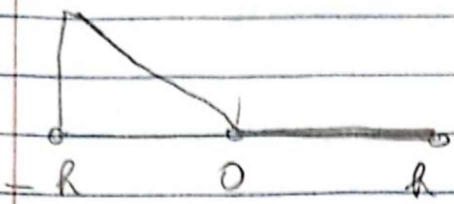
CSE 397 / EM 397

Stabilized and Variational Multiscale methods in CFD.

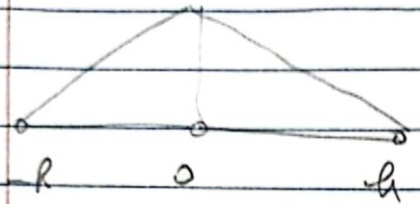
⇒ Renato Poli (rep2656) 02/16/2024

Homework #1

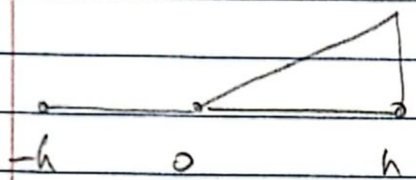
1.1 (1) $B \in \{A-1, A, A+1\}$



$$N_{A-1} = -x/h \rightarrow N_{A-1,x} = -1/h \Big|_{-h}^{0}$$



$$N_A = \begin{cases} 1 + x/h & , x < 0 \\ 1 - x/h & , x > 0 \end{cases} \Bigg|_{-h}^{h}$$



$$N_{A+1} = x/h \Big|_{0}^{h}$$

Diff - coeffs:

$$u_{A-1} : k\left[\int_\Omega N_{A,x} N_{A-1,x}\, dx\right] = k\left[\int_{-h}^{0} -1/h^2 \, dx\right] = -\frac{k}{h}$$

$$u_A : k\left[\int_\Omega N_{A,x} N_{A,x}\, dx\right] = k\left[\int_{-h}^{h} \frac{1}{h^2}\, dx\right] = \frac{k \cdot 2}{h}$$

$$u_{A+1} : k\left[\int_\Omega N_{A,x} N_{A+1,x}\, dx\right] = k\left[\int_{0}^{h} -1/h^2 \, dx\right] = \frac{-k}{h}$$

Hence:

$$C_{DIFF} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix} k/h \begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1} \end{bmatrix}$$

and $\underset{\sim}{S}_{DIFF} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$ //

Adv. coeff

$$u_{A-1} : -a \int_{-h}^{0} N_{A,x} N_{A-1}\, dx = -a \int_{-h}^{0} (+1/h)(-x/h)\, dx$$

$$= \frac{a}{h^2} \left(\frac{x^2}{2} \right)_{-h}^{0} = \frac{a}{h^2} \left[0 - \frac{h^2}{2} \right] = -a/2$$

$$u_A : -a \int_{-h}^{0} \frac{1}{h} \left(1 + \frac{x}{h} \right) dx + a \int_{0}^{h} \frac{+1}{h} \left(1 - \frac{x}{h} \right) dx$$

$$\frac{-a}{h} \left[x + \frac{x^2}{2h} \right]_{-h}^{0} + \frac{a}{h} \left[x - \frac{x^2}{2h} \right]_{0}^{h} =$$

$$-\frac{a}{h} \left[+h - \frac{h^{\cancel{2}}}{2\cancel{h}} \right] + \frac{a}{h} \left[h - \frac{h^{\cancel{2}}}{2\cancel{h}} \right] =$$

$$\frac{a}{h} \left[-h + \frac{h}{2} + h - \frac{h}{2} \right] = 0$$

$$u_{A+1} : -a \int_{0}^{h} N_{A,x} N_{A+1}\, dx = +a \int_{0}^{h} \left(+\frac{1}{h} \right) \frac{x}{h} dx =$$

$$= \frac{a}{h^2} \left[\frac{x^2}{2} \right]_{0}^{h} = \frac{a h^2}{2h^2} = \frac{a}{2}$$

