In [1]:

```
import numpy as np
from numpy import tanh
import sys

import matplotlib.pyplot as plt

plt.style.use('paper.mplstyle')
np.set_printoptions(threshold=200, linewidth=200)
plt.figure(figsize=(12,6))

def XI_KB() :
    global METHOD, A, H, KAPPA
    if METHOD == "GALERKIN" :
        return 0
    elif METHOD == "SUPG" :
        if KAPPA != 0 :
            alpha_h = A * H / 2 / KAPPA
            return 1/tanh(alpha_h) - 1/alpha_h
        else:
            return 1
    elif METHOD == "UPWIND" :
        return 1
    else :
        fail(f"XI(): Unknown method {METHOD}")

def XI_F( ) :
    global METHOD, A, H, KAPPA
    if METHOD == "GALERKIN" :
        return 0
    elif METHOD == "SUPG" :
        if KAPPA != 0 :
            alpha_h = A * H / 2 / KAPPA
            return 1/tanh(alpha_h) - 1/alpha_h
        else:
            return 1
    elif METHOD == "UPWIND" :
        return 0
    else :
        fail(f"XI(): Unknown method {METHOD}")


def Usolve() :
    global KAPPA, H, A, N, F, G0, G1
    if N <= 1 :
        return np.array( [ G0, G1 ] )

    Sdif = np.array( [ -1, 2, -1 ] )
    Sadv = np.array( [ -1, 0, 1] )

    K  = np.zeros( [ N-1, N-1 ] )
    B0 = np.zeros( N-1 )
    BN = np.zeros( N-1 )

    xi = XI_KB()
```

```
    B0[0]   = - ( KAPPA + A*H/2*xi )/H - A/2
    BN[N-2] = - ( KAPPA + A*H/2*xi )/H + A/2

    K[0,0]     = 2*( KAPPA + A*H/2*xi )/H
    K[N-2,N-2] = 2*( KAPPA + A*H/2*xi )/H
    if N > 2 :
        K[0,1]     = - ( KAPPA + A*H/2*xi )/H + A/2
        K[N-2,N-3] = - ( KAPPA + A*H/2*xi )/H - A/2

    for i in range( 1, N-2 ) :
        [ K[i,i-1],K[i,i],K[i,i+1] ] = ( KAPPA/H + A/2*xi ) * Sdif + A/2 * Sadv

    U = np.linalg.solve( K,  F - B0 * G0 - BN * G1 )
    U = np.append(U,G1)
    U = np.insert(U,G0,0)

    return U
```

```
<Figure size 864x432 with 0 Axes>
```

In [2]:

