Lecture #5, January 31, 2024

FEM. Weak form of BVP. ≡ variational eq.

Two spaces:

Trial solutions $\mathcal{S}$

Weighting functions $= \mathcal{V}$
= Test
= Variations

$$(W) \quad \int_0^L \Big(-w_{,x}\, a \underset{\mathcal{S}}{u} + w_{,x}\, \kappa\, \underset{\mathcal{S}}{u_{,x}} \Big)\, dx = \int_0^L w f\, dx \qquad \forall w \in \mathcal{V}$$

$$\mathcal{V} = \{ w \mid w(0) = 0,\ w(L) = 0,\ w \in H^1(0,L) \}$$

Sobolev sp.

$$\int_0^L w^2\, dx < \infty, \quad \int_0^L (w_{,x})^2\, dx$$

Sobolev embedding th'm:

$$C^0(0,L) \supset H^1(0,L)$$

continuous ftn's.

$$\mathcal{S} = \{ u \mid u(0) = g_0,\ u(L) = g_L,\ u \in H^1(0,L) \}$$

Given a, κ, f, g_0, g_L, find $u \in \mathcal{S} \ni (W)$ is satisfied.

Implications of the weak form:

Assume "smoothness": $w, u \in C^2(0,L)$ ← 2 continuous deriv's.

C^k ← k cont. deriv's.

$f \in C^0(0,L)$

Int.-by-parts:

(Euler-Lag. form)

$$0 = \int_0^L w\,(\underbrace{a\,u_{,x} - \kappa\,u_{,xx} - f}_{\text{res}(u)})\,dx$$

$$- \cancel{(w\,a\,u)}\Big|_0^L + \cancel{(w\,\kappa\,u_{,x})}\Big|_0^L \qquad \forall w \in \mathcal{V}$$

L^2 inner prod.

$$0 = (w, \text{res}(u)) \overset{\text{def.}}{=} \int_0^L w\,\text{res}(u)\,dx$$

$\Rightarrow$ AD eq sat. ← L^2 orthog. $\forall$

$\Leftrightarrow$ Fund. lemma of calc. of var's.

(W) $\Longleftrightarrow$ (S).

Galerkin: Modeled on (W), $\mathcal{S}, \mathcal{V}$.

Finite dimensional version of

$$\mathcal{V}^h = \{ w^h \mid w^h(0) = 0,\ w^h(L) = 0,\ w^h \in H^1(0,L) \}$$

$\subset \mathcal{V}$ ← subspace.

$$w^h(x) = \sum_{A=1}^{N-1} \underbrace{N_A(x)}_{\in\,\mathcal{V}^h}\, \underbrace{w_A}_{\in\,\mathbb{R}}$$

$$\mathcal{S}^h = \{ u^h \mid u^h(0) = g_0,\ u^h(L) = g_L,\ u^h \in H^1(0,L) \}$$

$$\subset \mathcal{S}$$

same.

$$\mathcal{S}^h = \mathcal{V}^h \oplus g^h$$

$$u^h = \sum_{B=1}^{N-1} N_B(x)\, \underset{\mathbb{R}}{\underset{\cap}{u_B}} + g^h(x)$$

$$g^h(x) = N_0(x)\, g_0 + N_N(x)\, g_L$$

Nodal FEM $\qquad N_A(x_B) = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{if } A \neq B \end{cases}$

$$= \delta_{AB} \qquad 0 \le A, B \le N$$

$$u^h(x_A) = \sum_{B=1}^{N-1} \underbrace{N_B(x_A)}_{\delta_{AB}} u_B + N_0(x_A)\, g_0 + N_N(x_A)\, g_L$$