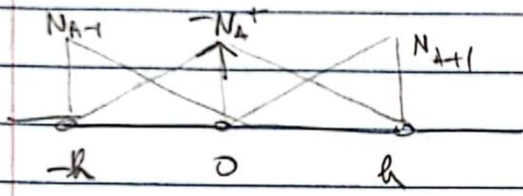
Hence : $C_{ADV} = \frac{a}{2} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1} \end{bmatrix}$

and $\underset{\sim}{S}_{ADV} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$ //

(2) $\frac{h}{2}\xi(\alpha_A)\sum_{e=1}^{N}\int_{\Omega_e} N_{A,x}\left(a u^h_{,x} - k u^h_{,xx} - f\right)dx = (*)$

$$\int_{\Omega_e} N_{A,x}\left(a u^h_{,x} - k u^h_{,xx}\right)dx = \sum_{B=0}^{N_e}\int_{\Omega_e}\left(N_{A,x}\, a\, N_{B,x} - k N_{A,x}\cancel{N_{B,xx}}^{\,0}\right)dx$$

$$= \sum_{B=0}^{N_e}\int_{\Omega_e}\left(N_{A,x} N_{B,x}\, a\right)dx$$



$N_{A-1,x} = -1/h$

$N_{A,x} = \begin{cases} 1/h, & x<0 \\ -1/h, & x>0 \end{cases}$

$N_{A+1,x} = x/h$

$u_{A-1}: \; a\int_{-h}^{0} N_{A,x}N_{A-1,x}\,dx = -a/h$

$u_A: \; a\int_{-h}^{h} N_{A,x}N_{A,x}\,dx = \frac{a}{h}\cdot 2$

$u_{A+1}: \; a\int_{0}^{h} N_{A,x}N_{A+1,x}\,dx = -a/h$

<u>Hence:</u>

$$(*) = \frac{h}{2}\xi(\alpha_h)\left\{\frac{a}{h}\underbrace{\begin{bmatrix}-1 & 2 & -1\end{bmatrix}}_{\tilde{T}^A}\begin{bmatrix}u_{A-1}\\ u_A \\ u_{A+1}\end{bmatrix} - \sum_{e=1}^{N}\int_{\Omega_e} N_{A,x} f\,dx\right\}$$

//

$\tilde{T}^A = \tilde{S}_{DIFF}$ //

③ $$\frac{\cancel{k}}{\cancel{h}}\left((1+\alpha_h\xi)\begin{bmatrix}-1 & 2 & -1\end{bmatrix} + \alpha_h\begin{bmatrix}-1 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}u_{A-1}\\ u_A\\ u_{A+1}\end{bmatrix} = 0$$

$$(1+\alpha\xi)(-u_{A-1}+2u_A-u_{A+1}) + \alpha(-u_{A-1}+u_{A+1}) = 0$$

$/u_{A-1}$:

$$(1+\alpha\xi)\left(-1+\frac{2u_A}{u_{A-1}}-\frac{u_{A+1}}{u_{A-1}}\right)+\alpha\left(\frac{u_{A+1}}{u_{A-1}}-1\right)=0$$

$$(1+\alpha\xi)\times\left(-1+\frac{2e^{2\alpha A}}{e^{2\alpha(A-1)}}-\frac{e^{2\alpha(A+1)}}{e^{2\alpha(A-1)}}\right)+\alpha\left(\frac{e^{2\alpha(A+1)}}{e^{2(A-1)}}-1\right)=0$$

$$(1+\alpha\xi)\left(-1+2e^{2\alpha}-e^{4\alpha}\right)+\alpha\left(e^{4\alpha}-1\right)=0$$

$$-(1+\alpha\xi)(e^{2\alpha}-1)^{\cancel{2}} \quad + \alpha\cancel{(e^{2\alpha}-1)}(e^{2\alpha}+1)$$

$$-(1+\alpha\xi)-\frac{\alpha(e^{2\alpha}+1)}{e^{2\alpha}-1}=0$$

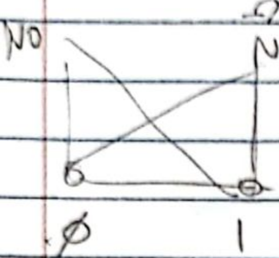
$$\xi=\frac{e^{2\alpha}+1}{e^{2\alpha}-1}-1/\alpha$$

$$\boxed{\xi=\coth(\alpha_h)-1/\alpha_h}$$

1.2

$$(\circ) = k + \frac{ah}{2}\xi$$

$$B\emptyset = \int_\Omega N_{1,x}\left((\circ) N_{0,x} - a N_\emptyset\right) dx$$



