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Fundamental Study

Petri nets and algebraic specifications*

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Abstract

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Petri nets gain a great deal of modelling power by representing dynamically changing items as structured tokens (instead of "black dots"). Algebraic specifications turned out adequate for dealing with structured items. We will use this formalism to construct Petri nets with structured tokens. Place- and transition-invariants are useful analysis techniques for conventional Petri nets. We derive corresponding formalisms for nets with structured tokens, based on term substitution.

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Introduction

Conventional Petri nets (place/transition nets) use "black dots" (*tokens*) for modelling dynamically changing items. A system state is then determined by the number of tokens on each place of a net. Other Petri net models use structured tokens. Data structures and any kind of structured items can be represented in this way. Their dynamic change is described by particular rules of change. The term *high level Petri net* usually refers to (the several versions of) such net models.

It is almost obvious and confirmed by experience that, for practical applications, high level Petri nets are much more useful than ordinary Petri nets. High level nets support the construction of concise, but nevertheless comprehensible and transparent models of real-world systems.

These advantages must be payed for by a more involved formalism. A formalism is needed coping with data structures and particularly with multisets over any kind of domains because multisets are a fundamental feature for high level Petri nets. This formalism furthermore should include a proper handling of *terms* (used as net inscriptions) and should clarify the step from nets and terms as syntactical objects to their interpretation (meaning) in concrete domains. Finally, the formalism should support analysis techniques for high level Petri nets, particularly place- and transition-invariants.

Algebraic specifications are a promising candidate for this purpose: They turned out to be an adequate and flexible instrument for handling structured items. Multisets over any domain can be specified by additional sorts, operations and equations. The concept of initial algebra semantics, relating (syntactical) terms to (semantical) interpretations, will appear to be directly applicable to high level Petri nets. Last but not least, the calculi of place- and transition-invariants can be based on term substitution.

The central concern of this paper is not the presentation of entirely new results. It is rather intended to present and to integrate known ideas in a—possibly adequate—new setting. A couple of concepts to cope with structured tokens, developed in various papers more or less completely from the scratch, will turn out to be representable by well established concepts of algebraic specifications.

After an introductory example in Section 1, in Section 2, we recall the—elementary—fundamentals of algebraic specifications to the extent needed in this paper. The particular case of abstractly specifying multisets, in fact a well-known standard construction, is discussed in Section 3. On this basis it is a simple step to define nets with structured tokens in Section 4: Markings and arc inscriptions are given by multiset ground terms and arbitrary multiset terms, respectively. Occurrences of transitions can then easily be formulated (depending on the equations of the underlying specification). Section 5 then deals with "place invariants" as a fundamental analysis tool. This technique is based on solutions of linear equations in the domain of multiset terms. Term substitution serves as product in this formalism. The central invariant theorem (5.6) is based on a well-known lemma of general algebra. In Section 6, we discuss some examples to derive system properties, using the technique of place invariants. An analysis technique dual to "place invariants" are the "transition invariants" introduced and investigated in Section 7. Systematic transformations of nets with structured tokens are considered in Sections 8 and 9. Some useful extensions of the formalism are discussed in Section 10. The conclusion finally relates the formalism to predicate/transition nets, coloured nets, and algebraic specifications.

1. An introductory example

As an example to explain the essential ideas of this paper, we consider a basic version of the well-known system of dining philosophers [6]. It is based on an algebra consisting of a set $P = \{p_0, \ldots, p_4\}$ of philosophers and a set $G = \{g_0, \ldots, g_4\}$ of forks. There are two operations *lf* and *rf*, assigning to each philosopher p_i his left and right forks $lf(p_i) = g_i$ and $rf(p_i) = g_{i+1}$ $(0 \leq i \leq 4,\ g_5 = g_0)$, respectively. The corresponding term algebra, generated by a set X of variables, includes terms such as $RF(p_1)$ and $LF(x)$. The embedding of this algebra into a multiset environment yields terms such as $RF(x) + LF(x)$ which occur as arc inscriptions in Fig. 1.

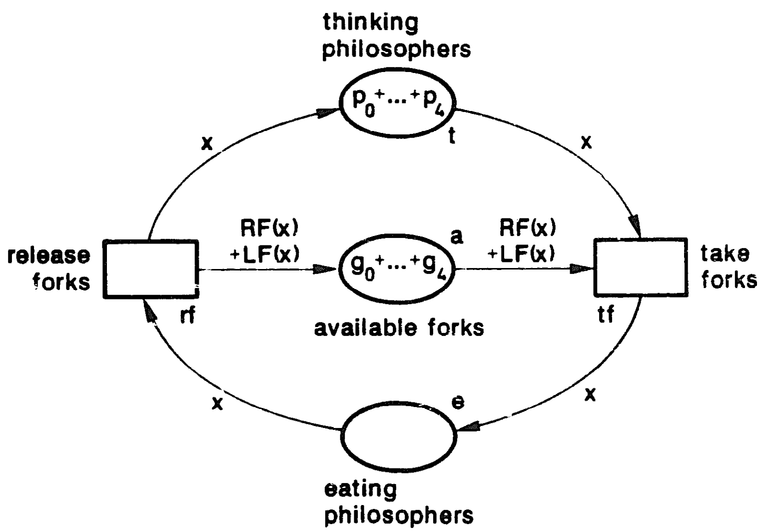

Fig. 1. The system of dining philosophers in its initial situation.

This figure shows a net which specifies the dynamic aspects of the philosophers system. It consists of three *places* (drawn as ellipses) called "thinking philosophers" (**t**), "available forks" (**a**) and "eating philosophers" (**e**), respectively. They can be inscribed by terms, representing objects such as philosophers or forks. Figure 1 shows the initial situation where all philosophers are thinking and all forks are available. The *transitions* (drawn as rectangles) "take forks" (**tf**) and "release forks" (**rf**) with their surrounding arcs and the arc inscriptions indicate the dynamics of the system.

System dynamics is based on the *occurrence of transitions in certain modes.* In Fig. 1, a mode is given by a substitution β of a philosopher p_i for the variable x, that is, $\beta(x)=p_0$. The arc inscription $RF(x)+LF(x)$ in this mode yields the term $RF(p_0)+LF(p_0)$. Assuming the equations $g_0=LF(p_0)$ and $g_1=RF(p_0)$, this term represents the set $\{g_0, g_1\}$, viz. the set of forks p_0 is to use.

The arc inscriptions around the transition "take forks" indicate that each philosopher p_i can start his meals only if his left and right forks g_i and g_{i+1} are available. The occurrence of "take forks" in mode $\beta(x)=p_i$ then removes these forks from "available forks", thus they are no longer available for the other philosophers during the meal of p_i. As an example, Fig. 2 shows the situation after p_0 and p_2 both having taken their forks. The occurrence of "release forks" in the mode $\beta(x)=p_0$ as well as in the mode $\beta(x)=p_2$ turns the situation shown in Fig. 2 again to the situation shown in Fig. 1.

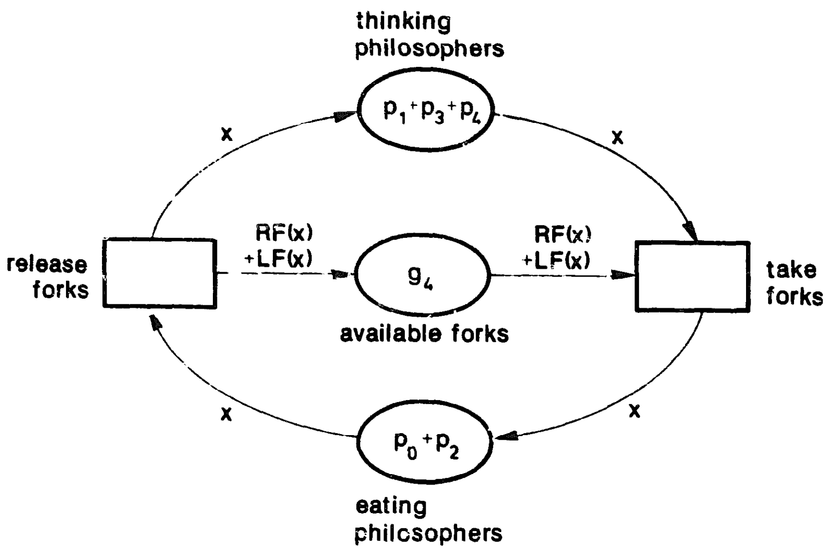

Fig. 2. The system of dining philosophers after p_0 and p_2 having taken their forks.

Notice that no multisets occur in this system. Ordinary sets suffice to describe its behaviour. It will nevertheless turn out that for the proof of system properties (e.g. to show that neighboured philosophers never eat concurrently), multisets are required.

2. Algebraic preliminaries

We recall here some fundamentals of algebraic specifications according to [7]. This serves fixing the—elementary—scope of algebraic notions used in this paper.

2.1. A *signature* $\Sigma=(S, OP)$ consists of a set S of *sorts* and of a family $OP=(OP_{w,s})_{w\in S^*, s\in S}$ of *operation symbols.* We particularly distinguish the sets $K_s := OP_{\lambda,s}$ of *constant symbols* (λ denotes the empty word over S).

2.2. A *Σ-algebra* $A = (S_A, OP_A)$ consists of a family $S_A = (A_s)_{s \in S}$ of *domains* and a family $OP_A = (N_A)_{N \in OP}$ of *operations* $N_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$ for all $N \in OP_{s_1 \ldots s_n, s}$. Clearly, $N_A \in A_s$ iff $N \in K_s$.

2.3. A set X of *Σ-variables* is a family $X = (X_s)_{s \in S}$ of *variables*, disjoint to OP.

2.4. The set $T_{OP,s}(X)$ of *(OP, X)-terms of sort* s is inductively defined by

(i) $X_s \cup K_s \subseteq T_{OP,s}(X)$, and

(ii) $N(u_1, \ldots, u_n) \in T_{OP,s}(X)$ for $N \in OP_{s_1 \ldots s_n, s}$ and $n \geq 1$, in case $u_1 \in T_{OP,s_1}(X), \ldots, u_n \in T_{OP,s_n}(X)$.

2.5. The set $T_{OP,s} = T_{OP,s}(\emptyset)$ contains the *ground terms of sort s*, $T_{OP}(X) := \bigcup_{s \in S} T_{OP,s}(X)$ is the set of *Σ-terms over X*, and $T_{OP} := T_{OP}(\emptyset)$ is the set of *Σ-ground terms*.

2.6. An *evaluation* is a mapping *eval*: $T_{OP} \to A$ of Σ-ground terms into a Σ-algebra A, inductively defined by

(i) $eval(N) = N_A$ for all constant symbols N, and

(ii) $eval(N(u_1, \ldots, u_N)) = N_A(eval(u_1), \ldots, eval(u_n))$ for all $N(u_1, \ldots, u_n) \in T_{OP}$.

2.7. *An assignment of Σ-variables X to a Σ-algebra A* is a mapping $ass : X \to A$ with $ass(x) \in A_s$ iff $x \in X_s$. *ass* is canonically extended to $\overline{ass}: T_{OP}(X) \to A$, inductively defined by $\overline{ass}(x) = ass(x)$ for $x \in X$, $\overline{ass}(N) = N_A$ for $N \in K_s$ and $\overline{ass}(N(u_1, \ldots, u_n)) = N_A(\overline{ass}(u_1), \ldots, \overline{ass}(u_n))$ for $N(u_1, \ldots, u_n) \in T_{OP}(X)$.

2.8. A *Σ-equation over X of sort s* is a pair (L, R) of terms $L, R \in T_{OP,s}(X)$.

2.9. A Σ-equation (L, R) over X is *valid in a Σ-algebra A* iff for all $ass: X \to A$, $\overline{ass}(L) = \overline{ass}(R)$.