$$= u_A + \delta_{0A}\, g_0 + \delta_{NA}\, g_L$$

$\forall$ ~~$A = 1, 2, \dots, N$.~~
$A = 0, 1, 2, \dots, N$

~~$A = 2, 3, \dots, n-$~~
$A = 1, 2, \dots, N-1$

$$u^h(x_A) = u_A + 0 + 0.$$

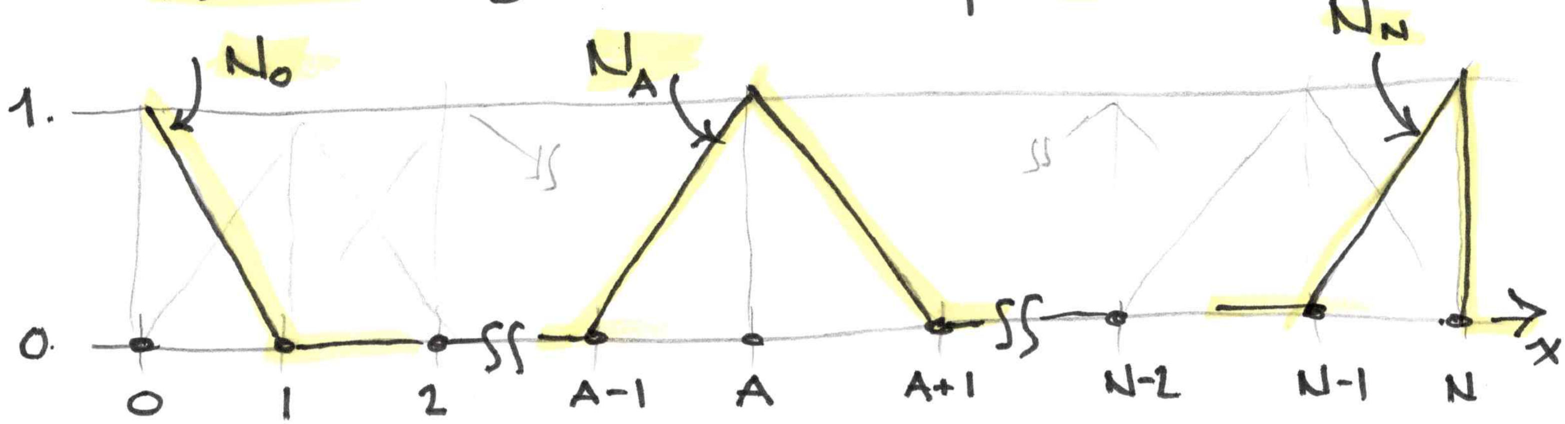
~~17~~ 18.

