In [1]:
```python
import sympy

def rat(expr):
    for i in expr.atoms(sympy.Float):
        r = sympy.Rational(str(i)).limit_denominator(1000)
        expr = expr.subs(i, r)
    return expr
```

In [2]:
```python
from IPython.display import display, Markdown
import sympy
from sympy import *
import numpy as np
x, xi, u = symbols(r'x \xi u', real=True)
d1, d2, d3 = symbols(r'd_1 d_2 d_3', real=True)

Xi = [ -1/sqrt(3), 1/sqrt(3) ]
W = [ 1, 1 ]

kappa = Function(r"K")(xi)
dkappa_du = Function(r"K_{,u}")(xi)

Na = [
1/2*xi*(xi-1),
1-xi*xi,
1/2*xi*(xi+1)
]

display( Markdown( f"$N_1 = {sympy.latex(Na[0])}$" ))
display( Markdown( f"$N_2 = {sympy.latex(Na[1])}$" ))
display( Markdown( f"$N_3 = {sympy.latex(Na[2])}$" ))
```

$N_1 = 0.5\xi\left(\xi - 1\right)$

$N_2 = 1 - \xi^2$

$N_3 = 0.5\xi\left(\xi + 1\right)$

In [3]:
```python
dN1_xi = diff(Na[0], xi)
dN2_xi = diff(Na[1], xi)
dN3_xi = diff(Na[2], xi)

display( Markdown( r"$dN_1/d\xi = "+ f"{sympy.latex(dN1_xi)}$" ))
display( Markdown( r"$dN_2/d\xi = "+ f"{sympy.latex(dN2_xi)}$" ))
display( Markdown( r"$dN_3/d\xi = "+ f"{sympy.latex(dN3_xi)}$" ))
```

$dN_1/d\xi = 1.0\xi - 0.5$

$dN_2/d\xi = -2\xi$

$dN_3/d\xi = 1.0\xi + 0.5$

In [4]:
```python
X = Na[1] * 0.5 + Na[2]
X = simplify(X)
dx_dxi = diff(X,xi)
dxi_dx = 1/dx_dxi

display( Markdown( r"$x(\xi) = "+ f"{sympy.latex(X)}$"))
```

```
display( Markdown( r"$dx/d\xi = "+ f"{sympy.latex(dx_dxi)}$" ))
display( Markdown( r"$d\xi/dx = "+ f"{sympy.latex(dxi_dx)}$" ))
```

$x(\xi) = 0.5\xi + 0.5$

$dx/d\xi = 0.5$

$d\xi/dx = 2.0$

In [5]:
```
dN1_x = dN1_xi * dxi_dx
dN2_x = dN2_xi * dxi_dx
dN3_x = dN3_xi * dxi_dx

display( Markdown( r"$dN_1/dx = "+ f"{sympy.latex(dN1_x)}$" ))
display( Markdown( r"$dN_2/dx = "+ f"{sympy.latex(dN2_x)}$" ))
display( Markdown( r"$dN_3/dx = "+ f"{sympy.latex(dN3_x)}$" ))
```

$dN_1/dx = 2.0\xi - 1.0$

$dN_2/dx = -4.0\xi$

$dN_3/dx = 2.0\xi + 1.0$

In [11]:
```
def build_N_x( xi_ ) :
    # Derivatives in X space
    N1_x = dN1_xi.subs(xi, xi_) * dxi_dx
    N2_x = dN2_xi.subs(xi, xi_) * dxi_dx
    N3_x = dN3_xi.subs(xi, xi_) * dxi_dx
    return N1_x, N2_x, N3_x

db = [ d1, d2, d3 ]
ne = [ 0, 0, 0 ]

f = symbols(r'f_1 f_2 f_3', real=True)
h = symbols(r'h', real=True)
fe = [ 0, 0, 0 ]
dna_ddb = zeros( 3, 3)
fcol = []
for xi_, W_ in zip( Xi, W ) :
    Na_x_ = build_N_x(xi_)
    x_ = X.subs(xi, xi_)
    kappa_ = kappa.subs( xi, xi_ )
    dkappa_du_ = dkappa_du.subs( xi, xi_ )
    fcol.append(kappa_)
    fcol.append(dkappa_du_)
    Na_ = [0, 0, 0]
    for a in range(3) : Na_[a] = simplify(Na[a].subs(xi, xi_))

    q = 0
    for b in range(3) : q += db[b] * Na_x_[b]
    for a in range(3) :
        ne[a] += W_ * dx_dxi * Na_x_[a] * q * kappa_
        fe[a] += W_ * dx_dxi * Na_[a] * f[a]
        fe[a] = simplify(fe[a])

    for a in range(3) :
        for b in range(3) :
            dna_ddb[a,b] += W_ * dx_dxi * Na_x_[a] * Na_[b] * q * dkappa_du_
            dna_ddb[a,b] += W_ * dx_dxi * kappa_ * Na_x_[a] * Na_x_[b]
```

```
    fe[0] += W_ * h * Na_[0]
    fe[0] = simplify(fe[0])

for a in range(3) :
    ne[a] = cancel(ne[a])
    for c in fcol : ne[a] = collect(ne[a],c)
    for d in db : ne[a] = collect( ne[a], d )
    ne[a] = simplify(ne[a],full=True)

for a in range(3) :
    fe[a] = nsimplify(fe[a])

for a in range(3) :
    for b in range(3) :
        dna_ddb[a,b] = cancel(dna_ddb[a,b])
        for c in fcol : dna_ddb[a,b] = collect(dna_ddb[a,b],c)
        for d in db : dna_ddb[a,b] = collect( dna_ddb[a,b], d )
        dna_ddb[a,b] = simplify(dna_ddb[a,b])
```

