Exercise 2 — Pg 143

$n_{int} = 3$ Derive gauss quadrature rule.

3 integration points $\Longrightarrow$ 5th order polynomial

$$g(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 + \alpha_4 \xi^4 + \alpha_5 \xi^5$$

The exact integral is:

$$\int_{-1}^{1} g(\xi)\, d\xi = \left[\alpha_0 \xi + \cancel{\frac{\alpha_1 \xi^2}{2}} + \frac{\alpha_2 \xi^3}{3} + \cancel{\frac{\alpha_3 \xi^4}{4}} + \frac{\alpha_4 \xi^5}{5} + \cancel{\frac{\alpha_5 \xi^6}{6}} \right]_{-1}^{1}$$

$$= 2\alpha_0 + \frac{2}{3}\alpha_2 + \frac{2}{5}\alpha_4$$

This is to be equal to: (Note: $\tilde{\xi}_1 = -\tilde{\xi}_3$, $\tilde{\xi}_2 = 0$, $W_1 = W_3$

$$\sum_{l=1}^{3} g(\tilde{\xi}_l) W_l =$$

$$= W_1 g(\tilde{\xi}_1) + W_2 g(\tilde{\xi}_2) + W_3 g(\tilde{\xi}_3) =$$

$$= W_1 \left[g(\tilde{\xi}_1) + g(-\tilde{\xi}_1) \right] + W_2\, g(0)$$

$$= W_1 \left\{ \begin{array}{l} \alpha_0 + \cancel{\alpha_1 \tilde{\xi}_1} + \alpha_2 \tilde{\xi}_1^2 + / + \alpha_4 \tilde{\xi}_1^2 + / \\ \alpha_0 - \cancel{\alpha_1 \tilde{\xi}_1} + \alpha_2 \tilde{\xi}_1^2 - / + \alpha_4 \tilde{\xi}_1^2 \,/ \end{array} \right\} + W_2 \alpha_0$$

$$= \alpha_0 (2W_1 + W_2) + \alpha_2 (2W_1 \tilde{\xi}_1^2) + \alpha_4 (W_1 \alpha_4 2) =$$

Hence:

$$\alpha_0 (2W_1 + W_2) + \alpha_2 (2W_1 \tilde{\xi}_1^2) + \alpha_4 (2 W_1 \tilde{\xi}_1^4)$$

$$= 2\alpha_0 + \frac{2}{3}\alpha_2 + \frac{2}{5}\alpha_4$$

$$\begin{cases} 2W_1 + W_2 = 2 \\ 2W_1 \tilde{\xi}_1^2 = 2/3 \quad \rightarrow \quad \tilde{\xi}_1^2 = 1/3W_1 \qquad W_1 \neq 0 \\ 2W_1 \xi_1^4 = 2/5 \end{cases}$$

$$\Rightarrow \cancel{2}\cancel{W_1} \frac{1}{9W_1^{\cancel{2}}} = \cancel{2}/5 \rightarrow \boxed{W_1 = 5/9}$$

$$\cancel{2} \times \frac{5}{9} + \frac{W_2}{2} = \cancel{2} \rightarrow \boxed{W_2 = \frac{8}{9}}$$

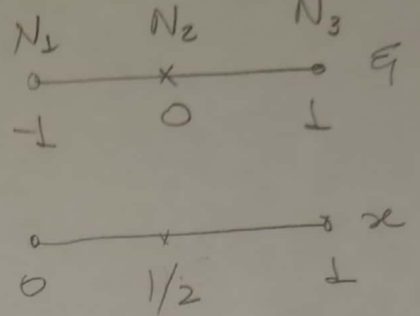
$$\tilde{\xi}_1 = \pm\sqrt{\frac{1}{3}\,\frac{9}{5}} = \pm\sqrt{\frac{3}{5}}$$

$$\rightarrow \boxed{\tilde{\xi}_1 = \sqrt{\frac{3}{5}}} \qquad \boxed{\tilde{\xi}_2 = -\xi_1 = -\sqrt{\frac{3}{5}}}$$

