Lecture #27, April 24, 2024.

A generalized entropy ftn., or a mathematical, entropy ftn.

1. $H(U) \in \mathbb{R}^+$

2. $\exists$ scalar val ftns called entropy fluxes $\sigma_i(U)$

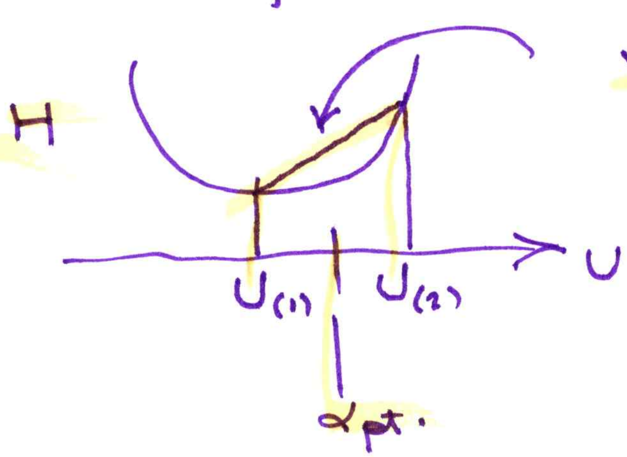
$$\ni \quad H_{,U} A_i = \sigma_{i,U}$$

So $H_{,t} + \sigma_{i,i} = H_{,U} U_{,t} + \sigma_{i,U} U_{,i}$

$= H_{,U} \left(U_{,t} + A_i U_{,i} \right)$

Choose $H = -\rho s$, with $\sigma_i = H u_i$

$s \uparrow \Leftrightarrow H \downarrow$ Note: H is a convex ftn of U.



$\forall \alpha \in [0,1]$

$$\alpha H(U_{(1)}) + (1-\alpha) H(U_{(2)}) \geq H\left(\alpha U_{(1)} + (1-\alpha) U_{(2)}\right)$$

We have a well-defined change of variables:

$$V = \frac{\partial H}{\partial U} \qquad V(U), \quad U(V)$$

by convex of $H(U)$ $\qquad V =$ "entropy variables"

Convexity can also be expressed as

$$\underbrace{H_{,UU}}_{5\times 5} \quad \text{pos. def.}$$

Why is V important? Let's see.

$$0 = R(U) = U_{,t} + A_i(U)U_{,i} - (K_{ij}(U)U_{,j})_{,i} - \mathcal{F}(U)$$

$$0 = \tilde{R}(V) = R(U(V))$$

$$= \underbrace{U_{,V}}_{\tilde{A}_0(V)} V_{,t} + \underbrace{A_i(U(V))\underbrace{U_{,V}}_{\tilde{A}_0(V)}}_{\tilde{A}_i(V)} V_{,i}$$

symm. > 0.

$$\underbrace{-(K_{ij}(U(V))U_{,V} V_{,j})_{,i}}_{-\tilde{K}_{ij}(V)V_{,j}} - \underbrace{\mathcal{F}(U(V))}_{\tilde{\mathcal{F}}(V)}$$

$$0 = \tilde{R}(V) = \overbrace{\underbrace{\tilde{A}_0(V)}_{\text{symm}, > 0}V_{,t}}^{U(V)_{,t}^{*}} + \overbrace{\tilde{A}_i(V)V_{,i}}^{\tilde{F}_i(V)_{,i}^{*}} - \overbrace{\left(\tilde{K}^{\text{diff}}_{ij}(V)V_{,j}\right)_{,i}}^{\tilde{F}_i^{\text{diff}}(V, \nabla V)_{,i}} - \tilde{\mathcal{F}}(V)$$

Properties

1. $\tilde{A}_0$ symm, pos def
2. Each $\tilde{A}_i$ is symm. and $\tilde{A}_i k_i$ has real eigenvalues for all $k_i \in \mathbb{R}$, $i = 1,2,3$.

Euler part is a symm. hyp. sys.

Friedrichs systems.

3. $$\underset{15\times 15}{\tilde{K}} = \underbrace{\begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \end{bmatrix}}_{15 \times 15}$$ is symm., pos-semi-def.

4. $$V \cdot \underbrace{\tilde{R}(V)}_{=0} = 0.$$

$$\boxed{H = -\rho s = -\rho \frac{\eta}{c_v}}$$

can be shown $$= \left(H_{,t} + (H u_i)_{,i} - \left(\frac{q_i}{c_v \theta}\right)_{,i} + \frac{\rho r}{c_v \theta} + \underbrace{(V_{,i})^T \tilde{K}_{ij}(V) V_{,j}}_{\geq 0} \right) = 0$$

$$\Rightarrow H_{,t} + (H u_i)_{,i} - \left(\frac{q_i}{c_v \theta}\right)_{,i} + \frac{\rho r}{c_v \theta} \leq 0.$$

$$\Rightarrow \boxed{\frac{\partial(\rho\eta)}{\partial t} + (\rho\eta u_i)_{,i} + \left(\frac{q_i}{\theta}\right)_{,i} - \frac{\rho r}{\theta}}$$

$$= c_v \big(\underbrace{V_{,i}^T K_{ij}(V) V_{,j}}_{\geq 0} \big)$$

$$\Lambda \geq 0$$

Clausius-Duhem ineq.

entropy prod. "

2nd law of thermo.

Toward our weak form.

$$0 = \int_Q W^{h\,T} \cdot \big(\underbrace{\tilde{R}(V^h)}_{=0} \big)\, dQ \Rightarrow \text{Clausius-Duhem ineq.}$$

V^h

sp-time

int-by-parts

$$\cancel{0} \neq \int_Q \Big(-W_{,t}^{h\,T} U(V^h) - W_{,i}^{h\,T} F_i(V^h)$$

$$+ W_{,i}^{h\,T} \cancel{\tilde{K}_{ij}} F_i^{\text{diff}}(V^h, \nabla V^h) - \tilde{\mathcal{F}}(V^h) \Big) dQ.$$

= bndy terms:

$$\int_{P_\sigma} W^T \mathcal{H} + \int_{P_\sigma^+ \text{ outflow}} W^T \widetilde{A_i n_i}^{+} V\, dP$$

$$+ \int_{\Omega \times \{t_{n+1}^-\}} W^T \tilde{A}_0 V^h d\Omega \quad \begin{bmatrix} 0 \\ h \\ h \cdot u_i + h_r \end{bmatrix} \quad K^{\text{heat source}}$$

$$- \int_{\Omega \times \{t_n^+\}} W^T \tilde{A}_0 V^h d\Omega$$

pos real eigen's zero out the neg real eigen.

discretize U^h

U^h

$\uparrow$

as a possible weighting fn

but you need V^h, so ?

$$\underbrace{V^h}_{\text{entropy}} * \left(\cancel{\mathbb{P}} - V^h + \cancel{U^h(V^h)} \right)$$

$\mathbb{P}^h(V^h)$: cons var.

$$\left(\mathbb{P}^h(V^h) - V^h \right)$$

$$V(U^h) + \underbrace{\mathbb{P}^h\left(V(U^h)\right) - V(U^h)}_{\text{entropy error.}}$$

h

X