4.1

$$U = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_4 \end{bmatrix} \Rightarrow \begin{array}{l} U_0 = \rho \\ U_1 = \rho u_1 \quad U_2 = \rho u_2 \quad U_3 = \rho u_3 \\ U_4 = \rho e \end{array}$$

$$s = \ln\left(\frac{p}{p_0} \left(\frac{\rho}{\rho_0}\right)^{-\gamma} \right) \qquad p = (\gamma - 1)\rho \iota$$

$$\iota = e - \frac{1}{2}\left(u_1^2 + u_2^2 + u_3^2\right) = \frac{U_4}{U_0} - \frac{1}{2U_0^2}\left(U_1^2 + U_2^2 + U_3^2\right)$$

$$\Rightarrow H = -sU_0 =$$

$$= -U_0 \ln\left[\frac{(\gamma-1)}{p_0} \underbrace{\left(U_4 - \frac{U_1^2 + U_2^2 + U_3^2}{2U_0} \right)}_{U_0 \iota} \frac{U_0^{-\gamma}}{\rho_0^{-\gamma}} \right]$$

$$\frac{dH}{dU_0} = -s - \frac{d \ln(\circ)}{dU_0}$$

$$\frac{d\ln(\cdot)}{dU_0} =$$

$$\frac{1}{\underbrace{\frac{(\gamma-1)}{\rho_0}\left[U_4 - \frac{U_1^2+U_2^2+U_3^2}{2U_0}\right]}_{U_0 \nu}\frac{U_0^{-\gamma}}{\rho_0^{-\gamma}}} \quad \frac{d(\cdot)}{dU_0} =$$

$$= \frac{1}{\cancel{\frac{(\gamma-1)}{\rho_0}} U_0 \nu \cancel{\frac{U_0^{-\gamma}}{\rho_0^{-\gamma}}}}\left[\cancel{\frac{(\gamma-1)}{\rho_0 \rho_0^{-\gamma}}}\left[\cancel{U_0 \nu}(-\gamma)\cancel{U_0^{-\gamma-1}} + \frac{U_1^2+U_2^2+U_3^2}{2U_0^2} U_0^{-\gamma}\right]\right]$$

$$= -\gamma + \frac{u_1^2+u_2^2+u_3^2}{2} = -\gamma + \frac{|u_+|}{2\nu}$$

$$\Rightarrow \frac{dH}{dU_0} = -s + \gamma - \frac{|u|}{2\nu} \quad \blacksquare$$

Note:

$$\frac{\nu - \frac{1}{2}|u|}{\nu} = \frac{\cancel{\nu} + \cancel{U_0}\overset{\cancel{(\gamma-1)U_0}}{} - \cancel{\nu}s - \frac{1}{2}|u|/\nu}{\nu}$$

$$= \gamma - s - \frac{1}{2}|u|/\nu$$

$$\Rightarrow \frac{dH}{dU_4} = -\frac{\cancel{\frac{(\gamma-1)}{p_0}}\cancel{\frac{U_0^{-\gamma}}{p_0^{-\gamma}}}\; U_0}{\cancel{\frac{(\gamma-1)}{p_0}}\Big(\underbrace{U_4 - \frac{U_1^2+U_2^2+U_3^2}{2U_0}}_{2U_0}\Big)\cancel{\frac{U_0^{-\gamma}}{p_0^{-\gamma}}}}$$

$$= -\frac{1}{2} \quad \square$$

$$\Rightarrow \frac{dH}{dU_1} = -U_0\,\frac{\cancel{\frac{(\gamma-1)}{p_0}}\left(\frac{-\cancel{2}U_1}{\cancel{2}U_0}\right)\cancel{\left(\frac{U_0^{-\gamma}}{p_0^{-\gamma}}\right)}}{\cancel{\frac{(\gamma-1)}{p_0}}\left(U_4 - \frac{U_1^2+U_2^2+U_3^2}{2U_0}\right)\cancel{\frac{U_0^{-\gamma}}{p_0^{-\gamma}}}}$$