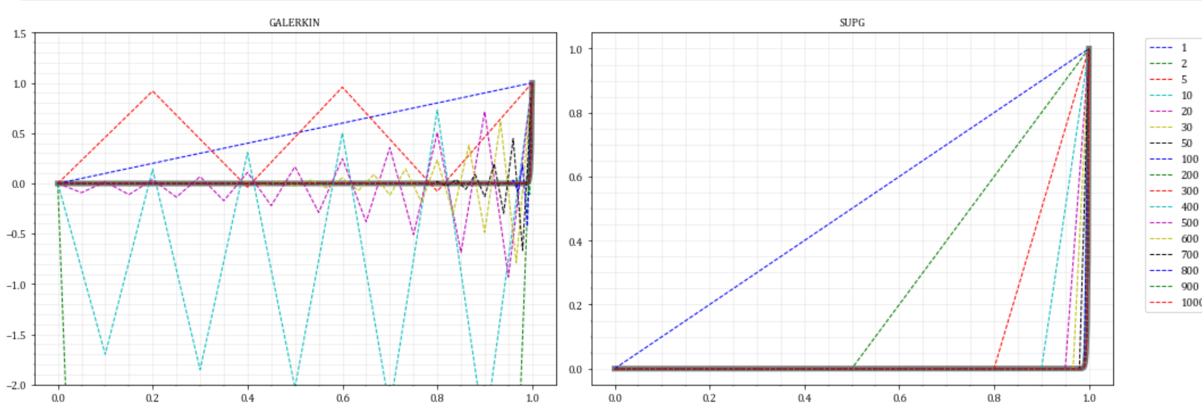
```
from scipy.stats import linregress

# GALERKIN or SUPG

KAPPA = 1
G0 = 0
G1 = 1
A=500

NN = [ 1, 2, 5, 10, 20, 30, 50, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000 ]
#NN = [2,10]
# X and solution vectors
XX = []
UUgal = []
UUsupg = []
# Errors
L2Ngal = []
L2Nsupg = []
H1SNgal = []
H1SNsupg = []
H1Ngal = []
H1Nsupg = []

# NUMERICAL SOLUTIONS
X_EXACT = np.linspace( 0, 1, 500 )
U_EXACT = ( np.exp( A*X_EXACT) - 1 ) / ( np.exp(A) - 1)
#
#
#
def _exact(X) :
    global A
    ret = []
    for x in X :
        ret.append( ( np.exp( A*x) - 1 ) / ( np.exp(A) - 1) )
    return ret
```

```
def _exact_dx(X) :
    global A
    ret = []
    for x in X :
        ret.append( A * ( np.exp( A*x) - 1 ) / ( np.exp(A) - 1) )
    return ret

#
# Calculate the errors
#
def _err( X, U ) :
    global H, A

    L2_NORM = 0
    H1_SEMINORM = 0
    H1_NORM = 0
    for i in range( len(X) - 1 ) :
        X0 = X[i]; X1 = X[i+1]; U0 = U[i]; U1 = U[i+1] ;
        n = 100                        # split the space in small pieces (brute force

        # H1 SEMINORM
        uxh = (U1-U0)/H
        c1 =  A/2 * ( np.exp(2*A*(X1-1)) - np.exp(2*A*(X0-1)) )
        c2 = uxh**2 * (X1 - X0 )
        c3 = -2*uxh*np.exp(-A)*(np.exp(A*X1) - np.exp(A*X0) )
        H1_SEMINORM += c1 + c2 + c3

        # L2 NORM
        dx = ( X1 - X0 ) / n
        _ex_an = U1/H - U0/H
        for j in range(n) :
            x0 = ( X0 * (n-j) + X1 * j ) /n
            x1 = ( X0 * (n-j-1) + X1 * (j+1) ) /n
            u0 = ( (n-j)*U0 + j*U1 ) / n
            u1 = ( (n-j-1)*U0 + (j+1)*U1 ) / n
            u = u0/2 + u1/2
            x = x0/2 + x1/2
            u_exact = _exact([ x0, x1, x ])

            # error
            _e = u_exact[2] - u
            L2_NORM += ( _e **2 ) * dx  # L2 nor: \int e^2 dx

    H1_NORM = ( L2_NORM + H1_SEMINORM ) ** 0.5
    L2_NORM = ( L2_NORM ) ** 0.5
    H1_SEMINORM = ( H1_SEMINORM ) ** 0.5

    return L2_NORM, H1_SEMINORM, H1_NORM

for i in range( len(NN) ) :
    N = NN[i]
    F = np.zeros( N-1 )
    H = 1/N
    X = np.linspace( 0, 1, N+1 )
    XX.append(X)
```

```
    METHOD = "GALERKIN"
    UUgal.append( Usolve() )
    ( l2n, h1sn, h1n ) = _err(X, UUgal[-1] )
    L2Ngal.append( l2n )
    H1SNgal.append( h1sn )
    H1Ngal.append( h1n )

    METHOD = "SUPG"
    UUsupg.append( Usolve() )
    ( l2n, h1sn, h1n ) = _err(X, UUsupg[-1] )
    L2Nsupg.append( l2n )
    H1SNsupg.append( h1sn )
    H1Nsupg.append( h1n )
```

In [10]:

```
## PLOT
fig, [ax1, ax2] = plt.subplots( 1,2, figsize=(15,5))

ax1.plot( X_EXACT, U_EXACT, c='k', lw=5, alpha=.5)
for i in range(len(UUgal)):
    U = UUgal[i]
    X=XX[i]
    ax1.plot( X, U, ls='--', lw=1, label=NN[i])

ax1.set_title(f"GALERKIN")
ax1.set_ylim( -2, 1.5)

###############
ax2.plot( X_EXACT, U_EXACT, c='k', lw=5, alpha=.5)
for i in range(len(UUsupg)):
    U = UUsupg[i]
    X=XX[i]
    ax2.plot( X, U,  ls='--', lw=1, label=NN[i])
ax2.set_title(f"SUPG")
ax2.legend(bbox_to_anchor=(1.05, 1))
fig.tight_layout()
fig.savefig("fig1.pdf")
```

