Spring 2024, Lecture #3 (first two ++ on slides), January 24, 2024

1d AD eq.

$$\overset{\text{const.}}{\underset{\geq 0}{a}}\, u_{,x} = \nabla \cdot \overset{\text{= const. diffusivity} > 0}{\alpha} \nabla u + f \quad \text{on } [0,L], \qquad \text{(AD)}$$

$$f : [0,L] \to \mathbb{R}$$

$$= \alpha \Delta u$$

$$= \alpha\, u_{,xx}$$

(S) Given f, a, α, g_0, g_L, consts.,

find $u : [0,L] \to \mathbb{R}$, s.t.

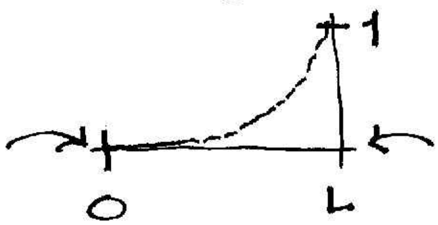
AD is satisfied and

BC $\quad u(0) = g_0, \quad u(L) = g_L$

Solve it exactly. ODE ✓

Introduce a FDM, solve it exactly too

Special case: $f = 0, \; g_0 = 0, \; g_L = 1.$



$$a \underbrace{u_{,x}}_{v} = \alpha \underbrace{u_{,xx}}_{v_{,x}}$$

$$\frac{a}{\kappa} = \frac{w_{,x}}{w} = (\ln w)_{,x}$$

$$\int: \qquad \frac{a}{\kappa} x + C_0 = \ln w$$

$$\exp: \qquad \exp\left(\frac{a}{\kappa} x + c_0\right) = w$$

$$\exp \frac{a}{\kappa} x \cdot \underbrace{\exp c_0}_{C_1} = w = u_{,x}$$

$$\int: \qquad \underbrace{\frac{\kappa}{a} \exp \frac{a}{\kappa} x \cdot C_1}_{C_3} + C_2 = u.$$

$$C_3 \exp \frac{a}{\kappa} x + C_2 = u \quad \leftarrow$$

BCs $\quad u(0) = 0$

$$C_3 \cdot 1 + C_2 = 0 \; ; \quad C_3 = -C_2$$

$$u(L) = 1$$

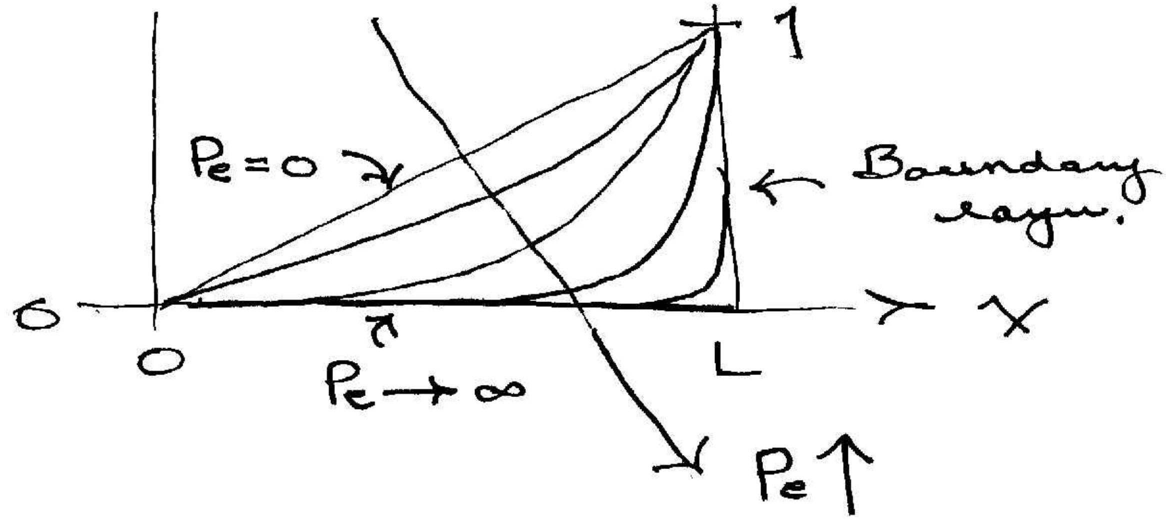
$$C_3 \exp \frac{a}{\kappa} L + C_2 = 1$$