$\underline{Q2}$ $\quad \tilde{\xi}_i = \begin{Bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix} \qquad W_i = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$N_a(\xi) = \begin{Bmatrix} \xi/2\,(\xi - 1) \\ 1 - \xi^2 \\ \xi/2\,(\xi + 1) \end{Bmatrix} \qquad N_{a,\xi} = \begin{Bmatrix} \xi - 1/2 \\ -2\xi \\ \xi + 1/2 \end{Bmatrix}$$

$$x(\xi) = \sum_a N_a(\xi)\, x_a^e$$



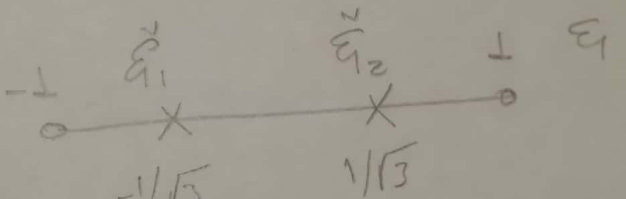
$$= \frac{1}{2} N_2 + N_3$$

$$= \frac{1}{2}(\xi + 1)$$

$$x_{,\xi} = 1/2 \qquad \xi_{,x} = 2$$

$$N_{a,x}^{(\xi)} = N_{a,\xi}\, \xi_{,x} = \begin{Bmatrix} 2\xi - 1 \\ -4\xi \\ 2\xi + 1 \end{Bmatrix}$$

— x —



$\underline{\text{Gauss points}}$

$$N_{a,x}(\tilde{\xi}_1) = \begin{Bmatrix} -\frac{2}{\sqrt{3}} - 1 \\ 4/\sqrt{3} \\ -2/\sqrt{3} + 1 \end{Bmatrix} \qquad N_{a,x}(\tilde{\xi}_2) = \begin{Bmatrix} 2/\sqrt{3} - 1 \\ -4/\sqrt{3} \\ 2/\sqrt{3} + 1 \end{Bmatrix}$$

$$N_a(\tilde{\xi}_1) = \begin{Bmatrix} \frac{1}{6}(1 + \sqrt{3}) \\ 2/3 \\ \frac{1}{6}(1 - \sqrt{3}) \end{Bmatrix} \qquad N_a(\tilde{\xi}_2) = \begin{Bmatrix} \frac{1}{6}(1 - \sqrt{3}) \\ 2/3 \\ \frac{1}{6}(1 + \sqrt{3}) \end{Bmatrix}$$

$$\underline{n}^e(\underline{d}^e) = \begin{Bmatrix} n^e_1(\underline{d}^e) \\ n^e_2(\underline{d}^e) \\ n^e_3(\underline{d}^e \end{Bmatrix} \qquad n^e_a(\underline{d}^e) = \int N_{a,x}\, k\, u_{,x}\, d\Omega$$

$$u_{,x} = \sum_1 N_B d_B$$

$$n^e_a = \int_{-1}^{1} N_{a,x}\, k(\xi)\, N_B\, d_B\, x_{,\xi}\, d\xi$$

$$= \sum_{l=1}^{2} W_l \left[N_{a,x}\, k\, N_B\, d_B\, x_{,\xi} \right]_{\tilde{\xi}_l}$$

$$n^e_a = \begin{Bmatrix} -2/\sqrt{3} - 1 \\ 4/\sqrt{3} \\ -2\sqrt{3} + 1 \end{Bmatrix} k(-1/\sqrt{3}) \left[\frac{1}{6}(1+\sqrt{3})d_1 + \frac{2}{3}d_2 + \frac{1}{6}(1-\sqrt{3})d_3 \right] \frac{1}{2} +$$

$$+ \begin{Bmatrix} 2/\sqrt{3} - 1 \\ -4/\sqrt{3} \\ 2/\sqrt{3} + 1 \end{Bmatrix} k(1/\sqrt{3}) \left[\frac{1}{6}(1-\sqrt{3})d_1 + \frac{2}{3}d_2 + \frac{1}{6}(1+\sqrt{3})d_3 \right] \frac{1}{2}$$

$$\underline{f}^{e} = \{ f_a^{e} \}$$

$$f_a^{e} = \int_{\Omega} N_a f \, d\Omega + h \, \delta_{a1} \delta_{e1} \quad , \; f = \sum_{b=1}^{3} N_b f_b^{e}$$

$$f_a^{e} = \int_{-1}^{1} N_a \sum_{b=1}^{3} N_b f_b^{e} \, x_{,\xi} \, d\xi + h \delta_{a1} \delta_{e1}$$

