PGE383L - CP5 Renato B-li
rep2656.

(2)

$$\frac{\partial u}{\partial t} = \alpha\left(u_{,xx} + u_{,yy}\right) - e^{t}\left(4\alpha + x^2 + y^2\right)$$

$\Delta x = \Delta y = 1/20 \qquad \alpha = 1 \qquad t_f = 0.2$

$\boxed{\frac{\partial u}{\partial t}} \quad \frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} \Rightarrow \begin{cases} LHS: u_{ij}^{n+1}\left(1/\Delta t\right) & \checkmark \\ RHS: u_{ij}^{n}\left(1/\Delta t\right) & \checkmark \end{cases}$

$\boxed{-\alpha u_{,xx}} \quad -\frac{\alpha}{2}(\circ)^{n+1} - \frac{\alpha}{2}(\circ)^{n} \quad \begin{cases} LHS: -\alpha/2\,(\circ) & \checkmark \\ RHS: +\alpha/2\,(\circ) \end{cases}$

$$(\circ) = \frac{1}{12\Delta x^2}\left(-u_{i-2,j} + 16u_{i-1,j} - 30u_{ij} + 16u_{i+1,j} - u_{i+2,j}\right)$$

$\boxed{-\alpha u_{,yy}} \quad -\frac{\alpha}{2}(\circ)^{n+1} - \frac{\alpha}{2}(\circ) \quad \begin{cases} LHS: -\alpha/2\,(\circ)^{n+1} \\ RHS: +\alpha/2\,(\circ)^{n} \end{cases}$

$$(\circ) = \frac{1}{12\Delta y^2}(---)$$

$\boxed{-e^{t}\left(4\alpha + x^2 + y^2\right)} \quad \text{———} \quad \begin{cases} LHS: 0 \\ RHS: -e^{t}\left(4\alpha + x^2 + y^2\right) \end{cases}$

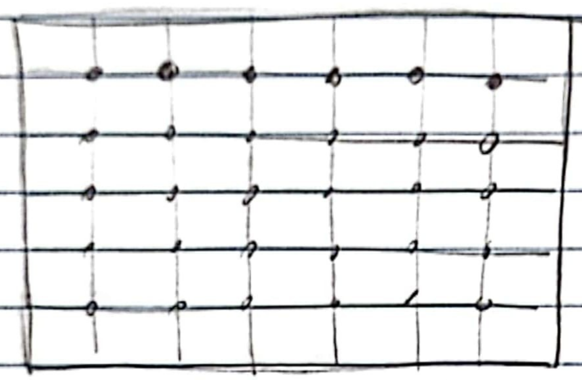
$0 < x < 1$
$0 < y < 1$

Initial :

$u(x, y, 0) = \alpha (x^2 + y^2) + 1$

BC : ...

Domain



$A u = b$

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where A_{11} are the internal nodes. u_2 is known by the BCs.

Solve :

$$\underline{A_{11}}\, u_1 = b_1 - \underline{A_{12}}\, u_2$$

Algorithm :

1. Build $\underline{A}$, b
2. Extract $\underline{A_{11}}$, $\underline{A_{12}}$
3. Solve for u_1
4. Build u.

CP5

March 8, 2024

PGE 382 - Numerical Methods in Petroleum and Geosystems Engineering

Renato Poli - rep2656

CP5 - Mar, 7th

a) Case 1

[1]:
```
from math import factorial, pi, sin, ceil
import numpy as np
np.set_printoptions(threshold=80, linewidth=80)

from numpy import exp, linspace, vectorize
import matplotlib.pyplot as plt

plt.style.use('paper.mplstyle')

B = np.diag(5*[-8]) + np.diag(4*[1],-1) + np.diag(4*[1],1)
B[0,1] = 2
I = np.eye( 5 )
Z = np.zeros( [5,5])

A = np.block( [ [ B-3/2*I, 6*I, Z, Z, Z ],
                [ 3*I , B, 3*I, Z, Z ],
                [ Z, 3*I , B, 3*I, Z ],
                [ Z, Z, 3*I , B, 3*I ],
                [ Z, Z, Z, 6*I , B ] ] )

b = -np.ones( 25 )

L = np.eye( 25 )
U = np.zeros( [ 25, 25 ] )
for i in range(25):
    for j in range(i,25):
        U[i,j] = A[i,j]
        for k in range( i ) :
            U[i,j] -= L[i,k] * U[k,j]

    for j in range(i+1,25):
        #print(f"i,j:{i},{j}")
        acc = 0
        for k in range( i ) :
            acc +=  L[j,k] * U[k,i]
        L[j,i] = ( A[j,i] - acc ) / U[i,i]

# Ly=b
y = np.zeros(25)
for i in range(25):
    y[i] = b[i]
    for k in range( i ) :
        y[i] -= L[i,k] * y[k]

# Ux=y
x = np.zeros(25)
for i in reversed(range(25)):
    acc = 0
    for k in range( i+1, 25 ) :
        acc += U[i,k] * x[k]
    x[i] = ( y[i] - acc ) / U[i,i]

print(f"L={L}")
print(f"\nU={U}")
print(f"\ny={y}")
print(f"\nx={x}")
```

```python
n = np.linalg.norm( A@x-b )
print(f"\n\nx holds the solution for the Ax=b problem. \nHence, the norm of Ax-b is near zero: ({n:.5e})")
```

```
L=[[ 1.          0.          0.         ...  0.          0.          0.        ]
 [-0.10526316  1.          0.         ...  0.          0.          0.        ]
 [-0.         -0.10764873  1.         ...  0.          0.          0.        ]
 ...
 [-0.         -0.         -0.         ...  1.          0.          0.        ]
 [-0.         -0.         -0.         ... -0.50222105  1.          0.        ]
 [-0.         -0.         -0.         ... -0.06644296 -0.44809461  1.        ]]

U=[[-9.5         2.          0.         ...  0.          0.          0.        ]
 [ 0.         -9.28947368  1.         ...  0.          0.          0.        ]
 [ 0.          0.         -9.39235127 ...  0.          0.          0.        ]
 ...
 [ 0.          0.          0.         ... -3.76126527  1.88898661  0.24990961]
 [ 0.          0.          0.         ...  0.         -3.94619127  1.76826703]
 [ 0.          0.          0.         ...  0.          0.         -4.30409793]]

y=[ -1.          -1.10526316  -1.11898017  -1.11913739  -1.11913917  -1.39998906
  -1.67855218  -1.77214186  -1.77346066  -1.73402964  -2.16491581  -2.61719031
  -2.8107788   -2.78167991  -2.55163875  -2.83993432  -3.42659194  -3.67751637
  -3.57488645  -3.11438658  -5.87069036  -7.87446238 -10.3008394 -11.11048807
  -9.9122708 ]

x=[3.72851307 3.61678469 3.26570222 2.62454228 1.59608112 4.53121747 4.39420655
 3.96380736 3.17856138 1.92303806 5.09192914 4.93609116 4.44686142 3.55600627
 2.13916659 5.42319947 5.25577302 4.73045728 3.77544601 2.26273741 5.5327541
 5.36141798 4.82395165 3.84745153 2.3029845 ]


x holds the solution for the Ax=b problem.
Hence, the norm of Ax-b is near zero: (2.57112e-14)
```

