3. Functional anal. bnd. Good enough for theorem proving.

$$\int_{\Gamma^e} (n \cdot \alpha \nabla w^h)^2 \, d\Gamma \leq \int_{\Omega^e} \alpha \, |\nabla w^h|^2 \, d\Omega$$

bound

(secretly λ^e_{max})

Trace ineq. ← relates the size of an integral over ~~the~~ a bndy to an integral (s) over the domain.

FEA context:

$\forall \zeta \in H^1(\Omega^e)$, non-dim. (think quasi-uniformity, $\alpha \leq h^e_{max}/h^e_{min} \leq \beta$)

$$\|\zeta\|^2_{\partial\Omega^e} \leq C_{trace} \left\{ h^e \|\nabla \zeta\|^2_{\Omega^e} + (h^e)^{-1} \|\zeta\|^2_{\Omega^e} \right\} \ldots$$

bndy of Ω^e

What we have to est. $\|\zeta\|^2_{\Gamma^e} \leq \|\zeta\|^2_{\partial\Omega^e}$ ...

First case of interest is $\zeta = \nabla w^h$ ← poly. $\in H^1(\Omega^e)$

$$\|\nabla w^h\|^2_{\partial\Omega^e} \leq C_{trace} \left\{ h^e \|\underbrace{\nabla \nabla w^h}_{\text{Hessian}}\|^2_{\Omega^e} + (h^e)^{-1} \|\nabla w^h\|^2_{\Omega^e} \right\}$$

Obviously, $\|n \cdot \nabla w^h\|^2_{\Gamma^e} \leq$ ✓

detour : the linear Δ.

$$\|n \cdot \nabla w^h\|^2_{\Gamma^e} \leq \underline{2\,(h^e_\perp)^{-1}}\,\|\nabla w^h\|^2_{\Omega^e}$$

cf. w. bnd for the trace ineq.

$$\|n \cdot \nabla w^h\|^2_{\Gamma^e} \leq c_{trace}\left\{h^e\,\|\nabla \cancel{\nabla w^h}\|^2_{\Omega^e} + (h^e)^{-1}\|\nabla w^h\|^2_{\Omega^e}\right\}$$

for lin. Δ, $\nabla(\underbrace{\nabla w^h}_{const}) \equiv 0$

$$\leq \underline{c_{trace}\,(h^e)^{-1}}\,\|\nabla w^h\|^2_{\Omega^e}$$

H.O. el's $\nabla\nabla w^h \neq 0$.

$$\|\nabla(\nabla w^h)\|^2_{\Omega^e} \leq \underbrace{c^2_{inv}\,(h^e)^{-2}}\,\|\nabla w^h\|^2_{\Omega^e}$$

sub in trace ineq.

(determine by another eigen.)

$$\|\nabla w^h\|^2_{\partial\Omega^e} \leq c_{trace}\left\{h^e\,\|\nabla\nabla w^h\|^2_{\Omega^e} + (h^e)^{-1}\,\|\nabla w^h\|^2_{\Omega^e}\right\}$$

sub.

$$\leq c_{trace}\left\{\cancel{h^e}\,c^2_{inv}\,(h^e)^{-\cancel{2}\,-1}\,\|\nabla w^h\|^2_{\Omega^e} + (h^e)^{-1}\,\|\nabla w^h\|^2_{\Omega^e}\right\}$$

$$= \underbrace{c_{trace}\,(c^2_{inv} + 1)}\,(h^e)^{-1}\,\|\nabla w^h\|^2_{\Omega^e}$$

$$= c_{trace/inv}\,(h^e)^{-1}\,\|\nabla w^h\|^2_{\Omega^e}$$

$$\frac{s+1}{2}\frac{1}{\varepsilon^e}\|n\cdot\alpha\nabla w^h\|^2_{\Gamma^e} \le \left[\frac{s+1}{\cancel{2}}\frac{1}{\varepsilon^e}\times \cancel{\alpha}\, C_{trace/inv}(h^e)^{-1}\right]\|\alpha^{1/2}\nabla w^h\|^2_{\Omega^e}$$

