

Factor

$$f(x)dx = g(y)dy$$

Exact

$$dU = M dx + N dy = 0$$

$$Exact \Leftrightarrow \partial M / \partial y = \partial N / \partial x$$

$$M = \frac{\partial U}{\partial x}, N = \frac{\partial U}{\partial y}$$

Partial Integration:

$$U = \int M dx = F + C(y) \quad \text{Treat } y \text{ as a constant}$$

$$\text{Find } C'(y) \rightarrow \frac{\partial F}{\partial y} + C'(y) = \frac{\partial U}{\partial y}$$

Then integrate to find $C(y)$ and $U(x, y)$

Linear, first order

$$y' + p(x)y = q(x)$$

$$\text{Integrating factor} \rightarrow m = e^{\int p(x)dx} \rightarrow d[ym] = m q(x) dx$$

Homogeneous Function

$$g(tx, ty) = t^n g(x, y)$$

Homogeneous DE

$$y' = f(x) \text{ and } f(tx, ty) = f(x, y)$$

$$y = vx \quad \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \text{or} \quad x = vy \quad \frac{dx}{dy} = v + y \frac{dv}{dy}$$

Separate variables and use integrating factor

Wronskian

$$W(f_1, f_2, \dots, f_n) = \det \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

Superposition for linear DE

If y and z are solutions of $y'' + ay' + by = 0$ so is $C_1 y + C_2 z$.

2nd order, constant coefficient

$$y'' + ay' + by = f(x)$$

Find y_h solving with $f(x) = 0$

Find roots m_1, m_2 of $m^2 + am + b = 0$

$$m_1 \neq m_2 \in \mathbb{R} \rightarrow \blacksquare = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$m = \alpha \pm i\beta \rightarrow \blacksquare = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

$$m \text{ repeated } n \text{ times: } \rightarrow (C_1 + C_2 x + \dots + C_n x^{n-1}) [\blacksquare]$$

Find y_p , then $y = y_h + y_p$

Undetermined coefficients

Convert to D-form $D^n = d^n/dx^n$

Find roots of the characteristic equation m_i

Find roots of the RHS by inverse inspection m_i'

Limitation: RHS must be such that we can find m_i'

Write y considering every m_i and m_i'

Identify y_h and y_p

Subs y_p in the DE to find the constants

y_h constants are found using the initial conditions

Variation of parameters

$$(D^n + a_{n-1}D^{n-1} + \dots + D + a_0)y = q(x)$$

Find y_h as in undetermined coefficients. Write:

$$y_h = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$$

$$y_p = v_1 y_1 + v_2 y_2 + \dots + v_n y_n$$

Find derivatives for y_i and solve the system for v_i'

$$\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(p)} & y_2^{(p)} & \dots & y_n^{(p)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ q(x) \end{bmatrix}$$

Find v_i integrating v_i'

Bernoulli

$$y' + p(x)y = q(x)y^n$$

$$y = v^{-1/(n-1)} \rightarrow \frac{dy}{dx} = \frac{dv}{dv} \frac{dv}{dx} = \frac{-1}{n-1} [v^{-n/(n-1)}] \frac{dv}{dx}$$

$$v' - (n-1)p(x)v = -(n-1)q(x)$$

Euler

$$b_n x^n y^{(n)} + b_{n-1} x^{n-1} y^{(n-1)} + \dots + b_1 x y' + b_0 y = 0 \quad \forall b_i = \text{const}$$

$$z = \ln(x) \rightarrow x = e^z$$

$$y^{(n)} = \frac{1}{x^n} D(D-1)(D-2) \dots (D-n+1)y, \text{ where } D = \frac{d}{dz}$$

Solve the resulting linear equation.

Power series solution

$f(x)$ is an **analytic function** if it has a power series represent. around x_0 .

Initial Value Problem (IVP)

$$y' + p(x)y = q(x) \quad y(x_0) = y_0 \quad p, q \text{ analytic about } x_0$$

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad a_n = \frac{1}{n!} y^{(n)}(x_0), \quad \text{in } (x_0 - h, x_0 + h)$$

Subs the IC to find $y'(x_0)$

Differentiate the DE and subs $y(x_0)$ and $y'(x_0)$ to find $y''(x_0)$

Drawback: differentiating the DE might not be practical.

