Lecture 18, March 25, 2024

$$u^h(t_{n+1}^-) = u^h(t_n^-) + \int_{t_n}^{t_{n+1}} f\,dt, \quad n = 0, 1, \ldots, N-1$$

start $n+1 = N$

$$u^h(\boxed{t_N^-}) = u^h(t_{N-1}^-) + \int_{t_{N-1}}^{t_N = T} f\,dt \qquad t_N^- = T^-$$

$$u^h(t_{N-1}^-) = u^h(t_{N-2}^-) + \int_{t_{N-2}}^{t_{N-1}} f\,dt$$

$$u^h(t_{N-2}^-) = u^h(t_{N-3}^-) + \int_{t_{N-3}}^{t_{N-2}} f\,dt$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$n+1 << N-2$

$$u^h(t_{n+1}^-) = u^h(t_n^-) + \int_{t_n}^{t_{n+1}} f\,dt$$

$$u^h(t_n^-) = u^h(t_{n-1}^-) + \int_{t_{n-1}}^{t_n} f\,dt$$

$$u^h(t_{n-1}^-) = u^h(t_{n-2}^-) + \int_{t_{n-2}}^{t_{n-1}} f\,dt$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$3 << n-1$

$$u^h(t_3^-) = u^h(t_2^-) + \int_{t_2}^{t_3} f\,dt$$

$$u^h(t_2^-) = u^h(t_1^-) + \int_{t_1}^{t_2} f\,dt$$

$$u^h(t_1^-) = \boxed{u^h(t_0^-)} + \int_{t_0 = 0}^{t_1} f\,dt$$

$$= u_0 . \checkmark$$

SUM

$$u^h(T^-) = u_0 + \int_0^T f\,dt$$

Global cons. ($\Leftarrow$ local cons.)

$$\left[\!\left[w^h \right]\!\right]^2 = \frac{1}{2}\Big(w^h(T^-)^2 + w^h(0^+)^2 + [\![w^h(t_n)]\!]^2 \Big)_x \quad (91)$$

Controlling: $w^h(T^-)$, $w^h(0^+)$

$\Rightarrow$ all $w^h(t_{n+1}^-)$ $\quad n = 0, 1, \dots, N-1$

$w^h(t_n^\pm) \Leftarrow$

$[\![w^h(t_n)]\!] \Rightarrow w^h(t_n^+)$.

all $n = 1, \dots, N-1$

No control: e.g. $u^h_{,t}$. $\Leftarrow$ require additional stab.

nothing in the interior

Stab. Th'm.

$$\mathbb{B}(w^h, w^h) = \left[\!\left[w^h \right]\!\right]^2$$

Pf: $\mathbb{B}(w^h, w^h) = \sum_{n=0}^{N-1} B(w^h, w^h)_n$

$=$

$$= \sum_{n=0}^{N-1} B(w^h; \underset{\uparrow}{w^h})_n \overset{\text{def}}{=} \sum_{n=0}^{N-1} \Bigg(\boxed{- \int_{t_n}^{t_{n+1}} w^h_{,t}\, w^h \, dt} \quad \text{(92.)}$$

$$\boxed{+ \; w^h(t_{n+1}^-)^2 - w^h(t_n^+)\, w^h(t_n^-)} \Bigg)$$

$$= \sum - \underbrace{\int_{t_n}^{t_{n+1}} \frac{1}{2}\left((w^h)^2\right)_{,t} dt}$$

$$= -\frac{1}{2}(w^h)^2 \Big|_{t_n}^{t_{n+1}} = \boxed{-\frac{1}{2}\left(w^h(t_{n+1}^-)^2 - w^h(t_n^+)^2 \right)}$$