$$= \sum_{l=1}^{2} W_l \left[N_a \sum_{b=1}^{3} N_b f_b^{e} \, x_{,\xi} \right]_{\xi = \tilde{\xi}_l}$$

$$f_a^{e} = \begin{Bmatrix} \frac{1}{6}(1+\sqrt{3}) \\ 2/3 \\ 1/6\,(1-\sqrt{3}) \end{Bmatrix} \left[\frac{1}{6}(1+\sqrt{3}) f_1^{e} + \frac{2}{3} f_2^{e} + \frac{1}{6}(1-\sqrt{3}) f_3^{e} \right] \frac{1}{2} +$$

$$+ \begin{Bmatrix} \frac{1}{6}(1-\sqrt{3}) \\ 2/3 \\ \frac{1}{6}(1+\sqrt{3}) \end{Bmatrix} \left[\frac{1}{6}(1-\sqrt{3}) f_1^{e} + \frac{2}{3} f_2^{e} + \frac{1}{6}(1+\sqrt{3}) f_3^{e} \right] \frac{1}{2} +$$

$$+ h \, \delta_{a1} \delta_{e1}$$

//

$$D\underline{n}^e(\underline{d}^e) = \left[\frac{\partial n_a^e}{\partial d_b^e}\right] = \int_{\Omega^e} N_{a,x}\, k_{,u}\, N_b \underbrace{\left(\sum N_{c,x}\, d_c^e\right)}_{(\circ)} dx + \int_{\Omega^e} N_{a,x}\, k\, N_{b,x}\, dx$$

$$(\circ)_{\tilde{\xi}_1} = \left(-\frac{2}{\sqrt{3}} - 1\right) d_1^e + \frac{4}{\sqrt{3}} d_2^e + \left(-\frac{2}{\sqrt{3}} + 1\right) d_3^e$$

$$(\circ)_{\tilde{\xi}_2} = \left(2/\sqrt{3} - 1\right) d_1^e - \frac{4}{\sqrt{3}} d_2^e + \left(\frac{2}{\sqrt{3}} + 1\right) d_3^e$$

$$D\underline{n}^e(\underline{d}^e) = \int_{-1}^{1} N_{a,x}\, k_{,u}\, N_b\, (\circ)\, x_{,\xi}\, d\xi + \int_{-1}^{1} N_{a,x}\, k\, N_{b,x}\, x_{,\xi}\, d\xi$$

$$= \sum_{l=1}^{2} W_l \left\{ \quad - - - \quad \right\}_{\xi = \tilde{\xi}_l}$$

Now subs the values of $N_{a,x}$, N_b, $(\circ)$ etc at $\tilde{\xi}_1 = -1/\sqrt{3}$ and $\tilde{\xi}_2 = 1/\sqrt{3}$ to obtain the full tensor.

I obtained the final expression using python "sympy".

See next page.

In [1]:
```python
import sympy

def rat(expr):
    for i in expr.atoms(sympy.Float):
        r = sympy.Rational(str(i)).limit_denominator(1000)
        expr = expr.subs(i, r)
    return expr
```

In [2]:
```python
from IPython.display import display, Markdown
import sympy
from sympy import *
import numpy as np
x, xi, u = symbols(r'x \xi u', real=True)
d1, d2, d3 = symbols(r'd_1 d_2 d_3', real=True)

Xi = [ -1/sqrt(3), 1/sqrt(3) ]
W = [ 1, 1 ]

kappa = Function(r"K")(xi)
dkappa_du = Function(r"K_{,u}")(xi)

Na = [
1/2*xi*(xi-1),
1-xi*xi,
1/2*xi*(xi+1)
]

display( Markdown( f"$N_1 = {sympy.latex(Na[0])}$" ))
display( Markdown( f"$N_2 = {sympy.latex(Na[1])}$" ))
display( Markdown( f"$N_3 = {sympy.latex(Na[2])}$" ))
```

$N_1 = 0.5\xi\left(\xi - 1\right)$

$N_2 = 1 - \xi^2$

$N_3 = 0.5\xi\left(\xi + 1\right)$

In [3]:
```python
dN1_xi = diff(Na[0], xi)
dN2_xi = diff(Na[1], xi)
dN3_xi = diff(Na[2], xi)

display( Markdown( r"$dN_1/d\xi = "+ f"{sympy.latex(dN1_xi)}$" ))
display( Markdown( r"$dN_2/d\xi = "+ f"{sympy.latex(dN2_xi)}$" ))
display( Markdown( r"$dN_3/d\xi = "+ f"{sympy.latex(dN3_xi)}$" ))
```

