Homology Theory
Notes & Exercises from my Independent Study

(Or: *If I could save Klein in a bottle ♪*)

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Introduction

What's this?

This document is a compendium of notes, exercises, and other miscellany from my independent study in Homology Theory. For this, I am working through the second half of *Topology Through Inquiry* by Michael Starbird and Francis Su (i.e., chapters 11-20), under supervision from Prof. Su himself. Rough topic coverage should be discernable from the table of contents, as I've tried to name each section identically to the corresponding title in the book.

Notation

Most notation I use is fairly standard. Here's a (by no means exhaustive) list of some stuff I do.

- "WTS" stands for "want to show," s.t. for "such that." WLOG, as usual, is without loss of generality.
- End-of-proof things: $\blacksquare$ is QED for exercises and theorems. $\square$ is used in recursive proofs (e.g., proving a Lemma within a theorem proof). If doing a proof with casework, ✓ will be used to denote the end of each case.
- $(\Rightarrow\Leftarrow)$ means contradiction
- $\mathcal{T}(U)$ will denote the topology of a topological space U.
- $\mathcal{P}(A)$ is the powerset of A. I don't like using 2^A.
- $\twoheadrightarrow$ denotes surjection.
- $\hookrightarrow$ denotes injection.
- Thus, $\hookrightarrow\!\!\!\!\!\twoheadrightarrow$ denotes bijection.
- **Important:** I use $f^{\rightarrow}(A)$ for the image of A under f, and $f^{\leftarrow}(B)$ for the inverse image of B under f.
- $\sim$ and $\equiv$ are used for equivalence relations. $\cong$ is used to denote homeomorphism and isomorphism of groups. $\simeq$ is for Homotopy equivalence.
- ϵ is for trivial elements (e.g., the trivial path), while ε is for small positive quantities.
- $\overline{U}$ denotes the closure of U, U° is the interior of U.
- A^c is A complement.
- $\{v_0 \cdots v_k\}$ denotes a simplex on $k+1$ vertices (that is, a k-simplex). $\{v_0 \cdots \widehat{v_i} \cdots v_k\}$ is the same simplex with the i^{th} vertex deleted.
- $[n] = \{i \mid i = 0, 1, \ldots, n\}$.

1. Manifolds, Simplexes Complexes, and Triangulability: Building Blocks

1.1 Manifolds

We define some basic Euclidean sets for use in homeomorphisms.

Definition 1.1.1 (n-cube)**.** The *n-dimensional cube*, denoted $\mathbb{D}^n$, is defined as

$$\begin{aligned}\mathbb{D}^n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1 \text{ for } i = 1, \dots, n\} \\ &= \underbrace{[0,1] \times [0,1] \times \cdots \times [0,1]}_{n \text{ times}} \subset \mathbb{R}^n.\end{aligned}$$

Definition 1.1.2 (n-ball)**.** The *standard n-ball*, denoted B^n, is

$$B^n = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; x_1^2 + \cdots + x_n^2 \le 1\right\}.$$

Definition 1.1.3 (n-sphere)**.** The *standard n-sphere*, denoted $\mathbb{S}^n$, is

$$\mathbb{S}^n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_0^2 + \cdots + x_n^2 = 1\right\}.$$

note that here, our indices start at 0.

Definition 1.1.4 (n-manifold)**.** An *n-dimensional manifold* or *n-manifold* is a separable metric space M such that $\forall p \in M$, $\exists U \in \mathcal{T}(M)$ s.t. $p \in U$ and $U \cong V \subset \mathbb{R}^n$.

15.8. If M is an n-manifold and U is an open subset of M, then U is also an n-manifold.

15.9. If M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold.

15.10. Let M^n be an n-dimensional manifold with boundary. Then ∂M^n is an $(n-1)$-manifold.

1.2 Simplicial Complexes

Definition 1.2.1 (Affine Independence)**.** Let $X = \{v_0, \dots, v_k\} \subset \mathbb{R}^n$. We say X is *affinely independent* if $\{v_1 - v_i, \dots, v_k - v_i\}$ is linearly independent for all v_i.

Example 1.2.1. $X = \left\{(0,1), (-\sqrt{3}/2, -1/2), (\sqrt{3}/2, -1/2)\right\}$ is affinely independent.

Definition 1.2.2 (Convex combination)**.** A *convex combination* of $v_0, \dots, v_k$ is a linear combination of these points whose coefficients are nonnegative and sum to 1.

Definition 1.2.3 (k-simplex)**.** A *k-simplex* is the set of all convex combinations of $k+1$ affinely independent points in $\mathbb{R}^n$. For affinely independent points $v_0, \dots, v_k$ in $\mathbb{R}^n$, $\{v_0 \cdots v_k\}$ denotes the

k-simplex

$$\left\{ \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k \;\middle|\; \forall i = 0, 1, \ldots, k;\ 0 \leq \lambda_i \leq 1 \text{ and } \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

each v_i is called a *vertex* of $\{v_0 \cdots v_k\}$. Any point x in the k-simplex is specified uniquely by the $k+1$ coefficients (λ_i); these coefficients are called the *barycentric coordinates of* x. The *barycentric coordinate of* x *with respect to vertex* v_i is the coefficient λ_i.