b) Case 2

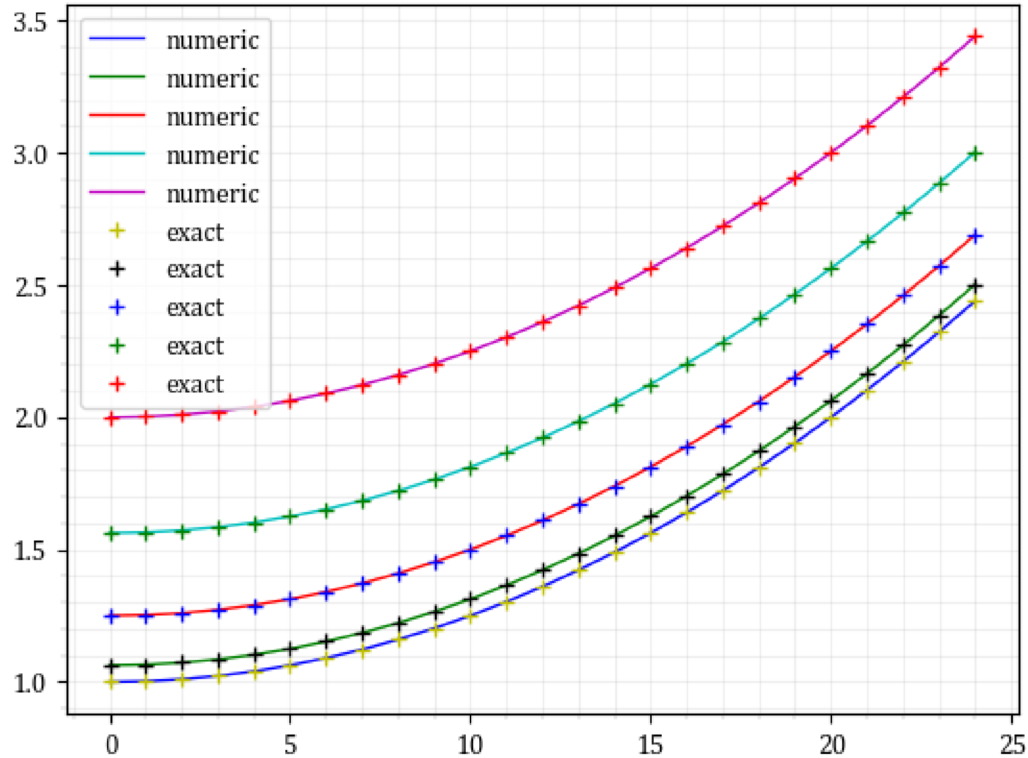
[2]:
```
from math import factorial, pi, sin, ceil
import numpy as np
np.set_printoptions(threshold=100000, linewidth=100000)

from numpy import exp, linspace, vectorize
import matplotlib.pyplot as plt

plt.style.use('paper.mplstyle')
# Index
def _( i, j ) :
      global nx
      return j * nx + i
# EXACT SOLUTION
def exact( x, y, t ) :
      global alpha
      return alpha * np.exp(-t)*(x**2+y**2)+1

tf = 0.2
dx = 1/20
alpha = 1
dt = 1/100

T = np.arange(0, tf + dt, dt)

# Indexing from -2 to N+2
nx = int(1/dx+5)
nt = len(T)
# Dimension of the full vectors and matrices
N = nx**2

# Assuming X=Y
X=np.zeros( nx )
for i in range(0,nx) : X[i] = dx * i

# MAPS OF UNKNOWNS - remove 3 unkwons from each side
UKN1 = np.zeros( N )
for i in range(2,nx-2) :
      for j in range(2,nx-2) :
            UKN1[_(i,j)] = 1
KN1 = ( UKN1 == 0 )
UKN1 = ( UKN1 == 1 )



# Feed exact solution
EXACTnk = np.zeros( [ nt, N ] )
for n in range(0,nt) :
      t = dt * n
      EXACTnk[n,:] = np.zeros( N )
      for i in range(0,nx) :
            for j in range(0,nx) :
                  k = _(i,j)
                  EXACTnk[n,k] = exact(X[i],X[j],t)

Unk = np.zeros( [ nt, N ] )
Unk[ 0, : ] = EXACTnk[0,:]
Unk[ :, KN1 ] = EXACTnk[ :, KN1 ]
for n in range(1,nt) :
      t = dt * n
      U = Unk[n-1,:]

      K = np.zeros([N,N])
      B = np.zeros( N )
      for i in range(2,nx-2) :
            for j in range(2,nx-2) :
                  k     = _(i,j)

                  # Diag
                  K[k,k] += 1/dt
                  B[k] += 1/dt * U[k]

                  # X
                  k1p = _(i+1,j)
```

```
            k2p = _(i+2,j)
            k1n = _(i-1,j)
            k2n = _(i-2,j)

            K[k,k2n] += (-alpha/2) * 1/12/dx/dx * ( -1 )
            K[k,k1n] += (-alpha/2) * 1/12/dx/dx * ( 16 )
            K[k,k]      += (-alpha/2) * 1/12/dx/dx * ( -30 )
            K[k,k1p] += (-alpha/2) * 1/12/dx/dx * ( 16 )
            K[k,k2p] += (-alpha/2) * 1/12/dx/dx * ( -1 )

            B[k] += (alpha/2) * 1/12/dx/dx * ( -1 )         * U[k2n]
            B[k] += (alpha/2) * 1/12/dx/dx * ( 16 )         * U[k1n]
            B[k] += (alpha/2) * 1/12/dx/dx * ( -30 ) * U[k]
            B[k] += (alpha/2) * 1/12/dx/dx * ( 16 )         * U[k1p]
            B[k] += (alpha/2) * 1/12/dx/dx * ( -1 )         * U[k2p]

            # Y
            k1p = _(i,j+1)
            k2p = _(i,j+2)
            k1n = _(i,j-1)
            k2n = _(i,j-2)

            K[k,k2n] += (-alpha/2) * 1/12/dx/dx * ( -1 )
            K[k,k1n] += (-alpha/2) * 1/12/dx/dx * ( 16 )
            K[k,k]      += (-alpha/2) * 1/12/dx/dx * ( -30 )
            K[k,k1p] += (-alpha/2) * 1/12/dx/dx * ( 16 )
            K[k,k2p] += (-alpha/2) * 1/12/dx/dx * ( -1 )

            B[k] += (alpha/2) * 1/12/dx/dx * ( -1 )         * U[k2n]
            B[k] += (alpha/2) * 1/12/dx/dx * ( 16 )         * U[k1n]
            B[k] += (alpha/2) * 1/12/dx/dx * ( -30 ) * U[k]
            B[k] += (alpha/2) * 1/12/dx/dx * ( 16 )         * U[k1p]
            B[k] += (alpha/2) * 1/12/dx/dx * ( -1 )         * U[k2p]

            # CONSTANTS
            x = X[i]
            y = X[j]

            B[k] += - np.exp(-t) * ( 4 * alpha + x**2 + y**2 )

    Kk = K[np.ix_(UKN1,KN1)]
    Ku = K[np.ix_(UKN1,UKN1)]
    Bu = B[UKN1] - Kk @ Unk[ n, KN1 ]
    Uu = np.linalg.solve( Ku, Bu )
    Unk[n,UKN1] = Uu
```

