Lecture #13, Feb. 28, 2024

SUPG: $B_{SUPG}(w^h, w^h) = B(w^h, w^h) + \int_{\Omega'} \tau a w^h_{,x} (a w^h_{,x} - \kappa w^h_{,xx})\, dx \quad \forall w^h \in \mathcal{V}^h.$

$$\underbrace{\left(\sum_{A=1}^{N} \int_{x_{A-1}}^{x_A} \dots dx\right)}_{\text{el. int's}} = \int_{\Omega'} \quad \Omega' \text{ el interiors}$$

$\Omega = [0, L]$

Find a stability norm.

$$\underbrace{|||w^h|||^2}_{?} \le B_{SUPG}(w^h, w^h)$$

$$= \quad = \kappa \|w^h_{,x}\|^2 + \tau a^2 \|w^h_{,x}\|^2$$

$$\ge \quad - \tau a \kappa (w^h_{,x}, \underbrace{w^h_{,xx}}_{})_{\Omega'}$$

$$\ge (\kappa + \tau a^2)\|w^h_{,x}\|^2 - \tau|a|\kappa \|w^h_{,x}\| \underbrace{\|w^h_{,xx}\|_{\Omega'}}_{}$$

$$\cancel{\ge} \quad \left(-\tau|a|\kappa \|w^h_{,x}\|\right)\underbrace{\left(c_{inv}\left(\frac{h}{L}\right)^{-1}\|w^h_{,x}\|\right)}_{}$$

$$\cancel{\ge (\kappa + \tau a^2 - \tau|a|\kappa\, c_{inv}\left(\frac{h}{L}\right)^{-1})\|w^h_{,x}\|^2}$$

$$\cancel{= (\kappa + \tau|a|^2 (1 - \underbrace{\frac{\kappa}{|a|} c_{inv}\left(\frac{h}{L}\right)^{-1}}_{\le \frac{1}{2}}))\|w^h_{,x}\|^2}$$

Peter-Paul

$$\ge (\kappa + \tau a^2)\|w^h_{,x}\|^2 - \tau|a|\kappa \frac{1}{2}\left(\frac{\|w^h_{,x}\|^2}{\varepsilon} + \varepsilon \|w^h_{,xx}\|^2_{\Omega'}\right)$$

$$\ge \underbrace{\left(\kappa - \tau|a|\frac{\kappa}{2}\frac{1}{\varepsilon}\right)}_{\le \frac{1}{2}\kappa,\ \varepsilon = \tau|a|}\|w^h_{,x}\|^2 + \left(\tau a^2 \|w^h_{,x}\|^2 - \tau|a|\frac{\kappa}{2}\varepsilon \|w^h_{,xx}\|^2_{\Omega'}\right)$$

$$\geq \left(\frac{\kappa}{2} \underbrace{\qquad}_{} \|w^h_{,x}\|^2 \right) \quad \text{inverse est.}$$

$$+ \underbrace{T a^2}_{} \|w^h_{,x}\|^2 - \underbrace{(T|a|)}_{}^2 \frac{\kappa}{2} \, c_{inv}^2 \left(\frac{h}{\cancel{L}}\right)^{-2} \|w^h_{,x}\|^2$$

$$= \frac{\kappa}{2} \|w^h_{,x}\|^2 + T|a|^{\cancel{2}} \Big(1 - \underbrace{T \cancel{|a|} \frac{\kappa}{2} c_{inv}^2 \left(\frac{h}{\cancel{L}}\right)^{-2}}_{\leq \frac{1}{2}} \Big) \|w^h_{,x}\|^2$$

$$T \cancel{|a|} \frac{\kappa}{\cancel{2}} c_{inv}^2 \left(\frac{h}{L}\right)^{-2} \leq \frac{1}{\cancel{2}}$$

$$T \leq \frac{1}{\kappa \, c_{inv}^2} \left(\frac{h}{\cancel{L}}\right)^{+2}$$

$$\boxed{T \leq \frac{h^2}{\kappa \, c_{inv}^2}}$$

$$\left(\frac{\kappa}{2} + \frac{T|a|^2}{2} \right) \|w^h_{,x}\|^2$$

$$\geq \frac{1}{2} (\kappa + T|a|^2) \|w^h_{,x}\|^2$$

$$\Rightarrow$$

$$T = \frac{h}{2|a|} \underbrace{\tilde{\xi}(\alpha_h)}_{} \leq \frac{h^2}{\kappa c_{inv}^2}$$

