Lecture #10, Feb. 19, 2024.

Equivalent norms.

$\|\cdot\| \sim |||\cdot|||$ if $\exists$ fixed const's $c_1, c_2 > 0$.

$\ni \forall w \in \mathcal{V}$. (sp. in question)

$$c_1 \|w\| \leq |||w||| \leq c_2 \|w\|$$

Equiv. norms on $\mathcal{V} = \{w \mid w \in H^1(0,L),\ w(0) = w(L) = 0\}$

are $\|w\|_1 \sim |w|_1$

$$\underbrace{c_1 \|w\|_1 \leq |w|_1}_{\text{LHS.}} \underbrace{\leq c_2 \|w\|_1}_{\text{RHS.}}$$

Lemma: (Poincaré - Friedrichs ineq.)

$$\|w\|_{L_2(0,L)} \leq \underbrace{L}_{\substack{\text{in gen.} \\ C.}} \|w_{,x}\|_{L_2(0,L)}$$

BC.s are the key issue.

$$\int_0^x w_{,x} = w(x) - \overset{0.}{\overset{\shortparallel}{w(0)}}$$

$$\underline{(w(x))^2} = \left(\int_0^x w_{,x}\right)^2$$

$$= \left(\int_0^x 1 \cdot w_{,x}\right)^2 = \left((1, w_{,x})_{L_2(0,x)}\right)^2$$

Cauchy. Schwarz ineq.

$$\leq \left(\|1\|_{L_2(0,x)} \; \|w_{,x}\|_{L_2(0,x)} \right)^2$$

$$= \left(\Big(\int_0^x \underbrace{1^2}_{>0} \Big)^{\cancel{1/2}} \Big(\int_0^x \underbrace{(w_{,x})^2}_{\geq 0.} \Big)^{\cancel{1/2}} \right)^{\cancel{2}}$$

$$\leq \Big(\int_0^L 1^2 \Big) \Big(\int_0^L (w_{,x})^2 \Big)$$

$0 \quad x \quad L$

$$= \underbrace{L \; \|w_{,x}\|^2_{L_2(0,L)}}_{\lambda} = \text{a no.}$$

$\int_0^L \dots$: $\quad \int_0^L w^2 = \int_0^L \lambda$

$$= L \; L \; \|w_{,x}\|^2_{L_2(0,L)}$$

$$\|w\|^2_{L_2(0,L)} \leq L^2 \; \|w_{,x}\|^2_{L_2(0,L)}$$

$$\boxed{\|w\| \leq L \; \|w_{,x}\|}$$

P.-F. ✓ $\square$.

Pf. of equiv. $\|\cdot\|_1 \sim |\cdot|_1$ on

$\mathcal{V}\int +$ w. BCs.

LHS $\quad \overset{?}{C_1} \|v\|_1 \leq |v|_1 \quad (?)$

$$|v|_1^2 = \int_0^L L^2 (v_{,x})^2 dx = L^2 \|v_{,x}\|^2$$

$$= \underbrace{\frac{1}{2} L^2 \|v_{,x}\|^2}_{\text{apply P.F.}} + \frac{1}{2} L^2 \|v_{,x}\|^2$$

$$\geq \frac{1}{2} \|v\|^2 + \frac{1}{2} L^2 \|v_{,x}\|^2$$

$$= \frac{1}{2} \int_0^L \left(v^2 + L^2 (v_{,x})^2\right)$$

$$= \frac{1}{2} \|v\|_1^2$$

$$\frac{1}{2} \|v\|_1^2 \leq |v|_1^2$$

$$\boxed{\frac{1}{\sqrt{2}} \|v\|_1 \leq |v|_1}$$ LHS ✓ $\quad C_1 = \frac{1}{\sqrt{2}}$.

RHS. $\quad |v|_1 \overset{?}{\leq} \overset{?}{C_2} \|v\|_1 \qquad$ ✓ $C_2 = 1$.

$$|v|_1^2 = \underline{\int_0^L L^2 (v_{,x})^2 dx} \overset{?}{\leq} \underbrace{\overset{?}{C_2^2}}_{= 1.} \int_0^L \underline{\left(v^2 + L^2 (v_{,x})^2\right)}$$

Error "orthog." $\equiv$ Gal. orthog.

$\equiv$ strong consistency of the method.

Our meth: $u^h \in \mathcal{S}^h$

$$B_{\dots}(w^h, u^h) = L_{\dots}(w^h) \qquad \forall w^h \in \mathcal{V}^h$$

All our methods are residual meth's; meaning u (exact sol.) satisfies our meth:

