ODEs

Factor

$f(x)dx = g(y)dy$

Exact

$dU = M\,dx + N\,dy = 0$ *Exact* $\Leftrightarrow \partial M/\partial y = \partial N/\partial x$

$M = \frac{\partial U}{\partial x}, N = \frac{\partial U}{\partial y}$

Partial Integration:

$U = \int M dx = F + C(y)$ *Treat y as a constant*

Find $C'(y) \rightarrow \frac{\partial F}{\partial y} + C'(y) = \frac{\partial U}{\partial y}$

Then integrate to find $C(y)$ and $U(x, y)$

Linear, first order

$y' + p(x)y = q(x)$

Integrating factor $\rightarrow m = e^{\int p(x)dx} \rightarrow d[ym] = m\,q(x)\,dx$

Homogeneous Function

$g(tx, ty) = t^n g(x, y)$

Homogeneous DE

$y' = f(x)$ and $f(tx, ty) = f(x, y)$

$y = vx$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$ or $x = vy$ $\frac{dx}{dy} = v + y\frac{dv}{dy}$

Separate variables and use integrating factor

Wronskian

$$W(f_1, f_2, \dots, f_n) = \det\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Superposition for linear DE

If y and z are solutions of $y'' + ay' + by = 0$ so is $C_1 y + C_2 z$.

2nd order, constant coefficient

$y'' + ay' + by = f(x)$

Find y_h solving with $f(x) = 0$

Find roots m_1,m_2 of $m^2 + am + b = 0$

$m_1 \neq m_2 \in \mathbb{R} \quad \rightarrow \quad ■ = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$m = \alpha \pm i\beta \quad \rightarrow \quad ■ = e^{\alpha x}(C_1 cos\beta x + C_2 sin\beta x)$

m repeated n times: $\rightarrow \quad (C_1 + C_2 x + \cdots + C_n x^n)[■]$

Find y_p, then $y = y_h + y_p$

Undetermined coefficients

Convert to D-form $D^n = d^n/dx$

Find roots of the characteristic equation m_i

Find roots of the RHS by inverse inspection m_i'

Limitation: RHS must be such that we can find m_i'

Write y considering every m_i and m_i'

Identify y_h and y_p

Subs y_p in the DE to find the constants

y_h constants are found using the initial conditions

Variation of parameters

$(D^n + a_{n-1}D^{n-1} + \cdots + D + a_0)y = q(x)$

Find y_h as in undetermined coefficients. Write:

$y_h = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$

$y_p = v_1 y_1 + v_2 y_2 + \cdots + v_n y_n$

Find derivatives for y_i and solve the system for v_i'

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(p)} & y_2^{(p)} & \cdots & y_n^{(p)} \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \\ \cdots \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ q(x) \end{bmatrix}$$

Find v_i integrating v_i'

Bernoulli

$y' + p(x)y = q(x)y^n$

$y = v^{-1/(n-1)} \rightarrow y' = \frac{-1}{n-1}\left[v^{-n/(n-1)}\right]\frac{dv}{dx}$

$v' - (n-1)p(x)v = -(n-1)q(x)$

Euler

$b_n x^n y^{(n)} + b_{n-1}x^{n-1}y^{n-1} + \cdots + b_1 xy' + b_0 y = 0$

$\forall b_i = cnt$

$z = \ln(x) \quad \rightarrow \quad x = e^z$

$y^{(n)} = \frac{1}{x^n}D(D-1)(D-2)\dots(D-n+1)y$, where $D = \frac{d}{dz}$

Solve the resulting linear equation.

Power series solution

$f(x)$ is an **analytic function** if it has a power series represent. around x_0.

Initial Value Problem (IVP)

$y' + p(x)y = q(x)$ $y(x_0) = y_0$ p, q analytic about x_0

$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ $a_n = \frac{1}{n!}y^{(n)}(x_0)$, in $(x_0 - h, x_0 + h)$

Subs the IC to find $y'(x_0)$

Differentiate the DE and subs $y(x_0)$ and $y'(x_0)$ to find $y''(x_0)$

Drawback: differentiating the DE might not be practical.

