Lecture # 26, April 22, 2024.

Intro. to Comp N.S. Flows.

(assume perfect gas)

$$U_{,t} + F_{i,i} = F^{visc}_{i,i} + F^{heat}_{i,i} + \mathcal{F}$$ all 5×1

all fns of U, the cons. variables

(d=3)

$$U = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho e \end{Bmatrix}$$

(5×1)

ρ = density

u_i = velocity

e = sp. total energy $= \underset{\text{internal energy}}{i} + \underbrace{\tfrac{1}{2}|u|^2}_{\text{kinetic energy}}$.

$F_i(U)$ = Euler flux

$$= u_i U + p \begin{Bmatrix} 0 \\ \hline \delta_i \\ \hline u_i \end{Bmatrix}$$

δ_i: 3×1

$$\delta_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \delta_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad \delta_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$\mathcal{F}(U) = \begin{Bmatrix} 0 \\ \rho b \\ \rho b \cdot u + r \end{Bmatrix}$$ r radiation

b = body force vector (e.g. g, gravity)

another way to express this:

$$\delta_i = \begin{Bmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{Bmatrix}$$, where δ_{ij} = identity tensor

$$F_i^{\text{heat}}(U) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -q_i \end{Bmatrix}$$ heat flux

where (Fourier law) $q_i = -\kappa\, \Theta_{,i}$

absolute temp

conductivity

constit eq.

$$F_i^{\text{visc}}(U) = \begin{Bmatrix} 0 \\ T_i \\ T_{ij} u_j \end{Bmatrix}$$

T_{ij} = viscous stress tensor 3×3

$$T_i = \begin{Bmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{Bmatrix} = \begin{Bmatrix} T_{1i} \\ T_{2i} \\ T_{3i} \end{Bmatrix}$$

symm.

Kronecker $\delta \equiv$ identity

$$T_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij}$$ const. eq.

symm.

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$ (symm)

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define all quantities in terms of U.

sp. heat at const vol.

(assumed const.)

~~eqs.~~

"Perfect gas"

$$\imath = c_v\, \Theta \qquad \text{E.O.S.}$$

$$p = (\gamma - 1)\rho\imath, \qquad \gamma = c_p / c_v$$

sp. heat at const press.

assumed const.

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entropy. prod., second law of thermo, Clausius - Duhem ineq.

$$(\rho\eta)_{,t} + (\rho u_i \eta)_{,i} + \left(\frac{q_i}{\Theta}\right)_{,i} - \rho r \geq 0.$$

thermo entropy

physical "

Thermo:

in general $\imath = \hat{\imath}(v, \eta)$, with $v = \rho^{-1}$ spec. vol.

$$\theta = \hat{\imath}_{,\eta}$$

$$p = -\hat{\imath}_{,v}$$

$$d\imath = \frac{\partial \hat{\imath}}{\partial v}\,dv + \frac{\partial \hat{\imath}}{\partial \eta}\,d\eta$$

$$= -p\,\underbrace{dv}_{-\frac{1}{\rho^2}d\rho} + \theta\,d\eta \qquad \text{(Gibbs relation)}$$

$$= \frac{p}{\rho^2}\,d\rho + \theta\,d\eta$$

$$d(C_v\theta) = (\gamma-1)\cancel{\rho}\,\overset{C_v\theta}{\imath}\,\frac{d\rho}{\rho^{\cancel{2}\,1}} + \theta\,d\eta$$

$$C_v\frac{d\theta}{\theta} = C_v(\gamma-1)\cancel{\theta}\,\frac{d\rho}{\rho} + \frac{\cancel{\theta}\,d\eta}{\cancel{C_v}}$$

$$ds\ \cancel{d\eta}^{\,k} = \cancel{C_v}\ \cancel{C_v}\left(\frac{d\theta}{\theta} - (\gamma-1)\frac{d\rho}{\rho}\right)$$

define non-dim entropy $s = \dfrac{\eta}{C_v}$

$$s = \ln\theta - (\gamma-1)\ln\rho + \text{const.}$$

$$= \ln\theta - \ln\rho^{(\gamma-1)} + \text{const}$$

$$= \ln\left(\frac{\rho\,\theta}{\cancel{\rho}\,\rho^{\cancel{\gamma-1}}}\right) + \text{const.}$$

$$p = (\gamma-1)\rho e$$
$$= (\gamma-1)\rho c_v \theta$$
$$\rho\theta = \frac{p}{(\gamma-1)c_v}$$

$$= \ln\left(\frac{p}{(\gamma-1)c_v \; \rho^{\gamma}}\right) + \text{const}$$

reference ρ_0, θ_0 constants

$$\rho_0\theta_0 = \frac{p_0}{(\gamma-1)c_v}$$

~~$\#$~~ let const $=$

$$-\ln\left(\frac{p_0}{(\gamma-1)c_v \, \rho_0{}^{\gamma}}\right)$$

$$= \ln\left(\left(\frac{p}{\cancel{(\gamma-1)c_v}\,\rho^{\gamma}}\right)\left(\frac{\cancel{(\gamma-1)c_v}\,\rho_0{}^{\gamma}}{p_0}\right)\right)$$

$$\boxed{s = \ln\left(\frac{p}{p_0}\left(\frac{\rho}{\rho_0}\right)^{-\gamma}\right)} \checkmark$$

$$s_0 = \ln\left(\cancel{\frac{p_0}{p_0}}\left(\cancel{\frac{\rho_0}{\rho_0}}\right)^{-\gamma}\right) = \ln 1 = 0.$$

ref. value $\equiv 0$.

Cons. laws. / Quasi-linear form.

$$F_i(U)_{,i} = \underbrace{\left(\frac{\partial F_i}{\partial U}\right)}_{} U_{,i} = \overset{5\times 5}{\overbrace{A_i(U)}} U_{,i}$$

matrix

generalized adv. flux

$$F_i^{visc}(U) = \overset{5\times 5}{\overbrace{K_{ij}^{visc}}}(U)\, U_{,j}$$

$$F_i^{heat}(U) = K_{ij}^{heat}(U)\, U_{,j}$$

$$\boxed{\overbrace{U_{,t} + A_i U_{,i}}^{Euler} = \overbrace{(K_{ij} U_{,j})_{,i}}^{NS} + \mathcal{F}(U)}$$

$$\underbrace{K_{ij}}_{5\times 5} = K_{ij}^{visc} + K_{ij}^{heat}$$

look familiar

Remarks. 1.) A_i's are <u>not</u> symm.

2.)

$$K = \underbrace{\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}}_{15\times 15}$$

gen diff matrix

K is not symm.

cannot talk about it definiteness.

connect w. entropy

and symm. hyp. systems.
Friedrich's

→ where we want to go:

symm adv-diff systems.
