Lecture #21, April 3, 2024.

$$\textcircled{1}+\textcircled{2} \;\cancel{BS}\; = \int_{Q_n} (-w_{,t}\, u)\, dQ \quad \text{int. by. parts}$$

$$\int_{t_n}^{t_{n+1}}\int_{\Omega} \dots \, d\Omega\, dt = \int_{\Omega}\left(\int_{t_n}^{t_{n+1}} \dots \, dt\right) d\Omega$$

$$+ \underset{\Omega_{n+1}}{\overset{\text{top}}{\int}} w(\cdot, t_{n+1}^{-})\, u(\cdot, t_{n+1}^{-})\, d\Omega$$

$$- \underset{\Omega_n}{\overset{\text{bottom}}{\int}} w(\cdot, t_n^{+})\, u(\cdot, t_n^{-})\, d\Omega$$

$$= + \int_{\Omega} \int_{t_n}^{t_{n+1}} w \underbrace{\boxed{u_{,t}}}_{\text{go to PDE res.}}\, dt\, d\Omega - \int_{\Omega} w u \Big|_{t_n^{+}}^{t_{n+1}^{-}} d\Omega$$

$$= + \dots - \left(\int_{\Omega} w(\cdot, t_{n+1}^{-})\, u(\cdot, t_{n+1}^{-})\, d\Omega - w(\cdot, t_n^{+})\, u(\cdot, t_n^{+})\, d\Omega\right)$$

$$= + \dots + \int_{\Omega} w(\cdot, t_n^{+}) \underbrace{\left(u(\cdot, t_n^{+}) - u(\cdot, t_n^{-})\right)}_{\overset{\text{def}}{=} [\![u(\cdot, t_n)]\!] \text{ res of I.C.}} d\Omega$$

$$\textcircled{3} = \int_{Q_n} \nabla w \cdot (\varkappa \nabla u)\, dQ = \int_{t_n}^{t_{n+1}} \left(\int_{\Omega} \nabla w \cdot (\varkappa \nabla u)\, d\Omega \right) dt$$

$$= \int_{t_n}^{t_{n+1}} \int_{\Omega} - w\, \nabla \cdot (\varkappa \nabla u)\, d\Omega\, dt \; \checkmark$$

$$+ \int_{t_n}^{t_{n+1}} \int_{\Gamma} + w\, n \cdot (\varkappa \nabla u)\, d\Gamma\, dt$$

$$= \int_{Q_n} -w \nabla\cdot(\alpha \nabla u)\, dQ \rightarrow \text{PDE res.}$$

$$+ \int_{P_n/\ell} \overset{w=0 \text{ on } P_n/g}{w\, n\cdot(\alpha \nabla u)}\, dP. \rightarrow \text{natural BC.}$$

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④ ⑤ :

$$④ = \int_{Q_n} -\nabla w \cdot a u \, dQ$$

$$= \int_{t_n}^{t_{n+1}} \left(\int_{\Omega} -\nabla w \cdot a u \, d\Omega \right) dt$$

PDE res.

$$= + \underbrace{\int_{t_n}^{t_{n+1}} \int_{\Omega}}_{\int_{Q_n}} w \underbrace{(\nabla\cdot(a u))}_{a\cdot\nabla u} \underbrace{d\Omega\, dt}_{dQ}$$

$$- \int_{t_n}^{t_{n+1}} \int_{\Gamma} w \overbrace{(n\cdot a)}^{a_n} u \, d\Gamma\, dt \overset{\ominus}{=} \int_{P_n} w\, a_n u \, dP$$

⑤. $\int_{P_n} w\, a_n^+ u \, dP$: ... ④+⑤

$$= \int_{P_n} w (\cancel{a_n^+} - \underset{a_n^+ + a_n^-}{a_n}) u \, dP$$

$$= \int_{P_n/\ell} \overset{0 \text{ on } P_n/g}{w} (-a_n^- u)\, dP.$$

Bndy res.

$1 + 2 + 3 + 4 + 5 - L(w^h)_n$

$0 = B(w, u)_n - L(w)_n \quad (W)$ — recall $\forall w \in \mathcal{V}_n$ so $\forall w^h \in \mathcal{V}_n^h \subset \mathcal{V}_n$

$$= \int_{Q_n} w \underbrace{\left(u_{,t} + a \cdot \nabla u - \nabla \cdot (\kappa \nabla u) - f \right)}_{\text{PDE residual}} dQ$$

$$+ \int_{\Omega_n} w(\cdot, t_n^+) \underbrace{[\![u(\cdot, t_n)]\!]}_{\text{IC residual.}} d\Omega$$

$$+ \int_{P_n/h} w \underbrace{\left(-a_n^- u + n \cdot (\kappa \nabla u) - h \right)}_{\text{natural BC residual.}} dP$$

$(W) \Longrightarrow (S)$ - variational work.

