Lecture #15, March 6, 2024.

Summary of Functional Anal. tools used in Gal & SUPG proofs of conv. and error est's

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Inner products
Norms
Semi-norms
Natural norms
Peter - Paul (Young's ineq.)
Poincaré ineq.
Norm equivalence
Inverse estimates v^h
Interpolation est's. $e = u^h - u$.
Bilinear forms / linear forms.

Sobolev embeddings.
Cauchy - Schwarz.
Sobolev norms.

Generalization to multi-D.

$\mathcal{L}u = f$ AD eq.

$$\mathcal{L}u = a\cdot\nabla u - \overbrace{\nabla\cdot(\kappa\nabla u)}^{\kappa\Delta u} \quad \text{PDE}$$

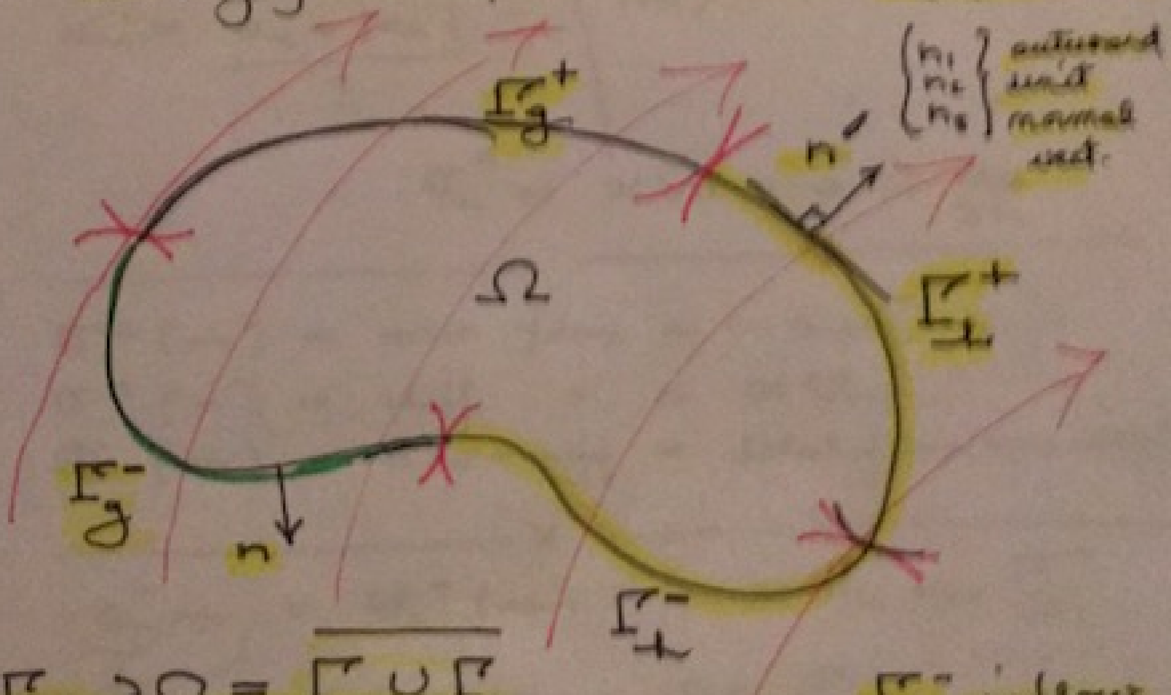
$a = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$, $\nabla\cdot a = 0$ if κ = const

Solenoidal

$x = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ $\nabla = \begin{Bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{Bmatrix}$

$$\Delta u = u_{,x_1x_1} + u_{,x_2x_2} + u_{,x_3x_3}$$

Strong form of BVP $= u_{,x_ix_i} = u_{,ii}$



$\Gamma = \partial\Omega = \overline{\Gamma_g \cup \Gamma_h}$ Γ^- inflow part of body.

$\Gamma_g^\pm = \Gamma_g \cap \Gamma^\pm$, $\Gamma_h^\pm = \Gamma_h \cap \Gamma^\pm$

$$a \cdot n = a_n(x)$$

$$\Gamma^- = \{x \mid a_n(x) < 0, \; x \in \Gamma\}$$

$$\Gamma^+ = \Gamma \setminus \Gamma^-$$

Dirichlet BCs $\quad u = g$ on Γ_g.

Flux BCs (Neumann/Robin)

$$-\underbrace{a_n^-}\, u + \sigma_n^d(u) = h^\pm \text{ on } \Gamma_h^\pm$$

$$a_n^- = \frac{a_n - |a_n|}{2} \qquad a_n = a_n^- + a_n^+$$

$$a_n^+ = \frac{a_n + |a_n|}{2} \qquad |a_n| = a_n^+ - a_n^-$$

$$\sigma_n^d = \alpha \nabla u \cdot n = \alpha \frac{\partial u}{\partial n}$$

$\sigma^a(u)$ = adv flux = $-au$

$\sigma^d(u)$ = diff. " = $\alpha \nabla u$

$\sigma(u)$ = adv + diff = total flux = $-au + \alpha \nabla u$

$$-\underbrace{a_n^-}_{<0}\, u + \sigma_n^d(u) = h^- \text{ on } \underline{\Gamma_h^-}$$

$\oplus$

total flux BC

Robin BC.

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$$\underbrace{-\bar{a}_n}_{0} u + \boxed{\sigma_n^d(u) = h^+ \text{ on } \Gamma_h^+}$$

diff flux BC.

Neumann BC.

Strong form of BVP.

