$$= \Big[\frac{1}{2} \int_\Omega \alpha \nabla(w) \cdot (\nabla(u + \cancel{\varepsilon w})) \, d\Omega$$

$$+ \frac{1}{2} \int_\Omega \alpha \nabla(u + \cancel{\varepsilon w}) \cdot \nabla w \, d\Omega \overset{②}{} - \int_\Omega w f \, d\Omega \overset{③}{}$$

$$+ \int_\Gamma \overset{④}{\mu} (u + \cancel{\varepsilon w} - g) \, d\Gamma + \int_\Gamma (\lambda + \cancel{\varepsilon \mu}) \overset{⑤}{w} \, d\Gamma$$

$$+ \int_\Gamma \frac{\alpha}{2\delta} \overset{⑥}{w} (u + \cancel{\varepsilon w} - g) \, d\Gamma + \int_\Gamma \frac{\alpha}{2\delta} (u + \cancel{\varepsilon w} - g) \overset{⑦}{w} \, d\Gamma \Big] \Big|_{\varepsilon = 0}$$

$$\boxed{0 = \underbrace{\int_\Omega \nabla w \cdot (\alpha \nabla u) \, d\Omega}_{①+②} - \underbrace{\int_\Omega w f \, d\Omega}_{③} + \underbrace{\int_\Gamma \mu (u-g) \, d\Gamma}_{④} + \underbrace{\int_\Gamma w \lambda \, d\Gamma}_{⑤} + \underbrace{\int_\Gamma \frac{\alpha}{\delta} w (u-g) \, d\Gamma}_{⑥+⑦}}$$

(W)

$$\boxed{0 = \int_\Omega \nabla w \cdot \alpha \nabla u \, d\Omega - \int_\Omega w f \, d\Omega + \int_\Gamma \mu (u-g) \, d\Gamma + \int_\Gamma w \left(\lambda + \frac{\alpha}{\delta} (u-g) \right) d\Gamma}$$

What don't we like? λ. $\longrightarrow \lambda^h$???

$u, w \in \mathcal{S} = \mathcal{V}$. $\quad \lambda, \mu \in \mathcal{M}$. $\quad \times$.

Avoid a mixed FE meth because of stab. prob's / eliminate both λ, μ

Determine the Euler-Lag equ., by int-by-parts:

$$0 = \int_\Omega -w(\nabla\cdot\alpha\nabla u)\,d\Omega - \int_\Omega wf\,d\Omega$$

unit out. normal v. to Γ

$$+ \int_\Gamma + w\,(n\cdot\alpha\nabla u)\,d\Omega \quad \checkmark$$

$$+ \int_\Gamma \mu(u-g)\,d\Gamma + \int_\Gamma w\left(\lambda + \frac{\alpha}{\delta}(u-g)\right)d\Gamma$$

$$= -\int_\Omega w\,(\underbrace{\nabla\cdot\alpha\nabla u + f}_{\text{PDE res.}})\,d\Omega$$

$$+ \int_\Gamma w\,(\underbrace{\lambda + n\cdot\alpha\nabla u + \frac{\alpha}{\delta}(u-g)}_{\text{bndy res}})\,d\Gamma$$

$$+ \int_\Gamma \mu\,(\underbrace{u-g}_{\text{bndy res.}})\,d\Gamma$$

$$\Longrightarrow \quad \nabla\cdot\alpha\nabla u + f \quad \text{on } \Omega \quad \checkmark$$

$$\lambda = -n\cdot\alpha\nabla u + \frac{\alpha}{\delta}\boxed{(u-g)}\cancel{} \quad \text{on } \Gamma$$

$$\boxed{u = g} \quad \text{on } \Gamma$$

$$\nabla \cdot \varkappa \nabla u + f = 0 \quad \text{on } \Omega$$

$$\lambda = -n \cdot \varkappa \nabla u \quad \text{on } \Gamma$$

$$u = g \quad \text{on } \Gamma$$

will enable eliminating λ in (W)

again you take the var der.

$$\frac{d}{d\varepsilon}\left(\lambda + \varepsilon \mu = -n \cdot \varkappa \nabla (u + \varepsilon w) \right)$$

$$\mu = -n \cdot \varkappa \nabla w$$

RHS.

$$(W) \quad 0 = \int_\Omega \varkappa \nabla w \cdot \nabla u \, d\Omega - \int_\Omega w f \, d\Omega$$

symm.

