Stabilize: unsteady, AD eq in space-time

$$\mathbb{B}_{stab}(w^h, u^h) = \mathbb{L}_{stab}(w^h), \ \forall w^h \in \mathcal{V}^h.$$

$$\mathbb{B}_{stab}(w^h, u^h) \overset{def.}{=} \mathbb{B}(w^h, u^h)$$

$$+ \sum_{n=0}^{N-1} \int_{Q_n} \tau \left(w^h_{,t} + a w^h_{,x}\right) \times$$

$$\times \left(\boxed{\mathcal{L} u^h} \right) dQ$$

unsteady, AD operator

$$\boxed{\mathcal{L} u^h = \frac{\partial}{\partial t} u^h + a \cdot \nabla u^h - \nabla \cdot (\kappa \nabla u^h)}$$

$$\mathbb{L}_{stab}(w^h) = \mathbb{L}(w^h) + \sum_{n=0}^{N-1} \int_{Q_n} \tau \left(w^h_{,t} + a w^h_{,x}\right) \times$$

PDE residual

$$\times \boxed{(f)} \, dQ$$

maintains Gal. orthog.:

$$\underbrace{\text{stab. norm}}_{\text{up to a const.}}{}^2 = [\![w^h]\!]^2_{\text{Gal}} +$$

Cauchy-Schwarz + Peter-Paul

$$\sum_{n=0}^{N-1} \int_{Q_n} \tau \left(w^h_{,t} + a \cdot \nabla w^h\right)^2 dQ$$

rough idea: 1D $\tau_{n,e} = \left\{ \frac{2}{\Delta t} + \frac{2|a|}{h} + 12 \frac{\kappa}{h^2} \right\}^{-1}$

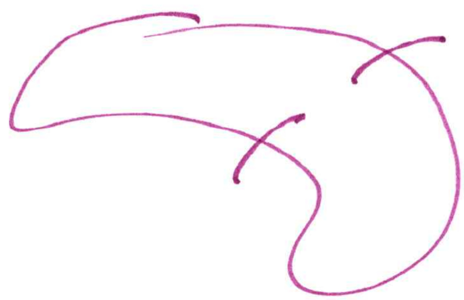
$$\leq \min \left\{ \frac{\Delta t}{2}, \frac{h}{2|a|}, \frac{h^2}{12\kappa} \right\}.$$

Nitsche's method. accommodate weakly enforced Dirichlet BCs.
(no longer built into spaces)

So : $\mathcal{S} = \mathcal{V} = \{ w \mid w \in H^1(\Omega) \}$

if in space-time,
replace $\Omega \leftarrow \bigoplus_{n=0}^{N-1} H^1(Q_n)$

$\mathcal{S}^h = \mathcal{V}^h =$ a usual FE space



Γ_g

all DOF/nodes are active.

simpler data structure.

Variational derivation, derive (symm) Nitsche's for the diffusion eq.

Begin w. the "augmented Lagrangian" approach. $\equiv$ 1. Lagrangian ($\equiv$ functional) with a Lagrange multiplier enforcing exact ✓ the constraint (i.e., the Dirichlet BC.)
+ 2. Penalty enforcement of the approx. ✓ constraint. (seems redundant but improves stability)

Focus on the Nitsche* fundamentals by assuming the diffusion prob.

$$0 = \underbrace{\nabla \cdot \kappa \nabla u}_{} + \overset{\text{given}}{f} \quad \text{on } \Omega$$

$\kappa = $ const. $\qquad u = \underset{\text{given}}{g} \quad \text{on } \Gamma = \partial\Omega$

(S)

The functional will generate the weak form (W) (or Lagrangian)

$$\Pi(u, \lambda) = \frac{1}{2} \int_\Omega \kappa |\nabla u|^2 d\Omega - \int_\Omega u f \, d\Omega$$

the exact sol of (S) — u; Lagrange mult. defined on Γ — λ

$$+ \int_\Gamma \lambda \underbrace{(u - g)}_{\text{Lag. mult.}} d\Gamma$$

$u = u(x),\ x \in \Omega$

$\lambda = \lambda(x),\ x \in \Gamma$

$$+ \underbrace{\int_\Gamma \frac{\kappa}{2\delta} (u-g)^2 d\Gamma}_{\text{Penalty}}$$

δ is a param. that comes to life in the discrete (will depend on the type of elements)

$\delta(x) > 0, \ \forall x \in \Gamma$

Consider variations: $\quad u \leftarrow u + \overset{\in \mathbb{R}}{\varepsilon} w \quad$ on Ω.

$\qquad \lambda \leftarrow \lambda + \varepsilon \mu \quad$ bndy

Compute the variational eqs.

$$0 = D\Pi(u,\lambda)\cdot(w,\mu)$$

$$= \left(\frac{d}{d\varepsilon}\Pi(u+\varepsilon w, \lambda+\varepsilon\mu)\right)\Bigg|_{\varepsilon=0.}$$

$$= \Bigg[\frac{d}{d\varepsilon}\Bigg(\frac{1}{2}\int_\Omega \kappa\, \nabla(u+\overset{(1)}{\varepsilon} w)\cdot\nabla(u+\overset{(2)}{\varepsilon} w)\, d\Omega$$

$$- \int_\Omega (u+\overset{(3)}{\varepsilon} w) f\, d\Omega$$

$$+ \int_\Gamma (\lambda+\overset{(4)}{\varepsilon}\mu)(u+\overset{(5)}{\varepsilon} w - g)\, d\Gamma$$

$$+ \int_\Gamma \frac{\kappa}{2\delta}(u+\overset{(6)}{\varepsilon} w - g)(u+\overset{(7)}{\varepsilon} w - g)\, d\Gamma\Bigg)\Bigg]_{\varepsilon=0}$$