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A NEW FINITE ELEMENT FORMULATION FOR COMPUTATIONAL FLUID DYNAMICS: I. SYMMETRIC FORMS OF THE COMPRESSIBLE EULER AND NAVIER–STOKES EQUATIONS AND THE SECOND LAW OF THERMODYNAMICS*

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Results of Harten and Tadmor are generalized to the compressible Navier–Stokes equations including heat conduction effects. A symmetric form of the equations is derived in terms of entropy variables. It is shown that finite element methods based upon this form automatically satisfy the second law of thermodynamics and that stability of the discrete solution is thereby guaranteed *ab initio*.

1. Introduction

In two recent papers Harten [1] and Tadmor [4] have discussed symmetrization of the conservation laws of gas dynamics and satisfaction of generalized entropy inequalities. These works are intriguing from the perspective of someone interested in the development of finite element methods because weighted residual formulations based upon the symmetrized systems automatically entail fundamental stability properties possessed by exact solutions of the governing equations. Harten [1] has also considered the compressible Navier–Stokes equations, neglecting heat conduction, and has derived a family of suitable generalized entropy functions. We consider herein the case in which heat conduction is accounted for. We show that for this case the only suitable member of Harten's family of generalized entropy functions is one which is at most trivially different from the physical entropy.

An outline of the paper follows: in Section 2 we recall the compressible Navier–Stokes equations in terms of so-called conservation variables. In Section 3 we review the relevant thermodynamical theory. In Section 4 we derive the symmetrized form of the compressible Navier–Stokes equations in terms of entropy variables and in Section 5 we discuss the significance of the results in the context of finite element methods. The detailed arrays for the entropy-variable form of the equations are presented in Appendix A.

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2. The Navier–Stokes equations in terms of conservation variables

In terms of so called conservation variables, the Navier–Stokes equations can be written as:

$$\boldsymbol{U}_{,t} + \boldsymbol{F}_{i,i} = \boldsymbol{F}^{v}_{i,i} + \boldsymbol{F}^{h}_{i,i} + \mathcal{F} \ , \tag{1}$$

where, in three dimensions,

$$\boldsymbol{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \rho \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ e \end{bmatrix} \quad \text{(conservation variables)} , \tag{2}$$

$$\boldsymbol{F}_i = u_i \boldsymbol{U} + p \begin{bmatrix} 0 \\ \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ u_i \end{bmatrix} \quad \text{(Euler flux)} , \tag{3}$$

$$\boldsymbol{F}^{v}_{i} = \begin{bmatrix} 0 \\ \tau_{1i} \\ \tau_{2i} \\ \tau_{3i} \\ \tau_{ij} u_j \end{bmatrix} \quad \text{(viscous flux)} , \tag{5}$$

$$\boldsymbol{F}^{h}_{i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -q_i \end{bmatrix} \quad \text{(heat flux)} , \tag{5}$$

$$\mathcal{F} = \rho \begin{bmatrix} 0 \\ b_1 \\ b_2 \\ b_3 \\ b_i u_i + r \end{bmatrix} \quad \text{(source vector)} . \tag{6}$$

The notation is as follows: ρ is the density; u_i is the velocity; e is the total energy density; p is the thermodynamic presure; δ_{ij} is the Kronecker delta (i.e., $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$); τ_{ij} is the viscous stress; q_i is the heat flux vector; b_i is the body force per unit mass; r is the heat supply per unit mass; an inferior comma denotes partial differentiation (e.g. $\boldsymbol{U}_{,t} = \partial \boldsymbol{U}/\partial t$, the partial derivative with respect to time); and repeated indices indicate summation (e.g. $\boldsymbol{F}_{i,i} = \partial \boldsymbol{F}_i/\partial x_i = \partial \boldsymbol{F}_1/\partial x_1 + \partial \boldsymbol{F}_2/\partial x_2 + \partial \boldsymbol{F}_3/\partial x_3$, in which x_i denotes the ith Cartesian coordinate). The total energy density may be expressed as