Definition 1.2.4 (Face and dimension)**.** Any simplex τ whose vertices are a nonempty subset of the vertices of a k-simplex σ is called a *face* of σ. If the number of vertices is $i+1$, then τ has *dimension* i and is called an i-face of σ and τ has *codimension* $k-i$, the number of dimensions below the top dimension.

Notational Note: if $\sigma = \{v_0 \cdots v_k\}$, the $(k-1)$-dimensional face of σ obtained by deleting the vertex v_j from the list of vertices of σ is denoted by $\{v_0 \cdots \widehat{v_i} \cdots v_k\}$.

15.11. Show that if σ is a simplex and τ is one of its faces, then $\tau \subset \sigma$.

Solution. This is fairly trivial, so we offer just a sketch. Suppose $\mathbf{v} \in \tau$. Then write $\mathbf{v}$ as an element of σ by taking $\lambda_i = 0$ for all those $v_i \notin \tau$. ∎

Definition 1.2.5 (Simplicial complex)**.** A *simplicial complex* K (in $\mathbb{R}^n$) is a collection of simplicies in $\mathbb{R}^n$ satisfying the following conditions.

1. If a simplex σ is in K, then each face of σ is also in K.
2. Any two simplices in K are either disjoint or their intersection is a face of each.

15.13. Exhibit a collection of simplices that satisfies condition (1) but not condition (2) in the definition of a simplicial complex.

Solution. Consider the following diagram, where the interior of each simplex is taken to be in the complex.

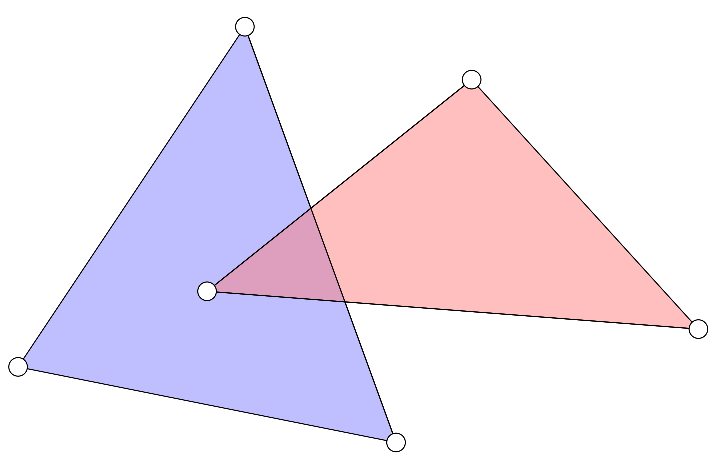


Figure 1.1: An unfortunate collision

Note that to fix this sorry situation, we can't just add two vertices at the points of intersections of the lines above (then the intersection of the resulting simplex with the two shone above would be non-trivial, but still not a face of the larger ones). We'd actually need something much more complicated.

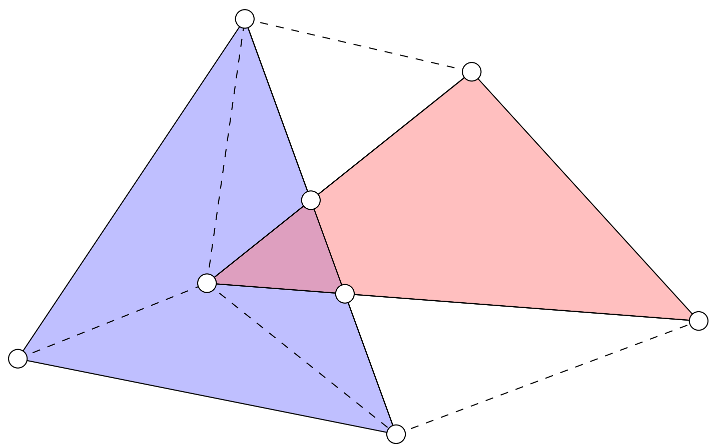


Figure 1.2: Constructing a resolution

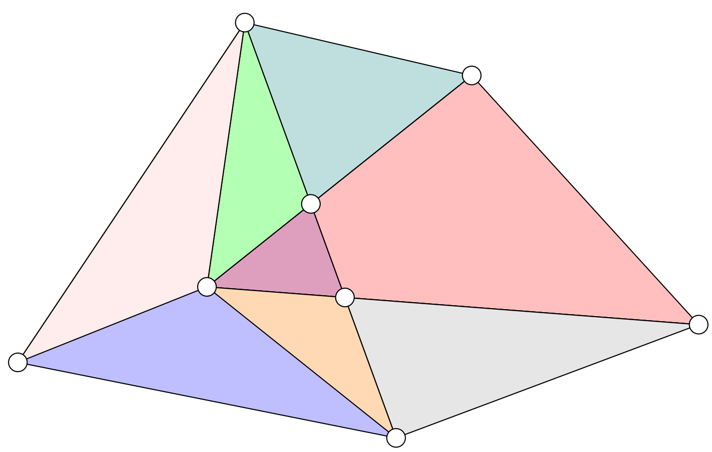


Figure 1.3: The completed resolution

■

Definition 1.2.6 (Underlying space)**.** The *underlying space* $|K|$ of a simplicial complex K is the set

$$|K| = \bigcup_{\sigma \in K} \sigma,$$

the union of all simplices in K, with a topology consisting of sets whose intersection with each simplex $\sigma \in K$ is open in σ. For finite simplicial complexes, this topology is the topology inherited as a subspace of $\mathbb{R}^n$.

