Time-dep., "unsteady", AD eq in multi-d.

$\boxed{\text{Lecture 20, April 1, 2024}}$

Recipe: Combine what did for steady AD eq with DG in time.

$\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ dim's.

$x \in \Omega$, $t \in (0, T)$, $\Gamma = \partial\Omega$

$\Gamma = \overline{\Gamma_g \cup \Gamma_h} = \overline{\Gamma^+ \cup \Gamma^-}$

Γ_g where Dirichlet BCs are imposed

Γ_h where flux BCs " "

(Neumann or Robin)

$\Gamma_h = \Gamma \setminus \Gamma_g$

$\Gamma^- =$ inflow boundary

$= \{ x \in \Gamma \mid a(x) \cdot n(x) = a_n(x) < 0 \}$

$\Gamma^+ = \Gamma \setminus \Gamma^- =$ outflow bndy

$\Gamma_g^\pm = \Gamma_g \cap \Gamma^\pm$

$\Gamma_h^\pm = \Gamma_h \cap \Gamma^\pm$

extruding through time: $Q = \Omega \times (0, T)$

$\Omega_n = \Omega \times \{t_n\}$. time slice, hypersurface in Q.

$\Omega_0 = \Omega \times \{0\}$, where our initial cond. will be specified.

$P^{\pm} = \Gamma^{\pm} \times (0,T)$ "lateral bndy"

$P_n^{\pm} = \Gamma^{\pm} \times (t_n, t_{n+1})$

$P_g^{\pm} = \Gamma_g^{\pm} \times (0,T)$

$P_{\cancel{g}\,h}^{\pm} = \Gamma_h^{\pm} \times (0,T)$

$P_{n/g}^{\pm} = \Gamma_g^{\pm} \times (t_n, t_{n+1})$

$P_{n/h}^{\pm} = \Gamma_h^{\pm} \times (t_n, t_{n+1})$

where our BCs are spec.

———— × ————

recall $a_n^+ \stackrel{\text{def}}{=} \max\{0, a_n\}$

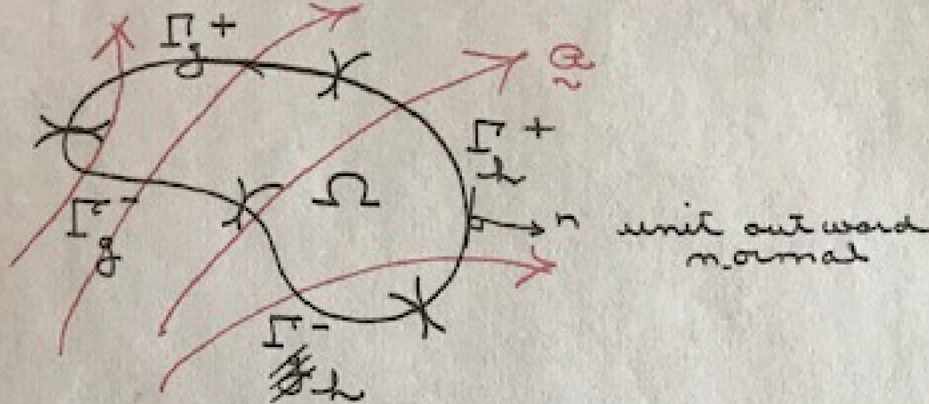
$a_n^- \stackrel{\text{def}}{=} \min\{0, a_n\}$

