Lecture # 8, Feb. 12, 2024

The method just described is known as SUPG ~ Petrov-Galerkin.

SUPG: streamline upwind

→ data.

First stabilized method.

Gal. Least-Squares

(PG)

$$\text{Gal. meth.} + \sum_{A=1}^{N} \int_{x_{A-1}^{+}}^{x_A^{-}} \tau \mathcal{L} w^h \underbrace{(\mathcal{L}u^h - f)}_{res(u^h)} dx$$

el int.

$$\mathcal{L}u^h = \underbrace{+ a u^h_{,x}}_{\mathcal{L}_{adv}} \underbrace{- \kappa u^h_{,xx}}_{\mathcal{L}_{diff}} .$$

$\mathcal{L}_{adv} + \mathcal{L}_{diff}$

SUPG → $\mathcal{L}_{adv}$

GLS → $\mathcal{L}_{adv} + \mathcal{L}_{diff}$

$$\mathcal{P}(u^h) = \sum_{A=1}^{N} \int_{x_{A-1}^{+}}^{x_A^{-}} \frac{\tau}{2} \underbrace{(\mathcal{L}u^h - f)}_{res(u^h)}{}^2 dx$$

$$0 = \delta \mathcal{P}(u^h) = D\mathcal{P}(u^h) \cdot w^h .$$

var. der.
Gateaux der.
directional der.

$$= \left(\frac{d}{d\varepsilon} \mathcal{P}(u^h + \underset{\in \mathbb{R}}{\varepsilon} w^h) \right) \Big|_{\varepsilon = 0} .$$

$$= \left(\frac{d}{d\varepsilon} \sum_{A=1}^{N} \int_{x_{A-1}^{+}}^{x_A^{-}} \frac{\tau}{2} \left(\mathcal{L}(u^h + \varepsilon w^h) - f \right)^2 dx \right)_{\varepsilon=0}$$

$$= \left(\sum \int \frac{\tau}{\cancel{2}} \cancel{2} \left(\mathcal{L}(u^h + \cancel{\varepsilon w^h}) - f \right) \mathcal{L} w^h \, dx \right)_{\varepsilon=0}$$

$$= \sum_{A=1}^{N} \int_{x_{A-1}^+}^{x_A^-} \tau \, \mathcal{L} w^h \, (\mathcal{L} u^h - f) \, dx$$ ↗ up above

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Why is good to exercise control of res. ?

$$r(u^h) = \mathcal{L} u^h - f$$
$$r(u) = \mathcal{L} u - f \equiv 0$$

$$r(u^h) - \cancel{r(u)} = \mathcal{L} u^h - \mathcal{L} u$$
$$= \mathcal{L}(\underbrace{u^h - u})$$
$$= \mathcal{L}\, e \leftarrow \text{error.}$$

a posteriori measure of error. , but...

Last of three stab. meth's. VMS

MS: $\quad \text{Gal} + \sum_{A=1}^{N} \int_{x_{A-1}^+}^{x_A^-} \tau \, (\underbrace{-\mathcal{L}^* w^h}) (\mathcal{L} u^h - f) \, dx$

← formal adj.

$$\mathcal{L}^* w^h = - a w^h_{,x} - \kappa w^h_{,xx}$$

$$\ominus \mathcal{L}_{adv} \ \oplus \mathcal{L}_{diff}$$

$$- \mathcal{L}^* w^h = + \mathcal{L}_{adv} - \mathcal{L}_{diff} . \leftarrow$$

MS	$-\mathcal{L}^* w^h$	$=$	$+ a w^h_{,x}$	$+ \; \kappa w^h_{,xx}$
SUPG	$\mathcal{L}_{adv} w^h$	$=$	$+ a w^h_{,x}$	$+ \; 0$
GLS	$\mathcal{L} w^h$	$=$	$+ a w^h_{,x}$	$- \; \kappa w^h_{,xx}$

add meth provide better control in the adv. limit than Gal.

and better control than Gal of the res.

Can prove everything for all three

Abstract notat.

Gal. Find $u^h \in \mathcal{S}^h$, $\ni \forall w^h \in \mathcal{V}^h$

$$B(w^h, u^h) = L(w^h)$$

$$B(w, u) = \int_0^L (-w_{,x}\, a\, u + w_{,x}\, \kappa\, u_{,x})\, dx$$

$$L(w) = \int_0^L w\, f\, dx$$

$\text{stab} = \{ \text{SUPG}, \text{GLS}, \text{MS} \}$

$$B_{stab}(w^h, u^h) = L_{stab}(w^h)$$

$$B_{stab}(w, u) = B(w, u) + \sum_{A=1}^{N} \int_{x_{A-1}^+}^{x_A^-} \tau \, \mathbb{L} w \, \underbrace{(\mathcal{L} u)}_{\text{AD op}}\, dx$$

$$L_{stab}(w) = L(w) + \sum_{A=1}^{N} \int_{x_{A-1}^+}^{x_A^+} \tau \, \mathbb{L} w \, (f) \, dx$$

✓

Gal $\mathbb{L} = 0$.