$$\le \frac{1}{\cancel{2}}$$

$$\frac{1}{\varepsilon^e} \le \frac{\alpha}{s+1} C_{trace/inv}(h^e)^{-1}.$$

$$\delta^e \le \frac{\alpha}{s+1}\frac{1}{\varepsilon^e}$$

$$s+1\ \frac{1}{\varepsilon^e} \le \frac{1}{\cancel{\alpha}}\frac{h^e}{C_{trace/inv}\ (s+1)}$$

$$\delta^e \le \frac{\cancel{\alpha}}{\cancel{(s+1)}}\frac{1}{\varepsilon^e}$$

$$\delta^e = \frac{h^e}{(s+1)^2\, C_{trace/inv}}$$

$$\|w^h\|^2_{Nitsche} = \frac{1}{2}\sum_e \left(\|\alpha^{1/2}\nabla w^h\|^2_{\Omega^e} + \left\|\left(\frac{\alpha}{\delta}\right)^{1/2} w^h\right\|^2_{\Gamma^e}\right)$$

stability

$$\le B(w^h, w^h)_{Nitsche}.$$

now everything goes as usual

recall : $e = u^h - u$ "interpolate"

we have est'n for η_{k+1-s}

$\|\eta\|_s \le c_{int} h \ \|u\|$

recall

$$= \underbrace{u^h - \tilde{u}^h} + \tilde{u}^h - u$$

$$= e^h + \eta$$

error of the method in $\mathcal{V}^h$

int error in $\mathcal{V}$

$$\|e^h\|^2_{Nitsche} \le B_{Nitsche}(e^h, e^h) \quad \text{stability}$$

$$= B_{Nitsche}(e^h, e - \eta) \quad \text{from above}$$

$$= \underbrace{\cancel{B_{Nitsche}(e^h, e)}}_{\text{error} \perp} ^{=0.} - B_{Nitsche}(e^h, \eta)$$

$$= |B_{Nitsche}(e^h, \eta)|$$

$$= \left| \sum_e \left(\overset{①}{\int_{\Omega^e} \nabla e^h \alpha \nabla \eta \, d\Omega} - \overset{②}{s \int_{\Gamma^e} (n \cdot \alpha \nabla e^h) \eta \, d\Gamma} \right. \right.$$

$$\left. \left. \overset{③}{- \int_{\Gamma^e} e^h (n \cdot \alpha \nabla \eta) \, d\Gamma} + \overset{④}{\int_{\Gamma^e} e^h \frac{\alpha}{\delta^e} \eta \, d\Gamma} \right) \right|$$

Cauchy - Schwarz.

$$
\begin{aligned}
\leq \Bigg| \sum_e \Bigg(& \|\alpha^{1/2} \nabla e^h\|_{\Omega^e} \, \|\alpha^{1/2} \nabla \eta\|_{\Omega^e} && (1) \\
& + s\, \|n \cdot \alpha \nabla e^h\|_{\Gamma^e} \, \|\eta\|_{\Gamma^e} && (2) \\
& + \|e^h\|_{\Gamma^e} \, \|n \cdot \alpha \nabla \eta\|_{\Gamma^e} && (3) \\
& + \underbrace{\Big\|\tfrac{\alpha}{\gamma^e} e^h\Big\|_{\Gamma^e}} \, \underbrace{\Big\|\Big(\tfrac{\alpha}{\gamma^e}\Big)^{1/2} \eta\Big\|_{\Gamma^e}} \Bigg) \Bigg| && (4)
\end{aligned}
$$

bndy terms
→ domain by trace ineq.
→ then int est.

use Peter-Paul.

$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$

hide behind

$$\frac{1}{2} \sum_e \|\alpha^{1/2} \nabla e^h\|^2_{\Omega^e}$$