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Effective Stress in Soils, Concrete and Rocks

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Introduction

The pore space in soils generally contains both air and water, the pressures in which may be denoted by u_a and u_w. If the 'total' stress acting in a given direction at any point in the soil is σ, then it is a fundamental problem in soil mechanics to determine in what manner the 'effective' stress is related to the three known stresses σ, u_a and u_w; where the effective stress, denoted by σ', is, by definition, the stress controlling changes in volume or strength of the soil.

For saturated soils in which the voids are filled with water, TERZAGHI (1923) showed experimentally that

$$\sigma' = \sigma - u_w \qquad \ldots.(1)$$

and subsequent work has confirmed this expression with a sufficiently high degree of accuracy for engineering purposes.

For partially saturated soils, however, tests carried out chiefly during the past decade have shown that this equation can be appreciably in error. And in the present paper experimental evidence will be given supporting a more general expression for effective stress, first suggested by BISHOP in 1955,

$$\sigma' = \sigma - [u_a - \chi(u_a - u_w)]$$

where χ is a parameter related to the degree of saturation and equalling unity when the soil is fully saturated. In this latter case Bishop's equation then becomes identical with Terzaghi's.

These equations can be used in all, or certainly in most, practical soil mechanics problems. But it is of philosophical interest to examine the fundamental principles of effective stress, since it would seem improbable that an expression of the form $\sigma' = \sigma - u_w$ is strictly true. And, indeed, when one turns to other porous materials such as concrete or limestone, this equation is not always adequate. We may therefore anticipate that, even for fully saturated porous materials, the general expression for effective stress is more complex, and that Terzaghi's equation has the status of an excellent approximation in the special case of saturated soils.

In fact this view has long been held; the common opinion being that the effective stress is actually the intergranular stress acting between the particles comprising the porous material. It can readily be shown that this stress is

$$\sigma_g = \sigma - (1 - a)u_w$$

where a is the area of contact between the particles, per unit gross area of the material. And hence, on this hypothesis,

$$\sigma' = \sigma - (1 - a)u_w$$

Now there can be little doubt that a is very small in soils, and this expression is therefore virtually identical with Terzaghi's equation. Hence the intergranular concept is superficially attractive. But recent high-pressure consolidation tests on lead shot have proved that the stress controlling volume change is by no means equal to σ_g; whilst triaxial compression tests on Marble indicate that the effective stress controlling shear strength changes in rocks is also not equal to σ_g.

Thus it is desirable to examine the physics of effective stress more closely in the hope of obtaining a theory which is reasonably consistent with all the available experimental data. Such a theory would have four advantages:

(a) it would provide a satisfactory explanation for the validity of Terzaghi's equation; although this advantage is admittedly of no more than academic interest in most soil problems;

(b) it would be of practical benefit in certain cases, such as the consolidation of deep beds of geological sediments, where the pressures are sufficiently great to establish appreciable contact areas between the particles;

(c) it would help to resolve the difficulties of interpreting triaxial and other tests on concrete which have been made with the express purpose of determining the contact area in this material;

(d) it would be relevant in various engineering and geophysical problems in rock mechanics.

PART I

Shear Strength of Saturated Materials

When porous materials are tested under the condition of zero pore pressure their shear strength τ_d is found to increase with increasing applied pressure σ' normal to the shear surface; and, within the range of stresses used in practice, the relationship between τ_d and σ' may be expressed by the Coulomb equation

$$\tau_d = c' + \sigma'.\tan\phi'$$

where c' and ϕ' are the apparent cohesion and the angle of shearing resistance. The problem now under consideration is to obtain an expression for σ' when the material is subjected to a pore-water pressure u_w as well as an applied pressure σ.

Three solutions to this problem will be given, but it is first necessary to set out the equations of equilibrium between the external forces and the stresses acting at the contact between any two particles or grains of the porous material.

Throughout the first two Parts of this paper the pore pressure will for simplicity be denoted by u, and this may be regarded as the pressure in any single-phase pore fluid.

Equilibrium Equations—With reference to Fig. 1 consider two particles in solid contact on a statistical plane of area A_s and occupying a gross area A in a plane parallel to the contact. Then the contact area ratio a is defined by the expression

$$a = \frac{A_s}{A}$$

If the total force normal to the contact plane is P and if the shear force is T, then the normal total stress σ and the shear stress τ are

$$\sigma = \frac{P}{A}$$

$$\tau = \frac{T}{A}$$

and the stresses at the interfacial contact are

$$\sigma_s = \frac{P_s}{A_s}\ \tau_s = \frac{T_s}{A_s}$$

where P_s and T_s denote the normal and the shear forces acting between the particles. In addition there is a pressure u in the pore water.

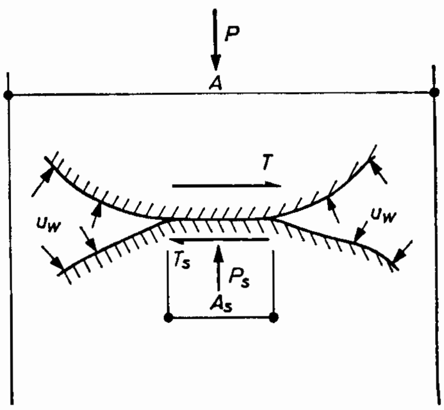

Fig. 1

For equilibrium normal to the plane

$$P = P_s + (A - A_s)u$$

Hence

$$\sigma = a.\sigma_s + (1 - a)u$$

and

$$a = \frac{\sigma - u}{\sigma_s - u}$$

For equilibrium parallel to the plane

$$T = T_s$$

Hence

$$\tau = a.\tau_s$$

In the above equation τ is the shear stress per unit gross area occupied by the particles. But in an assemblage of particles the shear strength τ_f per unit gross area of the material as a whole will be greater than the limiting value of τ. If, for example, the coefficient of friction at the interfacial content is μ, then CAQUOT (1934) has shown that in a non-cohesive granular material, sheared at constant volume,

$$\tan\phi' = \frac{\pi}{2}.\mu$$

and BISHOP (1954) has derived rather similar expressions for plane strain and triaxial stress conditions. Thus we may write

$$\tau_f = m.\tau = a.m.\tau_s$$

where m is a factor greater than unity.

Theory I—The shear strength at the contact is assumed to be

$$\tau_s = c_s + \sigma_s.\tan\phi_s$$

Thus, when this strength is fully mobilized, and failure takes place,

$$\tau = a.c_s + a.\sigma_s.\tan\phi_s$$

But

$$a.\sigma_s = \sigma - (1 - a)u$$

$$\therefore\quad \tau = a.c_s + [\sigma - (1 - a)u]\tan\phi_s$$

or

$$\tau_f = a.m.c_s + [\sigma - (1 - a)u]m.\tan\phi_s$$

Now, when $\sigma = u = 0, \tau_f = c'$

$$\therefore\quad c' = a.m.c_s$$

and in 'drained' tests (on soils) or 'jacketed' tests (on concrete and rocks), when $u = 0$,

$$\tau_d = c' + \sigma.\tan\phi',$$

$$\therefore\quad \tan\phi' = m.\tan\phi_s = m.\mu$$

Hence,

$$\tau_f = c' + [\sigma - (1 - a)u]\tan\phi'$$

and we see that the effective stress σ' is given by the expression

$$\underline{\sigma' = \sigma - (1 - a)u}$$

It is to be noted that the so-called 'intergranular stress' σ_g is defined as

$$\sigma_g = \frac{P_s}{A}$$

Thus

$$\sigma_g = \frac{P - (A - A_s)u}{A} = \sigma - (1 - a)u$$

Therefore Theory I leads to the conclusion that the effective stress is equal to the intergranular stress.

Theory II—In 1925 Terzaghi included in *Erdbaumechanik* a highly original treatment of the physical nature of friction. If two surfaces are lightly touching each other then, no matter how flat they may appear to be, the actual area of molecular contact at the interface is extremely small. If a load W is now applied normal to the surfaces the microscopic irregularities will yield and the total area of contact will attain a value A_s such that

$$W = A_s.\sigma_y$$

where σ_y is the yield stress of the material in these asperities. Terzaghi then makes the implicit assumption that the solid material is purely cohesive, with a shear strength k. The maximum tangential force which can be applied to the surfaces is therefore

$$F = A_s.k$$

But the coefficient of friction μ is defined by the ratio $\mu = F/W$. Hence

$$\mu = \frac{k}{\sigma_y}$$

Now for many substances μ is roughly 0·5. Hence σ_y is of the order $2k$. And this is the compression strength; which is not an unreasonable value for the yield stress of the asperities.

It will thus be seen that Terzaghi presented a concept of friction based on the hypothesis of a purely cohesive material. For many years the significance of the relevant page in *Erdbaumechanik* was overlooked. But, quite independently, the above equation was put forward by BOWDEN and TABOR in 1942 and broadly substantiated in a brilliant series of experiments, chiefly on metallic friction*.

Two modifications in this simple theory must be made, however, if a more accurate model is to be established (TABOR, 1959). Firstly, the shear strength at the interfacial contact is generally less than that of the body of the solid material. Thus we should write

$$\tau_s = \beta.k$$

where β is less than unity. Secondly, the yield stress σ_y is not an independent strength property but will decrease as the shear stress increases, probably in accordance with an equation of the type

$$\sigma_y^2 + \alpha^2.\tau_s^2 = \alpha^2.k^2$$

Hence when $\tau_s = 0$, $\sigma_y = \alpha.k$. But when failure takes place $\tau_s = \beta.k$ and

$$\sigma_y = \alpha\sqrt{1 - \beta^2}.k = N.k$$

where N is a parameter depending on the geometry of the asperities and on the stress-strain characteristics. If for example $\beta = \frac{4}{5}$, then $\sigma_y = \frac{3}{5}\alpha k$ and the contact area will increase

* The greater part of this work is described in BOWDEN and TABOR (1954).

by 30 per cent during shear. This is the phenomenon of 'junction growth'.