Power series solution using recurrence relation

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} (n-1) a_n (x - x_0)^{n-2}$$

Subs in the equation.

Shift indices in the summations so that the power of x is the same in all

Be careful not to loose terms!

The coefficients of each power x^k must be equal in LHS and RHS

Find recurrence relation for the each a_n

Subs a_n in $y(x)$ and expand the summations as needed.

$$\text{Shifting} \quad \sum_{n=0}^{\infty} C_n x^{n+a} = \sum_{n=a}^{\infty} C_{n-a} x^n$$

Singularities and the method of Frobenius

$$y'' + p(x)y' + q(x)y = f(x) \quad p, q \text{ analytical about } x_0; x_0 \text{ ordinary}$$

Singular point of the DE: p, q or f has zero denominator about x_0 .

Regular singular: $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytical

Ordinary: not singular. **Irregular:** Not regular

If equation has a regular singular point at x_0 , use a Frobenius series:

$$y = \sum_{n=0}^{\infty} C_n z^{n+r} \quad z = (x - x_0), \text{ where } r \in \mathbb{R}$$

$$y' = \sum_{n=0}^{\infty} (n+r) C_n z^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n z^{n+r-2}$$

Subs in the equation, shift indices etc. (same as pwr series)

Assume $C_0 \neq 0$ to find values for r_1, r_2 ($r_1 \geq r_2$).

Find the recurrence C_n using $r = r_1$. Write $y_1(x)$.

For the **second solution**, find the recurrence C_n^* as:

$$\begin{cases} \text{If } r_1 - r_2 \notin \mathbb{Z} & y_2 = \sum_{n=0}^{\infty} C_n^* z^{n+r_2} \\ \text{If } r_1 = r_2 & y_2 = y_1 \ln(z) + \sum_{n=0}^{\infty} C_n^* z^{n+r_1} \\ \text{If } r_1 - r_2 \in \mathbb{N} & y_2 = K y_1 \ln(z) + \sum_{n=0}^{\infty} C_n^* z^{n+r_2} \end{cases}$$

Subs y_2 into the DE and obtain an equation for K

¹ Subs y_2 into the DE and the \ln term vanishes as the terms it multiplies are equal to y_1 .

² We use $C_0 = 1$ for convenience

³ Subs the found solution y_1 after simplifications

Final solution: $y(x) = y_1 + y_2$

Bessel's equation of order ν

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$\nu \in \mathbb{R}, \nu \geq 0, x > 0$$

$$y(x) = C_0 J_\nu + C_1 Y_\nu$$

Bessel function of the 1st kind: $J_\nu = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n! (1+\nu)(2+\nu) \dots (n+\nu)} x^{2n+\nu}$

Bessel function of the 2nd kind: $Y_\nu = \dots$

Matrices and vectors

$$A = [a_{ij}] \rightarrow i: \text{row} \quad j: \text{column}$$

$$a_{1j}: \text{row matrix} \quad a_{i1}: \text{column matrix}$$

$$\frac{dA}{dt} = \dot{A} = \left[\frac{da_{ij}}{dt} \right] = [\dot{a}_{ij}] \quad \int A(\tau) d\tau = \left[\int a_{ij}(\tau) d\tau \right]$$

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta) \quad \|\mathbf{a}\| = \sqrt{a_{kk}^2}$$

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \mathbf{a} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|} \quad (\text{component of } \mathbf{a} \text{ in } \mathbf{b}) \quad \text{proj}_{\mathbf{b}} \mathbf{a} = \mathbf{a} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b} \quad (\text{projection of } \mathbf{a} \text{ in } \mathbf{b})$$

Definitions

$$\text{Conjugate: } \bar{A}: a_{ij} = \alpha_{ij} \pm i \beta_{ij} \rightarrow \overline{a_{ij}} = \alpha_{ij} \mp i \beta_{ij}$$

$$\text{Rank: largest non zero determinant; number of independent vectors;} \\ C = AB \rightarrow 0 \leq \text{rank}(C) \leq \min\{\text{rank}(A), \text{rank}(B)\}$$

$$\text{Trace: } \text{tr}(A) = \sum a_{ii} \quad \text{tr}(A+B) = \text{tr}(A) + \text{tr}(B) \quad \text{tr}(AB) = \text{tr}(BA) \\ \text{tr}(AB) \neq \text{tr}(BA)$$