Power series solution using recurrence relation

$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$

$y' = \sum_{n=1}^{\infty} n\,a_n(x - x_0)^{n-1}$

$y'' = \sum_{n=2}^{\infty}(n-1)\,a_n(x - x_0)^{n-2}$

Subs in the equation.

Shift indices in the summations so that the power of x is the same in all

Be careful not to loose terms!

The coefficients of each power x^k must be equal in LHS and RHS

Find recurrence relation for the each a_n

Subs a_n in $y(x)$ and expand the summations as needed.

Shifting $\sum_{n=0}^{\infty} C_n x^{n+a} = \sum_{n=a}^{\infty} C_{n-a}x^n$

Singularities and the method of Frobenius

$y'' + p(x)y' + q(x)y = f(x)$ p, q analytical about x_0; x_0 ordinary

Singular point of the DE: p, q or f has zero denominator about x_0.

Regular singular: $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytical

Ordinary: not singular. **Irregular:** Not regular

If equation has a regular singular point at x_0, use a Frobenius series:

$y = \sum_{n=0}^{\infty} C_n z^{n+r}$ $z = (x - x0)$, where $r \in \mathbb{R}$

$y' = \sum_{n=0}^{\infty}(n+r)\,C_n\,z^{n+r-1}$

$y'' = \sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,C_n\,z^{n+r-2}$

Subs in the equation, shift indices etc. (same as pwr series)

Assume $C_0 \neq 0$ to find values for r_1, r_2 ($r_1 \geq r_2$).

Find the recurrence C_n using $r = r_1$. Write $y_1(x)$.

For the **second solution**, find the recurrence C_n^* as:

If $r_1 - r_2 \notin \mathbb{Z}$ $y_2 = \sum_{n=0}^{\infty} C_n^*\,z^{n+r_2}$

If $r_1 = r_2$ $y_2 = y_1\ln(z) + \sum_{n=0}^{\infty} C_n^*\,z^{n+r_1}$

If $r_1 - r_2 \in \mathbb{N}$ $y_2 = Ky_1\ln(z) + \sum_{n=0}^{\infty} C_n^*\,z^{n+r_2}$

Subs y_2 into the DE and obtain an equation for K

[1] Subs y_2 into the DE and the $\ln$ term vanishes as the terms it multiplies are equal to y_1.

[2] We use $C_0 = 1$ for convenience

[3] Subs the found solution y_1 after simplifications

Final solution: $y(x) = y_1 + y_2$

Bessel's equation of order ν

$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$ $\nu \in \mathbb{R}, \nu \geq 0, x > 0$

$y(x) = C_0\,J_\nu + C_1\,Y_\nu$

Bessel function of the 1st kind: $J_\nu = C_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}\,n!(1+\nu)(2+\nu)\dots(n+\nu)}x^{2n+\nu}$

Bessel function of the 2nd kind: $Y_\nu = \cdots$

LIN ALG

Matrices and vectors

$A = [a_{ij}]$ $\rightarrow$ i: row j:column

a_{1j}: row matrix a_{i1}: column matrix

$\frac{dA}{dt} = \dot{A} = \left[\frac{da_{ij}}{dt}\right] = [\dot{a}_{ij}]$ $\int A(\tau)d\tau = [\int a_{ij}(\tau)d\tau$

$\boldsymbol{a} \cdot \boldsymbol{b} = ||\boldsymbol{a}||\,||\boldsymbol{b}|| \cos(\theta)$ $||\boldsymbol{a}|| = \sqrt{a_{kk}^2}$

$\text{comp}_b \boldsymbol{a} = \boldsymbol{a} \cdot \frac{\boldsymbol{b}}{||\boldsymbol{b}||}$ *(component of **a** in **b**)* $\text{proj}_b \boldsymbol{a} = \boldsymbol{a} \cdot \frac{\boldsymbol{b}}{||\boldsymbol{b}||^2}\boldsymbol{b}$ *(projection of **a** in **b**)*