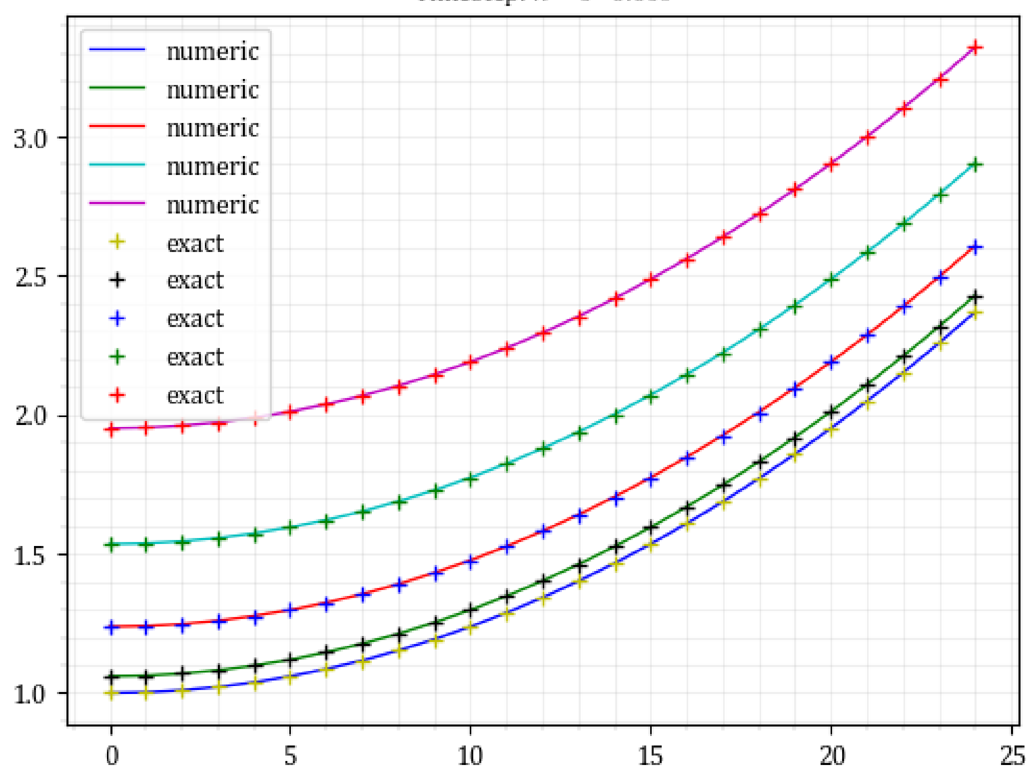
[3]:

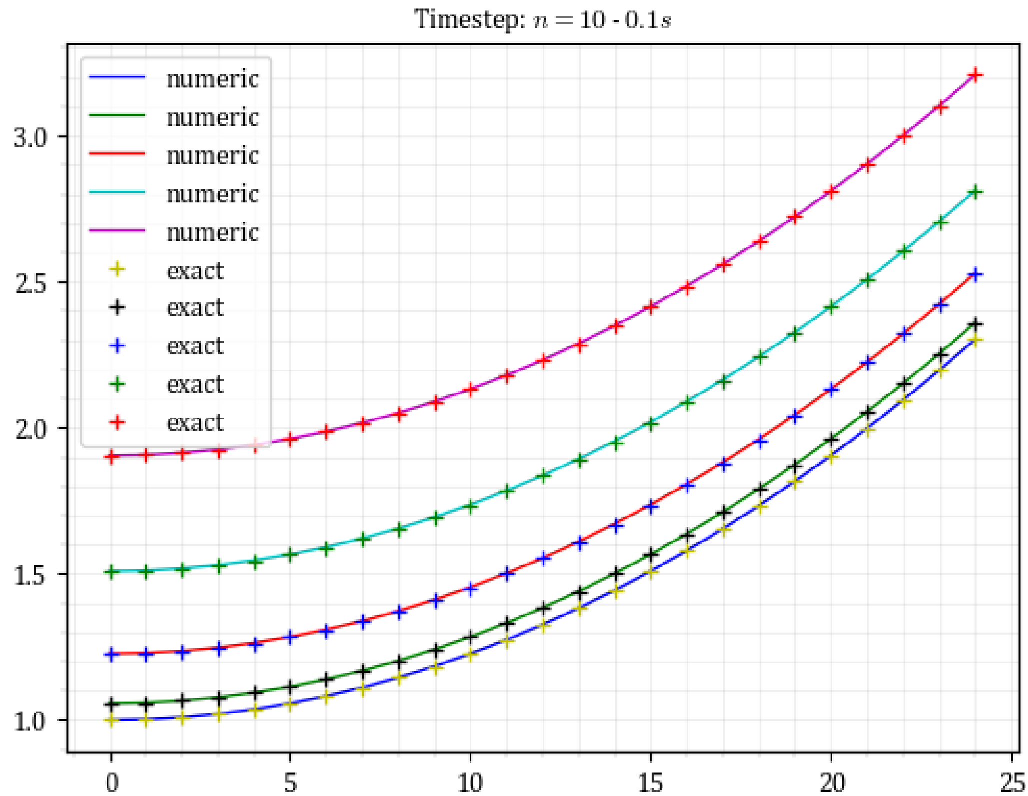
```
err = np.zeros( nt )
nxu = nx

for n in range( 0, nt, 5 ) :
    fig, ax1 = plt.subplots( 1, 1 );
    Uij = np.zeros( [ nxu, nxu] )
    Eij = np.zeros( [ nxu, nxu] )
    for i in range(0,nxu) :
        for j in range(0,nxu) :
            Uij[i,j] = Unk[n,_(i,j)]
            Eij[i,j] = EXACTnk[n,_(i,j)]

    ax1.plot( Uij[::5,:].transpose(), label='numeric' )
    ax1.plot( Eij[::5,:].transpose(), marker='+', lw=0, label="exact" )
    ax1.set_title(f"Timestep: $n={n}$ - ${T[n]} s$")
    ax1.legend()
```

