CSE 397 / EM 397 - Stabilized and Variational Multiscale Methods in CFD

Homework #1 - Solution

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Consider the following 1D advection-diffusion problem

$$au_{,x} = \kappa u_{,xx} + f, \quad x \in \Omega = [0, 1] \tag{1a}$$

$$u(0) = g_0 \tag{1b}$$

$$u(1) = g_1 \tag{1c}$$

For our purposes, assume that $a > 0$ and $\kappa > 0$ are constant.

Exercise 1.1

Recall that the A-th equation of the Galerkin form of the 1D advection-diffusion equation takes the form:

$$\sum_{B=0}^{N} \left(\kappa \int_{\Omega} N_{A,x} N_{B,x} \, dx - a \int_{\Omega} N_{A,x} N_B \, dx \right) u_B = \int_{\Omega} N_A \, f \, dx \tag{2}$$

where the domain $\Omega = [0, 1]$. The nodes x_A are equally spaced, yielding the mesh parameter $h = 1/N$. With the piecewise linear basis functions, the only non-zero entries occur at $B \in \{A-1, A, A+1\}$ and we can thus rewrite the above equation in a stencil form associated with the matrix equation that arises:

$$\left(\frac{\kappa}{h} \boldsymbol{S}^A_{\text{Diff}} + \frac{a}{2} \boldsymbol{S}^A_{\text{Adv}} \right) \begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1} \end{bmatrix} = \int_{\Omega} N_A \, f \, dx$$

where

$$\boldsymbol{S}^A_{\text{Diff}} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$$

$$\boldsymbol{S}^A_{\text{Adv}} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

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1. Verify that $\boldsymbol{S}^A_{\text{Diff}}$ and $\boldsymbol{S}^A_{\text{Adv}}$ do indeed take the form specified above for interior elements.

   Solution

   The basis function and its derivative are represented by

$$N_A(x) = \begin{cases} \frac{1}{h}(x - x_{A-1}), & \forall x \in (x_{A-1}, x_A] \\ \frac{1}{h}(x_{A-1} - x), & \forall x \in (x_A, x_{A+1}] \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

   and

$$N_{A,x}(x) = \begin{cases} \frac{1}{h}, & \forall x \in (x_{A-1}, x_A] \\ -\frac{1}{h}, & \forall x \in (x_A, x_{A+1}] \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

   The A-th equation with the linear basis functions in a stencil form reads

$$\begin{aligned} \Bigg(\kappa \left[\int_0^1 N_{A,x} N_{A-1,x} dx \quad \int_0^1 N_{A,x} N_{A,x} dx \quad \int_0^1 N_{A,x} N_{A+1,x} dx \right] \\ - a \left[\int_0^1 N_{A,x} N_{A-1} dx \quad \int_0^1 N_{A,x} N_A dx \quad \int_0^1 N_{A,x} N_{A+1} dx \right] \Bigg) \begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1} \end{bmatrix} = \int_0^1 N_A f \, dx \end{aligned} \tag{5}$$

   where

$$\begin{aligned} \int_0^1 N_{A,x} N_{A-1,x} dx &= \int_{x_{A-1}}^{x_A} \frac{1}{h} \left(-\frac{1}{h} \right) dx = -\frac{1}{h}, \\ \int_0^1 N_{A,x} N_{A,x} dx &= \int_{x_{A-1}}^{x_{A+1}} \frac{1}{h^2} dx = \frac{2}{h}, \\ \int_0^1 N_{A,x} N_{A+1,x} dx &= \int_{x_A}^{x_{A+1}} \frac{1}{h} \left(-\frac{1}{h} \right) dx = -\frac{1}{h}, \\ \int_0^1 N_{A,x} N_{A-1} dx &= \int_{x_{A-1}}^{x_A} \frac{1}{h^2} (x_A - x) dx = \frac{1}{2}, \\ \int_0^1 N_{A,x} N_A dx &= \int_{x_{A-1}}^{x_A} \frac{1}{h^2} (x - x_{A-1}) dx + \int_{x_A}^{x_{A+1}} -\frac{1}{h^2} (x_{A+1} - x) dx = 0, \\ \int_0^1 N_{A,x} N_{A+1} dx &= \int_{x_A}^{x_{A+1}} \frac{1}{h^2} (x - x_A) dx = -\frac{1}{2}. \end{aligned} \tag{6}$$