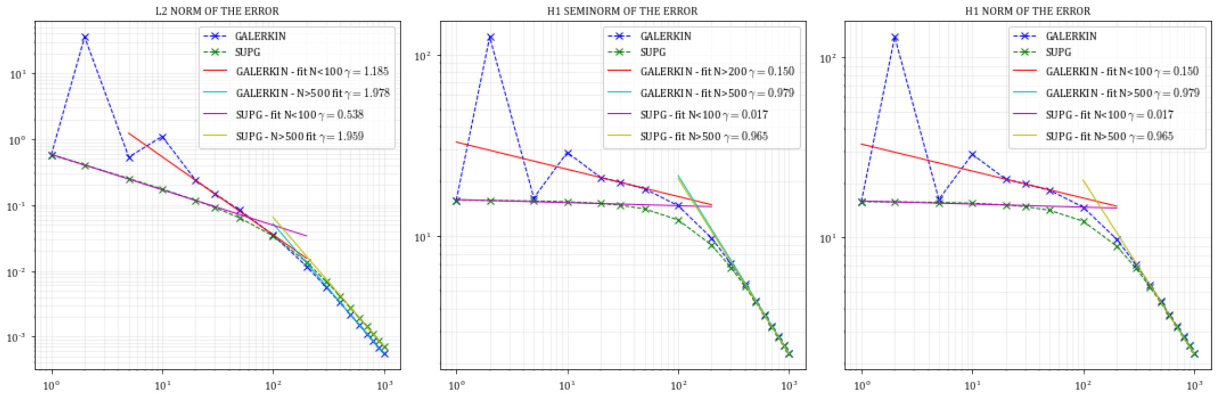


In [11]:

```
fig, [ax3, ax4,ax5] = plt.subplots( 1,3, figsize=(15,5))

## L2 NORMS
ax3.plot( NN, L2Ngal, marker='x', ls='--', lw=1,  label="GALERKIN")
ax3.plot( NN, L2Nsupg, marker='x', ls='--', lw=1,  label="SUPG")
```

```python
# Regressions
m, n, r, p, se = linregress(np.log(NN[4:6]), np.log(L2Ngal[4:6]))
ax3.plot( NN[2:9], np.exp(n) * NN[2:9]**m,label=f"GALERKIN - fit N<100 $\gamma={-m:
m, n, r, p, se = linregress(np.log(NN[13:]), np.log(L2Ngal[13:]))
ax3.plot( NN[7:], np.exp(n) * NN[7:]**m,label=f"GALERKIN - N>500 fit $\gamma={-m:.3

m, n, r, p, se = linregress(np.log(NN[0:6]), np.log(L2Nsupg[0:6]))
ax3.plot( NN[:9], np.exp(n) * NN[:9]**m,label=f"SUPG - fit N<100 $\gamma={-m:.3f}$"
m, n, r, p, se = linregress(np.log(NN[13:]), np.log(L2Nsupg[13:]))
ax3.plot( NN[7:], np.exp(n) * NN[7:]**m,label=f"SUPG - N>500 fit $\gamma={-m:.3f}$"

ax3.set_yscale('log')
ax3.set_xscale('log')
ax3.set_title(f"L2 NORM OF THE ERROR")
ax3.legend()

## H1 SEMINORMS
ax4.plot( NN, H1SNgal, marker='x', ls='--', lw=1, label="GALERKIN")
ax4.plot( NN, H1SNsupg, marker='x', ls='--', lw=1, label="SUPG")

m, n, r, p, se = linregress(np.log(NN[4:6]), np.log(H1SNgal[4:6]))
ax4.plot( NN[:9], np.exp(n) * NN[:9]**m,label=f"GALERKIN - fit N>200 $\gamma={-m:.3
m, n, r, p, se = linregress(np.log(NN[13:]), np.log(H1SNgal[13:]))
ax4.plot( NN[7:], np.exp(n) * NN[7:]**m,label=f"GALERKIN - fit N>500 $\gamma={-m:.3

m, n, r, p, se = linregress(np.log(NN[0:6]), np.log(H1SNsupg[0:6]))
ax4.plot( NN[:9], np.exp(n) * NN[:9]**m,label=f"SUPG - fit N<100 $\gamma={-m:.3f}$"
m, n, r, p, se = linregress(np.log(NN[13:]), np.log(H1SNsupg[13:]))
ax4.plot( NN[7:], np.exp(n) * NN[7:]**m,label=f"SUPG - fit N>500 $\gamma={-m:.3f}$"

ax4.set_yscale('log')
ax4.set_xscale('log')
ax4.set_title(f"H1 SEMINORM OF THE ERROR")
ax4.legend()

## H1 NORMS
ax5.plot( NN, H1Ngal, marker='x', ls='--', lw=1, label="GALERKIN")
ax5.plot( NN, H1Nsupg, marker='x', ls='--', lw=1, label="SUPG")

m, n, r, p, se = linregress(np.log(NN[4:6]), np.log(H1Ngal[4:6]))
ax5.plot( NN[:9], np.exp(n) * NN[:9]**m,label=f"GALERKIN - fit N<100 $\gamma={-m:.3
m, n, r, p, se = linregress(np.log(NN[13:]), np.log(H1Ngal[13:]))
ax5.plot( NN[13:], np.exp(n) * NN[13:]**m,label=f"GALERKIN - fit N>500 $\gamma={-m:

m, n, r, p, se = linregress(np.log(NN[0:6]), np.log(H1Nsupg[0:6]))
ax5.plot( NN[:9], np.exp(n) * NN[:9]**m,label=f"SUPG - fit N<100 $\gamma={-m:.3f}$"
m, n, r, p, se = linregress(np.log(NN[13:]), np.log(H1Nsupg[13:]))
ax5.plot( NN[7:], np.exp(n) * NN[7:]**m,label=f"SUPG - fit N>500 $\gamma={-m:.3f}$"

ax5.set_yscale('log')
ax5.set_xscale('log')
ax5.set_title(f"H1 NORM OF THE ERROR")
ax5.legend()
```

```python
fig.tight_layout()
fig.savefig("fig2.pdf")
```



