MAT 1955, 4/98 Finals Smith Sullivan

(a) Evaluate $\int_0^{\frac{1}{2}\pi} \sin^3 x \cos^3 x \, dx$.

(b) Evaluate $\int_0^1 (x + x^2)^{\frac{1}{2}} \, dx$.

(c) Integrate $\int \frac{dx}{x^{\frac{1}{3}} + x^{\frac{2}{3}}}$

(d) Integrate $\int \frac{dx}{e^x + 1}$

(e) Does the improper integral $\int_{-\infty}^{\infty} \frac{dx}{e^{x^2} + 1}$ exist?

Given a constant $\varphi > 1$, prove that the sequence $a_n = (\varphi)^{\frac{1}{n}} \to 1$ as $n \to \infty$.

(b) Calculate $\lim\limits_{n\to\infty} \left\{ n^2 \ln(1 + \frac{1}{n}) - n \right\}$

(c) Calculate $\lim\limits_{x\to 1} \left\{ \frac{1}{\ln x} - \frac{x}{x-1} \right\}$

(d) Use Taylor series expansions to calculate $\lim\limits_{x\to 0} \frac{\sin(x^2) - x^2}{(\sin x - x)^2}$

(a) Is the series $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ convergent ?

(b) For what values of x is the series $\sum_{k=1}^{\infty} \frac{x^k}{1+\frac{1}{2}+\cdots+\frac{1}{k}}$ convergent?

(c) Sum the series $\sum_{k=1}^{\infty} \frac{x^k}{k(k+2)}$ within its domain of convergence.

(d) Find the Taylor series for $x^{-\frac{1}{2}}$ about $x = 1$.

(a) Given a function $f(x)$ which satisfies $f'(x) = f(x) + x$, $f(0) = 3$, obtain a representation for $f(x)$ as the series $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Hence show that $f(x) = 4e^x - x - 1$.

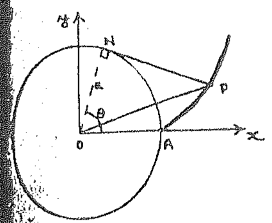
(b) Given the sequence of functions $f_n(x) = nxe^{-nx^2}$ for $n = 1, 2, 3, \ldots$ is it true that

$$\lim_{n \to \infty} \left\{ \int_0^1 f_n(x)dx \right\} = \int_0^1 \left\{ \lim_{n \to \infty} f_n(x) \right\} dx?$$

5. A thin string is wrapped around the circle $x^2 + y^2 = a^2$; one end is initiall at $A(a, 0)$, and then gradually unwound—always under tension.

[7] (a) When the length $a\theta$ of the string is unwound, show that the end P is given by

$$x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta).$$



[3] (b) Show that the length of the arc AP is $\frac{1}{2}a\theta^2$.

6. (a) If $f(x, y) = xF(x+y) + yG(x+y)$ for given functions F and G, show that $f_{xx} - 2f_{xy} + f_{yy} = 0$.

(b) If $f(x, y)$ satisfies $ff_{xy} = f_x f_y$, show that $f(x, y)$ must be given by $f(x, y) = F(x)G(y)$ for arbitrary F and G .

Let $r = f(\theta)$ be the equation of a curve in polar co-ordinates. Show that the curvature K is given by

$$K = \frac{|\, 2f'^2 - ff'' + f^2 \,|}{(f'^2 + f^2)^{\frac{3}{2}}}.$$

8. Find the tangent plane to the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ ($a > 0$ is constant) at the point (x_0, y_0, z_0). If this plane intercepts the axes at $(X, 0, 0)$, $(0, Y, 0)$ and $(0, 0, Z)$, show that $X^2 + Y^2 + Z^2$ is constant.

9. Find the dimensions of the open box (i.e. no lid) with given surface area S that contains the largest volume.

10. Three resistors which have resistances r_1, r_2, r_3 are connected in parallel so that the resistance of the circuit R is given by $R^{-1} = r_1^{-1} + r_2^{-1} + r_3^{-1}$. If the individual resistances are each subject to a small percentage error ϵ, find the corresponding percentage error in R.