   Therefore, The stencil form can be simplified as

$$\left(\frac{\kappa}{h} \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} + \frac{a}{2} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1} \end{bmatrix} = \int_0^1 N_A f dx \tag{7}$$

   which yields

$$\boldsymbol{S}^A_{\text{Diff}} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}, \quad \boldsymbol{S}^A_{\text{Adv}} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \tag{8}$$

2. Following the same procedure used to derive $\boldsymbol{S}^A_{\text{Diff}}$ and $\boldsymbol{S}^A_{\text{Adv}}$, convert the SUPG/GLS/MS stabilization contribution

$$\frac{h}{2} \xi(\alpha_h) \sum_{e=1}^{N} \int_{\Omega_e} N_{A,x} \left(a u^h_{,x} - \kappa u^h_{,xx} - f \right) dx$$

into a stencil form

$$\frac{h}{2}\tilde{\xi}(\alpha_h)\left(\frac{a}{h}\boldsymbol{T}^A\begin{bmatrix}u_{A-1}\\u_A\\u_{A+1}\end{bmatrix}-\sum_{e=1}^{N}\int_{\Omega_e}N_{A,x}\,f\,dx\right)$$

where $\alpha_h=\frac{ah}{2\kappa}$ is the element Péclet number and Ω_e denotes the interior element. In particular, verify that $\boldsymbol{T}^A=\boldsymbol{S}^A_{\text{Diff}}$.

Solution

For the 1D advection-diffusion problem, a general stabilization method including SUPG, GLS, and MS can be written

$$\int_0^1\left(-w^h_{,x}au^h+w^h_{,x}\kappa u^h_{,x}\right)dx+\sum_{e=1}^{N}\int_{\Omega_e}\tau\left(\mathbb{L}w^h\right)\left(au^h_{,x}-\kappa u^h_{,xx}-f\right)dx=\int_0^1 w^hfdx \tag{9}$$

where

$$\tau=\frac{h}{2|a|}\tilde{\xi}(\alpha_h) \tag{10}$$

and

$$\mathbb{L}w^h=aw^h_{,x},\qquad\text{for SUPG} \tag{11}$$

$$\mathbb{L}w^h=aw^h_{,x}-\kappa w^h_{,xx},\qquad\text{for GLS} \tag{12}$$

$$\mathbb{L}w^h=aw^h_{,x}+\kappa w^h_{,xx},\qquad\text{for MS} \tag{13}$$

In fact, $\mathbb{L}w^h$ is the same for SUPG, GLS, and MS due to the linear basis functions, for which $N_{A,xx}=0$. With the linear expansion for w^h and u^h with the linear basis functions, the A-th stabilization equation reads

$$\sum_{B=0}^{N}\left(\kappa\int_\Omega N_{A,x}N_{B,x}\,dx-a\int_\Omega N_{A,x}N_B\,dx+\sum_{e=1}^{N}\int_{\Omega_e}\tau aN_{A,x}aN_{B,x}\right)u_B$$
$$-\sum_{e=1}^{N}\int_{\Omega_e}\tau aN_{A,x}fdx=\int_\Omega N_A\,f\,dx \tag{14}$$

The A-th stencil form of the stabilization contribution term is written as

$$\begin{aligned}
&\sum_{B=0}^{N}\left(\sum_{e=1}^{N}\int_{\Omega_e}\tau aN_{A,x}aN_{B,x}\right)u_B-\sum_{e=1}^{N}\int_{\Omega_e}\tau aN_{A,x}fdx\\
&=\tau a\begin{bmatrix}\int_{x_{A-1}}^{x_A}aN_{A,x}N_{A-1,x}dx & \int_{x_{A-1}}^{x_{A+1}}aN_{A,x}N_{A,x}dx & \int_{x_A}^{x_{A+1}}aN_{A,x}N_{A+1,x}dx\end{bmatrix}\begin{bmatrix}u_{A-1}\\u_A\\u_{A+1}\end{bmatrix}\\
&\quad-\tau a\sum_{e=1}^{N}\int_{\Omega_e}N_{A,x}fdx\\
&=\tau a\left(a\begin{bmatrix}\int_{x_{A-1}}^{x_A}\frac{1}{h}\left(-\frac{1}{h}\right)dx & \int_{x_{A-1}}^{x_{A+1}}\frac{2}{h^2}dx & \int_{x_A}^{x_{A+1}}\left(-\frac{1}{h}\right)\frac{1}{h}dx\end{bmatrix}\begin{bmatrix}u_{A-1}\\u_A\\u_{A+1}\end{bmatrix}-\sum_{e=1}^{N}\int_{\Omega_e}N_{A,x}fdx\right)\\
&=\tau a\left(\frac{a}{h}\begin{bmatrix}-1 & 2 & 1\end{bmatrix}\begin{bmatrix}u_{A-1}\\u_A\\u_{A+1}\end{bmatrix}-\sum_{e=1}^{N}\int_{\Omega_e}N_{A,x}fdx\right)
\end{aligned} \tag{15}$$