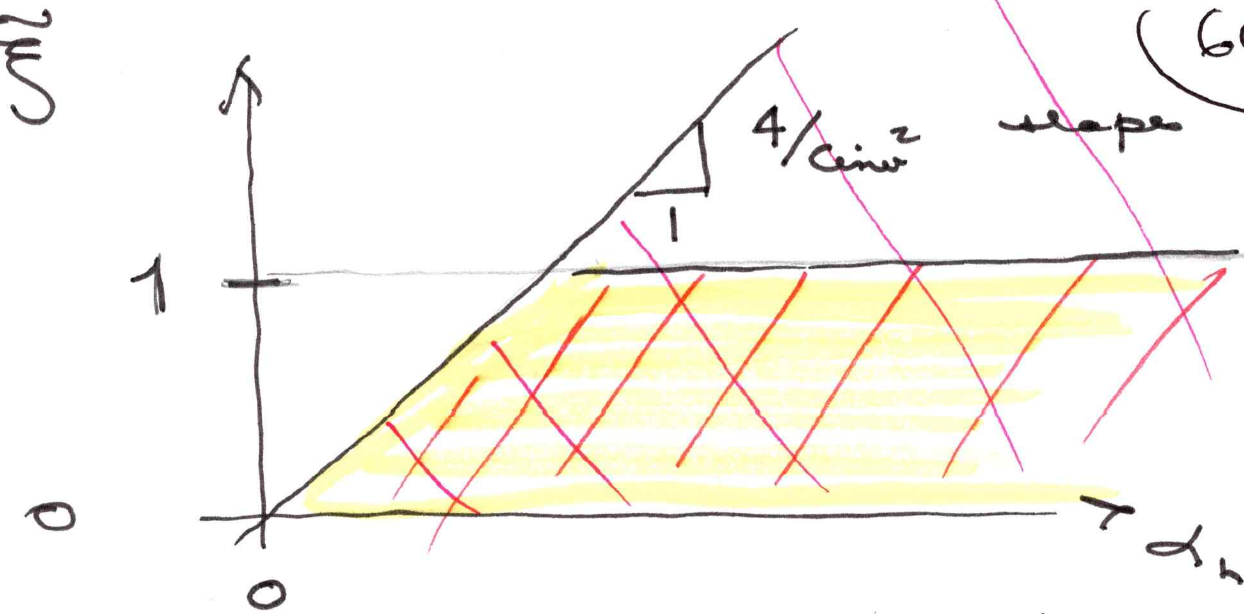
what are the conditions on $\tilde{\xi}(\alpha_h)$ that stability entails

$$\tilde{\xi}(\alpha_h) \leq \frac{h^{\cancel{2}}}{\kappa c_{inv}^2} \frac{2|a|}{\cancel{h}}$$