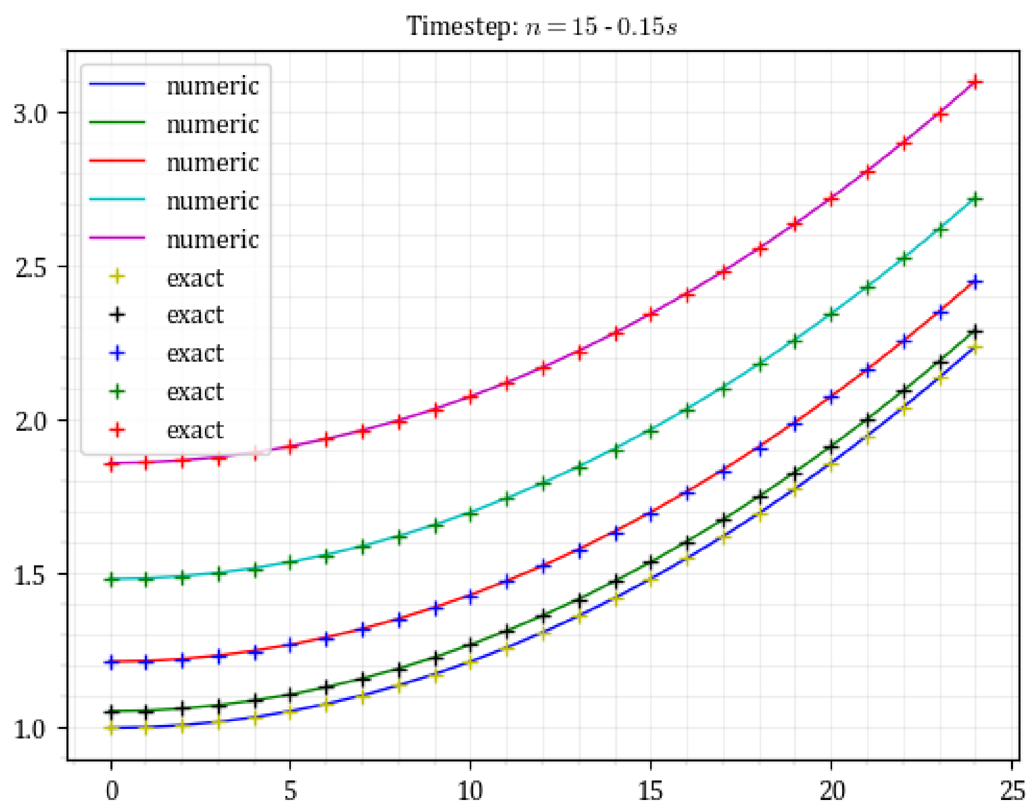
Timestep: $n = 0$ - $0.0s$

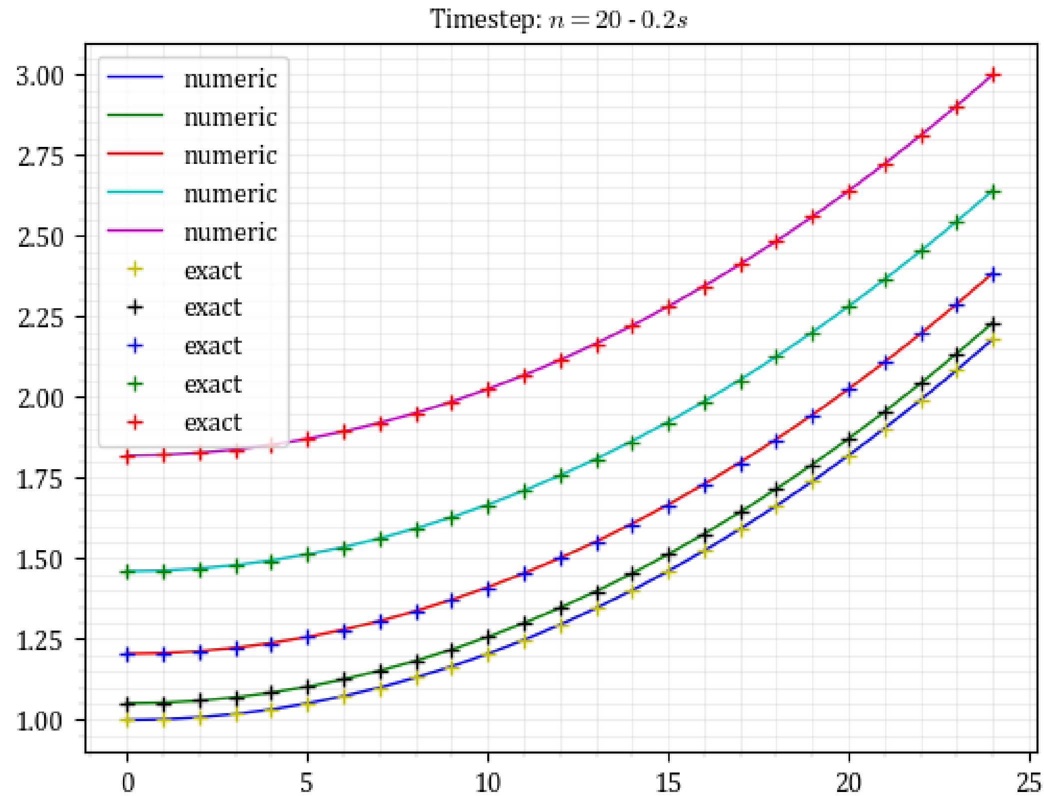


Timestep: $n = 5$ - $0.05s$

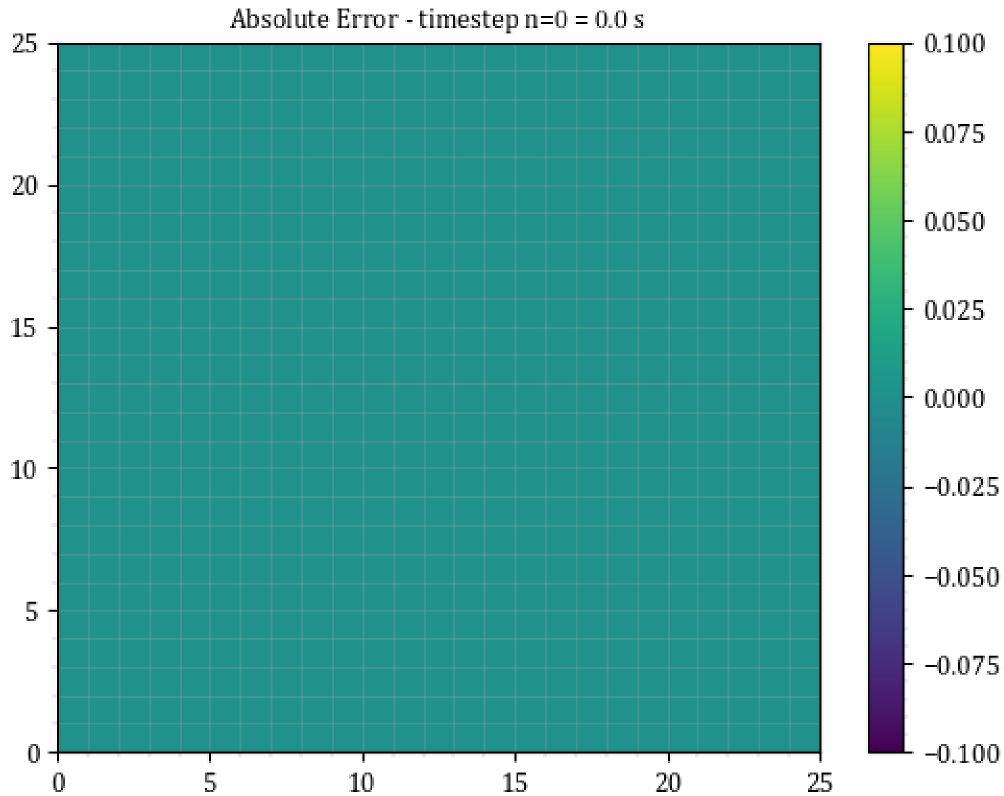


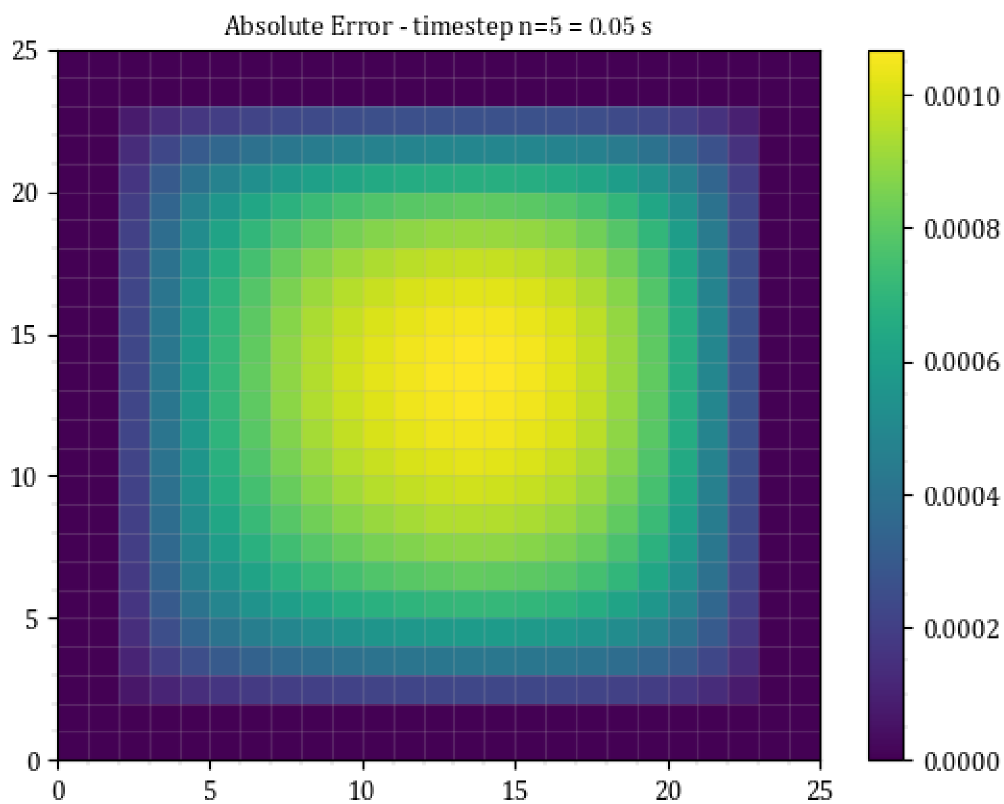

Timestep: $n = 10$ - $0.1s$
numeric
numeric