$$e = \iota + \tfrac{1}{2}u^2 , \tag{7}$$

where ι is the internal energy density and

$$\tfrac{1}{2}u^2 = \tfrac{1}{2}(u_1^2 + u_2^2 + u_3^2) \tag{8}$$

is the kinetic energy density. To (1) we append the following constitutive equations:

$$\iota = c_v \theta , \tag{9}$$

$$p = (\gamma - 1)\rho\iota , \tag{10}$$

$$\tau_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \tag{11}$$

$$q_i = -\kappa\theta_{,i} , \tag{12}$$

where c_v is the specific heat at constant volume; θ is the absolute temperature; $\gamma = c_p/c_v$, in which c_p is the specific heat at constant pressure; λ and μ are the viscosity coefficients; and κ is the conductivity. The specific heats are assumed to be positive constants.

3. The second law of thermodynamics

Equation (1) represents conservation of mass, balance of momenta and balance of energy. Entropy production is governed by the *second law of thermodynamics*, or *Clausius–Duhem inequality*, namely [3]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{U})} \rho\eta \,\mathrm{d}\Omega \geq \int_{\varphi_t(\mathcal{U})} \frac{\rho r}{\theta}\,\mathrm{d}\Omega - \int_{\varphi_t(\partial\mathcal{U})} \frac{q_i n_i}{\theta}\,\mathrm{d}\Gamma , \tag{13}$$

where η is the thermodynamic entropy density per unit mass; $\mathcal{U}$ is any open subset of material particles; $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$; $\varphi_t(\mathcal{U})$ is the current configuration of $\mathcal{U}$; n_i is the unit outward normal vector to $\partial\mathcal{U}$; $\mathrm{d}\Omega$ is the volume element; and $\mathrm{d}\Gamma$ is the surface area element. Assuming sufficient smoothness, a standard argument yields the *local form of the Clausius–Duhem inequality*:

$$(\rho\eta)_{,t} + (\rho\eta u_i)_{,i} + (q_i/\theta)_{,i} - \rho r/\theta \geq 0 . \tag{14}$$

Note that conservation of mass was *not* assumed in deriving (14). It is important to leave the Clausius–Duhem inequality in this form for our intended purpose.

It is convenient to introduce a nondimensional entropy

$$s = \eta/c_v . \tag{15}$$

Basic thermodynamical relations enable us to derive an expression for s in terms of pressure and density. For completeness we will sketch the derivation here. We begin with certain fundamental results which may be found, for example, in [2]:

$$\iota = \hat{\iota}(v, \eta)\,, \tag{16}$$

$$\theta = \hat{\iota}_{,\eta}\,, \tag{17}$$

$$p = -\hat{\iota}_{,v}\,, \tag{18}$$

where $v = 1/\rho$ is the specific volume. Note that (9) and (10) are special cases of (16)–(18). Taking the differential of (16) and employing (17) and (18) yields the so-called *Gibbs relation*:

$$\theta\, \mathrm{d}\eta = \mathrm{d}\iota + p\, \mathrm{d}v\,. \tag{19}$$

Employing (9), (10) and (15) in (19) leads to

$$s = \ln(p\rho^{-\gamma}) + \text{const.} \tag{20}$$

For the forms of the constitutive equations chosen, satisfaction of the Clausius–Duhem inequality requires [2]:

$$\mu \geqslant 0\,, \qquad \lambda + \tfrac{2}{3}\mu \geqslant 0\,, \qquad \kappa \geqslant 0\,. \tag{21}$$

The Clausius–Duhem inequality is seen to play two roles: one is to place restrictions on constitutive relations and the other is to discriminate between admissible and inadmissible discontinuities. In smooth regions, satisfaction of (1) implies that the Clausius–Duhem inequality is also satisfied as long as the constitutive equations respect the required restrictions. In other words, the Clausius–Duhem inequality may be *recovered* from (1). It is our aim here to develop a basis for numerical formulations such that the Clausius–Duhem inequality is respected *ab initio* for all numerical solutions.