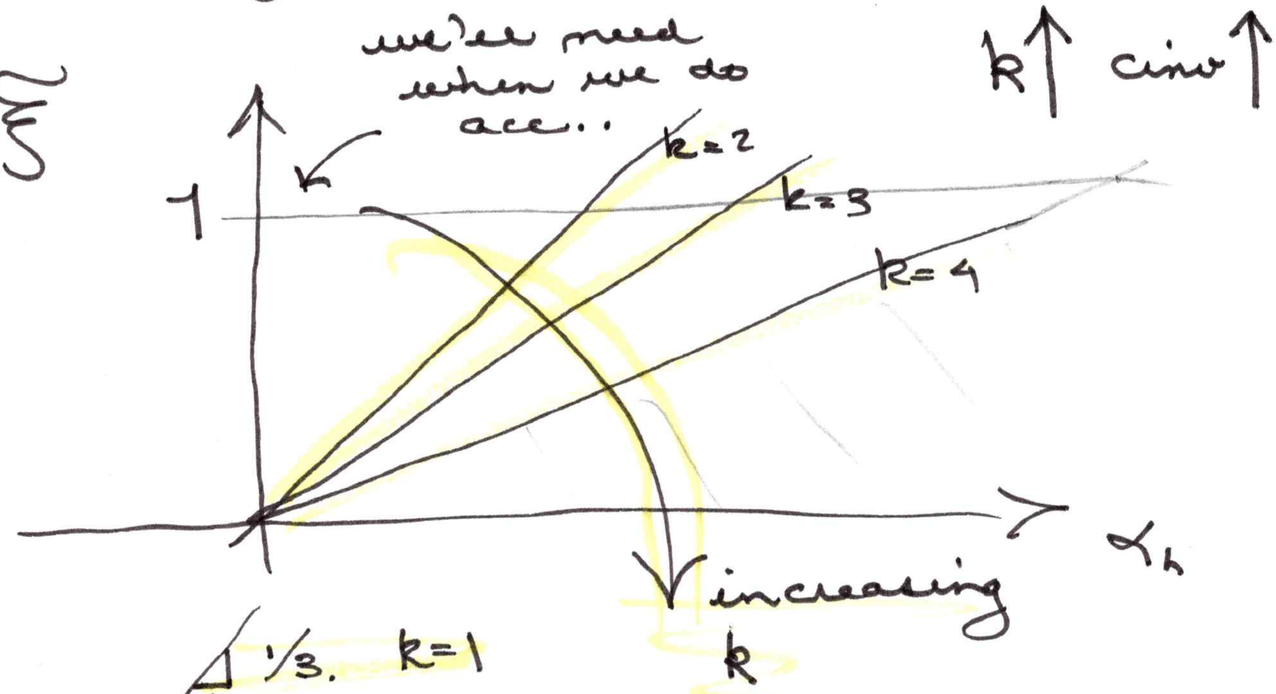
$$\Rightarrow$$

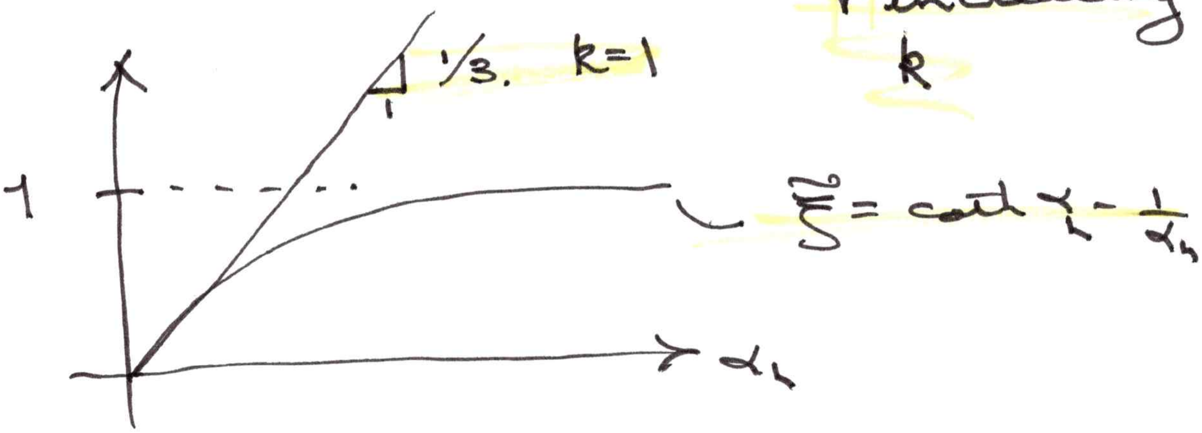
$$= 2 \cdot 2 \underbrace{\frac{h|a|}{2\kappa}}_{} \frac{1}{c_{inv}^2}$$

$$\leq 4 \, \alpha_h / c_{inv}^2$$


ξ̃
4/cinv² slope
1
1
0
0
α_h

k↑ cinv↑


ξ̃
we'll need when we do acc..
k=2
k=3
k=4
1
α_h
increasing k


1/3. k=1
1
1
ξ = coth α_h − 1/α_h
α_h

x

$$B_{SUPG}(w^h, w^h) \geq \frac{1}{2}(\kappa + \underbrace{\tau|a|^2}_{\frac{h}{2|a|}\tilde{\xi}})\|w^h_{,x}\|^2$$

