<u>Idea</u> of an "interp." est.

$\forall u \in \mathcal{S}, \ \exists \tilde{u}^h \in \mathcal{S}^h, \ni$

$\tilde{u}^h \to u$, as $h \to 0$.

Idea of an inverse est.

$\forall w^h \in \mathcal{V}^h, \quad \|w^h_{,x}\| \leq C_{inv}\, h^{-1} \|w^h\|$ (note, opposite of P-F. ineq.)

note: this in general is impossible

$\|w_{,x}\| \leq \ldots \|w\| \quad \forall w \in \mathcal{V}$ <u>NO!</u>

Try $w = x^{+1/2}, \ x \in [0,1]$

$\|w\|^2 = L^2/2$

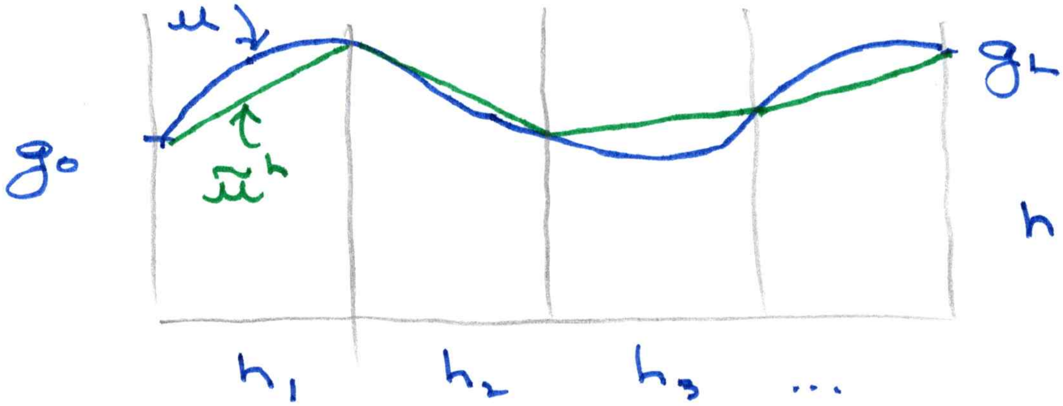
$\|w_{,x}\|^2 = \infty$.

Interp. est. (Original result for FEMs goes to Bramble - Hilbert)

Ciarlet.

Assume $u \in H^r(0,L) \cap \mathcal{S}, \ r \geq 1$ (r ← regularity)

Let $\tilde{u}^h$ be the piecewise linear interp., $\tilde{u}^h \in \mathcal{S}^h$



$h = \max\limits_{A=1\ldots N-1} h_A$

Interpolation error $\eta = \underset{\mathcal{S}^h}{\underset{\cap}{\tilde{u}^h}} - \underset{\mathcal{S}}{\underset{\cap}{u}} \in \mathcal{V}$

$\forall u \in \mathcal{S} \cap H^r(0,L)$, $\exists\, \tilde{u}^h \in \mathcal{S}^h$,

$\ni \quad \|\eta\|_s \le c_{\text{interp}} \left(\frac{h}{L}\right)^\alpha \|u\|_r$

s ← # weak der's of η

c_{interp} indep of h, u

r ← reg

$\alpha > 0, \Rightarrow \;\to 0$ as $h \to 0$

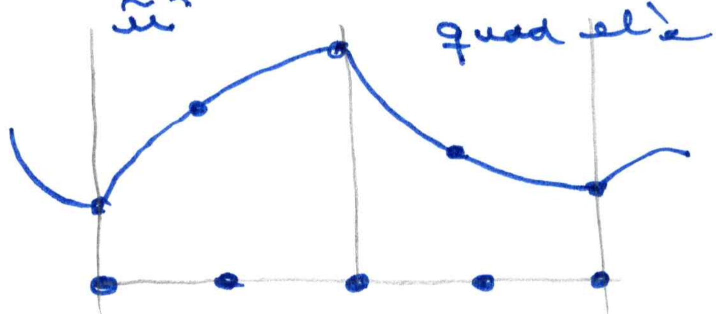
$$\boxed{\alpha = \min\{\cancel{\ } r - s,\; k+1-s\}}$$

$k+1$ ← order of complete poly.

lin. FE sp $k=1$

quad " $k=2$

$\vdots$



<u>Remarks</u> :

1.) If $r \geq k+1$, $\alpha = k+1-s$

opt. result.