4. Symmetrization of the Navier–Stokes equations

The flux terms in (1) can be written as

$$\boldsymbol{F}_{i,i} = \boldsymbol{F}_{i,\boldsymbol{U}}\boldsymbol{U}_{,i} = \boldsymbol{A}_i\boldsymbol{U}_{,i}\,, \tag{22}$$

$$\boldsymbol{F}_i^v = \boldsymbol{K}_{ij}^v\boldsymbol{U}_{,j}\,, \tag{23}$$

$$\boldsymbol{F}_i^h = \boldsymbol{K}_{ij}^h\boldsymbol{U}_{,j}\,, \tag{24}$$

where

$$\boldsymbol{A}_i = \boldsymbol{A}_i(\boldsymbol{U})\,, \tag{25}$$

$$\boldsymbol{K}_{ij}^v = \boldsymbol{K}_{ij}^v(\boldsymbol{U})\,, \tag{26}$$

$$\boldsymbol{K}_{ij}^h = \boldsymbol{K}_{ij}^h(\boldsymbol{U})\,. \tag{27}$$

With these, (1) becomes

$$\boldsymbol{U}_{,t} + \boldsymbol{A}_i \boldsymbol{U}_{,i} = (\boldsymbol{K}_{ij} \boldsymbol{U}_{,j})_{,i} + \mathcal{F}\,, \tag{28}$$

where

$$\boldsymbol{K}_{ij} = \boldsymbol{K}_{ij}^{v} + \boldsymbol{K}_{ij}^{h}\,. \tag{29}$$

Consider a change of variables:

$$\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{V})\,. \tag{30}$$

The new variables satisfy

$$\boldsymbol{A}_0 \boldsymbol{V}_{,t} + \tilde{\boldsymbol{A}}_i \boldsymbol{V}_{,i} = (\tilde{\boldsymbol{K}}_{ij} \boldsymbol{V}_{,j})_{,i} + \mathcal{F}\,, \tag{31}$$

where

$$\boldsymbol{A}_0 = \boldsymbol{U}_{,\boldsymbol{V}}\,, \tag{32}$$

$$\tilde{\boldsymbol{A}}_i = \boldsymbol{A}_i \boldsymbol{A}_0\,, \tag{33}$$

$$\tilde{\boldsymbol{K}}_{ij} = \boldsymbol{K}_{ij} \boldsymbol{A}_0\,. \tag{34}$$

We seek a change of variables satisfying the following conditions:
(i) $\boldsymbol{A}_0$ is symmetric and positive-definite and the $\tilde{\boldsymbol{A}}_i$ are symmetric.
(ii) The matrix

$$\tilde{\boldsymbol{K}} = \begin{bmatrix} \tilde{\boldsymbol{K}}_{11} & \tilde{\boldsymbol{K}}_{12} & \tilde{\boldsymbol{K}}_{13} \\ \tilde{\boldsymbol{K}}_{21} & \tilde{\boldsymbol{K}}_{22} & \tilde{\boldsymbol{K}}_{23} \\ \tilde{\boldsymbol{K}}_{31} & \tilde{\boldsymbol{K}}_{32} & \tilde{\boldsymbol{K}}_{33} \end{bmatrix} \tag{35}$$

is symmetric and positive-semidefinite.

Let us, for the moment, restrict our attention to the Euler equations:

$$\boldsymbol{U}_{,t} + \boldsymbol{A}_i \boldsymbol{U}_{,i} = 0\,. \tag{36}$$

It is well known that the $\boldsymbol{A}_i$ are nonsymmetric, but all linear combinations of the $\boldsymbol{A}_i$ possess real eigenvalues and a complete set of eigenvectors, and thus (36) constitutes a hyperbolic system of conservation laws [5]. If a change of variables satisfying (i) can be found, then the transformed system, namely

$$\boldsymbol{A}_0 \boldsymbol{V}_{,t} + \tilde{\boldsymbol{A}}_i \boldsymbol{V}_{,i} = 0 \tag{37}$$

is a *symmetric hyperbolic system*. Symmetric hyperbolic systems and notions of entropy are intimately linked.

