Third Year - Programs 5ce, 5e, 5bm
EXAMINER - W.M. Wonham

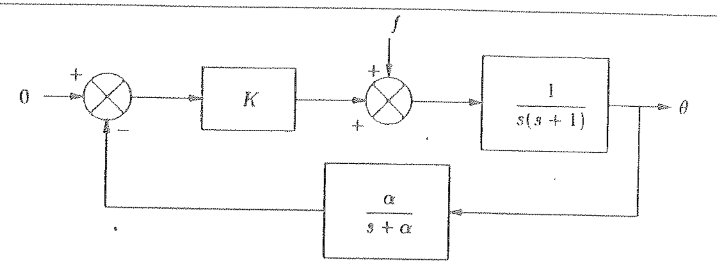
ECE 355F '97 FINAL

S+S CA

#1: 80 each

No aids permitted, other than a calculator.
PLEASE ANSWER EACH OF THE THREE MAIN QUESTIONS IN A SEPARATE BOOKLET.
Marking scheme: Each of the three main questions is worth 1/3 of the total mark.

1.



The system with block diagram displayed represents a regulator whose purpose is to hold the output angle variable θ approximately constant at the 'setpoint' value of 0, in the face of 'disturbances' represented by the input signal f. The parameters α, K are positive real numbers.

1.1 Find the transfer function $\hat{h}(s) := \hat{\theta}(s)/\hat{f}(s)$ (on the assumption that all internal initial values of the system are 0).

1.2 Find the range of K such that the system is BIBO stable.

1.3 Write $\hat{k}(s)$ for the limiting form of $\hat{h}(s)$ as the angle sensor (represented by the block in the feedback return path) becomes 'perfect'. What can be said about the stability of $\hat{k}(s)$?

1.4 Suppose f satisfies the differential equation and initial conditions

$$d^2 f(t)/dt^2 = 0, \quad t > 0; \qquad f(0) = f_o, \quad f'(0) = f_1$$

Find the Laplace transform $\hat{f}(s)$.

1.5 With f as in 1.4 what conditions must hold on the parameters K, α, f_o and f_1 to guarantee that $\lim \theta(t)$ $(t \to \infty)$ exists and is finite? In that case, evaluate the limit.

1.6 With $f \equiv 0$, find the critical frequency at which the system may oscillate spontaneously when it is just on the boundary between stability and instability.

1.7 For what range of K does the system have a steady-state frequency response? For such K, find the approximate amplitude and phase of the frequency response for very high frequencies ω.

1.8 Assume $\alpha = 2$, $K = 30$, and that $f \equiv 0$. Verify carefully that $\theta(t)$ may oscillate with an amplitude that grows like e^t as $t \to \infty$, and calculate the frequency of this oscillation.

2. Let $f(t)$, $-\infty < t < \infty$, be a real-valued signal with L_1 and L_2 norms both finite. Consider a linear time-invariant filter (called $\mathbb{H}$, say) with impulse response h, given by

$$h(u) = f(-u + \tau), \qquad -\infty < u < \infty$$

where τ is a fixed real number.

2.1 When the input to $\mathbb{H}$ is f, let the output signal be $g(t)$, $-\infty < t < \infty$. Calculate the Fourier transform $\hat{g}(\omega)$ in terms of the Fourier transform $\hat{f}(\omega)$.

2.2 With g as in 2.1, use the Fourier integral representation of g to calculate the specific output value $g(\tau)$, in terms of the energy of f.

2.3 With g as in 2.1, use 2.2 to show that $g(t)$ is maximized with respect to t $(-\infty < t < \infty)$ when $t = \tau$.

2.4 Under what condition on f is the filter $\mathbb{H}$ causal?

2.5 Specifically let

$$f(t) = \begin{cases} \sin t, & 0 \le t \le \pi \\ 0, & \text{otherwise} \end{cases}$$

and let $\tau = 2\pi$. Carefully define h and sketch the graphs of f and h.

2.6 From the result of 2.5, evaluate $g(\pi)$, $g(2\pi)$ and $g(3\pi)$. Sketch the graph of $g(t)$ for $0 \le t \le 4\pi$. **Hint:** It's easy and fun if you draw the pictures!

2.7 Consider the general case when the input to $\mathbb{H}$ is a real-valued signal $r(t)$, $-\infty < t < \infty$, with energy $\|r\|^2 = 1$. Denote the output signal by s. Use Schwarz to show that $s(\tau)$ is a positive maximum with respect to all such r when $r(t) = f(t)/\|f\|$ for all t, $-\infty < t < \infty$. Here $\|\cdot\|$ denotes the L_2 norm.

Let $\Phi = \{\phi_n | n = 1, 2, \ldots\}$ be an orthonormal system of functions in $L_2(I)$ for some subinterval I of the real line (so $\phi_n : I \to \mathbb{C}$). Let $f : I \to \mathbb{C}$, $f \in L_2(I)$.

3.1 Define the generalized Fourier series of f with respect to the system Φ.

3.2 If f_N is the Nth partial sum of the Fourier series of f, and $e_N := f - f_N$, calculate $\|f_N\|^2$ in terms of the Fourier coefficients of f, and show that e_N and f_N are orthogonal.

3.3 State Parseval's formula for f.

3.4 Define the meaning of the statement

$$\lim_{N \to \infty} f_N = f \quad \text{(m.s.)}$$

3.5 Show that the above statement is true only if Parseval's formula holds for f.

The system Φ is said to be **complete** if, whenever $f \in L_2(I)$ and $(f, \phi_n) = 0$ for all n, then necessarily $f = 0$ a.e.

3.6 Briefly explain in words what completeness means, and exhibit a system Φ that is not complete.

3.7 Show that if Parseval's formula holds for all $f \in L_2(I)$ then Φ is complete.