$(S) \Longrightarrow (W)$ ✓

$(G) = (W^h)$

✓ $0 = B(w^h, u^h)_n - L(w^h)_n \quad \forall w^h \in \mathcal{V}_n^h \leftarrow$

$\mathcal{V}_n^h = \{ w^h \in H^1(Q_n),\ w|_{Q_n^e} \in P^k(Q_n^e),$

$\forall e = 1, 2, \dots, n_{el},\ w^h \in C^0(Q_n) \}$

✓ $0 = B(w^h, u)_n - L(w^h)_n \quad \forall w^h \in \mathcal{V}_n^h \leftarrow$

$$\mathrm{B}(w^h, \underbrace{u^h - u}_{e})_n = 0$$

"Gal. orthog."

Global formulation.

$$w^h \in \mathcal{V}^h = \bigoplus_{n=0}^{N-1} \mathcal{V}_n^h$$

$$u^h \in \mathcal{S}^h = \bigoplus_{n=0}^{N-1} \mathcal{S}_n^h$$

sum of the (weak forms^h) on the slabs.

$$\rightarrow \mathbb{B}(w^h, u^h) = \mathbb{L}(w^h) \quad \forall w^h \in \mathcal{V}^h$$

$$\cap$$

$$\mathbb{B}(w, u) = \mathbb{L}(w) \quad \forall w \in \mathcal{V}.$$

$$\left.\begin{aligned} \mathcal{V} &= \bigoplus \mathcal{V}_n \\ \mathcal{S} &= \bigoplus \mathcal{S}_n \end{aligned}\right) \rightarrow \mathbb{B}(w^h, u) = \mathbb{L}(w^h)$$

$$\mathbb{B}(w^h, u^h - u) = 0$$

Local consis $\Rightarrow$ Global consis. "Gal orth."

Stability : $\mathbb{B}(w^h, w^h) \geq [\![w^h]\!]^2$

in $\mathcal{V}^h$ $\leftarrow$ not in all of $\mathcal{V}$.

$\Uparrow$

in our case $\equiv$ equality

$$\mathbb{B}(w^h, w^h) \equiv [\![w^h]\!]^2$$

$$[\![w^h]\!]^2 = \frac{1}{2}\left(\| w^h(\cdot, T^-) \|^2_{\Omega_T} + \| w^h(\cdot, 0^+) \|^2_{\Omega_0} + \sum_{n=1}^{N-1} \| \underbrace{[\![w^h]\!]}_{(\cdot, t_n)} \|^2_{\Omega_n} \right) + \| \kappa^{1/2} \nabla w^h \|^2_Q + \frac{1}{2} \| |a_n|^{1/2} w^h \|_{P/ch}$$

$= 0$ on P_g

$$\mathbb{B}(w^h, w^h) = \sum_{n=0}^{N-1} \left(\int_{Q_n} \Big(\boxed{-w^h_{,t}\, w^h}^{(1)} \cancel{\times} \; \cancel{- \nabla w^h \cdot (a w^h)}^{(4)} + \nabla w^h \cdot (\kappa \nabla w^h) ^{(3)} \Big) dQ \right.$$

$$+ \cancel{\int_{P_n} w^h a_n^+ w^h \, dP}^{(5)} + \int^{\text{top}}_{\Omega_{n+1}} w^h(\cdot, t^-_{n+1})\, w^h(\cdot, t^-_{n+1})\, d\Omega^{(2)}$$

$$\left. - \int^{\text{bottom}}_{\Omega_n} w^h(\cdot, t_n^+)\, w^h(\cdot, t_n^-)\, d\Omega \right) \checkmark$$