2.10. A *specification* $SPEC = (S, OP, E)$ consists of a signature $\Sigma = (S, OP)$ and a set E of Σ-equations.

2.11. For $SPEC = (S, OP, E)$, a *SPEC-algebra* $A = (S_A, OP_A)$ is a (S, OP)-algebra in which all equations in E are valid.

2.12. Two ground terms $u, v \in T_{OP}$ are *congruent* in $SPEC = (S, OP, E)$ (written $u \equiv_E v$) iff $eval_A(u) = eval_A(v)$ for all *SPEC*-algebras A. $\equiv_E$ is an equivalence on T_{OP}.

2.13. $\equiv_E$ is a congruence on T_{OP}, i.e. substitution of congruent terms retains congruence: $u \equiv_E v$ implies $N(\ldots u \ldots) \equiv_E N(\ldots v \ldots)$, cf. [7, Fact 3.11]. When E is clear from the context, $[u]$ denotes the congruence class of term u under $\equiv_E$.

2.14. The *quotient term algebra* $T_{SPEC} = ((Q_s)_{s \in S}, (N_Q)_{N \in OP})$ of a specification $SPEC = (S, OP, E)$ has the equivalence classes $Q_s = \{[u] \mid u \in T_{OP,s}\}$ of $\equiv_E$ as carrier sets and the operations N_Q defined by $N_Q([u_1], \ldots, [u_n]) := [N(u_1, \ldots, u_n)]$.

The semantics (meaning) of a specification *SPEC* is any algebra which is isomorphic to T_{SPEC}.

2.15. Congruence on ground terms is extended to terms $u, v \in T_{OP}(X)$: $u \equiv_E v$ iff for all $ass: X \to T_{OP}$, $\overline{ass}(u) \equiv_E \overline{ass}(v)$. Whenever it is clear from the context, the index E in $\equiv_E$ may be skipped.

2.16. A specification $SPEC1$ may consist of a given specification $SPEC = (S, OP, E)$ and additional sets of sorts $S1$, operation symbols $OP1$ and equations $E1$. Then the notation $SPEC1 = SPEC + (S1, OP1, E1)$ means $SPEC1 = (S + S1, OP + OP1, E + E1)$ where + stands for disjoint union of sets. For a signature $\Sigma = (S, OP)$, $\Sigma + (S1, OP1, E1)$ stands of course for the specification $(S + S1, OP + OP1, E1)$.

2.17. Compatibility of evaluation is preserved by extended evaluations: Commutativity of the following diagram (1) implies commutativity of diagram (2), cf. [7, Fact 1.12]:

$$\begin{tikzcd} X \arrow[rr, "ass"] \arrow[dr, "ass_1"'] \arrow[rr, phantom, "(1)"', shift right=4ex] & & T_{OP}(Y) \arrow[dl, "\overline{ass_2}"] \\ & A & \end{tikzcd}$$

$$\begin{tikzcd} T_{OP}(X) \arrow[rr, "\overline{ass}"] \arrow[dr, "\overline{ass_1}"'] \arrow[rr, phantom, "(2)"', shift right=4ex] & & T_{OP}(Y) \arrow[dl, "\overline{ass_2}"] \\ & A & \end{tikzcd}$$

Returning to the introductory example of Section 1, the system of dining philosophers can be constructed from the following signature:

2.18. **phils-base** =

sorts: *phils*
forks

opns: $p_0, \ldots, p_4 : \to phils$
$g_0, \ldots, g_4 : \to forks$
$LF, RF : phils \to forks$

The arc- and place-inscriptions of the net in Fig. 1 include terms of this signature over the variable x of sort philosopher. (The "+"symbol denotes union of multisets.) The intended meaning of the inscriptions is based on the following specification, extending the above signature by the following equations:

2.19. **phils** = **phils-base** +

eqns: $\left.\begin{array}{l} LF(p_i) = g_i \\ RF(p_i) = g_{i+1} \end{array}\right\} i = 0, \ldots, 4, \text{ with } g_5 := g_0$

$RF(p)$ and $LF(p)$ denote the right and left forks of philosophers p.

3. Multiset specifications

System modelling often requires different copies (items) of data or items which should not be distinguished in the model. This is reflected by *nonnegative multisets* (or *bags*), i.e. collections of elements, some of which may be undistinguishable.

Formally, a nonnegative multiset M over a given set D is a mapping $M: D \to \mathbb{N}$. The *empty multiset* ϑ_D over D is given by $\vartheta_D(d) = 0$ for all $d \in D$. Single elements $d \in D$ can be considered as one-elementary multisets m_d, defined by $m_d(x) = 1$ if $x = d$ and $m_d(x) = 0$, otherwise.

For nets with structured tokens we shall later on study analysis and representation techniques based on general multisets $M: D \to \mathbb{Z}$. The case of nonnegative multisets is thus extended to negative numbers $M(d)$. For general multisets M_1 and M_2, addition is defined component wise, by $(M_1 + M_2)(d) = M_1(d) + M_2(d)$. The inverse $-M$ of a multiset M is defined by $(-M)(d) = -(M(d))$.

Any specification $SPEC$ can be extended to its corresponding multiset specification **m**_$SPEC$.

To each constant symbol K_s of $SPEC$ a term $MAKE_s(K_s)$ is associated. If K_s is evaluated to the element d, $MAKE_s(K_s)$ is evaluated to the set $\{d\}$. A term ϑ_s denotes the empty multiset; addition and subtraction symbols are defined in the obvious way:

3.1. Definition. Given a specification $SPEC = (S, OP, E)$, let

$$
\begin{array}{lll}
\mathbf{m}\_SPEC = SPEC + & & \\
\quad \text{sorts:}\ m_s & & \\
\text{opns:} & \vartheta_s : \to m_s & \\
& MAKE_s : s \to m_s & \\
& +_s : m_s m_s \to m_s & \\
& -_s : m_s \to m_s & \\
\text{eqns:} & a \in s;\ p, q, r \in m_s & \text{for all } s \in S \\
& +_s(p, \vartheta_s) = p & \\
& +_s(p, q) = +_s(q, p) & \\
& +_s(p, +_s(q, r)) = +_s(+_s(p, q), r) & \\
& +_s((p, -_s(p)) = \vartheta &
\end{array}
$$

A couple of notations and shorthands supports handling the formalism:

3.2. Notations

(i) For a signature Σ, the specification **m**_Σ is defined in the obvious way, cf. 2.16.

(ii) Given a specification $SPEC = (S, OP, E)$ we denote the specification **m**_$SPEC$ by $(\hat{S}, \widehat{OP}, \hat{E})$, respectively.

(iii) Within multiset terms we often skip the sort indices s of operation symbols, and write ϑ instead of ϑ_s.

(iv) We use infix notations $u+v$ and $u-v$ for $+(u, v)$ and $-(u, v)$, respectively.
(v) We furthermore write a for $MAKE(a)$.
(vi) Whenever ambiguities are excluded, brackets may be skipped.

As an example, with constant symbols a and b of some sort s, $a-b$ stands for the term $+_s(MAKE_s(a), -_s(MAKE_s(b)))$.

Nonnegative multisets can be specified using (besides the operation symbols of the underlying specification) only the operation symbols ϑ_s, $MAKE_s$ and $+_s$. This motivates the following concepts:

3.3. Definition. Let $SPEC = (S, OP, E)$ be a specification.

(i) $NNS := \widehat{OP}\{-_s \mid s \in S\}$ is the set of *nonnegative operation symbols in m_SPEC.*

(ii) Let X be a family of *SPEC*-variables.

$$T_{OP^+}(X) := \{u \in T_{\widehat{OP}}(X) \mid u \equiv_{\hat{E}} v \text{ for some } v \in T_{NNS}(X)\}$$

is the set of *nonnegative terms of SPEC.*

(iii) Corresponding to 2.5, let $T_{OP^+} := T_{OP^+}(\emptyset)$, and for all $s \in S$, let $T_{OP^+,m_s}(X) := T_{OP^+}(X) \cap T_{\widehat{OP},m_s}(X)$ and $T_{OP^+,s} := T_{OP^+,m_s}(\emptyset)$.

(iv) Given two multiset terms $u, v \in T_{\widehat{OP}}(X)$, u is said to be *smaller or equal to v in SPEC*, written $u \leqslant_{\hat{E}} v$ if $v-u$ is nonnegative in *SPEC*.

Nonnegativity of terms depends in fact on the assumed equations. As an example, a term $a+a-b$ (with a and b constant symbols of some sort s) is nonnegative iff the equation $a=b$ is assumed.

The net in Fig. 1 can now entirely be explained: based on the specification **phils** $= (S, OP, E)$ of 2.19 and the set $X = X_{phils} = \{x\}$, the arc inscriptions of Fig. 1 are taken from $T_{OP^+}(X)$.

4. Nets with structured tokens

Nets with structured tokens can now be described in an algebraic framework. We start with the conventional definition of (uninscribed) nets:

4.1. Definition. A triple $N = (P, T, F)$ is called a *net* iff

(i) P and T are nonempty, finite, disjoint sets (the *places* and *transitions* of N, respectively), and

(ii) $F \subseteq (P \times T) \cup (T \times P)$ is a relation (the *arcs* of N).

Places, transitions, and arcs will graphically be represented as usual by circles, boxes and arrows, respectively.

Structured tokens are now introduced by inscribing a net w.r.t. a specification and a corresponding set of variables: Each place p of the net is assigned its sort $\varphi(p)$ and its initial marking $M_0(p)$ which is a nonnegative multiset ground term, and each arc f is inscribed by a nonnegative multiset term $\lambda(f)$. Both M_0 and λ should be sort-preserving, i.e. $M_0(p)$ is of sort $m_{\varphi(p)}$, and the multiset sort of $\lambda(f)$ corresponds to the sort of the place adjacent to f.

Based on the notations for multiset sorts of Definition 3.3 we have the following definition.

4.2. Definition. Let $N=(P, T, F)$ be a net, let $SPEC=(S, OP, E)$ be a specification, and let X be a family of Σ-variables

(i) A mapping $\varphi: P \to S$ is called a *sort assignment* of N. Assuming φ, for places $p \in P$ let $\tilde{p}$ denote the multiset sort $m_{\varphi(p)}$.

(ii) A mapping $M_0: P \to T_{OP^+}$ with $M_0(p) \in T_{OP^+,\tilde{p}}$ for each $p \in P$ is called a *φ-respecting initial marking of N.*

(iii) A mapping $\lambda: F \to T_{OP^+}(X)$ with $\lambda(f) \in T_{OP^+,\tilde{p}}(X)$ for each $f=(t,p)$ or $f=(p,t)$ is called a *φ-respecting arc inscription of N.*

(iv) A triple $ins=(\varphi, M_0, \lambda)$ of a sort assignment φ of N, a φ-respecting initial marking M_0 of N, and a φ-sorted arc inscription λ of N, is called a *SPEC-inscription of N*, and (N, ins, E) is a *SPEC-inscribed net.* As a shorthand, N is said to be *inscribed* assuming that *ins* and E can be understood from the context.

Within the **phils**-inscribed net of Fig. 1, the sort assignment φ is given by φ(thinking philosophers) $=\varphi$(eating philosophers) $=$ *phils*, and φ(available forks) $=$ *forks*. M_0 and λ are obvious in Fig. 1. (Entries $M(s)=\vartheta$ are skipped.)

The following notations will be useful when dealing with dynamics of inscribed nets:

4.3. Notations. Let (φ, M_0, λ) be a Σ-inscription of a net $N=(P, T, F)$ over X.

(i) For all $(x,y) \in (T \times P) \cup (P \times T)$ let

$$\xrightarrow{x,y} = \begin{cases} \lambda(x,y) & \text{iff } (x,y) \in F, \\ \vartheta & \text{otherwise.} \end{cases}$$

(ii) For each $t \in T$ we define the vector $\underline{t}: P \to T_{\widehat{OP}}(X)$ by

$$\underline{t}(p) = \xrightarrow{t,p} - \xrightarrow{p,t}.$$

Dynamics of inscribed nets is now defined as follows:

4.4. Definition. Let the net $N=(P, T, F)$ be inscribed over a specification $SPEC=(S, OP, E)$ and variables X.

(i) *Markings* of N are mappings $M: P \to T_{OP^+}$ with $M(p) \in T_{OP^+,\tilde{p}}$ for each $p \in P$.