Definitions

Conjugate: $\bar{A}$: $a_{ij} = \alpha_{ij} \pm i\,\beta_{ij} \rightarrow \overline{a_{ij}} = \alpha_{ij} \mp i\,\beta_{ij}$

Rank: largest non zero determinant; number of independent vectors;
$C = AB \rightarrow 0 \leq \text{rank}(C) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

Trace: $tr(A) = \sum a_{ii}$ $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$ $\text{tr}(AB) = \text{tr}(BA)$
$\text{tr}(AB) \neq \text{tr}(BA)$

Identities

$A^T = a_{ji}$ $[A^T]^T = A$ $[A \pm B]^T = A^T \pm B^T$

$[cA]^T = cA^T$ $[ABC]^T = C^T B^T A^T$

$[AB]^T = B^T A^T = BA \neq AB$

If A is symmetric, then so is $B^T AB$, $\forall B$

$a_{ij} \in \mathbb{R} \rightarrow A = \bar{A}$

Square matrices

Symmetric: $A = A^T$ Skew-Symmetric: $A = -A^T$

Positive definite: $x^T A\, x > 0$ Non negative definite: $x^T A\, x \geq 0$

Indefinite: $(x^T A\, x)(y^T A\, y) < 0$ $x, y \in \mathbb{R}^n$

Orthogonal $A^T = A^{-1}$ $\rightarrow$ $A^T A = I$

Nilpotent: $A^k = 0$ and $A^{k-1} \neq 0$ $k \in \mathbb{Z}$

Idempotent: $A^2 = A$

Involutory: $A^2 = I$ Unitary: $A^{-1} = \bar{A}^T$

Positive: $a_{ij} > 0$ $\forall i, j$ Non-negative: $a_{ij} \geq 0$ $\forall i, j$

Diagonal Dominant: $|a_{ii}| \geq \sum_{i \neq j} |a_{ij}|$ Strictly Diag Dom: $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$

Associate: $[\,\bar{A}\,]^T$ Hermitian: $A = \bar{A}^T$ Skew Hermitian: $A = -\bar{A}^T$

Determinants $det(A) = |A|$

A is a $(n \text{ x } n)$ matrix, then:

$$|A| = \frac{1}{(a_{11})^{n-2}} = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & \begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix} & \vdots & \vdots \\ \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{in} \end{vmatrix} & \cdots & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

Minor: $M_{ij} = \begin{vmatrix} \blacksquare & X & \blacksquare & \blacksquare \\ X & a_{ij} & X & X \\ \blacksquare & X & \blacksquare & \blacksquare \\ \blacksquare & X & \blacksquare & \blacksquare \end{vmatrix} = \begin{vmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{vmatrix}$ Cofactor: $C_{ij} = (-1)^{i+j}\, M_{ij}$

$|A| = \sum_{k=1}^{n} a_{ik} C_{ik}$, for any row i

Adjoint matrix C^T is the transpose of the cofactor's matrix.

Inverse

Inverses are unique

$A\,A^{-1} = I$ $A^{-1} = \frac{C^T}{|A|}$

Properties of determinants

$|A||B| = |AB|$ $|A| = |A^T|$

If any col or row is null, then $|A| = 0$

If operate columns or rows, then $|A|$ does not change

If swap columns or rows, then $|A|$ changes sign

If two columns or rows are proportional then $|A| = 0$

If one column or row is the linear combination of others then $|A| = 0$

Multiply column or row by α then $|B| = \alpha|A|$

If $|A| = 0$, A is singular and has no inverse.

Set of vectors

$\{\mathbf{v}_i\}$ are linearly independent $\Leftrightarrow \alpha_k \mathbf{v}_k = 0$ for at least one set of α_i.

