Lecture #9, February 14, 2024 ♡

Assume an i.p. sp. $\mathcal{V}$ with i.p. $(\cdot\,,\,\cdot)$, the nat norm is

$\|v\| \overset{\text{def.}}{=} (v,v)^{1/2}$, derive $\longrightarrow$

(obviously $(v,w) \le$) $\quad |(v,w)| \le \|v\|\,\|w\|$ Cauchy-Schwarz

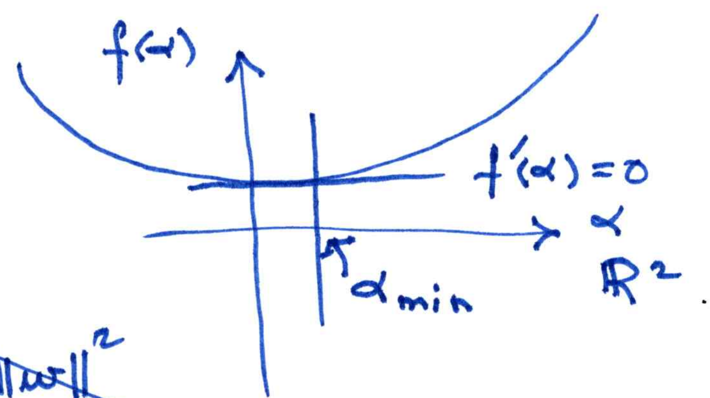
PF. $\quad 0 \le (\underbrace{v+\alpha w}, \underbrace{v+\alpha w}) \quad \forall \alpha \in \mathbb{R} \quad \forall v, w \in \mathcal{V}.$

$$= (v,v) + (v,\alpha w) + (\alpha w, v) + (\alpha w, \alpha w)$$
$$= \|v\|^2 + \alpha(v,w) + \alpha(w,v) + \alpha^2\|w\|^2$$

symm bilin ✓✓✓

$$= \|v\|^2 + 2\alpha(v,w) + \alpha^2\|w\|^2 \qquad (*)$$

$= f(\alpha)$



$$f'(\alpha) = 0 + \cancel{2}(v,w) + \cancel{2}\alpha\|w\|^2 = 0$$

$\alpha = -(v,w)/\|w\|^2$ minimizes $f(\alpha)$.

~~to~~ sub into $(*)$

$$0 \le \|v\|^2 + \frac{-2|(v,w)|^2}{\|w\|^2} + \frac{|(v,w)|^2}{\|w\|^{\cancel{4}\,2}}\cancel{\|w\|^2}$$

$$0 \leq \|v\|^2 - \frac{|(v,w)|^2}{\|w\|^2}$$

$$+ \frac{|(v,w)|^2}{1} \leq \|v\|^2 \|w\|^2$$

$$\to \quad |(v,w)| \leq \|v\| \, \|w\| \qquad \text{C.-S.} \ \checkmark$$

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i.p. space $\mathcal{V}$, with nat norm. $\to$

triangle ineq.

$$\begin{aligned} \|v+w\|^2 &= (v+w, v+w) \\ &= (v,v) + 2(v,w) + (w,w) \\ &\leq \|v\|^2 + \|w\|^2 + 2\|v\|\,\|w\| \qquad \text{C.S.} \\ &= (\|v\| + \|w\|)^2 \end{aligned}$$

$$\Rightarrow \quad \|v+w\| \leq \|v\| + \|w\| .$$

Semi-norms on a l.s. $\mathcal{V}$.

$$|\cdot| : \mathcal{V} \to \mathbb{R}$$

(i) pos. semi-def. $|v| \geq 0$. but there is at least one $v \neq 0$ $\ni |v| = 0$.

(ii) $$|c w| = \underbrace{|\overset{\mathbb{R}}{c}|}_{\text{abs val.}}\ \underbrace{|w|}_{\text{semi}}$$

(iii) $$|v + w| \leq |v| + |w|$$

$$\mathcal{V} = \{ w \mid w \in H^1(0,L),\ w(0) = w(L) = 0 \}$$

$\|v\| \overset{\text{def}}{=} \left(\int_0^L (v_{,x})^2 dx \right)^{1/2}$, surprisingly is a norm on $\mathcal{V}$. (BCs!)