Given $f, g, h^{\pm}, a, \alpha, \Omega, \Gamma_g^{\pm}, \Gamma_h^{\pm}$

Find, u satisfying, PDE, Dirichlet, Neumann, Robin BCs on the approp. portions of the bdry.

Weak Form $\mathcal{S}, \mathcal{V}$.

(W) $$B(w, u) = L(w) \quad \forall w \in \mathcal{V}.$$

$$u = \underset{\mathcal{V}}{\underset{\cap}{v}} + \underset{\mathcal{S}}{\underset{\in}{g}}^{ext}.$$

$$L(w) = (w, f)_\Omega + (w, h)_{\Gamma_h}$$

$$\mathcal{V} = \{w \mid w \in H^1(\Omega),\ w = 0 \text{ on } \Gamma_g\}$$

$$\mathcal{S} = \{u \mid u \in H^1(\Omega),\ u = g \text{ on } \Gamma_g.$$

$$B(w, u) = (\nabla w, \underset{-au + \alpha\nabla u}{\underset{\shortparallel}{\sigma}}(u)) + (\underset{\substack{0 \\ \text{on} \\ \Gamma_g}}{\underset{\shortparallel}{w}}, a_n^+ u)_{\Gamma_h^+ / \frac{\Gamma_h}{\Gamma}}$$

Conversion to Euler-Lag. form (assuming smoothness); expose consistency.

$$\int_{\Omega} w_{,i}\left(-a_i u + \kappa u_{,i}\right) d\Omega + \int_{\Gamma/\Gamma_h^+} w\, a_n^+ u \, d\Gamma \quad ✓$$

$B(w,u)$ int. by parts

$$= \int_{\Omega} w\Big(+\underbrace{(a_i u)_{,i}}_{\underbrace{a_{i,i}}_{=0} u + a_i u_{,i}} - (\kappa u_{,i})_{,i}\Big) d\Omega$$

$$+ \int_{\Gamma} w\Big(-\underbrace{a_i n_i}_{a_n} u + \underbrace{\kappa u_{,i} n_i}_{\kappa\, n\cdot\nabla u = \kappa \frac{\partial u}{\partial n}}\Big) d\Gamma$$

$$= \int_{\Omega} w\big(a\cdot\nabla u - \nabla\cdot(\kappa \nabla u)\big) d\Omega$$

$$+ \int_{\Gamma}\big(-w\, a_n u + w\,\kappa\, \partial u/\partial n\big) d\Omega$$

$$+ \int_{\Gamma}\left(w\, a_n^+ u\right) d\Gamma = \int_{\Gamma} w\,\underbrace{(a_n^+ - a_n)}_{\cancel{a_n^+} - (\cancel{a_n^+} + a_n^-)} u$$

$$= -\int_{\Gamma} w\, a_n^- u \quad ✓$$

$$0 = B(w,u) - L(w)$$

$$L(w) = \int_{\Omega} w f \, d\Omega + \int_{\Gamma/\Gamma_h} w h$$

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$$0 = \int_\Omega w \left(a \cdot \nabla u - \nabla \cdot (\kappa \nabla u) - f \right) d\Omega$$

$$+ \int_{\Gamma_h} \underset{\overset{\|}{0 \text{ on } \Gamma_g}}{w} \Big(\underbrace{-a_n^- u + \overbrace{\kappa\, \partial u / \partial n}^{\sigma_n^d(u)} - h}_{\text{PDE}} \Big) d\Gamma$$

$$+ \int_{\Gamma_h^+} w \Big(\underbrace{\underbrace{-a_n^-}_{0} u + \sigma_n^d(u) - h^+}_{\text{Neumann}} \Big) d\Gamma$$

$$+ \int_{\Gamma_h^-} w \Big(\underbrace{\underbrace{-a_n^-}_{+} u + \sigma_n^d(u) - h^-}_{\text{Robin}} \Big) d\Gamma$$

Dirichlet, built into $\mathcal{S}$.

Consis. $B(w, u) = L(w)$ for u (S)

Stability (coercivity)

$$B(w, w) \geq |||w|||^2 \leftarrow \text{stab. norm (?)}$$

$$B(w, w) = \int_\Omega w_{,i} (-a_i w + \kappa w_{,i}) d\Omega$$

$$+ \int_\Gamma w\, a_n^+ w \, d\Gamma$$

$$= \kappa \|\nabla w\|_\Omega^2$$

$$\int_\Omega \big(\underbrace{w_{,i} - a_i w}_{-\frac{1}{2}(w^2)_{,i}\, a_i} \big)\, d\Omega \qquad \underbrace{a_{i,i}}_{0}$$

$$\int_\Omega -\frac{1}{2}(w^2 a_i)_{,i}$$

$$= \int_\Gamma \Big(-\frac{1}{2}\, w^2 \underbrace{a_i n_i}_{a_n} + w^2 a_n^+ \Big)\, d\Gamma$$

$$a_n^+ - \frac{1}{2} a_n$$

$$\underbrace{a_n^+ - \frac{a_n^+ + a_n^-}{2}}$$

$$\frac{a_n^+ - a_n^-}{2} = |a_n| \text{ by def.}$$

$$\int_\Gamma |a_n|\, w^2 = \big\| \underbrace{w}_{0 \text{ on } \Gamma_g} |a_n|^{1/2} \big\|^2_{\Gamma/\Gamma_h}$$

$$B(w,w) = \alpha \|\nabla w\|^2_\Omega + \| |a_n|^{1/2} w \|^2_{\Gamma = \Gamma_h}$$

$$\overset{\text{def.}}{=} |||w|||^2$$

$$> 0. \quad \checkmark$$