$\{\mathbf{v}_i\}$ is a base if exists a unique choice of scalars for every vector $\boldsymbol{u}$. That is, $\{\mathbf{v}_i\}$ are independent

$\{\mathbf{v}_i\}$ is orthogonal $\Leftrightarrow \mathbf{v}_i^T \mathbf{v}_j = 0$, $\forall i \neq j$

$\{\mathbf{v}_i\}$ is ortho<u>normal</u> $\Leftrightarrow ||\mathbf{v}_i|| = 1$, $\forall i$

Normalization: $\tilde{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||}$

Gram-Schmidt orthogonalization of $\{\mathbf{v}_i\}$

$\{\mathbf{v}_i\}$ are linearly independent

$\boldsymbol{u}_1 = \mathbf{v}_1$ $\boldsymbol{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \boldsymbol{u}_1}{||\boldsymbol{u}_1||^2}\boldsymbol{u}_1$ $\boldsymbol{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \boldsymbol{u}_1}{||\boldsymbol{u}_1||^2}\boldsymbol{u}_1 - \frac{\mathbf{v}_3 \cdot \boldsymbol{u}_2}{||\boldsymbol{u}_2||^2}\boldsymbol{u}_2$

$\boldsymbol{u}_m = \mathbf{v}_m - \sum_{k=1}^{m-1} \text{proj}_{\boldsymbol{u}_k} \mathbf{v}_m = \mathbf{v}_m - \sum_{k=1}^{m-1} \frac{\mathbf{v}_m \cdot \boldsymbol{u}_k}{||\boldsymbol{u}_k||^2}\boldsymbol{u}_k$

Systems of linear equations

$A\,x = \boldsymbol{b}$

Cramer's rule: $x_j = \frac{\Delta_j}{|A|}$, where $\Delta_j = |A_j|$ and A_j is A with column j replaced by vector $\boldsymbol{b}$.

$|A| \neq 0, \boldsymbol{b} \neq 0 \rightarrow$ unique solution
$|A| \neq 0, \boldsymbol{b} = 0 \rightarrow \underline{x} = 0$
$|A| = 0, \boldsymbol{b} = 0 \rightarrow$ infinite solutions
$|A| \neq 0, \boldsymbol{b} \neq 0 \rightarrow$ infinite solutions $\Leftrightarrow \Delta_i = 0, \forall i$. Otherwise, no solution.

A is diagonal: system is called uncoupled or the variables are called separated

A and B are similar: $B = P^{-1}AP$ and $PB = AP$, whare P is the set of eigenvectors.

Gaussian Elimination

Operate equations/rows to *uptriangularize* the system.
Back substituition to find variables.

Augmented matrix

$A\,x = \boldsymbol{b} \rightarrow [A\!:\boldsymbol{b}]$

Operate rows to find $[I\!:\tilde{\boldsymbol{b}}]$

Operate $[A\!:\boldsymbol{b}\!:I]$ to find solution and inverse as $[I\!:\boldsymbol{x}\!:A^{-1}]$

LU Factorization

$A\,x = \boldsymbol{b}$

$A = LU \rightarrow U\boldsymbol{x} = \tilde{\boldsymbol{b}} \rightarrow LU\boldsymbol{x} = \boldsymbol{b} \rightarrow U\boldsymbol{x} = \boldsymbol{y} \rightarrow L\boldsymbol{y} = \boldsymbol{b}$

Iterative method: Jacobi

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Rightarrow \begin{cases} x = (c - by)/a \\ y = (f - dx)/e \end{cases} \Rightarrow \begin{cases} \text{Initial guess for x, y} \\ \text{Iterate} \end{cases}$$

Eigenvalues and Eigenvectors

$A\,\boldsymbol{x}_i = \lambda_i\,\boldsymbol{x}_i$ $\Rightarrow$ Characteristic equation (CE): $(A - \lambda_i I)\,\boldsymbol{x}_i = 0, \forall i$

Find eigenvalues λ_i as the scalar roots of the CE

Replace every λ_i in the CE to find eigenvectors $\boldsymbol{x}_i$ associated to each λ_i

$\rightarrow \boldsymbol{x}_i$ must be linearly independent

$\rightarrow A$ is a zero of its characteristic equation *(Cayley Hamilton)*.