15.14. Let K be the following simplicial complex:

(Omitted because it takes a long time to TeX out)

draw K and its underlying space.

Solution.

Figure 1.4: K (left) and its underlying space (right).

■

Definition 1.2.7 (Triangulable)**.** A topological space X is said to be *triangulable* if it is homeomorphic to the underlying space of a simplicial complex K. In that case, we say K is a *triangulation* of X.

15.15. Show that the space shown in Figure 15.2 (not included here) is triangulable by exhibiting a simplicial complex whose underlying space it is homeomorphic to.

Solution.

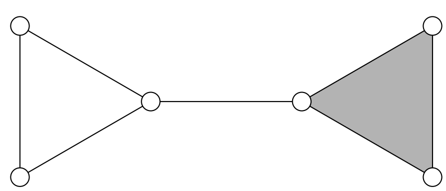


Figure 1.5: Such a simplicial complex. Note, the left triangle is unfilled.

■

15.6. For each $n \in \mathbb{N}$, $\mathbb{S}^n$ is triangulable.

Proof. We proceed by induction.

Base Case: Note that S^0 is trivially triangulable by taking $K = \{\{v_0\}, \{v_2\}\}$.

Inductive Hypothesis Suppose that for $k \in \mathbb{N} \cup \{0\}$, $\mathbb{S}^k$ is triangulable by a simplicial complex K.

Inductive Step: Take $v_{k+1} \in \mathbb{R}^{k+1}$ such that $v_{k+1} \in (\text{span}(K))^{\perp}$. Then

This proof is unfinished. Hey, future Forest — you should return to this later! ■

1.3 Simplicial Maps and PL Homeomorphisms

We now define structure-preserving maps bewteen simplicial concepts.

Definition 1.3.1 (Simplicial map)**.** Let X, Y be topological spaces. A function $f : X \to Y$ is called a *simplicial map* iff there exist simplicial complexes K and L such that $|K| = X$, $|L| = Y$, and f maps each simplex of K linearly onto a (possibly lower-dimensional) simplex in L.

Definition 1.3.2 (Simplicially homeomorphic)**.** A simplicial map f is a simplicial homomorphism iff it's a bijection; in that case, the two complexes are *simplicially homeomorphic*

15.17. A simplicial map from K to L is determined by the images of the vertices of K.

Solution. Apply linearity and show the analog of images of liner combinations being uniquely determined by the action on the basis. ■

15.18. A composition of simplicial maps is a simplicial map.

Solution. Simply plug in arbitrary points and verify the properties hold. ■

Definition 1.3.3 (Subdivision)**.** Let K be a simplicial complex. Then a simplicial complex K' is a *subdivision* of K iff each simplex of K' is a subset of a simplex of K and each simplex of K is the union of finitely many simplices of K'.

Definition 1.3.4 (Piecewise linear)**.** If K and L are complexes, a continuous map $f : |K| \to |L|$ is called *piecewise linear* or *PL* if and only if there are subdivisions K' of K and L' of L such that f is a simplicial map from K' to L'. If there exist subdivisions such that f is a simplicial homomorphism, then f is a *PL homeomorphism* and the spaces are *PL homeomorphic.*

15.21. A composition of PL maps is PL. A PL homeomorphism is an equivalence relation.

Solution. Let K, L, M be complexes, and let $g : |K| \to |L|$, $f : |L| \to |M|$ be continuous PL maps. WTS $h = f \circ g$ is a PL map.

Let K', L', M' be the corresponding subdivisions of K, L, and M, respectively. Then g is a simplicial map from K' to L', and f is a simplicial map from L' to M'. Then $\forall \sigma \in K'$, $g(\sigma) \in L'$, whence $f(g(\sigma)) \in M'$. It follows that $h = g \circ f$ is a simplicial map from K' to M'.

We give a sketch of the proof that PL homeomorphism is an equivalence relation. To verify reflexivity, take the identity map. Symmetry follows by inverting the simplicial homeomorphism. Transitivity follows by the above. Thus, the claim holds. ■

1.4 Simplicial Approximation

15.23. Let K be a complex consisting of the boundary of a triangle (three vertices and three edges) and L be an isomorphic complex. Both $|K|$ and $|L|$ are topologically circles. There is a continuous map that takes the circle $|K|$ and winds it twice around the circle $|L|$; however, show that there is no simplicial map from K to L that winds the circle $|K|$ twice around the circle $|L|$.

Solution. We offer a brief sketch. Basically, this would require each 1-simplex to map to two 1-simplices. Contradiction. ■

Definition 1.4.1 (Barycenter)**.** The *barycenter* of a k-simplex $\{v_0 \cdots v_k\}$ in $\mathbb{R}^n$ is the point $\frac{1}{k+1}(v_0 + \cdots + v_k)$.

