$$0 = \sum_{B=0}^{N} K_{AB}\, u_B - F_A \,, \quad A = 1, 2, \ldots, N-1$$

$$u_0 = g_0$$

$$u_N = g_L$$

---

Lecture #6, Feb. 5, 2024.

$$0 = \underbrace{\sum_{B=1}^{N-1} \underline{K_{AB}} \overset{?}{u_B}} + \underline{K_{A0}} \overset{\checkmark}{u_0} + \underline{K_{AN}} \overset{\checkmark}{u_N} - \underline{\overset{\checkmark}{F_A}} \qquad \underbrace{\forall A = 1, \ldots, N-1}$$

algebraic residual of given forces, or fluxes.

$$\underset{N-1 \times N-1}{\underset{\sim}{K}}\ \underset{N-1 \times 1}{\overset{?}{\underset{\sim}{U}}} = \underset{N-1 \times 1}{\underset{\sim}{R}}$$

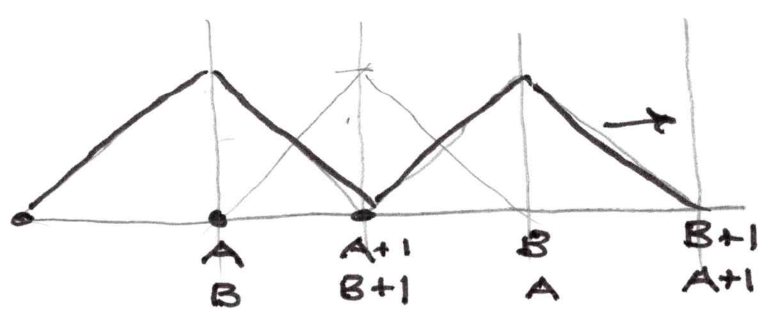
$$\underset{\sim}{K} = [K_{AB}]$$

$$\underset{\sim}{U} = \{U_B\} \checkmark$$

$$\underset{\sim}{R} = \{R_A\}$$

$$u^h_{,x} = \sum_{B=0}^{N} u_B\, N_{B,x}$$

element-by-element piecewise const.




if $B > A+1$, then $\int N_A N_B\, dx = 0$

if $A > B+1$ $\int N_A N_B\, dx = 0$.

$$
\begin{array}{c|ccccc}
A \backslash B & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & K_{11} & K_{12} & 0 & 0 & 0 \\
2 & K_{21} & K_{22} & K_{23} & 0 & 0 \\
3 & 0 & K_{32} & K_{33} & K_{34} & 0 \\
4 & 0 & 0 & \ddots & \ddots & \ddots \\
5 & & & & &
\end{array}
$$

Tridiag.

$$KU = R \qquad K^{-1}R = U$$

solve $O(N)$ $\qquad$ $O(N^3)$

---

Remarks:

1. Let $a(w, v) = \int_0^L w_{,x}\, \cancel{\kappa}\, \kappa\, \overset{v}{\cancel{w}}_{,x}\, dx \quad \forall w, \overset{v}{\cancel{w}} \in \mathcal{V}.$

Diffusion op.

$$= \int_0^L v_{,x}\, \kappa\, w_{,x}\, dx$$

$$= a(v, w) \quad \text{symm.}$$

Advection

$\hookrightarrow$ Let $b(w, v) = \int_0^L (-) w_{,x}\, a\, v\, dx \quad \forall w, v \in \mathcal{V}$

const.

int by parts

$$= \int_0^L + w\, a\, v_{,x}\, dx$$

$$- (w\, a\, v)\Big|_0^L \quad w = 0,\ v = 0$$

$$= \int_0^L + v_{,x}\, a\, w\, dx$$

$$b(w, v) = - b(v, w)$$ Skew-symm bilin. form.