where

$$\tau a = \frac{ah}{2|a|}\tilde{\xi}(\alpha_h) = \frac{h}{2}\tilde{\xi}(\alpha_h) \tag{16}$$

because $a > 0$. Hence,

$$\boldsymbol{T}^A = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \equiv \boldsymbol{S}^A_{\text{Diff}} \tag{17}$$

The A-th stabilization equation with linear basis functions in a stencil form reads

$$\left(\frac{\kappa}{h}\boldsymbol{S}^A_{\text{Diff}} + \frac{a}{2}\boldsymbol{S}^A_{\text{Adv}} + \frac{a}{2}\tilde{\xi}(\alpha_h)\boldsymbol{S}^A_{\text{Diff}}\right)\begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1}\end{bmatrix} = \int_0^1 N_A f dx + \frac{h}{2}\tilde{\xi}(\alpha_h)\sum_{e=1}^{N}\int_{\Omega_e} N_{A,x} f dx \tag{18}$$

3. With $f = 0$, the resulting stencil for SUPG takes the form

$$\frac{\kappa}{h}\Big((1+\alpha_h\xi(\alpha_h))\ \boldsymbol{S}^A_{\text{Diff}}\ +\ \alpha_h\,\boldsymbol{S}^A_{\text{Adv}}\Big)\begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1}\end{bmatrix} = 0 \tag{19}$$

Recall that the general solution to the homogeneous form of (1a) is $u = c_1 + c_2 \exp\left(\frac{ax}{\kappa}\right)$ and observe that $u_{A-1} = u_A = u_{A+1} = c_1$ will always satisfy (19). Find the function $\xi(\alpha_h)$ to ensure that (19) is satisfied when $u_B = \exp\left(\frac{a}{\kappa}Bh\right) = \exp(2\alpha_h B)$ for $B \in \{A-1,\ A,\ A+1\}$.[1]

Solution

By (18), the A-th stabilization equation in the stencil form can be simplified as

$$\begin{aligned} 0 &= \frac{\kappa}{h}\left(\boldsymbol{S}^A_{\text{Diff}} + \frac{ah}{2\kappa}\boldsymbol{S}^A_{\text{Adv}} + \frac{ah}{2\kappa}\tilde{\xi}(\alpha_h)\boldsymbol{S}^A_{\text{Diff}}\right)\begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1}\end{bmatrix} \\ &= \frac{\kappa}{h}\Big((1+\alpha_h\xi(\alpha_h))\ \boldsymbol{S}^A_{\text{Diff}}\ +\ \alpha_h\,\boldsymbol{S}^A_{\text{Adv}}\Big)\begin{bmatrix} u_{A-1} \\ u_A \\ u_{A+1}\end{bmatrix} \end{aligned} \tag{20}$$

where $\alpha_h = \frac{ah}{2\kappa}$.

If $u = c_1$,

$$\frac{\kappa}{h}\begin{bmatrix} -(1+\alpha_h\tilde{\xi}+\alpha_h) & 2(1+\alpha_h\tilde{\xi}) & -(1+\alpha_h\tilde{\xi}-\alpha_h)\end{bmatrix}\begin{bmatrix} c_1 \\ c_1 \\ c_1\end{bmatrix} = 0 \tag{21}$$

which satisfies (19).

