CSE 397 / EM 397 - Stabilized and Variational Multiscale Methods in CFD

Homework #3

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Multi-dimensional Pure Advection Problem

Let $\Omega = \bigcup_{e=1}^{n_{el}} \Omega^e$, where Ω^e is an element domain, $e = 1, 2, \ldots, n_{el}$ and let $\Gamma = \partial\Omega$ denote its boundary. Likewise, the boundary of element Ω^e is denoted $\Gamma^e = \partial\Omega^e$. The inflow and outflow boundaries are defined as follows:

$$\Gamma_{in} = \{\boldsymbol{x} : \boldsymbol{x} \in \Gamma,\, a_n(\boldsymbol{x}) = \boldsymbol{a}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) < 0\} \tag{1}$$

$$\Gamma_{out} = \{\boldsymbol{x} : \boldsymbol{x} \in \Gamma,\, a_n(\boldsymbol{x}) = \boldsymbol{a}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \geq 0\} \tag{2}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the unit outward normal vector to $\boldsymbol{x} \in \Gamma$. The element inflow and outflow boundaries, Γ^e_{in} and Γ^e_{out} are defined similarly. The given data are the source $f : \Omega \to \mathbb{R}$, $g : \Gamma_{in} \to \mathbb{R}$ and $\boldsymbol{a} : \Omega \to \mathbb{R}^d$ where $d \geq 2$ is the number of space dimensions. We assume $\boldsymbol{a} \in [C^1(\Omega)]^d$ and solenoidal, i.e. $\nabla \cdot \boldsymbol{a} = 0$.

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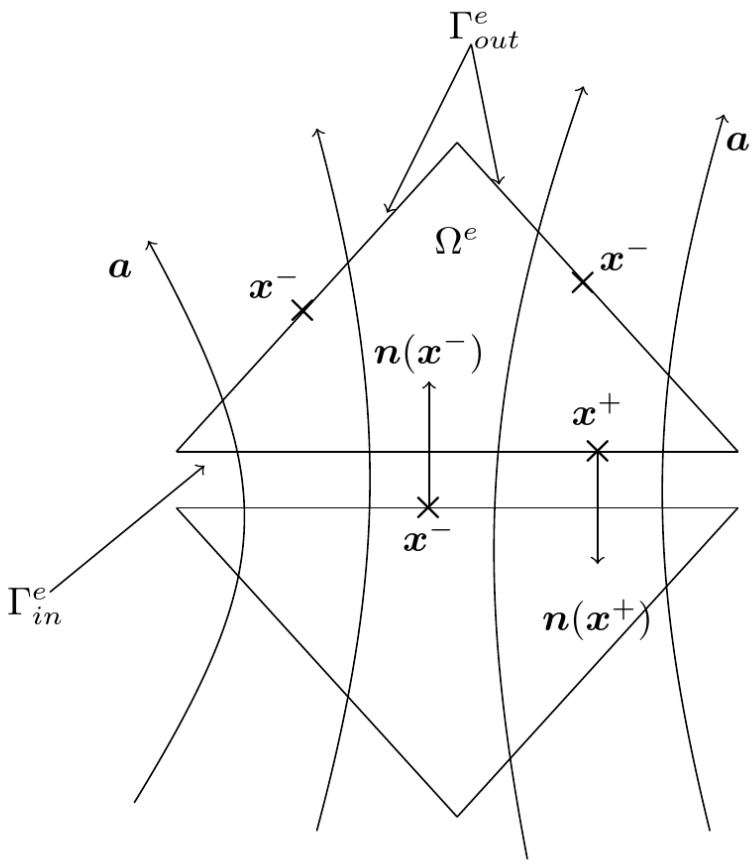

Figure 1: We use the notation $\boldsymbol{x}^+$ and $\boldsymbol{x}^-$ to denote locations on the inflow and outflow boundaries of a typical element Ω^e. A contiguous element to Ω^e is shown sharing its inflow boundary. The shared boundary is shown separately in the figure for clarity. In general, $w^h(\boldsymbol{x}^-) \neq w^h(\boldsymbol{x}^+)$ and $u^h(\boldsymbol{x}^-) \neq u^h(\boldsymbol{x}^+)$ along the interface. When we write $a_n(\boldsymbol{x}^+)$, we mean $\boldsymbol{a}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}^+)$, i.e. along the shared interface $\boldsymbol{n}(\boldsymbol{x}^+)$ points away from Ω^e. With this convention, $a_n(\boldsymbol{x}^-) = \boldsymbol{a}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}^-) = -a_n(\boldsymbol{x}^+)$.

