Exercise 2 - Pg 143

$n_{int} = 3$ Derive gauss quadrature rule.

3 integration points $\Longrightarrow$ 5th order polynomial

$$g(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 + \alpha_4 \xi^4 + \alpha_5 \xi^5$$

The exact integral is:

$$\int_{-1}^{1} g(\xi)\, d\xi = \Big[\alpha_0 \xi + \cancel{\frac{\alpha_1 \xi^2}{2}} + \frac{\alpha_2 \xi^3}{3} + \cancel{\frac{\alpha_3 \xi^4}{4}} + \frac{\alpha_4 \xi^5}{5} + \cancel{\frac{\alpha_5 \xi^6}{6}}\Big]_{-1}^{1}$$

$$= 2\alpha_0 + \frac{2}{3}\alpha_2 + \frac{2}{5}\alpha_4$$

This is to be equal to: (Note: $\tilde{\xi}_1 = -\tilde{\xi}_3$, $\tilde{\xi}_2 = 0$, $W_1 = W_3$)

$$\sum_{l=1}^{3} g(\tilde{\xi}_l) W_l =$$

$$= W_1 g(\tilde{\xi}_1) + W_2 g(\tilde{\xi}_2) + W_3 g(\tilde{\xi}_3) =$$

$$= W_1 \left[g(\tilde{\xi}_1) + g(-\tilde{\xi}_1) \right] + W_2 g(0)$$

$$= W_1 \left\{ \begin{matrix} \alpha_0 + \cancel{\alpha_1 \tilde{\xi}_1} + \alpha_2 \tilde{\xi}_1^2 - / + \alpha_4 \tilde{\xi}_1^2 + / \\ \alpha_0 - \cancel{\alpha_1 \xi_1} + \alpha_2 \xi_1^2 - / + \alpha_4 \xi_1^2 / \end{matrix} \right\} + W_2 \alpha_0$$

$$= \alpha_0 (2W_1 + W_2) + \alpha_2 (2W_1 \tilde{\xi}_1^2) + \alpha_4 (W_1 \alpha_4 2) =$$

Hence:

$$\alpha_0(2W_1+W_2)+\alpha_2(2W_1\tilde{\xi}_1^2)+\alpha_4(2W_1\tilde{\xi}_1^4)$$

$$=2\alpha_0+\frac{2}{3}\alpha_2+\frac{2}{5}\alpha_4$$

$$\begin{cases} 2W_1+W_2=2 \\ 2W_1\tilde{\xi}_1^2=2/3 \rightarrow \tilde{\xi}_1^2=1/3W_1 \\ 2W_1\tilde{\xi}_1^4=2/5 \end{cases}$$

$W_1 \neq 0$

$$\Rightarrow \cancel{2}\cancel{W_1}\frac{1}{9W_1^2}=\cancel{2}/5 \rightarrow \boxed{W_1=5/9}$$

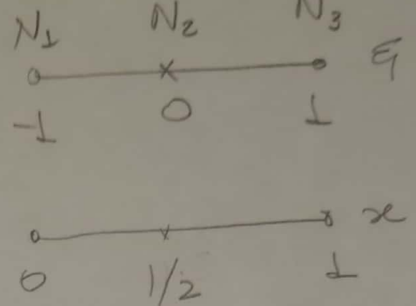
$$\cancel{2}\times\frac{5}{9}+\frac{W_2}{2}=\cancel{2} \rightarrow \boxed{W_2=\frac{8}{9}}$$

$$\tilde{\xi}_1=\pm\sqrt{\frac{1}{3}\frac{9}{5}}=\pm\sqrt{\frac{3}{5}}$$

$$\rightarrow \boxed{\tilde{\xi}_1=\sqrt{\frac{3}{5}}} \quad \boxed{\tilde{\xi}_2=-\xi_1=-\sqrt{\frac{3}{5}}}$$

Q2 $\tilde{\xi}_i = \begin{Bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix}$ $W_i = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$N_a(\xi) = \begin{Bmatrix} \xi/2\,(\xi - 1) \\ 1 - \xi^2 \\ \xi/2\,(\xi + 1) \end{Bmatrix} \qquad N_{a,\xi} = \begin{Bmatrix} \xi - 1/2 \\ -2\xi \\ \xi + 1/2 \end{Bmatrix}$$



$$x(\xi) = \sum N_a(\xi)\, x_a^e$$

$$= \frac{1}{2} N_2 + N_3$$

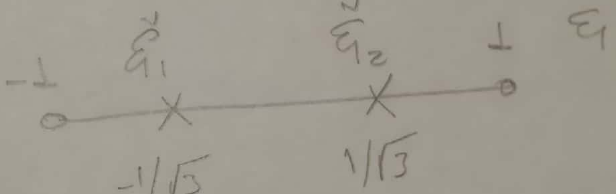
$$= \frac{1}{2}(\xi + 1)$$

$$x_{,\xi} = 1/2 \qquad \xi_{,x} = 2$$

$$N_{a,x}^{(\xi)} = N_{a,\xi}\, \xi_{,x} = \begin{Bmatrix} 2\xi - 1 \\ -4\xi \\ 2\xi + 1 \end{Bmatrix}$$