Definition 1.4.2 (First barycentric subdivision ($\mathsf{sd}\,\sigma$))**.** Let σ be an n-simplex. The *first barycentric subdivision of* σ, denoted $\mathsf{sd}\,\sigma$, is the complex of all simplices of the form $\{b_0 \cdots b_k\}$, where b_i is the barycenter of a face σ^i of σ from a chain of faces of σ,

$$\sigma^0 \subset \sigma^1 \subset \cdots \subset \sigma^k$$

of increasing (not necessarily consecutive) dimensions. The maximal simplices, that is, the n-simplices of $\mathsf{sd}\,\sigma$ each arise from a maximal sequence of faces, that is, from faces of consecutive dimensions starting with a vertex of σ. Thus an n-simplex of $\mathsf{sd}\,\sigma$ corresponds exactly to a permutation of the vertices of σ.

Definition 1.4.3 ($\mathsf{sd}\,K$)**.** Let K be a simplicial complex. The *first barycentric subdivision* of K, denoted $\mathsf{sd}\,K$, is the complex consisting of all the simplices in the barycentric subdivision of each simplex of K.

Definition 1.4.4 (m-th barycentric subdivision)**.** The *second barycentric subdivision*, denoted $\mathsf{sd}^2\,K$, is the first barycentric subdivision of $\mathsf{sd}\,K$. Proceeding in this way, the *m-th barycentric subdivision* is denoted $\mathsf{sd}^m\,K$.

Some diagrams:

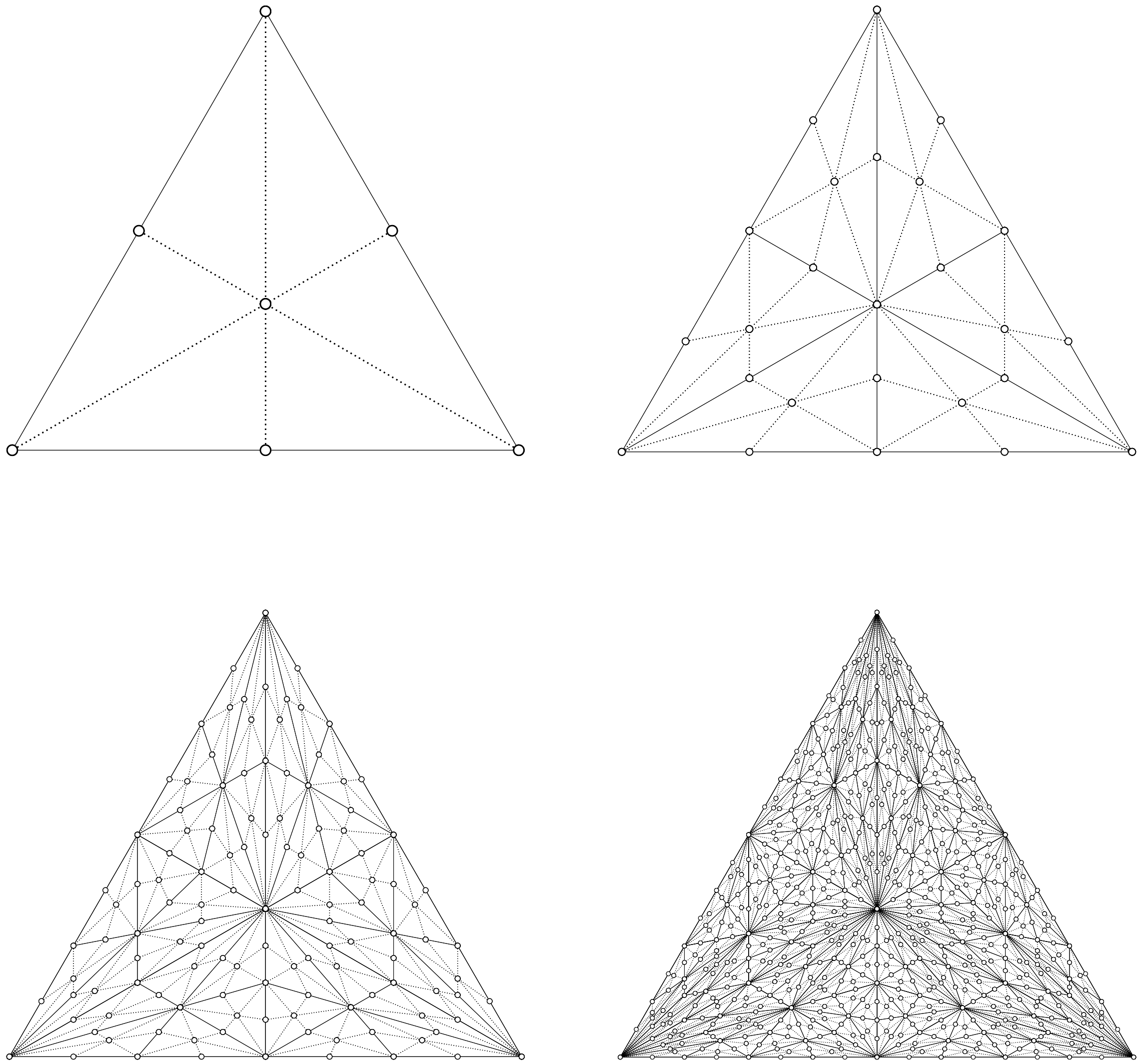


Figure 1.6: The first 4 barycentric subdivisions

15.24. How many n-simplices are there in the first barycentric subdivision of an n-simplex?

Solution. A simple induction shows there are 6^n n-simplices. ■

15.25. Convince yourself that the barycentric subdivision of a complex K is, in fact, a subdivision of K.

Solution. I'm convinced. ■

15.26. Let K be a finite simplicial complex and let a_n be the maximum among the diameters of simplices in $\mathsf{sd}^n K$. Then

$$\lim_{n\to\infty} a_n = 0.$$

Solution. First, we calculate the diameter of an n-simplex.