2.) Smoothness of the exact sol. u is crucial.

3.) $\underset{>0}{\alpha}$ ~~xxxx~~, $\|\eta\|_s \longrightarrow 0$

4.) $s \uparrow$ convergence rate (α) decreases

$\|\eta\|_0$ $s=0$ L_2 fastest

$\|\eta\|_1$ $s=1$ one power of h slower.

Ex. 1. $k=1$, lin FE sp, $r \geq \underset{2}{k+1}$

$\alpha = k+1-s$

$= \underbrace{1+1}-s$

$\|\eta\|_0 \leq \underbrace{\cancel{c} \text{ interp} \left(\frac{h}{L}\right)}^{\frac{1+1-0}{2}} \|u\|_{\underset{2}{k+1}}$

$\|\eta\|_1$ $\quad \cdots " \cdots \quad {\frac{1+1-1}{1}} \quad "$

$\| \; \|_0 \sim O(h^2)$

$\| \; \|_1 \sim O(h)$

Ex. 2 $k=2$, quad. FE sp. $r \geq k+1$, $r=3$

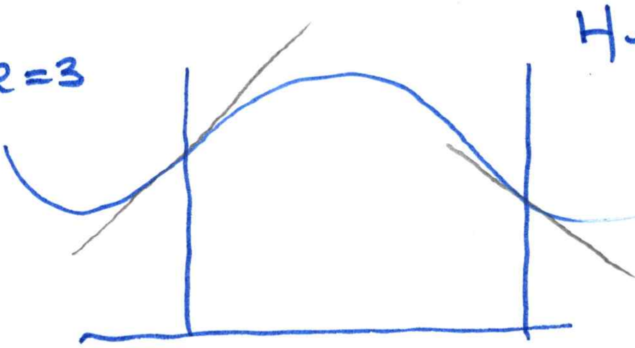
$\|\eta\|_0 \leq \quad \cdots \quad \left(\frac{h}{L}\right)^{\frac{2+1-0}{3}} \|u\|_3$

$\|\eta\|_1 \leq \quad \cdots \quad "^{\frac{2+1-1}{2}} \quad \|u\|_3$

$\|\cdot\|_0 \sim O(h^3)$

$\|\cdot\|_1 \sim O(h^2)$

$k=3$

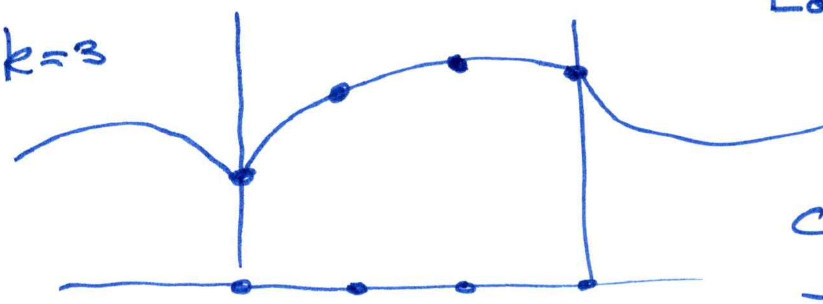


Hermite cubics

disp, slope are ~~not~~ continuous

C^1-cont.

