Lecture 19, March 27, 2024

exact prob. $u_{,t} = f$

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} f\,dt$$

$\forall n = 0, 1, \ldots, N-1.$

$$\rightarrow \quad u(t_1) = \overset{u_0}{u(t_0)} + \int_0^{t_1} f\,dt \;\checkmark$$

$$\longrightarrow \quad u(t_2) = \underline{u(t_1)}\checkmark + \int_{t_1}^{t_2} f\,dt \;\checkmark$$

$$u(t_3) = u(t_2) + \int_{t_2}^{t_3} f\,dt$$

$$\vdots$$

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our meth. is DG, or Stab. DG, ~~...~~

conservation.

$$u^h(t_{n+1}^-) = u^h(t_n^-) + \int_{t_n}^{t_{n+1}} f\,dt$$

$$\rightarrow \quad u^h(t_1^-) = \overset{u_0}{u^h(t_0^-)}\checkmark + \int_0^{t_1} f\,dt$$

$$\longrightarrow \quad u^h(t_2^-) = \underline{u^h(t_1^-)}\checkmark + \int_{t_1}^{t_2} f\,dt \;\checkmark$$

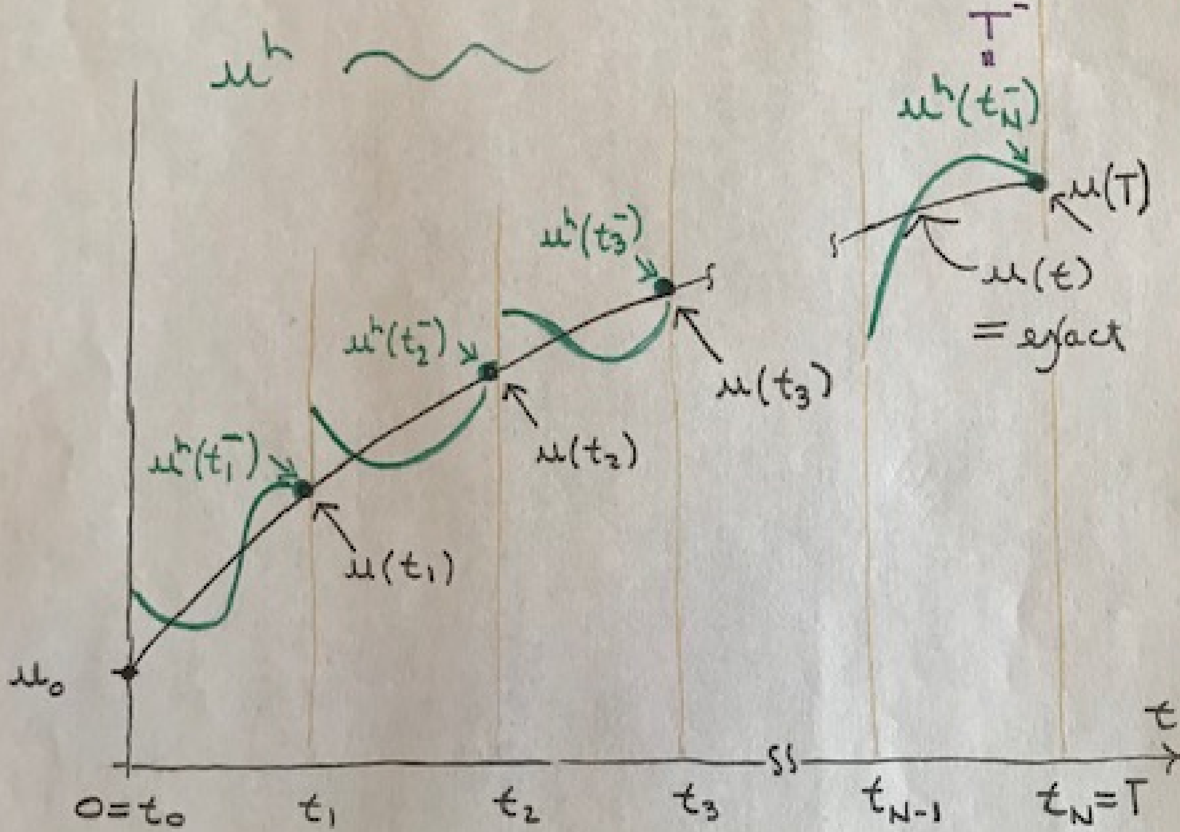
$$u^h(t_3^-) = u^h(t_2^-) + \int_{t_2}^{t_3} f\,dt$$