numeric
numeric
numeric
exact
exact
exact
exact
exact
3.0
2.5
2.0
1.5
1.0
0
5
10
15
20
25

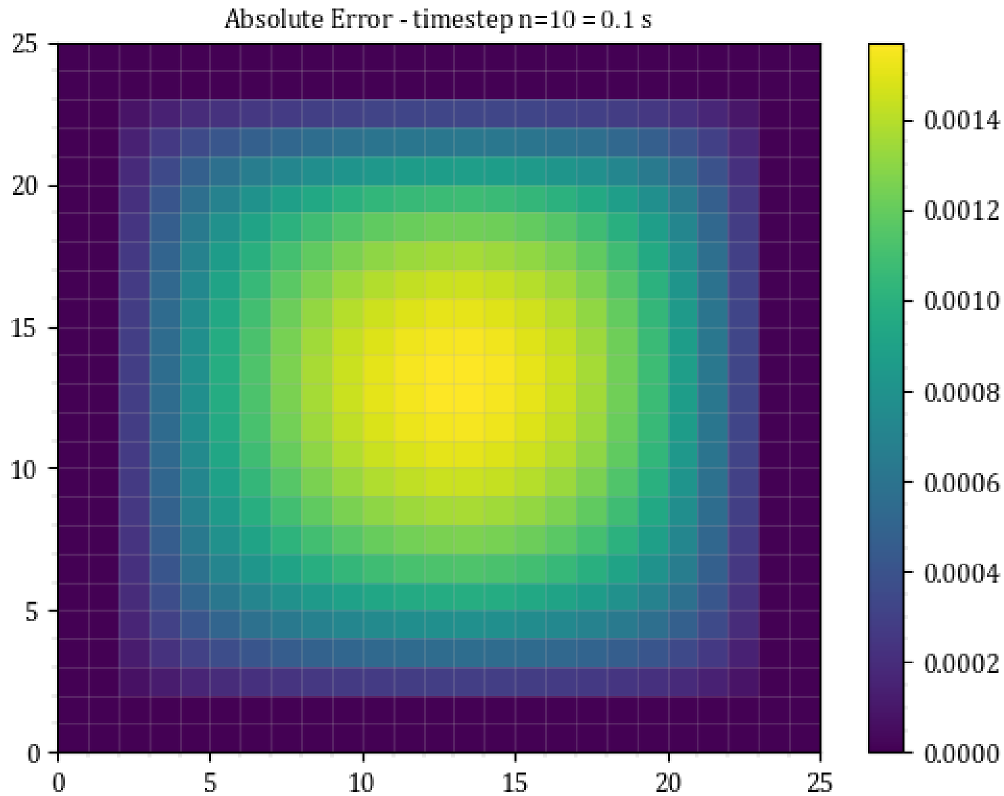

Timestep: $n = 15$ - $0.15s$
numeric
numeric
numeric
numeric
numeric
exact
exact
exact
exact
exact
3.0
2.5
2.0
1.5
1.0
0
5
10
15
20
25