$$= \sum_{n=0}^{N-1} + \frac{1}{2}\Big(w^h(t_{n+1}^-)^2 + w^h(t_n^+)^2$$

$$- 2 w^h(t_n^+)\, w^h(t_n^-) \Big)$$

$$w^h(t_0^-) = w^h(0^-) \overset{\text{def}}{=} 0.$$

expand the sum, term-by-term

each ~~row~~ line in what follows is

one of these $n = 0, 1, \ldots, N-1$

$$
\begin{aligned}
\frac{1}{2}\Big[\; & \overset{T^- \equiv}{w^h(t_N^-)^2} + w^h(t_{N-1}^+)^2 - 2w^h(t_{N-1}^+)\,w^h(t_{N-1}^-) \\
& \vdots \\
& + w^h(t_{n+1}^-)^2 + w^h(t_n^+)^2 - 2w^h(t_n^+)\,w^h(t_n^-) \\
& + w^h(t_n^-)^2 + w^h(t_{n-1}^+)^2 - 2w^h(t_{n-1}^+)\,w^h(t_{n-1}^-) \\
\text{Typ.}\quad & + w^h(t_{n-1}^-)^2 + w^h(t_{n-2}^+)^2 - 2w^h(t_{n-2}^+)\,w^h(t_{n-2}^-) \\
& \vdots \\
& + w^h(t_1^-)^2 + \overset{0^+ \equiv}{w^h(t_0^+)^2} - \cancel{2w^h(t_0^+)\,w^h(t_0^-)}^{\,0.} \;\Big]
\end{aligned}
$$

$$
= \frac{1}{2}\Big[w^h(T^-)^2 + w^h(0^+)^2 + \sum_{n=1}^{N-1} \underbrace{\left(w^h(t_n^+)^2 - 2w^h(t_n^+)\,w^h(t_n^-) + w^h(t_n^-)^2 \right)}_{\left(w^h(t_n^+) - w^h(t_n^-)\right)^2 \;\equiv\; [\![w^h(t_n)]\!]^2} \Big]
$$

Typ.

$$
\overset{\text{def.}}{=} [\![w^h]\!]^2 \qquad \text{QED}
$$

controlling: $\underbrace{w_{,t}}$ Stab w. "SUPG" in time

94.

(1) $B_{SUPG}(w^h, u^h)_n = L_{SUPG}(w^h)$

$$B_{SUPG}(w^h, u^h)_n \stackrel{def}{=} B(w^h, u^h)_n + \int_{t_n}^{t_{n+1}} \tau w^h_{,t} (u^h_{,t}) dt$$

$\tau = \frac{h}{2|a|} \tilde{\xi}$, ξ non-dim.

$\left[\frac{h}{v}\right] = [\tau]$

added res.

$$L_{SUPG}(w^h)_n \stackrel{def}{=} L(w^h)_n + \int_{t_n}^{t_{n+1}} \tau w^h_{,t} (f) dt$$

$$\mathbb{B}_{SUPG}(w^h, u^h) \stackrel{def}{=} \sum_{n=0}^{N-1} B_{SUPG}(w^h, u^h)_n$$

$$\mathbb{L}_{SUPG}(w^h) \stackrel{def.}{=} \sum_{n=0}^{N-1} L_{SUPG}(w^h)_n$$

Continuous meth weak form. w. $w^h, u^h \leftarrow w, u$

$$\mathbb{B}_{SUPG}(w, u) = \mathbb{L}_{SUPG}(w) \quad \forall w \in \mathcal{V}.$$

(2) $\mathbb{B}_{SUPG}(w^h, u) = \mathbb{L}_{SUPG}(w^h) \quad \forall w^h \in \mathcal{V}^h \subset \mathcal{V}.$

(1) − (2) $\mathbb{B}_{SUPG}(w^h, \underbrace{u^h - u}_{e}) = 0$

global consis. ✓

Stab. $\mathbb{B}_{Stab}(w^h, w^h) =$

$$\underbrace{\underbrace{\mathbb{B}(w^h, w^h)}_{\text{DG. part.}} + \underbrace{\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \tau (w^h_{,t})^2 \, dt}_{\text{additional stab.}}}_{\equiv |||w^h|||^2}$$

$$\equiv |||w^h|||^2_{Stab}.$$

Functional anal $\Rightarrow$ $\tau \Big|_{(t_n, t_{n+1})} = \dfrac{(t_{n+1} - t_n)}{2}$

$$= \frac{\Delta t}{2}. \checkmark$$

τ formula's in space

$\sim \dfrac{h}{2|a|} \tilde{\xi}$ ✓

adv limit $\tilde{\xi} \to 1$.

$a \leftarrow 1$.

$h \leftarrow \Delta t$

$$|||e|||_{stab} \sim \Delta t^{k + \frac{1}{2}}$$

$$|||e|||^2_{stab} \sim \Delta t^{2k+1}$$