$$\vdots$$

$\therefore$
$$u^h(t_1^-) = u(t_1)$$
$$u^h(t_2^-) = u(t_2)$$
$$u^h(t_3^-) = u(t_3)$$
$$\vdots$$

$\swarrow$ $u^h(t_n^-) = u(t_n)$ for $n = 1, 2, \ldots, N$

"Outflow" values are exact.



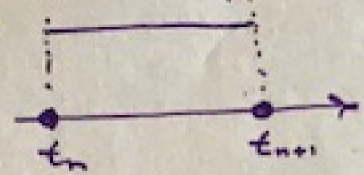
$$u^h(t_n^-) = u(t_n),$$

$$n = 1, 2, \ldots, N.$$

DG (+ stab.) in space-time, PDEs, outflow values are superconvergent

Consider some possibilities

$k=0$ constant

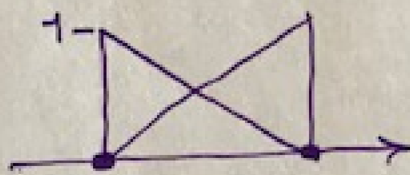


basis ftn

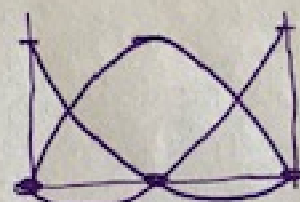
$w^h = N_A$

$\tau = 1$

$k=1$ linear

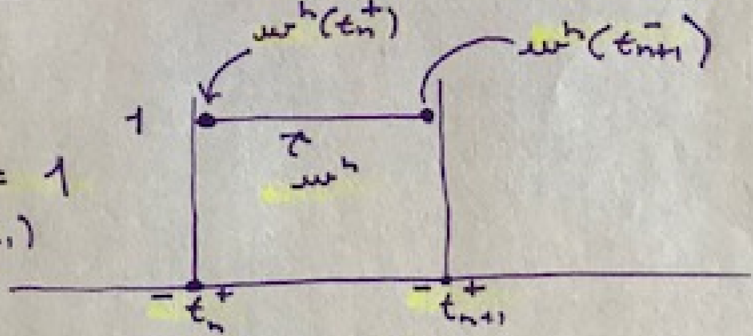


$k=2$ quadratic

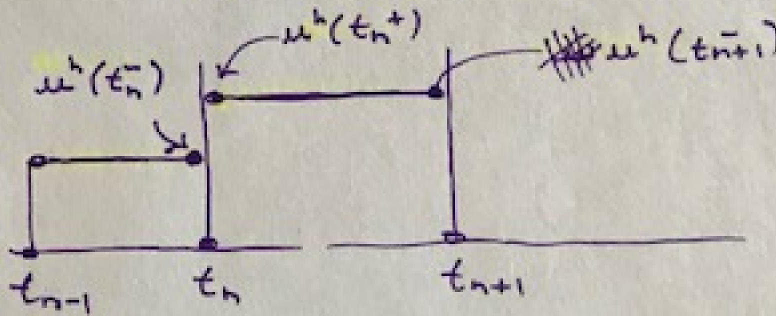


$k=0$.

$w^h\big|_{(t_n,t_{n+1})} = 1$



$w^h\big|_{(t_n,t_{n+1})}$ = const. in each interval.



