University of Toronto
Faculty of Applied Science and Engineering
FINAL EXAMINATIONS -- APRIL 2001

SECOND YEAR -- ENGINEERING SCIENCE
Program 5
AER 202H1 S -- FLUID MECHANICS

Examiner: P.A. Sullivan

Instructions:
(1) Closed book examination; except for a non-programmable calculator, no aids are permitted.
(2) Write your name and student number in the boxes below
(3) The questions are NOT assigned equal marks.
(4) Attempt as many questions as you can. Parts of questions may be answered.
(5) Marks are given for careful reasoning according to the basic principles, with algebraic errors being penalized lightly.
(6) The questions themselves contain formulae useful in other questions.
(7) Bold face quantities represent vectors.
(8) Use the overleaf side of the pages for additional or preliminary work
(9) Extra pages are provided at the end.

NAME	
STUDENT NUMBER	

Question	Marks	Earned
1	14	
2	6	
3	10	
4	14	
5	12	
6	14	
7	16	
8	10	
9	12	
10	14	
11	12	
12	14	
13	12	
14	14	
Total	174	

1) [14 MARKS] (a) The mean value theorem of the integral calculus states that, given a function $f(x)$ continuous for $x \in [a, b]$, there is at least one value of $x \in [a, b]$, x^* say, such that

$$\int_a^b f(x)dx = f(x^*)(b - a)$$

Contrast this with the differential approximation, explaining when the latter is valid.

(b) Given a surface S defined by the function $f(x, y, z) = 0$, explain why the quantity $\nabla f / |\nabla f|$ is a unit normal to the surface.

(c) State the Hydrostatic Postulate and explain, in words, what it means.

(d) The speed c of propagation of sound waves is derived as $c = \sqrt{(dp/d\rho)}$ where the pressure p is a function of the density ρ. Is this formula dimensionally homogeneous?

2) [6 MARKS] The following theorem converts certain volume integrals to surface integrals as follows: given a function $f(x, y, z)$ which is continuous over a volume V enclosed by a surface S, if $\boldsymbol{n} = n_1 \boldsymbol{i}_1 + n_2 \boldsymbol{i}_2 + n_3 \boldsymbol{i}_3$ is the unit normal on S pointing to the exterior of V, then

$$\int_V \frac{\partial f}{\partial x_j} dV = \int_S f n_j dS \quad \text{for } j = 1,\ 2, \text{ and } 3$$

Derive the two results fundamental to the development of subjects such as fluid mechanics and electrodynamics which follow from this theorem.

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3) [10 MARKS] For two dimensional incompressible inviscid flow with $\boldsymbol{v}(\boldsymbol{r}, t) = u(x,y,t)\boldsymbol{i}_x + v(x,y,t)\boldsymbol{i}_y$, the equation of motion $-\nabla p/\rho + \boldsymbol{g} = \boldsymbol{a}$ can be reduced to

$$\frac{D}{Dt}\left[(\nabla \times \boldsymbol{v})\cdot \boldsymbol{i}_z\right] = \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \quad \text{where} \quad \frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y}$$

(i) Show that the operation D/Dt is the rate of change with time of a property of a fluid particle for this flow.

(ii) Express the term in the [ ] brackets in terms of scalar quantities.

(iii) For terrestrial gravity, with $\boldsymbol{g} = -g\boldsymbol{i}_y$, state, in words what the equation asserts.

4) [14 MARKS] Given a surface S described parametrically by

$$\boldsymbol{r}(u,v) = x(u,v)\boldsymbol{i}_x + y(u,v)\boldsymbol{i}_y + z(u,v)\boldsymbol{i}_z$$

show that the area δS of a surface element generated by increments δu in u and δv in v is

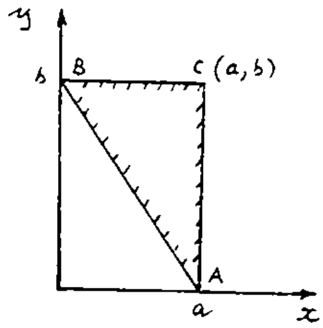
$$\delta S = \left| \det \begin{bmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} \right| \delta u \, \delta v$$

Hence show that, for the paraboloid of revolution $z = x^2 + y^2$, $0 \le z \le H$,

$$\delta S = \sqrt{(4r^2 + 1)}\, r \,\delta r\, \delta\theta$$

where (r, θ) are polar coordinates in the (x, y) plane. Finally, find the surface area of the paraboloid.