A scalar-valued function $H = H(\boldsymbol{U})$ is called a *generalized entropy function* for (36) if the following two conditions are satisfied:
(1) H is a convex function.

(2) There exists scalar-valued functions $\sigma_i = \sigma_i(U)$, $i = 1, 2, 3$, called *entropy fluxes* such that

$$H_{,U}A_i = \sigma_{i,U}\,. \tag{38}$$

We note that for every smooth solution of (36),

$$H_{,t} + \sigma_{i,i} = H_{,U}(U_{,t} + A_i U_{,i}) = 0\,. \tag{39}$$

The following theorems, proved in [1], delineate the relationship between symmetric hyperbolic systems and generalized entropy functions:

THEOREM 4.1 (Mock). A hyperbolic system of conservation laws possessing a generalized entropy function becomes symmetric hyperbolic under the change of variables

$$V^{t} = H_{,U}\,. \tag{40}$$

THEOREM 4.2 (Godunov). If a hyperbolic system can be symmetrized by introducing a change of variables, then a generalized entropy function and corresponding entropy fluxes exist for this system.

Harten [1] has proposed the following family of generalized entropy functions for the Euler equations:

$$H = -\rho\, g(s)\,, \tag{41}$$

where g is any function such that

$$\frac{g''}{g'} < \gamma^{-1}\,. \tag{42}$$

Equation (42) ensures satisfaction of the convexity condition[1]. The corresponding entropy flux is given by:

$$\sigma_i = H u_i\,. \tag{43}$$

The change of variables in defined by:

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \frac{g'}{\iota} \begin{bmatrix} \iota(\gamma - g/g') - \frac{1}{2}u^2 \\ u_1 \\ u_2 \\ u_3 \\ -1 \end{bmatrix}. \tag{44}$$

Harten [1] also derived the viscous contribution to the matrix $\tilde{K}$, but did not address its symmetry or positive-definiteness. It turns out that the viscous contribution to $\tilde{K}$ is both

[1] The convexity of H is equivalent to the positive-definiteness of A_0, viz., $A_0^{-1} = V_{,U} = H_{,UU}$.

symmetric and positive-semidefinite for any member of Harten's family of generalized entropy functions.

The heat flux term places more stringent requirements on the definition of generalized entropy function. Because

$$\tilde{\boldsymbol{K}}^h_{ij}\boldsymbol{V}_{,j} = \boldsymbol{F}^h_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \kappa\theta_{,i} \end{bmatrix}, \tag{45}$$

the only way symmetry of $\tilde{\boldsymbol{K}}$ will be maintained is if θ depends only on V_5. As

$$\theta(\boldsymbol{V}) = -\frac{g'}{c_v V_5}, \tag{46}$$

this requires that g' is a constant. In other words g must be an affine function of s. For simplicity, we may assume the additive constant is zero and take

$$H = -\rho s\,. \tag{47}$$

In this case, $\boldsymbol{U} \mapsto \boldsymbol{V}$ is defined by

$$\boldsymbol{V} = \frac{1}{\rho\iota}\begin{bmatrix} -U_5 + \rho\iota(\gamma + 1 - s) \\ U_2 \\ U_3 \\ U_4 \\ -U_1 \end{bmatrix}, \tag{48}$$

where

$$s = \ln\left[\frac{(\gamma - 1)\rho\iota}{U_1^\gamma}\right], \tag{49}$$

$$\rho\iota = U_5 - \tfrac{1}{2}(U_2^2 + U_3^2 + U_4^2)/U_1\,. \tag{50}$$

The inverse mapping, $\boldsymbol{V} \mapsto \boldsymbol{U}$, is given by

$$\boldsymbol{U} = \rho\iota\begin{bmatrix} -V_5 \\ V_2 \\ V_3 \\ V_4 \\ 1 - \tfrac{1}{2}(V_2^2 + V_3^2 + V_4^2)/V_5 \end{bmatrix}, \tag{51}$$