— x —

Gauss points



$$N_{a,x}(\tilde{\xi}_1) = \begin{Bmatrix} -\frac{2}{\sqrt{3}} - 1 \\ 4/\sqrt{3} \\ -2/\sqrt{3} + 1 \end{Bmatrix} \qquad N_{a,x}(\tilde{\xi}_2) = \begin{Bmatrix} 2/\sqrt{3} - 1 \\ -4/\sqrt{3} \\ 2/\sqrt{3} + 1 \end{Bmatrix}$$

$$N_a(\tilde{\xi}_1) = \begin{Bmatrix} \frac{1}{6}(1 + \sqrt{3}) \\ 2/3 \\ \frac{1}{6}(1 - \sqrt{3}) \end{Bmatrix} \qquad N_a(\tilde{\xi}_2) = \begin{Bmatrix} \frac{1}{6}(1 - \sqrt{3}) \\ 2/3 \\ \frac{1}{6}(1 + \sqrt{3}) \end{Bmatrix}$$

$$\underline{n}^e(\underline{d}^e) = \begin{Bmatrix} n^e_1(\underline{d}^e) \\ n^e_2(\underline{d}^e) \\ n^e_3(\underline{d}^e \end{Bmatrix} \qquad n^e_a(\underline{d}^e) = \int N_{a,x}\, k\, u_{,x}\, d\Omega$$

$$u_{,x} = \sum_1 N_B\, d_B$$

$$n^e_a = \int_{-1}^{1} N_{a,x}\, k(\xi)\, N_B\, d_B\, x_{,\xi}\, d\xi$$

$$= \sum_{l=1}^{2} W_l \left[N_{a,x}\, k\, N_B\, d_B\, x_{,\xi} \right]_{\tilde{\xi}_l}$$

$$n^e_a = \begin{Bmatrix} -2/\sqrt{3} - 1 \\ 4/\sqrt{3} \\ -2\sqrt{3} + 1 \end{Bmatrix} k(-1/\sqrt{3}) \left[\frac{1}{6}(1+\sqrt{3})d_1 + \frac{2}{3}d_2 + \frac{1}{6}(1-\sqrt{3})d_3 \right] \frac{1}{2} +$$

$$+ \begin{Bmatrix} 2/\sqrt{3} - 1 \\ -4/\sqrt{3} \\ 2/\sqrt{3} + 1 \end{Bmatrix} k(1/\sqrt{3}) \left[\frac{1}{6}(1-\sqrt{3})d_1 + \frac{2}{3}d_2 + \frac{1}{6}(1+\sqrt{3})d_3 \right] \frac{1}{2}$$

$$\underline{f}^e = \{ f_a^e \}$$

$$f_a^e = \int_\Omega N_a f \, d\Omega + h \, \delta_{a1} \delta_{e1} \quad , f = \sum_{b=1}^{3} N_b f_b^e$$

$$f_a^e = \int_{-1}^{+1} N_a \sum_{b=1}^{3} N_b f_b^e \, x,_\xi \, d\xi + h \, \delta_{a1} \delta_{e1}$$

$$= \sum_{l=1}^{2} W_l \left[N_a \sum_{b=1}^{3} N_b f_b^e \, x,_\xi \right]_{\xi = \tilde{\xi}_l}$$

$$f_a^e = \begin{Bmatrix} \frac{1}{6}(1+\sqrt{3}) \\ 2/3 \\ 1/6\,(1-\sqrt{3}) \end{Bmatrix} \left[\frac{1}{6}(1+\sqrt{3}) f_1^e + \frac{2}{3} f_2^e + \frac{1}{6}(1-\sqrt{3}) f_3^e \right] \frac{1}{2} +$$

$$+ \begin{Bmatrix} \frac{1}{6}(1-\sqrt{3}) \\ 2/3 \\ \frac{1}{6}(1+\sqrt{3}) \end{Bmatrix} \left[\frac{1}{6}(1-\sqrt{3}) f_1^e + \frac{2}{3} f_2^e + \frac{1}{6}(1+\sqrt{3}) f_3^e \right] \frac{1}{2} +$$

$$+ h \, \delta_{a1} \delta_{e1}$$

//

$$D\underline{n}^e(\underline{d}^e) = \left[\frac{dn_a^e}{dd_b^e}\right] = \int_{\Omega^e} N_{a,x}\, k_{,u}\, N_b \underbrace{\left(\sum N_{c,x}\, d_c^e\right)}_{(\circ)} dx + \int_{\Omega^e} N_{a,x}\, k\, N_{b,x}\, dx$$

$$(\circ)_{\tilde{\xi}_1} = \left(-\frac{2}{\sqrt{3}} - 1\right) d_1^e + \frac{4}{\sqrt{3}} d_2^e + \left(-\frac{2}{\sqrt{3}} + 1\right) d_3^e$$

$$(\circ)_{\tilde{\xi}_2} = \left(2/\sqrt{3} - 1\right) d_1^e - \frac{4}{\sqrt{3}} d_2^e + \left(\frac{2}{\sqrt{3}} + 1\right) d_3^e$$

$$D\underline{n}^e(\underline{d}^e) = \int_{-1}^{1} N_{a,x}\, k_{,u}\, N_b\, (\circ)\, x_{,\xi}\, d\xi + \int_{-1}^{1} N_{a,x}\, k\, N_{b,x}\, x_{,\xi}\, d\xi$$

$$= \sum_{l=1}^{2} W_l \left\{ \cdots \right\}_{\xi = \tilde{\xi}_l}$$

Now subs the values of $N_{a,x}$, N_b, $(\circ)$ etc at $\tilde{\xi}_1 = -1/\sqrt{3}$ and $\tilde{\xi}_2 = 1/\sqrt{3}$ to obtain the full tensor.

I obtained the final expression using python "sympy".

See next page.