2. $$K_{AB}^{diff} = a(N_A, N_B)$$
$$= a(N_B, N_A) \quad \text{symm.}$$
$$= K_{BA}^{diff}$$

$$K_{AB}^{adv} = b(N_A, N_B)$$
$$= -b(N_B, N_A)$$
$$= - K_{BA}^{adv.}$$

3. "Stability" / Coercivity

$$a(w,w) \overset{def}{=} \int_0^L \kappa (w_{,x})^2 dx \qquad \forall w \in \mathcal{V}.$$

$$= \| \underbrace{\kappa^{1/2}}_{>0} \underbrace{w_{,x}}\|^2_{L^2(0,L)} \underset{?}{\geq} 0$$

$L^2(0,L)$ = $L^2$

only zero if $w_{,x}=0$, $w$ = const., but $w(0)=w(L)=0$, $w \equiv 0$.

$> 0$ pos def. on $\mathcal{V}$.

is coercive

$$b(w,w) = -b(w,w) \qquad \forall w^h \in \mathcal{V}^h \text{ as well}$$

$$= 0. \quad \forall w \in \mathcal{V}^h$$

calc

$$\int_0^L -w_{,x}\, a\, w\, dx$$

$$= \int_0^L + w\, a\, w_{,x}\, dx \quad + \cancel{(\overset{0}{w} a \overset{0}{w})}\Big|_0^L$$

$$= a \int_0^L \frac{1}{2} (w^2)_{,x} dx$$

$$= a\, \frac{1}{2} w^2 \Big|_0^L \equiv 0.$$

$$\underbrace{a(w,w)}_{>0} + \underbrace{b(w,w)}_{0} > 0$$

op. is coercive. But not enough.

---

4. Gal FEM w. linear basis ftn's, A-th eq from $KU = R$

$$h\left\{ a\,\frac{(u_{A+1} - u_{A-1})}{2h} - \kappa\,\frac{(u_{A+1} - 2u_A + u_{A-1})}{h^2} \underbrace{- \frac{1}{h}\int_{x_{A-1}}^{x_{A+1}} N_A f \, dx}_{\text{FDM approx } f_A = f(x_A)} \right\} = 0$$

residual (u)

---

Euler-Lag. form of Gal. method.

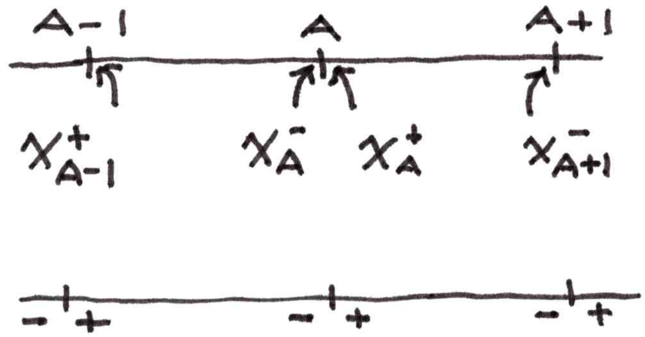
$$0 = \int_0^L \left(-w^h_{,x}\, a\, u^h + w^h_{,x}\, \kappa\, u^h_{,x}\right) dx \; - \int_0^L w^h f \, dx$$

$$\int_0^L = \quad 0 = \sum_{A=1}^{N} \int_{x_{A-1}}^{x_A} \left(-w^h_{,x}\, a\, u^h + w^h_{,x}\, \kappa\, u^h_{,x}\right) dx - \sum_{A=1}^{N} \int_{x_{A-1}}^{x_A} w^h f \, dx$$

int-by-parts element wise

$$0 = \sum_{A=1}^{N} \left\{ \int_{x_{A-1}}^{x_A} w^h \underbrace{\left( a u^h_{,x} - \cancel{w^h} \alpha u^h_{,xx} - f \right)}_{(\text{res}(u^h) \text{ on element interiors})} dx \right.$$

$$\left. - \left( w^h a u^h \right) \Big|_{x^+_{A-1}}^{x^-_A} + \left( w^h \alpha u^h_{,x} \right) \Big|_{x^+_{A-1}}^{x^-_A} \right\}$$