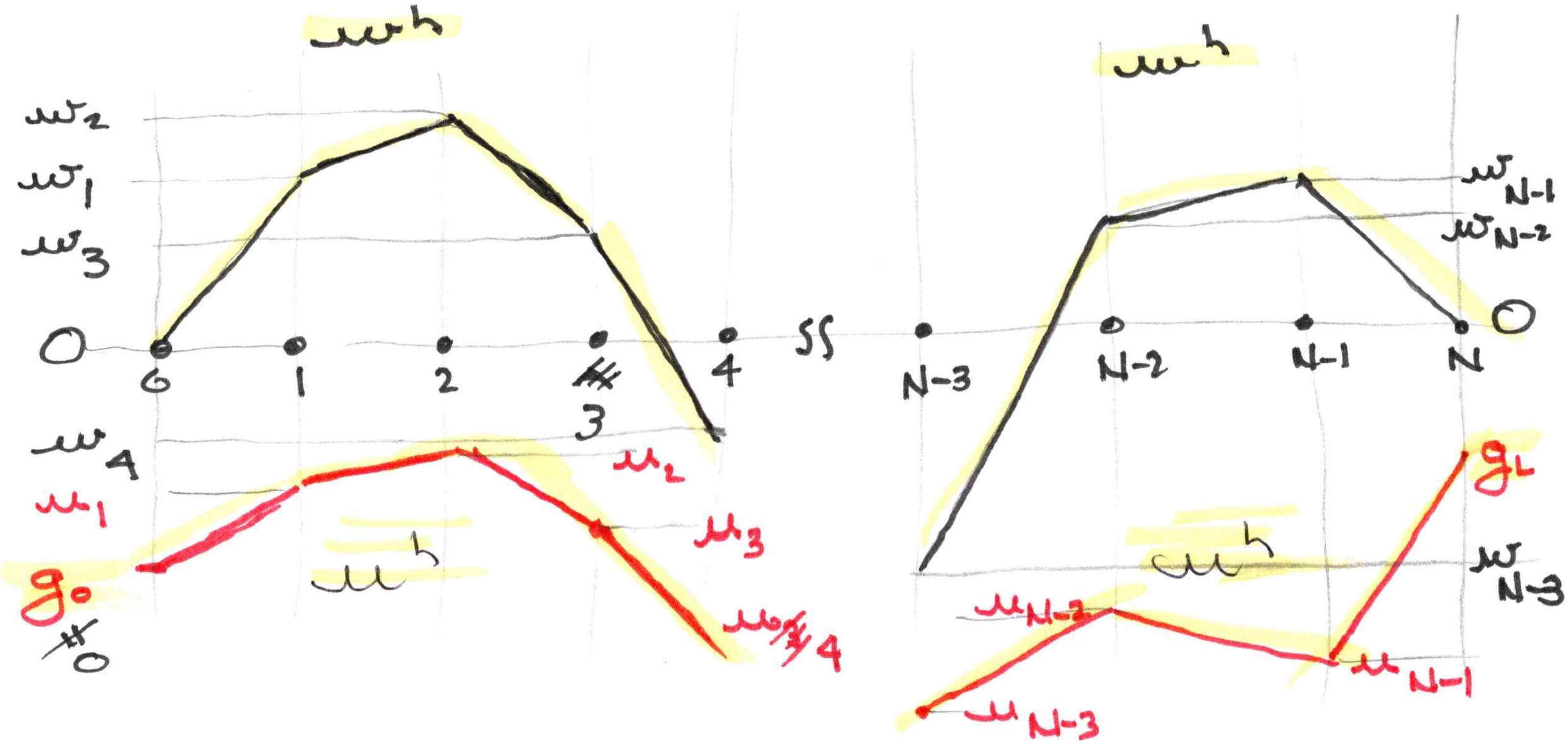
$$u^h(x_0) = 0 + \underbrace{\underbrace{N_0(x_0)}_{\delta_{00}}}_{=1} g_0 + \underbrace{\underbrace{N_N(x_0)}_{\delta_{N0}}}_{0} g_L$$

$$= g_0$$

$$u^h(x_N) = 0 + 0 + g_L .$$

Piecewise lin. FE space





Gal. form substitute our w^h, u^h ← into (W) : (G) = (W^h)

19.

$$0 = \int_0^L \left(-w^h_{,x}\, a\, u^h + w^h_{,x}\, x\, u^h_{,x} \right) dx - \int_0^L w^h f \, dx$$

$$0 = \sum_{A=1}^{N-1} \left\{ \int_0^L \left(-N_{A,x}\, w_A\, a \sum_{B=0}^{N} N_B u_B + N_{A,x}\, w_A\, x \sum_{B=0}^{N} N_{B,x} u_B \right) dx - \int_0^L N_A w_A f \, dx \right\}$$

$$0 = \sum_{A=1}^{N-1} w_A \left\{ \sum_{B=0}^{N} \underbrace{\int_0^L \left(-N_{A,x}\, a\, N_B + N_{A,x}\, x\, N_{B,x} \right) dx}_{K_{AB}=}\, u_B - \underbrace{\int_0^L N_A f \, dx}_{F_A} \right\}$$

$$0 = \sum_{A=1}^{N-1} w_A \left\{ \sum_{B=0}^{N} K_{AB} u_B - F_A \right\} \quad \forall w_A,\ A=1,2,\ldots N-1$$

$$0 = \sum_{B=0}^{N} K_{AB}\, u_B - F_A, \quad A = 1, 2, \ldots, N-1$$

$$u_0 = g_0$$

$$u_N = g_L$$