$|A| = \prod \lambda_i$ $tr(A) = \sum \lambda_i$

If, for any i, $\lambda_i = 0 \Rightarrow A$ is singular

If A is real and symmetric $\Rightarrow \lambda_i \in \mathbb{R}$

If A is diagonal $\Rightarrow A = diag(\lambda_i)$ and $A^{-1} = \text{diag}(1/\lambda_i)$

If A is upper or lower triangular $\Rightarrow \lambda_{ii} = a_{ii}$ and $|A| = \prod a_{ii}$

Companion Matrix

$p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0$

$\Rightarrow p(x) = x^n - (a_n - a_{n-1}x - \cdots - a_1 x^{n-1})$

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix} \Rightarrow \begin{cases} |C| = p(x) \\ \text{Eigenvalues are the roots of } p(x) \end{cases}$$

Partitioned Matrix

$$AB = C \Rightarrow \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} [B_1 \quad B_1] = \begin{bmatrix} A_1 B_1 & A_1 B_2 \\ B_2 B_1 & A_2 B_2 \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} \Rightarrow \det(A) = |A| = |A_1||A_2| \ldots |A_n|$$

LAPLACE

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

Workflow: IVP → $\mathcal{L}(t \to s)$ → Algebra → $\mathcal{L}^{-1}(s \to t)$

Properties - linearity

$\mathcal{L}\{f\} = F \quad \mathcal{L}\{g\} = G \quad f = f(t) \quad g = g(t) \quad F = F(s) \quad G = G(s)$

$\mathcal{L}\{f + g\} = F + G \quad \mathcal{L}^{-1}\{F + G\} = f + g$

$\mathcal{L}\{a\,f\} = a\,F \quad \mathcal{L}^{-1}\{a\,F\} = a\,f \quad a \in \mathbb{R}$

Operations

$\mathcal{L}\{f'\} = sF - f(0)$

$\mathcal{L}\{f^{(n)}\} = s^n F - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$

$\mathcal{L}\{e^{at} f\} = F(s-a) \quad \mathcal{L}\{f(t-a)\} = e^{-at} F$

$\mathcal{L}\{\int_0^t f(u)du\} = \frac{1}{s} F$

Periodic function:

$f(t+\omega) = f(t) \Rightarrow \mathcal{L}\{f\} = \frac{1}{1-e^{-s\omega}} \int_0^\omega f(t) e^{-st} du$

Partial fractions

$H = \frac{F(s)}{G(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_p s^p}{b_0 + b_1 s + b_2 s^2 + \cdots + b_r s^r} \quad \begin{cases} a_i, b_i \in \mathbb{R} \\ f \text{ and } g \text{ do not have common roots} \\ g \text{ is of higher degree than } f\ (r > p) \end{cases}$

Factor $F(s)$ in linear factors $(s-a)^m$ and quadratic factors $(s^2 + ps + q)^n$.

$H = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots \frac{A_m}{(s-a)^m} + \frac{B_1 s + C_1}{s^2+ps+q} + \frac{B_2 s + C_2}{(s^2+ps+q)^2} + \cdots + \frac{B_n s + C_n}{(s^2+ps+q)^n}$

Solve for A_i, B_i C_i

Remark:

$as^2 + bs + c = a(s+k)^2 + h^2 \quad k = \frac{b}{2a} \quad h = \sqrt{c - b^2/4a}$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+9}\right\} = \frac{1}{(s-1)^2 + \sqrt{8}^2} = \frac{1}{\sqrt{8}} \frac{\sqrt{8}}{(s-1)^2+\sqrt{8}^2} = \frac{1}{\sqrt{8}} e^x \sin\sqrt{8}x$

Ex: Avoid imaginary factors: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = \frac{1}{(s+1)^2+1} = e^{-x}\sin x$

Step Function

$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad \mathcal{L}\{u(t)\} = \frac{1}{s}$

$u(t-a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases} \quad \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}, \ a > 0$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s)$

$\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as} \mathcal{L}\{f(t+a)\}$

Impulse Function

$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \mathcal{L}\{\delta(t)\} = 1$