For $u = \exp(\frac{a}{\kappa}Bh) = \exp(2\alpha_h B)$, the stencil becomes

$$\frac{\kappa}{h}\begin{bmatrix} -(1+\alpha_h\tilde{\xi}+\alpha_h) & 2(1+\alpha_h\tilde{\xi}) & -(1+\alpha_h\tilde{\xi}-\alpha_h)\end{bmatrix}\begin{bmatrix} \exp(2\alpha_h(A-1)) \\ \exp(2\alpha_h(A)) \\ \exp(2\alpha_h(A+1))\end{bmatrix} = 0 \tag{22}$$

[1] Hint: Divide through by $\exp(2\alpha_h(A-1))$. Also

$$\coth(x) = \frac{\exp(2x)+1}{\exp(2x)-1}$$

Since $\exp(2\alpha_h(A-1)) > 0$, we have

$$(\alpha_h\tilde{\xi}(\alpha_h)+1)\Big(-1+2\exp(2\alpha_h)-\exp(4\alpha_h)\Big)+\alpha_h(\exp(4\alpha_h-1)=0 \tag{23}$$

Therefore, we have

$$\begin{aligned}\tilde{\xi}(\alpha_h) &= \frac{\alpha_h(\exp(2\alpha_h)+1)-(\exp(2\alpha_h)-1)}{\alpha_h(\exp(2\alpha_h)-1)} \\ &= \frac{\exp(2\alpha_h)+1}{\exp(2\alpha_h)-1}-\frac{1}{\alpha_h} \\ &= \coth(\alpha_h)-\frac{1}{\alpha_h}\end{aligned} \tag{24}$$

Exercise 1.2

The matrix equation derived from applying the SUPG method to (1a) takes the form

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} - g_0\boldsymbol{B}_0 - g_1\boldsymbol{B}_N \tag{25}$$

The $\boldsymbol{K}$ is a $(N-1)\times(N-1)$ matrix such that for $2 \le A \le N-2$

$$[\boldsymbol{K}_{A\,(A-1)} \quad \boldsymbol{K}_{A\,A} \quad \boldsymbol{K}_{A\,(A+1)}] = \left(\frac{\kappa}{h}+\frac{a}{2}\xi(\alpha_h)\right)\boldsymbol{S}^A_{\text{Diff}} + \frac{a}{2}\boldsymbol{S}^A_{\text{Adv}},$$

the non-zero entries of the first and last rows of the matrix:

$$[\boldsymbol{K}_{11} \quad \boldsymbol{K}_{12}] = \left[\int_\Omega N_{1,x}\left(\left(\kappa+\frac{ah}{2}\xi(\alpha_h)\right)N_{1,x}-aN_1\right)dx \qquad \int_\Omega N_{1,x}\left(\left(\kappa+\frac{ah}{2}\xi(\alpha_h)\right)N_{2,x}-aN_2\right)dx\right]$$

$$\boldsymbol{K}_{N-1\,(N-2)} = \int_\Omega N_{(N-1),x}\left((\kappa+\frac{ah}{2}\xi(\alpha_h))N_{(N-2),x}-aN_{N-2}\right)dx$$

$$\boldsymbol{K}_{(N-1)\,(N-1)} = \int_\Omega N_{(N-1),x}\left((\kappa+\frac{ah}{2}\xi(\alpha_h))N_{(N-1),x}-aN_{N-1}\right)dx$$

The vectors on the right hand side are

$$\boldsymbol{B}_0 = \begin{bmatrix}\int_\Omega N_{1,x}\left((\kappa+\frac{ah}{2}\xi(\alpha_h))N_{0,x}-aN_0\right)dx \\ 0 \\ \vdots \\ 0\end{bmatrix}$$

$$\boldsymbol{B}_N = \begin{bmatrix}0 \\ \vdots \\ 0 \\ \int_\Omega N_{(N-1),x}\left((\kappa+\frac{ah}{2}\xi(\alpha_h))N_{N,x}-aN_N\right)dx\end{bmatrix}$$

$$\boldsymbol{F} = \begin{bmatrix}\int_\Omega f\left(N_1+\frac{h}{2}\xi(\alpha_h)N_{1,x}\right)dx \\ \vdots \\ \int_\Omega f\left(N_{N-1}+\frac{h}{2}\xi(\alpha_h)N_{N-1,x}\right)dx\end{bmatrix},$$

and the unknown vector is

$$\boldsymbol{U} = \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

With $\kappa = 1$, $g_0 = 0$, $g_1 = 1$ and $f = 0$, the exact solution to (1a) will be