$N_\emptyset = 1 - x/h$

$\rightsquigarrow$

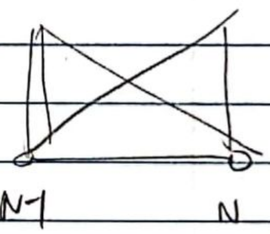
$N_1 = x/h$

$$B\emptyset = \int_0^h \frac{1}{h}(\circ)\left(-\frac{1}{h}\right) - a\left(\frac{x}{h}\right)\left(\frac{1}{h}\right)$$

$$-\frac{1}{h^2}(\circ)h - a\frac{h^2}{2h} = -\frac{1}{h}(\circ) - \frac{a}{2}$$

—— x ——

$$B_{N-1} = \int_\Omega N_{N-1,x}\left((\circ) N_{N,x} - a N_N\right)$$



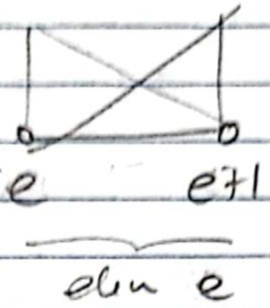
$$\int_0^h \left(-\frac{1}{h}\right)(\circ)\left(\frac{1}{h}\right) + a\left(+\frac{1}{h}\right)\left(\frac{x}{h}\right)$$

$$-\frac{(\circ)}{h^2}h + \frac{a}{\cancel{h^2}}\frac{\cancel{h^2}}{2} \Rightarrow B_{N-1} = -\frac{(\circ)}{h} + \frac{a}{2}$$

$$F_i = \int f\left(N_i + \frac{h}{2}\xi N_{i,x}\right) dx$$

$$N_e = 1 - \frac{x - x_0}{h}$$

$$N_{e+1} = \frac{x - x_0}{h}$$



$\underline{f = 1}$

Node e:

$$\int_{x_e}^{x_{e+1}} \left(1 - \frac{x - x_0}{h}\right) + \frac{h}{2}\xi\left(-\frac{1}{\cancel{h}}\right) dx$$

$$= \left[x - \frac{x^2}{2h} + \frac{x_0 x}{h} - \frac{\xi}{2}x\right]_{x_e}^{x_{e+1}} \quad //$$

Node e+1

$$\int_{x_e}^{x_{e+1}} \frac{x - x_0}{h} + \frac{h}{2}\xi\left(\frac{1}{\cancel{h}}\right) dx$$

$$\left[\frac{x^2}{2h} - \frac{x_0 x}{\cancel{h}} + \frac{\xi}{2}x\right]_{x_e}^{x_{e+1}} \quad //$$

$\underline{f = x}$

Node e:

$$\int_{x_e}^{x_{e+1}} x - \frac{x^2 - x_0 x}{h} - \frac{\xi x}{2} \; dx$$

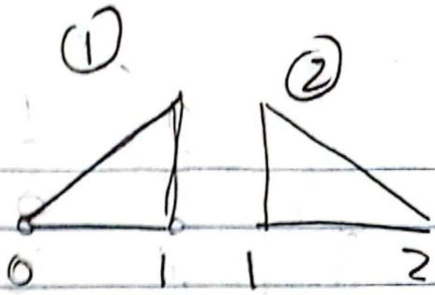
$$\left[\frac{x^2}{2} - \frac{x^3}{3h} + \frac{x^2 x_0}{2h} - \frac{\xi x^2}{4}\right]_{x_e}^{x_{e+1}} \quad //$$