where

$$\rho\iota = \left[\frac{\gamma - 1}{(-V_5)^\gamma}\right]^{1/(\gamma-1)} \exp\left[\frac{-s}{\gamma - 1}\right], \tag{52}$$

$$s = \gamma - V_1 + \tfrac{1}{2}(V_2^2 + V_3^2 + V_4^2)/V_5. \tag{53}$$

With this definition of the change of variables, conditions (i) and (ii) are satisfied. The flux vectors and coefficient matrices are explicitly defined for the V-variables in Appendix A. We refer to these V-variables as the (physical) *entropy variables*.

The present definition of V enables us to derive the following important result:

$$\begin{aligned} 0 &= V^{t}(A_0 V_{,t} + \tilde{A}_i V_{,i} - (\tilde{K}_{ij} V_{,j})_{,i} - \mathcal{F}) \\ &= H_{,t} + (Hu_i)_{,i} + V^{t}_{,i}\tilde{K}_{ij}V_{,j} - (V^{t}\tilde{K}_{ij}V_{,j})_{,i} - V^{t}\mathcal{F} \\ &= \frac{1}{c_v}\left(-(\rho\eta)_{,t} - (\rho\eta u_i)_{,i} + c_v V^{t}_{,i}\tilde{K}_{ij}V_{,j} - \left(\frac{q_i}{\theta}\right)_{,i} + \frac{\rho r}{\theta}\right). \end{aligned} \tag{54}$$

In going from the second to third lines in (54) we have used the following:

$$V^{t}F_i^{v} = 0\,, \tag{55}$$

$$V^{t}F_i^{h} = \frac{q_i}{c_v\theta}\,. \tag{56}$$

Equation (54) may be rearranged to verify satisfaction of the entropy production inequality:

$$(\rho\eta)_{,t} + (\rho\eta u_i)_{,i} + \left(\frac{q_i}{\theta}\right)_{,i} - \frac{\rho r}{\theta} = c_v V^{t}_{,i}\tilde{K}_{ij}V_{,j} \geq 0\,. \tag{57}$$

REMARK 4.3. We may write

$$\nabla V^{t}\tilde{K}\nabla V = V^{t}_{,i}K_{ij}V_{,j}\,, \tag{58}$$

where

$$\nabla V = \begin{bmatrix} V_{,1} \\ V_{,2} \\ V_{,3} \end{bmatrix}. \tag{59}$$

The matrix $\tilde{K}$ is symmetric and positive-semidefinite. Its size and rank[2] are given in Table 1.

Table 1

No. of space dimensions	Size of $\tilde{K}$	Rank of $\tilde{K}$	Rank of $\tilde{K}^{v}$	Rank of $\tilde{K}^{h}$
3	15×15	9	6	3
2	8×8	5	3	2
1	3×3	2	1	1

[2] In determining rank, we have assumed that inequalities (21) are strictly satisfied.

5. Discussion

The preceding results are of particular significance when the Navier–Stokes equations are discretized by finite element methods. For example, assume a semidiscrete Galerkin weighed residual formulation and, for simplicity, ignore boundary condition terms. The variational statement is: find a function $\boldsymbol{V}$ such that for all weighting functions $\boldsymbol{W}$,

$$0 = \int_{\Omega} \boldsymbol{W}^{\mathrm{t}}(\boldsymbol{A}_0 \boldsymbol{V}_{,t} + \tilde{\boldsymbol{A}}_i \boldsymbol{V}_{,i} - (\tilde{\boldsymbol{K}}_{ij} \boldsymbol{V}_{,j})_{,i} - \mathscr{F}) \, \mathrm{d}\Omega \,. \tag{60}$$