$k=0$ method. DG or stab. identical

$$B(1, u^h)_n = L(1)_n + \text{stab} \xcancel{}^{0}$$

$w^h = 1$

$$= \int_{t_n}^{t_{n+1}} -\cancel{\frac{1}{\partial t}}^{0} \, dt + 1 \cdot u^h(t_{n+1}^-) - 1\, u^h(t_n^{-})$$

$w^h(t_{n+1}^-) = 1 \qquad w^h(t_n^+) = 1$

$$= \int_{t_n}^{t_{n+1}} 1\, f \, dt$$

$$u^h(t_{n+1}^-) = u^h(t_n^-) + \int_{t_n}^{t_{n+1}} f \, dt$$

$k=0$, the meth $\equiv$ conservation law.
DG (or DG+stab)

"minus values", namely, $u^h(t_n^-) = u(t_n)$ ✓

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implicit Euler (aka Backward Differences)

$$u^h_{n+1} = u^h_n + \Delta t\, f_{n+1} \qquad f_{n+1} = f(t_{n+1})$$

converge $O(\Delta t)$.
first-order acc.

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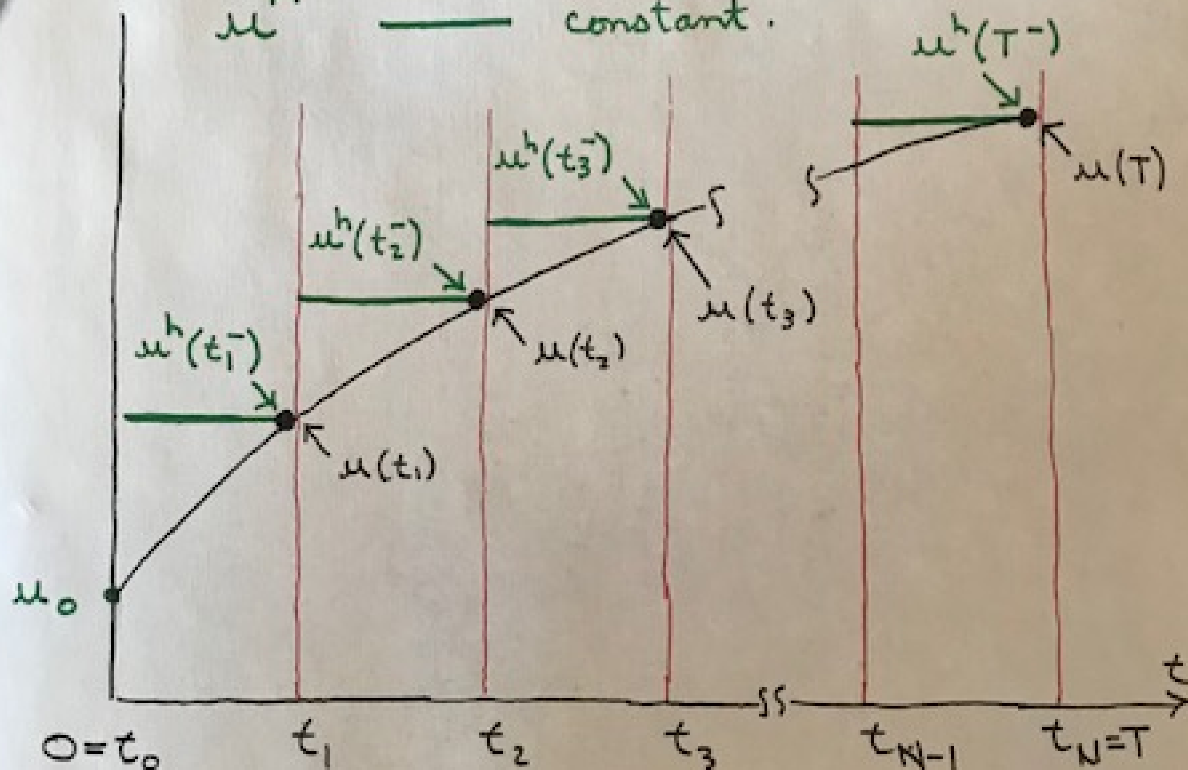
second ord difference method.

$$\int_{t_n}^{t_{n+1}} f \, dt \Rightarrow \Delta t\, f\left(\frac{t_n + t_{n+1}}{2} \right)$$

converge $O(\Delta t^2)$
second-order.