Node e+1

$$\int_{x_e}^{x_{e+1}} \frac{x^2 - x x_0}{h} + \frac{x\xi}{2} \; dx$$

$$\left[\frac{x^3}{3h} - \frac{x^2 x_0}{2h} + \frac{x^2\xi}{4}\right]_{x_e}^{x_{e+1}} \quad //$$

$$K_{11} = \int_{-h}^{h} N_{1,x}\left((0) N_{1,x} - a N_1\right) dx$$



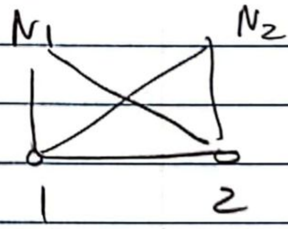
$$① \int_0^h \left(\frac{1}{h}\right)(0)\left(\frac{1}{h}\right) - a\left(\frac{1}{h}\right)\left(\frac{x}{h}\right) = \frac{(0)}{h} - \frac{a}{2}$$

$$② \int_0^h \left(-\frac{1}{h}\right)(0)\left(-\frac{1}{h}\right) + a\left(+\frac{1}{h}\right)\left(1 - \frac{x}{h}\right)$$

$$= \frac{(0)}{h} + \frac{a}{2}$$

$$① + ② = \frac{2(0)}{h} \quad /// \qquad (0) = \left(k + \frac{ah}{2} q\right)$$

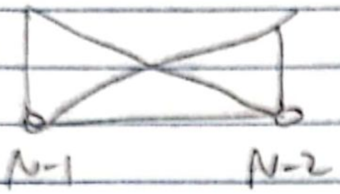
$$K_{12} = \int_0^h N_{1,x}\left((0) N_{2,x} - a N_2\right) dx$$



$$\int_0^h \left(-\frac{1}{h}\right)(0)\left(\frac{1}{h}\right) + a\left(+\frac{1}{h}\right)\left(\frac{x}{h}\right) dx$$

$$= -\frac{(0)}{h} + \frac{a}{2} \quad ///$$

$$K_{N-1,N-2} = \int N_{N-1,x}\left((\circ)\, N_{N-2,x} - a\, N_{N-2} \right) dx$$