The multi-dimensional version of the pure advection problem is given by

$$B(w^h, u^h)_{\Omega^e} = L(w^h)_{\Omega^e} \quad e = 1, 2, \ldots, n_{el}$$

where

$$B(w^h, u^h)_{\Omega^e} = -\int_{\Omega^e} \boldsymbol{a} \cdot \nabla w^h \, u^h d\Omega + \int_{\Gamma^e_{out}} a_n(\boldsymbol{x}^-) \, w^h(\boldsymbol{x}^-) \, u^h(\boldsymbol{x}^-) \, d\Gamma \tag{3}$$

$$L(w^h)_{\Omega^e} = \int_{\Omega^e} w^h f d\Omega + \int_{\Gamma^e_{in}} a_n(\boldsymbol{x}^-) \, w^h(\boldsymbol{x}^+) \, u^h(\boldsymbol{x}^-) \, d\Gamma \tag{4}$$

and

$$u^h(\boldsymbol{x}^-) = g(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Gamma_{in}$$

The plus and minus superscripts on $\boldsymbol{x}$ indicate on which side of the element boundary the function in question is to be evaluated (analogous to the one-dimensional case). See figure 1. The remaining results are given in the form of an exercise.

Exercise 3.1

Following the one-dimensional case, show that

(i) Euler-Lagrange form

$$0 = \int_{\Omega^e} w^h (\boldsymbol{a} \cdot \nabla u^h - f)\, d\Omega + \int_{\Gamma^e_{in}} w^h(\boldsymbol{x}^+)\, a_n(\boldsymbol{x}^-)\, (\, u(\boldsymbol{x}^+) - u(\boldsymbol{x}^-)\,)\, d\Gamma$$

(ii) Global formulation

$$\mathbb{B}(w^h, u^h) = \mathbb{L}(w^h) \qquad \forall w^h \in \mathcal{V}^h$$

where

$$\begin{aligned}
\mathbb{B}(w^h, u^h) &= \sum_{e=1}^{n_{el}} \int_{\Omega^e} -\boldsymbol{a} \cdot \nabla w^h u^h d\Omega + \int_{\Gamma^e_{out}} a_n(\boldsymbol{x}^-) w^h(\boldsymbol{x}^-) u^h(\boldsymbol{x}^-) d\Gamma - \int_{\Gamma^e_{in} \setminus \Gamma_{in}} a_n(\boldsymbol{x}^-) w^h(\boldsymbol{x}^+) u^h(\boldsymbol{x}^-) d\Gamma \\
\mathbb{L}(w^h) &= \sum_{e=1}^{n_{el}} \int_{\Omega^e} w^h\, f\, d\Omega + \int_{\Gamma^e_{in} \cap \Gamma_{in}} a_n(\boldsymbol{x}^-) w^h(\boldsymbol{x}^+) g(x)\, d\Gamma
\end{aligned}$$

(iii) Error orthogonality

$$\mathbb{B}(w^h, e) = 0 \qquad \forall w^h \in \mathcal{V}^h$$

(iv) Stability

$$\begin{aligned}
\mathbb{B}(w^h, w^h) &= |||w^h|||^2 \\
|||w^h|||^2 &:= \frac{1}{2} \int_{\Gamma_{out}} |a_n(\boldsymbol{x}^-)| \left(w^h(\boldsymbol{x}^-)\right)^2 d\Gamma + \frac{1}{2} \int_{\Gamma_{in}} |a_n(\boldsymbol{x}^+)| \left(w^h(\boldsymbol{x}^+)\right)^2 d\Gamma \\
&\quad + \frac{1}{2} \sum_{e=1}^{n_{el}} \int_{\Gamma^e \setminus \Gamma} |a_n(\boldsymbol{x})|\, [\![w^h]\!]^2\, d\Gamma
\end{aligned}$$