$$\frac{1}{2}\kappa(1 + \underbrace{\frac{h}{2\cancel{|a|}\kappa}\tilde{\xi}|a|^{\cancel{2}}}_{\frac{h|a|}{2\kappa}\tilde{\xi}(\alpha_h)})\|w^h_{,x}\|^2$$

$$\geq \frac{\kappa}{2}(1 + \underbrace{\alpha_h\tilde{\xi}(\alpha_h)})\|w^h_{,x}\|^2 \quad \forall w \in \mathcal{V}^h$$

Stab. ✓

$$\hookrightarrow \underbrace{\frac{\kappa}{2}(1 + \alpha_h \tilde{\xi}(\alpha_h))\|e^h_{,x}\|^2} \leq B_{SUPG}(e^h, \underset{e-\eta}{e^h})$$

$$= \underbrace{B_{SUPG}(e^h, e)}_{\substack{\text{"Gal. orth"} \\ \text{"error"}}} - B_{SUPG}(e^h, \eta)$$

$= 0,$

$$= +|B_{SUPG}(e^h, \eta)|$$

$$= \underbrace{B(e^h, \eta)}_{\text{Gal.}} + \int_{\Omega'} \tau a e^h_{,x}(a\eta_{,x} - \kappa\eta_{,xx})\,dx$$

$$\leq \|e^h_{,x}\|(|a|\,\|\eta\| + \kappa\|\eta_{,x}\|) \text{ (see p. 61)}$$

$$+ \tau \cancel{|a|}|a|^2(e^h_{,x}, \eta_{,x}) - \tau a\kappa(e^h_{,x}, \eta_{,xx})_{\Omega'}$$

Cauchy-Schwarz.

$$\leq \|e^h_{,x}\| \left(|a| \|\eta\| + \alpha \|\eta_{,x}\|\right) + T|a|^2 \|e^h_{,x}\| \|\eta_{,x}\| + T|a|\alpha \|e^h_{,x}\| \|\eta_{,xx}\|_{\Omega'}$$

$$\frac{\alpha}{2}(1+\alpha_h \tilde{\xi}) \|e^h_{,x}\| \leq \left(|a| \|\eta\| + \alpha \|\eta_{,x}\| + T|a|^2 \|\eta_{,x}\| + T|a|\alpha \|\eta_{,xx}\|_{\Omega'}\right.$$

$$= \underbrace{|a| \|\eta\|_0}_{s=0} + (\alpha + T|a|^2)\frac{1}{L}\underbrace{L\|\eta_{,x}\|}_{s=1} + T|a|\alpha \underbrace{\|\eta_{,xx}\|_{\Omega'}}_{s=2}$$

$s=1$: $|\eta|_1 \leq \|\eta\|_1$

$s=2$: $\frac{1}{L^2} L^2 \|\eta_{,xx}\|$, $|\eta|_{2,\Omega'} \leq \frac{1}{L^2}\|\eta\|_{2,\Omega'}$

$$\leq |a| \, c_{int}\left(\frac{h}{L}\right)^{k+1-0} \|u\|_{k+1} + (\alpha + T|a|^2)\frac{1}{L}\, c_{int}\left(\frac{h}{L}\right)^{k+\cancel{1}-\cancel{1}} \|u\|_{k+1} + T|a|\alpha \frac{1}{L^2}\, c_{int}\left(\frac{h}{L}\right)^{\underbrace{k+1-2}_{k-1}} \|u\|_{k+1}$$

$$\leq c_{int}\left(|a|\left(\frac{h}{L}\right)^{k+1} + (\alpha + T|a|^2)\frac{1}{L}\left(\frac{h}{L}\right)^{k} + T|a|\alpha\frac{1}{L^2}\left(\frac{h}{L}\right)^{k-1}\right)\|u\|_{k+1}$$

✓

(clean up next time.)

We are almost there.