We assume $\boldsymbol{W}$ and $\boldsymbol{V}$ are expanded in terms of typical C^0 finite element interpolation functions. This makes (60) well-defined in the sense of distributions. Selecting $\boldsymbol{W} = \boldsymbol{V}$ in (60) and proceeding as in (54) enables us to derive a global statement of the Clausius–Duhem inequality in terms of the discrete solution:

$$\int_{\Omega} \left((\rho\eta)_{,t} + (\rho\eta u_i)_{,i} + \left(\frac{q_i}{\theta}\right)_{,i} - \frac{\rho r}{\theta} \right) \mathrm{d}\Omega = c_v \int_{\Omega} \boldsymbol{V}^{\mathrm{t}}_{,i} \tilde{\boldsymbol{K}}_{ij} \boldsymbol{V}_{,j} \, \mathrm{d}\Omega \geqslant 0 \,. \tag{61}$$

This is an important result: *the discrete solution always satisfies the Clausius–Duhem inequality.*

Another interpretation of (61) is that the rate of growth of a convex function of the discrete solution, namely $H = -\rho s$, is appropriately bounded from above. *Thus, the fundamental stability property possessed by solutions of the Navier–Stokes equations is automatically inherited by discrete solutions.*

It is well known that the Galerkin finite element method is ineffective when flows contain shocks or sharp layers. It may be noted that, *for the case of the Euler equations, entropy will always be conserved by a Galerkin finite element solution*, that is

$$0 = \int_{\Omega} \boldsymbol{V}^{\mathrm{t}}(\boldsymbol{A}_0 \boldsymbol{V}_{,t} + \tilde{\boldsymbol{A}}_i \boldsymbol{V}_{,i}) \, \mathrm{d}\Omega = \frac{1}{c_v} \int_{\Omega} ((\rho\eta)_{,t} + (\rho\eta u_i)_{,i}) \, \mathrm{d}\Omega \,. \tag{62}$$

The entropy production associated with exact discontinuous solutions cannot be represented by a Galerkin formulation with C^0 functions. A Petrov–Galerkin formulation, in which $\boldsymbol{W}$ is altered to produce appropriate entropy production in the discrete solution, can be effectively used to rectify the situation. This will be discussed in future works.

We have done some Euler calculations with variables derived from different generalized entropy functions using otherwise identical numerical procedures. In particular, we have compared variables advocated by Harten [1], in which

$$H = -\rho a \exp\left(\frac{s}{\alpha + \gamma}\right), \tag{63}$$

where a and α are constants, with those advocated herein. The variables derived from (63) engender flux vectors which are homogeneous functions, whereas the physical entropy variables advocated herein do not. However, we have found that the latter set produces improved numerical results. These will also be reported in future works.

Appendix A

All the formulas in the appendix refer to $\boldsymbol{V}$ given by (48)–(50).
The following combinations of variables are introduced to simplify subsequent writing:

$$\begin{aligned}
&\bar{\gamma} = \gamma - 1\,, && k_1 = \tfrac{1}{2}(V_2^2 + V_3^2 + V_4^2)/V_5\,, && k_2 = k_1 - \gamma\,,\\
&k_3 = k_1^2 - 2\gamma k_1 + \gamma\,, && k_4 = k_2 - \bar{\gamma}\,, && k_5 = k_2^2 - \bar{\gamma}(k_1 + k_2)\,,\\
&c_1 = \bar{\gamma} V_5 - V_2^2\,, && d_1 = -V_2V_3\,, && e_1 = V_2V_5\,,\\
&c_2 = \bar{\gamma} V_5 - V_3^2\,, && d_2 = -V_2V_4\,, && e_2 = V_3V_5\,,\\
&c_3 = \bar{\gamma} V_5 - V_4^2\,, && d_3 = -V_3V_4\,, && e_3 = V_4V_5\,.
\end{aligned} \tag{A.1}$$

The Euler fluxes may be written as:

$$\boldsymbol{F}_1(\boldsymbol{V}) = \frac{\rho\iota}{V_5}\begin{bmatrix} e_1 \\ c_1 \\ d_1 \\ d_2 \\ k_2V_2 \end{bmatrix}, \quad \boldsymbol{F}_2(\boldsymbol{V}) = \frac{\rho\iota}{V_5}\begin{bmatrix} e_2 \\ d_1 \\ c_2 \\ d_3 \\ k_2V_3 \end{bmatrix}, \quad \boldsymbol{F}_3(\boldsymbol{V}) = \frac{\rho\iota}{V_5}\begin{bmatrix} e_3 \\ d_2 \\ d_3 \\ c_3 \\ k_2V_4 \end{bmatrix}. \tag{A.2}$$