$$C_2 \underbrace{\left(1 - \exp \frac{a}{\kappa} L\right)}_{} = 1$$

$$C_2 = (\qquad)^{-1}$$

$$\checkmark \quad u = \cancel{C_2} \frac{\left(1 - \exp \frac{a}{\kappa} x\right)}{\left(1 - \exp \frac{aL}{\kappa}\right)} \qquad Pe \frac{x}{L}$$

$$Pe = \frac{aL}{\kappa}$$



$\exp(\varepsilon) = 1 + \varepsilon + \dots$

limit case: $P_e \to 0$.

$$u \to \frac{\cancel{1} - \cancel{\exp}\left(\cancel{1} + P_e \frac{x}{L} + \dots\right)}{\cancel{1} - \left(\cancel{1} + P_e + \dots\right)}$$

$$\frac{\cancel{P_e} \frac{x}{L}}{\cancel{P_e}} \checkmark$$

$P \to \infty \qquad u \to \dfrac{\cancel{1} + \exp\left(P_e \frac{x}{L}\right)}{\cancel{1} + \exp P_e}$

$\bullet \leftarrow 1$

$$\to \exp\left(P_e \frac{x}{L} - P_e\right)$$

$$\exp P_e \left(\underbrace{\frac{x}{L} - 1}_{<0.}\right) \to 0.$$

0 ——— L

$\forall\, x < L$

$$x = L \quad \exp P_e \left(\underbrace{1 - 1}_{0}\right) = 1.$$

Simple FD meth's.

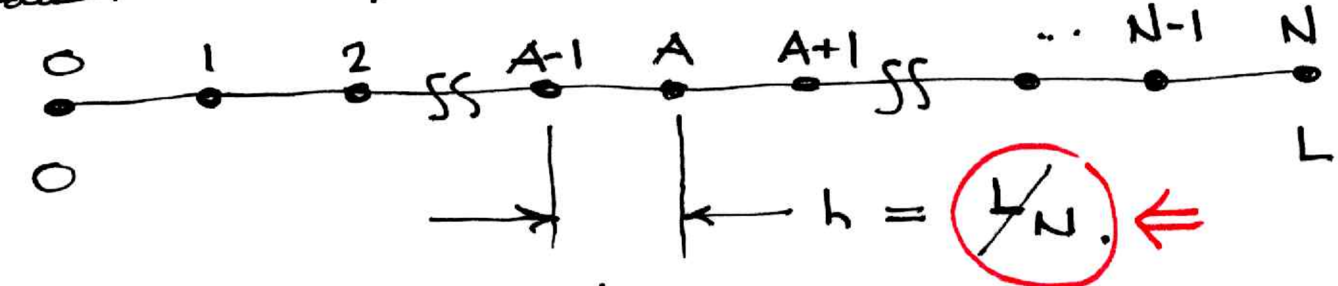
Method #1 : Central diff's

(Secretly for our model prob. $\equiv$ Galerkin w. linear FEs.)

Method #2: upwind diff's.

X

Nodes: $\longrightarrow a.$



uniform.

$u(x_A)$ approx by $u_A \in \mathbb{R}$

$$a\, u_{,x}(x_A) = \kappa\, u_{,xx}(x_A)$$

$$\approx \qquad\qquad \approx$$

$$a\left(\frac{\overset{u_{A+1}}{\cancel{u(x_{A+1})}} - \overset{\simeq u_{A-1}}{\cancel{u(x_{A-1})}}}{2h}\right) = \frac{\kappa}{h^2}\left(\overset{u_{A+1}}{\cancel{u(x_{A+1})}} - 2\overset{u_A}{\cancel{u(x_A)}} + \overset{u_{A-1}}{\cancel{u(x_{A-1})}}\right)$$

C.D. method.

eq sys. solve for all $A = 1, 2, \ldots, N-1$,

$u_0 = 0$, $u_N = 1$.