one-pt Gauss int

$k=0$ method



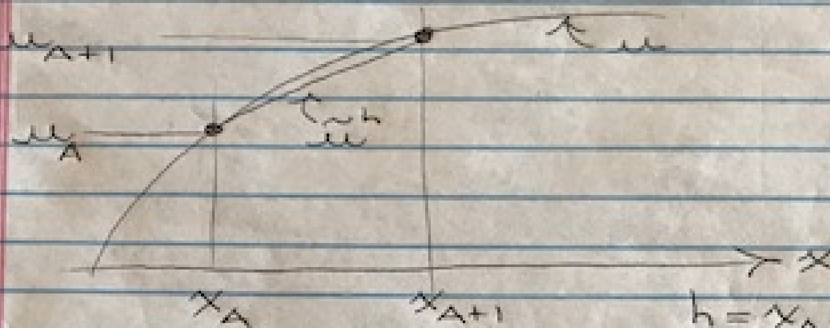
As usual $u^h(t_n^-) = u(t_n)$

$n = 1, 2, \ldots, N$

This is an HW prob. (1.

linear interpolate of u at nodes A and $A+1$

$$e = \tilde{u}^h - u$$



$$h = x_{A+1} - x_A$$

$$\tilde{u}^h(x) = \left(\frac{x_{A+1} - x}{h}\right) u_A + \frac{(x - x_A)}{h} u_{A+1}$$

note $u_A = u(x_A)$, $u_{A+1} = u(x_{A+1})$

expand $u(x_A)$ and $u(x_{A+1})$ about $x \in]x_A, x_{A+1}[$ in finite Taylor expansions.

see following pages for finite Taylor expansion notes.

$$u(x_A) = u(x) + (x_A - x)\,u'(x) + \frac{1}{2}(x_A - x)^2 u''(c_1)$$

$$u(x_{A+1}) = u(x) + (x_{A+1} - x)\,u'(x) + \frac{1}{2}(x_{A+1} - x)^2 u''(c_2)$$

$$e(x) = \tilde{u}^h(x) - u(x)$$

$$= \frac{(x_{A+1} - x)}{h}\left[\cancel{u(x)} + \cancel{(x_A - x)\,u'(x)} + \frac{1}{2}(x_A - x)^2 u''(c_1)\right]$$

$$+ \frac{(x - x_A)}{h}\left[\cancel{u(x)} + \cancel{(x_{A+1} - x)\,u'(x)} + \frac{1}{2}(x_{A+1} - x)^2 u''(c_2)\right]$$

$$- \cancel{u(x)}$$

$$= \frac{1}{2}\frac{(x_{A+1} - x)}{h}(x_A - x)^2 u''(c_1)$$

$$+ \frac{1}{2}\frac{(x - x_A)}{h}(x_{A+1} - x)^2 u''(c_2)$$

so $\sup_x |e(x)| \leq$

determine sup
$f(x)$; from $f'(x)=0$.

$$\sup_x \left[\frac{1}{2h} (x_{A+1} - x)(x_A - x)^2 \right] \sup_c |u''(c)|$$

$$+ \sup_x \left[\frac{1}{2h} (x_{\cancel{A+1}} - x_A)(x_{A+1} - x)^2 \right] \sup_c |u''(c)|$$

them
work ~~it~~ out

$$\sup_x |e(x)| \leq \alpha h^2 \sup_c |u''(c)|$$

(Can you do better than $\frac{4}{27}$?)

from (1.10.11) and using the bilinearity and symmetry of $a(\cdot, \cdot)$ yields the require result.

Lemma 2. $u(y) - u^h(y) = a(u - u^h, g)$, where g is the Green's functior

Proof

$$\begin{aligned} u(y) - u^h(y) &= (u - u^h, \delta_y) && \text{(definition of } \delta_y) \\ &= a(u - u^h, g) && \text{(by (1.10.10))} \end{aligned}$$

Note that line 2 is true since $u - u^h$ is in $\mathcal{V}$.

