2.1 $Pe \gg 1$

$\|u\|_2 \sim L^{1/2} Pe^{3/2}$ — $H^1$-norm

$a u_{,x} - K u_{,xx} = 0 \qquad f = 0 \qquad u(0) = 0$
$\qquad\qquad\qquad\qquad\qquad\qquad u(L) = 1$

$a u_{,x} = K u_{,xx}$

Set $v = u_{,x}$

$\frac{a}{K} = \frac{v_{,x}}{v} \Rightarrow \frac{a}{K} x + C_0 = \ln(v)$

$v = C_1 e^{a/K\, x} \Rightarrow u_{,x} = C_1 e^{a/K\, x}$

$u = C_1 \frac{K}{a} \exp\left(\frac{a}{K} x\right) + C_2 \Rightarrow u = C_3 \exp\left(\frac{a}{K} x\right) + C_2$

$u(0) = 0 \Rightarrow C_3 = -C_2$

$u(L) = 1 \Rightarrow 1 = -C_2 \exp\left(\frac{a}{K} L\right) + C_2$

$C_2 = \left(1 - \exp \frac{a}{K} L\right)^{-1}$

$u = \frac{-\exp \frac{a}{K} x + 1}{1 - \exp \frac{a}{K} L} \qquad$ Def $Pe = \frac{aL}{K}$

$$\Rightarrow \boxed{u = \frac{1 - \exp\left(Pe \frac{x}{L}\right)}{1 - \exp(Pe)}}$$

$$\|u\|_2 = (u,u)_2^{1/2}$$

$$(u,u)_2 = \int_0^L (u.u + L^2 u_{,x} u_{,x} + L^4 u_{,xx} u_{,xx})\, dx$$

for $Pe \gg 1 \Rightarrow u \approx \dfrac{\exp Pe \frac{x}{L}}{\exp Pe}$

let $C = \dfrac{1}{\exp Pe}$ $\quad D = \dfrac{Pe}{L}$ $\quad\Bigg\}\; u = C \exp(Dx)$

$u_{,x} = CD \exp Dx \qquad u_{,xx} = CD^2 \exp(Dx)$

$(u,u)_2 \Rightarrow \quad (u)^2 = C^2 \exp(2Dx)$

$$(u_{,x})^2 = C^2 D^2 \exp(2Dx)$$

$$(u_{,xx})^2 = C^2 D^4 \exp(2Dx)$$

as $Pe \gg 1$, $(u_{,xx})^2 \gg (u_{,x})^2 \gg (u)^2$

$$\int_0^L C^2 D^4 L^4 \exp(2Dx)\, dx = \int_0^L C^2 Pe^4 \exp(2Dx)\, dx$$

$$\Rightarrow \int_0^L \exp(2Dx)\, dx = \frac{1}{2D}\left[\exp(2Dx)\right]_0^L = \frac{1}{2D}\left[\exp 2DL - 1\right]$$

$$\exp\left(\frac{2Pe}{2}\right) \gg 1 \Rightarrow \exp 2DL - 1 \approx \exp(2DL)$$

$$(u,u)_2 = \frac{1}{\cancel{\exp(2Pe)}} Pe^4 \frac{L}{2Pe} \cancel{\exp(2Pe)}$$

$$(u,u)_2 = \frac{Pe^3 L}{2}$$

Hence:

$$\|u\| = \sqrt{(u,u)_2} \cong \sqrt{\frac{Pe^3 L}{2}} \approx Pe^{3/2} L^{1/2}.$$

when $Pe \gg 1$.

$$e = u - u^h$$

(22) $$\|e\|^2 = \int_0^L e^2\,dx$$

$$|e|_1^2 = \int_0^L e_{,x}^2\,dx$$

$$\|e\|_1^2 = \int_0^L \left(e^2 + L^2 e_{,x}^2\right)dx$$

$$\|e\| = \beta N_{el}^{-\gamma} \rightarrow \log\|e\| = \log\beta - \gamma \log N_{el}$$