Now it will be shown in the next section of this paper that the shear strength of solid substances can be expressed by the equation

$$\tau_i = k + \sigma . \tan \psi$$

where k is the intrinsic cohesion and ψ as the angle of intrinsic friction of the solid*. For most metals ψ is not more than 5° (or $\tan \psi = 0{\cdot}1$) whilst for minerals ψ appears to be typically in the range 3° to 10°. The physical reason for ψ being greater than zero is possibly associated in part with the closing up of internal flaws, under increasing pressure. And it may perhaps be considered that an ideal or 'perfect' solid is one in which $\psi = 0$.

In fact, the softer metals approximate to this case; but the minerals, with which we are concerned, depart appreciably from the perfect solid. Nevertheless it will be a matter of considerable theoretical interest to derive the equations for effective stress in porous materials comprised of ideal solid particles. The basic assumptions will therefore be

$$\tau_i = k \qquad \psi = 0$$

$$\tau_s = \beta . k$$

If under zero external pressure and zero pore pressure the contact area is A_0 then the corresponding shear strength of the porous material will be

$$\tau_0 = c' = a_0 . m . \beta . k$$

where $a_0 = A_0/A$. Under an external pressure σ and pore pressure u the contact area ratio will change, and if its value is a then

$$\tau_f = a . m . \beta . k$$

$$\therefore \ \tau_f = c' + (a - a_0) m . \beta . k$$

Now with zero pore pressure the yield stress at the contact will be $\sigma_y = Nk$. But a pore-water pressure will support the sides of the particles and the asperities, and just as in a triaxial test on a purely cohesive material,

$$\sigma_1 = 2c + \sigma_3$$

so in the present case

$$\sigma_y = N.k + u$$

The initial contact area A_0 may be associated with an internal force P_0 such that

$$\frac{P_0}{A_0} = N.k$$

And if the corresponding pressure per unit gross area is p_0 then

$$p_0 = \frac{P_0}{A} = \frac{A_0}{A} . \frac{P_0}{A_0} = a_0 . N . k$$

The addition of an external pressure σ and a pore-water pressure u will increase the normal load at the contact by P_s and thus

$$\frac{P_0 + P_s}{A_s} = N.k + u$$

$$\therefore \ p_0 + a . \sigma_s = a(N.k + u)$$

But

$$a . \sigma_s - a . u = \sigma - u$$

$$\therefore \ p_0 + \sigma - u = a . N . k$$

or

$$a = \frac{p_0 + \sigma - u}{N.k}$$

Hence

$$a - a_0 = \frac{\sigma - u}{N.k}$$

* The angle of intrinsic friction is a formal concept, merely expressing in a convenient manner the increase in strength with pressure. It might more properly be defined as the 'angle of intrinsic shearing resistance'.

and therefore

$$\tau_f = c' + \frac{\sigma - u}{N.k} \cdot m . \beta . k$$

or

$$\tau_f = c' + (\sigma - u) \frac{m.\beta}{N}$$

In drained (or jacketed) tests where $u = 0$

$$\tau_d = c' + \sigma \cdot \frac{m.\beta}{N}$$

Thus

$$\tan \phi' = \frac{m.\beta}{N} = m.\mu$$

$$\therefore \ \tau_f = c' + (\sigma - u) \tan \phi'$$

And, finally,

$$\underline{\sigma' = \sigma - u}$$

We therefore find that if the particles have the property of a perfect cohesive solid ($\psi = 0$), the effective stress in shear strength problems is given rigorously by Terzaghi's equation. The formal demonstration of this fact is here given for the first time, but the result may be considered as intuitively evident and has been previously stated by BISHOP (1955) and CHUGAEV (1958)*.

Theory III—Shear tests on many solids, under very high pressures, have been made by BRIDGMAN (1935, 1936). For metals, the tensile strength of the pure polycrystalline material is also available (SEITZ and READ 1941, Landolt-Bornstein tables etc.). Combining these data for aluminium and plotting the results in the form of Mohr circles (Fig. 2(a)) it is seen that the strength increases linearly with pressure, according to the equation

$$\tau_i = k + \sigma . \tan \psi$$

This type of relationship is found to hold good for many metals, at least as an approximation, although there are some in which a polymorphic transition under high stresses causes a change in ψ. We are not concerned with details, however, and the values of k and ψ given in Table 1 can be taken as representative.

Table 1

Intrinsic Shear Parameters

Solid	k kg/cm²	ψ
Lead	100	$\frac{1}{2}$°
Zinc	600	$1\frac{1}{2}$°
Aluminium	500	3°
Copper	1200	$4\frac{1}{2}$°
Nickel	1800	$7\frac{1}{2}$°
Rock Salt	450	$3\frac{1}{2}$°
Calcite	1900	8°
Quartz	9500	$13\frac{1}{2}$°

Strength data for minerals are far less abundant, and I have been able to determine k and ψ only for Calcite, Rock Salt and Quartz. In triaxial tests on Marble, which is a porous material consisting of practically pure Calcite, VON KARMAN (1911) found that under a cell pressure of 2500 kg/cm², when the deviatoric compression strength (corrected for area increase) was about 5000 kg/cm², the macroscopic voids were virtually eliminated by plastic flow. The corresponding stress circle, applicable essentially to solid Calcite, is shown in Fig. 2(b). Two sets of high-pressure tests have been made on Calcite by BRIDGMAN (1936, and in GRIGGS 1942), and the mean results are also

* And probably in *Trans. Res. Inst. Hydrotechnics* (U.S.S.R.), 1947, but I have not been able to check this reference.

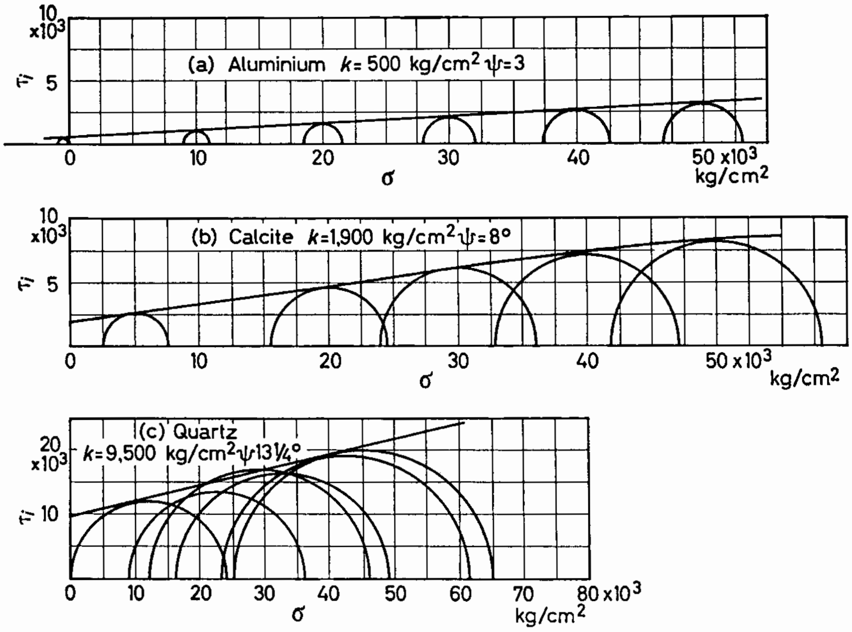

Fig. 2 Intrinsic strength data

plotted in this figure. Up to about 30,000 kg/cm^2 the strength increases linearly with pressure and $\psi = 8°$, but under greater pressures the slope of the failure envelope (or 'intrinsic line') decreases. From compression tests on discs of Rock Salt by King and Tabor (1954) a stress circle at comparatively low pressure can be obtained which, combined with Bridgman's tests (1936), shows that $\psi = 3\frac{1}{2}°$ for pressures up to about 20,000 kg/cm^2. For Quartz crystals, triaxial tests have been carried out by Griggs and Bell (1938) and by Bridgman (1941)*. The results are shown in Fig. 2(c) from which it will be seen that $\psi = 13\frac{1}{4}°$.

It is therefore evident, even from these fragmentary data, that ψ is likely to differ significantly from zero for soils, concrete and rocks. Hence for these materials we should assume that the intrinsic strength of the particles is given by the equation

$$\tau_i = k + \sigma.\tan\psi$$

The shear strength at the contact will then be

$$\tau_s = \beta.k + \sigma_s.\tan\psi_s$$

where ψ_s is the angle of friction at the interface which may, in general, be rather less than ψ.

Under zero external pressure and zero pore-water pressure the contact area ratio is a_0 and

$$\tau_0 = c' = a_0.m.\beta.k$$

When the external pressure is increased, and a pore-water pressure established, the contact area will change, and if the normal stress at the interface is σ_s then

$$\tau_f = a.m.\beta.k + a.\sigma_s.m.\tan\psi_s$$

But $$a\sigma_s = \sigma - u + au$$

$$\therefore\ \tau_f = c' + (a - a_0)m\beta k + (\sigma - u)m\tan\psi_s + aum\tan\psi_s$$

In order to obtain an expression for the change in contact area the yield stress of the solid material must be known. If the asperities have the form of a cylinder and if junction growth during shear is negligible, then

$$\sigma_y = k\cdot\frac{2\cos\psi}{1-\sin\psi} + u\cdot\frac{2\sin\psi}{1-\sin\psi} + u$$

or

$$\sigma_y = k\cdot\frac{2\cos\psi}{1-\sin\psi}\left(1 + \frac{u}{k}\cdot\tan\psi\right) + u$$

More generally, with any shape of asperities and with some junction growth, we may write

$$\sigma_y = k.M\left(1 + \frac{u}{k}\cdot\tan\psi\right) + u$$

where M is a coefficient similar to N in Theory II and numerically equal to N if $\psi = 0$. But as an approximation $u.\tan\psi/k$ may be neglected in comparison with unity. If, for example, k = 1000 kg/cm^2 and $\psi = 10°$, then even if u = 100 kg/cm^2 (which is equivalent to a head of 3000 ft of water), this term is equal to 0·018. Hence it is sufficiently accurate for most problems to assume that

$$\sigma_y = M.k + u$$

Thus, by direct analogy with the relevant passage in Theory II,

$$a - a_0 = \frac{\sigma - u}{M.k}$$

Hence

$$\tau_f = c' + (\sigma - u)\left(\frac{m.\beta}{M} + m.\tan\psi_s\right) + a.u.m.\tan\psi_s$$