X

upwind diff's $\qquad a\left(\frac{u_A - u_{A-1}}{h}\right) =$ C.D. diffusion

~~$a\, u(x_A) - u$~~

claim both meth'ds $\alpha u_{A+1} - 2\beta u_A + \gamma u_{A-1} = 0$

$$2\beta = \alpha + \gamma.$$

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cent. diff

α	-2β	γ
$\frac{a}{2h}$	0	$-\frac{a}{2h}$
$-\frac{\alpha}{h^2}$	$2\frac{\alpha}{h^2}$	$-\frac{\alpha}{h^2}$
$\frac{a}{2h} - \frac{\alpha}{h^2}$	$\frac{2\alpha}{h^2}$	$-\frac{a}{2h} - \frac{\alpha}{h^2}$

check $2\beta \overset{?}{=} \alpha + \gamma$

$$-\frac{2\alpha}{h^2} \overset{\checkmark}{=} -2\frac{\alpha}{h^2}$$

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upwind diff

α	-2β	γ
0	$\frac{a}{h}$	$-\frac{a}{h}$
$-\frac{\alpha}{h^2}$	$\frac{a}{h} + \frac{2\alpha}{h^2}$	$-\frac{a}{h} - \frac{\alpha}{h^2}$

check: $2\beta \overset{?}{=} \alpha + \gamma$

$$-\frac{a}{h} - \frac{2\alpha}{h^2} \overset{\checkmark}{=} -\frac{a}{h} - \frac{2\alpha}{h^2}$$

differential eq sub $u(x) \sim \exp ik^? x$.
difference " " $u_A \sim \zeta^A$ (6.

$$\alpha \zeta^{A+1} - 2\beta \zeta^A + \gamma \zeta^{A-1} = 0$$

$$\alpha \zeta^2 - 2\beta \zeta + \gamma = 0.$$

$$\zeta_{\pm} = \frac{2\beta \pm \sqrt{(2\beta)^2 - 4\alpha\gamma}}{2\alpha.}$$

$$2\beta = \alpha + \gamma.$$

$$= \frac{\alpha+\gamma \pm \sqrt{(\alpha+\gamma)^2 - 4\alpha\gamma}}{2\alpha}$$

$\alpha^2 + \gamma^2 + 2\alpha\gamma - 4\alpha\gamma$
$(\alpha-\gamma)^2$

≥ 0 ✓ don't worry.

$$= \frac{\alpha+\gamma \pm (\alpha-\gamma)}{2\alpha}$$

$$\zeta_+ = \frac{2\alpha}{2\alpha} = 1.$$

$$\zeta_- = \frac{2\gamma}{2\alpha} = \frac{\gamma}{\alpha} \checkmark$$

Central diff.

$$\zeta_+ = 1$$

$$\zeta_- = \frac{\gamma}{\alpha} = \frac{\left(-\frac{a}{2h} - \frac{\kappa}{h^2}\right)\cdot\frac{h^2}{\kappa}}{\left(+\frac{a}{2h} - \frac{\kappa}{h^2}\right)\cdot\frac{h^2}{\kappa}}$$ non-dim.

α_h = element Pe no.

$$\zeta_- = \frac{+\frac{ah}{2\kappa} + 1}{-\frac{ah}{2\kappa} + 1} = \left(\frac{1+\alpha_h}{1-\alpha_h}\right)$$

$$u_A = C_+ \zeta_+^A + C_- \zeta_-^A$$

$$= \underbrace{C_+ \cdot 1^A}_{C_+} + C_- \left(\frac{1+\alpha_h}{1-\alpha_h}\right)^A$$

BC's.

assume $\alpha_h < 1$, $\left(\frac{1+\alpha_h}{1-\alpha_h}\right) > 0$ monotone

" $\alpha_h > 1$, " < 0 oscillation

$\alpha_h \gg 1$, $\left(\frac{1+\alpha_h}{1-\alpha_h}\right)^A \propto (-1)^A$

———— X ————