Terzaghi's theory of consolidation, and the discovery of effective stress

Compiled from the work of K. Terzaghi and A. W. Skempton by C. R. I. Clayton, H. Müller Steinhagen and W. Powrie

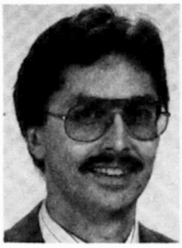
Proc. Instn Civ. Engrs Geotech. Engng, 1995, **113**, Oct., 191–205

Ground Board

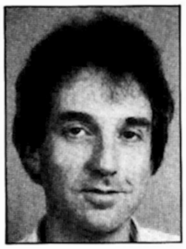
Geotechnical Engineering Advisory Panel
Paper 10843

Written discussion closes 15 December 1995

C. R. I. Clayton, Professor of Geotechnical Engineering, University of Surrey



H. Müller Steinhagen, Professor of Chemical and Process Engineering, University of Surrey



Professor W. Powrie, Department of Civil and Environmental Engineering, University of Southampton

Die Berechnung der Durchlässigkeitsziffer des Tones aus dem Verlauf der hydrodynamischen Spannungserscheinungen

A method of calculating the coefficient of permeability of clay from the variation of hydrodynamic stress with time

By Ing. Dr Karl v. Terzaghi

Sitzungsbrichte Akad. Wissen. Wien d. mathem.-naturw. Kl., part IIa, Vol. 132, pp. 125–138.

Introduction

By C. R. I. Clayton, H. Müller Steinhagen and W. Powrie

Every young civil engineer is surely now introduced to Terzaghi's theory of consolidation as part of an undergraduate course in basic soil mechanics, and most practising geotechnical engineers use Terzaghi's equations on a daily basis. The discovery of the principle of effective stress marks the start of modern soil mechanics, and that discovery was closely linked to Terzaghi's experimental work on the behaviour of clays, carried out between 1919 and 1925 at Robert College, in Istanbul, Turkey.

2. In the period between 1921 and 1925 Terzaghi published six classic papers, five of which are in German. One of his 1923 papers, *Die Berechnung der Durchlässigkeitsziffer des Tones aus dem Verlauf der hydrodynamischen Spannungsercheinungen*, is particularly significant, for several reasons

(*a*) the principle of effective stress is fully developed and understood by Terzaghi at this time
(*b*) the paper gives his consolidation theory for the first time, and illustrates the void ratio-effective vertical stress behaviour for three clays
(*c*) experimental results are produced which show that the flow of water through clay conforms to Darcy's law.

3. What Terzaghi was interested in showing by his work was that the permeability of clay could be determined from a test involving the squeezing out of the pore water within it. He had found that direct methods of permeability testing, using a constant head apparatus, did not give satisfactory results for 'viscoplastic and semi-rigid clays, whose permeabilities are much less than 0·06 cm/year'. He therefore searched for a different test method, and settled on the ingenious idea of applying an external pressure to the clay, and measuring the rate at which water was squeezed out. He notes that 'this method has the advantage of giving quick results which are not influenced by boundary effects (filter caking) because the flowing water is derived from the clay itself'.

4. Terzaghi's 1923 paper is not easy to follow in detail. There is a tendency towards economy of explanation and understatement, which masks the true complexity of the mathematical and physical arguments employed. Some errors and shortcomings have been discovered (e.g. Gibson *et al.* 1967;[1] Znidarcic and Schiffman, 1982;[2] Gibson *et al.* 1995[3]); however, these are quite subtle and tend to affect the generality of the analysis rather than the applicability of the result to the problem Terzaghi had in mind; that is, the development of small strains in a thin clay layer with an impermeable base, subjected to a constant load in comparison with which self-weight stresses are small. The significance of the paper is perhaps best summarized using the words of Gibson *et al.* (1995),[3] who wrote that the development of the theory of consolidation of soil by

Terzaghi 'acted as a catalyst which led to the emergence of a distinct sub-discipline of civil engineering, known initially as soil mechanics and now, more broadly, as geotechnical engineering .... Even now, Terzaghi's work forms the basis for all studies of consolidation as well as other subjects related to the deformation of and flow of fluids through porous media'.

5. There follows an historical account of Terzaghi's work by Professor A. W. Skempton, and a translation of Terzaghi's 1923 paper. The translation of the 1923 paper is accompanied by a commentary, which relates specifically to the points indicated in the main text by the superscripts A–P. The commentary is intended to elucidate the subtleties of Terzaghi's analysis, to compare it with the modern textbook approach, and to fill in some of the intermediate mathematical steps, which would otherwise require a very close inspection indeed of the 1923 paper. Minor comments are given in italics within the text of the translation.

Extract from Terzaghi's discovery of effective stress

By A. W. Skempton (1960)[4]

6. The concept of effective stress was first explicitly stated by Terzaghi in relation to the consolidation of clays. Geologists and civil engineers had long recognized that clay under load gradually consolidates as water escapes from the voids. As early as 1809 Telford preloaded a 55 ft (17 m) thick bed of soft clay, on which the eastern sea lock of the Caledonian Canal was to be founded, by building an embankment and allowing it to settle for about nine months 'for the purpose of squeezing out the water and consolidating the mud' before constructing the lock.

7. The first experimental work seems to have been carried out by Frontard in 1910. A sample 2 in. (50 mm) thick and 14 in. (355 mm) in diameter was placed in a metal container with a perforated base and loaded through a piston; the test was made in a room at a very high humidity to prevent drying out of the clay. Each increment of pressure was left in position until equilibrium was attained, and the results were plotted as a graph relating water content and pressure. In 1914 Forchheimer published a theoretical treatment in which the mathematics were based on the fact that the settlement of a clay layer is equal to the volume of water expelled during consolidation, and he derived an expression for the time required for a settlement of a given magnitude to take place. The solution was oversimplified, however, and in particular no account was taken of compressibility or pore-pressure distribution.

8. This was the state of the subject when Terzaghi commenced his research in 1919 at Robert College, Istanbul. During the next six years he carried out a series of masterly experiments, both on consolidation and shear strength, the results of which were given in five papers, and his book *Erdbaumechanik auf bodenphysikalischer Grundlage*, published in Vienna in 1925.

9. In consolidation tests, starting with clay at the liquid limit, increments of pressure up to about 1 kgf/cm^2 (100 kPa) were applied to the oedometer shown in Fig. 1(a). The apparatus was then dismantled and the 8 cm dia. bronze ring with its contained clay was removed and set up in the manner shown in Fig. 1(b), under a pressure of 2 kgf/cm^2 (200 kPa). After equi-

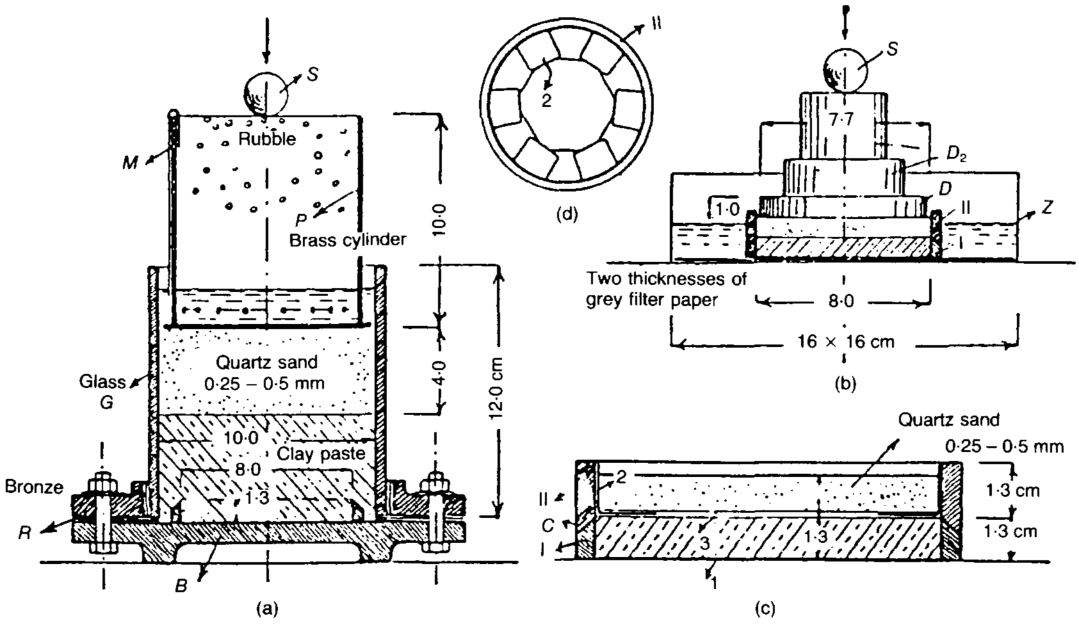


Fig. 1. Terzaghi's oedometer (a) during preparation of clay sample; (b) during actual test; (c) detail of clay sample and sand drainage layer during test; (d) cross section on plan [fig. 1 in Skempton, 1960;[4] after Terzaghi, 1924[5]]

librium had been attained, pressures up to 20 kgf/cm^2 (2000 kPa) were applied, each increment remaining constant for two days to allow consolidation to be completed before the next load was added. The pressures were then progressively decreased to zero, when the clay was again loaded to a pressure rather greater than the first maximum. The data were expressed as a relationship between void ratio e and pressure p; the first set of curves was published in 1921.