$$B_{\dots}(w^h, u) = L_{\dots}(w^h) \checkmark$$

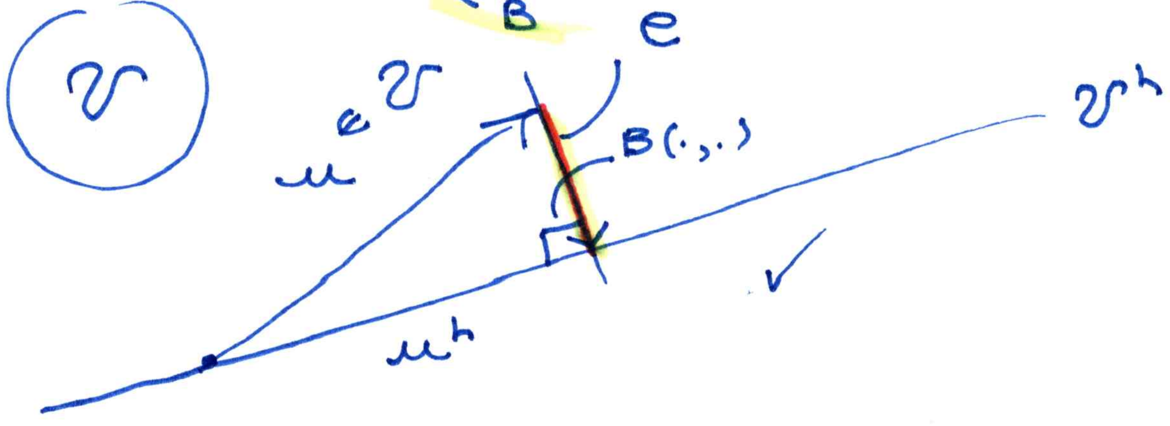
Subtract $\quad B(w^h, u^h) - B(w^h, u) = 0$

$$B(w^h, \underbrace{u^h - u}_{=e}) = 0 \qquad \text{bilin}$$

$$B(w^h, e) = 0$$

$$e \perp_{B_{\dots}(\cdot,\cdot)} w^h, \quad \forall w^h \in \mathcal{V}^h$$

$$e \perp_{B} \mathcal{V}^h$$



All our meth's possess the error orth. property: Gal, SUPG, GLS, MS.

Stability of the method.

Coercivity: $\exists$ a norm $|||\cdot|||$ $\ni$ (some norm on $\mathcal{V}^h$)

$$B_{\dots}(w^h, w^h) \geq \underset{\substack{>0 \\ \text{const.}}}{C_s} |||w^h|||^2$$

we can prove for all our meth's.

but Gal is unusual in that we can show

$$B(w,w) \geq C_s |||w|||^2_{Gal}.$$

Pf. Gal

$$\underline{B(w,w)} = \int_0^L (-w_{,x}\, a\, w + w_{,x}\, \alpha\, w_{,x})\,dx$$

$$= -\frac{a}{2}\int_0^L (w^2)_{,x} + \alpha \|w_{,x}\|^2$$

$$= -\frac{a}{2}\, \cancel{w^2}\Big|_0^L + \alpha \|w_{,x}\|^2$$

$$= \alpha \frac{1}{L^2}\int_0^L L^2 (w_{,x})^2\,dx$$

$$= \frac{\alpha}{L^2}\left(|w|_1^2\right)$$

no good in the limit case $\alpha \to 0$

$$\geq \frac{\alpha}{L^2}\left(\frac{1}{2}\|w\|_1^2\right) \quad \leftarrow \text{norm equivalence}$$

$$\hookrightarrow \quad B(w,w) \geq \left(\frac{\alpha}{2L^2}\right)\|w\|_1^2$$

Good but, not a strong enough stab. result.

Inspiration of stab. in the limit case:

$$a\, u_{,x} = f.$$

$$(a \cdot u_{,x})(a \cdot u_{,x}) = (a \cdot u_{,x})\, f$$

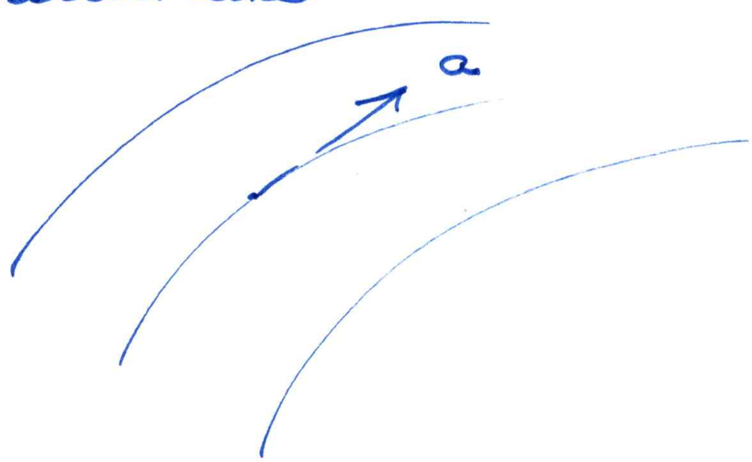
$\int_0^L$:
$$\int_0^L (a\, u_{,x})^2 = \int_0^L (a\, u_{,x} f)$$

$$\| a u_{,x} \|^2_{L_2(0,L)} = (a u_{,x}, f)_{L_2(0,L)}$$

$$\overset{C.S}{\le} \| a u_{,x} \| \, \| f \|$$

$$\| a u_{,x} \| \le \| f \| .$$

multi-d : $\| \underbrace{a \cdot \nabla u}_{\text{stab along the streamlines}} \| \le \| f \|$



Properties of our methods,

" " " spaces (finite el.)

	Method	Space $\mathcal{V}^h, \mathcal{S}^h$
✓ Consistency (accuracy)	Error "orthog."	"Interpolation" est's (approx.)
✓ Stability	Coercivity	Inverse est's.

✓ next time