Lecture 17, March 20, 2024.

(W) locally, one time-interval at a time.

Find $u^h\big|_{(t_n,t_{n+1})} \in \mathcal{V}^h(t_n,t_{n+1})$, $\ni$

$\forall\, w^h\big|_{(t_n,t_{n+1})}$

$$B(w^h,u^h)_n = L(w^h)_n \overset{\text{def}}{=} \int_{t_n}^{t_{n+1}} w^h f\,dt$$

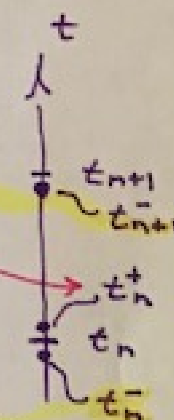
(a) int.-by-parts form

$$\overset{\text{def.}}{=} \int_{t_n}^{t_{n+1}} -1\, w^h_{,t}\, u^h\, dt \;+$$

"Bndy terms"

$$+\; 1^{(a)}\; w^h(t_{n+1}^-)\, u^h(t_{n+1}^-)$$

$$-\; 1\; w^h(t_n^+)\, u^h(t_n^-)$$



sol. from top of previous time int

To do:

Consistency with (S)
Euler-Lag. residuals correspond to (S).
int-by-parts

$$\to\; u^h(t_0^-) = u^h(0^-) \overset{\text{def.}}{=} u_0 = \text{given init. cond.}$$

$$w^h(t_0^-) = w^h(0^-) \overset{\text{def.}}{=} 0.$$

Remark DG takes values from neighbor elements.

$$0 = + \int_{t_n}^{t_{n+1}} w^h \overbrace{\left(u^h_{,t} - f\right)}^{\text{res. of the diff. eq.}} dt$$ 85.

$$- \left. w^h u^h \right|_{t_n^+}^{t_{n+1}^-}$$

$$+ \text{"Bndy terms"}$$

int by parts

$$\equiv \begin{pmatrix} - \cancel{w^h(t_{n+1}^-) u^h(t_{n+1}^-)} \\ + w^h(t_n^+) u^h(t_n^+) \\ + \cancel{w^h(t_{n+1}^-) u^h(t_{n+1}^-)} \\ - w^h(t_n^+) u^h(t_n^-) \end{pmatrix} \equiv$$

$$= + \int_{t_n}^{t_{n+1}} w^h \overbrace{\left(u^h_{,t} - f\right)}^{\text{res.}} dt$$

$$+ \; w^h(t_n^+) \overbrace{\left(u^h(t_n^+) - u^h(t_n^-)\right)}^{\text{init cond. res.}}$$

$$\underbrace{[\![u^h(t_n)]\!]}_{\text{jump op.}} = u^h(t_n^+) - u^h(t_n^-)$$

1.) Causality is built in. $\uparrow t$.

Global prob.

Find $u^h \in \mathcal{V}^h = \bigoplus_{n=0}^{N-1} \mathcal{P}^k(t_n, t_{n+1})$

$\ni \forall w^h \in \mathcal{V}^h,$

$$\mathbb{B}(w^h; u^h) = \mathbb{L}(w^h)$$

$$\sum_{n=0}^{N-1} B(w^h; u^h)_n = \sum_{n=0}^{N-1} L(w^h)_n$$

---

exact prob. $\mathbb{B}(w, u) = \mathbb{L}(w) \quad \forall w \in \mathcal{V}.$

$u \in \mathcal{V} = \bigoplus_{n=0}^{N-1} H^1(t_n, t_{n+1}) \leftarrow$ Broken Sobolev sp.

$$\Rightarrow \mathbb{B}(w^h; u) = \mathbb{L}(w^h) \quad \forall w^h \in \mathcal{V}^h \subset \mathcal{V}.$$

$\mathbb{B}(w^h, \underbrace{u^h - u}_{e \in \mathcal{V}.}) = 0.$ Gal. orth.

Global stab.

stab. norm $[\![w^h]\!]^2 = \frac{1}{2} \Big(w^h(T^-)^2 + w^h(0^+)^2 + \sum_{n=1}^{N-1} [\![w^h(t_n)]\!]^2 \Big)$

(DG norm.

wait for pf. next time.

Remarks: Prop's of DG.

1.) Local conservation $\Rightarrow$ Global

2.) " consistency ✓ $\Rightarrow$ " ✓

3.) " stab. ✓ $\Rightarrow$ " ✓

Local conservation = first integral.

set $w^h \equiv 1$ on (t_n, t_{n+1}); zero elsewhere

$w^h(t_n^+) = w^h(t_{n+1}^-) = 1$

$$B(1, u^h)_n = L(1)$$

$$= \int_{t_n}^{t_{n+1}} -\cancel{(1)_{,t}}^{\,0}\, u^h \, dt + 1 \cdot u^h(t_{n+1}^-) - 1 \cdot u^h(t_n^-)$$

$$= \int_{t_n}^{t_{n+1}} 1 \cdot f \, dt$$

$$\boxed{u^h(t_{n+1}^-) = u^h(t_n^-) + \int_{t_n}^{t_{n+1}} f \, dt} \leftarrow \text{local}$$

$$\sum_{n=0}^{N-1} (\ \ ) : \boxed{u^h(T^-) = u_0 + \int_0^T f \, dt} \leftarrow \text{global}$$

Some details :

$$u^h(t_{n+1}^-) = u^h(t_n^-) + \int_{t_n}^{t_{n+1}} f\,dt \quad \text{local cons}$$

$$u^h(t_n^-) = u^h(t_{n-1}^-) + \int_{t_{n-1}}^{t_n} f\,dt \quad \text{likewise}$$

add

$$u^h(t_{n+1}^-) + \cancel{u^h(t_n^-)} = \cancel{u^h(t_n^-)} + \int_{t_n}^{t_{n+1}} f\,dt$$

$$+ u^h(t_{n-1}^-) + \int_{t_{n-1}}^{t_n} f\,dt$$

$$u^h(t_{n+1}^-) = u^h(t_{n-1}^-) + \int_{t_{n-1}}^{t_{n+1}} f\,dt$$

---

now add:

$$u^h(t_{n-1}^-) = u^h(t_{n-2}^-) + \int_{t_{n-2}}^{t_{n-1}} f\,dt$$

$$u^h(t_{n+1}^-) = u^h(t_{n-2}^-) + \int_{t_{n-2}}^{t_{n+1}} f\,dt.$$

Keep doing this all the way to

$$u^h(t_1^-) = \underset{\substack{\equiv \\ u_0}}{u^h(t_0^-)} + \int_{t_0=0}^{t_1} f\,dt.$$

Then you have

$$u^h(t_{n+1}^-) = u_0 + \int_0^{t_{n+1}} f\,dt$$

Now start at the top and work down:

$$\underset{\substack{\parallel \\ T^-}}{u^h(t_N^-)} = u^h(t_{N-1}^-) + \int_{t_{N-1}}^{t_N} f\,dt.$$

and work down to

$$u^h(t_{n+1}^-) = u_0 + \int_0^{t_{n+1}} f\,dt.$$

write it out: until you see the result.

$$\boxed{u^h(T^-) = u_0 + \int_0^T f\,dt}$$

Global conv.