$$\sum_{A=1}^{N} (\dots) \Big|_{x^+_{A-1}}^{x^-_A}$$

$$= (\dots) \Big|_{\cancel{x^+_0}}^{x^-_1} \qquad x^+_0 = 0,\ w^h(0) = 0$$

$$+ (\dots) \Big|_{x^+_1}^{x^-_2}$$

$$+ (\dots) \Big|_{x^+_2}^{x^-_3}$$

$$\vdots$$

$$+ (\dots) \Big|_{x^+_{N-1}}^{\cancel{x^-_N}} \qquad x^-_N = L,\ w^h(L) = 0$$

$$= (\dots) \Big|_{x^+_1}^{x^-_1} + (\dots) \Big|_{x^+_2}^{x^-_2} + \dots + (\dots) \Big|_{x^+_{N-1}}^{x^-_{N-1}}$$

$$= \sum_{A=1}^{N-1} (\dots) \Big|_{x^+_A}^{x^-_A}$$

$$0 = \sum_{A=1}^{N} \int_{x_{A-1}^{+}}^{x_A^{-}} w^h \left(\mathrm{res}(u^h)\right) dx$$ ← el. interiors

$$+ \sum_{A=1}^{N-1} \left( -(w^h a u^h)\Big|_{x_A^{+}}^{x_A^{-}} + (w^h \kappa u^h_{,x})\Big|_{x_A^{+}}^{x_A^{-}} \right)$$

$$- w^h(x_A^-)\, a\, u^h(x_A^-) + w^h(x_A^+)\, a\, u^h(x_A^+)$$

$= 0$, because $w^h$ and $u^h$ are continuous at $x_A$

$$+ w^h(x_A^-)\, \kappa\, u^h_{,x}(x_A^-) - w^h(x_A^+)\, \kappa\, u^h_{,x}(x_A^+)$$

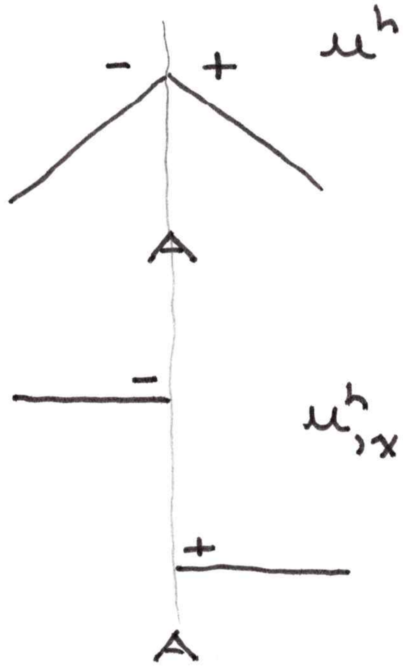
$$= w^h(x_A)\left(\kappa u^h_{,x}(x_A^-) - \kappa u^h_{,x}(x_A^+)\right)$$

$$= -w^h(x_A) [\![ \kappa u^h_{,x}(x_A) ]\!]$$

where the jump op. is defined as

$$[\![ \kappa u^h_{,x}(x_A) ]\!] = \kappa u^h_{,x}(x_A^+) - \kappa u^h_{,x}(x_A^-)$$

(Remark: Important in DG methods)



$$0 = \sum_{A=1}^{N} \int_{x_{A-1}^{+}}^{x_A^{-}} w^h \left( \text{res}\left(u^h\right) \right) dx$$

diff eq res. on el. int's.

$$- \sum_{A=1}^{N-1} w^h(x_A) \left[\!\left[ \kappa u^h_{,x}(x_A) \right]\!\right]$$

diffusive flux res.

$\equiv$ Euler-Lagrange form of Gal. FEM w. $C^0$-cont. modal FE$_s$.

$\equiv$ Residual form of the var. eq.

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