SUPG $\mathbb{L} = \mathcal{L}_{adv}$

GLS $\mathbb{L} = \mathcal{L}$

MS $\mathbb{L} = -\mathcal{L}^*$

	Cent Diff / Gal	Classical upwind diff's	Stab
Accuracy	Yes (formal acc.)	No	Yes
& Stability	No	Yes	Yes

competitors

not competitors

Implementation

$$0 = B_{stab}(w^h, u^h) - L_{stab}(w^h)$$

substitute

$$0 = \sum_{A=1}^{N-1} w_A \left(\sum_{B=0}^{N} \underbrace{B_{stab}(N_A, N_B)}_{K_{AB}} u_B - \underbrace{L_{stab}(N_A)}_{F_A} \right)$$

$\forall w_A \in \mathbb{R}, \; A = 1, \dots, N-1$

$$0 = \sum_{B=1}^{N-1} \left(\cancel{B_{stab}} \; K_{AB} u_B + K_{A0} \overset{g_0}{\overset{\parallel}{u_0}} + K_{AN} \overset{g_L}{\overset{\parallel}{u_N}} - F_A \right)$$

$$\sum_{B=1}^{N-1} K_{AB} u_B = R_A = \underline{F_A - K_{A0} g_0 - K_{AN} g_L}$$

$\forall A = 1, \dots, N-1$

$$\underbrace{K}_{N-1 \times N-1} \; \underbrace{U}_{N-1 \times 1} = \underbrace{R}_{N-1 \times 1}$$

x

$$B(w^h; u^h) + \int_\Omega \tau \underbrace{a w^h_{,x}}_{a \cdot \nabla w^h} (\mathcal{L} u^h - f)\, dx$$

Later

x

Intro to Funct. Anal.

Lin. space. $\mathcal{V}$ infinite dimensional function spaces

closed under addition and scalar mult.

$$w_1, w_2 \in \mathcal{V}, \Rightarrow \underset{\in \mathbb{R}}{c_1}\, \underset{\in \mathcal{V}}{w_1} + \underset{\in \mathbb{R}}{c_2}\, \underset{\in \mathcal{V}}{w_2} \in \mathcal{V}.$$

ex. 1 $L_2(0,L)$ square int ftns.

ex. 2 $H^1(0,L)$ " " " with " " derivatives

ex. 3 $\mathcal{V} = \{ w \mid w \in H^1(0,L),\ w(0) = 0,\ w(L) = 0 \}$

ex. 4 $\mathcal{V}^h = \{ w^h \mid w^h \in \mathcal{V},\ w^h(x) = \sum_{A=1}^{N-1} N_A(x) \underset{\in \mathbb{R}}{w_A} \}$

$\mathcal{S}, \mathcal{S}^h$ <u>not</u> lin sp.'s.

Inner prod. on a lin sp $\mathcal{V}$.

$$(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$$

1. symm. $(v, w) = (w, v) \quad \forall w, v \in \mathcal{V}$.
2. bilin. $(\underline{c_1 v_1 + c_2 v_2}, w) = c_1 (v_1, w) + c_2 (v_2, w)$

   $(v, c_1 w_1 + c_2 w_2) = c_1 (v, w_1) + c_2 (v, w_2)$
3. pos def. $(v, v) \geq 0 \quad = 0 \text{ iff } v = 0$

ex'n. $V = L_2(0,L)$

$$(v,w) = \int_0^L v\,w\,dx$$

$$(v,v) = \int_0^L v^2\,dx \ge 0, \; = 0 \text{ iff } v = 0.$$

$B_{Gal}(\cdot\,,\cdot)$ not symm

$B_{stab}(\cdot\,,\cdot)$ " "

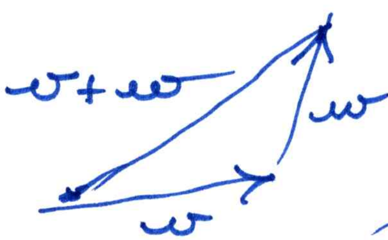
not i.p.'s.

<u>Norms</u> A norm on a l.s. V

$$\|\cdot\| : V \to \mathbb{R} \ni$$

1. pos def $\|v\| \ge 0$, $\|v\| = 0$ iff $v = 0$.
2. $\|\underset{\in \mathbb{R}}{c}\, v\| = |c|\,\|v\|$
3. Δ-ineq $\|v + w\| \le \|v\| + \|w\|$ ✓

pf for a <u>natural norm</u> by Cauchy-Schwarz ineq.



<u>Natural norm</u> for an i.p. sp.

$$\|v\| \overset{def}{=} (v,v)^{1/2}$$