<u>Fact</u>: The modified eq. with $\mathcal{K} \leftarrow \mathcal{K} + \tilde{\mathcal{K}}$ is <u>not</u> a <u>residual</u> <u>meth</u>.

$\tilde{\mathcal{K}}$: art diff.

---

We want to retain the good stab. provided by $\tilde{\mathcal{K}}$, but recapture the residual structure:

## A <u>Petrov-Galerkin</u> <u>meth</u>.

Classic (Bubnov -) Galerkin meth.

$$\int w^h (\text{res}(u^h))$$

emanate from the same sp. of ftns. (up to BCs)

$w^h \in \mathcal{V}^h \quad u^h \in \mathcal{S}^h$

$u^h = v^h + g^h$

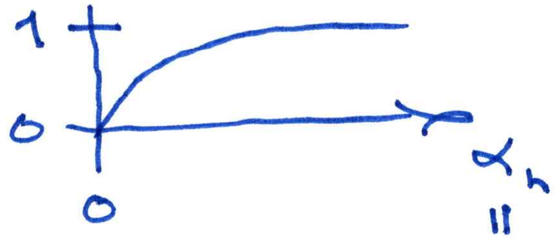
$v^h \in \mathcal{V}^h$, $g^h$: BC.

$$\int \tilde{w}^h (\text{res}(u^h))$$

$\tilde{w}^h \in \tilde{\mathcal{V}}^h \neq \mathcal{V}^h$, $u^h$ as usual stab. $\int w_{,x} (\kappa + \tilde{\kappa}) w_{,x}$

$$\tilde{\kappa} = \left(\frac{|a| h}{2}\right) \underbrace{\tilde{\xi}(\alpha_h)}$$

$$\kappa \int w_{,x} \Big(1 + \underbrace{\frac{|a|h}{2\kappa}}_{\alpha_h}\Big) w_{,x}$$




$$\alpha_h = \frac{|a|h}{2\kappa}$$

$\tilde{\xi}(\alpha_h) \to 1$, as $\alpha_h \to \infty$.

---

Our P.-G. meth. $\qquad a, \kappa = $ const's.

Find $u^h \in \mathcal{S}^h$ (as usual) $\ni \forall w^h \in \mathcal{V}^h$ (as usual)

$$\int_0^L \left( - w^h_{,x}\, a\, u^h + w^h_{,x}\, \kappa\, u^h_{,x} \right) dx$$

$$+ \sum_{A=1}^{N_{el}} \int_{x_{A-1}^{+}}^{x_A^{-}} p^h \underbrace{\left( a u^h_{,x} - \kappa u^h_{,xx} - f \right)}_{\text{res}(u^h) \text{ on el. int.}} dx$$

?

$$= \int_0^L w^h f \, dx$$

int-by-parts el by el

$$\sum_{A=1}^{N} \int_{x_{A-1}}^{x_A} (\overbrace{w^h + p^h}^{\tilde{w}^h \leftarrow \text{P.-G. interp.}})(a u^h_{,x} - \kappa u^h_{,xx} - f)\, dx$$

$$- \sum_{A=1}^{N-1} w^h(x_A) [\![ \kappa u^h_{,x}(x_A) ]\!] = 0.$$

candidate $p^h = \tau a w^h_{,x}$

claim $\tau = \dfrac{\tilde{\kappa}}{|a|^2} = \longrightarrow \dfrac{\cancel{|a|} h}{|a|^{\cancel{2}} 2} \tilde{\xi}(\alpha_h) = \dfrac{h}{2|a|} \tilde{\xi}(\alpha_h)$

$$\ldots + \underbrace{\tau a w^h_{,x} (a u^h_{,x}}\, - \ldots$$

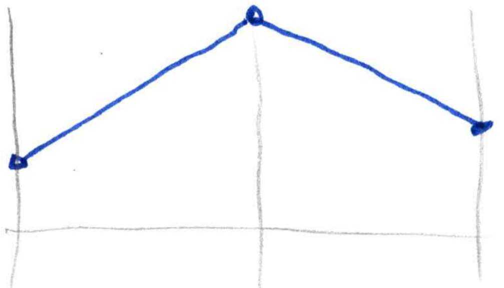
interaction: $\tau a^2 w^h_{,x} u^h_{,x}$