In drained (or jacketed) tests $u = 0$

$$\therefore\ \tau_d = c' + \sigma\left(\frac{m.\beta}{M} + m.\tan\psi_s\right)$$

Thus

$$\tan\phi' = \frac{m.\beta}{M} + m.\tan\psi_s = m.\mu$$

$$\therefore\ \tau_f = c' + (\sigma - u)\tan\phi' + a.u.m.\tan\psi_s$$

But since m is greater than unity ($m = \pi/2$ in Caquot's analysis) and ψ_s is less than ψ it is unlikely that any serious error will arise in writing $m.\tan\psi_s = \tan\psi$. In that case

$$\tau_f = c' + (\sigma - u)\tan\phi' + a.u.\tan\psi$$

or

$$\tau_f = c' + \left[\sigma - \left(1 - \frac{a.\tan\psi.}{\tan\phi'}\right)u\right]\tan\phi'$$

* At cell pressures of 13,000 and 19,500 kg/cm^2 the tests by Griggs and Bell show a very marked increase in strength, which was not found by Bridgman, using an improved technique, even with cell pressures of 25,000 kg/cm^2. These anomalous results were probably due to experimental difficulties, and they have been omitted from Fig. 2(c). Possibly ψ would be less, for Calcite as well as Quartz, if a gas was used as the cell fluid in triaxial tests.

Hence

$$\sigma' = \sigma - \left(1 - \frac{a.\tan\psi}{\tan\phi'}\right)u$$

The above equation is a general expression for effective stress in relation to the strength of saturated porous materials, and it includes the two previous theories as special cases.

Strength of porous materials—Before examining the foregoing theories in the light of experimental evidence, it is necessary to consider an aspect of the strength properties of porous material which does not appear hitherto to have been recognized. It is well known that the failure envelope for concrete and rocks, and to some extent for sands, becomes progressively flatter with increasing pressure. This is clearly shown in Fig. 3 where the whole series of triaxial (jacketed) tests on Marble by von Karman (*loc. cit.*) have been plotted. But it also becomes that when $\sigma = 0$

$$c' = a_0.m.\beta.k$$

and since $m.\beta$ is of the order unity, the initial area ratio is approximately

$$a_0 = \frac{c'}{k}$$

Further, if $m.\tan\psi_s$ is assumed to equal $\tan\psi$ then under a pressure σ the drained strength is approximately

$$\tau_d = c' + (a - a_0)k + \sigma.\tan\psi$$

or

$$\tau_d = a.k + \sigma.\tan\psi$$

And, for all positive values of σ

$$\tau_i = k + \sigma.\tan\psi$$

Hence

$$a = 1 - \frac{\tau_i - \tau_d}{k}$$

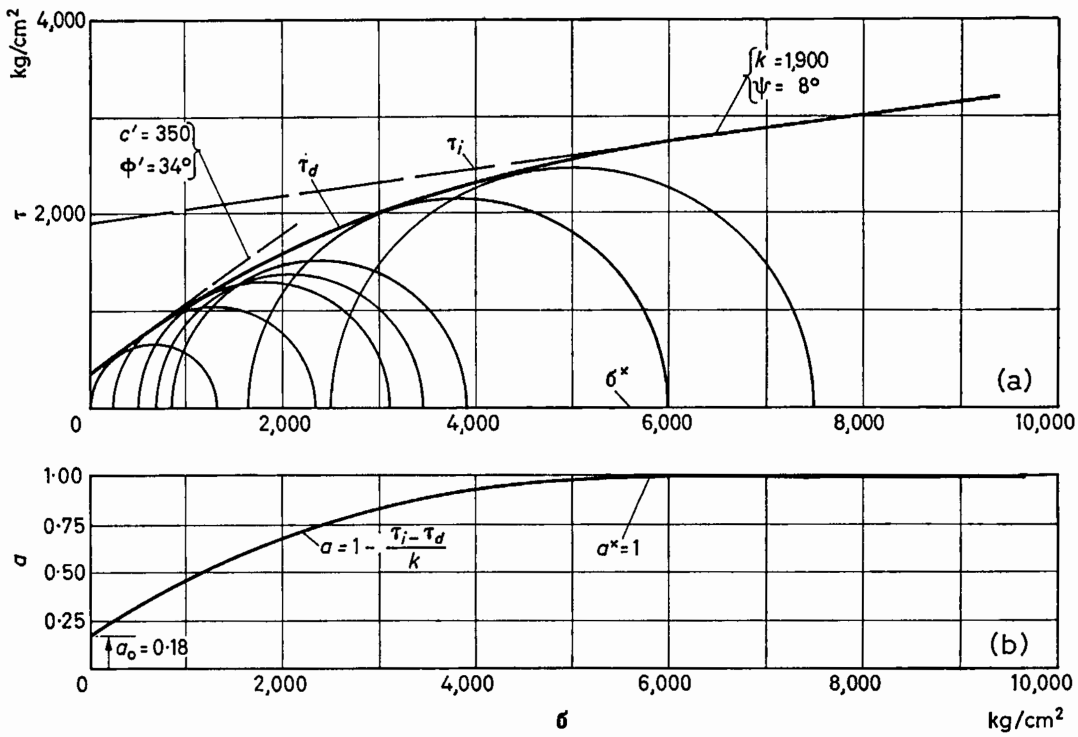

Fig. 3 Triaxial tests on Marble (after VON KARMAN, 1911)

clear that the failure envelope is tending towards the intrinsic line of the solid substance comprising the particles of the porous material and, finally, at a pressure sufficiently high to cause complete yield of the particles, when the voids are eliminated, the failure envelope becomes coincidental with the intrinsic line, as determined from Bridgman's high-pressure tests.

The shape of the failure envelope is therefore controlled, to an important extent, by changes in the contact area under pressure. When the porosity is comparatively high a given pressure increment $\Delta\sigma$ causes a correspondingly high increase in contact area Δa; but as the porosity is progressively reduced $\Delta a/\Delta\sigma$ also becomes smaller and eventually, at a pressure σ^* when the porosity is zero and $a = 1$, $\Delta a/\Delta\sigma$ becomes zero and the slope of the failure envelope falls to the value ψ. This behaviour is illustrated diagrammatically in Fig. 4 where τ_i is the intrinsic line for the solid, and τ_d is the failure envelope from drained (jacketed) tests on the porous material. When σ is small compared with σ^* the failure envelope may be considered as linear, and Coulomb's equation is applicable. Thus $\tau_d = c' + \sigma.\tan\phi'$ when σ/σ^* is small. But we have shown

For solids $a = 1$ throughout the entire pressure range whilst, at the other extreme, for cohesionless porous materials such as sand and lead shot, $a_0 = 0$ (see Fig. 4). The variation of a from $a_0 = 0{\cdot}18$ to $a^* = 1$ for von Karman's tests on Marble, is shown in Fig. 3. At a pressure equal to that on the shear plane in the unconfined compression test on this material, $a = 0{\cdot}25$.

It is probable that this concept of the strength of porous materials is capable of interesting development. But for our present purpose its importance lies in the fact that it provides a method for determining a in two calcitic limestones for which, from the results of unjacketed triaxial tests, the value of a can also be deduced from the theories of effective stress. And in this way at least a rough check can be obtained on the theories of effective stress.

Unjacketed tests on rocks and concrete—In these tests the specimen is not covered with a membrane and the cell fluid can penetrate fully into the voids of the material. Thus, with the usual notation, $\sigma_3 = u$ and the procedure consists of measuring the deviatoric compression strength under various cell pressures.

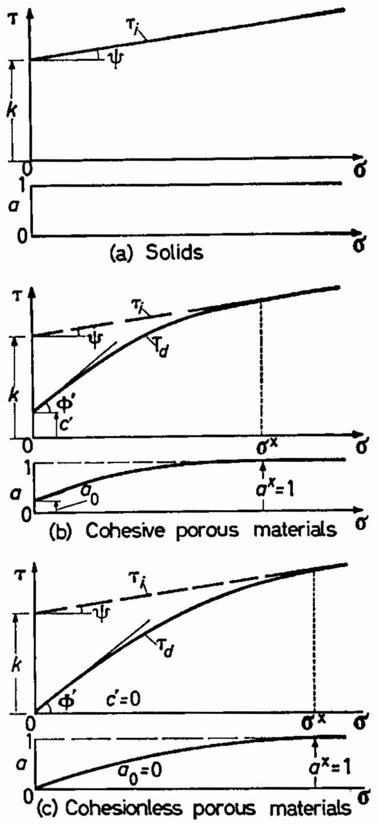

Fig. 4

Now according to Theory I

$$\tau_f = c' + [\sigma - (1 - a)u] \tan \phi'$$

This is the shear strength on a plane on which the normal applied stress is σ. In the triaxial test where, in general, the specimen is subjected to applied principal stresses σ_1 and σ_3 and there is a pore pressure u in the specimen,

$$\tau = \tfrac{1}{2}(\sigma_1 - \sigma_3) \sin 2\theta$$

and

$$\sigma = \sigma_3 + \tfrac{1}{2}(\sigma_1 - \sigma_3) \cos^2\theta$$

where θ is the inclination of the shear plane to the direction of σ_3.