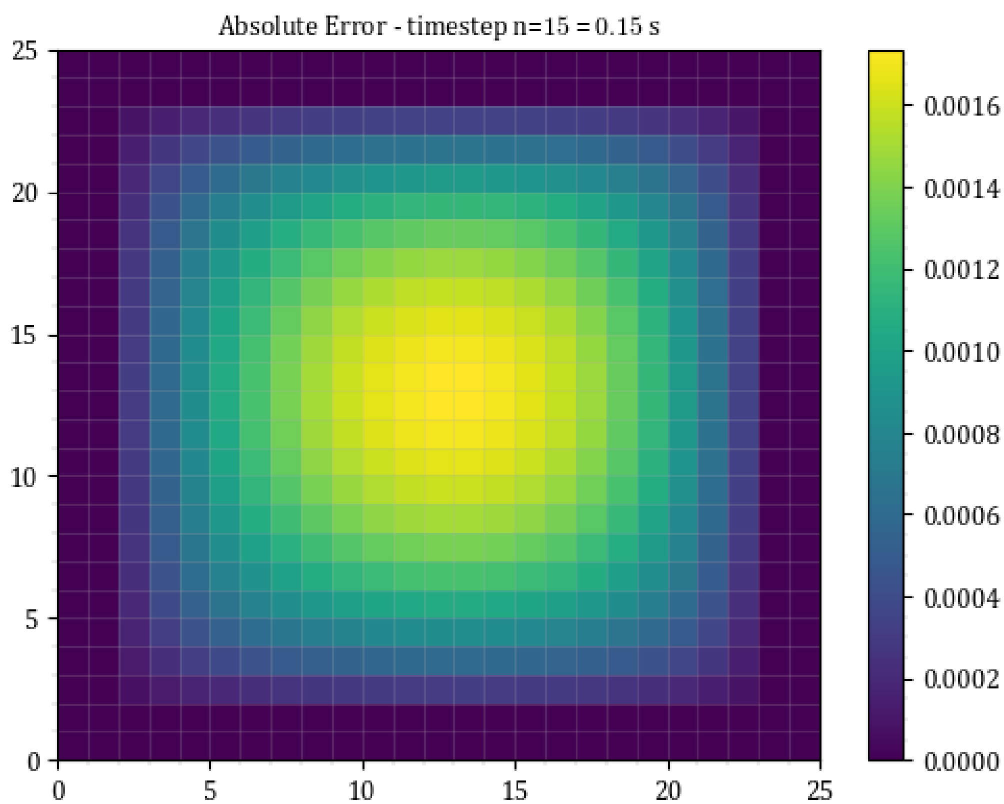
[4]:
```
nxu = nx
for n in range( 0, nt, 5 ) :
    fig, ax1 = plt.subplots( 1, 1);
    Uij = np.zeros( [ nxu, nxu] )
    Eij = np.zeros( [ nxu, nxu] )
    for i in range(0,nxu) :
        for j in range(0,nxu) :
            Uij[i,j] = Unk[n,_(i,j)]
            Eij[i,j] = EXACTnk[n,_(i,j)]
    c = ax1.pcolormesh( Uij-Eij,cmap='viridis' )
    ax1.set_title(f"Absolute Error - timestep n=${n}$ = ${T[n]}$ s")
    fig.colorbar(c)
```


Absolute Error - timestep n=0 = 0.0 s
0
5
10
15
20
25
0
5
10
15
20
25
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100

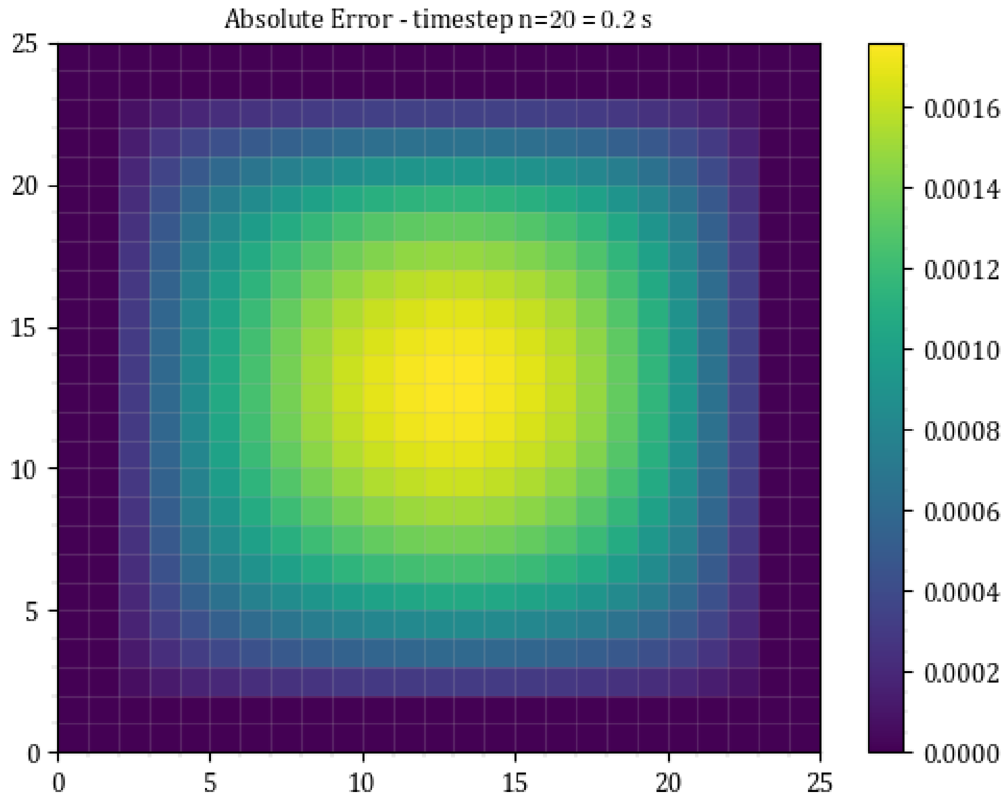

Absolute Error - timestep n=5 = 0.05 s
0
5
10
15
20
25
0
5
10
15
20
25
0.0010
0.0008
0.0006
0.0004
0.0002
0.0000