(ii) An *occurrence mode* of N is an assignment $\beta: X \to T_{OP}$.

(iii) Given a marking M, a transition $t \in T$ and an occurrence mode β, t is *β-enabled at M* (or *enabled at M in mode β*) iff, for all $p \in P$,

$$\bar{\beta}(\xrightarrow{p,t}) \leq_E M(p).$$

(iv) If t is β-enabled at M, t may *occur in mode* β. This returns the marking M' which is defined for each $p \in P$ by

$$M'(p) = M(p) - \bar{\beta}(\xrightarrow{p,t}) + \bar{\beta}(\xrightarrow{t,p})$$

We write $M \xrightarrow{t,\beta} M'$ in this case.

(v) For a marking M of N, the set $[M\rangle$ of *markings reachable from* M is the smallest set of markings such that $M \in [M\rangle$ and if $M' \in [M\rangle$ and $M' \xrightarrow{t,\beta} M''$ then $M'' \in [M\rangle$.

With Notation 4.3(ii) and 2.7 we get immediately the following corollary.

4.5. Corollary. *If* $M \xrightarrow{t,\beta} M'$ *then* $M'(p) \equiv_{\hat{E}} M(p) + \bar{\beta}(\underline{t}(p))$ *for all places* p.

Considering markings M and mappings $\underline{t}$ as P-indexed vectors and extending sum and extended assignments $\bar{\beta}$ component-wise to vectors, the above corollary reads as follows.

4.6. Corollary. *If* $M \xrightarrow{t,\beta} M'$ *then* $M' \equiv_{\hat{E}} M + \bar{\beta}(\underline{t})$.

As an example, the transition **tf** of Fig. 1 is enabled at M_0 in all modes $\beta : \{x\} \to \{p_0, \ldots, p_4\}$. There is however no mode β to enable **rf** at M_0. Assuming $\beta_0(x) = p_0$, let M_1 be reached by $M_0 \xrightarrow{tf,\beta_0} M_1$. Now in M_1, **tf** is enabled in both modes $\beta(x) = p_2$ and $\beta(x) = p_3$. The assignment β_0 additionally enables **rf** under M_1. With $M_1 \xrightarrow{rf,\beta_0} M_2$ we return to $M_0 \equiv_{\hat{E}} M_2$. Notice that without the equations of **phils**, each philosopher is assumed to have two forks of his own.

This completes the notion of *SPEC*-inscribed nets. Other versions of nets with structured tokens can be regarded as being based on particular specifications *SPEC*: *PrT*-nets with tuples of variables as arc inscriptions use the specification of tuples; place/transition nets do with one sort and one constant of this sort. Details of such classes will be discussed later.

Transitions may additionally be inscribed by logical formulae. Then a transition is enabled in a mode β only if in addition to the requirements of Definition 4.4(iii), its formula evaluates with β to *TRUE*. We skip this feature here, since it does not contribute to this paper's topic.

As a further example, let the specification **phils'** be given by

phils' =

sorts: *phils*
forks

opns: $g_0, \ldots, g_4 : \to phils$
$p_0, \ldots, p_4 : \to forks$
$RU : forks \to phils$
$RSF : forks \to forks$

eqns: $\left.\begin{array}{l} RU(g_i) = p_i \\ RSF(g_i) = g_{i+1}. \end{array}\right\} i = 0, \ldots, 4$, with $g_5 := g_0$

The operations RU and RSF are to return for each fork its right user and its right successor fork, respectively.

Figure 3 shows a **phils'**-inscribed net. Intuitively its behaviour is identical to the behaviour of the net in Fig. 1. We will discuss in Section 8 how algebraic specifications provide means to prove this formally.

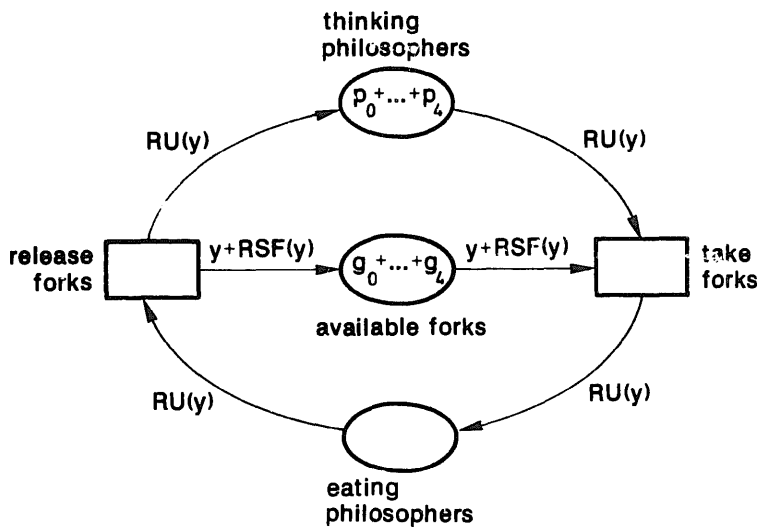

Fig. 3. A variant of the philosophers system.

5. Place invariants

Place invariants are one of the most important analysis tools for several versions of Petri nets. A place invariant provides a weight $W(M)$ to markings M in such a way that in each set $[M\rangle$ of reachable markings, the weighted markings $W(M')$ are constant for all $M' \in [M\rangle$. We will show that such weights can be represented by a place vector of multiset terms. The application of a weight function W to a marking M will be defined as a scalar product of the vectors W and M, with the product of components being defined as a term substitution. Products $W \cdot M$ thus amount to multiset ground terms. Safety properties can be derived from knowing the product $W \cdot M$ to remain constant.

A fundamental property of place invariants is their characterizability as solutions of homogeneous systems of linear equations: Each net N with places P and transitions T canonically defines a $P \times T$-matrix $\underline{N}$ with entries $\underline{N}(p, t) = \underline{t}(p)$ (cf. Notations 4.3). The product of matrix entries with invariant entries is again defined as term substitution, just as the above-mentioned product of markings and invariant entries. Place invariants will then be the solutions of $\underline{N}^T \cdot i \equiv_E \gamma_S$ ($\underline{N}^T$ denotes the transpose of matrix $\underline{N}$).

The term product for place invariants is based on a distinguished, quite simple kind of "constant" assignments, ass_u, of Σ-variables X. The range of ass_u is the set $T_{OP}(X)$ of Σ-terms over X. Given a sort s, ass_u maps all $x \in X_s$ to $u \in T_{OP,s}(X)$, leaving all other variables untouched.

5.1. Definition. Let $\Sigma = (S, OP)$ be a signature, let X be a set of Σ-variables with X_s the variables of sort $s \in S$, and let $u \in T_{OP,s}(X)$. Then the *constant u-assignment*, $ass_u : X \to T_{OP}(X)$ is defined as

$$ass_u(x) = \begin{cases} u & \text{iff } x \in X_s, \\ x & \text{iff } x \in X \setminus X_s. \end{cases}$$

Assignments ass_u are extended to multiset terms by $\overline{ass_u} : T_{\widehat{OP}}(X) \to T_{\widehat{OP}}(X)$ in the usual way (cf. 2.7). Based on constant assignment we define the following product for multiset terms:

5.2. Definition. Let (S, OP) be a signature and let X be a set of (S, OP)-variables. A *product* $u \cdot u' \in T_{\widehat{OP}}(X)$ for multiset terms $u, u' \in T_{\widehat{OP}}(X)$ is defined by induction over the structure of u: $MAKE(u) \cdot u' = \overline{ass_u}(u')$ iff $u \in T_{OP}(X)$, $\vartheta \cdot u' = \vartheta$, $(u_1 + u_2) \cdot u' = u_1 \cdot u' + u_2 \cdot u'$, and $(-u) \cdot u' = -(u \cdot u')$.

For vectors of multiset terms we extend this product furthermore to the usual inner product of linear algebra:

5.3. Definition. Let P be a finite set, let (S, OP) be a signature and let X be a set of (S, OP)-variables. For vectors $\underline{u}, \underline{u}' : P \to T_{\widehat{OP}}(X)$, let the *product* $\underline{u} \cdot \underline{u}' \in T_{\widehat{OP}}(X)$ be defined

$$\underline{u} \cdot \underline{u}' = \sum_{p \in P} \underline{u}(p) \cdot \underline{u}'(p).$$

This definition is (semantically) unique, due to associativity of addition (cf. Definition 3.1).

The central notion of *place invariants* is based on the above product:

5.4. Definition. Let a net $N = (P, T, F)$ be inscribed over a specification $SPEC = (S, OP, E)$ and a set X of variables. Let furthermore Y be a family of (S, OP)-variables, disjoint from X, and let $s \in S$ be freely chosen.

A vector $i : P \to T_{\widehat{OP},ms}(Y)$ is a *place invariant of sort s* of N iff for all markings M and M' of N with $M' \in [M\rangle$:

$$M \cdot i \equiv_E M' \cdot i.$$

We shall show that place invariants can be characterized as solutions of an equational system derived from N. The proof is based on the following lemma. It shows how the product of Definition 5.2 interacts with assignments:

5.5. Lemma. *Let $\Sigma = (S, OP)$ be a signature, let X and Y be disjoint sets of Σ-variables, let $u \in T_{\widehat{OP}}(X)$, $u' \in T_{\widehat{OP}}(Y)$ and let $\gamma : X \cup Y \to T_{OP}(Y)$ be an assignment with $\gamma(y) = y$ for all $y \in Y$. Then*

$$\bar{\gamma}(u \cdot u') = \bar{\gamma}(u) \cdot u'.$$

Proof. We proceed by induction over the structure of u.

(1) For $u \in T_{\widehat{OP}}(X)$ and all $u' \in T_{\widehat{OP}}(Y)$, we have to show: $\bar{\gamma}(\overline{ass_u}(u')) = \overline{ass_{\bar{\gamma}(u)}}(u')$. With 2.17 it suffices to show for all $y \in Y$: $\bar{\gamma}(ass_u(y)) = ass_{\bar{\gamma}(u)}(y)$. In a graphical representation, we have to show that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "ass_u"] \arrow[dr, "ass_{\bar{\gamma}(u)}"'] & & T_{\widehat{OP}}(X) \arrow[dl, "\bar{\gamma}"] \\ & T_{\widehat{OP}}(Y) & \end{tikzcd}$$

Let now s be the sort of u, i.e. $u \in T_{OP,s}(X)$. $\bar{\gamma}(u)$ has of course the same sort, i.e. $\bar{\gamma}(u) \in T_{OP,s}(Y)$.

In case $y \in Y_s$, we get $ass_{\bar{\gamma}(u)}(y) = \bar{\gamma}(u)$ (by Definition 5.1) $= \bar{\gamma}(ass_u(y))$ (by 5.1). Otherwise, for all $y \in Y \setminus Y_s$, $ass_{\bar{\gamma}(u)}(y) = y$ (by 5.1) $= \bar{\gamma}(y)$ (by construction of γ) $= \bar{\gamma}(ass_u(y))$ (by 5.1).

This completes the proof of $\bar{\gamma}(u) \cdot u' = \bar{\gamma}(u \cdot u')$ for $u \in T_{OP}(X)$. Thus the basis for the induction over the structure of u is given.

(2) To show the induction step over the structure of u, we distinguish three cases.