For the condition that $(\sigma_1 - \sigma_3)$ is a minimum $\theta = 45° + \phi'/2$ and it then follows that

$$(\sigma_1 - \sigma_3)_f = c' \cdot \frac{2 \cos \phi'}{1 - \sin \phi'} + (\sigma_3 - u) \frac{2 \sin \phi'}{1 - \sin \phi'} + a . u \cdot \frac{2 \sin \phi'}{1 - \sin \phi'}$$

Thus, in the unjacketed tests where $\sigma_3 = u$

$$\frac{\Delta(\sigma_1 - \sigma_3)_u}{\Delta u} = a \cdot \frac{2 \sin \phi'}{1 - \sin \phi'}$$

It is evident that according to Theory II there is no increase in strength in the unjacketed test, but in Theory III

$$\tau_f = c' + (\sigma - u) \tan \phi' + a . u . \tan \psi$$

As before $\theta = 45° + \phi'/2$ and hence

$$(\sigma_1 - \sigma_3)_f = c' \cdot \frac{2 \cos \phi'}{1 - \sin \phi'} + (\sigma_3 - u) \frac{2 \sin \phi'}{1 - \sin \phi'} + \text{a} . u \cdot \frac{\tan \psi}{\tan \phi'} \cdot \frac{2 \sin \phi'}{1 - \sin \phi'}$$

Thus in the unjacketed tests

$$\frac{\Delta(\sigma_1 - \sigma_3)_u}{\Delta u} = a . \frac{\tan \psi}{\tan \phi'} \cdot \frac{2 \sin \phi'}{1 - \sin \phi'}$$

Now in all three theories the shear strength in drained or jacketed tests, in which the pore-water pressure is zero, will be

$$\tau_d = c' + \sigma . \tan \phi'$$

Hence

$$\frac{\Delta(\sigma_1 - \sigma_3)_d}{\Delta \sigma_3} = \frac{2 \sin \phi'}{1 - \sin \phi'}$$

Therefore, provided ϕ' is known, the area ratio can be computed either from Theory I or Theory III, given the rate of increase in unjacketed strength with increasing cell pressure.

Griggs (1936) has carried out such tests on Marble and Solenhofen Limestone. At a cell pressure of 10,000 kg/cm^2 an extraordinary increase in strength was obtained in both materials, and to a lesser extent in the Solenhofen also at 8000 kg/cm^2. These results may be due to experimental errors as in the case of Quartz, previously mentioned; but the tests at smaller pressures show a definite increase in strength with pore pressure and there seems to be no reason to doubt this general trend; which is evidently at variance with Theory II. From Fig. 5 it will be seen that approximate values for $\Delta(\sigma_1 - \sigma_3)/\Delta u$

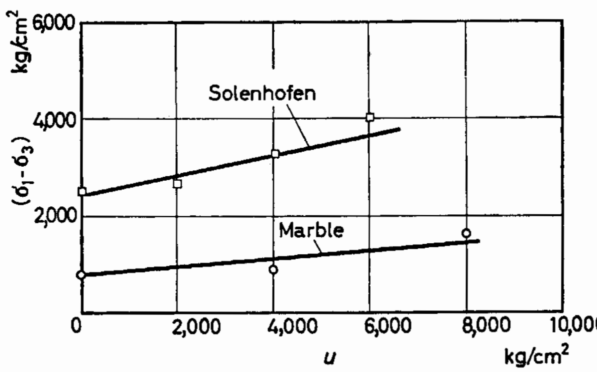

Fig. 5. Unjacketed triaxial tests on Marble and Solenhofen Limestone (Griggs, 1936)

are 0·08 and 0·14 for the Marble and Solenhofen respectively. Unfortunately Griggs does not give sufficient data to determine the failure envelope for drained or jacketed tests, but the stress circles for the unconfined compression tests are plotted in Fig. 6 together with the failure envelope for von Karman's tests. Since both of the materials in Griggs' tests consist of pure Calcite, the intrinsic line will presumably be the same, or very similar, to that previously used in analysing von Karman's tests. Thus it is not difficult to sketch the probable failure envelopes for the Marble and Solenhofen tested by Griggs. From Fig. 6 it will be seen that the corresponding values of ϕ are 34° and 24°; and ψ is 8°. The area ratios can then be calculated, see Table 2, and are found to be 0·03 and 0·14 (Theory I), or 0·15 and 0·45 (Theory III). But from the Mohr's circles in Fig. 6 and from the intrinsic line it is possible to estimate that the area ratios at the normal stress σ acting on the shear plane in the unconfined compression tests are 0·16 and 0·52 for the Marble and Solenhofen respectively. These latter values of a are not precise, but they tend to disprove Theory I and broadly to substantiate Theory III.

We may now turn to the tests which have been made on concrete with the object of determining the area ratio, and concerning which there has been for many years some uncertainty and controversy.

Triaxial tension tests on cement by Fillunger (1915), using unjacketed specimens, show little change in strength with increasing pore pressure, but more elaborate tests by Leliavsky (1945) on concrete prove the existence of a small but quite

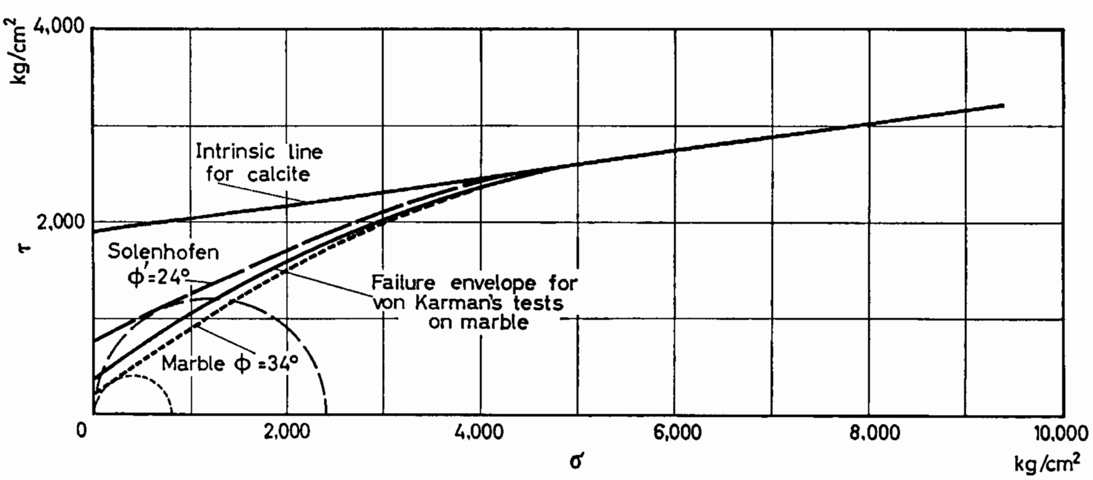

Fig. 6 Probable failure envelopes for Griggs' tests on Marble and Solenhofen Limestone

definite increase in strength. The analysis of tension tests, however, is beyond the scope of the present paper and it will merely be noted that Leliavsky concludes that an average value for the contact area ratio is 0·09 for concrete failing in tension.

Table 2
Evaluation of Contact Area Ratio

Parameter	*Marble*	*Solenhofen*
k (kg/cm²)	1900	1900
ψ	8°	8°
c' (kg/cm²)	210	780
ϕ'	34°	24°
q = unconfined comp. strength	800	2400
$\sigma = \frac{1}{2}q(1 - \sin\phi')$ } in unconfined	180	700
$\tau_d = \frac{1}{2}q.\cos\phi'$ } compression	330	1090
$\tau_i = k + \sigma.\tan\psi$ } test	1930	2000
$D_\sigma = \Delta(\sigma_1 - \sigma_3)/\Delta\sigma_3$	2·5	1·4
$D_u = \Delta(\sigma_1 - \sigma_3)/\Delta u$	0·08	0·20
$\tan\phi'/\tan\psi$	4·8	3·2
Theory I $a = \dfrac{D_u}{D_\sigma}$	0·03	0·14
Theory III $a = \dfrac{D_u}{D_\sigma}.\dfrac{\tan\phi'}{\tan\psi}$	0·15	0·45
$a = 1 - \dfrac{\tau_i - \tau_d}{k}$	0·16	0·52

Triaxial compression tests on concrete by Terzaghi and Rendulic (1934) also indicate small increases in strength in unjacketed specimens, although the scatter of individual results is too great for any exact deductions to be made. But a comprehensive series of triaxial compression tests on a 1:2½:2½ concrete has been reported by McHenry (1948) and the results are plotted in Fig. 7, each point being the average of at least three tests. The unconfined compression strength appears to be somewhat anomalous and if this is neglected we find from the jacketed tests that $\phi' = 52°$ or $2\sin\phi'/(1 - \sin\phi') = 7{\cdot}4$. The increase in unjacketed strength with increasing pore pressure is expressed by the ratio $\Delta(\sigma_1 - \sigma_3)/\Delta u = 0{\cdot}25$. Thus according to Theory I, $a = 0{\cdot}25/7{\cdot}4 = 0{\cdot}035$.

If the unconfined test is included $\phi' = 48°$, but statistically the unjacketed strength then decreases with increasing pore pressure. This is not physically possible, however, and since the unjacketed tests under the eight different pressures show a fairly steady increase (Fig. 7) it seems more reasonable to accept the foregoing value of $\Delta(\sigma_1 - \sigma_3)/\Delta u$. Whether ϕ' is taken as 48° or 52° is of little consequence and, for consistency, the latter will be used: this being equivalent to neglecting the unconfined strength in both types of test.

