CSE 397 / EM 397 - Stabilized and Variational Multiscale Methods in CFD

Homework #4

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The compressible Navier-Stokes equations can be written as

$$\boldsymbol{U}_{,t} + \boldsymbol{F}_{i,i} = \boldsymbol{F}^{\text{visc}}_{i,i} + \boldsymbol{F}^{\text{heat}}_{i,i} + \mathcal{F}$$

where

$$\boldsymbol{U} = \rho \begin{Bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ e \end{Bmatrix}, \quad \boldsymbol{F}_i = u_i \boldsymbol{U} + p \begin{Bmatrix} 0 \\ \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ u_i \end{Bmatrix}$$

$$\boldsymbol{F}^{\text{visc}}_i = \begin{Bmatrix} 0 \\ \tau_{1i} \\ \tau_{2i} \\ \tau_{3i} \\ \tau_{ij} u_j \end{Bmatrix}, \quad \boldsymbol{F}^{\text{heat}}_i = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -q_i \end{Bmatrix}, \quad \mathcal{F} = \begin{Bmatrix} 0 \\ \rho b_1 \\ \rho b_2 \\ \rho b_3 \\ \rho \left(b_i u_i + r \right) \end{Bmatrix}$$

For calorically perfect gases, the constitutive relations are

$$\begin{aligned}
e &= \iota + \frac{1}{2} |\boldsymbol{u}|^2, \\
\iota &= c_v \theta, \\
p &= (\gamma - 1) \rho \iota, \\
\gamma &= c_p / c_v \\
\tau_{ij} &= 2 \mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij}, \\
\epsilon_{ij} &= \frac{1}{2} \left(u_{i,j} + u_{j,i} \right) \\
q_i &= -\kappa \theta_{,i}, \\
s &= \ln \left(\frac{p}{p_0} \left(\frac{\rho}{\rho_0} \right)^{-\gamma} \right).
\end{aligned}$$

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Exercise 4.1

The entropy variables $\boldsymbol{V}$ are defined as

$$\boldsymbol{V}^T := \frac{\partial H}{\partial \boldsymbol{U}},$$

where $H := -\rho s$. Show that the entropy variables can be written as

$$\boldsymbol{V} = \frac{1}{\iota}\begin{Bmatrix} \nu - \frac{1}{2}|\boldsymbol{u}|^2 \\ u_1 \\ u_2 \\ u_3 \\ -1 \end{Bmatrix},$$

where $\nu := \iota + p/\rho - \iota s$ is the electrochemical potential. Note that there are many different but equivalent ways to express V_1.

Exercise 4.2

Show that

$$\begin{aligned} \boldsymbol{V} \cdot \boldsymbol{F}_{i,i} &= (Hu_i)_{,i}, \\ \boldsymbol{V} \cdot \boldsymbol{F}_i^{\text{visc}} &= 0, \\ \boldsymbol{V} \cdot \boldsymbol{F}_i^{\text{heat}} &= \frac{q_i}{c_v \theta}, \\ \boldsymbol{V} \cdot \mathcal{F}_i &= -\frac{\rho r}{c_v \theta}. \end{aligned}$$

Exercise 4.3

Show that

$$\begin{aligned} \mathbb{D} :=& \boldsymbol{V}_{,i} \cdot \left(\boldsymbol{F}_i^{\text{visc}} + \boldsymbol{F}_i^{\text{heat}}\right) \\ =& a\epsilon_{ij}^{\text{dev}}(\boldsymbol{u})\epsilon_{ij}^{\text{dev}}(\boldsymbol{u}) + b(\nabla \cdot \boldsymbol{u})^2 + c\left|\nabla\theta\right|^2, \end{aligned}$$

where

$$\epsilon_{ij}^{\text{dev}}(\boldsymbol{u}) := \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3}u_{k,k}\delta_{ij}.$$

Express a, b, and c in terms of the constitutive coefficients μ, λ, and κ. Determine the sign of $\mathbb{D}$.

Exercise 4.4

If we define $\boldsymbol{R}$ as the residual of the compressible Navier-Stokes equations:

$$\boldsymbol{R} := \boldsymbol{U}_t + \boldsymbol{F}_{i,i} - \boldsymbol{F}_{i,i}^{\text{visc}} - \boldsymbol{F}_{i,i}^{\text{heat}} - \mathcal{F},$$

show that

$$0 = \boldsymbol{V} \cdot \boldsymbol{R} = H_{,t} + (Hu_i)_{,i} - \left(\frac{q_i}{c_v\theta}\right)_{,i} + \frac{\rho r}{c_v \theta} + \mathbb{D}$$

or

$$0 = c_v \boldsymbol{V} \cdot \boldsymbol{R} = (-\rho\eta)_{,t} + (-\rho\eta u_i)_{,i} - \left(\frac{q_i}{\theta}\right)_{,i} + \frac{\rho r}{\theta} + c_v\mathbb{D}.$$

The entropy inequality follows from this result.