CSE 397 / EM 397 - Stabilized and Variational Multiscale Methods in CFD

Homework #1

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Consider the following 1D advection-diffusion problem

$$au_{,x} = \kappa u_{,xx} + f, \quad x \in \Omega = [0,1] \tag{1a}$$

$$u(0) = g_0 \tag{1b}$$

$$u(1) = g_1 \tag{1c}$$

For our purposes, assume that $a > 0$ and $\kappa > 0$ are constant.

Exercise 1.1

Recall that the A-th equation of the Galerkin form of the 1D advection-diffusion equation takes the form:

$$\sum_{B=0}^{N} \left(\kappa \int_\Omega N_{A,x} N_{B,x} \, dx - a \int_\Omega N_{A,x} N_B \, dx \right) u_B = \int_\Omega N_A \, f \, dx \tag{2}$$

where the domain $\Omega = [0,1]$. The nodes x_A are equally spaced, yielding the mesh parameter $h = 1/N$. With the piecewise linear basis functions, the only non-zero entries occur at $B \in \{A-1, A, A+1\}$ and we can thus rewrite the above equation in a stencil form associated with the matrix equation that arises:

$$\left(\frac{\kappa}{h} \boldsymbol{S}^A_{\text{Diff}} + \frac{a}{2} \boldsymbol{S}^A_{\text{Adv}} \right) \begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1} \end{bmatrix} = \int_\Omega N_A \, f \, dx$$

where

$$\boldsymbol{S}^A_{\text{Diff}} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$$

$$\boldsymbol{S}^A_{\text{Adv}} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

1. Verify that $\boldsymbol{S}^A_{\text{Diff}}$ and $\boldsymbol{S}^A_{\text{Adv}}$ do indeed take the form specified above for interior elements.

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2. Following the same procedure used to derive $\boldsymbol{S}^A_{\text{Diff}}$ and $\boldsymbol{S}^A_{\text{Adv}}$, convert the SUPG/GLS/MS stabilization contribution
$$\frac{h}{2}\xi(\alpha_h)\sum_{e=1}^{N}\int_{\Omega_e} N_{A,x}\left(au^h_{,x} - \kappa u^h_{,xx} - f\right)\,dx$$
into a stencil form
$$\frac{h}{2}\xi(\alpha_h)\left(\frac{a}{h}\boldsymbol{T}^A\begin{bmatrix}u_{A-1}\\ u_A\\ u_{A+1}\end{bmatrix} - \sum_{e=1}^{N}\int_{\Omega_e} N_{A,x}\,f\,dx\right)$$
where $\alpha_h = \frac{ah}{2\kappa}$ is the element Péclet number and Ω_e denotes the interior element. In particular, verify that $\boldsymbol{T}^A = \boldsymbol{S}^A_{\text{Diff}}$.

3. With $f = 0$, the resulting stencil for SUPG takes the form
$$\frac{\kappa}{h}\left((1+\alpha_h\xi(\alpha_h))\ \boldsymbol{S}^A_{\text{Diff}} + \alpha_h\,\boldsymbol{S}^A_{\text{Adv}}\right)\begin{bmatrix}u_{A-1}\\ u_A\\ u_{A+1}\end{bmatrix} = 0 \tag{3}$$
Recall that the general solution to the homogeneous form of (1a) is $u = c_1 + c_2\exp\left(\frac{ax}{\kappa}\right)$ and observe that $u_{A-1} = u_A = u_{A+1} = c_1$ will always satisfy (3). Find the function $\xi(\alpha_h)$ to ensure that (3) is satisfied when $u_B = \exp\left(\frac{a}{\kappa}Bh\right) = \exp(2\alpha_h B)$ for $B \in \{A-1,\ A,\ A+1\}$[1].