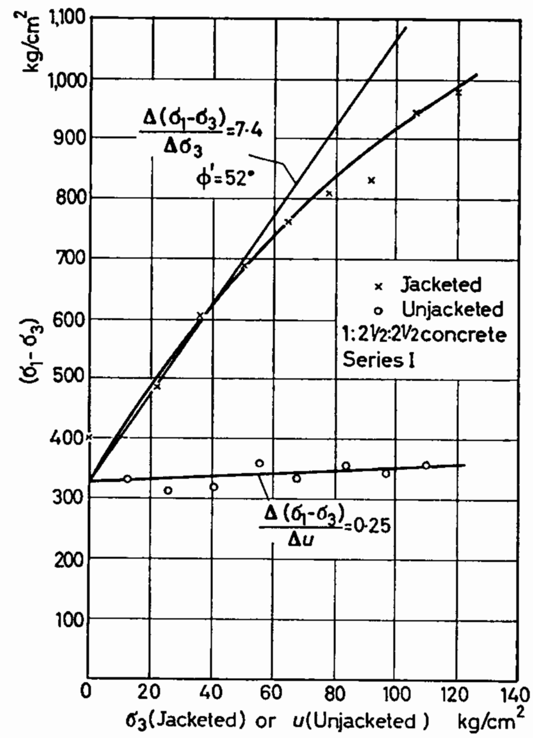

Fig. 7 Triaxial tests on concrete (McHenry, 1948)

Now an area ratio of 0·035 (it merely rises to 0·045 if $\phi' = 48°$) appears to be remarkably small for a concrete which, in this series of tests, has a compression strength of about 5000 lb/in². Unfortunately there is no evidence from an intrinsic line which would enable a to be estimated independently, as with the calcitic limestones. But it may readily be assumed that ψ can hardly be greater than the value for Quartz, namely about 13°. In that case $\tan\phi'/\tan\psi$ is 5·5 and hence according to Theory III a would be not less than $5{\cdot}5 \times 0{\cdot}035 = 0{\cdot}19$. Any further refinement would not be justified and, in round figures, this analysis of McHenry's tests may be summarized by saying that the contact area ratio is 0·04 or 0·2 depending on whether Theory I or Theory III is adopted. The Marble and Solenhofen tests have already shown that the latter is to be preferred and it seems that Theory III also leads to a more reasonable result for concrete.

Soils—So far as soils are concerned no critical tests, necessarily involving high pore pressures, have been carried out. But there is ample evidence, from work by RENDULIC (1937), TAYLOR (1944), BISHOP and ELDIN (1950) and others, that within the range of pore pressures encountered in practice Terzaghi's equation $\sigma' = \sigma - u$ involves no significant error for fully saturated sands and clays. If Theory III is correct, however, the effective stress is more properly given by the expression

$$\sigma - \left(1 - \frac{a.\tan\psi}{\tan\phi'}\right)u$$

and the validity of Terzaghi's equation must therefore depend upon the small magnitude of the term $a.\tan\psi/\tan\phi'$.

For sands, which typically consist chiefly of Quartz particles, ψ will be about 13°. And ϕ' is usually within the range 30° to 40°. Thus $\tan\psi/\tan\phi'$ may be expected to be of the order 0·3. This is far from negligible, as compared with unity, and hence the validity of the equation $\sigma' = \sigma - u$ in shear strength problems in sands must be dependent upon the small magnitude of the contact area ratio at normal engineering pressures. Owing to the high-yield stress of Quartz, and probably of other minerals found in sands, there is no difficulty in accepting this conclusion.

For clay minerals it is likely that ψ has quite low values, but clay soils contain appreciable quantities of silt and often some fine sand as well. The average values of ψ for clays may therefore be very roughly in the range 5° to 10°. In these soils ϕ' usually lies between 20° and 30°. Thus $\tan\psi/\tan\phi'$ is probably not less than about 0·15 for clays, and possibly a more representative value is about 0·25. Thus it seems that in clays, as well as in sands, the physical basis for Terzaghi's equation in shear strength problems is the small area of contact between the particles.

Lead Shot—Triaxial tests on lead shot show that $\phi' = 24°$ (BISHOP 1954)*. But as we have seen, ψ for lead is only $\frac{3}{4}$°. Thus $\tan\psi/\tan\phi' = 0{\cdot}03$. Consequently even at pressures sufficiently high virtually to eliminate the voids, when the particles will approximate to polyhedra and a is approaching unity, an error of not more than 3 per cent would be involved in using the expression $(\sigma - u)$ for effective stress in relation to shear strength. It may therefore be anticipated that Terzaghi's equation would be proved valid in shear tests on lead shot over a very wide pressure range; but, in this case, owing to the fact that $\tan\psi/\tan\phi'$ is small.

PART II

Compressibility of Saturated Materials

When porous materials are subjected to an increase in all-round pressure, under the condition of zero pore pressure (i.e. in jacketed or drained consolidation tests), their volume decreases, and if for any comparatively small increase in pressure from p' to $p' + \Delta p'$ the volume changes from V to $V + \Delta V$, where ΔV is negative, then the compressibility C of the material for this particular pressure increment is defined by the equation

$$-\left(\frac{\Delta V}{V}\right)_d = C.\Delta p'$$

Compressibility is not a constant, but decreases with increasing pressure and eventually, under a pressure sufficiently high to eliminate the voids, C will fall to the value C_s where C_s is the compressibility of the solid particles. This behaviour is analogous to the progressive flattening of the slope of the failure envelope with increasing pressure, until it is eventually equal to ψ, and is clearly illustrated by the jacketed tests on Vermont Marble and a Quartzitic sandstone (ZISMAN, 1933) shown in Fig. 8. It will be seen that, under the highest pressure in these tests, namely 600 kg/cm², the compressibilities have decreased to values only about 15 per cent greater than the compressibilities of Calcite and Quartz, respectively.

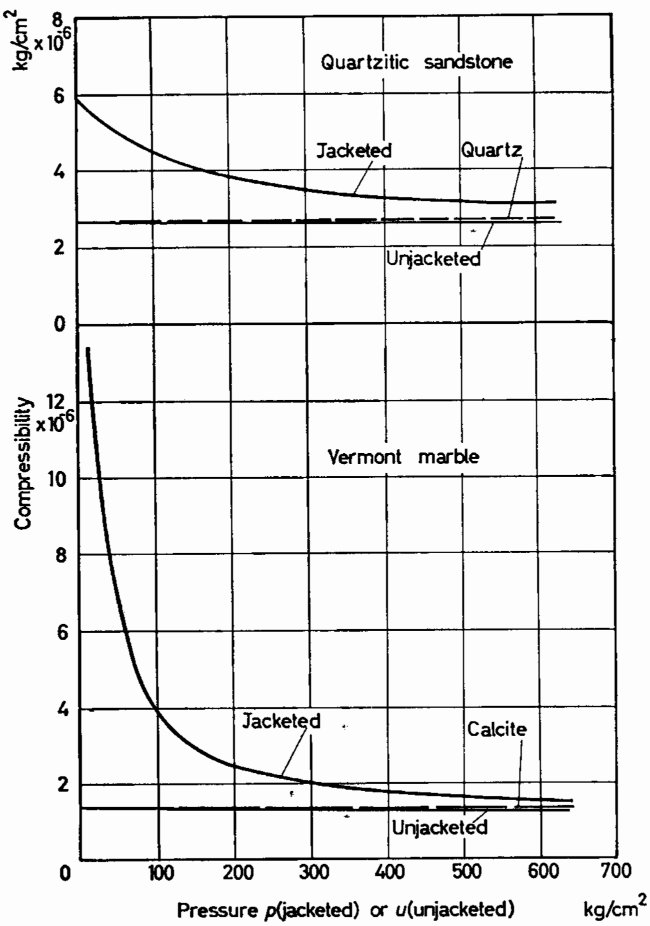

Fig. 8 Compressibility tests (ZISMANN, 1933 and BRIDGMAN, 1928)

The problem under consideration, however, is to obtain an expression for $\Delta p'$ when a saturated porous material is subjected to a pore-pressure change Δu as well as a change in the applied pressure Δp.

Theory I—As in the case of shear strength, the usually accepted theory is based on the assumption that the effective stress is the intergranular pressure

$$\sigma_g = \sigma - (1 - a)u$$

On this hypothesis it therefore follows at once that

$$-\frac{\Delta V}{V} = C[\Delta p - (1 - a)\Delta u]$$

and

$$\Delta p' = \Delta p - (1 - a)\Delta u$$

Theory II—It has already been suggested that in a 'perfect' solid $\psi = 0$. We may now add that such a solid will also be incompressible, or $C_s = 0$. Thus in a porous material consisting of perfect solid particles a change in applied stress together with an equal change in pore pressure will cause no volume change or deformation in the particles, nor any change in the contact area and shear strength. Consequently a decrease in volume, just as an increase in shear strength, can only result from an increase in $(\sigma - u)$ or, in the present case, from

* Bishop has also shown that for lead $\mu = 0{\cdot}26$. Thus in the equation $\tan\phi' = m.\mu$ the coefficient $m = 1{\cdot}70$. This may be compared with Caquot's value $m = \pi/2 = 1{\cdot}57$.

an increase in $(\Delta p - \Delta u)$. Therefore

$$-\frac{\Delta V}{V} = C[\Delta p - \Delta u]$$

and

$$\Delta p' = \Delta p - \Delta u$$

Theory III—In fact, however, the solid particles have a finite compressibility C_s and an angle of intrinsic friction ψ. If the porous material is subjected to equal increases in applied pressure and pore pressure, as in an unjacketed test with an equal all-sided cell pressure and no additional axial stress, then each particle will undergo cubical compression under a hydrostatic pressure Δu. And the material will decrease in volume by an amount*

$$-\left(\frac{\Delta V}{V}\right)_s = C_s.\Delta u$$

If now, for the moment, we assume that ψ is zero, or negligible, then a net pressure increment $(\Delta p - \Delta u)$ will cause a volume change exactly equal to that which would result from an application of an identical pressure in the absence of pore pressure. Thus the total volume change of the porous material must be

$$-\Delta V/V = C(\Delta p - \Delta u) + C_s.\Delta u$$

and it will be noted that the contact area ratio is not involved.

A similar line of reasoning has been expressed by CHUGAEV (1958 and probably 1947—see previous footnote). But the above equation was first given, so far as the author is aware, by A. W. Bishop in a letter to A. S. Laughton (24 Nov. 1953) following discussions at Imperial College between Dr Laughton, Dr Bishop and the author.