$$= \frac{U_1}{U_0\,2} = \frac{u_1}{2} \quad \square$$

Similar to dH/dU_2 and dH/dU_3. $\square$

Hence, we derived

$$\frac{dH}{dU} = \begin{bmatrix} dH/dU_0 \\ dH/dU_1 \\ dH/dU_2 \\ dH/dU_3 \\ dH/dU_4 \end{bmatrix} = \underline{\underline{V}} \quad \square$$

$$F_i = u_i U + p G_i \qquad V = \frac{dH}{dU}$$

4.2/

$$V \cdot F_{i,i} = H_{,i} u_i + H u_{i,i}$$

$$V \cdot F_{i,i} = \left(u_{i,i} U + u_i U_{,i} + p_{,i} G_i + p\, G_{i,i} \right) \cdot V$$

$$\quad ① \qquad\qquad ② \qquad\qquad ③ \qquad\qquad ④$$

② $$u_i \frac{\cancel{dU}}{dx_i} \frac{dH}{\cancel{dU}} = u_i \quad H_{,i} \quad V$$

③ $$V \cdot (p_{,i} G_i) = V \cdot (p_{,1} G_1 + p_{,2} G_2 + p_{,3} G_3)$$

$$\frac{1}{T} \begin{bmatrix} Y - \frac{1}{2}|u|^2 \\ u_1 \\ u_2 \\ u_3 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ p_{,1} \\ p_{,2} \\ p_{,3} \\ p_{,i} u_i \end{bmatrix} = 0$$

$$G_1 = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ u_1 \end{Bmatrix} \qquad G_2 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ u_2 \end{Bmatrix} \qquad G_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ u_3 \end{Bmatrix}$$

④ $\nabla \cdot (p\, G_{i,i}) = \frac{1}{2} \begin{bmatrix} - \\ u_1 \\ u_2 \\ u_3 \\ -1 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ u_{i,i} \end{bmatrix} p$

$G_{i,i} = G_{1,1} + G_{2,2} + G_{3,3}$

$= \frac{1}{2} (-u_{i,i})\, p$

① $\nabla \cdot (u_{i,i}\, U) = u_{i,i}\; \nabla \cdot U = u_{i,i} \left[\frac{p}{2} + H \right]$

$\frac{p}{2} \begin{bmatrix} r - \frac{1}{2}|u|^2 \\ u_1 \\ u_2 \\ u_3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ e \end{bmatrix} =$

$= \frac{p}{2} \left[r - \cancel{\frac{1}{2}|u|^2} + \cancel{u_k u_k} - 2 - \cancel{\frac{1}{2}|u|^2} \right]$

$= \frac{p}{2} \left[\cancel{r} + \frac{p}{\rho} - 2s - \cancel{r} \right] = \frac{p}{2} - \rho s$

$= \frac{p}{2} + H$

Hence: $(1 + 2 + 3 + 4) \cdot V$

$$= \left(\cancel{\frac{P}{2}} + H\right) u_{i,i} + u_i H_{,i} + 0 - \cancel{\frac{P}{2} u_{i,i}}$$

$$= H u_{i,i} + u_i H_{,i} = (H u_i)_{,i} \qquad \blacksquare$$

4.2 (cont)

$$V \cdot F_i^{visc} = \begin{bmatrix} - \\ u_j \\ -1 \end{bmatrix} \begin{bmatrix} \tau_{ji} \\ \\ \tau_{ij} u_j \end{bmatrix} \frac{1}{2}$$

$$= \frac{u_j \tau_{ji} - \tau_{ij} u_j}{2} = 0 \,. \quad \blacksquare$$

$$V \cdot F_i^{H} = \begin{bmatrix} \\ \\ -q_i \end{bmatrix} \begin{bmatrix} \\ \\ -1 \end{bmatrix} \frac{1}{2} = \frac{q_i}{2} = \frac{q_i}{c_v \theta} \quad \blacksquare$$

$$V \cdot \mathcal{F}_i = \begin{bmatrix} - \\ u_j \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ \rho b_j \\ \rho (b_i u_i + r) \end{bmatrix} \frac{1}{2}$$

$$= \frac{u_j \rho b_j - \rho b_i u_i - \rho r}{2} = -\frac{\rho r}{c_v \theta} \quad \blacksquare$$