Identities

$$A^T = a_{ji} \quad [A^T]^T = A \quad [A \pm B]^T = A^T \pm B^T$$

$$[cA]^T = cA^T \quad [ABC]^T = C^T B^T A^T$$

$$[AB]^T = B^T A^T = BA \neq AB$$

$$\text{If } A \text{ is symmetric, then so is } B^T A B, \forall B$$

$$a_{ij} \in \mathbb{R} \rightarrow A = \bar{A}$$

Square matrices

$$\text{Symmetric: } A = A^T \quad \text{Skew-Symmetric: } A = -A^T$$

$$\text{Positive definite: } x^T A x > 0 \quad \text{Non negative definite: } x^T A x \geq 0$$

$$\text{Indefinite: } (x^T A x)(y^T A y) < 0 \quad x, y \in \mathbb{R}^n$$

$$\text{Orthogonal } A^T = A^{-1} \rightarrow A^T A = I$$

$$\text{Nilpotent: } A^k = 0 \quad \text{and } A^{k-1} \neq 0 \quad k \in \mathbb{Z}$$

$$\text{Idempotent: } A^2 = A$$

$$\text{Involutory: } A^2 = I \quad \text{Unitary: } A^{-1} = \bar{A}^T$$

$$\text{Positive: } a_{ij} > 0 \quad \forall i, j \quad \text{Non-negative: } a_{ij} \geq 0 \quad \forall i, j$$

$$\text{Diagonal Dominant: } |a_{ii}| \geq \sum_{i \neq j} |a_{ij}| \quad \text{Strictly Diag Dom: } |a_{ii}| > \sum_{i \neq j} |a_{ij}|$$

$$\text{Associate: } [\bar{A}]^T \quad \text{Hermitian: } A = \bar{A}^T \quad \text{Skew Hermitian: } A = -\bar{A}^T$$

Determinants

$$\det(A) = |A|$$

A is a (n x n) matrix, then:

$$|A| = \frac{1}{(a_{11})^{n-2}} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$\text{Minor: } M_{ij} = \begin{vmatrix} \begin{matrix} \blacksquare & X & \blacksquare & \blacksquare \\ X & a_{ij} & X & X \\ \blacksquare & X & \blacksquare & \blacksquare \\ \blacksquare & X & \blacksquare & \blacksquare \end{matrix} \end{vmatrix} = \begin{vmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{vmatrix} \quad \text{Cofactor: } C_{ij} = (-1)^{i+j} M_{ij}$$

$$|A| = \sum_{k=1}^n a_{ik} C_{ik}, \text{ for any row } i$$

Adjoint matrix C^T is the transpose of the cofactor's matrix.

Inverse

Inverses are unique

$$A A^{-1} = I \quad A^{-1} = \frac{C^T}{|A|}$$

Properties of determinants

$$|A||B| = |AB| \quad |A| = |A^T|$$

If any col or row is null, then $|A| = 0$

If operate columns or rows, then $|A|$ does not change

If swap columns or rows, then $|A|$ changes sign

If two columns or rows are proportional then $|A| = 0$

If one column or row is the linear combination of others then $|A| = 0$

Multiply column or row by α then $|B| = \alpha|A|$

If $|A| = 0$, A is singular and has no inverse.

LIN ALG

Set of vectors

$\{\mathbf{v}_i\}$ are linearly independent $\Leftrightarrow \alpha_k \mathbf{v}_k = 0$ for at least one set of α_i .

$\{\mathbf{v}_i\}$ is a base if exists a unique choice of scalars for every vector \mathbf{u} . That is,

$\{\mathbf{v}_i\}$ are independent

$\{\mathbf{v}_i\}$ is orthogonal $\Leftrightarrow \mathbf{v}_i^T \mathbf{v}_j = 0, \forall i \neq j$

$\{\mathbf{v}_i\}$ is orthonormal $\Leftrightarrow \|\mathbf{v}_i\| = 1, \forall i$

$$\text{Normalization: } \tilde{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Gram-Schmidt orthogonalization of $\{\mathbf{v}_i\}$

$\{\mathbf{v}_i\}$ are linearly independent

$$\mathbf{u}_1 = \mathbf{v}_1 \quad \mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 \quad \mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2} \mathbf{u}_2$$

$$\mathbf{u}_m = \mathbf{v}_m - \sum_{k=1}^{m-1} \text{proj}_{\mathbf{u}_k} \mathbf{v}_m = \mathbf{v}_m - \sum_{k=1}^{m-1} \frac{\mathbf{v}_m \cdot \mathbf{u}_k}{\|\mathbf{u}_k\|^2} \mathbf{u}_k$$

Systems of linear equations

$$A \mathbf{x} = \mathbf{b}$$

Cramer's rule: $x_j = \frac{\Delta_j}{|A|}$, where $\Delta_j = |A_j|$ and A_j is A with column j replaced by vector \mathbf{b} .