10. In 1923 sets of p–e curves were shown for three more clays, and the first statement of the classic theory of consolidation were given, in which the concepts of pore pressure and effective stress are fundamental. The process of consolidation is now described as follows*

> The local change of the governing pressure (by which Terzaghi means 'effective stress') implies a change of water content in the same place. This change of water content is caused by an outflow of water, which at very low permeability requires a high pressure gradient. Therefore, the result of a change of applied stress (by which Terzaghi means 'total stress') is to give local differences of pressure in the porewater. These pressure differences are dependent on time only, and their rate of dissipation, and the rate of stress equalisation is determined by the permeability of the material.

11. In deriving the theory of consolidation Terzaghi considers a clay layer originally in equilibrium under p_0. A pressure increment p_1 is then applied, and, in an accompanying diagram, it is shown that the effective pressure increment at any depth, at a time t after the load application, is p and the pore pressure is

$$u = p_1 - p$$

12. This equation is given in the text; the physical significance of the effective pressure increase p is made evident from the expression for the change in void ratio

$$\Delta e = -a_v p$$

where a_v is the *coefficient of compressibility* as deduced from the relationship between pressure and void ratio for the condition of 'hydraulic equilibrium'. We thus see that the principle of effective stress is fully comprehended in the 1923 paper.

13. It may be noted that the apparatus in Fig. 1 and the method of carrying out the consolidation tests were first described in 1924. The well-known time–consolidation curves were not published until 1927, however, and in 1923 the rate of consolidation was studied by means of an experiment in which observations were made on the decrease in pressure in a loaded clay specimen held at constant strain, rather than measuring the compression under constant pressure. In spite of the rather considerable difficulties inherent in this test Terzaghi was able to show that the permeability as deduced from the theory of consolidation was practically identical with the results obtained in 1919 (published in 1921) from direct measurements on the same clay in the combined falling-head permeameter and oedometer shown in Fig. 2. The basic assumptions of the theory were therefore confirmed.

* This text has been altered from that given by Skempton, to conform with the translation which follows.

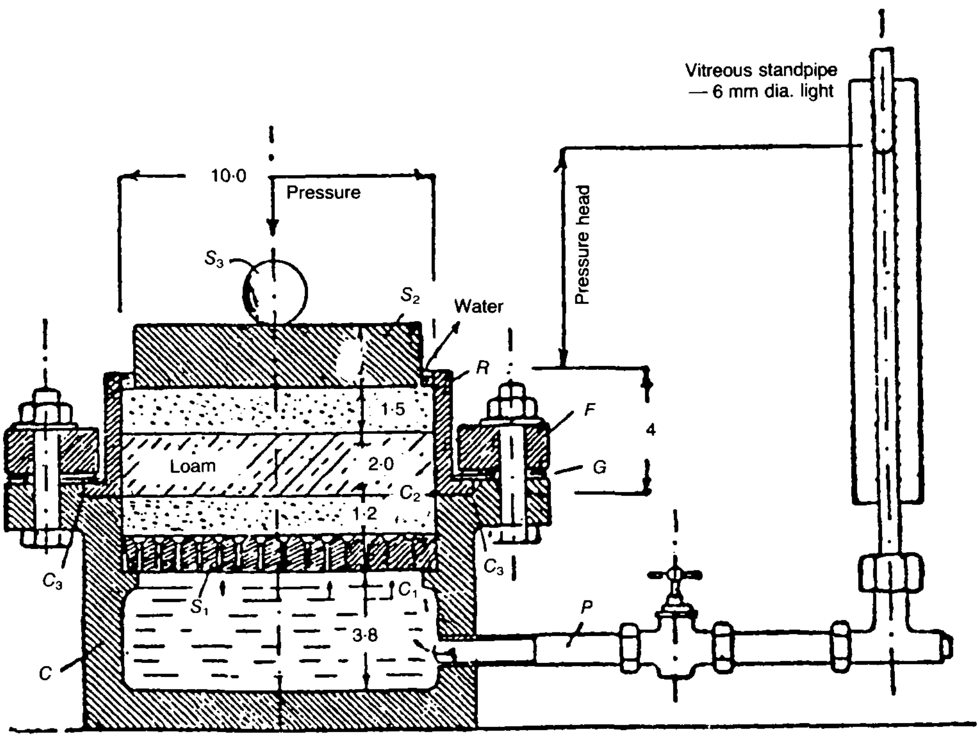


Fig. 2. Combined oedometer and falling head permeameter [fig. 2 in Skempton, 1960;[4] after Terzaghi, 1921[6]]

A method of calculating the coefficient of permeability of clay from the variation of hydrodynamic stress with time

By Ing. Karl von Terzaghi (1923)

Translated by C. R. I. Clayton and H. Müller Steinhagen

Introduction

14. The permeability of clay depends on its water content, which in turn depends on the pressure applied to the clay. In order to maintain constant water content in the clay during permeability testing, the author constructed an apparatus, in 1919, in which the clay under test was kept under constant pressure between two layers of sand filter. Water flowed from the bottom of the sample to the top under a hydraulic gradient produced by the head difference between an overflow and a standpipe (6 mm dia.). The permeability of the sample was calculated from the rate at which the water level in the standpipe fell.

15. With the aid of this apparatus it could be seen that in clay with voids ratio less than 2·00 and permeability greater than 0·06 cm/year there were no noticeable departures from Darcy's law. However, this direct method of testing did not work for viscoplastic and semi-rigid clays, whose permeabilities are much less than 0·06 cm/year. Under these conditions it became necessary to find a testing method that would overcome the unavoidable disadvantages of the direct method. Such a method was found in the observation of the variation of hydrodynamic stress effects in clay with time.

Hydrodynamic stresses

16. By hydrodynamic stress effects the author means the delays which are experienced in the stresses set up in a clay during the application of an external force, as a result of the resistance to the outflow of pore water. The local change of the governing pressure [*by which Terzaghi means 'effective stress'*] implies a change of water content at the same place. This change of water content is caused by an outflow of water, which at the very low permeability of clay requires a high pressure gradient. Therefore, the initial result of a change of applied pressure [*total stress*] is to give local differences of pressure in the pore water. These pressure differences disappear with time and the rate of stress equalization is determined by the permeability of the material. The permeability of the material can therefore be calculated from the time-dependent rate of their equalization, provided that all the other important parameters are known. This method has the advantage of giving quick results which are not influenced by boundary effects (filter caking) because the flowing water is derived from the clay itself.

The influence of constant pressure and the thermodynamic analogy

17. In Fig. 3(a) the curve of C_p, which has to be derived experimentally for each clay, represents the relationship between the applied pressure p which acts on the surface of the clay mass, and the void ratio [*literally, pore value*] e of this material,[A] for a loading process in which p is increased. The curve C_k gives the relationship between void ratio and the permeability k. The surface of a submerged clay layer, thickness h, is initially loaded with p_0/unit area. Lateral expansion is prevented by a rigid boundary. If we increase the pressure by p_1, then the void ratio of the clay drops from e_0 to e_1, and the permeability from k_0 to k_1. Under the influence of the additional pressure, a certain amount of water will be expelled from the clay. If the additional pressure p_1 is less than approximately 2 kgf/cm^2, then the distance $(p_0 - p_1)$ on the C_p curve can be replaced with sufficient accuracy by the tangent T_p and the decrease in permeability, from k_0 to k_1, can be approximated to by the mean value of k.

18. To simplify the mathematical treatment of the problem, a prism of 1 cm^2 cross-section and 1 cm length, containing dry clay, is considered. The prism can change volume by expanding in the direction of applied pressure, which is also the direction of flow. The actual length is always equal to $1 + e$, where e is the void ratio of the clay.

19. The reduced thickness of the layer h_r indicates the thickness that the clay would have if its void ratio were zero. The permeability coefficient k_r is defined as the velocity of flow of water through a layer of clay of real thickness $(1 + e)$, under a head difference of 1 cm. This definition may seem to be strange. It is, however, identical to the usage adopted in the theory of non-steady state heat conduction where changes in length due to temperature change are ignored, and has the advantage that the boundary conditions necessary for the solution of the partial differential equation are simplified, and that the error due to assuming a constant permeability is reduced.[B]

20. If the pressure on the clay is increased from p_0 to $p_0 + p$ [*where p is a general pressure increase*], the water contained per unit volume of dry clay changes (Fig. 3(a)) as follows

$$q = (e - e_0) = -\frac{e_0}{p_2}p = -a_v p \qquad (1)^{C}$$

where a_v is the coefficient of compressibility,[D] which changes with the position of tangent T_p.