$k=3$



Lag. cubics

Conv. rates are the same for Lag & Hermite cubics

Some ineq's :

1. let $z_i \in \mathbb{R}$, $i = 1, \dots, n$, $z = \sum_{i=1}^{n} z_i$

$$|z| \leq |z_1| + |z_2| + \dots + |z_n|$$

2. Young's ineq. (sp. case)

$$\boxed{|ab| \leq \frac{1}{2}(a^2+b^2)}$$

(i) if $ab > 0$, $0 \leq (a-b)^2 = a^2 + b^2 - 2ab$

$$\cancel{2}ab = \cancel{2}|ab| \leq \frac{a^2+b^2}{2}$$

(ii) if $ab < 0$, $0 \leq (a+b)^2 = a^2+b^2+2ab$

$$-\cancel{2}ab = \cancel{2}|ab| \leq \frac{a^2+b^2}{2}$$

3.) <u>Peter - Paul ineq.</u>

$$|ab| \le \frac{1}{2}\left(\frac{a^2}{\varepsilon} + \varepsilon b^2\right) \qquad \varepsilon > 0.$$

Let $a' = a/\sqrt{\varepsilon}$, $b' = b\sqrt{\varepsilon}$

you use
Young on a', b'

$$|a'b'| = |ab| \le \frac{1}{2}\left(\frac{a^2}{\varepsilon} + \varepsilon b^2\right)$$

4.) $\forall v, w \in V$,

nat norm conesp. to $(\cdot,\cdot)$

$$(v,w) \le |(v,w)| \le \frac{1}{2}\left(\frac{\|v\|^2}{\varepsilon} + \varepsilon\|w\|^2\right)$$

Cauchy - Schwarz

$\cancel{a = \|v\|}$
$\cancel{b = \|w\|}$

$$|(v,w)| \le \underbrace{\|v\|}_{a}\ \underbrace{\|w\|}_{b}$$

Peter-Paul

$$\le \frac{1}{2}\left(\frac{a^2}{\varepsilon} + \varepsilon b^2\right)$$

$$= \frac{1}{2}\left(\frac{\|v\|^2}{\varepsilon} + \varepsilon\|b\|^2\right)$$

Anal. of Gal. meth.

Assume $B(w^h; \underset{\shortparallel}{e}) = 0$ ✓ $\forall w^h \in \mathcal{V}^h$

"orthog" $u^h - u$

" stab. in FE space $\mathcal{V}^h$ $B(w^h; w^h) \geq \alpha \|w^h_{,x}\|^2$

$\geq \underbrace{\frac{\alpha}{2L^2}}_{\text{by equiv norms.}} \|w^h\|_1^2$

reg. " $u \in H^{r = k+1}(0,L) \cap \mathcal{S}$

int. " $\eta = \underset{\substack{\uparrow \\ \exists}}{\tilde{u}^h} - u$, $\exists$ $\|\eta\|_1 \leq c_{int} \left(\frac{h}{L}\right)^{\overset{k+1-s}{\underbrace{k+1-1}_{k}}} \|u\|_{k+1}$

$$\begin{array}{ccccccc} & \mathcal{S}^h \times \mathcal{S}^h & & \mathcal{S}^h & & \mathcal{S} \\ & \Psi \quad\quad \Psi & & \Psi & & \Psi \\ e = u^h - u = & \underbrace{u^h - \tilde{u}^h} & + & \underbrace{\tilde{u}^h} & - & u \end{array}$$

est. hard part ? $\rightarrow e^h \in \mathcal{V}^h$ $\quad$ η ✓ $\in \mathcal{V}$

$|||e||| = |||e^h + \eta|||$

tri ineq

$\leq \underbrace{|||e^h|||}_{?} + |||\eta|||$ ✓

(60.

Stab $B(w^h; w^h) \geq \frac{\alpha}{2L^2} \|w^h\|_1^2 \quad \forall w^h \in \mathcal{V}^h$

select $w^h = e^h$.

weighting ftn slot

sol. ftn slot

$e = e^h + \eta$

$$\frac{\alpha}{2L^2} \|e^h\|_1^2 \leq B(e^h; e^h)$$

$$\leq B(e^h, e - \eta)$$

$$= \underbrace{B(e^h; e)} - B(e^h; \eta) \quad \text{bilin}$$

these steps are the same

$$= \quad 0 - B(e^h; \eta) \quad \text{by error orthog.}$$

$$= |B(e^h; \eta)|$$

now expanding next time