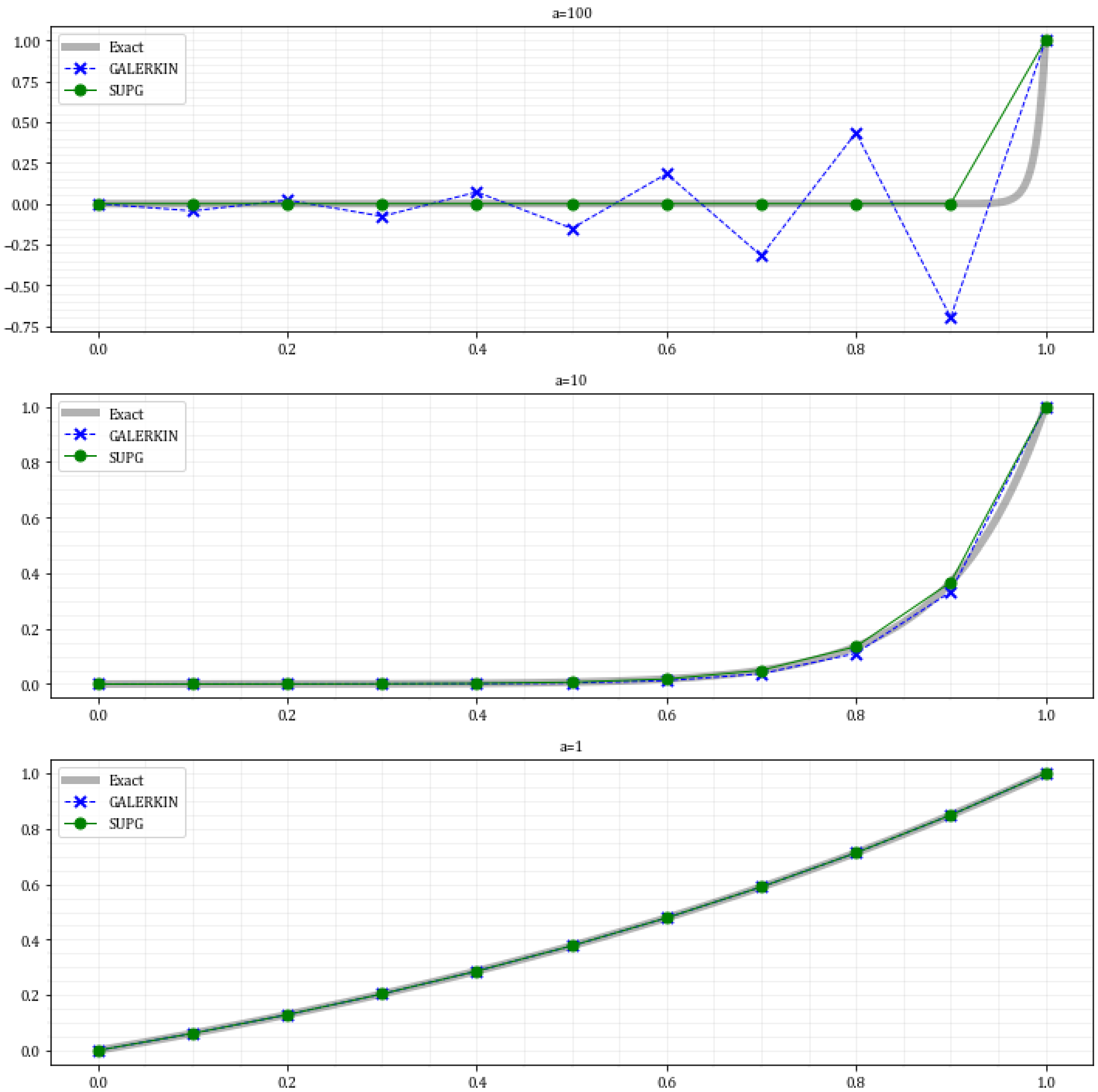


$$= \int_0^h \left(\frac{1}{h}\right)(\circ)\left(-\frac{1}{h}\right) - a\left(\frac{1}