RHS

$$+ \int_\Gamma \underbrace{-n \cdot \varkappa \nabla w}_{\mu} (u - g) \, d\Gamma$$

somewhat arbitrary

consistency term

RHS

$$+ \int_\Gamma w \big(\underbrace{-n \cdot \varkappa \nabla u}_{\lambda} + \int_\Gamma \frac{\varkappa}{\delta} (u - g) \, d\Gamma$$

mandatory.

symm. Nitsche : $B_{\text{Nitsche}}(w,u) = L_{\text{Nitsche}}(w)$, $\forall w \in \mathcal{V}$.

$$B_{\text{Nitsche}}(w,u) = \int_\Omega \nabla w \cdot \kappa \nabla u \, d\Omega$$

Options:
$s \in [-1,+1]$
$s = +1$, symm N.
$= 0$, orig. N.
$= -1$, skew N.
stab. adv.

$$- s \int_\Gamma (n \cdot \kappa \nabla w) u \, d\Gamma \qquad (W)$$

$$- \int_\Gamma w (n \cdot \kappa \nabla u) \, d\Gamma$$

$$+ \int_\Gamma w \frac{\kappa}{\delta} u \, d\Gamma$$

w.o. penalty, symm, orig are unstable

$$L_{\text{Nitsche}}(w) = \int_\Omega w f \, d\Omega - \int_\Gamma (n \cdot \kappa \nabla w) g \, d\Gamma$$

$$+ \int_\Gamma w \frac{\kappa}{\delta} g \, d\Gamma$$

(G) $B_{\text{Nitsche}}(w^h, u^h) = L_{\text{Nitsche}}(w^h) \quad \forall w^h \in \mathcal{V}^h$

$w^h, u^h \in \mathcal{V}^h = \mathcal{S}^h \subset \mathcal{V} = \mathcal{S}$.

also $B_{\text{Nitsche}}(w^h, u) = L_{\text{Nitsche}}(w^h) \quad \forall w^h \in \mathcal{V}^h \subset \mathcal{V}$.

$B_{\text{Nitsche}}(w^h, \underbrace{u^h - u}_{e}) = 0$ Gal. orthog.

---

stab. $B_{\text{Nitsche}}(w^h, w^h) = \int_\Omega \kappa |\nabla w^h|^2 \, d\Omega$

$$- (\underbrace{s+1}_{\geq 0}) \int_\Gamma (n \cdot \kappa \nabla w^h) \, w^h \, d\Gamma \quad \text{bad guy.}$$

$$+ \int_\Gamma \frac{\kappa}{\delta} (w^h)^2 \, d\Gamma$$

hunting for our norm.

$$\geq \int_\Omega \kappa |\nabla w^h|^2 d\Omega + \int_\Gamma \frac{\kappa}{\delta} (w^h)^2 d\Gamma \Big) \text{ good guys}$$

Cauchy-Schwarz.

$$- (s+1) \| n \cdot \kappa \nabla w^h \|_\Gamma \, \| w^h \|_\Gamma$$

$$\geq \text{good guys} - (s+1) \frac{1}{2} \left(\frac{1}{\varepsilon} \| n \cdot \kappa \nabla w^h \|_\Gamma^2 + \varepsilon \| w^h \|_\Gamma^2 \right)$$

$$= \int_\Omega \kappa |\nabla w^h|^2 d\Omega - \frac{(s+1)}{2} \frac{1}{\varepsilon} \| n \cdot \kappa \nabla w^h \|_\Gamma^2$$

hide behind

$\| \kappa^{1/2} \nabla w^h \|_\Omega^2$

Takes some work to def ε.

$$+ \| (\tfrac{\kappa}{\delta})^{1/2} w^h \|_\Gamma^2 - \frac{(s+1)}{2} \varepsilon \| w^h \|_\Gamma^2$$

simplicity assume $\frac{\kappa}{\delta}$ is const.

$$+ \int_\Gamma \left(\frac{\kappa}{\delta} - \frac{(s+1)}{2} \varepsilon \right) (w^h)^2 d\Gamma$$

once you have ε

to def δ.