$|A| \neq 0, \mathbf{b} \neq 0 \rightarrow$ unique solution

$|A| \neq 0, \mathbf{b} = 0 \rightarrow \mathbf{x} = 0$

$|A| = 0, \mathbf{b} = 0 \rightarrow$ infinite solutions

$|A| \neq 0, \mathbf{b} \neq 0 \rightarrow$ infinite solutions $\Leftrightarrow \Delta_i = 0, \forall i$. Otherwise, no solution.

A is diagonal: system is called uncoupled or the variables are called separated

A and B are similar: $B = P^{-1}AP$ and $PB = AP$, where P is the set of eigenvectors.

Gaussian Elimination

Operate equations/rows to *uptriangularize* the system.

Back substitution to find variables.

Augmented matrix

$$A \mathbf{x} = \mathbf{b} \rightarrow [A: \mathbf{b}]$$

Operate rows to find $[I: \tilde{\mathbf{b}}]$

Operate $[A: \mathbf{b}: I]$ to find solution and inverse as $[I: \mathbf{x}: A^{-1}]$

LU Factorization

$$A \mathbf{x} = \mathbf{b}$$

$$A = LU \rightarrow U \mathbf{x} = \tilde{\mathbf{b}} \rightarrow LU \mathbf{x} = \mathbf{b} \rightarrow U \mathbf{x} = \mathbf{y} \rightarrow L \mathbf{y} = \mathbf{b}$$

Iterative method: Jacobi

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Rightarrow \begin{cases} x = (c - by)/a \\ y = (f - dx)/e \end{cases} \Rightarrow \begin{cases} \text{Initial guess for } x, y \\ \text{Iterate} \end{cases}$$

Eigenvalues and Eigenvectors

$A \mathbf{x}_i = \lambda_i \mathbf{x}_i \Rightarrow$ Characteristic equation (CE): $(A - \lambda_i I) \mathbf{x}_i = 0, \forall i$

Find eigenvalues λ_i as the scalar roots of the CE

Replace every λ_i in the CE to find eigenvectors \mathbf{x}_i associated to each λ_i

$\rightarrow \mathbf{x}_i$ must be linearly independent

$\rightarrow A$ is a zero of its characteristic equation (Cayley Hamilton).

$$|A| = \prod \lambda_i \quad \text{tr}(A) = \sum \lambda_i$$

If, for any $i, \lambda_i = 0 \Rightarrow A$ is singular

If A is real and symmetric $\Rightarrow \lambda_i \in \mathbb{R}$

If A is diagonal $\Rightarrow A = \text{diag}(\lambda_i)$ and $A^{-1} = \text{diag}(1/\lambda_i)$

If A is upper or lower triangular $\Rightarrow \lambda_{ii} = a_{ii}$ and $|A| = \prod a_{ii}$

Companion Matrix

$$p(x) = x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$$

$$\Rightarrow p(x) = x^n - (a_n - a_{n-1} x - \dots - a_1 x^{n-1})$$

$$C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & \dots & -a_2 & -a_1 \end{bmatrix} \Rightarrow \begin{cases} |C| = p(x) \\ \text{Eigenvalues are the roots of } p(x) \end{cases}$$

Partitioned Matrix

$$AB = C \Rightarrow \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix} \Rightarrow \det(A) = |A| = |A_1| |A_2| \dots |A_n|$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)$$

LAPLACE

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

Workflow: IVP $\rightarrow \mathcal{L}(t \rightarrow s) \rightarrow \text{Algebra} \rightarrow \mathcal{L}^{-1}(s \rightarrow t)$