$dN_1/d\xi = 1.0\xi - 0.5$

$dN_2/d\xi = -2\xi$

$dN_3/d\xi = 1.0\xi + 0.5$

In [4]:
```python
X = Na[1] * 0.5 + Na[2]
X = simplify(X)
dx_dxi = diff(X,xi)
dxi_dx = 1/dx_dxi

display( Markdown( r"$x(\xi) = "+ f"{sympy.latex(X)}$"))
```

```
display( Markdown( r"$dx/d\xi = "+ f"{sympy.latex(dx_dxi)}$" ))
display( Markdown( r"$d\xi/dx = "+ f"{sympy.latex(dxi_dx)}$" ))
```

$x(\xi) = 0.5\xi + 0.5$

$dx/d\xi = 0.5$

$d\xi/dx = 2.0$

In [5]:
```
dN1_x = dN1_xi * dxi_dx
dN2_x = dN2_xi * dxi_dx
dN3_x = dN3_xi * dxi_dx

display( Markdown( r"$dN_1/dx = "+ f"{sympy.latex(dN1_x)}$" ))
display( Markdown( r"$dN_2/dx = "+ f"{sympy.latex(dN2_x)}$" ))
display( Markdown( r"$dN_3/dx = "+ f"{sympy.latex(dN3_x)}$" ))
```

$dN_1/dx = 2.0\xi - 1.0$

$dN_2/dx = -4.0\xi$

$dN_3/dx = 2.0\xi + 1.0$

In [11]:
```
def build_N_x( xi_ ) :
    # Derivatives in X space
    N1_x = dN1_xi.subs(xi, xi_) * dxi_dx
    N2_x = dN2_xi.subs(xi, xi_) * dxi_dx
    N3_x = dN3_xi.subs(xi, xi_) * dxi_dx
    return N1_x, N2_x, N3_x

db = [ d1, d2, d3 ]
ne = [ 0, 0, 0 ]

f = symbols(r'f_1 f_2 f_3', real=True)
h = symbols(r'h', real=True)
fe = [ 0, 0, 0 ]
dna_ddb = zeros( 3, 3)
fcol = []
for xi_, W_ in zip( Xi, W ) :
    Na_x_ = build_N_x(xi_)
    x_ = X.subs(xi, xi_)
    kappa_ = kappa.subs( xi, xi_ )
    dkappa_du_ = dkappa_du.subs( xi, xi_ )
    fcol.append(kappa_)
    fcol.append(dkappa_du_)
    Na_ = [0, 0, 0]
    for a in range(3) : Na_[a] = simplify(Na[a].subs(xi, xi_))

    q = 0
    for b in range(3) : q += db[b] * Na_x_[b]
    for a in range(3) :
        ne[a] += W_ * dx_dxi * Na_x_[a] * q * kappa_
        fe[a] += W_ * dx_dxi * Na_[a] * f[a]
        fe[a] = simplify(fe[a])

    for a in range(3) :
        for b in range(3) :
            dna_ddb[a,b] += W_ * dx_dxi * Na_x_[a] * Na_[b] * q * dkappa_du_
            dna_ddb[a,b] += W_ * dx_dxi * kappa_ * Na_x_[a] * Na_x_[b]
```

```
    fe[0] += W_ * h * Na_[0]
    fe[0] = simplify(fe[0])

for a in range(3) :
    ne[a] = cancel(ne[a])
    for c in fcol : ne[a] = collect(ne[a],c)
    for d in db : ne[a] = collect( ne[a], d )
    ne[a] = simplify(ne[a],full=True)

for a in range(3) :
    fe[a] = nsimplify(fe[a])

for a in range(3) :
    for b in range(3) :
        dna_ddb[a,b] = cancel(dna_ddb[a,b])
        for c in fcol : dna_ddb[a,b] = collect(dna_ddb[a,b],c)
        for d in db : dna_ddb[a,b] = collect( dna_ddb[a,b], d )
        dna_ddb[a,b] = simplify(dna_ddb[a,b])
```