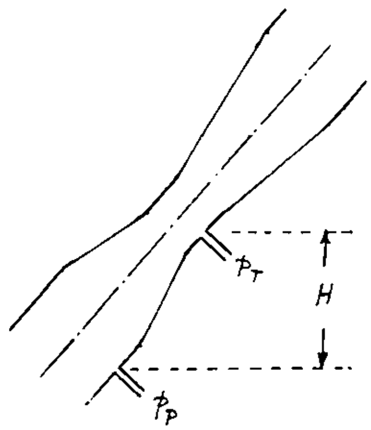
5) [12 MARKS] One result fundamental to the development of the equations of continuum fluid mechanics is the expression for the pressure force δF_p acting on a fluid particle, $\delta F_p = -\nabla p \delta V$. Verify this by determining the pressure forces acting in the x-direction on the surfaces of the triangular volume ABC depicted to the right for which the local pressure field may be taken as $p = p_0 + Cx + Dy$. The volume has unit length in the z-direction.



6) [14 MARKS] The paraboloid of revolution $z = x^2 + y^2$ forms an open vessel which is completely filled with a liquid of density ρ to a height $z = H$. Assuming that this liquid is acted on by terrestrial gravity $g = -g\boldsymbol{i}$, and is subject to atmospheric pressure P_a at the top, by direct integration of the pressure distribution, show that the resultant force acting vertically on the vessel interior is $F_{pz} = -\pi P_a H - (\pi/2)\rho g H^2$. Explain the contributions of the two terms.

7) [16 MARKS] For gas bubbles of mean radius D_0 undergoing spherically symmetric oscillations in a liquid of density ρ_L subjected to pressure P_0 the oscillation frequency f (= $1/T_P$, the period) can be expected to have the following functional dependence: $f = f(\rho_L, D_0, P_A, \gamma, \mu, g)$, where γ is the specific heat ratio, μ is the viscosity, and g is the acceleration due to gravity. Perform a dimensional analysis of the problem using ρ_L, D_0 and P_0 as reference quantities. If the liquid is water at room temperature and subject to atmospheric pressure $P_A = 10^5$ N/m^2, for which $\rho_L = 1000$ kg/m^3 and $\mu = 10^{-3}$ N·s/m^2, and if $g = 9.81$ m/s^2, what would you infer about the importance of viscous and gravitational effects for $D_0 = 4.00$ mm? If measurements for a bubble with $D_0 = 4.0$ mm give $f = 1600$ Hz, what would be the expected frequency for the same size bubble subjected to $P_0 = 5\times10^4$ N/m^2 in mercury, for which $\rho_L = 13{,}550$ kg/m^3?

8) [10 MARKS] A venturi flow meter is installed in an inclined pipe of cross-sectional area A_P as depicted in the diagram to the right. The throat has cross-sectional area A_T. If the throat pressure tap, which measures pressure p_T, is at a height H above the pipe tap, which measures pressure p_P, by assuming that the flow is incompressible, frictionless and may be treated as a filament, find an expression for the volume flux Q. For what value of A_T/A_P does the formula for $A_T/A_P = 0$ produce an error of one percent?