$$\frac{\tilde{\kappa} \cancel{a^2}}{\cancel{|a|^2}} w^h_{,x} u^h_{,x}$$

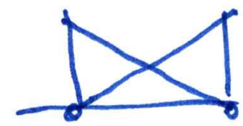
---

interaction of $\tau a w^h_{,x} (\ldots - \kappa u^h_{,xx} - \ldots$

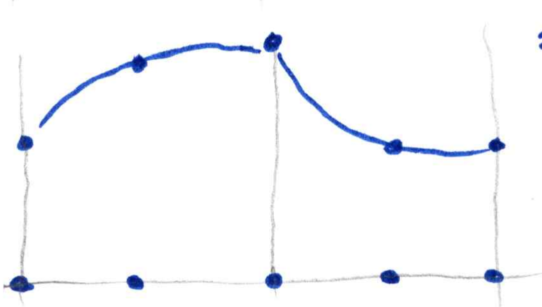
1. For lin FE sp., $u^h_{,xx} = 0$.
2. For higher-~~xxxx~~order el's, $u^h_{,xx} \neq 0$
   later, we are saved by "inverse est's"
   no prob.



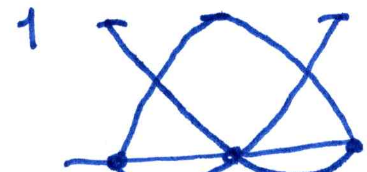
p.w. lin FE sp



$N_A$'s on el.



p.w. quad. FE sp (nodal, Lag.)



$N_{A,s}$.

---

$$\tilde{w}^h = w^h + p^h \qquad w^h = \sum N_A w_A$$

$$= w^h + \tau a w^h_{,x}$$

$$= \sum_A \big(\underbrace{N_A + \tau a N_{A,x}}\big) w_A$$

$$= \sum_A \tilde{N}_A w_A$$

---

$$\tilde{N}_A = N_A + \tau a N_{A,x}$$

$$= N_A + \frac{h}{2|a|}\tilde{\xi}(\alpha_h)\, a N_{A,x}$$

$$= N_A + \mathrm{sign}(a)\frac{h}{2}\tilde{\xi}(\alpha_h) N_{A,x}$$

diff dom case, $\alpha_h \to 0$, $\tilde{\xi}(\alpha_h) \longrightarrow 0$

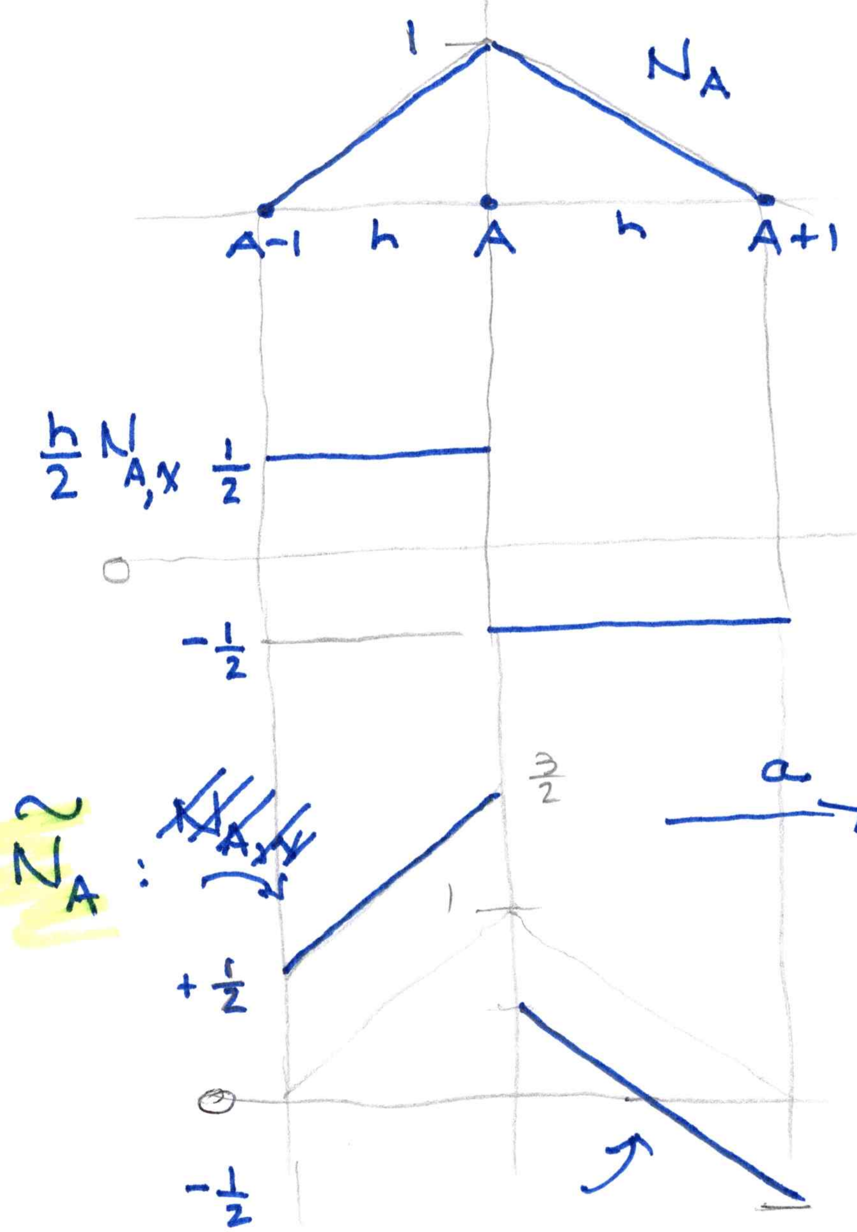
but in adv dom case $\alpha_h \to \infty$, $\tilde{\xi}(\alpha_h) \to 1$