Proof of Theorem. As we have remarked previously, if $y = x_A$, a nod $g \in \mathcal{V}^h$. Let us take this to be the case. Then

$$\begin{aligned} u(x_A) - u^h(x_A) &= a(u - u^h, g) && \text{(Lemma 2)} \\ &= 0 && \text{(Lemma 1)} \end{aligned}$$

The theorem is valid for $A = 1, 2, \ldots, n + 1$. Strang and Fix [6] attribute th argument to Douglas and Dupont. Results of this kind, embodying exceptional acc racy characteristics, are often referred to as ***superconvergence*** phenomena. Howeve the reader should appreciate that, in more complicated situations, we will not be abl in practice, to guarantee nodal exactness. Nevertheless, as we shall see later or weighted residual procedures provide a framework within which optimal accurac properties of some sort may often be guaranteed.

Accuracy of the Derivatives

In considering the convergence properties of the derivatives, certain elementary nc tions of numerical analysis arise. The reader should make sure that he or she has complete understanding of these ideas as they subsequently arise in other contexts. W begin by introducing some preliminary mathematical results.

Taylor's Formula with Remainder

Let $f : [0, 1] \rightarrow \mathbb{R}$ possess k continuous derivatives and let y and z be two points i $[0, 1]$. Then there is a point c between y and z such that

$$\boxed{\begin{aligned} f(z) = f(y) &+ (z - y)f_{,x}(y) + \frac{1}{2}(z - y)^2 f_{,xx}(y) \\ &+ \frac{1}{3!}(z - y)^3 f_{,xxx}(y) + \cdots + \\ &+ \frac{1}{k!}(z - y)^k f_{,\underbrace{x\ldots x}_{k \text{ times}}}(c) \end{aligned}} \qquad (1.10.12$$

The proof of this formula may be found in [7]. Equation (1.10.12) is sometimes called a ***finite Taylor expansion***.

Mean-Value Theorem

The mean-value theorem is a special case of (1.10.12) which is valid as long as $k \geq 1$ (i.e., f is continuously differentiable):

$$\boxed{f(z) = f(y) + (z - y)f_{,x}(c)} \tag{1.10.13}$$

Consider a typical subinterval $[x_A, x_{A+1}]$. We have already shown that u^h is exact at the endpoints (see Fig. 1.10.3). The derivative of u^h in $]x_A, x_{A+1}[$ is constant:

$$u^h_{,x}(x) = \frac{u^h(x_{A+1}) - u^h(x_A)}{h_A}, \qquad x \in \,]x_A, x_{A+1}[\tag{1.10.14}$$

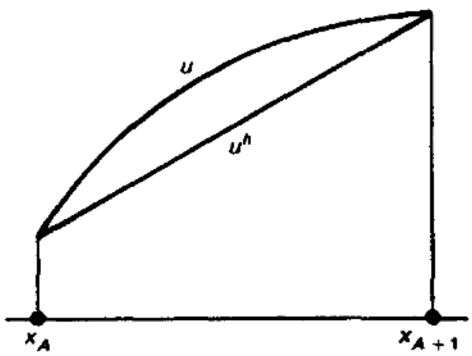


Figure 1.10.3

Theorem. Assume u is continuously differentiable. Then there exists at least one point in $]x_A, x_{A+1}[$ at which (1.10.14) is exact.

Proof. By the mean value theorem, there exists a point $c \in \,]x_A, x_{A+1}[$ such that

$$\frac{u(x_{A+1}) - u(x_A)}{h_A} = u_{,x}(c) \tag{1.10.15}$$

(We have used (1.10.13) with u, x_A, and x_{A+1}, in place of f, y, and z, respectively.) Since $u(x_A) = u^h(x_A)$ and $u(x_{A+1}) = u^h(x_{A+1})$, we may rewrite (1.10.15) as

$$\frac{u^h(x_{A+1}) - u^h(x_A)}{h_A} = u_{,x}(c) \tag{1.10.16}$$

Comparison of (1.10.16) with (1.10.14) yields the desired result. ∎

Remarks

1. This result means that the constant value of $u^h_{,x}$ must coincide with $u_{,x}$ somewhere on $]x_A, x_{A+1}[$; see Fig. 1.10.4.

