# CSE 397 / EM 397 - Stabilized and Variational Multiscale Methods in CFD

## Homework #2

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### Exercise 2.1 (25 points)

Assume that Pe $\gg 1$. Show that $\|u\|_2 \sim L^{1/2}\mathrm{Pe}^{3/2}$ ($H^2$-norm) where $u$ is the solution of the advection diffusion equation, assuming $f = 0$, $u(0) = 0$, and $u(L) = 1$.

### Exercise 2.2 (25 points)

Assume Pe $= 500$, $u(0) = 0$, $u(L) = 1$, and $f = 0$. Plot the $L_2$-norms of the error $\|e\|$, the $H^1$ semi-norms $|e|_1$, and the $H^1$-norms $\|e\|_1$ versus the number of elements $N_{el} = 1, 2, 5, 10, 10^2, 10^3$ for SUPG and Galerkin with linear finite elements. Note that $|||e||| \approx \beta N_{el}^{-\gamma}$. Determine $\gamma$ for each discernible branch, comment on its value, and interpret the difference of branches.

### Exercise 2.3 (25 points) Some simple interpolation estimates in the "max norm"

Consider piecewise linear finite elements. Given $u \in C^2(]0, L[)$, obtain a bound for the interpolation error $\eta = \tilde{u}^h - u$ and its derivative $\eta_{,x}$.

### Exercise 2.4 (25 points) Some simple inverse estimates

Show that

$$\left\| w^h_{,x} \right\|_{\Omega^e} \leq C_{inv} h^{-1} \left\| w^h \right\|_{\Omega^e}$$

for the linear element ($k = 1$) with two nodes and for the quadratic element $k = 2$ with three nodes, where we assume equally spaced nodes. Determine the smallest $C_{inv}$ in each case.

Furthermore, show that

$$\left\| w^h_{,xx} \right\|_{\Omega^e} \leq C_{inv} h^{-1} \left\| w^h_{,x} \right\|_{\Omega^e} \qquad \text{and} \qquad \left\| w^h_{,xx} \right\|_{\Omega^e} \leq C_{inv} h^{-2} \left\| w^h \right\|_{\Omega^e}$$

for the quadratic element with three nodes, and determine the smallest $C_{inv}$.

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