Lemma 1.4.1. *Let σ_n be an n-simplex. Then the diameter of σ_n*

$$D = \sup_{\mathbf{x},\,\mathbf{y} \in \sigma_n} \|\mathbf{x} - \mathbf{y}\|_2$$

is given by the maximum distance between vertices in the simplex:

$$D = \sup_{\mathbf{v}_i, \mathbf{v}_j} \|\mathbf{v}_i - \mathbf{v}_j\|_2$$

Proof of Lemma: Let $\mathbf{x}, \mathbf{y} \in \sigma_n$ be arbitrary. It will suffice to show that $\mathbf{y}$ is not a vertex in σ_n, then there exists a vertex $\mathbf{v} \in \sigma_n$ such that $\|\mathbf{x} - \mathbf{y}\|_2 < \|\mathbf{x} - \mathbf{v}\|_2$.

Write $\mathbf{y}$ as convex combinations by

$$\mathbf{y} = \sum_{i=0}^{n} \mu_i \mathbf{v}_i.$$

and observe that since $\sum_{i=0}^{n} \mu_i = 1$, we have

$$\mathbf{x} = \sum_{i=0}^{n} \mu_i \mathbf{x} = \mathbf{x}.$$

Then

$$\begin{aligned}
\|\mathbf{x} - \mathbf{y}\|_2 &= \left\| \sum_{i=0}^{n} \mathbf{x} - \mu_i \mathbf{v}_i \right\|_2 \\
&= \left\| \sum_{i=0}^{n} \mu_i (\mathbf{x} - \mathbf{v}_i) \right\|_2 \\
&\le \sum_{i=0}^{n} \mu_i \|\mathbf{x} - \mathbf{v}_i\|_2 \\
&\le \sum_{i=0}^{n} \mu_i \sup_{\mathbf{v}_i} \|\mathbf{x} - \mathbf{v}_i\|_2 \\
&= \sup_{\mathbf{v}_i} \|\mathbf{x} - \mathbf{v}_i\|_2.
\end{aligned}$$

Hence, we see for arbitrary $\mathbf{y}$, $\|\mathbf{x} - \mathbf{y}\|_2 \le \sup_{\mathbf{v}_i} \|\mathbf{x} - \mathbf{v}_i\|_2$. Now, apply the same result to $\mathbf{x}' = \arg\max_{\mathbf{v}_i} \|\mathbf{x} - \mathbf{v}_i\|_2$ and $\mathbf{y}' = \mathbf{x}$ to obtain

$$\|\mathbf{x} - \mathbf{y}\|_2 \le \sup_{\mathbf{v}_i} \|\mathbf{x} - \mathbf{v}_i\|_2 \le \sup_{\mathbf{v}_j} \left(\sup_{\mathbf{v}_i} \|\mathbf{v}_j - \mathbf{v}_i\| \right) = \sup_{\mathbf{v}_i, \mathbf{v}_j} \|\mathbf{v}_j - \mathbf{v}_i\|_2$$

as desired.

By the lemma, a_n is given by the maximal side length of a 2-simplex in $\mathsf{sd}^n K$. Hence

$$0 \le a_n \le \frac{1}{2^n} \frac{2}{\sqrt{3}}$$ This bound is incorrect. How can I fix it?

and so by the squeeze theorem,

$$\lim_{n\to\infty} a_n = 0$$

as desired. ■

Definition 1.4.5 (Minimal face)**.** Let K be a simplicial complex. The *minimal face* of $x \in |K|$ is the simplex of K of smallest dimension that contains x.

Definition 1.4.6 (Star of vertex)**.** The *star of a vertex* v in K, denoted $\mathrm{St}(v)$, is the set of all points whose minimal face contains v.

Remark. The definition of the star of a vertex is basically the interior of the union of all simplices containing v.

15.27. The star of a vertex v in a complex K is an open set of $|K|$, and the collection of all vertex stars covers $|K|$.

Solution. Let $v \in K$ be a vertex. Let $x \in \mathrm{St}(v)$ be arbitrary. WTS $\exists \varepsilon > 0$ s.t. $B_\epsilon(x) \subset \mathrm{St}(v)$. We have the following cases:

(1) Suppose $x = v$. Then taking $\epsilon = \frac{1}{2}\inf_{\mathbf{v}_i} |v - \mathbf{v}_i|_2$ we get the desired result.

(2) Suppose $x \neq v$. Then taking $v = v_0$, write x in the barycentric coordinates

$$x = \lambda_0 v + \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

Since $x \in \mathrm{St}(v)$, $\lambda_0 \neq 0$.

■

15.28. If the simplex $\sigma = \{v_0 \cdots v_k\}$ in K is the minimal face of a point $x \in |K|$, then

$$x \in \bigcap_{i=0}^{n} \mathrm{St}(v_0)$$

Solution.