The matrix $\boldsymbol{A}_0$ and its inverse are given by

$$\boldsymbol{A}_0 = \boldsymbol{U}_{,V} = \frac{\rho\iota}{\bar{\gamma}V_5}\begin{bmatrix} -V_5^2 & e_1 & e_2 & e_3 & V_5(1-k_1) \\ & c_1 & d_1 & d_2 & V_2k_2 \\ & & c_2 & d_3 & V_3k_2 \\ & & & c_3 & V_4k_2 \\ \text{Symm.} & & & & -k_3 \end{bmatrix} \tag{A.3}$$

and

$$\boldsymbol{A}_0^{-1} = \boldsymbol{V}_{,U} = \frac{-1}{\rho\iota V_5}\begin{bmatrix} k_1^2 + \gamma & k_1V_2 & k_1V_3 & k_1V_4 & (k_1+1)V_5 \\ & V_2^2 - V_5 & -d_1 & -d_2 & e_1 \\ & & V_3^2 - V_5 & -d_3 & e_2 \\ & & & V_4^2 - V_5 & e_3 \\ \text{Symm.} & & & & V_5^2 \end{bmatrix}. \tag{A.4}$$

The Jacobians of the Euler fluxes are:

$$\tilde{\boldsymbol{A}}_1 = \boldsymbol{F}_{1,V} = \frac{\rho\iota}{\bar{\gamma}V_5^2}\begin{bmatrix} e_1V_5 & c_1V_5 & d_1V_5 & d_2V_5 & k_2e_1 \\ & -(c_1 + 2\bar{\gamma}V_5)V_2 & -c_1V_3 & -c_1V_4 & c_1k_2 + \bar{\gamma}V_2^2 \\ & & -c_2V_2 & -d_1V_4 & k_4d_1 \\ & & & -c_3V_2 & k_4d_2 \\ \text{Symm.} & & & & k_5V_2 \end{bmatrix}, \tag{A.5}$$

$$\tilde{A}_2 = F_{2,V} = \frac{\rho\iota}{\bar{\gamma}V_5^2}\begin{bmatrix} e_2V_5 & d_1V_5 & c_2V_5 & d_3V_5 & k_2e_2 \\ & -c_1V_3 & -c_2V_2 & -d_1V_4 & k_4d_1 \\ & & -(c_2+2\bar{\gamma}V_5)V_3 & -c_2V_4 & c_2k_2+\bar{\gamma}V_3^2 \\ & & & -c_3V_3 & k_4d_3 \\ \text{Symm.} & & & & k_5V_3 \end{bmatrix}, \quad \text{(A.6)}$$

$$\tilde{A}_3 = F_{3,V} = \frac{\rho\iota}{\bar{\gamma}V_5^2}\begin{bmatrix} e_3V_5 & d_2V_5 & d_3V_5 & c_3V_5 & k_2e_3 \\ & -c_1V_4 & -d_2V_3 & -c_3V_2 & k_4d_2 \\ & & -c_2V_4 & -c_3V_3 & k_4d_3 \\ & & & -(c_3+2\bar{\gamma}V_5)V_4 & c_3k_2+\bar{\gamma}V_4^2 \\ \text{Symm.} & & & & k_5V_4 \end{bmatrix}. \quad \text{(A.7)}$$

The velocity and temperature can be written as

$$u_i(V) = -V_{i+1}/V_5\,, \quad i = 1, 2, 3\,, \quad \text{(A.8)}$$

$$\theta(V) = -1/(c_v V_5)\,. \quad \text{(A.9)}$$

The gradients of the viscous and heat fluxes may be computed with the aid of

$$u_{i,j} = \frac{-V_5V_{i+1,j} + V_{i+1}V_{5,j}}{V_5^2}\,, \quad \text{(A.10)}$$

$$\kappa\theta_{,i} = \frac{\gamma\mu}{\Pr}\frac{1}{V_5^2}V_{5,i}\,, \quad \text{(A.11)}$$

where $\Pr = \mu c_p/\kappa$ is the Prandtl number.