Properties - linearity

$$\mathcal{L}\{f\} = F \quad \mathcal{L}\{g\} = G \quad f = f(t) \quad g = g(t) \quad F = F(s) \quad G = G(s)$$

$$\mathcal{L}\{f + g\} = F + G \quad \mathcal{L}^{-1}\{F + G\} = f + g$$

$$\mathcal{L}\{a f\} = a F \quad \mathcal{L}^{-1}\{a F\} = a f \quad a \in \mathbb{R}$$

Operations

$$\mathcal{L}\{f'\} = sF - f(0)$$

$$\mathcal{L}\{f^{(n)}\} = s^n F - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

$$\mathcal{L}\{e^{at} f\} = F(s - a) \quad \mathcal{L}\{f(t - a)\} = e^{-at} F$$

$$\mathcal{L}\left\{\int_0^t f(u) du\right\} = \frac{1}{s} F$$

Periodic function:

$$f(t + \omega) = f(t) \Rightarrow \mathcal{L}\{f\} = \frac{1}{1 - e^{-s\omega}} \int_0^\omega f(t) e^{-st} du$$

Partial fractions

$$H = \frac{F(s)}{G(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \dots + a_p s^p}{b_0 + b_1 s + b_2 s^2 + \dots + b_r s^r} \begin{cases} a_i, b_i \in \mathbb{R} \\ f \text{ and } g \text{ do not have common roots} \\ g \text{ is of higher degree than } f \text{ (} r > p \text{)} \end{cases}$$

Factor $F(s)$ in linear factors $(s - a)^m$ and quadratic factors $(s^2 + ps + q)^n$.

$$H = \frac{A_1}{s - a} + \frac{A_2}{(s - a)^2} + \dots + \frac{A_m}{(s - a)^m} + \frac{B_1 s + C_1}{s^2 + ps + q} + \frac{B_2 s + C_2}{(s^2 + ps + q)^2} + \dots + \frac{B_n s + C_n}{(s^2 + ps + q)^n}$$

Solve for A_i, B_i, C_i

Remark:

$$as^2 + bs + c = a(s + k)^2 + h^2 \quad k = \frac{b}{2a} \quad h = \sqrt{c - b^2/4a}$$

$$\text{Ex: } \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 2s + 9}\right\} = \frac{1}{(s-1)^2 + \sqrt{8}^2} = \frac{1}{\sqrt{8}} \frac{\sqrt{8}}{(s-1)^2 + \sqrt{8}^2} = \frac{1}{\sqrt{8}} e^x \sin \sqrt{8} x$$

$$\text{Ex: Avoid imaginary factors: } \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2s + 2}\right\} = \frac{1}{(s+1)^2 + 1} = e^{-x} \sin x$$

Step Function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad \mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$u(t - a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases} \quad \mathcal{L}\{u(t - a)\} = \frac{e^{-as}}{s}, \quad a > 0$$

$$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as} F(s)$$

$$\mathcal{L}\{f(t)u(t - a)\} = e^{-as} \mathcal{L}\{f(t + a)\}$$

Impulse Function

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \mathcal{L}\{\delta(t)\} = 1$$

$$\delta(t - a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases} \quad \mathcal{L}\{\delta(t - a)\} = e^{-as}$$

Convolution

$$\mathcal{L}\{F \cdot G\} = f * g = \int_0^t f(u)g(t - u)du$$

$$f * g = g * f$$

$$\text{Ex: } \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} = \int_0^t u e^{a(t-u)} du = \frac{1}{a^2}(e^{at} - at - 1)$$

Polynomial coefficients

$$\mathcal{L}\{t^n \cdot f(t)\} = (-1)^n F^{(n)}(s) \quad n \in \mathbb{N}$$

Let $y(0) = y_0$ and $y'(0) = w_0$

$$\mathcal{L}\{ty'\} = -\frac{d}{ds} \mathcal{L}\{y'\} = -\frac{d}{ds} [sY - y_0] = -Y - sY' + y_0'$$

$$\mathcal{L}\{ty''\} = -\frac{d}{ds} \mathcal{L}\{y''\} = -\frac{d}{ds} [s^2 Y - sw_0 - y_0] = -2sY - s^2 Y' + sw_0' + w_0 + y_0'$$

$$\mathcal{L}\{t^2 y'\} = \frac{d^2}{ds^2} \mathcal{L}\{y'\} = \frac{d^2}{ds^2} [sY - y_0] = 2Y' + sY'' - y_0''$$

$$\text{Ex: } y'' + 2ty' - 4y = 1 \quad y(0) = y'(0) = 0 \quad s^2 Y - 2Y - 2sY' - 4Y = \frac{1}{s} \quad \text{New linear DE! Solve with Integration Factor to find } Y(s).$$