■

2. Simplicial $\mathbb{Z}_2$-Homology: Physical Algebra

2.1 Intro

This chapter, we'll talk about *homology*, which captures holes in a much more satisfying way than higher homotopy groups do.

> **Remark.** Although not exactly accurate, a good way to start to understand homology for a space X is to view an n-manifold in X that is not the boundary of an $(n+1)$-manifold-with-boundary as capturing some geometry of X while an n-manifold that is the boundary of an $(n+1)$-dimensional manifold-with-boundary is not detecting any hole or structure.

2.2 Chains, Cycles, Boundaries, and the Homology Groups

Definition 2.2.1 (n-chain)**.** An *n-chain* of K is a finite formal sum

$$\sum_{i=1}^{k} \sigma_i$$

of distinct n-simplices in K. Note that the dimensions of the simplices must be the same. So *chain* will mean n-chain whenever the dimension is either unimportant or understood.

Definition 2.2.2 (n-chain group)**.** The *n-chain group of K* (with coefficients in $\mathbb{Z}/2\mathbb{Z}$), denoted $\mathsf{C}_n(K)$, is the collection of n-chains in K under formal addition modulo 2. If there are no n-simplices in K, the n-chain group of K is defined to be trivial (containing the "empty" chain).

16.1. Check that $\mathsf{C}_n(K)$ is an abelian group.

Solution.

(1) $\epsilon = \sum_{i \in \varnothing} \sigma_i$.

(2) Associativity inherited from $\cup$.

(3) Closure inherited from $\cup$ over the domain given.

(4) Existence of inverses — since we're taking formal linear combinations over $\mathbb{Z}/2\mathbb{Z}$, then every element is its own inverse.

Finally, to see that $\mathsf{C}_n(K)$ is abelian, observe that $+$ in $\mathsf{C}_n(K)$ inherits commutativity from $\cup$. ■

Definition 2.2.3 ($\mathbb{Z}/2\mathbb{Z}$ boundary of a simplex)**.** The *$\mathbb{Z}/2\mathbb{Z}$-boundary of an n-simplex* $\sigma = \{v_0 \cdots v_n\}$ is defined by

$$\partial\sigma = \sum_{i=0}^{n} \{v_0 \cdots \widehat{v_i} \cdots v_n\}$$

the formal sum of the $(n-1)$-faces of σ.

For a 0-simplex, the $\mathbb{Z}/2\mathbb{Z}$ boundary is defined to be $0 \in \mathsf{C}_{-1}(K)$.

Definition 2.2.4 ($\mathbb{Z}/2\mathbb{Z}$ boundary of an n-chain)**.** The $\mathbb{Z}/2\mathbb{Z}$ *boundary of an n-chain* is the sum of the boundaries of the simplices. That is, $\partial_n : \mathsf{C}_n(K) \to \mathsf{C}_{n-1}(K)$ is given by

$$\partial\left(\sum_{i=1}^{k} \sigma_i\right) = \sum_{i=1}^{k} \partial(\sigma_i)$$

16.2. Verify that ∂ is a homomorphism, and use the definition to compute the $\mathbb{Z}/2\mathbb{Z}$ boundary of $\sigma_1 + \sigma_2$ in Figure 16.1

Solution. We want to show ∂ is a homomorphism.

(a) Let $\epsilon_n \in \mathsf{C}_n(K)$ be identity. We want to show $\partial(\epsilon_n) = \epsilon_{n-1}$. Taking the empty sum to be identity, we see

$$\begin{aligned}\partial(\epsilon_n) &= \partial\left(\sum_{i\in\varnothing} \sigma_i\right) \\ &= \sum_{i\in\varnothing} \partial(\sigma_i) \\ &= \epsilon_{n-1}\end{aligned}$$

as desired.

(b) That ∂ respects addition is definitional.

We have $\partial(\sigma_1 + \sigma_2) = e_1 + e_2 + e_4 + e_5$. ■

Definition 2.2.5 (n-cycle and n-boundary)**.** An *n-cycle* is an n-chain of K whose boundary is zero. The set of all n-cycles on K is denoted $\mathsf{Z}_n(K)$. An *n-boundary* is an n-chain that is the boundary of an $(n+1)$-chain of K. The set of all n-boundaries is denoted $\mathsf{B}_n(K)$.

16.4. Both $\mathsf{Z}_n(K)$ and $\mathsf{B}_n(K)$ are subgroups of $\mathsf{C}_n(K)$. Moreover,

$$\partial \circ \partial = 0.$$

In other words, $\partial_n \circ \partial_{n+1} = 0$ for each index $n \geq 0$. Hence, $\mathsf{B}_n(K) \subset \mathsf{Z}_n(K)$.

Solution. Let $\sigma_1, \sigma_2 \in \mathsf{Z}_n(K)$. Then by linearity of ∂_n, we have

$$\begin{aligned}\partial_n(\sigma_1 + \sigma_2) &= \partial_n(\sigma_1) + \partial_n(\sigma_2) \\ &= 0\end{aligned}$$

and hence $\mathsf{Z}_n(K) < \mathsf{C}_n(K)$.