where $[\![w^h]\!] = w^h(\boldsymbol{x}^+) - w^h(\boldsymbol{x}^-)$. Note that $-a_n(\boldsymbol{x}^+) = |a_n(\boldsymbol{x}^+)| > 0$ on Γ_{in} and $a_n(\boldsymbol{x}^-) = |a_n(\boldsymbol{x}^-)| > 0$ on Γ_{out}.

(v) Stabilized Global Formulation

$$\mathbb{B}_{Stab}(w^h, u^h) = \mathbb{L}_{Stab}(w^h) \tag{5}$$

where

$$\begin{aligned}
\mathbb{B}_{Stab}(w^h, u^h) &= \mathbb{B}(w^h, u^h) + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \boldsymbol{a} \cdot \nabla w^h\, \tau\, \boldsymbol{a} \cdot \nabla u^h\, d\Omega \\
\mathbb{L}_{Stab}(w^h) &= \mathbb{L}(w^h) + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \boldsymbol{a} \cdot \nabla w^h\, \tau\, f\, d\Omega
\end{aligned}$$

where τ may be defined element-wise by

$$\tau(\boldsymbol{x}) = \tau^e(\boldsymbol{x}) = \frac{h^e}{2|\boldsymbol{a}(\boldsymbol{x})|} \qquad \boldsymbol{x} \in \Omega^e$$

Error orthogonality

$$\mathbb{B}_{Stab}(w^h, e) = 0 \qquad \boldsymbol{w}^h \in \mathcal{V}^h$$

Stability

$$\mathbb{B}_{Stab}(w^h, w^h) = |||w^h|||^2_{Stab} := |||w^h|||^2 + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau \left(\boldsymbol{a} \cdot \nabla w^h\right)^2 d\Omega$$

(vi) Local conservation

$$\int_{\Gamma^e_{out}} a_n(\boldsymbol{x}^-) u^h(\boldsymbol{x}^-)\, d\Gamma = \int_{\Gamma^e_{in}} a_n(\boldsymbol{x}^-) u^h(\boldsymbol{x}^-)\, d\Gamma + \int_{\Omega^e} f d\Omega$$

Global conservation

$$\int_{\Gamma_{out}} a_n(\boldsymbol{x}^-) u^h(\boldsymbol{x}^-)\, d\Gamma = \int_{\Gamma_{in}} a_n(\boldsymbol{x}^-) g(\boldsymbol{x})\, d\Gamma + \int_{\Omega} f d\Omega$$

Exercise 3.2

Consider the multi-dimensional problem of pure advection:

$$\begin{aligned} \boldsymbol{a} \cdot \nabla u &= f \qquad \forall \boldsymbol{x} \in \Omega \\ u(\boldsymbol{x}) &= g(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Gamma_{in} \end{aligned}$$

where $\boldsymbol{a} : \Omega \to \mathbb{R}^d$ is a smooth, solenoidal vector field such that $\boldsymbol{a}(\boldsymbol{x}) \neq 0$, $\forall x \in \Omega \cup \Gamma_{in}$. Assume the set of integral curves of $\boldsymbol{a}$ emanating from Γ_{in} cover Ω. The integral curves of $\boldsymbol{a}$ are defined by

$$\begin{aligned} \frac{d\boldsymbol{x}}{ds}(s) &= \boldsymbol{a}(\boldsymbol{x}(s)) \\ \boldsymbol{x}(0) &\in \Gamma_{in} \end{aligned}$$

Another way to say this is that for each $\boldsymbol{x} \in \Omega$, there is a unique integral curve of $\boldsymbol{a}$ through $\boldsymbol{x}$ originating at some point in Γ_{in}.

Show that

$$u(\boldsymbol{x}(s)) = u(\boldsymbol{x}(0)) + \int_0^s f(\boldsymbol{x}(t))\, dt$$

and interpret this result by way of a sketch.