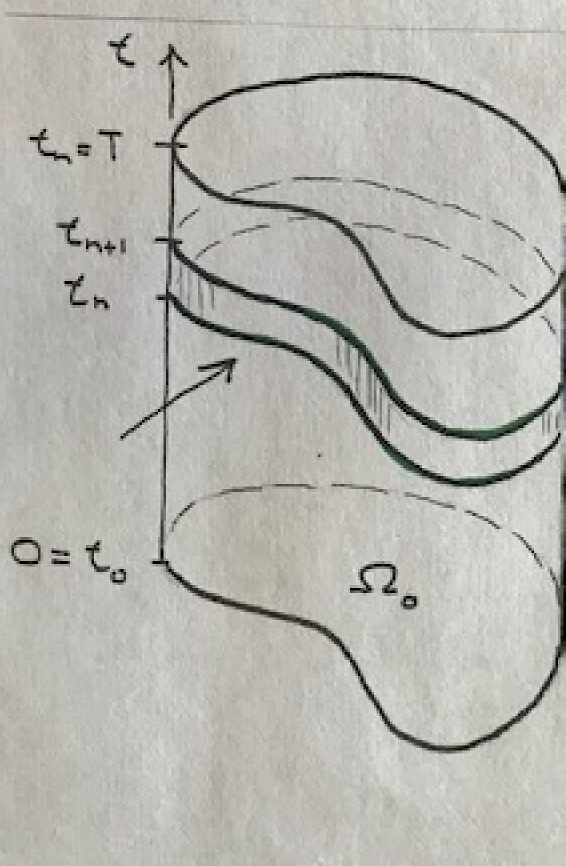
$a_n^+ > 0$ on Γ^+, $a_n^- < 0$ on Γ^-.

$\therefore$ on P^+ $\qquad$ $\therefore$ on P^-

$|a_n| = a_n^+ - a_n^-$ ✓

———— × ————


Γ_g^+
Γ_g^-
Γ_h^+
Γ_h^-
Ω
n
unit outward normal
$\tilde{a}$


t
$t_N = T$
t_{n+1}
t_n
$0 = t_0$
Ω_0

$$(S)_h \Leftrightarrow (W)_h \approx (W^h)_h = (G)_h$$

consistent ?

+ stab.

+ inherent consistency

+ global $\bigoplus_{n=0}^{N-1}$

global stab ...

(S)

PDE + BC's + IC.

strong form of our initial / bndy-val prob.