$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases} \quad \mathcal{L}\{\delta(t-a)\} = e^{-as}$

Convolution

$\mathcal{L}\{F.G\} = f * g = \int_0^t f(u) g(t-u) du$

$f * g = g * f$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\}$ $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\frac{1}{s-a}\right\} = \int_0^t u e^{a(t-u)} du =$

$\frac{1}{a^2}(e^{at} - at - 1)$

Polynomial coefficients

$\mathcal{L}\{t^n . f(t)\} = (-1)^n F^{(n)}(s) \quad n \in \mathbb{N}$

Let $y(0) = y_0$ and $y'(0) = w_0$

$\mathcal{L}\{ty'\} = -\frac{d}{ds}\mathcal{L}\{y'\} = -\frac{d}{ds}[sY - y_0] = -Y - sY' + y_0'$

$\mathcal{L}\{ty''\} = -\frac{d}{ds}\mathcal{L}\{y''\} = -\frac{d}{ds}[s^2 Y - s w_0 - y_0] =$

$-2sY - s^2 Y' + s w_0' + w_0 + y_0'$

$\mathcal{L}\{t^2 y'\} = \frac{d^2}{ds^2}\mathcal{L}\{y'\} = \frac{d^2}{ds^2}[sY - y_0] = 2Y' + sY'' - y_0''$

Ex: $y'' + 2ty' - 4y = 1$ $\quad y(0) = y'(0) = 0$ $\quad s^2 Y - 2Y - 2sY' - 4Y = \frac{1}{s}$ New linear DE! Solve with Integration Factor to find $Y(s)$.

Systems of DE using Laplace Transforms

Ex: $\begin{cases} \frac{dx}{dt} = 2x - 3y & x(0) = 8 \\ \frac{dy}{dt} = y - 2x & y(0) = 3 \end{cases}$ $\begin{cases} sX - x(0) = 2X - 3Y \\ sY - y(0) = Y - 2X \end{cases}$

Solve for X and Y. Invert to find $x(t)$ and $y(t)$.

Integrals (bizuzario)

$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln|x^2+a^2|$

$\int \frac{dx}{ax+b} dx = \frac{1}{a}\ln|ax+b|$

$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}} \operatorname{atan}\frac{2ax+b}{\sqrt{4ac-b^2}}$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{atan}\frac{x}{a}$

$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln|x^2+a^2|$

$\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(x+a)(x-a)} = \frac{1}{2a}\left[\int\frac{dx}{x-a} - \int\frac{dx}{x+a}\right] = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

Integral by parts:

$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$

Example:

$\int_{t_0}^{t_1} e^{-st}\cos(at)\,dt$

$u = e^{-st} \to du = -se^{-st}dt \qquad dv = \cos(at) \to v = \frac{1}{a}\sin(at)$

$\int_{t_0}^{t_1} e^{-st}\cos(at)\,dt = \frac{1}{a}[e^{-st}\sin(at)]_{t_0}^{t_1} + \frac{s}{a}\int_{t_0}^{t_1} e^{-st}\sin(at)\,dt$

$u = e^{-st} \to du = -se^{-st}dt \qquad dv = \sin(at) \to v = \frac{-1}{a}\cos(at)$

$\int_{t_0}^{t_1} e^{-st}\cos(at)\,dt = \frac{1}{a}[e^{-st}\sin(at)]_{t_0}^{t_1} + \frac{s}{a^2}[e^{-st}\sin(at)]_{t_0}^{t_1} - \frac{s^2}{a^2}\int_{t_0}^{t_1} e^{-st}\cos(at)\,dt$

$\int_{t_0}^{t_1} e^{-st}\cos(at)\,dt = \frac{a}{a^2+s^2}[e^{-st}\sin(at)]_{t_0}^{t_1} + \frac{s}{a^2+s^2}[e^{-st}\sin(at)]_{t_0}^{t_1}$

Euler

$e^{\pm ix} = \cos x \pm i\sin x \quad \cos x = \frac{e^{ix}+e^{-ix}}{2} \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$

Baskhara

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Solution of Bessel's equation