$$u(x) = \frac{\exp(ax) - 1}{\exp(a) - 1}$$

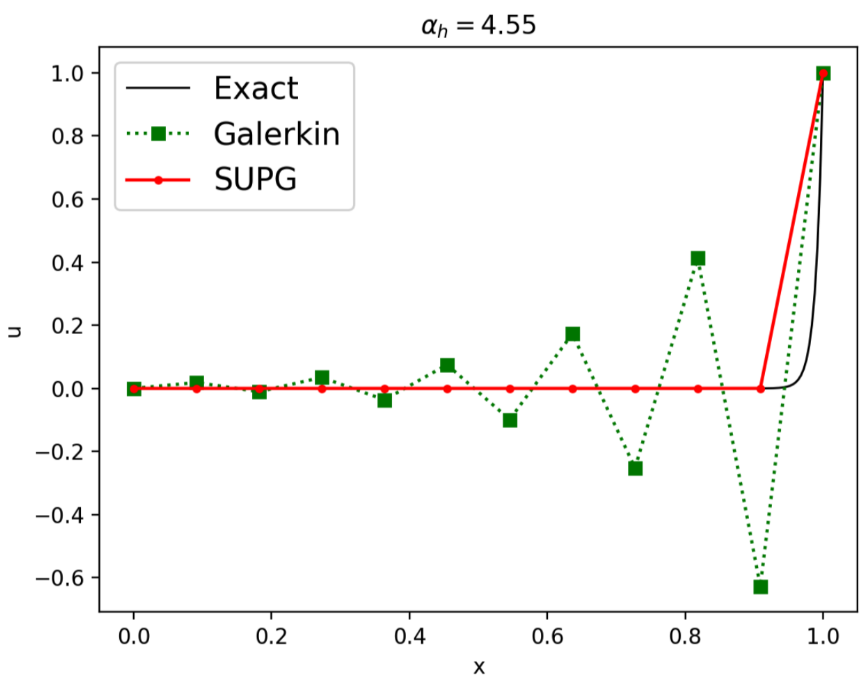
1. With $h = 0.1$, write a function `Usolve(a)` that assembles the matrix $\boldsymbol{K}$ and the vectors $\boldsymbol{F}$, $\boldsymbol{B}_0$, $\boldsymbol{B}_N$ and solve for $\boldsymbol{U}$ in (25) using SUPG and the Galerkin method. The Galerkin method is obtained from (25) by setting $\xi(\alpha_h) = 0$. For SUPG, use $\xi(\alpha_h) = \coth(\alpha_h) - \frac{1}{\alpha_h}$.

   Solution

   The codes written in `python` is uploaded in here.

2. Plot the exact solution vs. the SUPG solution vs. the Galerkin solution for $a = 100^2$.

   Solution

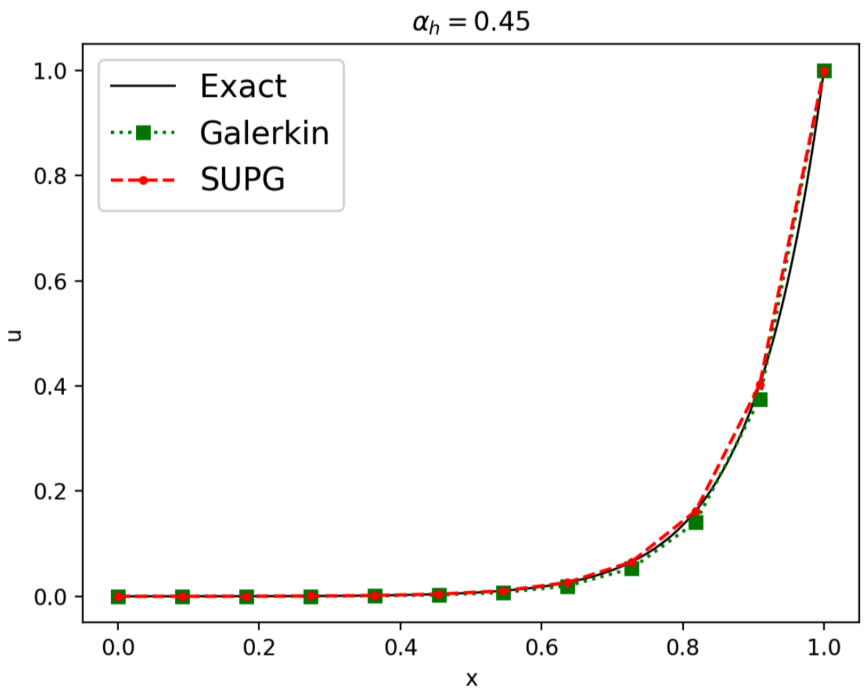


   The Galerkin solution oscillates, and SUPG interpolates the exact solution.