2. Without knowledge of u we have no way of determining the locations at which the derivatives are exact. The following results are more useful in that they tell us that the midpoints are, in a sense, optimally accurate, independent of u.

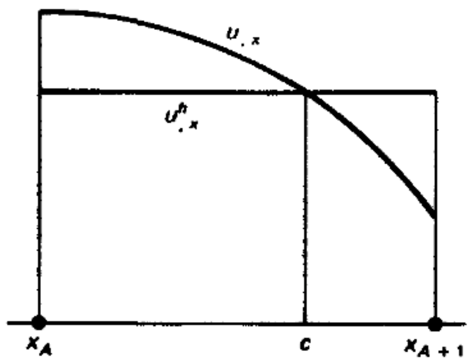


Figure 1.10.4

Let

$$e_{,x}(\alpha) \overset{\text{def.}}{=} u^h_{,x}(\alpha) - u_{,x}(\alpha) = \frac{u^h(x_{A+1}) - u^h(x_A)}{h_A} - u_{,x}(\alpha)$$

the ***error in the derivative*** at $\alpha \in [x_A, x_{A+1}]$. To establish the superiority of the midpoints in evaluating the derivatives, we need a preliminary result.

Lemma. Assume u is three times continuously differentiable. Then

$$\begin{aligned} e_{,x}(\alpha) &= \left(\frac{x_{A+1} + x_A}{2} - \alpha\right)u_{,xx}(\alpha) \\ &\quad + \frac{1}{3!\,h_A}[(x_{A+1} - \alpha)^3 u_{,xxx}(c_1) - (x_A - \alpha)^3 u_{,xxx}(c_2)] \end{aligned} \tag{1.10.17}$$

where c_1 and c_2 are in $[x_A, x_{A+1}]$.

Proof. Expand $u(x_{A+1})$ and $u(x_A)$ in finite Taylor expansions about $\alpha \in [x_A, x_{A+1}]$, viz.,

$$\begin{aligned} u(x_{A+1}) &= u(\alpha) + (x_{A+1} - \alpha)u_{,x}(\alpha) + \frac{1}{2}(x_{A+1} - \alpha)^2 u_{,xx}(\alpha) \\ &\quad + \frac{1}{3!}(x_{A+1} - \alpha)^3 u_{,xxx}(c_1), \qquad c_1 \in [\alpha, x_{A+1}] \end{aligned}$$

$$\begin{aligned} u(x_A) &= u(\alpha) + (x_A - \alpha)u_{,x}(\alpha) + \frac{1}{2}(x_A - \alpha)^2 u_{,xx}(\alpha) \\ &\quad + \frac{1}{3!}(x_A - \alpha)^3 u_{,xxx}(c_2), \qquad c_2 \in [x_A, \alpha] \end{aligned}$$

Subtracting and dividing through by h_A yields

$$\begin{aligned} \frac{u(x_{A+1}) - u(x_A)}{h_A} &= u_{,x}(\alpha) + \left(\frac{x_{A+1} + x_A}{2} - \alpha\right)u_{,xx}(\alpha) \\ &\quad + \frac{1}{3!\,h_A}[(x_{A+1} - \alpha)^3 u_{,xxx}(c_1) - (x_A - \alpha)^3 u_{,xxx}(c_2)] \end{aligned}$$

Replacing $u(x_{A+1})$ by $u^h(x_{A+1})$ and $u(x_A)$ by $u^h(x_A)$ in the left-hand side and rearranging terms completes the proof. ■

Discussion

To determine what (1.10.17) tells us about the accuracy of the derivatives, we wish to think of the situation in which the mesh is being systematically refined (i.e., we let h_A approach zero). In this case h_A^2 will be much smaller than h_A. Thus, for a given u, if the right-hand side of (1.10.17) *is* $O(h_A^2)$,[3] the error in the derivatives will be much smaller than if the right-hand side is only $O(h_A)$. The exponent of h_A is called the ***order of convergence*** or ***order of accuracy***. In the former case we would have second-order convergence of the derivative, whereas in the latter case we would have only first-order convergence.

As an example, assume $\alpha \to x_A$. Then

$$e_{,x}(x_A) = \frac{h_A}{2}u_{,xx}(x_A) + \frac{h_A^2}{3!}u_{,xxx}(c_1) = O(h_A)$$

As $h_A \to 0$, the first term dominates. (We have seen from the example calculations in Sec. 1.8 that the endpoints of the subintervals are not very accurate for the derivatives.)

Clearly any point $\alpha \in [x_A, x_{A+1}]$ achieves first-order accuracy. We are thus naturally led to asking the question, are there any values of α at which higher-order accuracy is achieved?

Corollary. Let $x_{A+1/2} \equiv (x_A + x_{A+1})/2$ (i.e., the midpoint). Then

$$\begin{aligned} e_{,x}(x_{A+1/2}) &= \frac{h_A^2}{24}u_{,xxx}(c), \qquad c \in [x_A, x_{A+1}] \\ &= O(h_A^2) \end{aligned}$$

Proof. By (1.10.17)

$$e_{,x}(x_{A+1/2}) = \frac{h_A^2}{48}[u_{,xxx}(c_1) + u_{,xxx}(c_2)]$$

By the continuity of $u_{,xxx}$, there is at least one point c between c_1 and c_2 such that

$$u_{,xxx}(c) = \frac{1}{2}[u_{,xxx}(c_1) + u_{,xxx}(c_2)]$$

Combining these facts completes the proof. ■

Remarks

1. From the corollary we see that the derivatives are second-order accurate at the midpoints.

[3]A function $f(x)$ is said to be $O(x^k)$ (i.e., order x^k) if $f(x)/x^k \to$ a constant as $x \to 0$. For example, $f(x) = x^k$ is $O(x^k)$, as is $f(x) = \sum_{j=k}^{k+l} x^j$, $l \geq 0$. But neither is $O(x^{k+1})$. (Verify.)

2. If the exact solution is quadratic (i.e., consists of a linear combination of the monomials $1, x, x^2$), then $u_{,xxx} = 0$ and—by (1.10.17)—the derivative is exact at the midpoints. This is the case when $\ell(x) = \rho =$ constant.

3. In linear elastic rod theory, the derivatives are proportional to the stresses. The midpoints of linear "elements" are sometimes called the ***Barlow stress points,*** after Barlow [8], who first noted that points of optimal accuracy existed within elements.

Exercise 1. Assume the mesh length is constant (i.e., $h_A = h, A = 1, 2, \ldots, n$). Consider the standard finite difference "stencil" for $u_{,xx} + \ell = 0$ at a typical internal node, namely,

$$\frac{u_{A+1} - 2u_A + u_{A-1}}{h^2} + \ell_A = 0 \tag{1.10.18}$$

Assuming ℓ varies in piecewise linear fashion and so can be expanded as

$$\ell = \sum_{A=1}^{n+1} \ell_A N_A \tag{1.10.19}$$

where the ℓ_A's are the nodal values of ℓ, set up the finite element equation associated with node A and contrast it with (1.10.18). Deduce when (1.10.18) will also be capable of exhibiting superconvergence phenomena. (That is, what is the restriction on ℓ?) Set up the finite element equation associated with node 1, accounting for nonzero h. Discuss this equation from the point of view of finite differences. (For further comparisons along these lines, the interested reader is urged to consult [6], Chapter 1.)

Summary. The Galerkin finite element solution u^h, of the problem (S), possesses the following properties:

i. It is exact at the nodes.
ii. There exists at least one point in each element at which the derivative is exact.
iii. The derivative is second-order accurate at the midpoints of the elements.

1.11 INTERLUDE: GAUSS ELIMINATION; HAND-CALCULATION VERSION

It is important for anyone who wishes to do finite element analysis to become familiar with the efficient and sophisticated computer schemes that arise in the finite element method. It is felt that the best way to do this is to begin with the simplest scheme, perform some hand calculations, and gradually increase the sophistication as time goes on.

To do some of the problems we will need a fairly efficient method of solving matrix equations by hand. The following scheme is applicable to systems of equations