$=$

$$= N_A + \mathrm{sign}(a)\,\frac{h}{2}\cdot 1\, N_{A,x}$$

$\mathrm{sign}\, a = +1 \quad \xrightarrow{a}$

$\quad '' \quad = -1 \quad \xleftarrow{a}$




$N_{A,x} = \pm \frac{1}{h}$

$\xrightarrow{a}$ upwind weighted

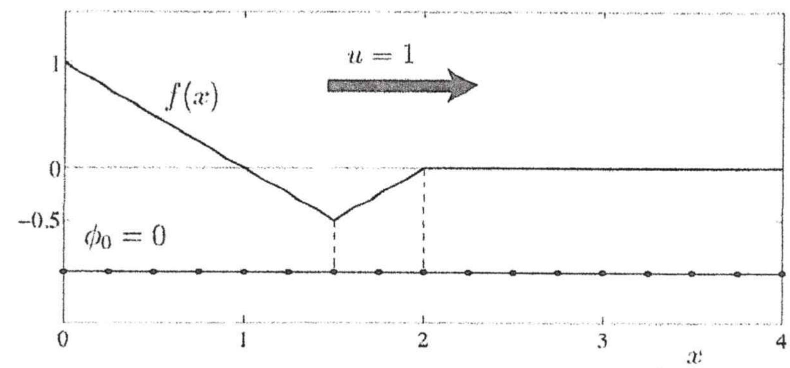
$\xleftarrow{a}$ upwind weighted

reverse picture

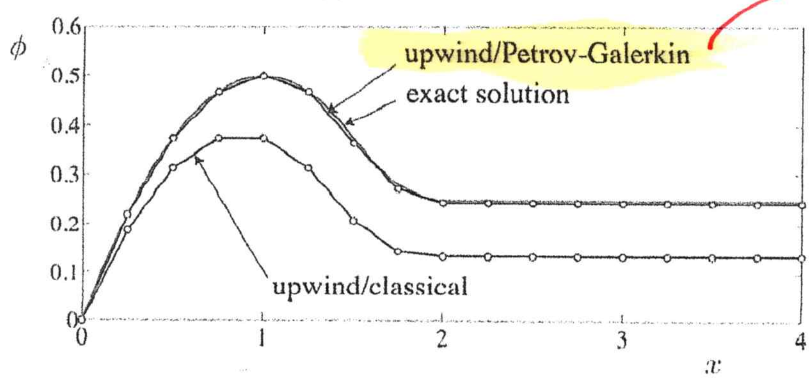
$\int \tilde{N}_A f \, dy$

interaction w. $f$

34.




(a) Problem definition.




SUPG ✓
GLS ✓
MS ✓

(b) Comparison of results.