3. Plot the exact solution vs. the SUPG solution vs. the Galerkin solution for $a = 10$.

   Solution

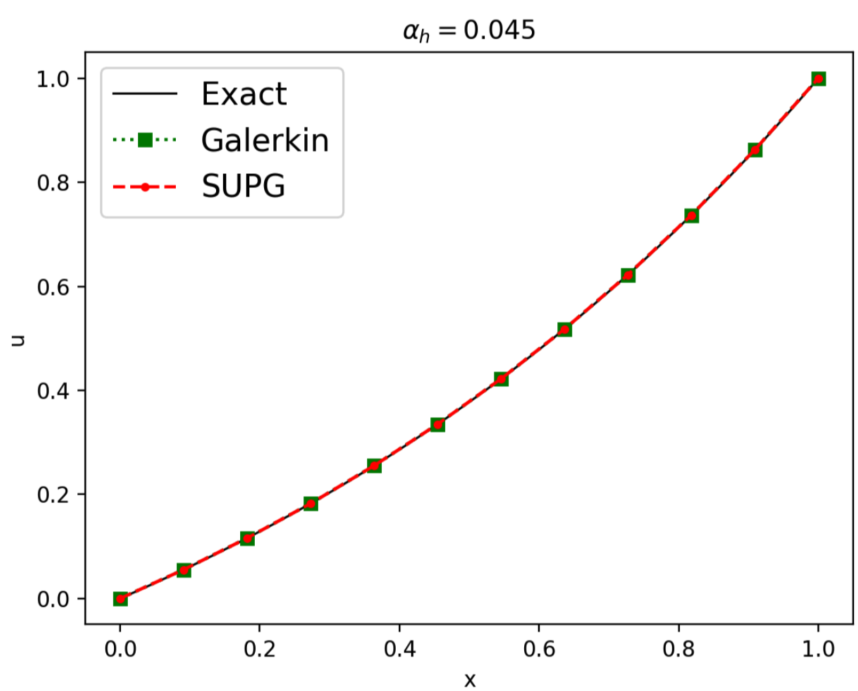
[2]Recall that with the piecewise linear basis, the entries of $\boldsymbol{U}$ are the nodal values of the solution.



Although the oscillation of the Galerkin is disappeared because $\alpha_h < 1$, the Galerkin slightly deviates from the exact solution, whereas the SUPG exactly interpolates the exact solution.

4. Plot the exact solution vs. the SUPG solution vs. the Galerkin solution for $a = 1$.

Solution



This is a diffusion-dominated case, in which case the Galerkin almost perfectly approximates the exact solution.

Exercise 1.3

With $a = 1$, $\kappa = 0$, $g_0 = 0$, $g_1 = 1$[3] and

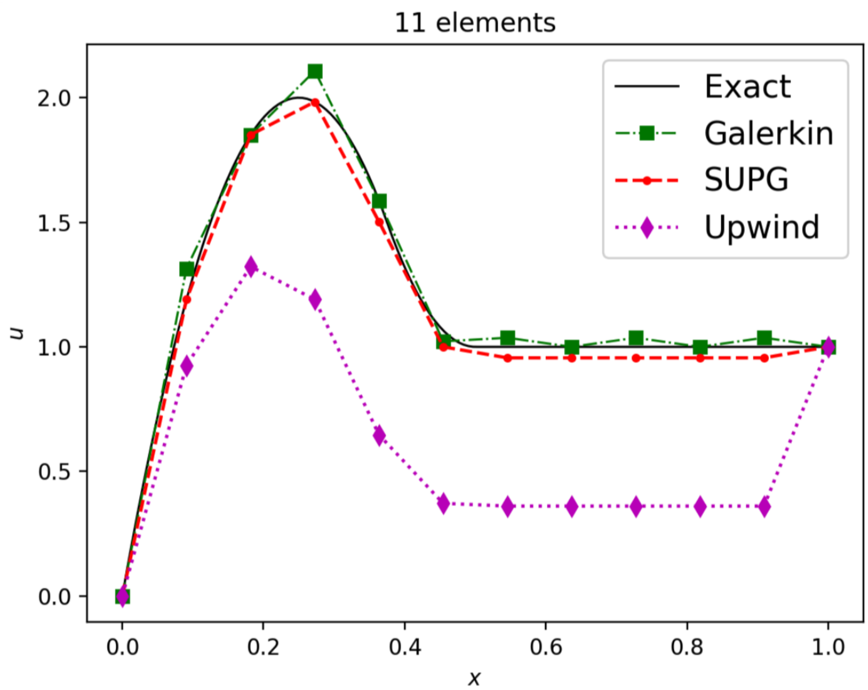
$$f(x) = \begin{cases} 16a\,(1-4x) & \text{for } 0 \leq x \leq \frac{3}{8} \\ 16a\,(-2+4x) & \text{for } \frac{3}{8} \leq x \leq \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases},$$