21. In the clay layer at a [*reduced*] depth $(h - z)_r$ (Fig. 3(b)), all the water expelled from

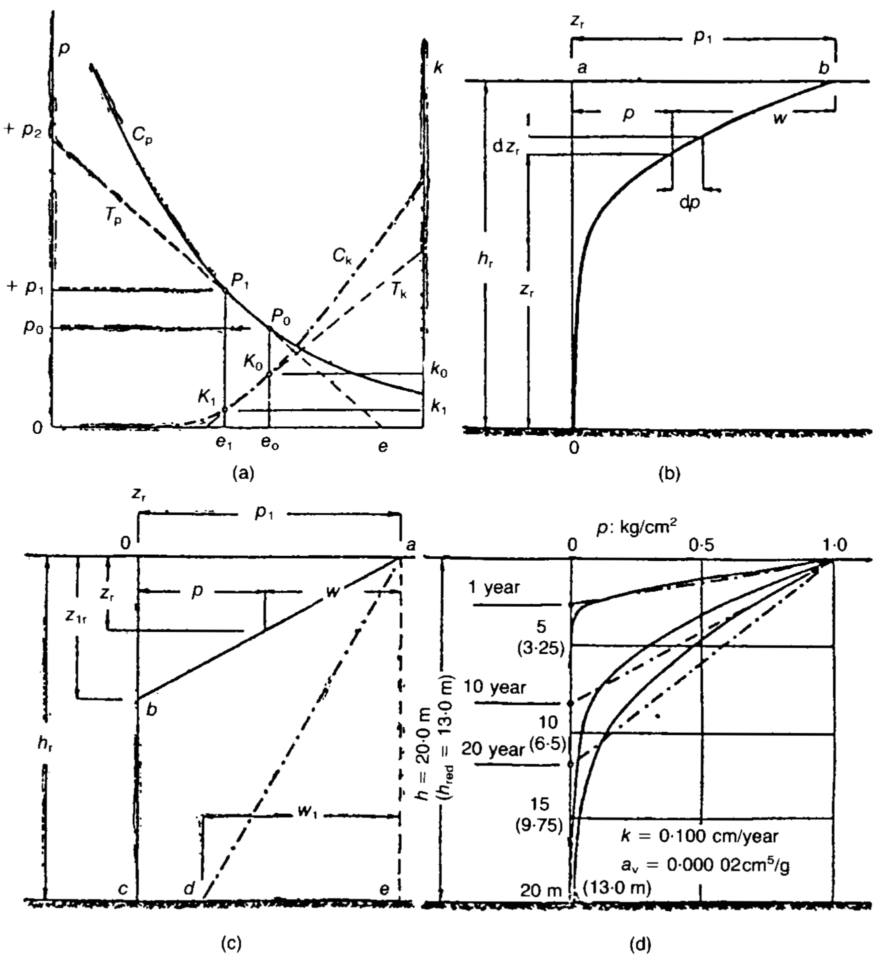


Fig. 3. (a) Relationships between void ratio e and (i) effective stress p (C_p) and (ii) permeability k (C_k). (b) Isochrone showing increase in vertical effective stress p, and (excess) pore-water pressure w, as functions of the reduced depth co-ordinate z_r, at a time t after the start of consolidation. (c) Idealized linear isochrones. (d) Comparison between isochrones using approximate and exact solutions (fig. 1 in Terzaghi, 1923)

the region $O - z_r$ flows through the differential element with thickness dz_r. The production of this flow points to a head difference in the pore water.

22. Denoting Q as the water flow rate through a unit cross-section at the depth $(h-z)_r$, and $w = p_1 - p$ as the hydrostatic pressure (above atmospheric) in the pore water at $(h-z)_r$, then the necessary head difference to produce flow, given by Darcy's law is

$$i = \frac{Q}{k_r} = -\frac{dw}{dz_r} \qquad (2)^{E}$$

23. Because water is flowing out under constant [*total*] load, $p_0 + p_1$, over a period of time this proves that there is an increase of effective pressure in the clay. If this pressure were constant then the water content would also remain constant. The increase of the pressure in the clay from $(p_0 + p)$ to $(p_0 + p + dp)$ at the depth $(h-z)_r$ causes an increase in flow in the region $z_r \rightarrow (z_r - dz_r)$ (from Equation (1)) of

$$\frac{\partial Q}{\partial z_r} = a_v \frac{\partial p}{\partial t} = -a_v \frac{\partial w}{\partial t} \qquad (3)^{F}$$

where $\partial p/\partial t$ is the increase of the pressure $(p_0 + p)$ per unit time at depth $(h-z)_r$.

24. The value of Q is given by Equation (2) and the differential equation governing flow is therefore

$$\frac{k_r}{a_v}\frac{\partial^2 w}{\partial z_r^2} = \frac{\partial w}{\partial t} \qquad (4)$$

25. Equation (4) is identical to the fundamental equation for one-dimensional unsteady-state heat conduction in isotropic media. The transfer of problems from this theory to that of hydrodynamic stress change with time in clay can be achieved using the following substitutions

Heat content (enthalpy)
= water content
Specific heat
= coefficient of compressibility a_v
Temperature
= hydrostatic head w of the pore water
Thermal conductivity
= coefficient of permeability k_r

Furthermore, the following correspondences exist in the physical values of these variables

Thermodynamics

(*a*) Specific heat increases with increasing heat content (i.e. temperature).

(*b*) A body contracts with decreasing heat content (i.e. temperature).

Consolidation of clays

(*a*) Coefficient of compressibility increases with increasing water content.
(*b*) The clay shrinks with decreasing water content.

26. The boundary conditions for the solution of Equation (4) are as follows, for the case of a clay layer loaded at the surface from p_0 to p_1

$$p_0 + p = f(z_r)$$

[*where p is the actual increase in effective stress at a general depth and time.* Strictly, $p_0 + p = f(z_r, t)$.].

When

$$t = 0 \text{ and for } 0 < z_r < h_r : f(z_r) = p_0$$
$$t = \infty \text{ and for } 0 < z_r < h_r : f(z_r) = p_0 + p_1$$

For

$$z_r = h_r \text{ and } 0 < t < \infty : f(z_r) = p_0 + p_1;$$
$$w = p_1 - p$$
$$z_r = 0 \text{ and } 0 < t < \infty:\ \partial p/\partial z_r = 0$$

27. The equation can be solved with the aid of Fourier series and the result is as follows

$$(p_0 + p) = p_0 + p_1 \times \left[1 - \frac{2}{n\pi}\sum_{n=1}^{\infty}\left[\exp\left[-\frac{k_r}{a_v}\frac{n^2\pi^2 t}{4h_r^2}\right] \times \sin\left(\frac{n\pi(z_r + h_r)}{2h_r}\right)(1 - \cos n\pi)\right]\right] \quad (5)^{G}$$

where n is an integer from 1 to ∞.

28. Figure 3(d) shows the results calculated for a particular case. The full lines, which give the relationship between pressure and depth (from Equation (5)) are so flat that they can be approximated to by a straight line with sufficient accuracy. This fact forms the basis of an extremely simple approach to the numerical treatment of stress equalization (Fig. 3(c)).[H] If $\partial p/\partial t$ is the increase of the pressure, $p_0 + p$ per unit time at a depth z_r below the surface of the clay, then the flow rate of water per unit area at this depth is

$$Q = a_v \int_{z_r}^{z_{r1}} \frac{\partial p}{\partial t}\,dz_r \quad (6)^{I}$$

29. From Fig. 3(c) we have

$$p = p_1 - p_1\frac{z_r}{z_{r1}} \quad (7)$$

and [*by differentiating Equation (7)*]

$$\frac{\partial p}{\partial t} = p_1 \frac{z_r}{z_{r1}^2}\frac{\partial z_{r1}}{\partial t} \quad (8)$$

30. The pressure difference $p_1 = w_1$ represents the head under which the water from depth z_1 flows to the surface. Therefore using Equation (6) and for $z = 0$

$$w_1 = p_1 = \int_0^{z_{r1}} \frac{Q\,dz_{r1}}{k_r} = \frac{a_v}{k_r}\int_0^{z_{r1}}\left\{\int_{z_r}^{z_{r1}} p_1 \frac{z_r}{z_{r1}^2}\frac{\partial z_{r1}}{\partial t}\,dz_r\right\}dz_r$$

31. By integrating three times we obtain

$$z_{r1} = \sqrt{\frac{k_r}{a_v}6t} \quad (9)^{J}$$

which when substituted in Equation (7) gives

$$p = p_1\left[1 - \frac{z_r}{\sqrt{t}}\sqrt{\frac{a_v}{6k_r}}\right] \quad (10)$$

32. As soon as z_{r1} becomes equal to h_r Equation (9) becomes invalid, and we get a pressure trapezoid, *Oadc*, in place of the triangle, *Oab*, while the total head [*at the base of the sample*] w_1 becomes less than p_1.

33. From the same line of reasoning used to develop Equation (10) we get

$$p = p_1\left[1 - \exp\left[-\left(\frac{3k_r t}{a_v h_r^2} - \frac{1}{2}\right)\right]\right] \quad (11)$$

so that for $t = \infty$, $p = p_1$.[K]

34. Equation (10) gave the dotted and dashed lines for the example on Fig. 3(d), for $t = 1$, 10 and 20 years. The coincidence of results from the rigorous and approximate methods is very good.

35. The approach explained above with the help of a simple example was used by the author with minor changes to solve problems where the solution of the differential Equation (4) is troublesome due to complicated boundary conditions, or for cases where the assumption of constant coefficients is not valid due to too large a value of p_1. The author took advantage of the fact that the product of the permeability and the pressure for plastic clay is almost constant. The following problems could be solved approximately

(*a*) formation of a crust by evaporation at the surface
(*b*) variation of the degree of saturation with time after flooding a plastic clay layer
(*c*) rate of compression of a clay layer being deposited at a river mouth as a function of the rate of deposition
(*d*) the effect of boring into a sandy layer in between two clay strata.

36. Equations (10) and (11) can be used to determine the permeability of the clay with the aid of compression tests. Several centimeters of visco-plastic clay are placed in a flat circular metal box which has thick walls and base, then

covered with filter paper and a thin layer of quartz sand, submerged in water and loaded with a pressure of p_0 in an oedometer [*literally, 'strength machine'*]. After the pressure p_0 has taken effect (i.e. when the clay has consolidated), the pressure is increased by p_1. The time-dependent deformation of the clay layer under constant pressure is observed by means of a dial gauge. The quantity of water flowing out of the clay per unit time per unit area is given by

$$Q = a_v \int_0^{z_{r1}} \frac{\partial p}{\partial t}\, dz_r \qquad (6)$$

37. Combining this with Equation (10) [*and substituting for z_{r1} in terms of t using Equation 9*] we obtain

$$Q = \frac{p_1}{4}\sqrt{\frac{6k_r a_v}{t}}$$

whence

$$k_r = \frac{8}{3}\frac{tQ^2}{a_v p_1^2} \qquad (12)$$

38. The coefficient of compressibility a_v is determined from the experimental curve C_p in Fig. 3(a), and Q is determined from the tangent to the curve which represents the relationship between time and compression (change in void ratio e) under the application of the constant pressure $p_0 + p_1$ [*because* $dQ = (de/dt) \times$ *volume*]. The oedometer must be capable of exerting a constant pressure despite the increase of compression.

Calculation of permeability from the adiabatic change in state

39. The author's apparatus is designed in such a way that the height of the clay can be kept constant, while the pressure changes. The permeability is then calculated from an observed change in state which is analogous to the adiabatic temperature equalization in a locally heated body. The above-described clay layer, enclosed in a metal box and covered with a filter, is submerged and fairly quickly loaded to pressure p_1. The deformation is then kept constant. The pressure acting against the piston of the machine decreases with time and asymptotically approaches a limit pressure p_∞.[L]

40. Pore-water flow occurs only within the clay layer. Therefore, the total water content of the clay layer remains constant during the decrease of pressure, although the pressure decrease points to a decrease in hydrostatic pressure.

41. At a time t, the pressure distribution in the clay is approximated by a straight line $a'b'$, Fig. 4(a). After some time increment dt, the pressure at the surface decreases by dp, while the point b' is displaced towards the bottom by

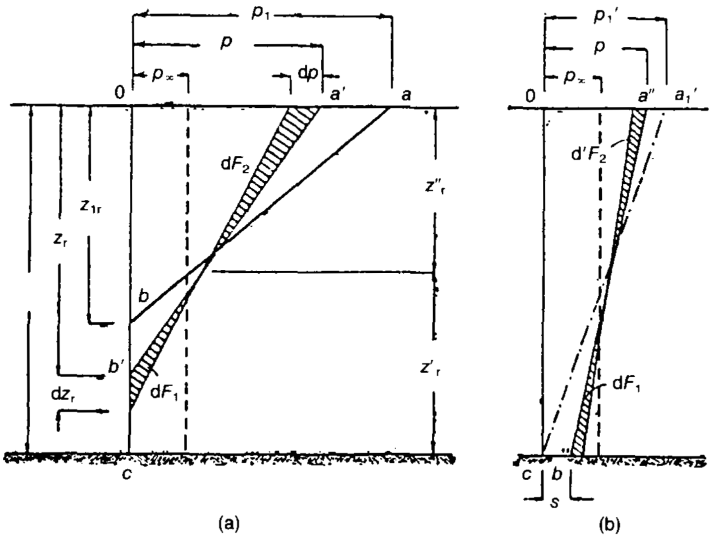


Fig. 4. Approximate linear isochrones during pore-water pressure equalization in a clay sample with no overall volume change permitted. (a) $z_r \le h_r$; (b) $z_r \ge h_r$ (fig. 2 in Terzaghi, 1923)

dz_r. Therefore, the water content in the depth range z_r' decreases, while the upper part of the clay in the region z_r'' swells. As soon as the value of z_r becomes equal to h_r we have a trapezoid $Oa''b''c$ instead of the triangle $Oa'b'$ (Fig. 4(b)). A decrease of the pressure p at the surface of the clay by dp is coupled with an increase of pressure s at the bottom of the clay by ds. When $t = \infty$, $p = s = p_\infty$.

42. The shaded areas dF_1 and dF_2, in Fig. 4(a), give the increase and decrease, respectively, of the pressure in the clay.[M]

43. With an increase in pressure dp the water content of the material decreases by $a_1\, dp$. But with an equal decrease in pressure dp the water content of the clay increases by $a_2\, dp$ where $a_1 > a_2$, because the gradient of the swelling curve A_1' (Fig. 5) is considerably flatter than the main curve A_1. The gradient under which water flows from the bottom of the clay to the top is $(p - s)/h_r$. The quantity of water flowing out of the domain dF_1 must be equal to the quantity flowing into dF_2. Therefore, we have

$$\frac{dF_1}{dF_2} = \frac{a_{v1}}{a_{v2}}$$

and

$$s = (p_1' - p)\sqrt{\frac{a_{v2}}{a_{v1}}}$$ [N]

44. From this we obtain the following expression

$$\frac{dp}{dt} = \frac{2k_r\left(1 + \sqrt{\frac{a_{v1}}{a_{v2}}}\right)^2}{h_r^2 a_{v1}}(p - p_\infty)$$ [O]

and therefore

$$k_r = \frac{dp}{dt}\frac{a_{v1} h_r^2}{2\left(1 + \sqrt{\frac{a_{v1}}{a_{v2}}}\right)^2 (p - p_\infty)} \qquad (13)$$

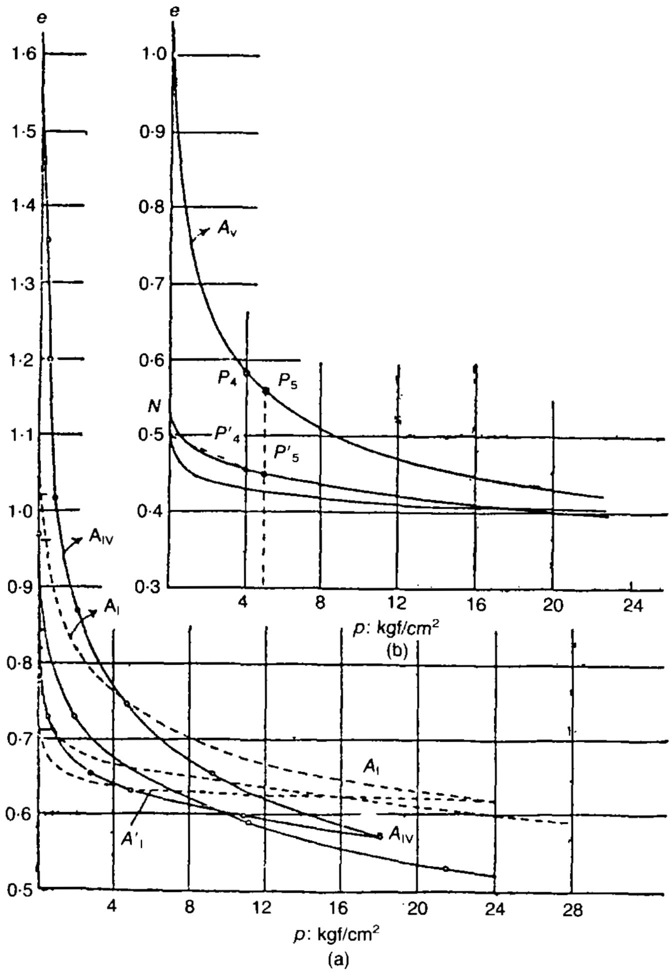


Fig. 5. Experimental relationships between void ratio e and vertical effective stress p in compression (A_I, etc.) and swelling (A'_I, etc.) (fig. 3 in Terzaghi, 1923)

45. The compression curve A_I and swelling curve A'_I for the residual clay I examined by the author are shown in Fig. 5. The clay layer had a diameter of 8 cm. The initial water content was 32·8% and the grain-specific gravity of the dry material was 2·93. The initial void ratio e_0 was 0·967. The true thickness of the layer was 4 cm at the beginning of the test. The reduced thickness h_r for a void ratio $e = 0$ is therefore

$$h_r = \frac{h}{1 + e_0} = \frac{h}{1{\cdot}967} = 2{\cdot}03 \text{ cm}$$

46. The pressure was increased step by step from zero to 14·1, 27·3, 57·2, 119·3, 207·0 kgf/cm². After each increment the deformation of the clay was kept constant for 48 h. The pressure in the compression machine decreased quickly at first, but the rate of decrease slowed with time. The relationships between pressure and time are shown in Fig. 6 (curves C_1–C_4). (The curve C_5 for the highest pressure (207·0 kgf/cm²) falls outside the graph).

47. From the pressure against time curves (marked C on Fig. 6), the curves which represent the rate of decrease in pressure dp/dt as a function of the difference $(p - p_\infty)$ (marked (C) on Fig. 6) were derived. p is the pressure noted during the test and p_∞ is the limit pressure approached asymptotically during each load increment. The dp/dt lines are curved because the coefficients of compressibility a'_{v1} and a'_{v2} are related to different pressures, as $s < p_\infty < p$ (Fig. 4(b)) before equilibrium is reached. The lines asymptotically approach limit values of a_{v1} and a_{v2}, corresponding to the values of p_∞, shown in Fig. 6.[P]

48. After the test the clay was removed from the box, cut into slices, and the void ratio of each slice was determined. The distribution of void ratio found over the whole thickness of the layer is represented by the [*curved*] line in Fig. 6(b). The low void ratio of the upper part of the curve shows that this part of the clay has swollen after consolidation, since the gradient of the swelling curve is much less steep than the gradient of the consolidation curve. The sequence of changes of state can be seen from the curve; the upper part of the layer was first under a high pressure and subsequently expanded, while the lower part was exposed only to compression.

49. The method of calculating the permeability will be explained with the aid of curve C_1. The average void ratio of the clay was calculated from the compression measured with the aid of a dial gauge and was found to be $e = 0{\cdot}84$. The value of the asymptotically approached pressure $p_{\infty 1}$ is given in Fig. 6(a) by the ordinate of the point of intersection between curve C_1 and the ordinate axis, and is $p_{\infty 1} = 1{\cdot}72$ kgf/cm². The values of the coefficients a_{v1} and a_{v2} were obtained from the lines A_I and A'_I, determined separately from the permeability test (Fig. 5(a)). They were determined as in Fig. 3(a) from the tangents to the compression curve A_I and the swelling curve A'_I at pressure $p_{\infty 1}$ $(= 1{\cdot}72$ kgf/cm²).

50. Their values are

$$a_{v1} = 0{\cdot}425 \times 10^{-4} \text{ cm}^5/\text{gf}$$
$$a_{v2} = 0.110 \times 10^{-4} \text{ cm}^5/\text{gf}.$$

51. In Equation (13) h_r is the reduced thickness of the clay layer for a void ratio $e = 0$. Therefore, to obtain the true value of k we must multiply the expression by $(1 + e_1)$ [*strictly, by* $(1 + e_1)\gamma_w$*: see notes E and G*] as previously mentioned. The tangent at the point $p = p_{\infty 1}$ on the curve C_1 has the equation

$$\frac{dp}{dt} = 5710(p - p_{\infty 1}) \quad (\text{gf/cm}^2/\text{year}) \tag{14}$$

52. For $(p - p_{\infty 1}) = 0$ the coefficients of compressibility a_{v1} and a_{v2} have the previously noted values, and so

$$\sqrt{\frac{a_{v1}}{a_{v2}}} = 1{\cdot}96$$

53. We obtain the permeability from the expression

$$k_1 = \frac{\mathrm{d}p}{\mathrm{d}t}\frac{1}{(p - p_{\infty 1})}\frac{a_{v1}h_r^2(1 + e_1)}{2\left(1 + \sqrt{\dfrac{a_{v1}}{a_{v2}}}\right)^2}$$

$$= 5710 \times \frac{0{\cdot}425 \times 10^{-4} \times 2{\cdot}03^2 \times 1{\cdot}84}{2 \times 2{\cdot}96^2}$$

$$= 0{\cdot}105 \text{ cm/year} = 2{\cdot}0 \times 10^{-7} \text{ cm/min}$$

54. The test was performed at a temperature of 22°C. For a temperature $T_0 = 15$°C we obtain, from the ratio of viscosities

$$k_0 = k_1 \times \frac{(1 + 0{\cdot}033T_0 + 0{\cdot}00022T_0^2)}{(1 + 0{\cdot}033T_1 + 0{\cdot}00022T_1^2)}$$

$$= 2{\cdot}0 \times 10^{-7} \times \frac{1{\cdot}544}{1{\cdot}834}$$

$$= 1{\cdot}69 \times 10^{-7} \text{ cm/min}$$

55. In 1919 the permeability was determined for the same material by the direct method.[6] With a void ratio $e = 0{\cdot}86$ at a temperature of 15°C the permeability k was found to be $1{\cdot}65 \times 10^{-7}$ cm/min.

56. From the evaluation of further test data we obtained

e	0·790	0·742	0·680	0·600
k_0	0·917	0·610	0·156	0·0473

$\times 10^{-7}$ cm/min

57. The shape of the curve (C) can be approximated theoretically by calculating the pressure s (Fig. 4(b)) at the bottom of the clay layer with different values of $(p - p_\infty)$. The value a'_{v2} can be found from the tangent to the compression curve A_I (Fig. 5(a)) at pressure s, and the value of a'_{v1} can be found from the tangent to the swelling curve A'_I, at pressure p.

58. With increasing $(p - p_\infty)$ the value of a'_{v1} increases, the value a'_{v2} decreases and therefore the value of $\mathrm{d}p/\mathrm{d}t$ increases more quickly than expected using Equation (14). The calculated values of $\mathrm{d}p/\mathrm{d}t$, for all the curves approximated in this manner agree with the experimentally obtained curves. For the permeability of the clay in the range used in the compression tests (for $e = 0{\cdot}86$) the validity of Darcy's law has been experimentally verified. Therefore one can assume that this law is also valid for clays of semi-stiff consistency. If this were not so, the departures from Darcy's law would be seen in the shape of the $\mathrm{d}p/\mathrm{d}t$ curves.

Notes on the mathematics of Terzaghi's 1923 paper

By W. Powrie

59. A. In his 1923 paper, Terzaghi uses the symbol ε for void ratio, and the symbol a instead of a_v. For the reasons explained in note B, the symbols used by Terzaghi for sample depth (z and h) and coefficient of permeability k have been changed to z_r, h_r and k_r, respectively.

60. B. The content of these two paragraphs is highly significant, albeit somewhat understated. Terzaghi is using a reduced or material co-ordinate system, which moves with the soil skeleton as consolidation progresses. This can be explained, following Gibson *et al.* (1967),[1] with reference to Fig. 7.

61. Figure 7(a) shows an element of the soil skeleton, of unit cross-sectional area, which at time $t = 0$ is defined by planes A_0B_0 and C_0D_0, embedded in the soil skeleton at distances a and $(a + \delta a)$ from an embedded datum plane. At some later time t, following consolidation of the soil, these planes will have moved in absolute terms. In an absolute frame of reference using a conventional or Eulerian co-ordinate system, the new positions of the element faces AB and CD are at unknown distances $\xi(a, t)$ and $\xi(a + \delta a, t)$ from the embedded datum plane (Fig. 7(b)). In a relative frame of

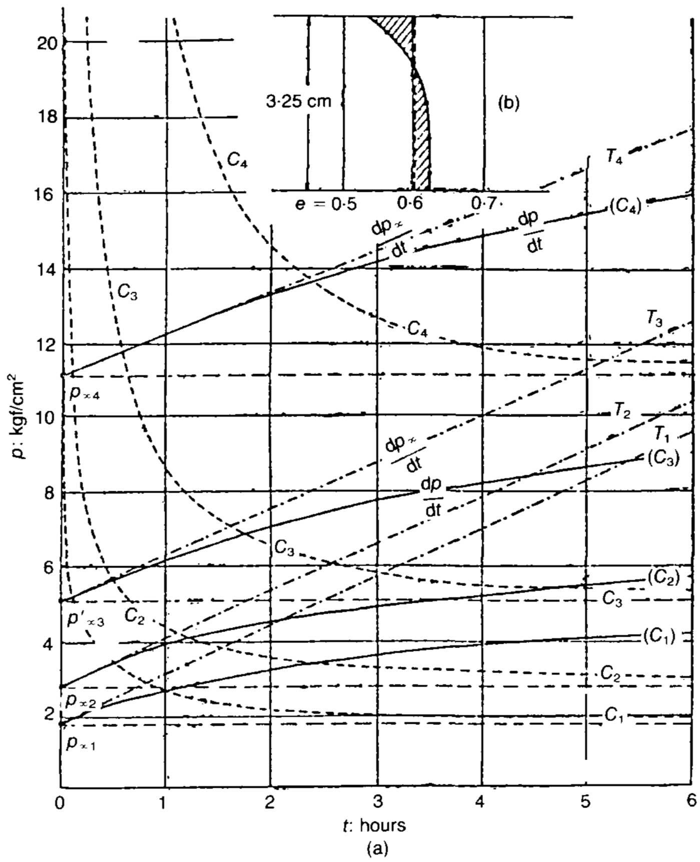


Fig. 6. (a) Applied vertical stress p and rate of change of stress with time dp/dt as functions of time, measured during pore-water pressure equalization in a clay sample with no overall volume change. (b) Void ratio as a function of depth as measured at the end of the test (fig. 4 in Terzaghi, 1923)

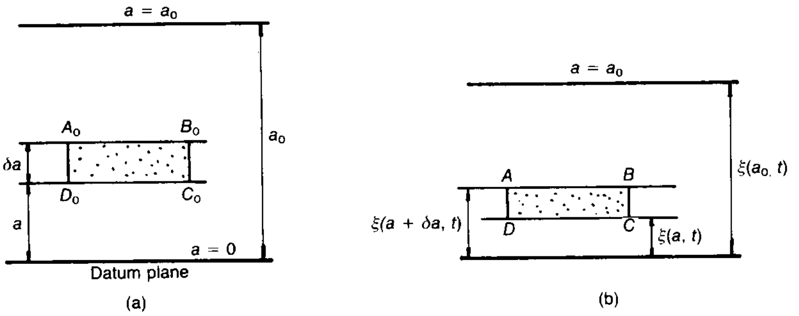


Fig. 7. Material (Lagrangian) and fixed (Eulerian) co-ordinate systems a and ξ (from Gibson et al., 1967).[1] (a) Initial configuration, t = 0; (b) current configuration at time t (note B refers)

reference using a Lagrangian or material co-ordinate system, a plane is identified by its initial distance from the datum: thus the Lagrangian or material co-ordinate does not change. In the Lagrangian or material co-ordinate system, the plane AB remains at $(a + \delta a)$, and the plane CD remains at a. The element $ABCD$ always contains the same volume of soil solids, and the reduced material co-ordinate z_r is defined as the volume of solids (per unit area) contained within the volume bounded by the datum plane and a parallel plane through the actual Lagrangian co-ordinate point (McNabb, 1960)[7]

$$z_r = \int_0^a \frac{da}{1 + e_0(a)}$$

where $e_0(a)$ is the void ratio at $t = 0$, which will in general be a function of position within the clay layer.

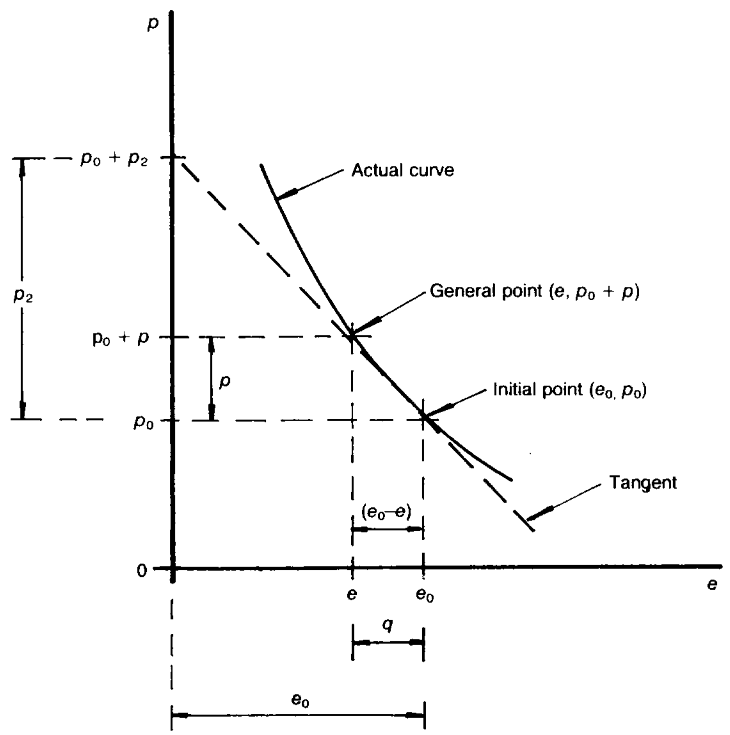


Fig. 8. Geometry used in the derivation of Terzaghi's Equation (1) re-drawn from Fig. 3(a) (note C refers)

62. The use of Lagrangian co-ordinate systems might be viewed as an unnecessary complication if the strains are small. However, in large strain problems, the introduction of the Lagrangian co-ordinate a allows a moving boundary to be treated as a fixed boundary, which often leads to a useful degree of simplification. As a further step, the subsequent use of the reduced Lagrangian co-ordinate z_r enables problems in which the initial void ratio $e_0(a)$ varies with depth to be analysed as easily as problems in which the initial void ratio $e_0(a)$ is uniform (Gibson *et al.* 1967;[1] Gibson *et al.* 1995[3]).

63. Quantities related to the reduced co-ordinate system are denoted in this translation by a subscript r: these subscripts were not used by Terzaghi in the original 1923 paper.

64. C. Equation (1) is derived from the geometry of Fig. 3(a), the relevant parts of which have been re-drawn as Fig. 8. By similar triangles

$$(e_0 - e)/p = e_0/p_2$$

Hence

$$q = (e - e_0) = -(e_0/p_2) \times p = -a_v p$$

where q is the change in volume, per unit volume of the clay.

65. D. a_v is defined as $-de/dp$. The negative sign is needed to make a_v positive, because an increase in pressure (dp +ive) causes a reduction in void ratio (de −ive), so that dp/de is in itself inherently negative.

66. For a clay element of reduced thickness δh_r and a uniform void ratio which changes from e to $(e - de)$, the initial total volume $V_T = \delta h_r(1 + e)$, and the reduction in total volume is $dV_T = \delta h_r de$. Since $a_v = -de/dp$

$$a_v = (1/\delta h_r) \times (-dV_T/dp), \text{ or}$$
$$a_v = (1/\delta h_r) \times (dV_T/dp) \tag{15}$$

taking compression as positive.

67. The one-dimensional modulus E_0' (= $1/m_v$, where m_v is the compressibility) is defined as the ratio of an increment of vertical effective stress dp to the increment of vertical compressive strain it causes, $d\varepsilon_v$

$$E_0' = dp/d\varepsilon_v$$

68. In one-dimensional compression, $d\varepsilon_v = dV_T/V_T$, so that

$$E_0' = V_T\, dp/dV_T \tag{16}$$

69. Comparing Equations (15) and (16)

$$\mathrm{d}V_T/\mathrm{d}p = \delta h_r a_v = V_T/E'_0$$
$$= \mathrm{d}h_r(1+e)/E'_0\text{, hence}$$
$$a_v = (1+e)/E'_0 = (1+e)m_v$$

70. E. Having introduced permeability in terms of a head difference, Terzaghi now proceeds with the analysis in terms of a pressure difference. In general, Terzaghi seems to use the terms 'head' and 'pressure' interchangeably. This is probably a consequence of the practice, sloppy but apparently common in Europe at that time, of omitting the unit weight of water γ_w from engineering formulae because its numerical value in the cgs (centimetre–gram–second) system is unity (Bjerrum *et al.*, 1960).[8] It is clear, however, from the text immediately preceding Equation (2) that the units of p and w are the same, so that the version of Darcy's law given in Equation (2) is in terms of a pressure gradient $\mathrm{d}w/\mathrm{d}z_r$, rather than a head gradient. The negative sign in Equation (2) is needed because the flow Q is in the direction of decreasing w; that is, Q and $\mathrm{d}w/\mathrm{d}z_r$ are of opposite sign.

71. It might also be noted that, if the flowrate Q in Equations (2) and (3) were to be expressed in terms of a seepage velocity, the appropriate seepage velocity would be the velocity of the pore water relative to the soil solids, $v_w - v_s$ (Znidarcic and Schiffman, 1982).[2] The essence of a consolidation problem is that the soil skeleton is not rigid; that is, that the velocity of the solids v_s is non-zero. This point is discussed by Gibson (1967)[1] and Gibson *et al.* (1995).[3]

72. Terzaghi does not obviously distinguish between pore-water pressure u_w and excess pore-water pressure u_e. The excess pore-water pressure may be defined as the pressure over and above hydrostatic, or the pressure over and above some other equilibrium condition (Gibson *et al.*, 1989).[9] Using the first definition, differences in excess pore-water pressure are indicated by differences in hydraulic total head; that is, the standpipe rise above some arbitrary datum level (Fig. 9). It is a difference in excess pore-water pressure (i.e. a difference in pressure over and above hydrostatic) which causes pore-water flow, and which should therefore feature in Darcy's law (Equation (2)). From the definition of w

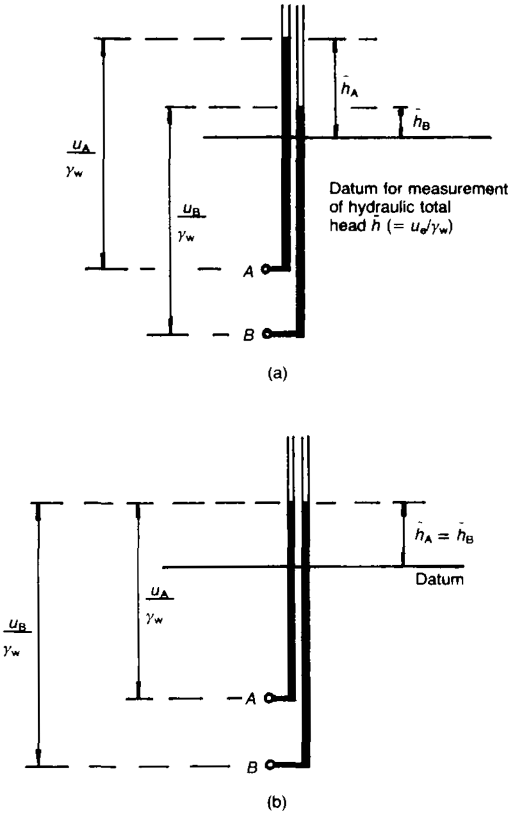


Fig. 9. Illustration of excess and absolute pore-water pressures. (a) Pore-water pressures at A and B are the same, but excess pore-water pressures u_e are different and flow occurs from A to B; (b) pore-water pressures at A and B are different due to hydrostatic variation with depth, but excess pore-water pressures are the same so there is no flow (note E refers)

($w = p_1 - p$) given in the text which precedes Equation (2), w is an excess pore-water pressure expressed with reference to the initial equilibrium condition, which is probably in this case hydrostatic. Terzaghi's use of the term 'hydrostatic pressure' for w is therefore inappropriate: if the pore-water pressures were merely hydrostatic, there would be no flow from a sample with an impermeable base.

73. F. Equation (3) is derived as follows. For an elemental volume of unit cross-sectional area and reduced thickness $\mathrm{d}z_r$, the flowrate in is Q, and the flowrate out is $Q + \mathrm{d}Q$. The net outward flowrate is therefore $\mathrm{d}Q$, which must be equal to the rate of change of volume of the element with time. $\mathrm{d}q$ is the change in volume per unit volume of clay, hence the rate of change of volume with time is $\mathrm{d}q/\mathrm{d}t \times$ volume, or

$$\mathrm{d}Q = -(\mathrm{d}q/\mathrm{d}t) \times \mathrm{d}z_r$$

(the negative sign is needed because a positive outward flowrate corresponds to a reduction in volume; that is, $\mathrm{d}q$ −ive).

74. In the limit as $\mathrm{d}z_r \to 0$

$$\partial Q/\partial z_r = -\partial q/\partial t$$

75. From Equation (1), $\partial q/\partial t = -a_v\,\partial p/\partial t$ and, as $w = p_1 - p$, $-\partial p/\partial t = \partial w/\partial t$. Hence

$$\partial Q/\partial z_r = a_v\,\partial p/\partial t = -a_v\,\partial w/\partial t \qquad (3)$$

76. It is pointed out by Gibson *et al.* (1995)[3] that the derivation of Equation (3) in this way involves a degree of legerdemain concerning absolute and excess pore-water pressures and boundary conditions, and is not necessarily applicable in a general case. However, Equation (4) is correct for the particular circumstance being analysed by Terzaghi, viz. a clay layer with an impermeable base.

77. G. Strictly, the term $(2/n\pi)$ should be inside the summation sign, rather than outside. In comparing Terzaghi's 1923 analysis with the small strain formulation usually presented in modern soil mechanics textbooks, the following points should be borne in mind.

(*a*) The actual thickness h of a sample of reduced thickness h_r and uniform void ratio e is given by $h = h_r \times (1 + e)$.

(*b*) The permeability coefficient k_r used by Terzaghi is defined as the Darcy flow velocity under an hydraulic gradient of $1/(1 + e)$. This is different from the conventional definition, which is the Darcy flow velocity v $(= v_f - v_s$: see note E) under a total head gradient or hydraulic gradient i of unity (that is, $k = v/i$ with $i = 1$). Using the conventional definition, the Darcy seepage velocity under a total head gradient of $1/(1 + e)$ is $v = k/(1 + e)$. k_r must also be divided by the unit weight of water γ_w to account for the use of a pressure gradient (rather than a head gradient) in Equation (2). Thus, for a soil element of uniform void ratio e, $k_r = k/\{\gamma_w(1 + e)\}$.

(*c*) From note D, $a_v = (1 + e)/E'_0$. Substituting $k_r = k/\{\gamma_w(1 + e)\}$, $a_v = (1 + e)/E'_0$, and $h = h_r(1 + e)$, the expression

$$(k_r/a_v)(t/h_r^2)$$

reduces to the more familiar time factor

$$T = c_v t/h^2$$

where c_v is the consolidation coefficient, $c_v = kE'_0/\gamma_w$. Also, z_r is measured from the base of the sample (Fig. 3(b)), rather than from the surface. If the actual space ordinate z is measured from the surface

$$z = (1 - e)(h_r - z_r)$$

(assuming a uniform void ratio), and it can be shown (using standard trigonometrical identities) that

$$\sin\{[n\pi(z_r + h_r)]/2h_r\} = \sin(n\pi z/2h)$$

78. Thus Equation (5) is effectively identical to the solution for the excess pore-water pressure presented in modern texts (e.g. Whitlow, 1990;[10] Craig, 1992[11])

$$u_e = \sum_{n=1}^{\infty} \frac{2p_1}{n\pi} \times \left[e^{-n^2\pi^2 T/4} \sin\left(\frac{n\pi z}{2h}\right)(1 - \cos n\pi) \right]$$

79. H. The lines giving the relationships between effective pressure and depth at given times are termed isochrones. The approximation suggested by Terzaghi in Fig. 3(c) is that the isochrones are straight lines, and his assertion (following Equation (11)) that the coincidence results from the rigorous and approximate analyses is perhaps open to debate. The now well-known assumption that the isochrones are parabolic—which for one-dimensional consolidation is actually a much closer approximation to their true shape—appears in Terzaghi and Frohlich (1936).[12]

80. In Fig. 3(c), z_r is measured positive downward from the surface of the soil sample. This is different from Fig. 3(b), in which z_r is measured from the bottom up. In the ensuing analysis, the second sign convention, in which z_r is measured positive downward from the surface, is used.

81. I. Equation (6) is obtained by integrating Equation (3) between limits of $Q = 0$ at $z_r = z_{r1}$, and general values of $Q = -Q$ at $z_r = z_r$. The negative sign denotes upward flow; that is, in the z-negative direction.

82. J. The steps involved in the derivation of Equation (9) are as follows. Integrating Equation (2) ($i = Q/k_r = -\mathrm{d}w/\mathrm{d}z_r$) between limits of $w = w_1$ at $z_r = z_{r1}$, and $w = 0$ at $z_r = 0$

$$\int_0^{z_{r1}} \frac{Q}{k_r}\,\mathrm{d}z_r = \int_0^{w_1} -\,\mathrm{d}w \qquad (17)$$

83. Substituting for $\mathrm{d}p/\mathrm{d}t$ from Equation (8) into Equation (6), and substituting

for Q from Equation (6) into Equation (17) (Equation (17) being the integrated form of Equation (2)), with limits of $w = w_1$ at $z_r = z_{r1}$ and $w = 0$ at $z = 0$

$$\int_0^{w_1} -\,\mathrm{d}w = \int_0^{z_{r1}} \frac{a_v}{k_r} \times \left(\int_{z_r}^{z_{r1}} \frac{-p_1 z_r}{z_{r1}^2} \frac{\mathrm{d}z_{r1}}{\mathrm{d}t}\, \mathrm{d}z_r \right) \mathrm{d}z_r$$

$$\Rightarrow w_1 = \frac{a_v}{k_r} \int_0^{z_{r1}} \times \left(\frac{p_1}{2z_{r1}^2} (z_{r1}^2 - z_r^2) \frac{\mathrm{d}z_{r1}}{\mathrm{d}t} \right) \mathrm{d}z_r$$

$$\Rightarrow w_1 = \frac{a_v p_1 z_{r1}}{3k_r} \frac{\mathrm{d}z_{r1}}{\mathrm{d}t} \tag{18}$$

84. Integrating Equation (18) between limits of $z_{r1} = 0$ at $t = 0$ and general values $z_{r1} = z_{r1}$ at $t = t$ and noting that $w_1 = p_1$

$$t = \frac{a_v}{3k_r} \frac{z_{r1}^2}{2}$$

or

$$z_{r1} = \sqrt{\frac{k_r}{a_v} 6t} \tag{9}$$

85. Substituting $k_r = k/\{(1+e)\gamma_w\}$, $a_v = (1+e)/E'_0$ and $z_1 = (1+e)z_{r1}$

$$z_1 = \sqrt{\frac{6kE'_0}{\gamma_w} t} = \sqrt{6c_v t}$$

86. It is interesting to note that Terzaghi does not use the straight isochrone to calculate the pore-water pressure gradient i (Equation (2)). From Fig. 3(c), the pore-water pressure gradient according to the approximate straight isochrone is

$$i = -\frac{\mathrm{d}w}{\mathrm{d}z_r} = \frac{p_1}{z_{r1}} \tag{19}$$

87. Substitution of Equation (19) into Equation (2) gives

$$Q = \frac{k_r p_1}{z_1}$$

88. Substituting this expression for Q into Equation (6)

$$k_r \frac{p_1}{z_{r1}} = a_v \int_0^{z_{r1}} \frac{\mathrm{d}p}{\mathrm{d}t}\, \mathrm{d}z_r$$

and then, substituting for $\mathrm{d}p/\mathrm{d}t$ from Equation (8)

$$k_r \frac{p_1}{z_{r1}} = a_v \int_0^{z_{r1}} \left(p_1 \frac{z_r}{z_{r1}^2} \frac{\mathrm{d}z_{r1}}{\mathrm{d}t} \right) \mathrm{d}z_r$$

$$\Rightarrow \frac{k_r}{a_v} = \frac{z_{r1}}{2} \frac{\mathrm{d}z_{r1}}{\mathrm{d}t}$$

89. Integrating again

$$\int_0^t \frac{k_r}{a_v}\, \mathrm{d}t = \int_0^{z_{r1}} \frac{z_{r1}}{2}\, \mathrm{d}z_{r1}$$

$$\Rightarrow z_{r1} = \sqrt{\frac{4k_r t}{a_v}}$$

or

$$z_1 = \sqrt{4c_v t}$$

90. If the isochrones are assumed to be parabolic, and the shape of the isochrone is used to determine the hydraulic gradient

$$z_1 = \sqrt{12c_v t}$$

(e.g. Bolton, 1991).[13]

91. K. In Equation (11), p is the increase in effective stress at the base of the sample. Equation (11) may be rewritten as

$$p = p_1 \times [1 - \exp(\tfrac{1}{2} - 3T)]$$

where $T = c_v t/h^2$

92. This may be compared with

$$p = p_1 \times [1 - \exp(\tfrac{1}{2} - 2T)]$$

if the straight isochrones are used to compute the hydraulic gradient, and

$$p = p_1 \times [1 - \exp(\tfrac{1}{4} - 3T)]$$

if the parabolic isochrone approximation is used (e.g. Bolton, 1991).[13]

93. L. From the discussion centred on Fig. 6(b), it is apparent that some consolidation is allowed to occur before deformation is prevented. This results in some dissipation of excess pore-water pressure near the drainage boundary (Fig. 10).

94. M. Actually, the shaded ares $\mathrm{d}F_1$ and $\mathrm{d}F_2$ represent $1/a_v \times$ the incremental change in volume of the lower and upper parts of the sample, respectively. By rearranging and integrating Equation (15)

$$\Delta(\mathrm{vol}) = a_v \int \Delta p\, \mathrm{d}z_r, \text{ or}$$

$$\Delta(\mathrm{vol}) = a_{v1}\, \mathrm{d}F_1 = a_{v2}\, \mathrm{d}F_2, \text{ hence}$$

$$\frac{\mathrm{d}F_1}{\mathrm{d}F_2} = \frac{a_{v2}}{a_{v1}}$$

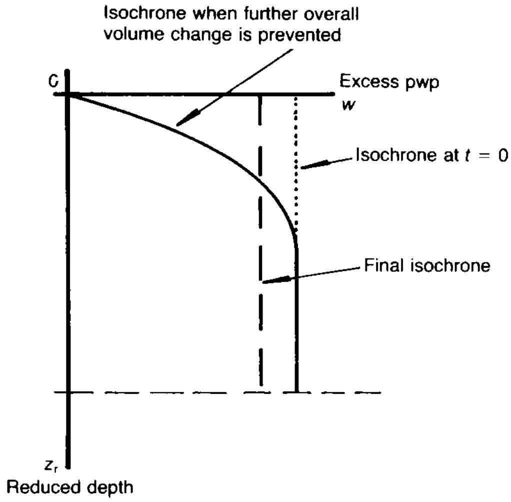


Fig. 10. Variations in pore-water pressure distribution with time during constant volume oedometer test (note L refers)

95. N. The expression for s is obtained as follows. From the geometry of Fig. 4(b) (redrawn for clarity as Fig. 11) for the second phase of pore-water pressure equilibration

$$A_1 = \tfrac{1}{2} s z'_r \tag{20}$$

$$A_2 = \tfrac{1}{2}[p'_1 - p] z''_r \tag{21}$$

and, using the same reasoning as above

$$\frac{A_1}{A_2} = \frac{a_{v2}}{a_{v1}} \tag{22}$$

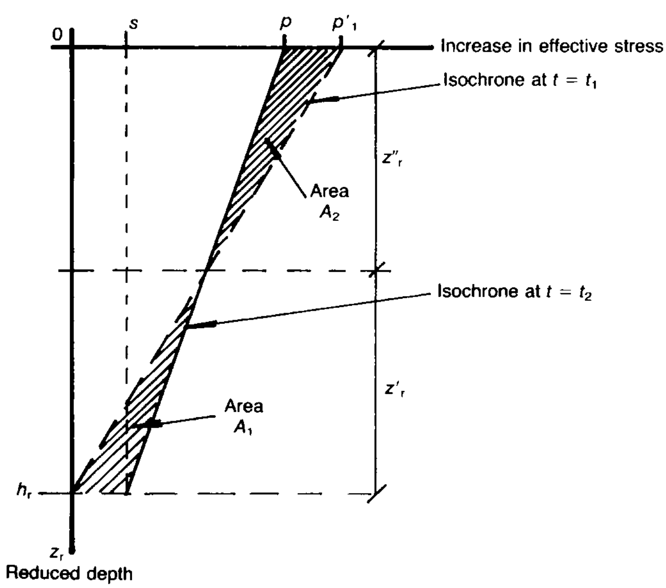
96. Also

$$z'_r \times \frac{p'_1}{h_r} = s + \left[\frac{z'_r \times (p - s)}{h_r}\right] \tag{23}$$

where

$$z''_r = h_r - z'_r \tag{24}$$

Fig. 11. Geometry used in the derivation of Terzaghi's expression for s, re-drawn from Fig. 4(b) (note N refers)



97. Using Equations (23) and (24) to express z'_r and z''_r in terms of p, p'_1, s and h_r, substituting these expressions into Equations (20) and (21), and substituting for A_1 and A_2 in Equation (22), we obtain

$$s = (p'_1 - p) \times \sqrt{\frac{a_{v2}}{a_{v1}}} \tag{25}$$

98. O. The steps used to obtain this expression for $\mathrm{d}p/\mathrm{d}t$ are as follows. From the geometry of Fig. 4

$$\mathrm{d}F_2 = \tfrac{1}{2} z''_r \,\mathrm{d}p$$

99. From Equations (23) and (25), it may be shown that, for the second phase of pore-water pressure equilibration

$$z''_r = \frac{h_r}{1 + \sqrt{\dfrac{a_{v2}}{a_{v1}}}} = \text{constant}$$

(assuming that a_{v2}/a_{v1} = constant), so that the rate of increase in volume of the upper part of the sample is

$$a_{v2} \times \frac{\mathrm{d}F_2}{\mathrm{d}t} = a_{v2} \times \frac{1}{2} z''_r \frac{\mathrm{d}p}{\mathrm{d}t} \tag{26}$$

100. The rate of increase in volume of the upper part of the sample is also given by

$$\frac{\mathrm{d}V_T}{\mathrm{d}t} = k_r i \tag{27}$$

per unit area, where the pore-water pressure gradient i is given by

$$i = \frac{(p - s)}{h_r} \tag{28}$$

101. Substituting for s from Equation (25) into Equation (28)

$$i = \frac{1}{h_r} \times \left[p\left(1 + \sqrt{\frac{a_{v2}}{a_{v1}}}\right) - p'_1 \sqrt{\frac{a_{v2}}{a_{v1}}}\right] \tag{29}$$

102. Now when $p = p_\infty$

$$s = p_\infty = (p'_1 - p_\infty) \times \sqrt{\frac{a_{v2}}{a_{v1}}}$$

from Equation (25). Hence

$$p'_1 = p_\infty \times \left(1 + \sqrt{\frac{a_{v1}}{a_{v2}}}\right) \tag{30}$$

and, substituting for p'_1 in Equation (29) from Equation (30)

$$i = \frac{1}{h_r} \times \left(1 + \sqrt{\frac{a_{v2}}{a_{v1}}}\right) \times (p - p_\infty) \tag{31}$$

103. Substituting Equation (31) into Equation (27), and setting the result equal to Equation (26), we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{2k_r\left(1 + \sqrt{\frac{a_{v2}}{a_{v1}}}\right)^2}{h_r^2 a_{v2}}(p - p_\infty)$$

$$= \frac{2k_r\left(1 + \sqrt{\frac{a_{v1}}{a_{v2}}}\right)^2}{h_r^2 a_{v1}}(p - p_\infty)$$

This expression applies to the second phase of pore-water pressure equilibration, indicated in Figs 4(b) and 11.

104. P. From Equation (13), a graph of $\mathrm{d}p/\mathrm{d}t$ against $(p - p_\infty)$ should be a straight line, provided that k_r, a_{v1} and a_{v2} are constant. Terzaghi ascribes the curvature of the graphs of $\mathrm{d}p/\mathrm{d}t$ against $(p - p_\infty)$ shown in Fig. 6 to the variation in a_{v1} and a_{v2} with effective stress. He also introduces at this stage the notation a'_{v1} and a'_{v2} to describe the general (variable) values of the coefficient or compressibility in the lower and upper parts of the sample, respectively, and begins to reserve the symbols a_{v1} and a_{v2} for the equilibrium values when $p = p_\infty$. He returns to this point in the last paragraph of his paper, where he states that if the variation in a'_{v1} and a'_{v2} with effective stress is taken into account, the calculated values of $\mathrm{d}p/\mathrm{d}t$ agree with the experimental curves.

References

1. Gibson R. E. *et al.* The theory of one-dimensional consolidation of saturated clays. I. Finite non-linear consolidation of thin homogeneous layers. *Géotechnique*, 1967, **17**, No. 3, 261–273.
2. Znidarcic D. and Schiffman R. L. On Terzaghi's concept of consolidation. *Géotechnique*, 1982, **32**, No. 4, 387–389.
3. Gibson R. E. *et al.* Some aspects of one-dimensional consolidation and contaminant transport in wastes. *Proceedings of an international symposium on compression and consolidation of clayey soils: IS—Hiroshima, 1995.*
4. Skempton A. W. Significance of Terzaghi's concept of effective stress. *From theory to practice in soil mechanics.* Wiley, New York, 1960, pp. 42–53.
5. Terzaghi K. Die Theorie der hydrodynamischen Spannungserscheinungen und ihr erdbautechnisches. Anwendungsgebiet. *Proc. Int. Cong. App. Mech. (Delft)*, 1924, 288–294.
6. Terzaghi K. Die physikalischen Grundlagen des technisch-geologischen Gutachtens-Zeitschr. *Österr. Ing. u. Arch.-Verein*, 1921, Sept., Table V.
7. McNabb A. A mathematical treatment of one-dimensional soil consolidation. *Q. Appl. Math.*, 1960, **17**, 337–347.
8. Bjerrum L. *et al. From theory to practice in soil mechanics.* Wiley, New York, 1960.
9. Gibson R. E. *et al.* On two definitions of excess pore water pressure. *Géotechnique*, 1989, **39**, No. 1, 169–171.
10. Whitlow R. *Basic soil mechanics* (2nd edn). Longman, Harlow, 1990.
11. Craig R. F. *Soil mechanics* (5th edn). Chapman & Hall, London, 1992.
12. Terzaghi K. von and Frohlich O. K. *Theorie der Setzung von Tonschichten: eine Einführung in die analytische Tonmechanik.* Franz Deuticke, Leipzig, 1936.
13. Bolton M. D. *A guide to soil mechanics.* M. D. & K. Bolton, Cambridge, 1991.