In [12]:
```
for a in range(3) :
    display( Markdown( f"$n_{a+1}^e(d^e) = {sympy.latex(rat(ne[a]))}$" ))
```

$$n_1^e(d^e) = \left(d_1\left(\frac{7}{6} - \frac{2\sqrt{3}}{3}\right) - d_2\left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right) + \frac{d_3}{6}\right)K\left(\frac{\sqrt{3}}{3}\right) + \left(d_1\left(\frac{2\sqrt{3}}{3} + \frac{7}{6}\right) - d_2\left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right) + \frac{d_3}{6}\right)K\left(-\frac{\sqrt{3}}{3}\right)$$

$$n_2^e(d^e) = -\left(d_1\left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right) - \frac{8d_2}{3} + d_3\left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right)\right)K\left(\frac{\sqrt{3}}{3}\right) - \left(d_1\left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right) - \frac{8d_2}{3} + d_3\left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right)\right)K\left(-\frac{\sqrt{3}}{3}\right)$$

$$n_3^e(d^e) = \left(\frac{d_1}{6} - d_2\left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right) + d_3\left(\frac{7}{6} - \frac{2\sqrt{3}}{3}\right)\right)K\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{d_1}{6} - d_2\left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right) + d_3\left(\frac{2\sqrt{3}}{3} + \frac{7}{6}\right)\right)K\left(\frac{\sqrt{3}}{3}\right)$$

In [13]:
```
for a in range(3) :
    display( Markdown( f"$f_{a+1}^e = {sympy.latex(rat(fe[a]))}$" ))
```

$$f_1^e = \frac{f_1}{6} + \frac{h}{3}$$

$$f_2^e = \frac{2f_2}{3}$$

$$f_3^e = \frac{f_3}{6}$$

In [15]:
```
for a in range(3) :
    for b in range(3) :
        display( Markdown( r"$\frac{\partial n_"+f"{a+1}"+r"^e}{\partial d_"+f"{b+1}"+r"^e} $ ="+ f"${sympy.latex(rat(dna_ddb[a,b]))}$" ))
    print("\n\n")
```

$$\frac{\partial n_1^e}{\partial d_1^e} = \left(d_1\left(\frac{19}{36} - \frac{11\sqrt{3}}{36}\right) - d_2\left(\frac{5}{9} - \frac{\sqrt{3}}{3}\right) + \frac{d_3\left(1-\sqrt{3}\right)}{36}\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(d_1\left(\frac{19}{36} + \frac{11\sqrt{3}}{36}\right) - d_2\left(\frac{5}{9} + \frac{\sqrt{3}}{3}\right) + \frac{d_3\left(1+\sqrt{3}\right)}{36}\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{2\sqrt{3}}{3} + \frac{7}{6}\right)K\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{7}{6} - \frac{2\sqrt{3}}{3}\right)K\left(\frac{\sqrt{3}}{3}\right)$$

$$\frac{\partial n_1^e}{\partial d_2^e} = \left(d_1\left(\frac{7}{9} - \frac{4\sqrt{3}}{9}\right) - d_2\left(\frac{8}{9} - \frac{4\sqrt{3}}{9}\right) + \frac{d_3}{9}\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(d_1\left(\frac{4\sqrt{3}}{9} + \frac{7}{9}\right) - d_2\left(\frac{4\sqrt{3}}{9} + \frac{8}{9}\right) + \frac{d_3}{9}\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right)K\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right)K\left(\frac{\sqrt{3}}{3}\right)$$

$$\frac{\partial n_1^e}{\partial d_3^e} = \left(-d_1\left(\frac{5}{36} - \frac{\sqrt{3}}{12}\right) + d_2\left(\frac{1}{9} - \frac{\sqrt{3}}{9}\right) + \frac{d_3\left(1+\sqrt{3}\right)}{36}\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(-d_1\left(\frac{5}{36} + \frac{\sqrt{3}}{12}\right) + d_2\left(\frac{1}{9} + \frac{\sqrt{3}}{9}\right) + \frac{d_3\left(1-\sqrt{3}\right)}{36}\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right) + \frac{K\left(-\frac{\sqrt{3}}{3}\right)}{6} + \frac{K\left(\frac{\sqrt{3}}{3}\right)}{6}$$