Case (a) For $u = \vartheta$ we get

$$\begin{aligned} \bar{\gamma}(\vartheta \cdot u') &= \bar{\gamma}(\vartheta) && \text{(by Definition 5.2)} \\ &= \vartheta && \text{(by 2.7)} \\ &= \vartheta \cdot u' && \text{(by Definition 5.2)} \\ &= \bar{\gamma}(\vartheta) \cdot u' && \text{(by 2.7).} \end{aligned}$$

Case (b) For $u = u_1 + u_2$ we show $\bar{\gamma}((u_1 + u_2) \cdot u') = \bar{\gamma}(u_1 + u_2) \cdot u'$ as follows:

$$\begin{aligned} \bar{\gamma}((u_1 + u_2) \cdot u') &= \bar{\gamma}(u_1 \cdot u' + u_2 \cdot u') && \text{(by Definition 5.2)} \\ &= \bar{\gamma}(u_1 \cdot u') + \bar{\gamma}(u_2 \cdot u') && \text{(by 2.7)} \\ &= \bar{\gamma}(u_1) \cdot u' + \bar{\gamma}(u_2) \cdot u' && \text{(by induction hypothesis)} \\ &= (\bar{\gamma}(u_1) + \bar{\gamma}(u_2)) \cdot u' && \text{(by 5.2)} \\ &= \bar{\gamma}(u_1 + u_2) \cdot u' && \text{(by 2.7).} \end{aligned}$$

Case (c) For $u=-u_1$ we show $\bar{\gamma}((-u_1)\cdot u')=\bar{\gamma}(-u_1)\cdot u'$ as follows:

$$\begin{aligned}\bar{\gamma}((-u_1)\cdot u') &= \bar{\gamma}(-(u_1\cdot u')) \quad \text{(by 5.2)}\\ &= -\bar{\gamma}(u_1\cdot u') \quad \text{(by 2.7)}\\ &= -(\bar{\gamma}(u_1)\cdot u') \quad \text{(by induction hypothesis)}\\ &= (-\bar{\gamma}(u_1)\cdot u') \quad \text{(by 5.2)}\\ &= \bar{\gamma}(-u_1)\cdot u' \quad \text{(by 2.7)}\end{aligned}$$

This completes the proof of Lemma 5.5. □

The following theorem states the central property of place invariants:

5.6. Theorem. *Let a net $N=(P,T,F)$ be inscribed over a specification $SPEC=(S,OP,E)$ and a set X of variables. Let furthermore Y be a family of (S,OP)-variables disjoint from X, and let $i:P\to T_{\widehat{OP},m_s}(Y)$ for some $s\in S$. If $\underline{t}\cdot i\equiv_{\hat{E}}\vartheta_s$ for all $t\in T$, then i is a place invariant.*

Proof. Assume $\underline{t}\cdot i\equiv_{\hat{E}}\vartheta_s$ for all $t\in T$. (a) We first show for all $t\in T$ and all assignments $\beta:X\to T_{OP}(Y)$: $\sum_{p\in P}\bar{\beta}(\underline{t}(p))\cdot i(p)\equiv_{\hat{E}}0_s$. To do so, we extend β to $\gamma:X\cup Y\to T_{OP}(Y)$ by $\gamma(x)=\beta(x)$ for $x\in X$ and $\gamma(y)=y$ for $y\in Y$. Now,

$$\begin{aligned}\sum_{p\in P}\bar{\beta}(\underline{t}(p))\cdot i(p) &= \sum_{p\in P}\bar{\gamma}(\underline{t}(p))\cdot i(p) \quad \text{(by definition of } \gamma)\\ &= \sum_{p\in P}\bar{\gamma}(\underline{t}(p)\cdot i(p)) \quad \text{(by Lemma 5.5)}\\ &= \bar{\gamma}\Big(\sum_{p\in P}\underline{t}(p)\cdot i(p)\Big) \quad \text{(by 2.7)}\\ &\equiv_{\hat{E}} \bar{\gamma}(\underline{t}\cdot i) \quad \text{(by 5.3)}\\ &\equiv_{\hat{E}} \bar{\gamma}(\vartheta_s) \quad \text{(by 2.13 and the assumption on } i)\\ &= \vartheta_s \quad \text{(by 2.7).}\end{aligned}$$

(b) To show the theorem, it is sufficient to show $M\cdot i=M'\cdot i$ for all $M\xrightarrow{t,\beta}M'$. So we get

$$\begin{aligned}M'\cdot i &= \sum_{p\in P}M'(p)\cdot i(p) \quad \text{(by Definition 5.3)}\\ &= \sum_{p\in P}(M(p)+\bar{\beta}(\underline{t}(p)))\cdot i(p) \quad \text{(by Corollary 4.6)}\\ &= \sum_{p\in P}(M(p)\cdot i(p)+\bar{\beta}(\underline{t}(p))\cdot i(p)) \quad \text{(by Definition 5.2)}\\ &\equiv_{\hat{E}} \sum_{p\in P}(M(p)\cdot i(p)+\vartheta_s) \quad \text{(by part (a) of this proof and 2.13)}\\ &\equiv_{\hat{E}} \sum_{p\in P}M(p)\cdot i(p) \quad \text{(by 3.1)}\\ &= M\cdot i \quad \text{(by Definition 5.3).} \qquad □\end{aligned}$$

The inverse of Theorem 5.6 is also valid:

5.7. Theorem. *Let N, SPEC, X, Y and i be as in the assumption of Theorem 5.6. If i is a place invariant of N, then $\underline{t} \cdot i \equiv_{\hat{E}} \vartheta$ for all $t \in T$.*

Proof. Let $t \in T$, $\beta : X \to T_{OP}(Y)$ and let M, M' be markings of N with $M \xrightarrow{t,\beta} M'$. (Choose e.g. $M(p) = \bar{\beta}(\xrightarrow{p,t})$.) Then, we get

$$M' \cdot i = (M + \bar{\beta}(\underline{t})) \cdot i \quad \text{(by Corollary 4.6)}$$
$$= M \cdot i + \bar{\beta}(\underline{t}) \cdot i \quad \text{(by Definitions 5.3 and 5.2).}$$

The assumption $M' \cdot i \equiv_{\hat{E}} M \cdot i$ now implies $\bar{\beta}(\underline{t}) \cdot i \equiv_{\hat{E}} \vartheta$.

Now we extend β to $\gamma : X \cup Y \to T_{OP}(Y)$ by $\gamma(x) = \beta(x)$ for $x \in X$ and $\gamma(y) = y$ for $y \in Y$. Clearly, $\bar{\gamma}(\underline{t}) \cdot i \equiv_{\hat{E}} \vartheta$, and with Lemma 5.5 we get $\bar{\gamma}(\underline{t} \cdot i) \equiv_{\hat{E}} \vartheta$. As this holds for all assignments β, Definition 2.15 implies $\underline{t} \cdot i \equiv_{\hat{E}} \vartheta$. □

From Definition 5.2 it follows directly that place invariants are additive:

5.8. Corollary. *If i_1 and i_2 are place invariants of some inscribed net N, then $i_1 + i_2$ and $-i_1$ are also place invariants of N.*

6. Application examples for place invariants

6.1. Properties of the dining philosophers system

Figure 4 shows the matrix, the initial marking and some invariants for the dining philosophers system in Fig. 1. (Entries ϑ are skipped.) The invariants are useful for proving some properties of this system, avoiding the consideration of its runs. These properties include:

(1) *Each philosopher always is either eating or thinking*: For each reachable marking $M \in [M_0\rangle$ we get with invariant i_1: $M(\mathbf{t}) + M(\mathbf{e}) = M(\mathbf{t}) \cdot y + M(\mathbf{e}) \cdot y = M \cdot i_1 = M_0 \cdot i_1 = p_0 + \cdots + p_4$. Hence, each philosopher occurs exactly once in $M(\mathbf{t}) + M(\mathbf{e})$.

	tf	rf	M_0	i_1	i_2	i_3
t	$-x$	x	$p_0 + \ldots + p_4$	y	$RF(y) + LF(y)$	
a	$-(RF(x) + LF(x))$	$RF(x) + LF(x)$	$g_0 + \ldots + g_4$		$-z$	z
e	x	$-x$		y		$RF(y)+ LF(y)$

$$M_0 \cdot i_1 = p_0 + \ldots + p_4$$
$$M_0 \cdot i_2 = M_0 \cdot i_3 = g_0 + \ldots + g_4$$

Fig. 4. Matrix, initial marking and three place invariants to Fig. 1 with $y \in Y_{phils}$ and $z \in Y_{forks}$.

(2) *If a fork is available, then both of its potential users are thinking*: for each reachable marking $M \in [M_0\rangle$, we get from i_2:

$$-M(\mathbf{a}) + RF(M(\mathbf{t})) + LF(M(\mathbf{t}))$$

$$= M(\mathbf{a}) \cdot (-z) + M(\mathbf{t}) \cdot (RF(y) + LF(y))$$

$$= M \cdot i_2 = M_0 \cdot i_2$$

$$= (g_0 + \cdots + g_4) \cdot (-z) + (p_0 + \cdots + p_4) \cdot (RF(y) + LF(y))$$

$$= (-g_0 - \cdots - g_4) + (g_0 + \cdots + g_4) + (g_0 + \cdots + g_4) = g_0 + \cdots + g_4.$$

So we get $M(\mathbf{a}) = RF(M(\mathbf{t})) + LF(M(\mathbf{t})) - (g_0 + \cdots + g_4)$. Now, if $g_i \in M(\mathbf{a})$, it follows $g_i \in RF(M(\mathbf{t}))$ as well as $g_i \in LF(M(\mathbf{t}))$. This yields p_i and p_{i+1} in $M(t)$.

(3) *Neighboured philosophers never eat at the same time*: Neighbours p_i, p_{i+1} both eating at the same time are represented by a marking $\bar{M}$ with $\bar{M}(\mathbf{e}) \geqslant p_i + p_{i+1}$. Then, $\bar{M} \cdot i_3 \geqslant \bar{M}(\mathbf{e}) \cdot i_3(\mathbf{e}) = RF(\bar{M}(\mathbf{e})) + LF(\bar{M}(\mathbf{e})) \geqslant RF(p_i) + LF(p_{i+1}) = g_i + g_i$. But for each reachable marking $M \in [M_0\rangle$ we get with i_3: $M(\mathbf{a}) + RF(M(\mathbf{e})) + LF(M(\mathbf{e})) = M(\mathbf{a}) \cdot z + M(\mathbf{e}) \cdot (RF(y) + LF(y)) = M \cdot i_3 = M_0 \cdot i_3 = (g_0 + \cdots + g_4) \cdot z$ $= g_0 + \cdots + g_4$.

Notice that the invariants of Fig. 4 rely on the multiset equations only. They make no use of the particular equations of **phils** and therefore might be denoted *Σ-invariants.*

6.2. N-Tuples as net inscriptions

Nets with structured tokens often include pairs or generally n-tuples of constants and variables as markings and arc inscriptions, respectively. Pairs and generally n-tuples can easily be specified in the algebraic framework discussed in this paper, providing all universal properties and constructs of cartesian products. A most elementary specification is the following one:

pair =
sorts: s
$\quad$ *pair*

opns: $a_1, a_2 : \rightarrow s$
$\quad PAIR : s\ s \rightarrow pair$
$\quad PR1 : pair \rightarrow s$
$\quad PR2 : pair \rightarrow s$

eqns: $x_1, x_2 \in s$
$\quad PR1(PAIR(x_1, x_2)) = x_1$
$\quad PR2(PAIR(x_1, x_2)) = x_2$

It is obvious how to construct pairs with components of different sorts or general n-tuples, or how to use predefined sorts as component sorts of such tuples.

Figure 5 shows a **pair**-inscribed net, and one of its invariants is discussed in Fig. 6. With $\beta(x_1) = a_1$ and $\beta(x_2) = a_2$, we get $M \xrightarrow{t,\beta} M'$. As usual, we write (x_1, x_2) for $PAIR(x_1, x_2)$.

Notice that the invariant i of Fig. 6 makes use of the particular equations of **pair** (whereas the invariants of Fig. 4 rely on the multiset equations only). i of Fig. 6 might therefore be called a (proper) *SPEC*- (or **pair**-)*invariant.*

6.3. A database maintaining scheme

As a last example, we refer to the often considered scheme for maintaining multiple copies of a database [11, 14].

