CSE 397 / EM 397

Stabilized and Variational Multiscale methods in CFD.

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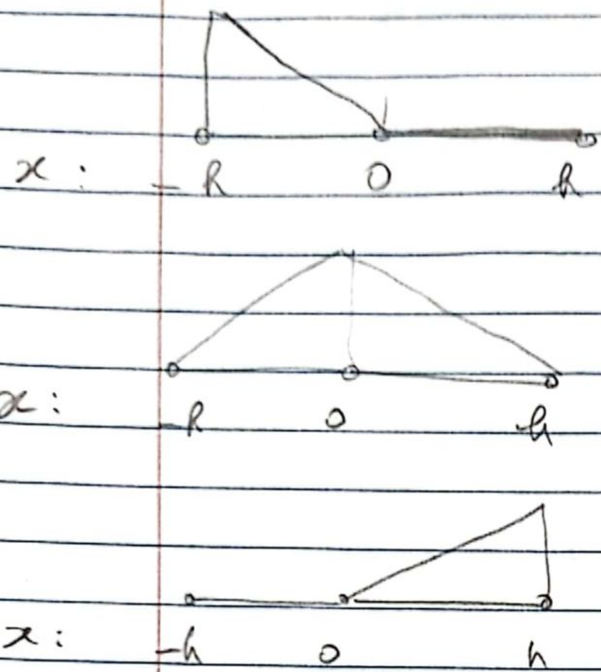
Homework #1

1.1 BG {A-1, A, A+1}



$$N_{A-1} = -x/h \rightarrow N_{A-1,x} = -1/h \Big|_{-h}^{0}$$

$$N_A = \begin{cases} 1 + x/h & , x < 0 \\ 1 - x/h & , x > 0 \end{cases} \Bigg|_{-h}^{h}$$

$$N_{A+1} = x/h \Big|_{0}^{h}$$

Diff - coefs:

$$u_{A-1} : k\left[\int_\Omega N_{A,x} N_{A-1,x}\, dx\right] = k\left[\int_{-h}^{0} -1/h^2\, dx\right] = -\frac{k}{h}$$

$$u_A : k\left[\int_\Omega N_{A,x} N_{A,x}\, dx\right] = k\left[\int_{-h}^{h} \frac{1}{h^2}\, dx\right] = \frac{k \cdot 2}{h}$$

$$u_{A+1} : k\left[\int_\Omega N_{A,x} N_{A+1,x}\, dx\right] = k\left[\int_{0}^{h} -1/h^2\, dx\right] = \frac{-k}{h}$$

Hence:

$$C_{DIFF} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix} k/h \begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1} \end{bmatrix}$$

and $\underset{\sim}{S}_{DIFF} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$ //

Adv. coeff

$$u_{A-1} : -a\int_{-h}^{0} N_{A,x} N_{A-1}\, dx = -a\int_{-h}^{0} (+1/h)\,(-x/h)\, dx$$

$$= \frac{a}{h^2}\left(\frac{x^2}{2}\right)_{-h}^{0} = \frac{a}{h^2}\left[0 - \frac{h^2}{2}\right] = -a/2$$

$$u_A : -a\int_{-h}^{0} \frac{1}{h}\left(1+\frac{x}{h}\right) dx + a\int_{0}^{h} \frac{+1}{h}\left(1-\frac{x}{h}\right) dx$$

$$\frac{-a}{h}\left[x + \frac{x^2}{2h}\right]_{-h}^{0} + \frac{a}{h}\left[x - \frac{x^2}{2h}\right]_{0}^{h} =$$

$$-\frac{a}{h}\left[+h - \frac{h^{\cancel{2}}}{2\cancel{h}}\right] + \frac{a}{h}\left[h - \frac{h^{\cancel{2}}}{2\cancel{h}}\right] =$$

$$\frac{a}{h}\left[-h + \frac{h}{2} + h - \frac{h}{2}\right] = 0$$

$$u_{A+1} : -a\int_{0}^{h} N_{A,x} N_{A+1}\, dx = +a\int_{0}^{h} \left(+\frac{1}{h}\right)\frac{x}{h}\, dx =$$

$$= \frac{a}{h^2}\left[\frac{x^2}{2}\right]_{0}^{h} = \frac{a h^2}{2h^2} = \frac{a}{2}$$