4.3

$$D = V_{,i}\left(F_i^v + F_i^h\right) \rightarrow F_i^v + F_i^h = \left\{ \begin{array}{c} - \\ \sigma_{ki} \\ \sigma_{ij} u_j + \kappa \theta_{,i} \end{array} \right\}$$

$$V = \left\{ \begin{array}{c} - \\ u_k \\ -1 \end{array} \right\} \frac{1}{\tau}$$

$$D = \left(\frac{u_k}{\tau}\right)_{,i} \left(\sigma_{ki}\right) + \left(\frac{-1}{\tau}\right)_{,i} \left(\sigma_{ij} u_j + \kappa \theta_{,i}\right)$$

$$= \left(\frac{u_{k,i}}{\tau} - \cancel{\frac{\tau_{,i}}{\tau^2} u_k}\right) \sigma_{ki} + \frac{\tau}{\tau^2} \left(\cancel{\sigma_{ij} u_j} + \kappa \theta_{,i}\right)$$

$$= \frac{u_{k,i}}{\tau} \sigma_{ki} + \frac{c_v \theta_{,i}}{c_v^2 \theta^2} \kappa \theta_{,i} \rightarrow \frac{\kappa}{c_v \theta^2} \theta_{,i} \theta_{,i}$$

$$c = \kappa / c_v \theta^2$$

$$\frac{u_{k,i}}{\tau} \left(2\mu \varepsilon_{ki} + \lambda \varepsilon_{jj} \delta_{ki}\right)$$

$$= \frac{2\mu \varepsilon_{ki} u_{k,i}}{\tau} + \frac{\lambda}{\tau} \varepsilon_{jj} u_{i,i}$$

Note that:

$$u_{k,i} \varepsilon_{ki} = \left\{ \overbrace{\varepsilon_{ki} - \frac{\varepsilon_{nn}}{3} \delta_{ki}}^{\varepsilon^{dev} \text{ (traceless)}} + \frac{\varepsilon_{nn}}{3} \delta_{ik} \right\} u_{k,i}$$

$$= \varepsilon_{ij}^{dev} \varepsilon_{ij}^{dev} + \frac{1}{3} \varepsilon_{nn} \varepsilon_{kk}.$$

Hence, putting everything back together :

$$\textcircled{D} = \varepsilon_{ij}^{dev}\, \varepsilon_{ij}^{dev} \left(\frac{2\mu}{t} \right) + \frac{2\mu/3 + \lambda}{t} (\nabla \cdot u)$$

$$+ \frac{\kappa}{c_v \theta^2} |\nabla\theta|^2$$

and we can see that $\textcircled{D} \geq 0$ as long as $\frac{2\mu}{3} + \lambda > 0$, which is true for the majority of materials.

4.4

$$R = U_{,t} + F_{i,i} - F^{v}_{i,i} - F^{h}_{i,i} - \tilde{F}$$

$$V\cdot R = V\cdot U_{,t} + \underbrace{V\cdot F_{i,i}}_{(Hu_i)_{,i}} - V\cdot F^{v}_{i,i} - V\cdot F^{h}_{i,i} - \underbrace{V\cdot \tilde{F}}_{-\frac{\rho r}{c_v \theta}}$$

Where:

$$V\cdot U_{,t} = \frac{\partial H}{\partial U}\cdot\frac{\partial U}{\partial t} = \frac{\partial H}{\partial t}$$

$$\left[V\cdot\left(F^{H}_{i} + F^{V}_{i}\right)\right]_{,i} = \underbrace{V_{,i}\left(F^{H}_{i} + F^{V}_{i}\right)}_{\mathbb{D}} + \underbrace{V\left(F^{H}_{i} + F^{V}_{i}\right)_{,i}}_{\left(\frac{-q_i}{c_v\theta}\right)_{,i}}$$

Hence:

$$\boxed{V\cdot R = H_{,t} + (Hu_i)_{,i} - \left(\frac{q_i}{c_v\theta}\right)_{,i} + \frac{\rho r}{c_v\theta} + \mathbb{D}\ .}$$