Systems of DE using Laplace Transforms

$$\text{Ex: } \begin{cases} \frac{dx}{dt} = 2x - 3y & x(0) = 8 \\ \frac{dy}{dt} = y - 2x & y(0) = 3 \end{cases} \quad \begin{cases} sX - x(0) = 2X - 3Y \\ sY - y(0) = Y - 2X \end{cases}$$

Solve for X and Y . Invert to find $x(t)$ and $y(t)$.

Integral by parts:

$$\int_a^b u dv = uv|_a^b - \int_a^b v du$$

$$\int_{t_0}^{t_1} e^{-st} \cos(at) dt = ?$$

$$u = e^{-st} \rightarrow du = -se^{-st} dt \quad dv = \cos(at) \rightarrow v = \frac{1}{a} \sin(at)$$

$$\int_{t_0}^{t_1} e^{-st} \cos(at) dt = \frac{1}{a} [e^{-st} \sin(at)]_{t_0}^{t_1} + \frac{s}{a} \int_{t_0}^{t_1} e^{-st} \sin(at) dt$$

$$u = e^{-st} \rightarrow du = -se^{-st} dt \quad dv = \sin(at) \rightarrow v = -\frac{1}{a} \cos(at)$$

$$\int_{t_0}^{t_1} e^{-st} \cos(at) dt = \frac{1}{a} [e^{-st} \sin(at)]_{t_0}^{t_1} + \frac{s}{a^2} [e^{-st} \sin(at)]_{t_0}^{t_1} - \frac{s^2}{a^2} \int_{t_0}^{t_1} e^{-st} \cos(at) dt$$

$$\int_{t_0}^{t_1} e^{-st} \cos(at) dt = \frac{a}{a^2 + s^2} [e^{-st} \sin(at)]_{t_0}^{t_1} + \frac{s}{a^2 + s^2} [e^{-st} \sin(at)]_{t_0}^{t_1}$$

$$\int t \cos t dt = t \sin t + \cos t \quad u = t ; \quad du = dt ; \quad dv = \cos t dt ; \quad v = \sin t$$

Euler

$$e^{\pm ix} = \cos x \pm i \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Baskhara

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

INTEGRALS

$$\int \frac{x \, dx}{x^2 + a^2} = \frac{1}{2} \ln |x^2 + a^2|$$

$$\int \frac{dx}{ax+b} dx = \frac{1}{a} \ln |ax + b|$$

$$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}} \operatorname{atan} \frac{2ax+b}{\sqrt{4ac-b^2}}$$

$$\int \frac{x \, dx}{(x+a)^2+b^2} = \frac{1}{2} \ln | (x+a)^2 + b^2 | - \frac{a}{b} \operatorname{atan} \frac{x+a}{b}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \operatorname{atan} \frac{x}{a}$$

$$\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(x+a)(x-a)} = \frac{1}{2a} \left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right] = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$$

$$\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}[(n+1)x-a]}{(n+1)(n+2)}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}$$

$$\int \frac{x^2}{x^2+a^2} dx = x - a \left(\operatorname{atan} \frac{x}{a} \right)$$

$$\int \frac{x^3}{x^2+a^2} dx = \frac{1}{2} x^2 - \frac{1}{2} a^2 \ln |a^2 + x^2|$$

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a} \ln \left(\frac{a+x}{b+x} \right) \quad a \neq b$$

$$\int \frac{x \, dx}{ax^2+bx+c} = \frac{1}{2a} \ln |ax^2 + bx + c| - \frac{b}{a} \frac{1}{\sqrt{4ac-b^2}} \operatorname{atan} \frac{2ax+b}{\sqrt{4ac-b^2}}$$

$$\int \sqrt{x-a} \, dx = \frac{2}{3} (x-a)^{3/2}$$

$$\int \frac{dx}{\sqrt{x \pm a}} = 2\sqrt{x \pm a}$$

$$\int \frac{dx}{\sqrt{a-x}} = -2\sqrt{a-x}$$

$$\int \sqrt{ax+b} \, dx = \left(\frac{2b}{3a} + \frac{2x}{3} \right) \sqrt{ax+b}$$