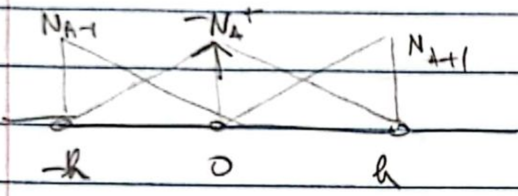
Hence : $C_{ADV} = \frac{a}{2}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1} \end{bmatrix}$

and $\underset{\sim}{S}_{ADV} = [-1 \ \ 0 \ \ 1]$ //

② $\frac{h}{2}\xi(\alpha_A)\sum_{e=1}^{N}\int_{\Omega_e} N_{A,x}\left(a u^h_{,x} - k u^h_{,xx} - f\right)dx = ⊛$

$$\int_{\Omega_e} N_{A,x}\left(a u^h_{,x} - k u^h_{,xx}\right)dx = \sum_{B=0}^{N_e}\int_{\Omega_e}\left(N_{A,x}\, a\, N_{B,x} - k N_{A,x}\overset{0}{\cancel{N_{B,xx}}}\right)dx$$

$$= \sum_{B=0}^{N_e}\int_{\Omega_e}\left(N_{A,x}N_{B,x}\,a\right)dx$$



$N_{A-1,x} = -1/h$

$N_{A,x} = \begin{cases} 1/h, & x<0 \\ -1/h, & x>0 \end{cases}$

$N_{A+1,x} = x/h$

$u_{A-1} : a\int_{-h}^{0} N_{A,x}N_{A-1,x}\,dx = -a/h$

$u_{A} : a\int_{-h}^{h} N_{A,x}N_{A,x}\,dx = \frac{a}{h}\cdot 2$

$u_{A+1} : a\int_{0}^{h} N_{A,x}N_{A+1,x}\,dx = -a/h$

Hence :

$$⊛ = \frac{h}{2}\xi(\alpha_h)\left\{\frac{a}{h}\underbrace{\begin{bmatrix}-1 & 2 & -1\end{bmatrix}}_{\underset{\sim}{T}^A}\begin{bmatrix}u_{A-1}\\ u_A\\ u_{A+1}\end{bmatrix} - \sum_{e=1}^{N}\int_{\Omega_e} N_{A,x} f\,dx\right\}$$ //

$\underset{\sim}{T}^A = \underset{\sim}{S}_{DIFF}$ //

③ $\frac{\cancel{k}}{\cancel{h}}\left((1+\alpha_h \xi)\begin{bmatrix}-1 & 2 & -1\end{bmatrix} + \alpha_h\begin{bmatrix}-1 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}u_{A-1}\\ u_A \\ u_{A+1}\end{bmatrix} = 0$

$$(1+\alpha\xi)\left(-u_{A-1}+2u_A-u_{A+1}\right) + \alpha\left(-u_{A-1}+u_{A+1}\right)=0$$

$/u_{A-1}$:

$$(1+\alpha\xi)\left(-1+\frac{2u_A}{u_{A-1}}-\frac{u_{A+1}}{u_{A-1}}\right)+\alpha\left(\frac{u_{A+1}}{u_{A-1}}-1\right)=0$$

$$(1+\alpha\xi)\times\left(-1+\frac{2e^{2\alpha A}}{e^{2\alpha(A-1)}}-\frac{e^{2\alpha(A+1)}}{e^{2\alpha(A-1)}}\right)+\alpha\left(\frac{e^{2\alpha(A+1)}}{e^{2(A-1)}}-1\right)=0$$

$$(1+\alpha\xi)\left(-1+2e^{2\alpha}-e^{4\alpha}\right)+\alpha\left(e^{4\alpha}-1\right)=0$$

$$-(1+\alpha\xi)\left(e^{2\alpha}-1\right)^{\cancel{2}} \qquad +\alpha\cancel{\left(e^{2\alpha}-1\right)}\left(e^{2\alpha}+1\right)$$

$$-(1+\alpha\xi)-\frac{\alpha\left(e^{2\alpha}+1\right)}{e^{2\alpha}-1}=0$$

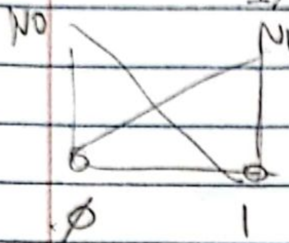
$$\xi=\frac{e^{2\alpha}+1}{e^{2\alpha}-1}-1/\alpha$$

$$\boxed{\xi=\coth(\alpha_h)-1/\alpha_h}$$

$$(0) = k + \frac{ah}{2}\xi$$

1.2

$$B\emptyset = \int_\Omega N_{1,x}\left((0)N_{0,x} - a N_\emptyset\right) dx$$



$$N_\emptyset = 1 - x/h$$

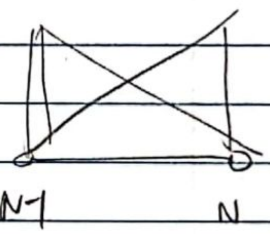
$$N_1 = x/h$$

$$B\emptyset = \int_0^h \frac{1}{h}(0)\left(-\frac{1}{h}\right) - a\left(\frac{x}{h}\right)\left(\frac{1}{h}\right)$$

$$-\frac{1}{h^2}(0)h - a\frac{h^2}{2h} = -\frac{1}{h}(0) - \frac{a}{2}$$

— x —

$$B_{N-1} = \int_\Omega N_{N-1,x}\left((0)N_{N,x} - aN_N\right)$$



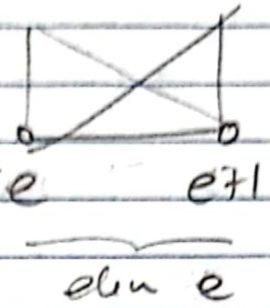
$$\int_0^h \left(-\frac{1}{h}\right)(0)\left(\frac{1}{h}\right) + a\left(+\frac{1}{h}\right)\left(\frac{x}{h}\right)$$