A set D of sites is assumed, each of which being able to send update requests to all other sites. Upon receiving an update request, a site performs the required update of its database and returns an acknowledgement to the sender.

We assume a successor function SUC on the set $D = \{a_1, \ldots, a_n\}$ of sites, by $a_{i+1} = SUC(a_i)$ and $a_1 = SUC(a_n)$. We furthermore assume the specification **pair** considered above. This leads to the following specification, assuming a given natural number n:

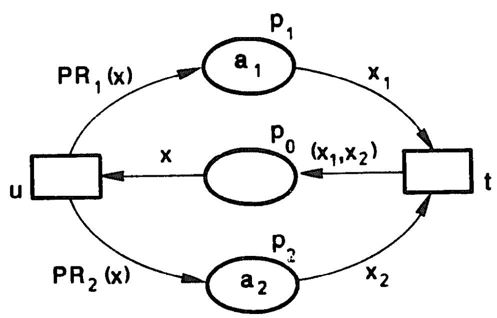

Fig. 5. A **pair**-inscribed net with $x_1, x_2 \in X_s$ and $x \in X_{pair}$.

	t	u	i	t · i	u · i	M	$M \cdot i$	M'	$M' \cdot i$
p_0	(x_1, x_2)	$-x$	$PR_1(y)$	x_1	$-PR_1(x)$			(a_1, a_2)	a_1
p_1	$-x_1$	$PR_1(x)$	z	$-x_1$	$PR_1(x)$	a_1	a_1		
p_2	$-x_2$	$PR_2(x)$				a_2			
				θ	θ		a_1		a_1

Fig. 6. The matrix to Fig. 5, a place invariant i with $y \in Y_{pair}$ and $z \in Y_s$, and two markings M, M' with $M' \in [M\rangle$.

number n:

database$^{(n)}$ = **pair** +
sorts: *const*

opns: $a_1, \ldots, a_n : \to s$
$dot : \to const$
$SUC : s \to s$
$d_1 : s \to const$
$d_2 : pair \to const$

eqns: $x, y \in s$;
$SUC(a_i) = a_{i+1}\ (i = 1, \ldots, n-1)$
$SUC(a_n) = a_1$
$d_1(x) = d_2(x, y) = dot$

Figure 7 shows the database-maintaining scheme as a **data-base**$^{(n)}$-inscribed net over $X_s = \{q, r\}$. The constant "*dot*" is as usual represented as a black dot. As shorthands for the initial marking, let $D = a_1 + \cdots + a_n$ and let N be the sum of all pairs (a_i, a_j) with $i \neq j$. In Fig. 7 as well as in Fig. 8, for variables x in $X_s \cup Y_s$ we use the shorthand N_x for $(x, SUC(x)) + \cdots + (x, SUC^{n-1}(x))$.

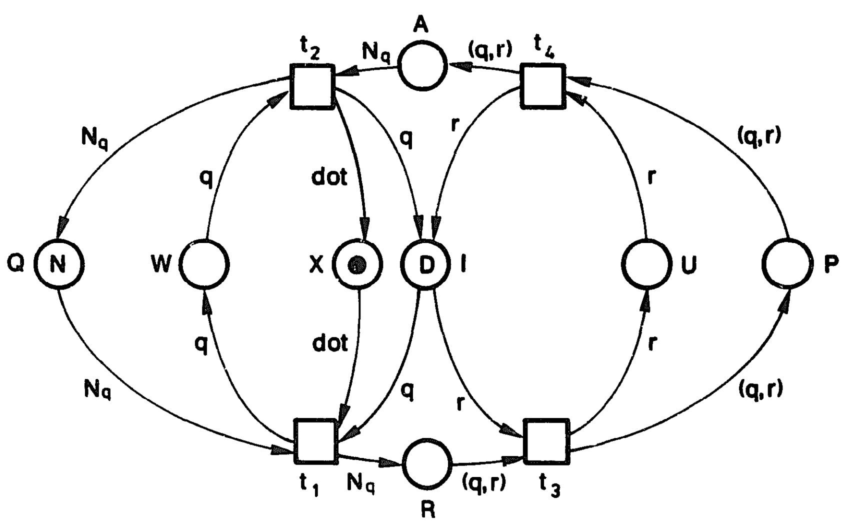

Fig. 7. Scheme for maintaining multiple copies of a database.

Initially, all sites are idle (marking D on I), all "envelopes" for messages are empty (marking N on Q) and no update requests are under way (dot on X). Transition t_1 models the dispatch of update requests by some site q, t_3 the reception of such requests by each single recipient, t_4 the dispatch of acknowledgements, and t_2 the reception of all acknowledgements by q.

Figure 8 gives the matrix, the initial marking and some place invariants over $Y = Y_s \cup Y_{pair} \cup Y_{const}$ with $Y_s = \{x\}$, $Y_{pair} = \{z\}$ and $Y_{const} = \{c\}$. For terms u and

	$\mathbf{t}_1$	$\mathbf{t}_2$	$\mathbf{t}_3$	$\mathbf{t}_4$	M_o	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9
X	$-dot$	dot			dot			c			$(n-1)c$	$(n-1)(ABS(c))$	$(n-1)(ABS(c))$	
I	$-q$	q	$-r$	r	D	x								
W	q	$-q$				x		$d_1(x)$		N_z				N_z
U			r	$-r$		x			$-x$			$ABS(x)$		
Q	$-N_q$	N_q			N		z			z	$-d_2(z)$			
R	N_q		$-(q,r)$				z					$ABS(z)$	$ABS(z)$	$-z$
P			(q,r)	$-(q,r)$			z		$pr_2(z)$				$ABS(z)$	$-z$
A		$-N_q$		(q,r)			z					$ABS(z)$	$ABS(z)$	$-z$

Fig. 8. Matrix, initial marking and nine invariants for the net in Fig. 7.

naturals n, we write nu as a shorthand for $u+\cdots+u$ (n times). A new operation symbol ABS is used in the invariants of Fig. 8. $ABS(u)$ is to give the cardinality of the multiset represented by the term u. To cover this formally, the multiset specification of 3.1 is extended to

$$\mathbf{m\_}SPEC' = \mathbf{m\_}SPEC + \mathbf{int} +$$

$$\text{opns:}\quad ABS_s : m_s \to int$$

$$\text{eqns:}\quad \left.\begin{array}{l} a \in s;\ p, q \in m_s \\ ABS_s(\vartheta_s) = 0 \\ ABS_s(MAKE(a)) = 1 \\ ABS_s(+_s(p,q)) = ABS_s(p) + ABS_s(q) \\ ABS_s(-_s(p)) = -ABS_s(p) \end{array}\right\}, \text{ for all } s \in S$$

assuming any reasonable specification **int** of the natural numbers.

7. Transition invariants

The previous section showed solutions of $\underline{N}^T \cdot i \equiv_{\hat{E}} \vartheta$ to represent properties of inscribed nets N. Symmetry and duality of the net calculus suggest to study also equations formed $\underline{N} \cdot j \equiv_{\hat{E}} \vartheta$. A solution j then assigns an object—of whatever kind—to each transition. As in the case of place invariants we have to specify which kinds of solutions we are interested in and how to derive system properties from them.

Transition invariants of place/transition nets return a natural number j_t for each transition t. Each occurrence sequence $M_0[t_1\rangle M_1 \ldots [t_n\rangle M_n$ with each t occurring j_t times, reproduces the initial marking (i.e. $M_0 = M_n$). We will here obtain a similar result, but we have of course to take into consideration the modes of transition occurrences.

Formally we must be capable of summing up several modes in which a transition may occur. This is captured by the following notion of *multi-assignments*:

7.1. Definition. Let $\Sigma = (S, OP)$ be a signature and let X, Y be sets of Σ-variables. The set $MA_\Sigma(X, Y)$ of *multiassignments over Σ, X and Y* is the smallest set of mappings $\Gamma: T_{\widehat{OP}}(X) \to T_{\widehat{OP}}(Y)$ such that

(i) for each assignment $ass: X \to T_{OP}(Y)$, $\overline{ass} \in MA_\Sigma(X, Y)$;

(ii) if $\Gamma_1, \Gamma_2 \in MA_\Sigma(X, Y)$ then $(\Gamma_1 + \Gamma_2) \in MA_\Sigma(X, Y)$ and $(-\Gamma_1) \in MA_\Sigma(X, Y)$, inductively given by $(\Gamma_1 + \Gamma_2)(u) = \Gamma_1(u) + \Gamma_2(u)$ and $(-\Gamma_1)(u) = -(\Gamma_1(u))$, respectively, for each $u \in T_{\widehat{OP}}(X)$.

A multiassignment Γ is *constant* iff for all u, $\Gamma(u) \in T_{\widehat{OP}}$.

There is a particular multiassignment ϑ_{MA}, definable as $\Gamma + (-\Gamma)$ for any Γ. Obviously, $\vartheta_{MA}(u) = \vartheta$ for each $u \in T_{\widehat{OP}}(X)$.

The product of multiset terms defined in Definition 5.2 can be considered as a special case of the above definition, based on constant (instead of arbitrary) assignments: $(u_1 + u_2) \cdot u'$ and $(-u_1) \cdot u'$ read now $(\overline{ass_{u_1}} + \overline{ass_{u_2}})u'$ and $(-\overline{ass_{u_1}})(u')$, respectively.

The occurrence count of each transition in a sequence of transition occurrences can now be defined as a constant multiassignment:

7.2. Definition. Let $N = (P, T, F)$ be an inscribed net and let $\sigma = M_0 \xrightarrow{t_1, \beta_1} \cdots \xrightarrow{t_n, \beta_n} M_n$ be an occurrence sequence in N. For each $t \in T$, the *occurrence count* Γ_t of t in σ is defined by $\Gamma_t = \Sigma\{\overline{\beta_i} \mid t_i = t\}$. ($\Sigma$ of course denotes the sum of multiset assignments.)

Initial and final markings of occurrence sequences can now be related by occurrence counts:

7.3. Theorem. *Let $N = (P, T, F)$ be an inscribed net and for each $t \in T$, let Γ_t be the occurrence count of t in $\sigma = M_0 \xrightarrow{t_1, \beta_1} \cdots \xrightarrow{t_n, \beta_n} M_n$. Then for all $p \in P$ it holds:*

$$M_n(p) = M_0(p) + \sum_{t \in T} \Gamma_t(\underline{t}(p)).$$

Proof. By induction over the length n of σ.

If $n = 0$, for all $t \in T$, the occurrence count of t in σ is $\Gamma_t = \vartheta_{MA}$. Then we get for all $p \in P$: $M_n(p) = M_0(p) = M_0(p) + \vartheta = M_0(p) + \sum_{t \in T} \vartheta_{MA}(\underline{t}(p)) = M_0(p) + \sum_{t \in T} \Gamma_t(\underline{t}(p))$.

To show the induction step, for each $t \in T$ let Γ'_t be the occurrence count of $M_0 \xrightarrow{t_1, \beta_1} \cdots \xrightarrow{t_{n-1}, \beta_{n-1}} M_{n-1}$. The definition of occurrence counts in Definition 7.2 implies

(*) $\Gamma_{t_n} = \Gamma'_{t_n} + \overline{\beta_n}$ and

(**) $\Gamma_t = \Gamma'_t$ for all $t \neq t_n$.

For each $p \in P$, let $u'_p := \sum_{t \in T \setminus \{t_n\}} \Gamma'_t(\underline{t}(p))$ and $u_p := \sum_{t \in T \setminus \{t_n\}} \Gamma_t(\underline{t}(p))$. From (**) it follows that for each $p \in P$: $u'_p = u_p$.

