	α	-2β	γ
adv	0	$+\frac{a}{h}$	$-\frac{a}{h}$
$\oplus$			
diff	$-\frac{\varkappa}{h^2}$	$\frac{2\varkappa}{h^2}$	$-\frac{\varkappa}{h^2}$

$$\alpha = -\frac{\varkappa}{h^2} \qquad \text{—} \qquad \gamma = -\frac{a}{h} - \frac{\varkappa}{h^2}$$

$$\zeta_+ = 1$$

$$\zeta_- = \frac{\gamma}{\alpha} = \frac{+\frac{a}{h} + \frac{\varkappa}{h^2}}{+\frac{\varkappa}{h^2}} \cdot \frac{\frac{h^2}{\varkappa}}{\frac{h^2}{\varkappa}} = \frac{ah}{\varkappa} + 1$$

$$= \underline{2\alpha_h + 1}$$

$$u_A = C_+ \zeta_+^A + C_- \zeta_-^A$$

$$= C_+ + C_- (1 + 2\alpha_h)^A \quad \text{monotone increasing}$$

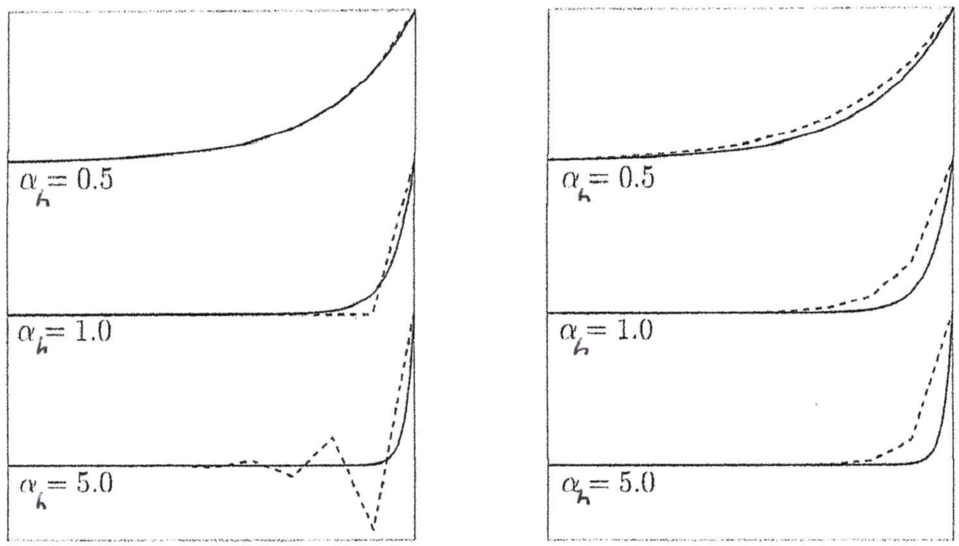
Artificial Diffusion interp of upwind diff's.

$$\Longleftrightarrow$$

u_A :

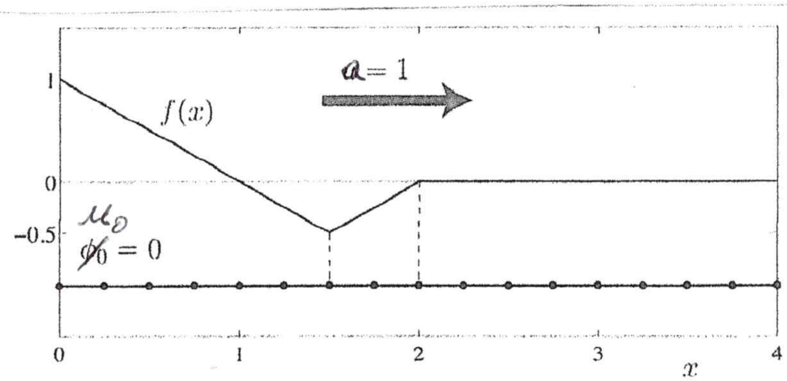
$$\overbrace{\frac{a}{h}(u_A - u_{A-1})}^{\text{bwd. diff.}} = \frac{\varkappa}{h^2} \overbrace{(u_{A+1} - 2u_A + u_{A-1})}^{\text{cent.}}$$

write as C.D. + (?).

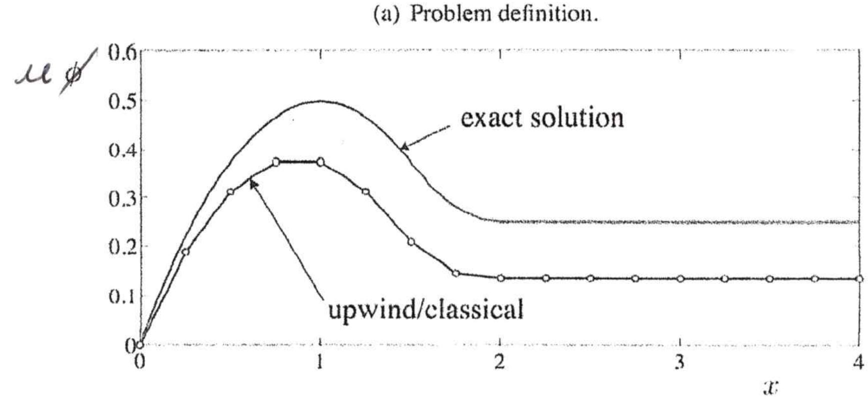


(a) Central differences (overly anti-diffusive). (b) Upwind differences (overly diffusive).