{h}\right)\left(\frac{x}{h}\right)$$

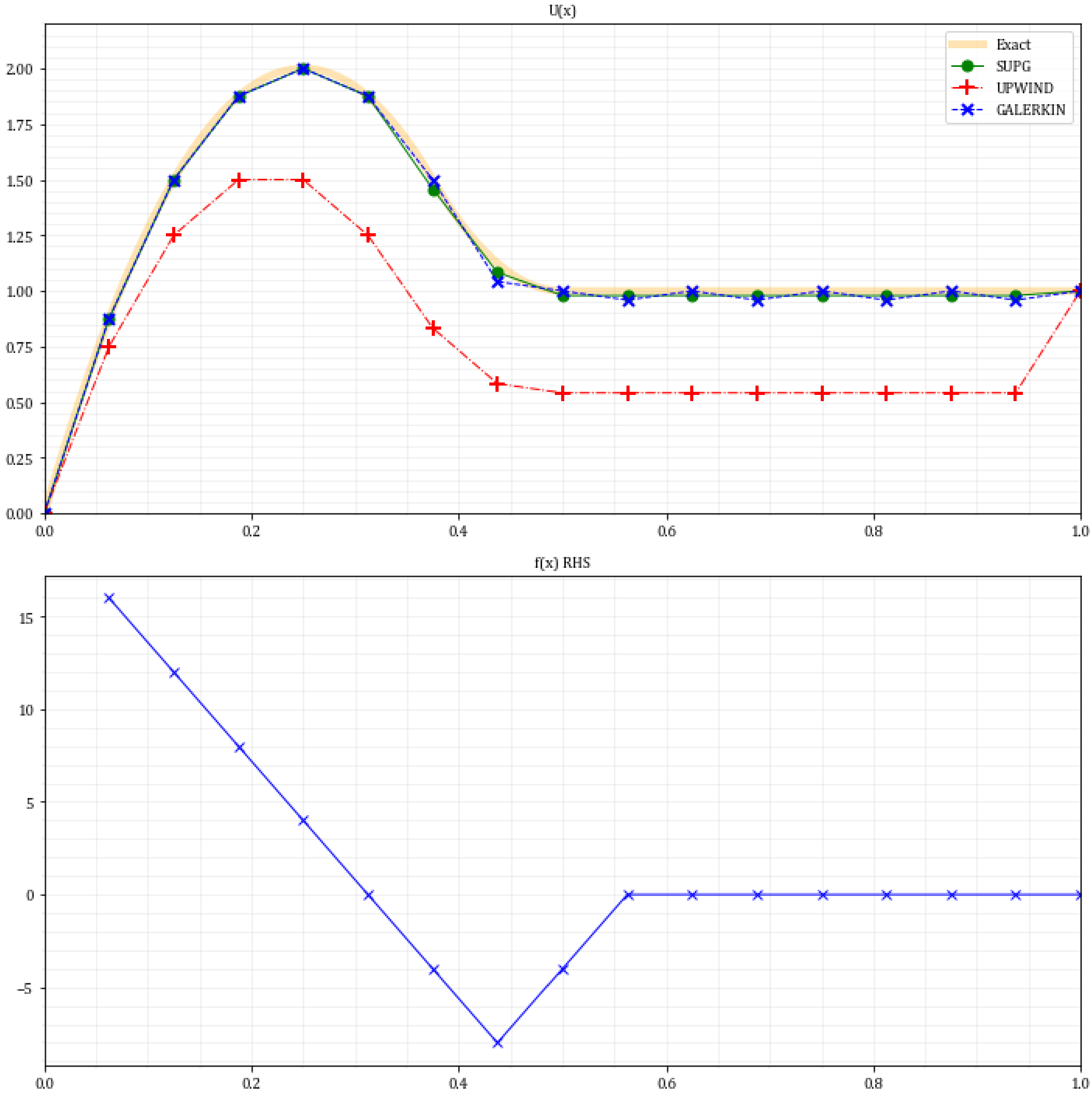
$$= \frac{-(\circ)}{h} - \frac{a}{2} \quad ///$$