Now we get for each $p \in P$:

$$\begin{aligned} M_n(p) &= M_{n-1}(p) + \overline{\beta_n}(\underline{t_n}(p)) \quad \text{(by Corollary 4.6)} \\ &= M_0(p) + u'_p + \Gamma'_{t_n}(\underline{t_n}(p)) \\ &\quad + \overline{\beta_n}(\underline{t_n}(p)) \quad \text{(by the induction assumption)} \\ &= M_0(p) + u'_p + \Gamma_{t_n}(\underline{t_n}(p)) \\ &= M_0(p) + u_p + \Gamma_{t_n}(\underline{t_n}(p)) \\ &= M_0(p) + \sum_{t \in T} \Gamma_t(\underline{t}(p)). \qquad \square \end{aligned}$$

We now turn to the notion of "transition invariants" and obtain their essential properties as a corollary to the above Theorem 7.3:

7.4. Definition. Let a net $N = (P, T, F)$ be inscribed over (S, OP, E) and X. Let $\sigma = M_0 \xrightarrow{t_1, \beta_1} \cdots \xrightarrow{t_n, \beta_n} M_n$ be an occurrence sequence of N with $M_0 \equiv_{\hat{E}} M_n$. For each $t \in T$, let j_t be the occurrence count of t in σ. Then the vector $(j_t)_{t \in T}$ is a *transition invariant* of N.

7.5. Corollary. *Let a net $N = (P, T, F)$ be inscribed over (S, OP, E), and let $(j_t)_{t \in T}$ be a transition invariant of N. Then for each $p \in P$,*

$$\sum_{t \in T} j_t(\underline{t}(p)) \equiv_{\hat{E}} \vartheta.$$

Proof. This follows from Theorem 7.3 and the above definition. $\square$

To achieve in Corollary 7.5 a product notation comparable to the product of Definition 5.3, we have to define a product $j \cdot v$ for $MA_\Sigma(X, Y)$-vectors j with $T_{\widehat{OP}}(X)$-vectors v:

7.6. Definition. Let T be a finite set, let $\Sigma = (S, OP)$ be a signature and let X be a set of Σ-variables. For $j : T \to MA_\Sigma(X, Y)$ and $v : T \to T_{\widehat{OP}}(X)$ we define a product $j \cdot v \in T_{\widehat{OP}}(Y)$ by

$$j \cdot v = \sum_{t \in T} j(t)(v(t)).$$

Based on Definitions 7.4 and 7.6 we then get the following corollary.

7.7. Corollary. *Let $N = (P, T, F)$ be an inscribed net. For each $p \in P$, let $\underline{p} : T \to T_{OP}(X)$ be defined by $\underline{p}(t) := \underline{t}(p)$. If a vector $j : T \to MA_\Sigma(X, Y)$ is a transition invariant of N, then $j \cdot \underline{p} \equiv_{\hat{E}} \vartheta$.*

We close this section with some examples:

Transition invariants of the system of dining philosophers (Fig. 1) are quite simple: For each assignment ass of the only variable x, $j_{lf} = j_{rf} = \overline{ass}$ yields a transition invariant. This shows that initial markings are retained upon firing of both transitions equally often in the same modes.

For Fig. 5, we get a transition invariant (j_t, j_u) with $j_t(x_1) = a_1$, $j_t(x_2) = a_2$ and $j_u(x) = (a_1, a_2)$. Thus the initial marking is retained by t occurring in mode $\beta(x_i) = a_i$ $(i = 1, 2)$ and u occurring in mode $\beta(x) = (a_1, a_2)$.

Transition invariants for Fig. 7 are somewhat more involved: For $i = 1, \ldots, n$, let $\beta_i : \{q, r\} \to \{a_1, \ldots, a_n\}$ be defined by $\beta_i(q) = a_1$ and $\beta_i(r) = a_i$. Then we define a transition invariant $(j_{t_i})_{i=1,\ldots,4}$ by $j_{t_1} = j_{t_2} = \beta_1$ and $j_{t_3} = j_{t_4} = \beta_2 + \cdots + \beta_n$. This invariant describes an update cycle of the data base, initiated by a_1: Firing t_1 in mode β_1 describes a_1 sending messages to $a_2, \ldots, a_n$. Each of $a_2, \ldots, a_n$ then performs its local update (occurrences of t_3 and t_4 in modes $\beta_2, \ldots, \beta_n$). a_1 finally collects all commitments (occurrence of t_2 in mode β_1) and releases a dot to X, allowing for a further update cycle.

8. Homomorphic transformations of arc inscriptions

Here we investigate the effect of transforming (by extended assignments) arc inscriptions. It turns out that the overall behaviour will in general be restricted but never be extended under this kind of transformation. The behaviour is retained by the special case of bijectively renaming variables in the environment of a transition. Place invariants are retained and transition invariants are transformed by extended assignment transformations.

8.1. Definition. Let $ins = (\varphi, M_0, \lambda)$ be an inscription of a net $N = (P, T, F)$.

(i) For $t \in T$ and $ass : X \to T_{OP}(X)$, let $\lambda_{t,ass}$ be defined by

$$\lambda_{t,ass}(x, y) = \begin{cases} \overline{ass} \circ \lambda(x, y) & \text{iff } x = t \text{ or } y = t, \\ \lambda(x, y) & \text{otherwise.} \end{cases}$$

(ii) Let the inscription $ins_{t,ass}$ of N be defined as $ins_{t,ass} = (\varphi, M_0, \lambda_{t,ass})$.

As an application example for homomorphic transformation we consider the relationship among the two versions of the dining philosophers system in Fig. 1 and Fig. 3: To this end we combine both underlying specifications and augment two obvious equations: Let

phils″ = **phils** + **phils′** +
  eqns: $y \in fork$
    $LF(RU(y)) = y$
    $RF(RU(y)) = RSF(y)$

Then the assignment $ass(x)=RU(y)$, applied to both transitions of Fig. 1, yields the inscriptions of Fig. 3. One likewise transforms Fig. 3 to Fig. 1 by e.g. an assignment ass' with $ass'(y)=LF(x)$, assuming the equations $RU(LF(x))=x$ and $RSF(LF(x))=RF(x)$.

Next we investigate the behaviour of transformed nets:

8.2. Lemma. *Let ins be an inscription of a net N over (S, OP, E) and X, and let M, M' be markings of N. Let $ass: X \to T_{OP}(X)$ be an assignment, and let $\beta: X \to T_{OP}$ be an occurrence mode. Then $M \xrightarrow{t,\beta\circ ass} M'$ in (N, ins, E) iff $M \xrightarrow{t,\beta} M'$ in $(N, ins_{t,ass}, E)$.*

Proof. $M \xrightarrow{t,\beta\circ ass} M'$ for *ins* iff for each $p \in P$,

$$\begin{aligned} M'(p) &= M(p) - \bar{\beta}\circ\overline{ass}(\lambda(p,t)) + \bar{\beta}\circ\overline{ass}(\lambda(t,p)) \\ &= M(p) - \bar{\beta}(\overline{ass}\circ\lambda(p,t)) + \bar{\beta}(\overline{ass}\circ\lambda(t,p)). \end{aligned}$$

This holds iff $M \xrightarrow{t,\beta} M'$ for $ins_{t,ass}$. □

Hence each step in the transformed net corresponds to a step in the original net. Vice versa, only $\overline{ass}$-prefixed assignments in the original net correspond to steps in the transformed net. Consequently, reachability sets $[M\rangle$ under $ins_{t,ass}$ are subsets of $[M\rangle$ under *ins*.

Above we have shown that both systems in Fig. 1 and Fig. 3 can mutually be transformed by the assignments $ass(x)=RF(y)$ and $ass'(y)=LF(x)$, respectively. So Lemma 8.2 implies that both nets in fact behave equally, i.e. a step $M \xrightarrow{t,\beta} M'$ can occur in Fig. 1 if and only if a step $M \xrightarrow{t,\tilde{\beta}} M'$ can occur for some assignment $\tilde{\beta}$ in Fig. 3.

Next we show that place invariants are retained by homomorphic transformations of arc inscriptions:

8.3. Theorem. *Let ins be an inscription of a net N, let t be a transition of N and let ass be an assignment of the involved variables. Then each place invariant i of (N, ins, E) is also a place invariant of $(N, ins_{t,ass}, E)$.*

Proof. For all $u \in T$, if $u \neq t$, the vectors $\underline{u}$ are equal for both nets. So, according to Definitions 5.4 and 5.3, we have to show: If $\sum_{p\in P}\underline{t}(p)\cdot i(p) \equiv_E \vartheta$ for *ins*, then $\sum_{p\in P}\underline{t}(p)\cdot i(p) \equiv_E \vartheta$ for $ins_{t,ass}$. With Notation 4.3, $\underline{t}(p)=\lambda(t,p)-\lambda(p,t)$ for *ins* and $\underline{t}(p)=\overline{ass}\circ\lambda(t,p)-\overline{ass}\circ\lambda(p,t)$ for $ins_{t,ass}$. Now we get for $ins_{t,ass}$:

$$\begin{aligned} \sum_{p\in P}\underline{t}(p)\cdot i(p) &= \sum_{p\in P}(\overline{ass}\circ\lambda(t,p)-\overline{ass}\circ\lambda(p,t))\cdot i(p) \\ &= \sum_{p\in P}(\overline{ass}(\lambda(t,p))-\overline{ass}(\lambda(p,t)))\cdot i(p) \\ &= \sum_{p\in P}\overline{ass}(\lambda(t,p)-\lambda(p,t))\cdot i(p) \quad \text{(by 2.7)} \\ &= \sum_{p\in P}\overline{ass}((\lambda(t,p)-\lambda(p,t))\cdot i(p)) \end{aligned}$$

(by Lemma 5.5, extending *ass* to Y by identity)

$$=\overline{ass}\left(\sum_{p\in P}(\lambda(t,p)-\lambda(p,t))\cdot i(p)\right)\quad\text{(by 2.7)}$$

$$=\overline{ass}(\vartheta)\quad\text{(by } i \text{ being a place invariant of } N \text{ for } ins)$$

$$=\vartheta.\qquad\square$$

As Fig. 1 and Fig. 3 can be mutually transformed, the above theorem implies that the place invariants of both systems coincide.

Next we consider the effect of transformations to transition invariants. In contrast to place invariants, they are not retained but transformed by the assignment applied to the underlying net:

8.4. Theorem. *Let ins be an inscription of a net $N=(P,T,F)$, let t be a transition of N and let ass be an assignment of the involved variables. If $(j_{t'})_{t'\in T}$ is a transition invariant of (N, ins, E) with $j_t=j\circ\overline{ass}$ for some assignment j, the vector $(j'_{t'})_{t'\in T}$ is a transition invariant of $(N, ins_{t,ass}, E)$, with $j'_{t'}=j_{t'}$ for $t'\neq t$ and $j_t=\bar{j}$.*

Proof. It is sufficient to show

$$\overline{j\circ ass}(\underline{t}(p)) \text{ in } (N, ins, E)$$
$$=\overline{j\circ ass}(\lambda(t,p)-\lambda(p,t))$$
$$=\bar{j}(\overline{ass}\circ\lambda(t,p)-\overline{ass}\circ\lambda(p,t))$$
$$=\bar{j}(\underline{t}(p)) \text{ in } (N, ins_{t,ass}, E).\qquad\square$$

In case two inscribed nets can be mutually transformed, the above lemma and theorems imply entirely identical behaviour:

8.5. Corollary. *Let ins be an inscription of a net $N=(P,T,F)$ over (S, OP, E) and X and let ass_1, $ass_2: X\to T_{OP}(X)$ be two assignments such that $\overline{ass}_2\circ ass_1\equiv_E\overline{ass}_1\circ ass_2\equiv_E id$.*

(i) *For each marking $M: P\to T_{OP^+}$, the reachability sets $[M\rangle$ are identical for both (N, ins, E) and (N, ins_{t,ass_1}, E).*

(ii) *Both ins and ins_{t,ass_1} yield identical sets of place invariants for N.*

(iii) *If $(j_t)_{t\in T}$ is a transition invariant for (N, ins, E) then $(j_t\circ ass_2)_{t\in T}$ is a transition invariant for (N, ins_{t,ass_1}, E), and if $(j_t)_{t\in T}$ is a transition invariant for (N, ins_{t,ass_1}, E) then $(j_t\circ ass_1)_{t\in T}$ is a transition invariant for (N, ins, E).*

Proof. (i) If $M\xrightarrow{t,\beta}M'$ for *ins*, then $M\xrightarrow{t,\beta\circ ass_2\circ ass_1}M'$ for *ins* (by assumptions of the theorem). Then $M\xrightarrow{t,\beta\circ ass_2}M'$ for ins_{t,ass_1} (by Lemma 8.2).