$$-\frac{(0)}{h^2}h + \frac{a}{\cancel{h^2}}\frac{\cancel{h^2}}{2} \Rightarrow B_{N-1} = -\frac{(0)}{h} + \frac{a}{2}$$

$$F_i = \int f\left(N_i + \frac{h}{2}\xi N_{i,x}\right)dx$$

$N_e = 1 - \frac{x-x_0}{h}$

$N_{e+1} = \frac{x-x_0}{h}$



<u>$f=1$</u>

<u>Node e:</u>

$$\int_{x_e}^{x_{e+1}} \left(1-\frac{x-x_0}{h}\right) + \frac{\cancel{h}}{2}\xi\left(-\frac{1}{\cancel{h}}\right)dx$$

$$= \left[x - \frac{x^2}{2h} + \frac{x_0 x}{h} - \frac{\xi}{2}x\right]_{x_e}^{x_{e+1}} \quad //$$

<u>Node e+1</u>

$$\int_{x_e}^{x_{e+1}} \frac{x-x_0}{h} + \frac{\cancel{h}}{2}\xi\left(\frac{1}{\cancel{h}}\right)dx$$

$$\left[\frac{x^2}{2h} - \frac{x_0 x}{h} + \frac{\xi}{2}x\right]_{x_e}^{x_{e+1}} \quad //$$

<u>$f=x$</u>

Node e:

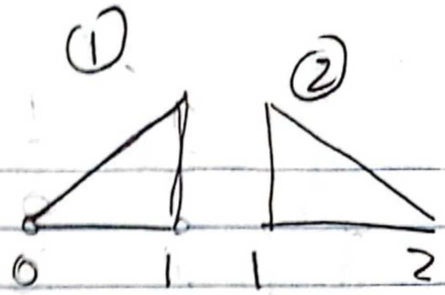
$$\int_{x_e}^{x_{e+1}} x - \frac{x^2 - x_0 x}{h} - \frac{\xi x}{2}\, dx$$

$$\left[\frac{x^2}{2} - \frac{x^3}{3h} + \frac{x^2 x_0}{2h} - \frac{\xi x^2}{4}\right]_{x_e}^{x_{e+1}} \quad //$$

<u>Node e+1</u>

$$\int_{x_e}^{x_{e+1}} \frac{x^2 - x x_0}{h} + \frac{x\xi}{2}\, dx$$

$$\left[\frac{x^3}{3h} - \frac{x^2 x_0}{2h} + \frac{x^2\xi}{4}\right]_{x_e}^{x_{e+1}} \quad //$$



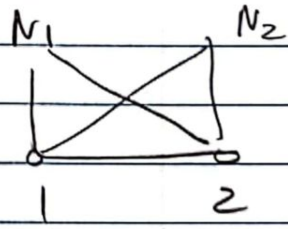
$$K_{11} = \int_{-h}^{h} N_{1,x}\left((o)N_{1,x} - aN_1\right) dx$$

$$① \int_0^h \left(\frac{1}{h}\right)(o)\left(\frac{1}{h}\right) - a\left(\frac{1}{h}\right)\left(\frac{x}{h}\right) = \frac{(o)}{h} - \frac{a}{2}$$

$$② \int_0^h \left(-\frac{1}{h}\right)(o)\left(-\frac{1}{h}\right) + a\left(+\frac{1}{h}\right)\left(1 - \frac{x}{h}\right)$$

$$= \frac{(o)}{h} + a/2$$

$$① + ② = \frac{2(o)}{h} \quad // \qquad (o) = \left(k + \frac{ah}{2} q\right)$$

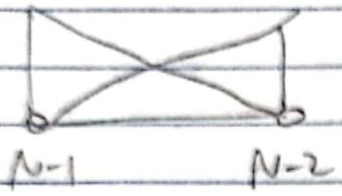


$$K_{12} = \int_0^h N_{1,x}\left((o)\, N_{2,x} - a N_2\right) dx$$

$$\int_0^h \left(-\frac{1}{h}\right)(o)\left(\frac{1}{h}\right) + a\left(+\frac{1}{h}\right)\left(\frac{x}{h}\right) dx$$

$$= -\frac{(o)}{h} + \frac{a}{2} \quad //$$

$$K_{N-1,N-2} = \int N_{N-1,x}\left((\circ)\, N_{N-2,x} - a N_{N-2} \right) dx$$



$$= \int_0^h \left(\frac{1}{h}\right)(\circ)\left(-\frac{1}{h}\right) - a\left(\frac{1}{h}\right)\left(\frac{x}{h}\right)$$

$$= \frac{-(\circ)}{h} - \frac{a}{2} \quad ///$$

$$K_{N-2,N-2} = K_{11} \quad ///$$