Matrices $\tilde{K}_{ij}$ appearing in (34) are given below:

$$\tilde{K}_{11} = \frac{1}{V_5^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -(\lambda+2\mu)V_5^2 & 0 & 0 & (\lambda+2\mu)e_1 \\ 0 & 0 & -\mu V_5^2 & 0 & \mu e_2 \\ 0 & 0 & 0 & -\mu V_5^2 & \mu e_3 \\ 0 & (\lambda+2\mu)e_1 & \mu e_2 & \mu e_3 & -[(\lambda+2\mu)V_2^2+\mu(V_3^2+V_4^2)-\gamma\mu V_5/\Pr] \end{bmatrix}, \quad \text{(A.12)}$$

$$\tilde{K}_{12} = \frac{1}{V_5^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda V_5^2 & 0 & \lambda e_2 \\ 0 & -\mu V_5^2 & 0 & 0 & \mu e_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu e_2 & \lambda e_1 & 0 & (\lambda+\mu)d_1 \end{bmatrix}. \quad \text{(A.13)}$$

$$\tilde{\boldsymbol{K}}_{13}=\frac{1}{V_5^3}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\lambda V_5^2 & \lambda e_3\\ 0 & 0 & 0 & 0 & 0\\ 0 & -\mu V_5^2 & 0 & 0 & \mu e_1\\ 0 & \mu e_3 & 0 & \lambda e_1 & (\lambda+\mu)d_2\end{bmatrix}, \tag{A.14}$$

$$\tilde{\boldsymbol{K}}_{22}=\frac{1}{V_5^3}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & -\mu V_5^2 & 0 & 0 & \mu e_1\\ 0 & 0 & -(\lambda+2\mu)V_5^2 & 0 & (\lambda+2\mu)e_2\\ 0 & 0 & 0 & -\mu V_5^2 & \mu e_3\\ 0 & \mu e_1 & (\lambda+2\mu)e_2 & \mu e_3 & -[(\lambda+2\mu)V_3^2+\mu(V_2^2+V_4^2)-\gamma\mu V_5/\mathrm{Pr}]\end{bmatrix}, \tag{A.15}$$

$$\tilde{\boldsymbol{K}}_{23}=\frac{1}{V_5^3}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\lambda V_5^2 & \lambda e_3\\ 0 & 0 & -\mu V_5^2 & 0 & \mu e_2\\ 0 & 0 & \mu e_3 & \lambda e_2 & (\lambda+\mu)d_3\end{bmatrix}, \tag{A.16}$$

$$\tilde{\boldsymbol{K}}_{33}=\frac{1}{V_5^3}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & -\mu V_5^2 & 0 & 0 & \mu e_1\\ 0 & 0 & -\mu V_5^2 & 0 & \mu e_2\\ 0 & 0 & 0 & -(\lambda+2\mu)V_5^2 & (\lambda+2\mu)e_3\\ 0 & \mu e_1 & \mu e_2 & (\lambda+2\mu)e_3 & -[(\lambda+2\mu)V_4^2+\mu(V_2^2+V_3^2)-\gamma\mu V_5/\mathrm{Pr}]\end{bmatrix}, \tag{A.17}$$

$$\tilde{\boldsymbol{K}}_{21}=\tilde{\boldsymbol{K}}^{\mathrm{t}}_{12}\,,\qquad \tilde{\boldsymbol{K}}_{31}=\tilde{\boldsymbol{K}}^{\mathrm{t}}_{13}\,,\qquad \tilde{\boldsymbol{K}}_{32}=\tilde{\boldsymbol{K}}^{\mathrm{t}}_{23}\,. \tag{A.18}$$

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