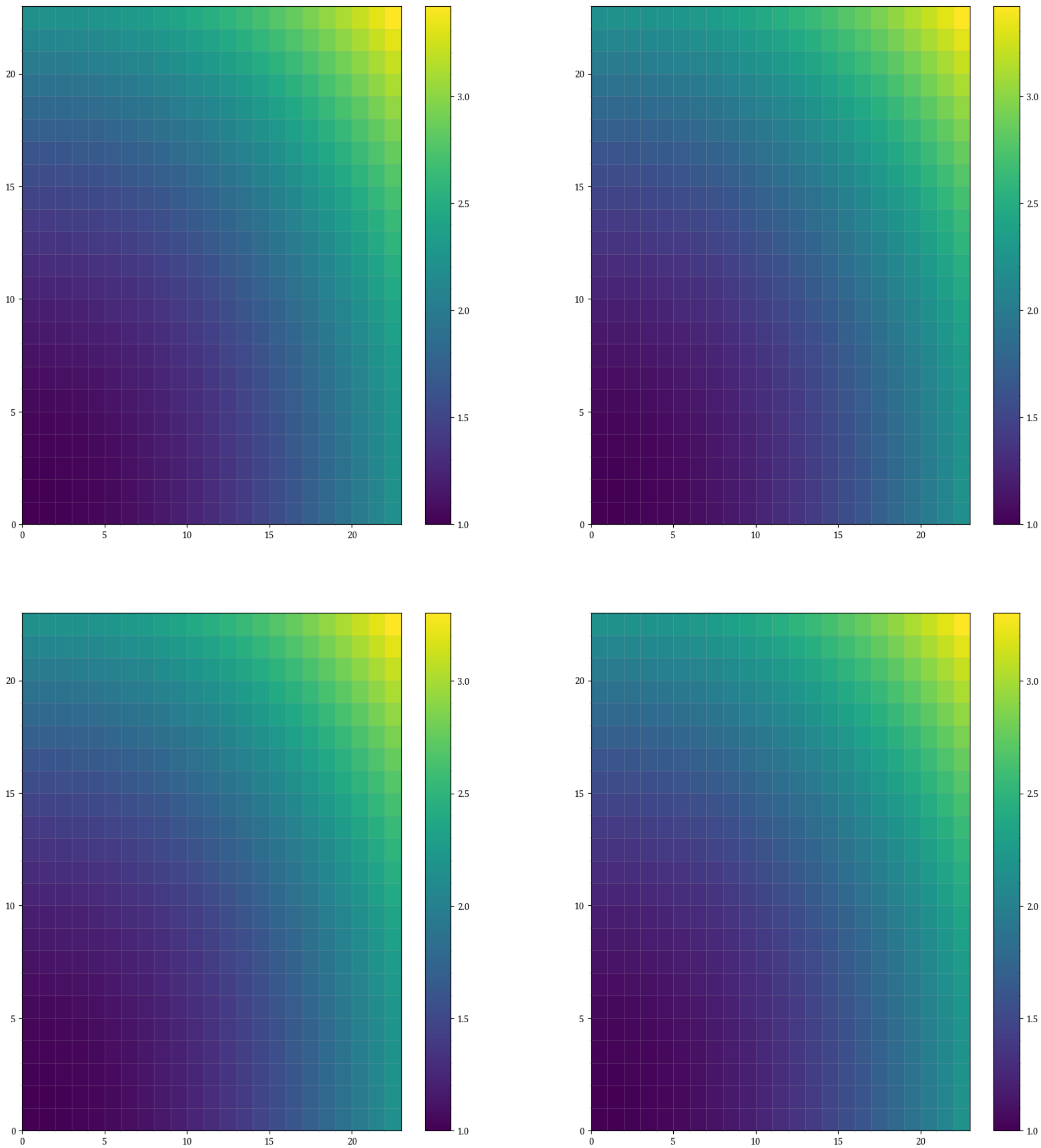
Absolute Error - timestep n=10 = 0.1 s
0
5
10
15
20
25
0.0014
0.0012
0.0010
0.0008
0.0006
0.0004
0.0002
0.0000

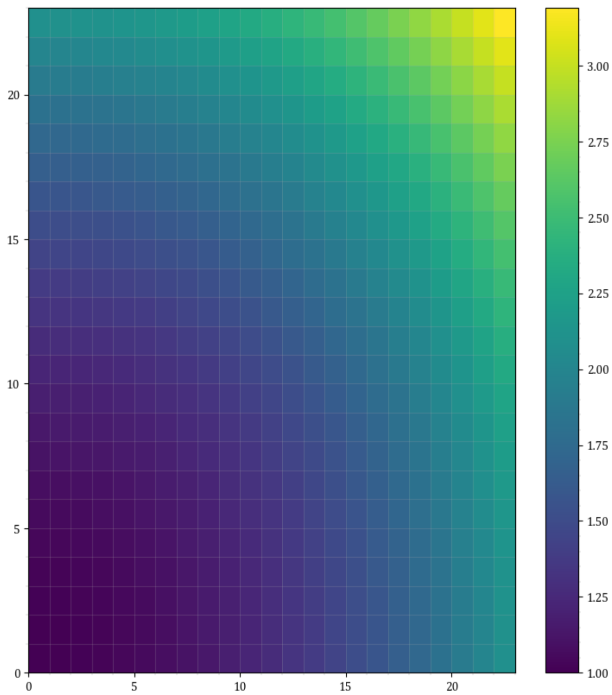
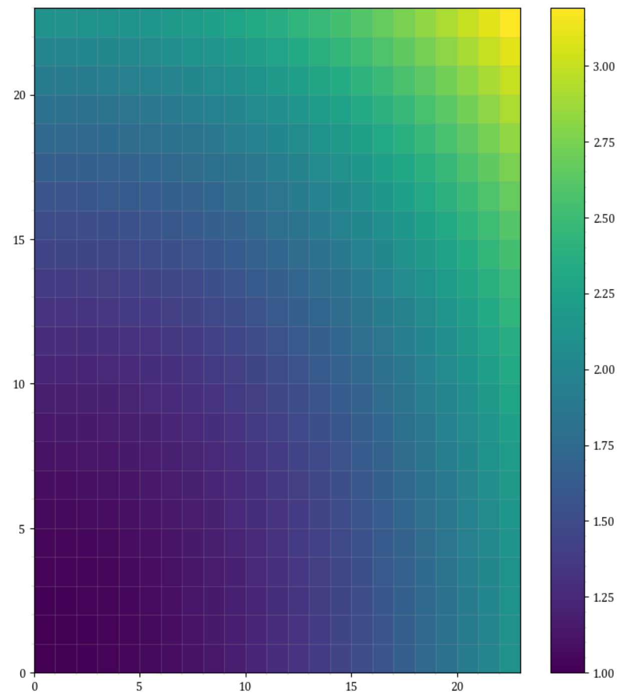
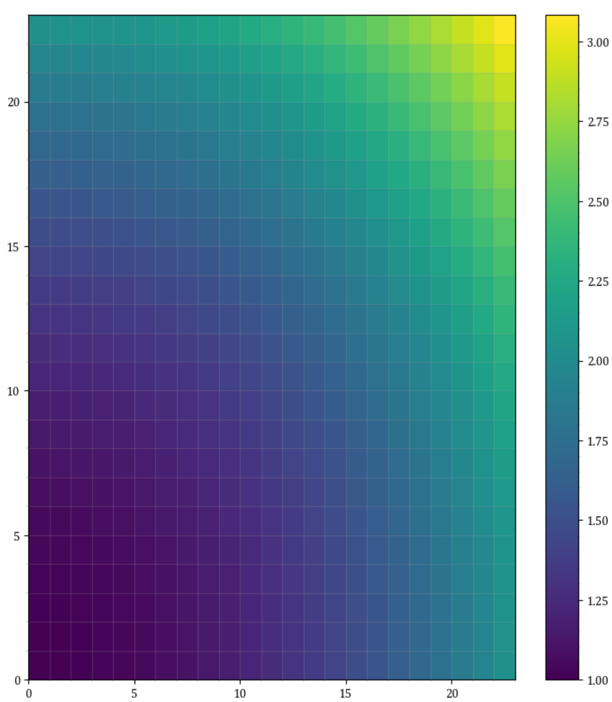
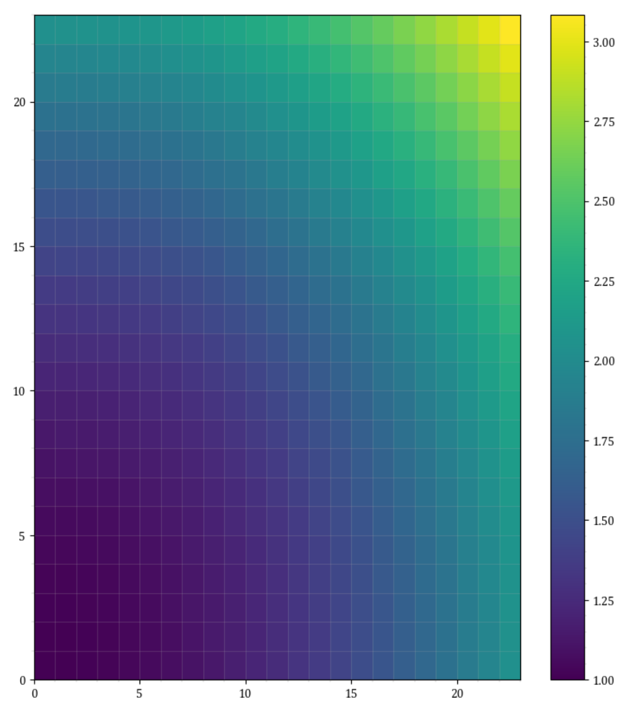