Now, let $\sigma_1, \sigma_2 \in \mathsf{B}_n(K)$. Then $\exists \tau_1, \tau_2 \in \mathsf{Z}_{n+1}(K)$ such that $\partial_{n+1}(\tau_1) = \sigma_1, \partial_{n+1}(\tau_2) = \sigma_2$. Since $\mathsf{Z}_{n+1}(K) < \mathsf{C}_{n+1}(K)$, then $\tau_1 + \tau_2 \in \mathsf{Z}_{n+1}(K)$. Now, by linearity of ∂, we have

$$\begin{aligned}\partial_{n+1}(\tau_1 + \tau_2) &= \partial_{n+1}(\tau_1) + \partial_{n+1}(\tau_2) \\ &= \sigma_1 + \sigma_2\end{aligned}$$

hence $\mathsf{B}_n(K)$ is a subset closed under the operation, so we have $\mathsf{B}_n(K) < \mathsf{C}_n(K)$.

It remains to show $\partial_n \circ \partial_{n+1} = 0$. Let $\sigma \in \mathsf{C}_{n+1}(K)$. Then

$$\begin{aligned}\partial_{n+1}(\sigma) &= \partial_{n+1}\left(\sum_{i\in I}\left\{v_0^{(i)}\cdots v_{n+1}^{(i)}\right\}\right)\\ &= \sum_{i\in I}\partial_{n+1}\left(\left\{v_0^{(i)}\cdots v_{n+1}^{(i)}\right\}\right)\\ &= \sum_{i\in I}\sum_{j\in[n+1]}\left\{v_0^{(i)}\cdots \widehat{v_j^{(i)}}\cdots v_{n+1}^{(i)}\right\}\end{aligned}$$

and so

$$\begin{aligned}\partial_n(\partial_{n+1}(\sigma)) &= \sum_{i\in I}\sum_{j\in[n+1]}\partial_n\left(\left\{v_0^{(i)}\cdots \widehat{v_j^{(i)}}\cdots v_{n+1}^{(i)}\right\}\right)\\ &= \sum_{i\in I}\sum_{j\in[n+1]}\sum_{\substack{k\in[n+1]\\ k\neq j}}\left\{v_0^{(i)}\cdots \widehat{v_k^{(i)}}\cdots \widehat{v_j^{(i)}}\cdots v_{n+1}^{(i)}\right\}\end{aligned}$$

hence all the terms cancel, and we're left with $\mathbf{0}$. So $\partial_n \circ \partial_{n+1} = 0$, as desired.

Since every $\sigma \in \mathsf{B}_n(K)$ is of the form $\partial_{n+1}(\tau)$ where $\tau \in \mathsf{C}_{n+1}(K)$, it follows that $\partial_n^{\rightarrow}(\mathsf{B}_n(K)) = (\partial_n \circ \partial_{n+1})(\mathsf{C}_{n+1}(K)) = 0$. Thus $\mathsf{B}_n(K) < \mathsf{Z}_n(K)$. ■

Definition 2.2.6 (Homologous cycles)**.** Two n-cycles α and β in K are *equivalent* or *homologous* iff $\alpha - \beta = \partial(\gamma)$ for some $(n+1)$-chain γ. In other words, α and β are homologous iff they differ by an element of the subgroup $\mathsf{B}_n(K)$, denoted by

$$\alpha \sim_{\mathbb{Z}/2\mathbb{Z}} \beta.$$

The equivalence class of α is denoted by enclosing it in brackets thusly: $[\alpha]$. For $\mathbb{Z}/2\mathbb{Z}$ n-chains, observe that $\alpha - \beta = \alpha + \beta$. So we see that two n-cycles are equivalent if together they bound an $(n+1)$-chain.

Definition 2.2.7 (n^{th} Homology group)**.** The *n^{th}-homology group* (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) of a finite simplicial complex K, denoted $\mathsf{H}_n(K)$, is the additive group whose elements are equivalence classes of cycles under the $\mathbb{Z}/2\mathbb{Z}$-equivalence defined above, with $[\alpha] + [\beta] = [\alpha + \beta]$. I.e.,

$$\mathsf{H}_n(K) = \mathsf{Z}_n(K)/\mathsf{B}_n(K)$$

F1. Consider the simplicial complex given below in Figure **??**. Then for $n = 0, 1, 2$,
(a) describe elements of $\mathsf{C}_n(K)$,
(b) compute $\mathsf{Z}_n(K)$,
(c) compute $\mathsf{B}_n(K)$, and
(d) compute $\mathsf{H}_n(K)$.

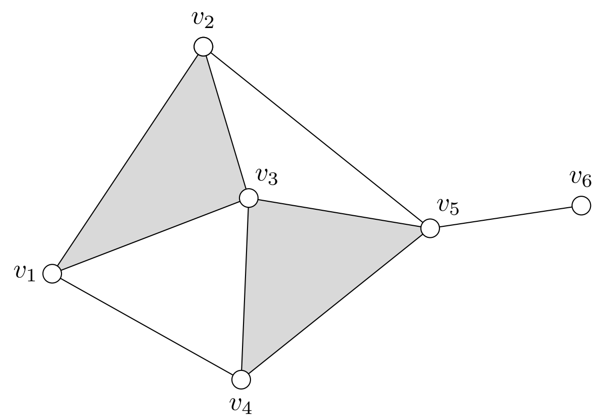


Figure 2.1: Simplicial complex K

Solution. First, we redraw the simplicial complex as follows:

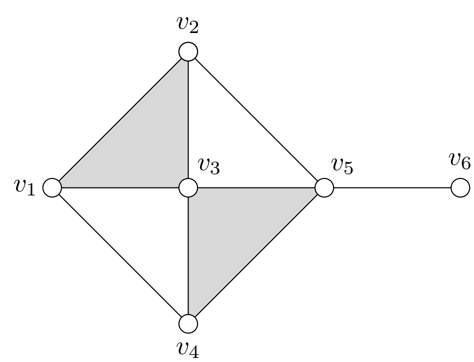


Figure 2.2: Simplicial complex K, straightened out

For the purposes of this problem, take angled brackets indicate span. We have

(i) We calculate the $k = 0$ case.