In [12]:
```
for a in range(3) :
    display( Markdown( f"$n_{a+1}^e(d^e) = {sympy.latex(rat(ne[a]))}$" ))
```

$n_1^e(d^e) = \left(d_1\left(\frac{7}{6} - \frac{2\sqrt{3}}{3}\right) - d_2\left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right) + \frac{d_3}{6}\right) K\left(\frac{\sqrt{3}}{3}\right) + \left(d_1\left(\frac{2\sqrt{3}}{3} + \frac{7}{6}\right) - d_2\left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right) + \frac{d_3}{6}\right) K\left(-\frac{\sqrt{3}}{3}\right)$

$n_2^e(d^e) = -\left(d_1\left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right) - \frac{8d_2}{3} + d_3\left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right)\right) K\left(\frac{\sqrt{3}}{3}\right) - \left(d_1\left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right) - \frac{8d_2}{3} + d_3\left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right)\right) K\left(-\frac{\sqrt{3}}{3}\right)$

$n_3^e(d^e) = \left(\frac{d_1}{6} - d_2\left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right) + d_3\left(\frac{7}{6} - \frac{2\sqrt{3}}{3}\right)\right) K\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{d_1}{6} - d_2\left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right) + d_3\left(\frac{2\sqrt{3}}{3} + \frac{7}{6}\right)\right) K\left(\frac{\sqrt{3}}{3}\right)$

In [13]:
```
for a in range(3) :
    display( Markdown( f"$f_{a+1}^e = {sympy.latex(rat(fe[a]))}$" ))
```

$f_1^e = \frac{f_1}{6} + \frac{h}{3}$

$f_2^e = \frac{2f_2}{3}$

$f_3^e = \frac{f_3}{6}$

In [15]:
```
for a in range(3) :
    for b in range(3) :
        display( Markdown( r"$\frac{\partial n_"+f"{a+1}"+r"^e}{\partial d_"+f"{b+1}"+r"^e} $ ="+ f"${sympy.latex(rat(dna_ddb[a,b]))}$" ))
    print("\n\n")
```

$\frac{\partial n_1^e}{\partial d_1^e}$ = $\left(d_1\left(\frac{19}{36} - \frac{11\sqrt{3}}{36}\right) - d_2\left(\frac{5}{9} - \frac{\sqrt{3}}{3}\right) + \frac{d_3\left(1-\sqrt{3}\right)}{36}\right) K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(d_1\left(\frac{19}{36} + \frac{11\sqrt{3}}{36}\right) - d_2\left(\frac{5}{9} + \frac{\sqrt{3}}{3}\right) + \frac{d_3\left(1+\sqrt{3}\right)}{36}\right) K_{,u}\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{2\sqrt{3}}{3} + \frac{7}{6}\right) K\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{7}{6} - \frac{2\sqrt{3}}{3}\right) K\left(\frac{\sqrt{3}}{3}\right)$

$\frac{\partial n_1^e}{\partial d_2^e}$ = $\left(d_1\left(\frac{7}{9} - \frac{4\sqrt{3}}{9}\right) - d_2\left(\frac{8}{9} - \frac{4\sqrt{3}}{9}\right) + \frac{d_3}{9}\right) K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(d_1\left(\frac{4\sqrt{3}}{9} + \frac{7}{9}\right) - d_2\left(\frac{4\sqrt{3}}{9} + \frac{8}{9}\right) + \frac{d_3}{9}\right) K_{,u}\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right) K\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right) K\left(\frac{\sqrt{3}}{3}\right)$

$\frac{\partial n_1^e}{\partial d_3^e}$ = $\left(-d_1\left(\frac{5}{36} - \frac{\sqrt{3}}{12}\right) + d_2\left(\frac{1}{9} - \frac{\sqrt{3}}{9}\right) + \frac{d_3\left(1+\sqrt{3}\right)}{36}\right) K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(-d_1\left(\frac{5}{36} + \frac{\sqrt{3}}{12}\right) + d_2\left(\frac{1}{9} + \frac{\sqrt{3}}{9}\right) + \frac{d_3\left(1-\sqrt{3}\right)}{36}\right) K_{,u}\left(-\frac{\sqrt{3}}{3}\right) + \frac{K\left(-\frac{\sqrt{3}}{3}\right)}{6} + \frac{K\left(\frac{\sqrt{3}}{3}\right)}{6}$