The existence of an angle of intrinsic friction implies, however, that when a porous material is subjected to an increase in pressure $(\Delta p - \Delta u)$ there will be a small increase in resistance to particle rearrangement and deformation, associated with the increase in pore pressure. An approximate analysis suggests that a more complete expression is of the form

$$-\frac{\Delta V}{V} = C(\Delta p - \Delta u)[1 - \eta.\Delta u.\tan\psi] + C_s.\Delta u$$

when η is a function depending on the relative contributions by particle rearrangement and deformation to the total volume change, and involving the parameters a, $1/\tan\phi'$ and $1/k$. But this analysis also indicates that the η term is numerically unimportant in most cases†. Thus it is sufficiently accurate to write

$$-\frac{\Delta V}{V} = C\left[\Delta p - \left(1 - \frac{C_s}{C}\right)\Delta u\right]$$

or

$$\Delta p' = \Delta p - \left(1 - \frac{C_s}{C}\right)\Delta u$$

Some representative values of C_s/C for rocks (from data by Zisman *loc. cit.*) and for soils (data from author's files) are given in Table 3.

It will at once be seen that for soils the ratio C_s/C is negligible and hence Terzaghi's equation is acceptable to a high degree of approximation. In contrast, the compressibility ratio is so considerable for rocks and concrete that important errors could be involved in using the equation $\Delta p' = \Delta p - \Delta u$ for such materials.

Unjacketed tests on rocks—In these tests the specimen is not covered with a membrane and the cell fluid can penetrate fully into the voids of the material. Thus $\Delta p = \Delta u$, and the procedure consists of measuring the volume changes under various cell pressures.

Table 3

Compressibilities at $p = 1$ kg/cm²
Water $C_w = 48 \times 10^{-6}$ per kg/cm²

Material	*Compressibility per* kg/cm² × 10^{-6}		$\frac{C_s}{C}$
	C	C_s	
Quartzitic sandstone	5·8	2·7	0·46
Quincy granite (100 ft deep)	7·5	1·9	0·25
Vermont marble	17·5	1·4	0·08
Concrete (approx. values)	20	2·5	0·12
Dense sand	1,800	2·7	0·0015
Loose sand	9,000	2·7	0·0003
London Clay (over-cons.)	7,500	2·0	0·00025
Gosport Clay (normally-cons.)	60,000	2·0	0·00003

According to Theory I the volume change in an unjacketed test is

$$-(\Delta V/V)_u = C.a.\Delta u$$

or

$$-\frac{(\Delta V/V)_u}{\Delta u} = C.a$$

According to Theory II the volume change in these tests will be zero, but from Theory III even in its most general form including the η term,

$$-\frac{(\Delta V/V)_u}{\Delta u} = C_s$$

Unjacketed tests on a number of rocks have been made by Zisman (*loc. cit.*) and those on Marble and on a Quartzitic sandstone are especially valuable since the particles of each of these materials consist essentially of a pure mineral of known compressibility; namely Calcite and Quartz respectively. In Fig. 8 the results of the unjacketed tests are plotted, together with the compressibility of the relevant mineral, and it will be seen that the observed values of $-(\Delta V/V)/\Delta u$ are almost indistinguishable from the values of C_s (as determined by BRIDGMAN, 1925, 1928).

This evidence is strongly in favour of Theory III, but it is not sufficient to rule out Theory I since the values of a which could be deduced from these tests are quite reasonable.

Lead shot—It is therefore fortunate that further information is available from a special oedometer test on lead shot (grade 10) carried out by LAUGHTON (1955 and personal communication). An increment of total pressure was applied together with a known increment of pore pressure, and after a sufficient time for equilibrium to be attained the volume change was noted. The pore pressure was then reduced to zero and, after the same time interval, the further consolidation was observed. From a number of such increments a curve can be obtained (Fig. 9) relating void ratio and pressure, with zero pore pressure. And since this curve gives the effective pressure for a known void ratio, the effective pressure corresponding to any of the particular combinations of total pressure and pore pressure can be deduced. Moreover, at the end of the test the lead shot was removed and by measurement with the microscope of the facets where the particles had been pressed together, an approximate value of a was obtained. Separate tests were also made to determine a in this way for two of the intermediate stages of loading.

The author is indebted to Dr Laughton for the results of these experiments, from which the data set out in Table 4 have been computed. Perhaps the most striking point to note is

* This equation may be confirmed by analysing a formal arrangement of spherical particles. It is to be noted that the total volume change of the particles themselves, not of the porous material, is equal to $-C_s(1 - n)V.\Delta u$ where n is the porosity.

† For example, the η term will, in effect, typically reduce C by the order of 3 per cent for Marble and for Lead Shot.

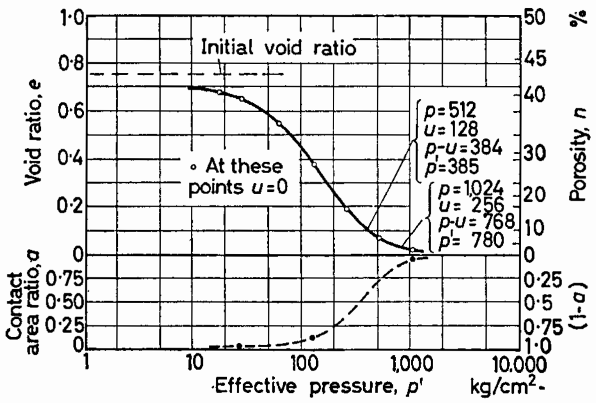

Fig. 9 Consolidation test on lead shot (Laughton)

that the use of Terzaghi's equation would involve quite small errors over the entire pressure range, in spite of the fact that the particles had been so distorted under the highest pressure that the area ratio was approaching unity. But the last four increments in this test provide the most critical evidence for checking the validity of the various theories of effective stress. The pressures and void ratios, and the changes in these quantities during the penultimate increment, are given for convenience in Table 5, together with approximate values of a obtained by interpolation from the microscope measurements.

Table 4

Consolidation Test on Lead Shot

Total pressure p kg/cm²	*Pore pressure* u kg/cm²	*Void ratio* e	*Effective pressure* p' kg/cm²		$p - u$ kg/cm²	a *Observed*
			Observed	*Deduced*		
17	0	0·675	17			
27	8	0·670	—	19	19	
27	0	0·650	27			0·03
60	16	0·601	—	42	44	
60	0	0·549	60			
128	32	0·452	—	94	96	
128	0	0·379	128			0·11
256	64	0·262	—	195	192	
256	0	0·190	256			
512	128	0·105	—	385	384	
512	0	0·069	512			
1024	256	0·032	—	780	768	
1024	0	0·020	1024			0·95

Table 5

p	u	e	p'	a
512	0	0·069	512	0·7
1024	256	0·032	780	0·9
$\Delta p = 512$	$\Delta u = 256$	$\Delta e = 0{\cdot}037$	$\Delta p' = 268$	$a = 0.8$

The volume change is

$$-\frac{\Delta V}{V} = \frac{\Delta e}{1 + e} = \frac{0{\cdot}037}{1{\cdot}069} = 0{\cdot}035$$

and this is caused by an increase in effective stress of 268 kg/cm². Thus, since the three-dimensional compressibility C can be taken to be of the same order as the one-dimensional compressibility (Skempton and Bishop, 1954),

$$C = \frac{0{\cdot}035}{268} = 130 \times 10^{-6} \text{ per kg/cm}^2$$

The compressibility of lead, as determined by Ebert (1935), is. $C_s = 2{\cdot}45 \times 10^{-6}$ per kg/cm.² Hence $C_s/C = 0{\cdot}02$.

Before calculating the effective stress it should be mentioned that high precision is not possible in the oedometer, owing to friction between the specimen and the walls of the apparatus, and owing to the unknown lateral pressures. These factors will, however, reduce both Δp and $\Delta p'$. Consequently the usefulness of the test as a comparative check will not be greatly impaired.

Now according to Theory I

$$\Delta p' = \Delta p - (1 - a)\Delta u$$

$$\therefore \Delta p' = 512 - (1 - 0{\cdot}8)\,256 = \underline{460} \text{ kg/cm}^2$$

But the value of $\Delta p'$ deduced from the test is 268 kg/cm. In this case Theory I therefore involves a discrepancy of 70 per cent, which is far in excess of experimental error.

According to Theory II

$$\Delta p' = \Delta p - \Delta u$$

$$\therefore \Delta p' = 512 - 256 = \underline{256} \text{ kg/cm}^2$$

and in the simpler form of Theory III

$$\Delta p' = \Delta p - \left(1 - \frac{C_s}{C}\right)\Delta u$$

$$\therefore \Delta p' = 512 - (1 - 0{\cdot}02)\,256 = \underline{261} \text{ kg/cm}^2$$

These two results differ from 268 kg/cm² by less than 5 per cent, which is probably insignificant in view of the inaccuracies associated with the test.

Similar calculations have been made for the other high-pressure increments, and the results are summarized in Table 6. The apparently excellent agreement between Theory III and the test values is to some extent fortuitous, but there can be no doubt that Theory I is altogether unacceptable; whilst Terzaghi's equation (Theory II) is a good approximation even when the contact-area ratio exceeds 50 per cent.

Table 6

p kg/cm²	u kg/cm²	a (approx.)	$\frac{C_s}{C}$	$\Delta p'$ kg/cm²			
				Theory I	*Theory II*	*Theory III*	*Experimental*
256	0						
		0·35	0·005	170	128	129	129
512	128						
		0·6	0·01	50	128	127	127
512	0						
		0·8	0·02	460	256	261	268
1024	256						
		0·9	0·05	20	256	243	244
1024	0						

The foregoing experiments, together with the unjacketed compressibility tests on Marble and Quartzite, therefore confirm that the effective stress controlling volume changes in porous materials is given with sufficient accuracy by the equation

$$\Delta p' = \Delta p - \left(1 - \frac{C_s}{C}\right)\Delta u$$

And since this expression does not include the contact-area ratio, it follows that this parameter cannot be determined from volume change tests.

Globigerina ooze—In the past there has been some doubt as to the effective pressure in very deep beds of sediment. This uncertainty has arisen from the use of Theory I and the probability that under high-pressures the contact area ratio must be considerable. But it will now be seen that the only relevant question is whether the ratio C_s/C becomes significant. From compressibility data on clays (SKEMPTON, 1953) it would seem that at pressures of the order 200 kg/cm² C is unlikely to be less than about 250×10^{-6} per kg/cm² and C_s/C is therefore still not more than 0·01. In other words Terzaghi's equation should be applicable to depths of several thousand feet of sediments.