Vice versa, if $M \xrightarrow{t,\beta} M'$ for ins_{t,ass_1}, then $M \xrightarrow{t,\beta\circ ass_1 \circ ass_2} M'$ for ins_{t,ass_1} (by assumptions of the theorem). Then, by Lemma 8.2, $M \xrightarrow{t,\beta\circ ass_1} M'$ for $ins_{t,ass_1\circ ass_2} = ins$.

(ii) follows from Theorem 8.3, as $ins_{t,ass_1\circ ass_2} = ins$.

(iii) $(j_t)_{t\in T}$ is a transition invariant of N for ins iff $(j_t \circ ass_2 \circ ass_1)_{t\in T}$ is one. Then by Theorem 8.4, $(j_t \circ ass_2)_{t\in T}$ is a transition invariant for ins_{t,ass_1}. Vice versa, $(j_t)_{t\in T}$ is a transition invariant of N for ins_{t,ass_1} iff $(j_t \circ ass_1 \circ ass_2)_{t\in T}$ is one. Then by Theorem 8.4, $(j_t \circ ass_1)_{t\in T}$ is a transition invariant of N for $ins_{t,ass_1\circ ass_2} = ins$. □

The bijective renaming of variables in the environment of transitions is a special case of this theorem.

Homomorphic transformations of arc inscriptions preserve dead transitions. If a net N has a dead transition t_0 under an inscription ins, then t_0 remains dead under each $ins_{t,ass}$. This follows directly from Lemma 8.2. Homomorphic transformations may produce additional dead transitions as Fig. 9 shows.

The situation is slightly different if we consider dead markings. Call a marking M of a net N *dead* iff M has no successor marking M', i.e. for no t and no β,

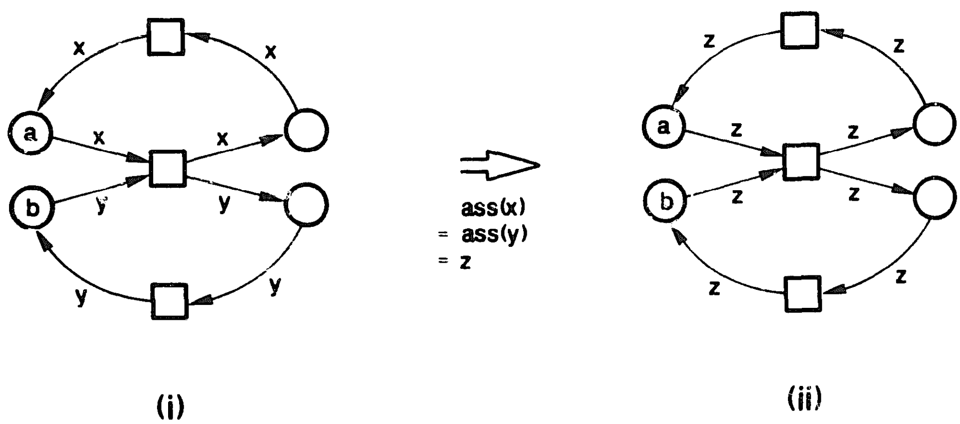

Fig. 9. A homomorphic transformation, producing dead transitions and dead markings.

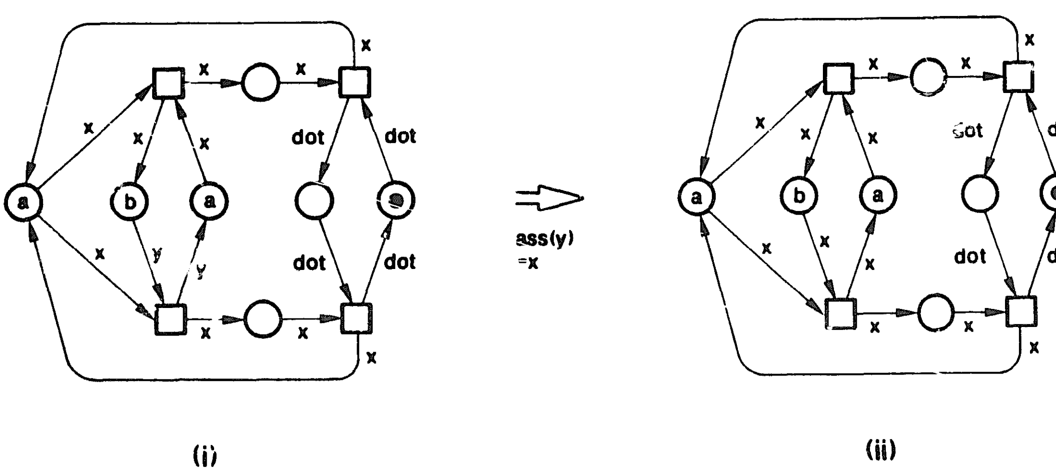

Fig. 10. A homomorphic transformation, preventing dead markings to be reachable.

$M \xrightarrow{t,\beta} M'$. It is almost obvious that homomorphic transformations may produce additional reachable dead markings, and Fig. 9 shows an example. But homomorphic transformations additionally may prevent dead markings to be reachable any more; Fig. 10 shows an example for this fact (assuming no equations).

9. Enhancing equations

Let a net N be inscribed over a specification $SPEC = (S, OP, E)$, and let E_1 be an additional set of (S, OP)-equations. Here we consider properties retained or lost by interpreting N over $SPEC1 = (S, OP, E + E1)$.

Transition occurrences, place and transition invariants, and the absence of dead transitions turn out to be retained under additional equations:

9.1. Proposition. *Let $SPEC = (S, OP, E)$ and $SPEC1 = (S, OP, E + E1)$ be two specifications and let ins be a SPEC-inscription of a net N.*

(i) *If $M \xrightarrow{t,\beta} M'$ in (N, ins, E), then $M \xrightarrow{t,\beta} M'$ also in $(N, ins, E + E_1)$.*

(ii) *For each marking M of N, the set $[M\rangle$ in (N, ins, E) is a subset of $[M\rangle$ in $(N, ins, E + E_1)$.*

Proof. (i) follows directly from Corollary 4.5.

(ii) is an immediate consequence of (i). □

The reverse of this proposition is not valid under the assumption of initial semantics. As an example, in Fig. 9 (ii) no step $M \xrightarrow{t,\beta} M'$ is possible at all, whereas the additional equation "$a = b$" leads to a live system.

9.2. Proposition. *Let SPEC, SPEC1 and ins be as in Proposition 9.1.*

(i) *Each place invariant of (N, ins, E) is also a place invariant of $(N, ins, E + E_1)$.*

(ii) *Each transition invariant of (N, ins, E) is also a transition invariant of $(N, ins, E + E_1)$.*

Proof. *If* $\underline{t} \cdot i \equiv_{\hat{E}} \vartheta$, then also $\underline{t} \cdot i \equiv_{\widehat{E+E_1}} \vartheta$. Likewise, if $j \cdot \underline{p} \equiv_{\hat{E}} \vartheta$, then $j \cdot \underline{p} \equiv_{\widehat{E+E_1}} \vartheta$. □

The reverse of this theorem does not hold. As an example, let $SPEC$ be the specification **pair** of Subsection 6.2 without the two equations given there, and let $SPEC1 =$ **pair**. Then the vector i of Fig. 6 is no invariant under $SPEC$, but under $SPEC1$.

Additional equations preserve the absence of dead transitions: If a net N has no dead transition under a specification $SPEC$, it has also no dead transition if additional equations are assumed to be valid. Additional equations may turn dead transitions into nondead ones. As an example, in Fig. 9(ii) all transitions are dead, whereas the additional equation "$a = b$" makes the net entirely live. This step shows that dead markings may turn into nondead ones, but additional equations also can lead to dead markings. As an example, Fig. 10(ii) has no reachable dead markings, whereas, with the equation "$a = b$", dead markings become reachable.

10. Extending the formalism

This paper strives at the basic notions and techniques for construction and analysis of Petri nets with structured tokens. A number of extensions make the formalism more handy for practical applications. We start with short look at using ordinary sets instead multisets. Then five generalizations of the formalism are glanced over; place capacities, transition inscriptions, extended arc inscriptions, schematic markings and more general equivalence transformations.

10.1. Place capacities

Capacity functions K_p may be assigned to places p, indicating for each item a maximum number of copies in allowable markings. Transition occurrences $M \xrightarrow{t,\beta} M'$ are discharged if in M', the multiplicity of some item d in some place $p \in t^{\bullet}$ exceeds the capacity $K_p(d)$.

In the setting of the above term calculus, an item may be represented by different ground terms u, $u' \in T_{OP}$. A capacity function for place p, $K_p : T_{OP,\varphi(p)} \to \mathbb{N} \cup \{\omega\}$ must therefore assign equal multiplicities $K_p(u) = K_p(v)$ in case $u \equiv_E v$.

In case of finitely many $\equiv_E$-equivalence classes and finite capacities for all items, the well-known construct of complements as outlined in Fig. 11 renders capacity functions superfluous. More general capacity functions cannot be implemented this way as infinite multisets cannot be represented in the term calculus.

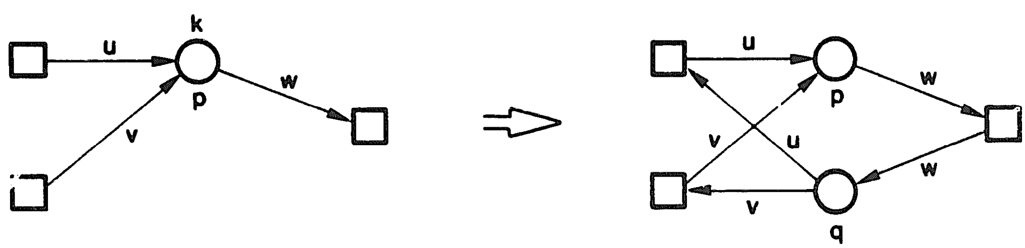

Fig. 11. Place complementation: Let $M(q)$ be such that the number of items in $M(p)+M(q)$ is just the item's capacity.

The calculi of place and transition invariants remain valid under the introduction of capacities. This holds likewise for the theorems on homomorphic transformations and additional equations.

10.2. Strict nets

A further variant of nets with structured tokens assumes markings to represent ordinary sets (instead of multisets). Transitions are prevented from occurrence in case an item to be put to a place does already belong to the place's marking. Places p in this setting represent predicates $\tilde{p}$ with variable extensions. The overall maximal extension of $\tilde{p}$ is the set of all items of sort $\varphi(p)$. The respective actual extension of $\tilde{p}$ is given by the actual markings $M(p)$. The denotation of "Predicate/Transition Nets" is due to those predicates.

This model can be considered as being based on the capacity function $K_p(d)=1$ for all places p and all d of sort $\varphi(p)$. In our formalism a specification **set**_*SPEC* could describe this model (**set**_*SPEC* should of course provide the usual operations on sets).

10.3. Transition inscriptions

Coming back to the remark following Corollary 4.6, additional predicates can be assigned to transitions. This provides a means for formulating additional requirements to the enabling of transitions. In our setting, terms of sort *bool* will do this job. Assuming in Definition 4.4 a further component $\eta: T \to T_{OP,bool}(X)$, we define in Definition 4.4(iii) a transition to be enabled in a mode β if additionally $\bar{\beta}(\eta(t)) \equiv_E TRUE$.