$$K_{N-2,N-2} = K_{11} \quad ///$$

Exercise 1.2


a=100
Exact
GALERKIN
SUPG
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
0.0
0.2
0.4
0.6
0.8
1.0
a=10
Exact
GALERKIN
SUPG
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
a=1
Exact
GALERKIN
SUPG
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Exercise 1.3


U(x)
Exact
SUPG
UPWIND
GALERKIN
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.0
0.2
0.4
0.6
0.8
1.0
f(x) RHS
15
10
5
0
−5
0.0
0.2
0.4
0.6
0.8
1.0

```python
import numpy as np
from numpy import tanh
import sys

import matplotlib.pyplot as plt

plt.style.use('paper.mplstyle')
np.set_printoptions(threshold=200, linewidth=200)
plt.figure(figsize=(12,6))

def XI_KB() :
    global METHOD, A, H, KAPPA
    if METHOD == "GALERKIN" :
        return 0
    elif METHOD == "SUPG" :
        if KAPPA != 0 :
            alpha_h = A * H / 2 / KAPPA
            return 1/tanh(alpha_h) - 1/alpha_h
        else:
            return 1
    elif METHOD == "UPWIND" :
        return 1
    else :
        fail(f"XI(): Unknown method {METHOD}")

def XI_F( ) :
    global METHOD, A, H, KAPPA
    if METHOD == "GALERKIN" :
        return 0
    elif METHOD == "SUPG" :
        if KAPPA != 0 :
            alpha_h = A * H / 2 / KAPPA
            return 1/tanh(alpha_h) - 1/alpha_h
        else:
            return 1
    elif METHOD == "UPWIND" :
        return 0
    else :
        fail(f"XI(): Unknown method {METHOD}")


def Usolve() :
    global KAPPA, H, A, N, F, G0, G1

    Sdif = np.array( [ -1, 2, -1 ] )
    Sadv = np.array( [ -1, 0, 1] )

    K  = np.zeros( [ N-1, N-1 ] )
    B0 = np.zeros( N-1 )
    BN = np.zeros( N-1 )

    xi = XI_KB()
    B0[0]   = - ( KAPPA + A*H/2*xi )/H - A/2
    BN[N-2] = - ( KAPPA + A*H/2*xi )/H + A/2

    K[0,0]     = 2*( KAPPA + A*H/2*xi )/H
    K[0,1]     = - ( KAPPA + A*H/2*xi )/H + A/2
    K[N-2,N-3] = - ( KAPPA + A*H/2*xi )/H - A/2
    K[N-2,N-2] = 2*( KAPPA + A*H/2*xi )/H

    for i in range( 1, N-2 ) :
        [ K[i,i-1],K[i,i],K[i,i+1] ] = ( KAPPA/H + A/2*xi ) * Sdif + A/2 * Sadv

    U = np.linalg.solve( K,  F - B0 * G0 - BN * G1 )
    U=np.append(U,G1)
    U=np.insert(U,G0,0)

    return U
```

```
# GALERKIN or SUPG
METHOD = "SUPG"
KAPPA = 1
N = 10
H = 1/N
G0 = 0
G1 = 1
F = np.zeros( N-1 )

fig, [ax1, ax2, ax3] = plt.subplots( 3,1, figsize=(10,10))

# NUMERICAL SOLUTIONS
X = np.linspace( 0, 1, N+1 )
X_EXACT = np.linspace( 0, 1, 500 )
evr = 1 #int(N/20)

ax=ax1
A=100
U_EXACT = ( np.exp( A*X_EXACT) - 1 ) / ( np.exp(A) - 1)
ax.plot( X_EXACT, U_EXACT, c='k', lw=5, alpha=.3, label='Exact' )
METHOD = "GALERKIN"
U = Usolve()
ax.plot( X[::evr], U[::evr], marker='x',markeredgewidth=2, ls='--', ms=7, lw=1, label=METHOD,
c='blue' )
METHOD = "SUPG"
U = Usolve()
ax.plot( X[::evr], U[::evr],marker='o', ms=7, ls='-', lw=1, label=METHOD, c='green' )
ax.set_title(f"a={A}")
ax.legend()

ax=ax2
A=10
U_EXACT = ( np.exp( A*X_EXACT) - 1 ) / ( np.exp(A) - 1)
ax.plot( X_EXACT, U_EXACT, c='k', lw=5, alpha=.3, label='Exact' )
METHOD = "GALERKIN"
U = Usolve()
ax.plot( X[::evr], U[::evr], marker='x',markeredgewidth=2, ls='--', ms=7, lw=1, label=METHOD,
c='blue' )
METHOD = "SUPG"
U = Usolve()
ax.plot( X[::evr], U[::evr],marker='o', ms=7, ls='-', lw=1, label=METHOD, c='green' )
ax.set_title(f"a={A}")
ax.legend()

ax=ax3
A=1
U_EXACT = ( np.exp( A*X_EXACT) - 1 ) / ( np.exp(A) - 1)
ax.plot( X_EXACT, U_EXACT, c='k', lw=5, alpha=.3, label='Exact' )
METHOD = "GALERKIN"
U = Usolve()
ax.plot( X[::evr], U[::evr], marker='x',markeredgewidth=2, ls='--', ms=7, lw=1, label=METHOD,