$$\int (ax+b)^{3/2} dx = \frac{2}{5a} (ax+b)^{5/2}$$

$$\int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3} (x \mp 2a) \sqrt{x \pm a}$$

$$\int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x(a-x)} - a \operatorname{atan} \left(\frac{\sqrt{x(a-x)}}{x-a} \right)$$

$$\int \sqrt{\frac{x}{a+x}} dx = \sqrt{x(a+x)} - a \ln [\sqrt{x} + \sqrt{x+a}]$$

$$\int x\sqrt{ax+b} \, dx = \frac{2}{15a^2} (-2b^2 + abx + 3a^2x^2) \sqrt{ax+b}$$

$$\int \sqrt{x^2 \pm a^2} \, dx = \frac{1}{2} x \sqrt{x^2 \pm a^2} \pm \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right|$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \operatorname{atan} \frac{x}{\sqrt{a^2 - x^2}}$$

$$\int x \sqrt{x^2 \pm a^2} \, dx = \frac{1}{3} (x^2 \pm a^2)^{3/2}$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right|$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2}$$

$$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2}$$

$$\int \ln ax \, dx = x \ln ax - x$$

$$\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \frac{x^2}{4}$$

$$\int x^n \ln x \, dx = x^{n+1} \left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2} \right) \quad n \neq -1$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2} (\ln ax)^2$$

$$\int \frac{\ln ax}{x^2} dx = -\frac{1}{x} - \frac{\ln x}{x}$$

$$\int \ln(ax+b) \, dx = \left(x + \frac{b}{a} \right) \ln(ax+b) - x \quad a \neq 0$$

$$\int x \ln(ax+b) \, dx = \frac{bx}{2a} - \frac{1}{4} x^2 + \frac{1}{2} \left(x^2 - \frac{b^2}{a^2} \right) \ln(ax+b)$$

$$\int \ln(x^2 + a^2) \, dx = x \ln(x^2 + a^2) + 2a \operatorname{atan} \frac{x}{a} - 2x$$

$$\int \ln(x^2 - a^2) \, dx = x \ln(x^2 - a^2) + a \ln \frac{x+a}{x-a} - 2x$$

$$\int \ln(ax^2 + bx + c) \, dx = \frac{1}{\sqrt{4ac-b^2}} \operatorname{atan} \frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a} + x \right) \ln(ax^2 + bx + c)$$

$$\int (\ln x)^2 dx = 2x - 2x \ln x + x (\ln x)^2$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2} \right) e^{ax}$$

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right) e^{ax}$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax$$

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \sin^3 ax \, dx = -\frac{3 \cos ax}{4a} - \frac{\cos 3ax}{12a}$$

$$\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \cos^3 ax \, dx = \frac{3 \sin ax}{4a} + \frac{\sin 3ax}{12a}$$

$$\int \cos x \sin x \, dx = \frac{1}{2} \sin^2 x$$

$$\int \sin^2 x \cos x \, dx = \frac{1}{3} \sin^3 x$$

$$\int \cos^2 ax \sin ax \, dx = -\frac{1}{3a} \cos^3 ax$$

$$\int \cos^2 ax \sin^2 ax \, dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax$$

$$\int \sec x \tan x \, dx = \sec x$$

$$\int \sec^n x \tan x \, dx = \frac{1}{n} \sec^n x$$

$$\int \sec x \csc x \, dx = \ln |\tan x|$$

$$\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$$

$$\int x^2 \cos ax \, dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax$$

$$\int x \sin ax \, dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2}$$

$$\int x^2 \sin ax \, dx = \frac{(2-a^2x^2)}{a^3} \cos ax + \frac{2x \sin ax}{a^2}$$

$$\int x \cos^2 x \, dx = \frac{x^2}{4} + \frac{1}{8} \cos 2x + \frac{1}{4} x \sin 2x$$

$$\int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{1}{8} \cos 2x - \frac{1}{4} x \sin 2x$$

$$\int x \tan^2 x \, dx = -\frac{x^2}{2} + \ln (\cos x) + x \tan x$$

$$\int x \sec^2 x \, dx = \ln (\cos x) + x \tan x$$

$$\int x e^x \sin x \, dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x)$$

$$\int x e^x \cos x \, dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x)$$