— x —

H1 - seminorm

$$u = \frac{e^{ax}-1}{e^a-1} \longrightarrow u_{,x} = \frac{a}{e^a-1}\left(e^{ax}-1\right)$$

let $a = 500$. $\boxed{u_{,x} \cong a\,e^{ax-a}}$

$$u^h_{,x} = \frac{1}{h}\left(u_{A+1} - u_A\right) = C \qquad e_{,x} = u_{,x} - u^h_{,x}$$

$$e_{,x}^2 = C^2 + a^2 e^{2ax} e^{-2a} - 2aCe^{ax}e^{-a}$$

$$\int_{x_0}^{x_1} e_{,x}^2\,dx = \begin{cases} C^2(x_1 - x_0) \\ \frac{a}{2} e^{-2a}\left(e^{2ax_1} - e^{2ax_0}\right) \\ -2C e^{-a}\left(e^{ax_1} - e^{ax_0}\right) \end{cases}$$

//

2.2

As expected, L2-norm of the error convergence rate is $\gamma \approx 2$, and $H^1$ norms and seminorms convergence rate is $\gamma \approx 1$.

The theoretical convergence rate is observed for Galerkin and SUPG methods only when using a large number of elements (> 200).

Galerkin oscillates with a small number of elements. SUPG is nodally exact in all cases.

2.3 $e = \tilde{u} - u$

$$\tilde{u}^h = \frac{x_{A+1} - x}{h} u_A + \frac{x - x_A}{h} u_{A+1}$$

$$u_{A+1} = u + (x_{A+1} - x) u' + \frac{1}{2} (x_{A+1} - x)^2 u''$$

$$u_A = u + (x_A - x) u' + \frac{1}{2} (x_A - x)^2 u''$$

$$h(\tilde{u} - u) = (x_{A+1} - x)\left[\cancel{u} + \cancel{(x_A - x)u'} + \frac{1}{2}(x_A - x)^2 u''\right]$$
$$+ (x - x_A)\left[\cancel{u} + \cancel{(x_{A+1} - x)u'} + \frac{1}{2}(x_{A+1} - x)^2 u''\right] - \cancel{uh}$$

$$2h(e) = (x_{A+1} - x)(x_A - x)^2 u''$$
$$+ (x - x_A)(x_{A+1} - x)^2 u''$$

$$e = u''\left(\frac{1}{2h}\right)\left[(x_{A+1} - x)(x_A - x)^2 + (x - x_A)(x_{A+1} - x)^2\right]$$

$$\sup(e) \Rightarrow x = h/2 = \frac{x_{A+1} + x_A}{2}$$

$$\sup(e) = u''\left(\frac{1}{2h}\right)\left[\left(\frac{x_{A+1} - x_A}{2}\right)\left(\frac{x_A - x_{A+1}}{2}\right)^2 + \left(\frac{x_{A+1} - x_A}{2}\right)\left(\frac{x_{A+1} - x_A}{2}\right)\right.$$

$$= u''\left(\frac{1}{2h}\right)\left[\frac{h^3}{8} + \frac{h^3}{8}\right] = \frac{u'' h^2}{8} \quad //$$

For the derivative:

$$\tilde{u}^h_{,x} = \frac{u_{A+1} - u_A}{h}$$

$$e = \tilde{u}^h_{,x} - u_{,x} =$$

$$\frac{1}{h}\Big(\cancel{u} + \cancel{(x_{A+1} - x)u'} + (x_{A+1} - x)^2 u''$$

$$- \cancel{u} - \cancel{(x_A - x)u'} - (x_A - x)^2 u''\Big) - \cancel{u_{,x}}$$

$$e = \frac{u''}{h}\left[(x_{A+1} - x)^2 - (x_A - x)^2\right]$$

$$\sup(|e|) = u''\left(\frac{1}{h}\right)\left[h^2\right] = u'' h$$