$$u_{,t} + a \cdot \nabla u - \nabla \cdot (\kappa \nabla u) - f = 0$$

on Q = sp-time

$u(x,t)$, $f(x,t)$, $a(x)$, $\nabla \cdot a = 0$,

$\kappa(x) > 0$, $g(x,t)$, $h(x,t)$

BCs $u = g$ on P_g

$$-a_n^- u + n \cdot (\kappa \nabla u) = h \text{ on } P_h$$

Robin

$$\Rightarrow \; -a_n^- u + n \cdot (\kappa \nabla u) = h^- \text{ on } P_h^-$$

$$0 + n \cdot (\kappa \nabla u) = h^+ \text{ on } P_h^+$$

Neumann

IC: $u(x,0) = u_0(x)$ = given

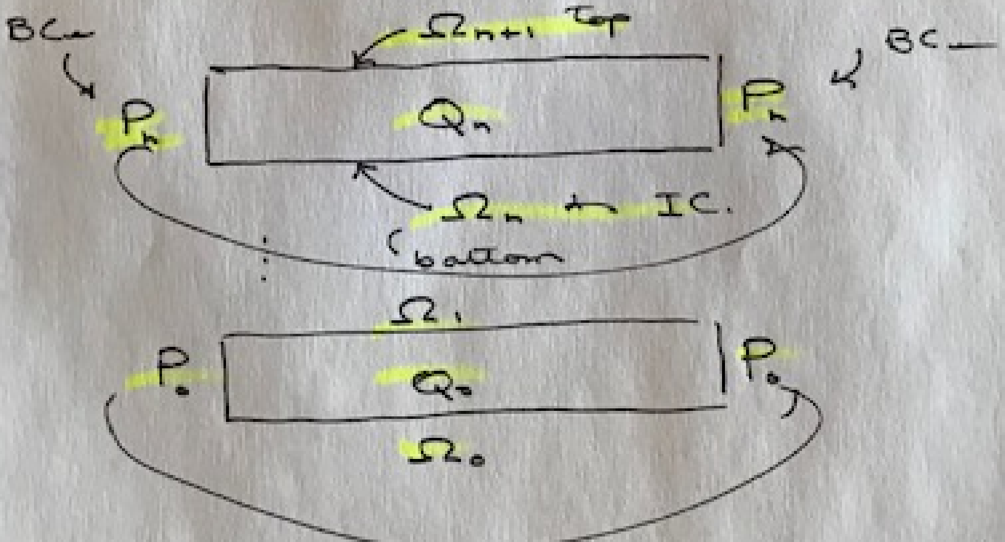
x

Weak form (W) on a typ. sp-time slab

$$\int_{Q_n} \dots dQ = \int_{t_n}^{t_{n+1}} \int_{\Omega} \dots d\Omega\, dt$$

$$\int_{P_n^{\pm}/g \text{ or } h} \dots dP = \int_{t_n}^{t_{n+1}} \int_{\Gamma^{\pm}/g \text{ or } h} \dots d\Gamma\, dt.$$

Simplified picture of a sp-time slab.

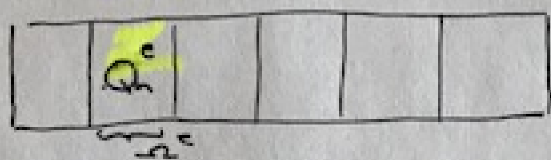


$$\mathcal{V}_n = \{ w \in H^1(Q_n) \mid w(x,t) = 0,\ \forall x \in P_g,\ t \in (t_n, t_{n+1}) \}$$

$$\mathcal{S}_n = \{ u \in H^1(Q_n) \mid u(x,t) = g(x,t),\ \forall x \in P_{n/g} \} = P_g \times (t_n, t_{n+1})$$

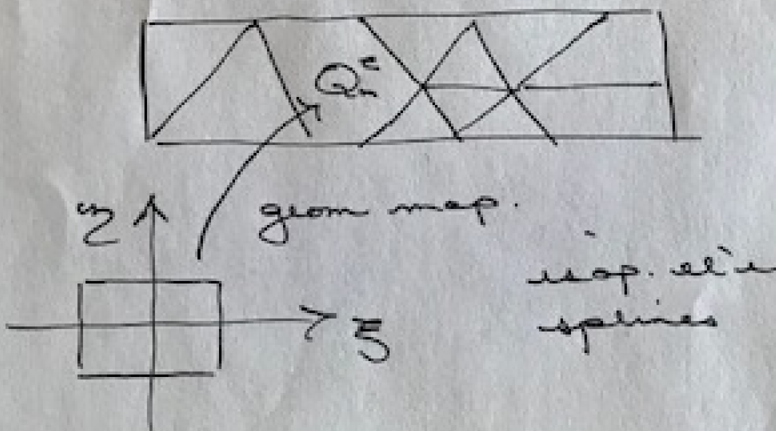
Broken Sob. sp's.

$$\mathcal{V} = \bigoplus_{n=\cancel{1}0}^{N-1} \mathcal{V}_n \quad , \quad \mathcal{S} = \bigoplus_{n=0}^{N-1} \mathcal{S}_n$$



$Q_n^e = \Omega^e \times (t_n, t_{n+1})$ extruded el in time.

however



$$B(w, u)_n = L(w)_n$$

& define.

int by parts form

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$$B(w, u)_n \overset{\text{def.}}{=} \int_{Q_n} \Big(\boxed{-w_{,t}\, u} \; \textcircled{1} \; \textcircled{4} \; - \nabla w \cdot a u \; \textcircled{3} \; + \nabla w \cdot (\varkappa \nabla u) \Big)\, dQ$$

$\textcircled{2}$

top

$$\boxed{+ \int_{\Omega_{n+1}} w(\cdot, t_{n+1}^-)\, u(\cdot, t_{n+1})\, d\Omega \; - \int_{\Omega_n} w(\cdot, t_n^+)\, u(\cdot, t_n^-)\, d\Omega}$$

bottom

$\textcircled{5}$

$$+ \int_{P_n^+} w\, a_n^+\, u\, dP \quad \text{(outflow term)}$$

$$L(w)_n \overset{\text{def}}{=} \int_{Q_n} w f\, dQ + \int_{P_n/h} w\, h\, dP .$$

First thing is to check consistency, int-by-parts, to attain Euler-Lag. form (residual form)