9) [12 MARKS] For incompressible steady flow, find the constant a such that the velocity field

$$\boldsymbol{v}(\boldsymbol{r},t) = 2x\boldsymbol{i}_x + ay\boldsymbol{i}_y + (3z + 2)\boldsymbol{i}_z$$

satisfies the equation of continuity. Show that, for the right circular cylinder $x^2 + y^2 = R^2$, with $0 \leq z \leq H$ this implies

$$Q = \int_S \boldsymbol{v} \cdot \boldsymbol{n} dS = 0$$

10) [14 MARKS] For steady inviscid adiabatic flow along a streamline in the presence of terrestrial gravity, the equation of conservation of energy takes the form

$$e(p,\rho) + \frac{p}{\rho} + \frac{1}{2}q + gz = constant$$

where e is the inernal enegy, and the other symbols have their usual meaning. Starting from the theorem stated in Question 5 derive the equation of motion for this flow, and use it to show that $de/ds - (p/\rho^2)\,d\rho/ds = 0$. What can you conclude about the relationship between p and ρ in the event that the gas is perfect with constant specific heats, so that $p = \rho RT$ and $e = C_v T$?

11) [12 MARKS] Unsteady frictionless incompressible flow in an open rectangular horizontal channel is assumed to have a velocity field of the form $\mathbf{v}(\mathbf{r}, t) = u(x, t)\mathbf{i}_x + v(x, y, t)\mathbf{i}_y$ with $v \ll u$ as in Russel's solitary waves. The water surface profile is given by $y_{surface} = h(x, t)$. By considering the motion of a fluid particle in the surface, show that

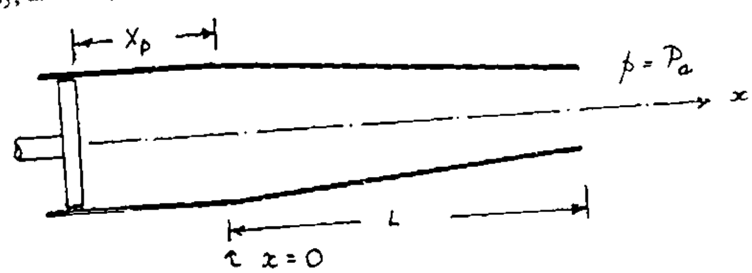
$$v|_{h(x,t)} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}$$

Hence or otherwise show that the equation of continuity for this flow is

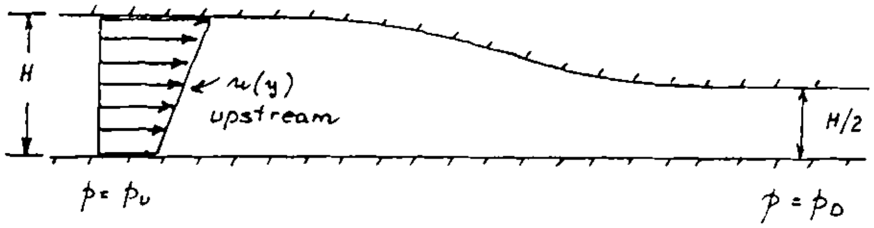
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

12) [14 MARKS] Derive an equation of motion for the flow in Question 11 under the assumption that the vertical acceleration $a_y \ll g$ and show that, for small amplitude waves propagating on water at rest and having a mean depth H, the speed c of propagation is given by $c = (gH)^{½}$

13) [12 MARKS] A piston in a cylinder having cross-sectional area A moves with time t according to the law $x = X_p(t) = -D + Kt^2$, where the origin of coordinates is depicted in the diagram below. At $x = 0$, the cylinder is connected to a slowly converging nozzle having cross-sectional area $a(x) = A\{1 - x/(2L)\}$. The assembly is filled with an incompressible frictionless fluid which is expelled to atmosphere at pressure P_a. Assuming that the flow is one-dimensional and unsteady, that is $v(r, t) = u(x, t)i_x$, find the pressure at the nozzle entrance as a function of t.



14) [14 MARKS] A channel of width H is connected to a contraction which reduces the width to $H/2$ as depicted in the diagram below. The incompressible fluid approaching the contraction has the velocity distribution $\mathbf{v}(\mathbf{r}, t) = u(y)\mathbf{i}_x$, where u increases linearly from U_B at $y = 0$ to U_T at $y = H$. By assuming that the flow is frictionless and steady and that, downstream of the contraction, the streamlines are parallel to $\mathbf{i}_x$, find the pressure decrease $p_U - p_D$ through the contraction.



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