2.4 - SIMPLE INVERT ESTIMATES

In [5]:

```python
import numpy as np
from sympy import *
from IPython.display import display

h_, l_, x_ = symbols('h \lambda x', real=True)

def calc_cinv( Nlhs, Nrhs ) :
    NN1 = Nlhs * Nlhs.transpose()
    NN2 = Nrhs * Nrhs.transpose()

    K = integrate( NN1, (x_, 0, h_) )
    M = integrate( NN2, (x_, 0, h_) )

    L = solve( (K-l_*M).det() , l_ )
    return L

X = [ 0, h_/2, h_ ]
Na_quad = Matrix(
            [ (x_-X[1])*(x_-X[2])/(X[0]-X[1])/(X[0]-X[2]),
              (x_-X[0])*(x_-X[2])/(X[1]-X[0])/(X[1]-X[2]),
              (x_-X[0])*(x_-X[1])/(X[2]-X[0])/(X[2]-X[1])
            ])

Na_lin = Matrix( [ 1 - x_/h_, x_/h_ ] )
```

In [6]:

```python
Na = Na_lin
Na_x = diff( Na, x_ )
Lambda = calc_cinv( Na_x, Na )

print("Linear shape function - C/h=");
display(sqrt(Lambda[1]))
```

Linear shape function - C/h=

$$\frac{2\sqrt{3}}{|h|}$$

In [7]:
```
Na = Na_quad
Na_x = diff( Na, x_ )

Lambda = calc_cinv( Na_x, Na )

print("Quadratic shape function - C/h=");
display(sqrt(Lambda[2]))
```

Quadratic shape function - C/h=

$$\frac{2\sqrt{15}}{|h|}$$

In [8]:
```
Na = Na_quad
Na_xx = diff( Na, x_, x_ )

Lambda = calc_cinv( Na_xx, Na )
print("Quadratic shape function (||w_xx|| < C ||w||)- C/h^2=");
display(sqrt(Lambda[1]))
```

Quadratic shape function (||w_xx|| < C ||w||)- C/h^2=

$$\frac{12\sqrt{5}}{h^2}$$

In [9]:
```
Na = Na_quad
Na_xx = diff( Na, x_, x_ )
Na_x = diff( Na, x_ )
Lambda = calc_cinv( Na_xx, Na_x )

print("Quadratic shape function (||w_xx|| < C ||w_x||)- C/h= 0");
Lambda
```

Quadratic shape function (||w_xx|| < C ||w_x||)- C/h= 0

Out[9]: []