c='blue' )
METHOD = "SUPG"
U = Usolve()
ax.plot( X[::evr], U[::evr],marker='o', ms=7, ls='-', lw=1, label=METHOD, c='green' )
ax.set_title(f"a={A}")
ax.legend()

fig.tight_layout()
```

```python
def F_update() :
    global F, H, N, A
    F = np.zeros(N+1)
    XI = XI_F()
    for EL in range(0,N) :
        N0 = EL
        N1 = EL+1
        x0 = N0 * H
        x1 = N1 * H

        def f_01(x) : return x - x**2/2/H + x0*x/H - XI*x/2
        def f_0x(x) : return x**2/2 - x**3/3/H + x0*x**2/2/H - XI*x**2/4
        def f_11(x) : return x**2/2/H - x0*x/H + XI*x/2
        def f_1x(x) : return x**3/3/H - x0*x**2/2/H + XI*x**2/4

        a01 = 0; a0x = 0; a11 = 0; a1x = 0
        b01 = 0; b0x = 0; b11 = 0; b1x = 0

        if x0 <= 3/8 :
            if x1 <= 3/8 :
                a01 = f_01(x1) - f_01(x0)
                a0x = f_0x(x1) - f_0x(x0)
                a11 = f_11(x1) - f_11(x0)
                a1x = f_1x(x1) - f_1x(x0)
            else : # partially in (3/8 - 1/2)
                a01 = f_01(3/8) - f_01(x0)
                a0x = f_0x(3/8) - f_0x(x0)
                a11 = f_11(3/8) - f_11(x0)
                a1x = f_1x(3/8) - f_1x(x0)

                b01 = f_01(x1) - f_01(3/8)
                b0x = f_0x(x1) - f_0x(3/8)
                b11 = f_11(x1) - f_11(3/8)
                b1x = f_1x(x1) - f_1x(3/8)
        elif x0 <= 1/2 :
            if x1 <= 1/2 :  # completely in (3/8 - 1/2)
                b01 = f_01(x1) - f_01(x0)
                b0x = f_0x(x1) - f_0x(x0)
                b11 = f_11(x1) - f_11(x0)
                b1x = f_1x(x1) - f_1x(x0)
            else :   # partially in (3/8 - 1/2)
                b01 = f_01(1/2) - f_01(x0)
                b0x = f_0x(1/2) - f_0x(x0)
                b11 = f_11(1/2) - f_11(x0)
                b1x = f_1x(1/2) - f_1x(x0)

        F[N0] += 16*A*( a01 - 4*a0x )
        F[N1] += 16*A*( a11 - 4*a1x )

        F[N0] += 16*A*( -2*b01 + 4*b0x )
        F[N1] += 16*A*( -2*b11 + 4*b1x )

    F = F[1:N]
```

```
KAPPA = 0
N = 16
H = 1/N
A = 1
G0 = 0
G1 = 1
F = np.zeros(N-1)
F_ = np.zeros(N-1)

fig,[ax1,ax2] = plt.subplots( 2,1 , figsize=(10,10))

X_EXACT = np.linspace( 0, 1, 500 )
U_EXACT = np.zeros( 500 )
for i in range(0,500) :
    x = X_EXACT[i]
    if   x <= 3/8 :
        U_EXACT[i] = 16*x*( 1 - 2*x )
    elif x <= 1/2 :
        U_EXACT[i] = 9 + 32*x*(x-1)
    else :
        U_EXACT[i] = 1
    # print(f"x:{x} => F:{F[i]}")
ax1.plot( X_EXACT, U_EXACT, c='orange', lw=5, alpha=.3, label='Exact' )

# NUMERICAL SOLUTIONS
X = np.linspace( 0, 1, N+1 )
for i in range(0,N-1) :
    x = X[i]
    if   x <= 3/8 :
        F_[i] = 16 * A * ( 1 - 4*x )
    elif x <= 1/2 :
        F_[i] = 16 * A * (-2 + 4*x )
    else :
        F_[i] = 0
ax2.plot( X, np.insert(np.append(F_,0),0,None), marker='x', label="f(x)"  )
ax2.set_title("f(x) RHS")
evr = 1 #int(N/20)
# plt.scatter( X[::evr], U[::evr], marker='+', s=50, lw=1 )

METHOD = "SUPG"
F_update()
U = Usolve()
ax1.plot( X[::evr], U[::evr],marker='o', ms=7, ls='-', lw=1, label=METHOD, c='green' )

METHOD = "UPWIND"
F_update()
U = Usolve()
ax1.plot( X[::evr], U[::evr], marker='+', markeredgewidth=2, ms=10, ls='-.', lw=1, label=METHOD,
c='red' )
#ax1.legend()

METHOD = "GALERKIN"
KAPPA = 1e-15
F_update()
U = Usolve()
ax1.plot( X[::evr], U[::evr], marker='x',markeredgewidth=2, ls='--', ms=7, lw=1, label=METHOD,
c='blue' )

ax1.set_title("U(x)")
ax1.set_xlim(0,1)
ax1.set_ylim(0,2.2)
ax1.legend()

ax2.set_xlim(0,1)

fig.tight_layout()
```