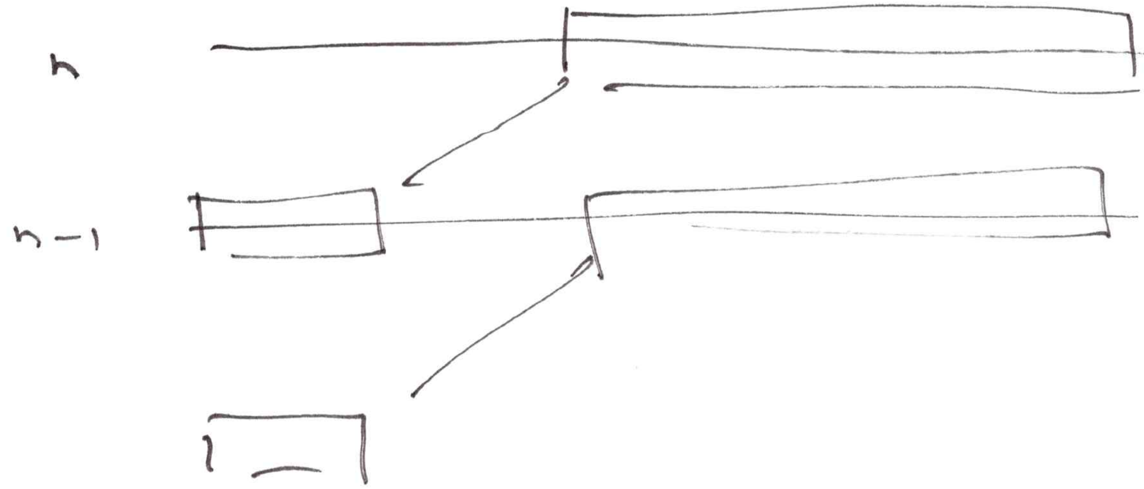
$$(1): \quad \int_{Q_n} -\frac{1}{2}\left((w^h)^2\right)_{,t} dQ = \int_{\Omega} \left(\int_{t_n}^{t_{n+1}} -\frac{1}{2}\left((w^h)^2\right)_{,t} dt \right) d\Omega$$

$$\int_{\Omega} -\frac{1}{2}(w^h)^2 \Big|_{t_n^+}^{t_{n+1}^-} d\Omega = -\frac{1}{2}\left(\int_{\Omega} \left(w^h(\cdot, t_{n+1}^-)\right)^2 d\Omega - \int_{\Omega} \left(w^h(\cdot, t_n^+)\right)^2 d\Omega \right)$$

$$(1)+(2) \;\; \cancel{=} \; + \frac{1}{2}\left(\int_{\Omega} w^h(\cdot, t_{n+1}^-)^2 d\Omega + \int_{\Omega} w^h(\cdot, t_n^+)^2 d\Omega \right.$$

$$\left. - 2 \int_{\Omega} w^h(\cdot, t_n^+)\, w^h(\cdot, t_n^-)\, d\Omega \right)$$

remind you of how to combine these ① + ②



→ exactly $\frac{1}{2}\Big\{ \|w^h(\cdot,T^-)\|^2_{\Omega_T}$

$+ \|w^h(\cdot,0^+)\|^2_{\Omega_0}$

$+ \sum_{n=1}^{N-1} \| [\![w^h(\cdot,t_n)]\!] \|^2_{\Omega_n} \Big\}$

$\equiv [\![w^h]\!]^2$ ← part of this

③ $\int_Q \nabla w^h \cdot \kappa \nabla w^h \, dQ = \|\kappa^{1/2}\nabla w^h\|^2_Q$ ✓

$(\kappa^{1/2}\nabla w^h)\cdot(\kappa^{1/2}\nabla w^h)$

④ $\int_{Q_n} -\nabla w^h \cdot a w^h \, dQ$ $-w^h_{,i}\, a_i\, w^h$

$= \int_{P_n/h} -\frac{1}{2} \underbrace{n\cdot a}_{a_n} (w^h)^2 \, dP$ $= -\frac{1}{2}((w^h)^2)_{,i}\, a_i$

$= -\frac{1}{2}(a_i (w^h)^2)_{,i}$

⊕

⑤ $+ \int_{P_n/a} a_n (w^h)^2 \, dP$ $= -\frac{1}{2}\nabla\cdot(a(w^h)^2)$

$$\int_{P_n/h} (w^h)^2 \Big(a_n^+ - \underbrace{\tfrac{1}{2} a_n}_{\tfrac{1}{2}(a_n^+ + a_n^-)} \Big) \, dP$$

$$= \frac{1}{2} (a_n^+ - a_n^-)$$

$$= \frac{1}{2} |a_n|$$

$$\int_{P_n/h} \frac{1}{2} |a_n| (w^h)^2 \, d\Omega .$$

$$\mathbb{B}(w^h, w^h) = [\![w^h]\!]^2 \checkmark$$