Figure 2.3: Steady one-dimensional advection-diffusion.



(a) Problem definition.



(b) Comparison of results.

Figure 2.5: Pure advection with a non-constant source term.
($\Rightarrow$ no diffusion)

$$\frac{a}{h}(u_A - u_{A-1}) = \frac{a}{2h}(u_{A+1} - u_{A-1}) + ?$$

$$\frac{a}{h}(u_A - u_{A-1}) - \frac{a}{2h}(u_{A+1} - u_{A-1}) = ?$$

$$-\left(\frac{a}{h} - \frac{a}{2h}\right)u_{A-1}$$

$$+\frac{a}{2h}$$

$$+\left(-\frac{a}{2h} + \cancel{\frac{a}{h}}\right)u_{A+1}$$

$$? \quad -\frac{a}{2h}\underbrace{\left(u_{A+1} - 2u_A + u_{A-1}\right)}\frac{h^2}{h^2}$$

$$-\frac{ah}{2}\left(\frac{\quad\text{"}\quad}{h^2}\right)$$

Artificial diffusion of upwind differences

$$\frac{a}{2h}(u_{A+1} - u_{A-1}) = \left(\varkappa + \frac{ah}{2}\right)\frac{(u_{A+1} - 2u_A + u_{A-1})}{h^2}$$

$$\boxed{\varkappa\left(1 + \frac{ah}{2\varkappa}\right) = \varkappa(1 + \alpha_h)}$$

$\frac{L^2}{T}$ $\quad \frac{L}{T}\cdot h$

num. or art. diffusivity

CD $\quad \varkappa(1 + 0)$

UD $\quad \varkappa(1 + \alpha_h)$

CD "under diffuse"

UD "over "

First step on a journey to a v. good FEM:

Sharpen the art. diff., will help us get there but it's not enough.

$$\kappa(1+\alpha_h^{\delta}) = \kappa + \tilde{\kappa}.$$

$\tilde{\kappa}$ ← art. diff.

float.

As we will see.

$$\tilde{\alpha}_h \overset{def}{=} \frac{ah}{2(\kappa+\tilde{\kappa})}$$

X

C.D $\quad \zeta_+ = 1, \; \zeta_- = \dfrac{1+\alpha_h}{1-\alpha_h}$

C.D + art diff $\quad \zeta_- = \dfrac{1+\tilde{\alpha}_h}{1-\tilde{\alpha}_h}$

$$u_A = \frac{1 - \left(\frac{1+\tilde{\alpha}_h}{1-\tilde{\alpha}_h}\right)^A}{1 - \left(\frac{1+\tilde{\alpha}_h}{1-\tilde{\alpha}_h}\right)^N}$$

+ art. diff.

what is $\tilde{\alpha}_h$?

ask that $u_A = u(x_A)$

$$u(\overset{x}{x_A}) = \frac{1 - \exp Pe \frac{\overset{x}{x_A}}{L}}{1 - \exp Pe.}$$

$x_A = A \cdot h$

$x_N = N \cdot h = L$

12.

$$\exp\left(Pe\,\frac{x_A}{L}\right) = \exp\left(\overset{\cancel{\frac{aL}{\alpha}}}{Pe}\,\frac{Ah}{\cancel{L}}\right)$$

$$= \exp\left(\underbrace{\frac{ah}{\alpha}}_{2\alpha_h}\,A\right)$$

$$= \left(\exp 2\alpha_h\right)^{\cancel{A}}$$

$$\left(\frac{1+\tilde{\alpha}_h}{1-\tilde{\alpha}_h}\right)^{\cancel{A}} \quad \text{set equal} \Rightarrow u_A = u(x_A)$$

$$(1+\tilde{\alpha}_h) = (1-\tilde{\alpha}_h)\,e^{2\alpha_h}$$

?

$$(1+e^{2\alpha_h})\tilde{\alpha}_h = e^{2\alpha_h}-1$$

$$\tilde{\alpha}_h = \frac{(e^{2\alpha_h}-1)}{(e^{2\alpha_h}+1)}\cdot\frac{e^{-\alpha_h}}{e^{-\alpha_h}}$$

def

$$\frac{ah}{2(\alpha+\tilde{\alpha})} = \frac{e^{\alpha_h}-e^{-\alpha_h}}{e^{\alpha_h}+e^{-\alpha_h}} = \tanh\alpha_h$$

?

$$\frac{2(\alpha+\tilde{\alpha})}{ah} = \coth\alpha_h$$

$$\tilde{\alpha} = \frac{ah}{2}\left(\coth\alpha_h\right) - \cancel{\tilde{\alpha}}\,\alpha$$

$$= \frac{ah}{2}\left(\coth\alpha_h - \underbrace{\frac{2\alpha}{ah}}_{\frac{1}{\alpha_h}}\right)$$

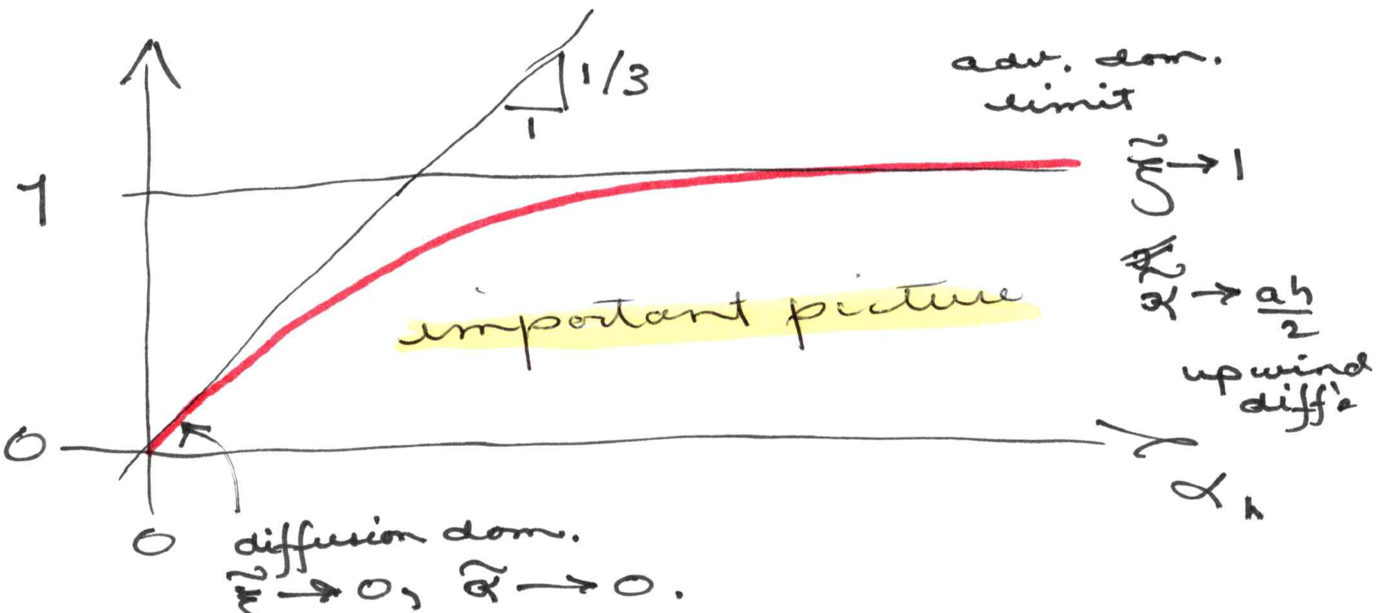
$$= \frac{ah}{2}\left(\coth\alpha_h - \frac{1}{\alpha_h}\right)$$

$$\tilde{\xi}(\alpha_h) = \coth \alpha_h - \frac{1}{\alpha_h}$$

$$\boxed{\tilde{\alpha} = \tilde{\xi} \cdot \frac{ah}{2}}$$

$$\alpha_h \to 0, \quad \tilde{\xi} \to \left(\cancel{\frac{1}{\alpha_h}} + \frac{\alpha_h}{3} + \dots \right) - \cancel{\frac{1}{\alpha_h}}$$

$$\alpha_h \to \infty, \quad \tilde{\xi} \to 1. \qquad 0$$



truth (when $f=0$) is somewhere in between C.D. & U.D.

X

check dim $\quad 1 - \left(\frac{1+\tilde{\alpha}_h}{1-\tilde{\alpha}_h}\right)^N = 1 - \exp \underbrace{P_e}_{2\alpha_h N}$

$$\left(\frac{1+\tilde{\alpha}_h}{1-\tilde{\alpha}_h}\right)^A = (\exp 2\alpha_h)^A .$$

set $A = N$. ✓

This is where the FDM died.

Remarks:

1) This so-called exact art. diff. meth. is a C.D. method for the modified eq.

assumed const

$$\widetilde{a u_{,x}} \cancel{\#} = (\widetilde{\kappa + \tilde{\kappa}}) u_{,xx} + f$$ ✓

2) Suppose $a < 0$, just replace a with $|a|$ in $\alpha_h = \dfrac{|a| h}{2\kappa}$.

"Exact" when $f = 0$.

Artificial diffusion method

$$\frac{a}{2h}\left(u_{A+1} - u_{A-1}\right) = (\kappa + \tilde{\kappa})\left(\frac{u_{A+1} - 2u_A + u_{A-1}}{h^2}\right)$$

where $\tilde{\kappa} = \dfrac{|a| h}{2} \tilde{\xi}(\alpha_h)$

$$\tilde{\xi}(\alpha_h) = \coth \alpha_h - \frac{1}{\alpha_h}$$

$$\alpha_h = \frac{|a| h}{2\kappa}$$

but adding the source term in does not provide accurate results when advection dominates, as seen in the example on P. 9.