$\frac{\partial n_2^e}{\partial d_1^e}$ = $\left(-d_1\left(\frac{5}{9} - \frac{\sqrt{3}}{3}\right) + \frac{4d_2\left(1-\sqrt{3}\right)}{9} + d_3\left(\frac{1}{9} + \frac{\sqrt{3}}{9}\right)\right) K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(-d_1\left(\frac{5}{9} + \frac{\sqrt{3}}{3}\right) + \frac{4d_2\left(1+\sqrt{3}\right)}{9} + d_3\left(\frac{1}{9} - \frac{\sqrt{3}}{9}\right)\right) K_{,u}\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right) K\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right) K\left(\frac{\sqrt{3}}{3}\right)$

$$\frac{\partial n_2^e}{\partial d_2^e} = -\left(d_1\left(\frac{8}{9}-\frac{4\sqrt{3}}{9}\right)-\frac{16d_2}{9}+d_3\left(\frac{4\sqrt{3}}{9}+\frac{8}{9}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right)-\left(d_1\left(\frac{4\sqrt{3}}{9}+\frac{8}{9}\right)-\frac{16d_2}{9}+d_3\left(\frac{8}{9}-\frac{4\sqrt{3}}{9}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right)+\frac{8K\left(-\frac{\sqrt{3}}{3}\right)}{3}+\frac{8K\left(\frac{\sqrt{3}}{3}\right)}{3}$$

$$\frac{\partial n_2^e}{\partial d_3^e} = \left(d_1\left(\frac{1}{9}-\frac{\sqrt{3}}{9}\right)+\frac{4d_2\left(1+\sqrt{3}\right)}{9}-d_3\left(\frac{5}{9}+\frac{\sqrt{3}}{3}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right)+\left(d_1\left(\frac{1}{9}+\frac{\sqrt{3}}{9}\right)+\frac{4d_2\left(1-\sqrt{3}\right)}{9}-d_3\left(\frac{5}{9}-\frac{\sqrt{3}}{3}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right)-\left(\frac{4}{3}-\frac{2\sqrt{3}}{3}\right)K\left(-\frac{\sqrt{3}}{3}\right)-\left(\frac{2\sqrt{3}}{3}+\frac{4}{3}\right)K\left(\frac{\sqrt{3}}{3}\right)$$

$$\frac{\partial n_3^e}{\partial d_1^e} = \left(\frac{d_1\left(1-\sqrt{3}\right)}{36}+d_2\left(\frac{1}{9}+\frac{\sqrt{3}}{9}\right)-d_3\left(\frac{5}{36}+\frac{\sqrt{3}}{12}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right)+\left(\frac{d_1\left(1+\sqrt{3}\right)}{36}+d_2\left(\frac{1}{9}-\frac{\sqrt{3}}{9}\right)-d_3\left(\frac{5}{36}-\frac{\sqrt{3}}{12}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right)+\frac{K\left(-\frac{\sqrt{3}}{3}\right)}{6}+\frac{K\left(\frac{\sqrt{3}}{3}\right)}{6}$$

$$\frac{\partial n_3^e}{\partial d_2^e} = \left(\frac{d_1}{9}-d_2\left(\frac{8}{9}-\frac{4\sqrt{3}}{9}\right)+d_3\left(\frac{7}{9}-\frac{4\sqrt{3}}{9}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right)+\left(\frac{d_1}{9}-d_2\left(\frac{4\sqrt{3}}{9}+\frac{8}{9}\right)+d_3\left(\frac{4\sqrt{3}}{9}+\frac{7}{9}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right)-\left(\frac{4}{3}-\frac{2\sqrt{3}}{3}\right)K\left(-\frac{\sqrt{3}}{3}\right)-\left(\frac{2\sqrt{3}}{3}+\frac{4}{3}\right)K\left(\frac{\sqrt{3}}{3}\right)$$

$$\frac{\partial n_3^e}{\partial d_3^e} = \left(\frac{d_1\left(1-\sqrt{3}\right)}{36}-d_2\left(\frac{5}{9}-\frac{\sqrt{3}}{3}\right)+d_3\left(\frac{19}{36}-\frac{11\sqrt{3}}{36}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right)+\left(\frac{d_1\left(1+\sqrt{3}\right)}{36}-d_2\left(\frac{5}{9}+\frac{\sqrt{3}}{3}\right)+d_3\left(\frac{19}{36}+\frac{11\sqrt{3}}{36}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right)+\left(\frac{7}{6}-\frac{2\sqrt{3}}{3}\right)K\left(-\frac{\sqrt{3}}{3}\right)+\left(\frac{2\sqrt{3}}{3}+\frac{7}{6}\right)K\left(\frac{\sqrt{3}}{3}\right)$$

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In [ ]:
```