Sobolev sp's. ← useful in analyzing PDEs & var approx's.

ex. 1 $L_2(0,L) = H^0(0,L)$ $\qquad (v,w) = \int_0^L v w \, dx$

$$\|v\| = \left(\int_0^L v^2 dx \right)^{1/2}$$

ex. 2 $H^1(0,L)$ = sq. int. ftns, w. sq. int. 1st deriv's

$$(v,w)_{\underset{H^1(0,L)}{1}} = \int_0^L \left(v w + L^2 \, v_{,x} w_{,x} \right) dx$$

$$\|v\|_1 = (v,v)_1^{1/2}.$$

H^1-seminorm $\quad |v|_1 = \left(\int_0^L L^2 (v_{,x})^2 \, dx \right)^{1/2}$

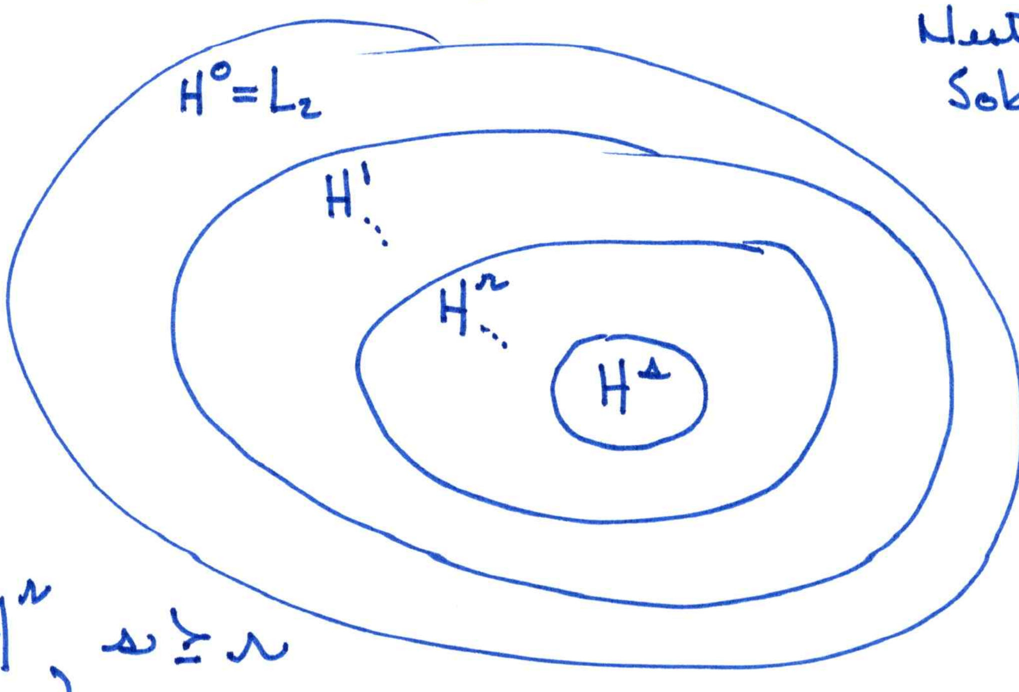
ex 3 $\quad H^s(0,L)$ $\qquad$ s an int., pos $\geq$ 1.

$$(v,w)_s = \int_0^L \left(vw + L^2 v_{,x} w_{,x} + \dots + L^{2s} v_{,\underbrace{x\dots x}_{s \text{ times}}} w_{,\underbrace{x\dots x}_{s \text{ times}}} \right) dx$$

$$\|v\|_s = (v,v)_s^{1/2}$$

$$|v\|_s = \int_0^L L^{2s} v_{,x\dots x} w_{,x\dots x} \, dx$$

Nesting of Sobolev sp's



$$H^s \subset H^r, \; s \geq r$$

imbedding Sobolev

$$v \in H^1(\underbrace{0,L}_{\subset \mathbb{R}}) , \quad v \in \underbrace{C^0}_{\text{cont. fns.}}(0,L)$$

not true in 2D or higher dim. dom. spaces Ω.

$$v \in H^s(0,L) , \quad v \in \cancel{H}^{s-1}(0,L)$$
$$C$$

$$\cancel{\in} = \in C^j(0,L)$$

$$0 \le j \le s-1 .$$

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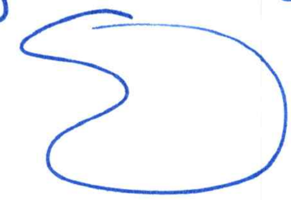
if $v \in H^s(\Omega)$, Ω nice open region in $\mathbb{R}^n$

then $v \in C^k(\Omega)$

not a ~~k~~ hyper-surf.



$k \ge 0$. when $s > \frac{n}{2} + k$

ex. $v \in H^2(\underbrace{\Omega}_{\subset \mathbb{R}^2})$, $v \in C^0(\Omega)$.