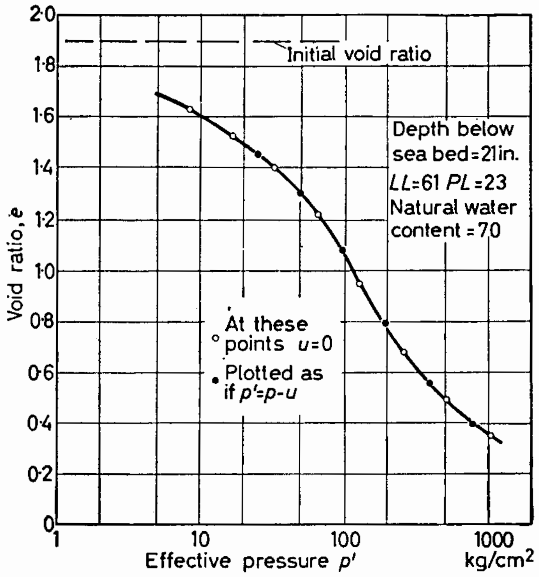

Fig. 10 Consolidation test on Globigerina ooze (LAUGHTON, 1955)

In order to confirm this conclusion Laughton (*op. cit.*) carried out a consolidation test on a sample of Globigerina ooze in the same apparatus as that used for the lead shot tests, and with the same procedure. The sample was obtained from the bed of the eastern Atlantic ocean, during the 1952 cruise of *R.R.S. Discovery II* at Station 2994 (Lat. 49° 08′ N; Long. 17° 37′ W). The results are plotted in Fig. 10 and it will be seen that up to pressures of at least 800 kg/cm² the effective stress is given within reasonably close limits by the expression $(p - u)$. This would be expected, since at 800 kg/cm² the compressibility of the material is about 150×10^{-6} per kg/cm² and C_s/C is therefore less than 2 per cent.

PART III

Partially Saturated Soils

When the pore space of a soil contains both air and water, the soil is said to be partially saturated, and the degree of saturation is defined by the ratio

$$S_r = \frac{\text{vol. water}}{\text{vol. pore space}}$$

Owing to surface tension the pore-water pressure u_w is always less than the pore-air pressure, u_a. And if we take the case of a soil with a rather low degree of saturation, the pore water will be present chiefly as menisci in the vicinity of the interparticle contacts, as sketched in Fig. 11. It is then possible to consider that the pore-water pressure acts over an area χ per unit gross area of the soil*, and that the pore-air pressure acts over an area $(1 - \chi)$. The equivalent pore pressure will thus be

$$\chi . u_w + (1 - \chi)u_a$$

When the degree of saturation is small, and the voids are chiefly filled with air, this expreession is better written in the

* Strictly, χ is not an area (Aitchison and Donald 1956). But this assumption leads to a simple model of the problem and a correct *form* of the expression for equivalent pore pressure.

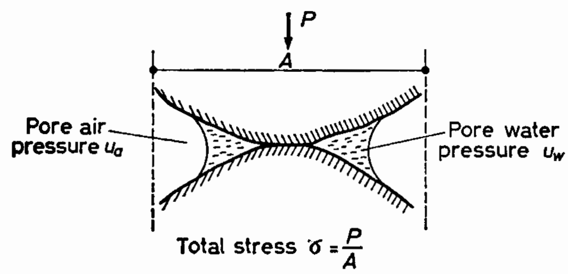

Fig. 11

form

$$u_a - \chi(u_a - u_w)$$

since this represents directly the concept that the equivalent pore pressure is less than the pore-air pressure only by the pressure difference $(u_a - u_w)$ acting over a small area χ. Similarly when the soil is almost fully saturated, and χ is approaching unity, the equivalent pore pressure is better expressed in the form

$$u_w + (1 - \chi)(u_a - u_w)$$

Now it has been shown that for fully saturated soils the effective stress is given to a very close approximation by Terzaghi's equation

$$\sigma' = \sigma - u_w$$

Thus it is reasonable to presume that for partially saturated soils the effective stress would similarly be given by the equation, first suggested by BISHOP in 1955,

$$\sigma' = \sigma - [u_a - \chi(u_a - u_w)]$$

There is no reason, however, why the coefficient χ should be identical in problems of shear strength and consolidation, and for a given degree of saturation the value of χ must be determined experimentally in both types of test. Nevertheless it is evident that $\chi = 1$ when $S_r = 1$ and that $\chi = 0$ when $S_r = 0$, for all conditions of test. Consequently, at these two limits, we have

$$\sigma' = \sigma - u_w \quad \text{when} \quad S_r = 1$$
$$\sigma' = \sigma - u_a \quad \text{when} \quad S_r = 0$$

An important special case arises when u_a is equal to atmospheric pressure. For, since all pressures are normally expressed in relation to atmospheric pressure as a base, u_a is then zero and

$$\sigma' = \sigma - \chi . u_w$$

It is to be noted that in most engineering problems the degree of saturation is more nearly unity than zero; and since it is the pore-water pressure that is usually measured, in the laboratory and in the field, a more convenient form of the effective stress equation for partially saturated soils is

$$\sigma' = \sigma - [u_w + (1 - \chi)(u_a - u_w)]$$

or

$$\sigma' = \sigma - \left[1 + (1 - \chi)\frac{u_a - u_w}{u_w}\right]u_w$$

If we write

$$S_\chi = 1 + (1 - \chi)\frac{u_a - u_w}{u_w}$$

then

$$\sigma' = \sigma - S_\chi . u_w$$

and as $S_r \to 1$, so does $S_\chi \to 1$. Further, if $u_a = 0$, then $S_\chi = \chi$.

Moreover the equations for effective stress derived in Parts I and II of the present paper may now be expressed in the general forms

$$\sigma' = \sigma - \left(1 - \frac{a . \tan \psi}{\tan \phi'}\right) S_\chi . u_w$$

$$\sigma' = \sigma - \left(1 - \frac{C_s}{C}\right) S_\chi . u_w$$

Experimental verification of Bishop's equation—In order to check the validity of the equation

$$\sigma' = \sigma - u_a + \chi(u_a - u_w)$$

a triaxial test was carried out recently at Imperial College under Dr Bishop's direction by Mr Ian Donald on a specimen of silt with a degree of saturation of about 45 per cent. In this test the cell pressure, the pore-water pressure and the pore-air pressures could be varied, and measured, independently but simultaneously. Throughout the early part of the test σ_3, u_a and u_w were held at steady values. When failure was being approached, however, and the slope of the stress-strain curve was therefore becoming comparatively small, all three pressures were varied; but in such a manner that $(\sigma_3 - u_a)$ and $(u_a - u_w)$ remained constant. If Bishop's equation is correct, then these variations should cause no change in the effective stress and, consequently, there should be no change in the stress-strain curve. The results are plotted in Fig. 12, from which it is seen that this prediction is verified within the limits of experimental accuracy.

Equally important, however, is the fact that the effective stress is not equal to $(\sigma - u_w)$; as can be proved from the following data. The sample for which the results in Fig. 12 were

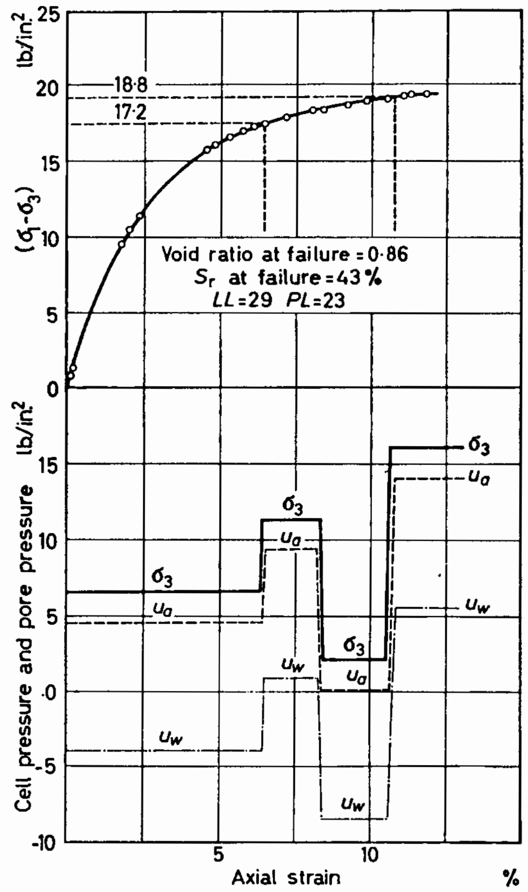

Fig. 12 Triaxial test on partially saturated silt (Bishop and Donald)
$\sigma_3 - u_a = 2{\cdot}0$ lb/in^2 throughout test
$u_a - u_w = 8{\cdot}5$ lb/in^2 throughout test

obtained was a normally consolidated silt, from Braehead ($LL = 29$, $PL = 23$, clay fraction = 6 per cent). At failure the deviatoric compression strength $(\sigma_1 - \sigma_3)_f = 19{\cdot}2$ lb/in^2, the void ratio = 0·86, and the degree of saturation $S_r = 43$ per cent. Now a series of four drained triaxial tests on the same material and with the same void ratio at failure, but in the fully saturated condition, gave the results $c' = 0$ and $\phi' = 33\frac{1}{2}°$. There is no reason why this value of ϕ' should not apply closely to the partially saturated test, since the void ratio was identical, and hence the effective minor principal stress σ_3' in the partially saturated test can be calculated from the equation

$$(\sigma_1 - \sigma_3)_f = \sigma'_3 \cdot \frac{2 \sin\phi'}{1 - \sin\phi'}$$

With $\phi' = 33\frac{1}{2}°$ and $(\sigma_1 - \sigma_3)_f = 19{\cdot}2$ lb/in^2 it is at once found that $\sigma'_3 = 7{\cdot}8$ lb/in^2.

