CSE 397 / EM 397 - Stabilized and Variational Multiscale Methods in CFD

Homework #2 - Solution

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Exercise 2.3 (25 points) Some simple interpolation estimates in the "max norm"

Consider piecewise linear finite elements. Given $u \in C^2(]0, L[)$, obtain a bound for the interpolation error $\eta = \tilde{u}^h - u$ and its derivative $\eta_{,x}$.

Solution

Let us the mesh parameter $h_A = x_{A+1} - x_A$ for all $0 \leq A \leq N-1$ in which N is the number of nodes. The nodally exact piecewise linear element for $x \in]x_A, x_{A+1}[$ can be written as

$$\tilde{u}^h(x) = \frac{x_{A+1} - x}{h_A} u_A + \frac{x - x_A}{h_A} u_{A+1} \tag{1}$$

Since $\tilde{u}^h(x)$ is nodally exact, $u_A = u(x_A)$ and $u_{A+1} = u(x_{A+1})$. Let us consider $u \in C^2(]x_A, x_{A+1}[)$, we can use the finite Taylor expansion at $x \in]x_A, x_{A+1}[$; that is, there exist $\xi_A \in]x_A, x[$ and $\xi_{A+1} \in]x, x_{A+1}[$ such that

$$\begin{aligned} u_A = u(x_A) &= u(x) + u'(x)(x_A - x) + \frac{1}{2}u''(\xi_A)(x_A - x)^2 \\ u_{A+1} = u(x_{A+1}) &= u(x) + u'(x)(x_{A+1} - x) + \frac{1}{2}u''(\xi_{A+1})(x_{A+1} - x)^2 \end{aligned} \tag{2}$$

Therefore, the piecewise interpolation can be rewritten as

$$\begin{aligned} \tilde{u}^h(x) &= \frac{x_{A+1} - x}{h_A}\left[u(x) + u'(x)(x_A - x) + \frac{1}{2}u''(\xi_A)(x_A - x)^2\right] \\ &\quad + \frac{x - x_A}{h_A}\left[u(x) + u'(x)(x_{A+1} - x) + \frac{1}{2}u''(\xi_{A+1})(x_{A+1} - x)^2\right] \\ &= u(x)\frac{x_{A+1} - x_A}{h_A} + u'(x)\frac{(x_{A+1} - x)(x_A - x) + (x - x_A)(x_{A+1} - x)}{h_A} \\ &\quad + \frac{1}{2h}u''(\xi_A)(x_{A+1} - x)(x - x_A)^2 + \frac{1}{2h}u''(\xi_{A+1})(x_{A+1} - x)^2(x - x_A) \\ &= u(x) + \frac{(x - x_A)(x_{A+1} - x)}{2}\left[u''(\xi_A)\frac{x - x_A}{h_A} + u''(\xi_{A+1})\frac{x_{A+1} - x}{h_A}\right] \end{aligned} \tag{3}$$

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Since $u'' \in C^0\left(]x_A, x_{A+1}[\right)$ and

$$\min\left(u''(\xi_A), u''(\xi_{A+1})\right) < u''(\xi_A)\frac{x - x_A}{h_A} + u''(\xi_{A+1})\frac{x_{A+1} - x}{h_A} < \max\left(u''(\xi_A), u''(\xi_{A+1})\right), \tag{4}$$

we can use the intermediate value theorem, i.e., there exists $\xi \in]\xi_A, \xi_{A+1}[\subset]x_A, x_{A+1}[$ such that

$$u''(\xi_A)\frac{x - x_A}{h_A} + u''(\xi_{A+1})\frac{x_{A+1} - x}{h_A} = u''(\xi) \tag{5}$$

Hence, the interpolation error for $x \in]x_A, x_{A+1}[$ becomes

$$\begin{aligned} \eta(x) &= \tilde{u}^h(x) - u(x) \\ &= \frac{(x - x_A)(x_{A+1} - x)}{2}\left[u''(\xi_A)\frac{x - x_A}{h_A} + u''(\xi_{A+1})\frac{x_{A+1} - x}{h_A}\right] \\ &= \frac{(x - x_A)(x_{A+1} - x)}{2} u''(\xi), \end{aligned} \tag{6}$$

The L^∞-norm of the interpolation error is

$$\sup_{x \in]x_A, x_{A+1}[} |\eta(x)| \leq \sup_{x \in]x_A, x_{A+1}[} \frac{|x - x_A||x_{A+1} - x|}{2} \sup_{\xi \in]x_A, x_{A+1}[} |u''(\xi)| \tag{7}$$

By the Young's inequality,

$$\frac{|x - x_A| + |x_{A+1} - x|}{2} = \frac{h_A}{2} \geq \sqrt{|x - x_A|\,|x_{A+1} - x|} \tag{8}$$

we have

$$|x - x_A|\,|x_{A+1} - x| \leq \frac{h_A^2}{4} \tag{9}$$

Therefore, the interpolation error becomes

$$\sup_{x \in]x_A, x_{A+1}[} |\eta(x)| \leq \frac{h_A^2}{8} \sup_{\xi \in]x_A, x_{A+1}[} |u''(\xi)| \tag{10}$$

In other words, for $x \in]x_A, x_{A+1}[$ there exists $\xi \in]x_A, x_{A+1}[$ such that

$$\|\eta(x)\|_{L^\infty(]x_A, x_{A+1}[)} \leq \frac{h_A^2}{8} \|u''(\xi)\|_{L^\infty(]x_A, x_{A+1}[)} \tag{11}$$

Likewise, we can obtain the interpolation error of the other elements. The global L^∞-norm of the interpolation error can be obtained as follows:

$$\|\eta(x)\|_{L^\infty([0,L])} \leq \frac{1}{8} \max_{0 \leq A \leq N-1} h_A^2 \|u''(\xi)\|_{L^\infty(]x_A, x_{A+1}[)} \tag{12}$$