$$\frac{\partial n_2^e}{\partial d_1^e} = \left(-d_1\left(\frac{5}{9} - \frac{\sqrt{3}}{3}\right) + \frac{4d_2\left(1-\sqrt{3}\right)}{9} + d_3\left(\frac{1}{9} + \frac{\sqrt{3}}{9}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(-d_1\left(\frac{5}{9} + \frac{\sqrt{3}}{3}\right) + \frac{4d_2\left(1+\sqrt{3}\right)}{9} + d_3\left(\frac{1}{9} - \frac{\sqrt{3}}{9}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right)K\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right)K\left(\frac{\sqrt{3}}{3}\right)$$

$$\frac{\partial n_2^e}{\partial d_2^e} = -\left(d_1\left(\frac{8}{9} - \frac{4\sqrt{3}}{9}\right) - \frac{16d_2}{9} + d_3\left(\frac{4\sqrt{3}}{9} + \frac{8}{9}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right) - \left(d_1\left(\frac{4\sqrt{3}}{9} + \frac{8}{9}\right) - \frac{16d_2}{9} + d_3\left(\frac{8}{9} - \frac{4\sqrt{3}}{9}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right) + \frac{8K\left(-\frac{\sqrt{3}}{3}\right)}{3} + \frac{8K\left(\frac{\sqrt{3}}{3}\right)}{3}$$

$$\frac{\partial n_2^e}{\partial d_3^e} = \left(d_1\left(\frac{1}{9} - \frac{\sqrt{3}}{9}\right) + \frac{4d_2\left(1+\sqrt{3}\right)}{9} - d_3\left(\frac{5}{9} + \frac{\sqrt{3}}{3}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(d_1\left(\frac{1}{9} + \frac{\sqrt{3}}{9}\right) + \frac{4d_2\left(1-\sqrt{3}\right)}{9} - d_3\left(\frac{5}{9} - \frac{\sqrt{3}}{3}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right)K\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right)K\left(\frac{\sqrt{3}}{3}\right)$$

$$\frac{\partial n_3^e}{\partial d_1^e} = \left(\frac{d_1\left(1-\sqrt{3}\right)}{36} + d_2\left(\frac{1}{9} + \frac{\sqrt{3}}{9}\right) - d_3\left(\frac{5}{36} + \frac{\sqrt{3}}{12}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(\frac{d_1\left(1+\sqrt{3}\right)}{36} + d_2\left(\frac{1}{9} - \frac{\sqrt{3}}{9}\right) - d_3\left(\frac{5}{36} - \frac{\sqrt{3}}{12}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right) + \frac{K\left(-\frac{\sqrt{3}}{3}\right)}{6} + \frac{K\left(\frac{\sqrt{3}}{3}\right)}{6}$$

$$\frac{\partial n_3^e}{\partial d_2^e} = \left(\frac{d_1}{9} - d_2\left(\frac{8}{9} - \frac{4\sqrt{3}}{9}\right) + d_3\left(\frac{7}{9} - \frac{4\sqrt{3}}{9}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{d_1}{9} - d_2\left(\frac{4\sqrt{3}}{9} + \frac{8}{9}\right) + d_3\left(\frac{4\sqrt{3}}{9} + \frac{7}{9}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right) - \left(\frac{4}{3} - \frac{2\sqrt{3}}{3}\right)K\left(-\frac{\sqrt{3}}{3}\right) - \left(\frac{2\sqrt{3}}{3} + \frac{4}{3}\right)K\left(\frac{\sqrt{3}}{3}\right)$$

$$\frac{\partial n_3^e}{\partial d_3^e} = \left(\frac{d_1\left(1-\sqrt{3}\right)}{36} - d_2\left(\frac{5}{9} - \frac{\sqrt{3}}{3}\right) + d_3\left(\frac{19}{36} - \frac{11\sqrt{3}}{36}\right)\right)K_{,u}\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{d_1\left(1+\sqrt{3}\right)}{36} - d_2\left(\frac{5}{9} + \frac{\sqrt{3}}{3}\right) + d_3\left(\frac{19}{36} + \frac{11\sqrt{3}}{36}\right)\right)K_{,u}\left(\frac{\sqrt{3}}{3}\right) + \left(\frac{7}{6} - \frac{2\sqrt{3}}{3}\right)K\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{2\sqrt{3}}{3} + \frac{7}{6}\right)K\left(\frac{\sqrt{3}}{3}\right)$$

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