the exact solution will be

$$u(x) = \begin{cases} 16\,x\,(1-2x) & \text{for } 0 \leq x \leq \frac{3}{8} \\ 9+32x(x-1) & \text{for } \frac{3}{8} \leq x \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}.$$

To obtain the upwind differences solution from your function `Usolve`, set $\xi(\alpha_h) = 1$ in $\boldsymbol{K}$, $\boldsymbol{B}_0$ and $\boldsymbol{B}_N$ and set $\xi(\alpha_h) = 0$ in $\boldsymbol{F}$. Use $h = 0.1$ as before. Plot the exact solution vs. the Galerkin solution vs. SUPG solution vs Upwind differences solution[4].

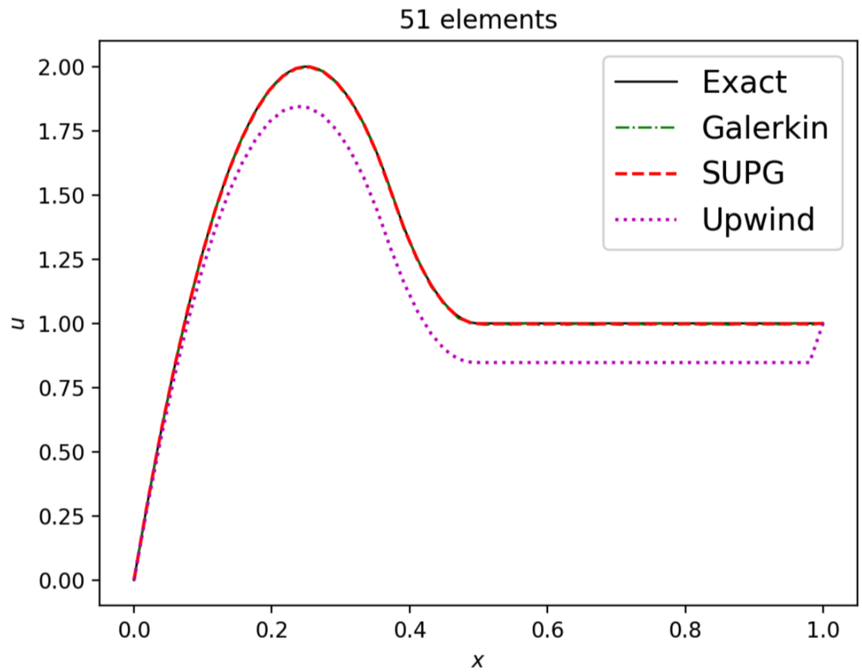
Solution



Since there is no diffusion, the Galerkin method is unstable with a small oscillation. This figure shows poor results for the classical upwind method. The upwind result is actually a very accurate solution to a diffuse problem with $\alpha_h = 1$. The SUPG, on the other hand, is nearly exact.

[3] For a pure advection problem, g_1 does not affect the exact solution because it is consistent with the exact solution.

[4] The matrix you obtain for the Galerkin method is only invertible for certain number of elements (It has to do with the fact that Galerkin is inherently unstable for highly advective problems). Unfortunately, the value of h given for that problem leads to a singular matrix. As a simple fix, I suggest that you use $h = 1/11$ instead.



The upwind method slowly converges to the exact solution. The Galerkin is essentially exact as is SUPG.

The L_2-errors for Galerkin, SUPG, and Upwind differences: