is the area in the (x,y) plane enclosed by C, by completing the line integral show that this reduces to

$$\boldsymbol{F}_p = - bA\boldsymbol{i}_x - cA\boldsymbol{i}_y$$

(6c) [6 MARKS] Show how this result may also be obtained by use of an appropriate form of Gauss' Theorem.

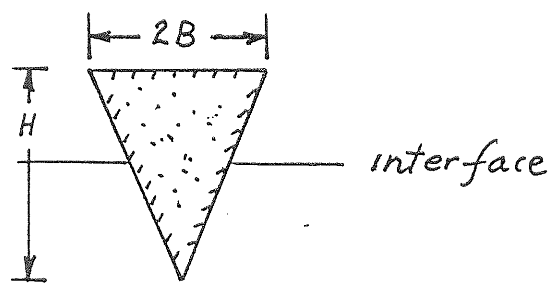
(7a) [6 MARKS] Given a fluid at rest having density and pressure fields $\rho(\boldsymbol{r})$ and $p(\boldsymbol{r})$ respectively, and given a body force field $\boldsymbol{g}(\boldsymbol{r})$, show that a necessary condition of hydrostatic equilibrium is $\nabla p = \rho \boldsymbol{g}$.

(7b) [6 MARKS] Hence show that the resultant pressure force $\boldsymbol{F}_p$ acting on a solid body immersed in a constant density fluid subject to terrestrial gravity is given by

$$\boldsymbol{F}_p = \rho g V \boldsymbol{i}_z$$

where V is the volume of the body, and $\boldsymbol{g} = -g\boldsymbol{i}_z$.

(7c) [6 MARKS] A triangular beam having base width $2B$, height H, and relative density 0.9 floats apex down at the interface of two immiscible liquids having relative densities 1.0 and 0.8 respectively; this is shown in the diagram below. Find the distance which the apex penetrates the lower liquid. Express your result as a fraction of H.



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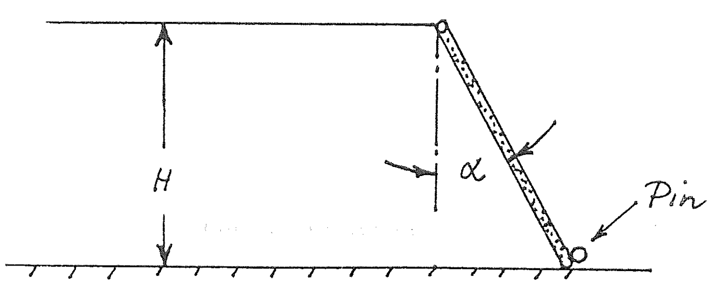
(8) [8 MARKS] A rectangular channel for conveying water has width B and depth H. It has a control gate depicted in the diagram below which is hinged at the top of the channel, inclined at an angle α to the vertical, and retained in place by a pin at the bottom. If water of density ρ can rise to the top of the channel, calculate the magnitude of the force acting on the pin. Ignore the weight of the gate.



(9a) [8 MARKS] Apply Newton's law for the motion of a particle to steady frictionless flow in a fluid filament, and thus obtain an equation for the motion of a particle along the filament. Does this relationship apply to compressible or incompressible flow?

(9b) [2 MARKS] Show how this may be integrated to obtain Bernoulli's Theorem for incompressible flow in the presence of terrestrial gravity; which is

$$\frac{p}{\rho} + gz + \frac{1}{2}q^2 = E_T$$

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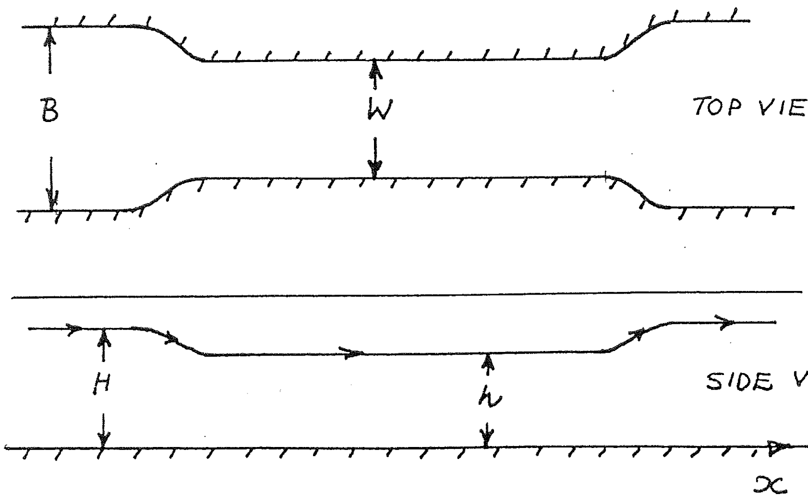
(10) [12 MARKS] Water flows in a horizontal rectangular channel of width B, and then passes through a measuring station in the form of a constriction which reduces the channel width to W as shown in the diagrams below. The height of the water level changes from H to h. By assuming that the flow is frictionless and steady and that, in both the upstream channel and the constriction, the streamlines are straight and parallel to the x-axis shown in the diagram below, find an expression for the volume flux Q through the channel as a function of B, W, H and h.

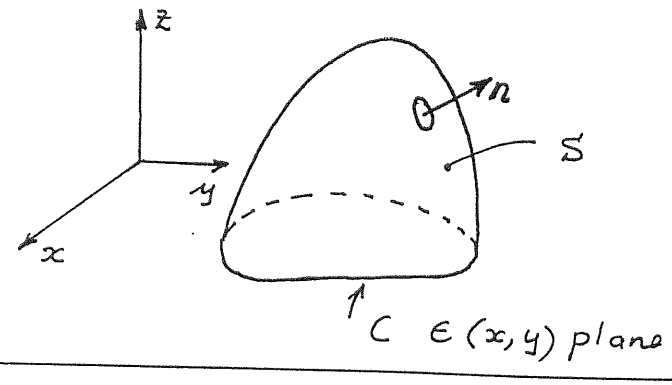


(11a) [4 MARKS] A fluid velocity field $\boldsymbol{v}(\boldsymbol{r})$ rotates about the $\boldsymbol{i}_z$ axis with angular speed Ω as if it were a solid body. Show that the velocity field takes the form

$$v(r,t) = -\Omega y i_x + \Omega x i_y$$

(11b) [8 MARKS] By carrying out the surface integral, for any surface S bounded by a curve C lying in the (x,y) plane as depicted in the diagram below. Show that, if A is the area of the planar region bounded by C,

$$\int_S (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{n} dS = 2\Omega A$$



(12a) [4 MARKS] Given a property of a fluid ϕ per unit mass, the Reynolds Transport Theorem gives the rate of change of the total amount of that property Φ for a moving body of fluid of mass M contained in a moving volume V_M:

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_{V_M} \phi \rho dV \doteq \frac{D}{Dt}\int_{V_c} \phi \rho dV = \int_{V_C} \frac{\partial}{\partial t}(\phi\rho) dV + \int_{S_C} \phi \rho \boldsymbol{v} \cdot \boldsymbol{n} dS$$

where $\rho\,(\boldsymbol{r},t)$ and $\boldsymbol{v}\,(\boldsymbol{r},t)$ are, respectively, the density and velocity fields. Explain the physical meaning of the two terms on the right-hand side of the above expression.

(12b) [7 MARKS] Use it to derive the differential form of the equation of continuity, which is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

(12c) [7 MARKS] Show that, for steady flow in a fluid filament, the Reynolds Transport Theorem reduces to

$$\frac{d\Phi}{dt} = \dot{m}_f(\phi_2 - \phi_1)$$

and explain the meaning of the terms on the right hand side of this expression.

(13a) [6 MARKS] A one-dimensional unsteady flow field, such as that induced by a piston in a cylinder, has the functional form

$$\boldsymbol{v}(\boldsymbol{r},t) = u(x,t)\boldsymbol{i}_x$$

where the spatial co-ordinate x is aligned along the cylinder axis. Show that the acceleration a_x of a fluid particle is given by

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

(13b) [9 MARKS] By evaluating the fluxes out of a control volume V_C consisting of a section of the cylinder of length Δx, show directly that the equation of continuity takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

(14a) [8 MARKS] For surface waves propagating in a horizontal channel which are long compared with the depth H of the undisturbed water, vertical speeds and accelerations are very small, so that the velocity field has the approximate form

$$\boldsymbol{v}(\boldsymbol{r},t) \approx u(x,t)\boldsymbol{i}_x$$

where x is the distance along the channel. Show that the equation of motion takes the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

(14b) [8 MARKS] If $h = h(x,t)$ is the height of the water surface above the floor of the channel, <u>given</u> that the equation of continuity takes the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

Show that the speed c of propagation of small disturbances is given by $c = \sqrt{(gH)}$.

(15a) [8 MARKS] For a moving body of fluid contained within a volume $V_M(t)$ which coincides at some instant with a fixed control volume V_C, show that the rate dW_p/dt at which pressure forces acting at the boundary S of V_C do work on this body of fluid is

$$\frac{dW_p}{dt} = - \int_S p\boldsymbol{v}\cdot\boldsymbol{n}\,dS$$

where $\boldsymbol{n}$ is the unit outward normal on S.

(15b) [6 MARKS] Show that, in the case of incompressible flow, this reduces to

$$\frac{dW_p}{dt} = - \int_{V_C} \boldsymbol{v}\cdot\nabla p\,dV$$

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(16) [12 MARKS] For compressible inviscid adiabatic steady flow, use the principle of conservation of energy in the presence of terrestrial gravity to show that, with h being the *enthalpy* of the fluid, an equation having the form

$$h + gz + \frac{1}{2}q^2 = constant,$$

applies along streamlines.