Given a transition inscription u, the set $U := \{\bar{\beta}(u) \mid \beta : X \to T_{OP}\}$ may decompose into finitely many $\equiv_{\hat{E}}$-equivalence classes $U = [u_1] \cup \cdots \cup [u_n]$ for some $u_1, \ldots, u_n \in T_{OP^+}$. In this case, transition inscriptions may equivalently be replaced by the construction of a loop as Fig. 12 outlines.

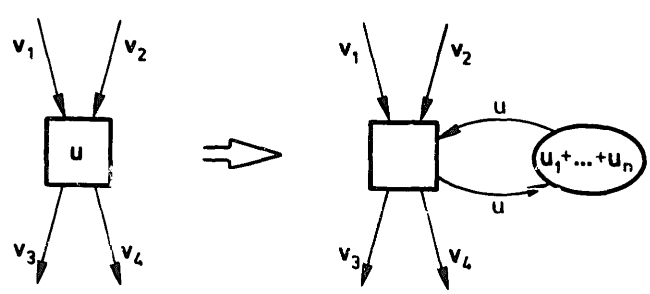

Fig. 12. Replacement of transition inscriptions using additional predicates.

The general case of *fool*-typed transition inscriptions can not be replaced this way. Below we shall discuss a further transformation for this case.

Like the above introduction of place capacities, additional transition inscriptions at most limit the overall behaviour of a net. Therefore, even in this case, the calculi of invariants as well as the theorems on homomorphisms and additional equations remain valid.

10.4. Flexible arc inscriptions

In this paper the operations on multisets are limited to the operations introduced by **m_**$SPEC$, viz. addition, negation and the constant empty multisets. Of course, one could think of more general operations on multisets, and also of using variables of multiset sorts m_s.

Introducing multiplication of terms with natural numbers is a nearby extension: As long as n represents a constant natural, with $u \in T_{\widehat{OP}}(X)$, the term $n * u$ may be considered just as a shorthand for the n-fold sum $u + \cdots + u$. A corresponding algebraic specification should then specify the items of sort s to yield a (left written) module over the integers as discussed in [21]. This extension does not principally exceed the formalism and could be considered must a "syntactic sugaring".

A more general formalism is obtained with terms $v \in T_{OP,nat}(X)$ where nat denotes the natural numbers. Then an arc inscription $v * u$ yields a "flexible throughput": At event occurrences $M \xrightarrow{t,\beta} M'$, with an arc $f = (p, t)$ or $f = (t, p)$ inscribed $v * u$, the "number of tokens flowing through f" essentially depends on $\bar{\beta}(v)$ (and hence on the chosen occurrence mode β), whereas in the nets of this paper the throughput of each arc is constant for all occurrence modes.

In a formal setting, for a given specification $SPEC$, one may consider some extended multiset specification including any kind of operations over multisets (and other sorts), particularly the product with integers. Such specifications may be formed by

$$
\begin{array}{l}
\textbf{extm\_}SPEC = \textbf{m\_}SPEC + \textbf{int} + \\
\quad \text{opns:} \; * : int\ m_s \to m_s \\
\qquad\qquad \vdots \\
\qquad\qquad F_i : m_s \ldots m_s \to m_s \\
\qquad\qquad \vdots
\end{array}
$$

With **extm_**$SPEC = (S, \widehat{\widehat{OP}})$ one can then inscribe arcs by terms in $T_{\widehat{\widehat{OP}}}(X)$ with X including variables of sort m_s.

The extension from **m_**$SPEC$ to **extm_**$SPEC$ influences of course the invariant calculi and the theorems on homomorphic transformations. Place invariants can easily be generalized if the product with terms of sort integer is the only additional operator. In this case, the theory resembles invariants for self-modifying nets [23]. Details are beyond the scope of this paper.

Products with integer terms can be used for moving the above considered transition inscriptions to arcs: With the additional operation $[\ ] : bool \to nat$, defined for $u \in T_{OP,bool}$ by $[u] = 1$ iff $u \equiv_{\hat{E}} TRUE$ and $[u] = 0$ iff $u \equiv_{\hat{E}} FALSE$, the scheme of Fig. 13 outlines a meaning-preserving transformation. This extension has been suggested in [9].

10.5. Marking schemes

The intended scheme of using nets implies markings to represent distributions of items in systems. In our formalism such items are adequately represented as ground

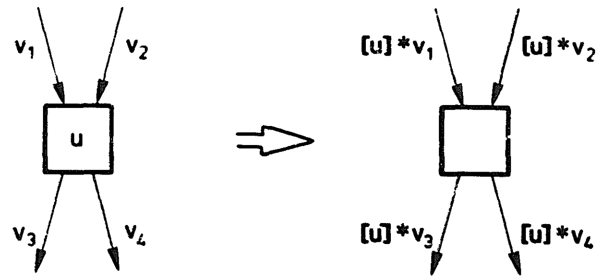

Fig. 13. Replacement of transition inscriptions using flexible arc inscriptions.

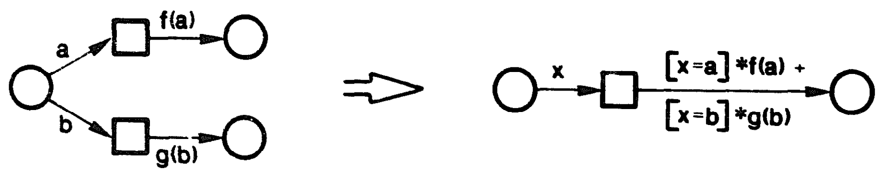

Fig. 14. Equivalent reduction of the net structure.

terms. If only partial knowledge of markings is available or if one is interested in relationships or properties of structured sets of markings, it may nevertheless be useful to consider terms with variables as markings. A mapping $M: P \to T_{OP^+}(Z)$ then can be considered as a *scheme* for all markings gained from M by assignments $ass: Z \to T_{OP}$. (Marking schemes have been suggested in [20] in a different context.) The invariant calculi (particularly Theorems 5.6, 5.7 and 7.5) remain valid, provided the set Z of variables is disjoint from the set $X \cup Y$ of variables appearing in arc inscriptions and invariant components, respectively. Under this assumption, also the theorems on homomorphic transformations and on additional equations remain.

10.6. Equivalence transformations

In the context of transition inscriptions (Subsection 10.3) and flexible arc inscriptions (Subsection 10.4), we discussed already some equivalence transformations, keeping the underlying net structure untouched. Equivalence transformations may change the net structure itself. Figure 14 outlines an example. Genrich [10] introduces a complete list of such transformations for inscriptions based on n-tuples (cf. Subsection 6.2).

A systematic approach to such transformations in the style of this paper might include an algebraic specification of the underlying net structure itself (as e.g., in [18]). Net transformations can then be formulated as algebra homomorphisms.

11. Conclusion

11.1. The formalisms of Predicate/Transition nets and coloured nets

The several versions and variants of Petri nets with structured tokens can roughly be divided into two groups. The first kind of formalisms is based on the idea of "dynamizing" predicate logic, using predicates with changing extensions. We denote them in the following "Predicate/Transition nets" (PrT-nets) in accordance with their introduction in [11, 12]. PrT-nets are schemes of system models with n-tuples of expressions (including variables) as arc inscriptions and as invariant entries. The product of such expressions is assumed to be commutative. Due to the use of variables, formal expressions, products and sums, PrT-nets are essentially a syntax-based formalism.

A second, more semantically oriented line of models is based on "coloured tokens". We call such models in the following "coloured nets", according to their first introduction in [14]. Coloured nets have been motivated as shorthands for conventional Petri nets. Sums of functions serve as arc inscriptions and as invariant entries. Their product is based on the composition of functions.

A greater number of papers relate or reformulate various versions and aspects of Petri nets with structured tokens: Two different types of place invariants have been suggested in [13, 19] for PrT-nets; techniques to easier construct place invariants for coloured nets are discussed in [15] and for special PrT-nets in [25]. Different versions of high level nets are compared and interrelated in [16, 19, 21]. Recent reformulations of PrT-nets and coloured nets include [8, 17].

In [21] we aimed at a transparent mathematical treatment of Petri nets with structured tokens, suggesting the set of multirelations over some given set to be taken as the underlying domain of the formalism. Multirelations form a homogeneous semantical domain, in fact an integer module. This approach in the above classification is totally semantics-oriented, and close to coloured Petri nets.

In this paper we follow the opposite way, suggesting a heterogeneous, syntax-oriented approach which resembles PrT-nets. The formalism for (place) invariants differs however substantially from PrT-nets: We replaced the formal, commutative product by—in general—noncommutative term substitution. As term substitution is essentially the syntactical analogon to the composition of functions, our (place) invariants resemble those of coloured nets. In [24], term substitution was first suggested to base the invariant calculus upon; cf. also [22].

Syntax oriented approaches are useful because each system model needs a syntactical representation for being communicated or implemented. The explicit use of terms including variables is common mathematics. We use them by means of a lot of concepts which are standard notions in general algebra, hence algebraic specifications are a natural basis for a formalism to deal with structured tokens. This idea has been suggested several times and will be considered next.

11.2. Combining Petri nets and algebraic specifications

A couple of papers combine Petri nets and algebraic specifications. The specification language SEGRAS [18] includes an abstract specification of strict nets (also called Predicate/event nets) in the sense of Subsection 10.2 and uses partial operations. Based on the specification language OBJ2, Battiston et al. [2] abstractly specify a particular class of nets, consisting of superposed automata with special requirements to the terms occurring as arc inscriptions. Tokens have a constant individuality and a variable data part. Place invariants are based on a product which essentially tests equality of terms, in the style of [13]. A special product is defined for transition invariants according to [19]. Berthomien et al. [3] and Vautherin [24] suggest multisets of variables and of terms, respectively (whereas we use particular terms to handle multisets). Vautherin [26] additionally introduces a formalism of place invariants, called "type1-semiflows", based on multiple occurrences of terms. The tokens of PROT nets [1], are essentially Pascal records to be transformed according to transition inscriptions.

An important aspect in all those papers concerns the border between the abstract data type formalism and what is formulated in usual mathematics. In SEGRAS and OBJSA [18, 30, 2], everything under consideration is abstractly specified, including the involved nets and the occurrence rule. On the other hand, Berthomien et al. [3] and Vautherin [26] keep the formally specified parts quite limited: arcs are inscribed by multisets of terms, and the occurrence rule is formulated with respect to interpreted net schemes. Billington [4] presents a similar approach, adding threshold inhibitor inscriptions and capacities.

Reference [27] suggests net inscriptions over specifications similar to our formalism. For analysis purposes they derive place/transition nets from given high-level nets, and they consider reachability trees. A couple of papers besides algebraic techniques also employ category theory. Reference [29] defines parameterized net schemes and structuring concepts, [30] introduces a notion of "implementation", and [28] gives several versions of semantics of the same schematic inscription.

We saw in Definition 4.4 that enabledness of a transition is already a semantical notion in the sense that enabledness depends on the validity of equations: On the purely syntactical, uninterpreted level, a transition may appear not enabled, whereas equations may cause enabledness. In [22], this was covered by a mixture of syntactical and semantical concepts: Arc inscriptions were multiset terms, i.e. syntactical constructs, whereas multisets over concrete algebras were taken as markings.

In this paper we consequently apply corresponding concepts of initial algebra semantics: Ground terms are to represent markings. They are to be considered equivalent if and only if they belong to one equivalence class induced by the involved equations.

We aimed at obtaining at a particularly adequate border and integration of general algebra and nets. In fact, it turned out that the essential aspects of nets with structured tokens can concisely be represented this way and, vice versa, that a lot of elementary concepts of abstract data types have been applied. This includes ground terms for markings, general terms as arc inscriptions, assignments and initiality for the occurrence rule and term substitution for place and transition invariants.

On this basis, one might hope that more involved problems of nets with structured tokens will be adequately solvable by more involved but well-known algebraic techniques. This includes particularly systematic net transformations and equivalence notions.

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