But throughout the test on the partially saturated silt, $(\sigma_3 - u_w)$ was held constant at 10·5 lb/in^2. Thus, if Terzaghi's equation were used, the effective stress would be given as 10·5 lb/in^2 instead of the actual value of 7·8 lb/in.^{2}

Also, throughout the test, $(\sigma_3 - u_a) = 2{\cdot}0$ lb/in^2 and $(u_a - u_w) = 8{\cdot}5$ lb/in^2. Hence, since $\sigma'_3 = 7{\cdot}8$ lb/in^2, it follows from the equation

$$\sigma'_3 = \sigma_3 - u_a + \chi(u_a - u_w)$$

that $\chi = 0{\cdot}68$ at the particular degree of saturation of the sample.

Summary

It is usually assumed that the effective stress controlling changes in shear strength and volume, in saturated porous materials, is given by the equation

$$\sigma' = \sigma - (1 - a)u_w \qquad \dots (1)$$

where a is the area of contact between the particles, per unit gross area of the material. Experimental evidence is presented however, which shows that equation 1 is not valid; and theoretical reasoning leads to the conclusion that more correct expressions for effective stress in fully saturated materials are

(i) for shear strength

$$\sigma' = \sigma - \left(1 - \frac{a.\tan\psi}{\tan\phi'}\right)u_w \qquad \dots (2)$$

(ii) for volume change

$$\sigma' = \sigma - \left(1 - \frac{C_s}{C}\right)u_w \qquad \dots (3)$$

where ψ and C_s are the angle of intrinsic friction and the compressibility of the solid substance comprising the particles, and ϕ' and C are the angle of shearing resistance and the compressibility of the porous material.

For soils $\tan\psi/\tan\phi'$ may be about 0·15 to 0·3 but a is very small at pressures normally encountered in engineering and geological problems. Also, under these low pressures, C_s/C is extremely small. Thus, for fully saturated soils, equations 2 and 3 both degenerate into the form

$$\sigma' = \sigma - u_w \qquad \dots (4)$$

which is Terzaghi's equation for effective stress.

If we define as a 'perfect' solid a substance which is incompressible and purely cohesive, then $C_s = 0$ and $\psi = 0$. In that case Terzaghi's equation is rigorously true. And the comparison may be made with Boyle's Law for a 'perfect' gas; when equations 2 and 3 become analogous to the more complex expression derived by Van der Waals in which the attractions between the molecules, and the volume of the molecules, are taken into account.

But if Terzaghi's equation has the status of an excellent approximation for saturated soils, this cannot be said to be generally true for saturated rocks and concrete. For in these materials C_s/C is typically in the range 0·1 to 0·5, whilst $\tan\psi/\tan\phi'$ may be of the order 0·1 to 0·3 and a is not negligible*. Equation 1 is, however, also not correct and where this expression has been used to determine the area ratio from triaxial

* Nevertheless it must be emphasized that in most practical shear strength problems, where u rarely exceeds $\frac{1}{4}\sigma$, little error will also be involved in taking $\sigma' = \sigma - u$.

compression tests on concrete, the value of a may be about five times too small. Tests on two calcitic limestones show that equation 1 leads to a similar underestimation of the area ratio although equation 2 is in reasonable agreement with the experimental results. Moreover, equation 3 is substantiated by high-pressure consolidation tests on lead shot, but equation 1 is in serious error.

For partially saturated soils Bishop has suggested the following expression for effective stress

$$\sigma' = \sigma - [u_a - \chi(u_a - u_w)] \qquad \ldots. (5)$$

where χ is a coefficient to be determined experimentally and, at a given degree of saturation, has not necessarily the same values in relation to shear strength and volume change. Nevertheless χ is always equal to unity in saturated soils and is always equal to zero for dry soils. In both these limiting conditions equation 5 therefore becomes identical with equation 4, since this latter equation can apply to any single-phase pore fluid with a pressure u; when

$$\sigma' = \sigma - u$$

Bishop's equation, which has been confirmed experimentally, can be written in the form

$$\sigma' = \sigma - S_\chi . u_w \qquad \ldots. (6)$$

where

$$S_\chi = 1 + (1 - \chi)\frac{u_a - u_w}{u_w} \qquad \ldots. (7)$$

and in the important special condition of $u_a = 0$

$$\sigma' = \sigma - \chi . u_w$$

Since $S_\chi . u_w$ can be considered as an equivalent pore pressure, the effective stresses in partially saturated porous materials may be expressed by the general equations

$$\sigma' = \sigma - \left(1 - \frac{a . \tan\psi}{\tan\phi'}\right) S_\chi . u_w \qquad \ldots. (8)$$

$$\sigma' = \sigma - \left(1 - \frac{C_s}{C}\right) S_\chi . u_w \qquad \ldots. (9)$$

These equations define the influence of pore pressure on the stress controlling shear strength and volume change in porous materials.

References

AITCHISON, G. D. and DONALD, I. B. (1956). Effective Stresses in unsaturated soils. *Proc. 2nd Australia–New Zealand Conf. Soil Mech.* pp. 192–199

BISHOP, A. W. (1954). Letter in *Geotechnique*, **4**, 43–45

BISHOP, A. W. (1955). Lecture delivered in Oslo, entitled 'The principle of effective stress'. Printed in *Tek. Ukeblad*, No. 39 (1959).

BISHOP, A. W. and ELDIN, G. (1950). Undrained triaxial tests on saturated sands and their significance in the general theory of shear strength. *Geotechnique*, **2**, 13–32.

BOWDEN, F. P. and TABOR, D. (1942). The mechanism of metallic friction. *Nature*, **150**, 197–199.

BOWDEN, F. P. and TABOR, D. (1954). *Friction and Lubrication of Solids*. Oxford.

BRIDGMAN, P. W. (1925). The linear compressibility of fourteen natural crystals. *Amer. J. Sci.*, **10**, 483–498.

BRIDGMAN, P. W. (1928). The compressibility of thirteen natural crystals. *Amer. J. Sci.*, **15**, 287–296.

BRIDGMAN, P. W. (1935). Effects of high shearing stress combined with high hydrostatic pressure. *Phys. Rev.*, **48**, 825–847.

BRIDGMAN, P. W. (1936). Shearing phenomena at high pressures. *Proc. Amer. Acad. Arts Sci.*, **71**, 387–459.

BRIDGMAN, P. W. (1941). Explorations towards the limit of utilizable pressures. *J. Appl. Phys.*, **12**, 461–469.

CAQUOT, A. (1934). *Equilibre des Massifs à Frottement Interne*, Paris.

CHUGAEV, R. R. (1958). Influence of uplift pressure on deformation and stressed conditions of dams. *Proc. 6th Cong. Large Dams*, Paper 129.

EBERT, H. (1935). Versuche bis zu Drucken von 5000 kg/sq. cm. *Phys. Zeits.*, **36**, 385–392.

FILLUNGER, P. (1915). Versuche über die Zugfestigkeit bei allseitigem Wasserdruck. *Osterr. Wochenschr. Offentlich Baudienst. Wien.*, 443–448.

GRIGGS, D. T. (1936). Deformation of rocks under high confining pressure. *J. Geol.*, **44**, 541–577.

GRIGGS, D. T. (1942). Strength and plasticity. *Handbook of Physical Constants*. Geol. Soc. America. Special Paper, No. 36, 107–130.

GRIGGS, D. T. and BELL, J. F., (1938). Experiments bearing on the orientation of quartz in deformed rocks. *Bull. Geol. Soc. Amer.*, **49**, 1723–1746.

KARMAN, TH. VON (1911). Festigkeitsversuche unter allseitigem Druck. *Ziet. Vereines Deutsch. Ing.*, **55**, 1749–1757

KING, R. F. and TABOR, D. (1954). The strength properties and frictional behaviour of brittle solids. *Proc. Roy. Soc. A*, **223**, 225–238

LAUGHTON, A. S. (1955). *The Compaction of Ocean Sediments*. Ph.D. Thesis, University of Cambridge.

LELIAVASKY, S. (1945). Experiments on effective uplift area in gravity dams. *Proc. Amer. Soc. Civ. Engrs.*, **71**, 1474–1517

MCHENRY, D. (1948). The effect of uplift pressure on the shearing strength of concrete. *Proc. 6th Congr. Large Dams*, Paper 48.

RENDULIC, L. (1937). Ein Grundgesetz der Tonmechanik. *Bauing.*, **18**, 459–467

SEITZ, F. and READ, T. A. (1941). The theory of the plastic properties of solids. *J. Appl. Phys.*, **12**, 538–554.

SKEMPTON, A. W. (1953). Soil mechanics in relation to geology. *Proc. Yorkshire Geol. Soc.*, **29**, 33–62.

SKEMPTON, A. W. and BISHOP, A. W. (1954). Soils. *Building Materials: their Elasticity and Inelasticity*, ed. M. Reiner, Amsterdam (see p. 441)

TABOR, D. (1959). Junction growth in metallic friction. *Proc. Roy. Soc. A.*, **251**, 378–393.

TAYLOR, D. W. (1944). Tenth progress report on shear research. *Mass. Inst. Tech.*

TERZAGHI, K. (1923). Die Berechnung der Durchlässigkeitsziffer des Tones aus dem Verlauf der hydrodynamischen Spannungserscheinungen. *Sitz. Akad. Wissen. Wien Math-naturw Kl. Abt. IIa*, **132**, 105–124

TERZAGHI, K. (1925). *Erdbaumechanik auf bodenphysikalischer Grundlage*. Leipzig (see pp. 50–52).

TERZAGHI, K. and RENDULIC, L. (1934). Die wirksame Flächenporositat des Betons. *Zeit. österr. Ing. Arch. Ver.*, **86**, 1–9

ZISMAN, W. A. (1933). Compressibility and anisotropy of rocks at and near the earth's surface. *Proc. Nat. Acad. Sci.*, **19**, 666–679