Exercise 1.2

The matrix equation derived from applying the SUPG method to (1a) takes the form
$$\boldsymbol{KU} = \boldsymbol{F} - g_0\boldsymbol{B}_0 - g_1\boldsymbol{B}_N \tag{4}$$
The $\boldsymbol{K}$ is a $(N-1)\times(N-1)$ matrix such that for $2 \le A \le N-2$
$$[\boldsymbol{K}_{A\,(A-1)}\quad \boldsymbol{K}_{A\,A}\quad \boldsymbol{K}_{A\,(A+1)}] = \left(\frac{\kappa}{h} + \frac{a}{2}\xi(\alpha_h)\right)\boldsymbol{S}^A_{\text{Diff}} + \frac{a}{2}\boldsymbol{S}^A_{\text{Adv}},$$
the non-zero entries of the first and last rows of the matrix:
$$[\boldsymbol{K}_{11}\quad \boldsymbol{K}_{12}] = \left[\int_\Omega N_{1,x}\left(\left(\kappa + \frac{ah}{2}\xi(\alpha_h)\right)N_{1,x} - aN_1\right)\,dx \qquad \int_\Omega N_{1,x}\left(\left(\kappa + \frac{ah}{2}\xi(\alpha_h)\right)N_{2,x} - aN_2\right)\,dx\right]$$
$$\boldsymbol{K}_{N-1\,(N-2)} = \int_\Omega N_{(N-1),x}\left((\kappa + \frac{ah}{2}\xi(\alpha_h))N_{(N-2),x} - aN_{N-2}\right)\,dx$$
$$\boldsymbol{K}_{(N-1)\,(N-1)} = \int_\Omega N_{(N-1),x}\left((\kappa + \frac{ah}{2}\xi(\alpha_h))N_{(N-1),x} - aN_{N-1}\right)\,dx$$
The vectors on the right hand side are
$$\boldsymbol{B}_0 = \begin{bmatrix}\int_\Omega N_{1,x}\left((\kappa + \frac{ah}{2}\xi(\alpha_h))N_{0,x} - aN_0\right)\,dx\\ 0\\ \vdots\\ 0\end{bmatrix}$$

[1]Hint: Divide through by $\exp(2\alpha_h(A-1))$. Also
$$\coth(x) = \frac{\exp(2x)+1}{\exp(2x)-1}$$

$$\boldsymbol{B}_N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \int_\Omega N_{(N-1),x} \left((\kappa + \frac{ah}{2}\xi(\alpha_h)) N_{N,x} - a N_N \right) \, dx \end{bmatrix}$$

$$\boldsymbol{F} = \begin{bmatrix} \int_\Omega f \, \left(N_1 + \frac{h}{2}\xi(\alpha_h) N_{1,x} \right) \, dx \\ \vdots \\ \int_\Omega f \, \left(N_{N-1} + \frac{h}{2}\xi(\alpha_h) N_{N-1,x} \right) \, dx \end{bmatrix},$$

and the unknown vector is

$$\boldsymbol{U} = \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

With $\kappa = 1$, $g_0 = 0$, $g_1 = 1$ and $f = 0$, the exact solution to (1a) will be

$$u(x) = \frac{\exp(ax) - 1}{\exp(a) - 1}$$

1. With $h = 0.1$, write a function `Usolve(a)` that assembles the matrix $\boldsymbol{K}$ and the vectors $\boldsymbol{F}$, $\boldsymbol{B}_0$, $\boldsymbol{B}_N$ and solve for $\boldsymbol{U}$ in (4) using SUPG and the Galerkin method. The Galerkin method is obtained from (4) by setting $\xi(\alpha_h) = 0$. For SUPG, use $\xi(\alpha_h) = \coth(\alpha_h) - \frac{1}{\alpha_h}$.
2. Plot the exact solution vs. the SUPG solution vs. the Galerkin solution for $a = 100$[2].
3. Plot the exact solution vs. the SUPG solution vs. the Galerkin solution for $a = 10$.
4. Plot the exact solution vs. the SUPG solution vs. the Galerkin solution for $a = 1$.

Exercise 1.3

With $a = 1$, $\kappa = 0$, $g_0 = 0$, $g_1 = 1$[3] and

$$f(x) = \begin{cases} 16a\,(1 - 4x) & \text{for } 0 \le x \le \frac{3}{8} \\ 16a\,(-2 + 4x) & \text{for } \frac{3}{8} \le x \le \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \le x \le 1 \end{cases},$$

the exact solution will be

$$u(x) = \begin{cases} 16\,x\,(1 - 2x) & \text{for } 0 \le x \le \frac{3}{8} \\ 9 + 32x(x-1) & \text{for } \frac{3}{8} \le x \le \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \le x \le 1 \end{cases}.$$

To obtain the upwind differences solution from your function `Usolve`, set $\xi(\alpha_h) = 1$ in $\boldsymbol{K}$, $\boldsymbol{B}_0$ and $\boldsymbol{B}_N$ and set $\xi(\alpha_h) = 0$ in $\boldsymbol{F}$. Use $h = 0.1$ as before. Plot the exact solution vs. the Galerkin solution vs. SUPG solution vs Upwind differences solution. [4]

[2] Recall that with the piecewise linear basis, the entries of $\boldsymbol{U}$ are the nodal values of the solution.

[3] For a pure advection problem, g_1 does not affect the exact solution because it is consistent with the exact solution.

[4] The matrix you obtain for the Galerkin method is only invertible for certain number of elements (It has to do with the fact that Galerkin is inherently unstable for highly advective problems). Unfortunately, the value of h given for that problem leads to a singular matrix. As a simple fix, I suggest that you use $h = 1/11$ instead.