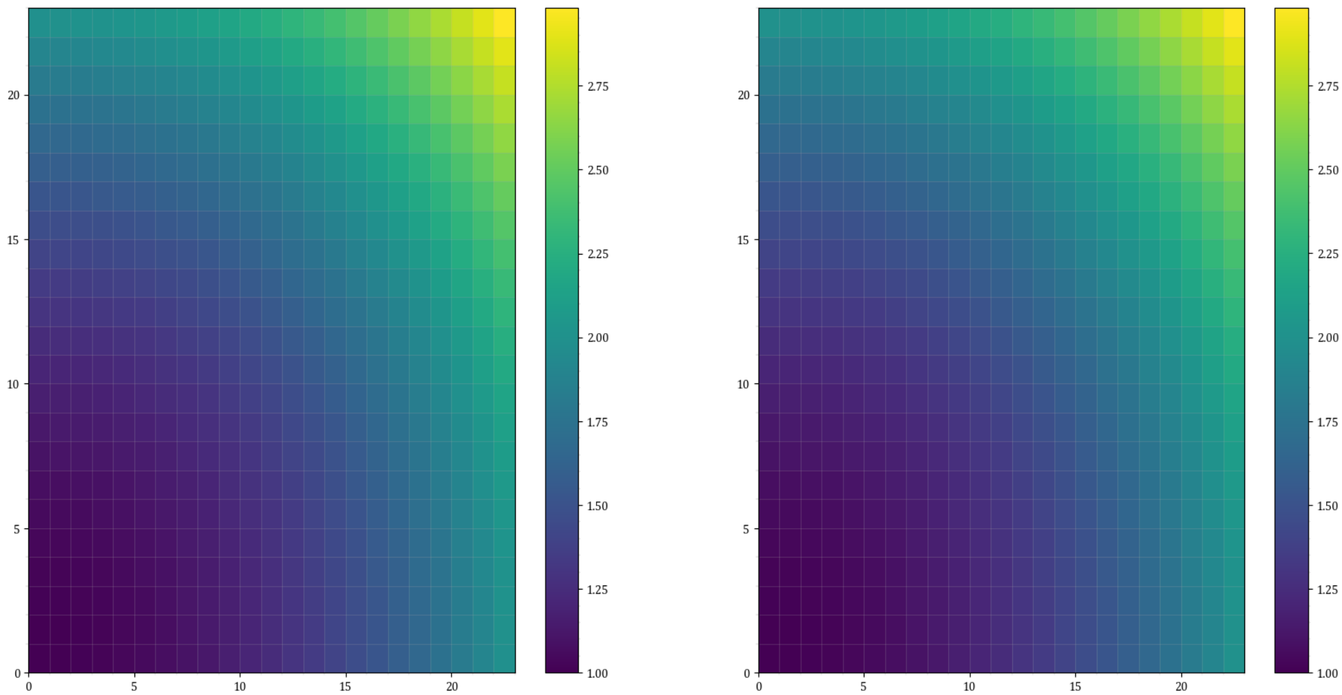
Absolute Error - timestep n=15 = 0.15 s
0
5
10
15
20
25
0.0016
0.0014
0.0012
0.0010
0.0008
0.0006
0.0004
0.0002
0.0000



[5]:
```
nxu = nx - 2
for n in range( 0, nt, 5 ) :
    fig, [ax1, ax2] = plt.subplots( 1, 2, figsize=[20,10] );
    Uij = np.zeros( [ nxu, nxu] )
    Eij = np.zeros( [ nxu, nxu] )
    for i in range(0,nxu) :
        for j in range(0,nxu) :
            Uij[i,j] = Unk[n,_(i,j)]
            Eij[i,j] = EXACTnk[n,_(i,j)]
    c = ax1.pcolormesh( Uij,cmap='viridis' )
    fig.colorbar(c)
    c = ax2.pcolormesh( Eij, cmap='viridis' )
    fig.colorbar(c)
```


0
5
10
15
20
1.0
1.5
2.0
2.5
3.0


0 5 10 15 20
0 5 10 15 20
1.00 1.25 1.50 1.75 2.00 2.25 2.50 2.75 3.00

0 5 10 15 20
0 5 10 15 20
1.00 1.25 1.50 1.75 2.00 2.25 2.50 2.75 3.00

0 5 10 15 20
0 5 10 15 20
1.00 1.25 1.50 1.75 2.00 2.25 2.50 2.75 3.00

0 5 10 15 20
0 5 10 15 20
1.00 1.25 1.50 1.75 2.00 2.25 2.50 2.75 3.00


2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00