(a) Elements of $\mathsf{C}_0(K)$ are formal linear combinations over the set $\{v_1, v_2, \ldots, v_6\}$. Then

$$\mathsf{C}_0(K) = \left\langle v_1, v_2, v_3, v_4, v_5, v_6 \right\rangle$$

that is, collections of points in $\mathsf{C}_0(K)$.

(b) Let $\sigma_1, \ldots, \sigma_k \in \mathsf{C}_0(K)$. Then by definition,

$$\begin{aligned} \partial\left(\sum_{i=1}^{k} \sigma_i\right) &= \sum_{i=1}^{k} \partial(\sigma_i) \\ &= \sum_{i=1}^{k} 0 \\ &= 0 \end{aligned}$$

hence $\mathsf{Z}_n(K) = \mathsf{C}_n(K)$.

(c) A $\sigma \in \mathsf{C}_0(K)$ is an n-boundary if $\exists \tau \in \mathsf{C}_1(K)$ with $\partial(\tau) = \sigma$. Note, for any 1-dimensional face $\{v_i v_j\} \in K$,

$$\begin{aligned}\partial(\{v_i v_j\}) &= \{v_i \widehat{v_j}\} + \{\widehat{v_i} v_j\} \\ &= \{v_i\} + \{v_j\} \\ &= \delta_{ij}.\end{aligned}$$

Hence, any edge formed of a pair of two distinct vertices yields a nonempty boundary. We first count all edges:

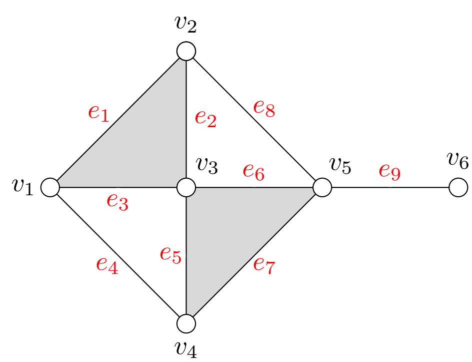
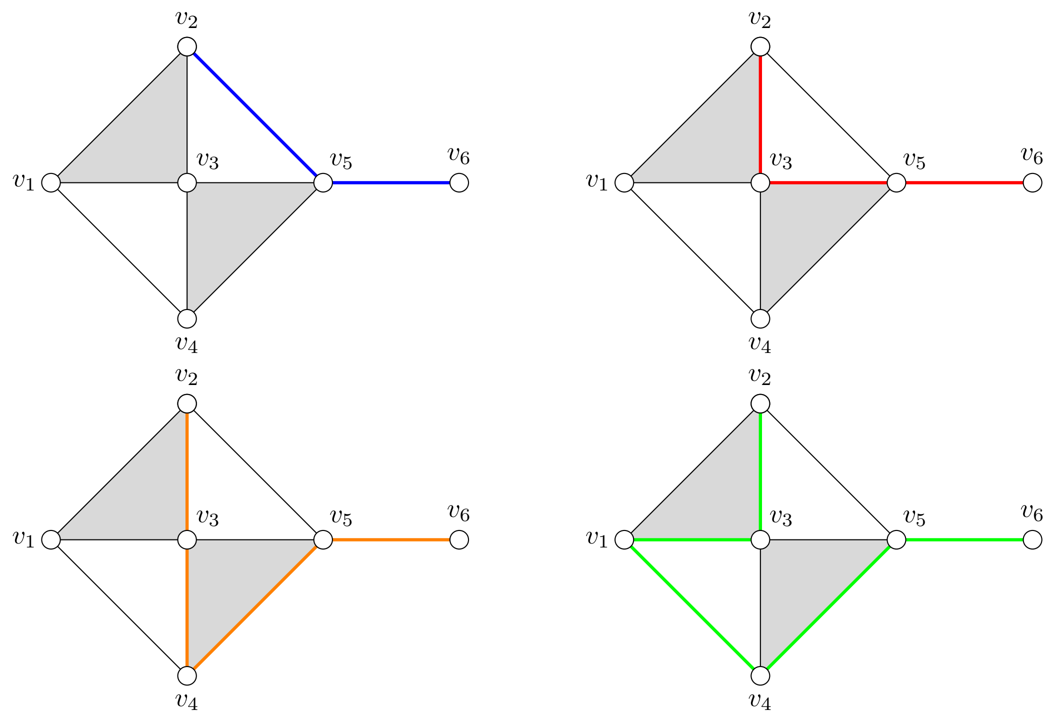


Figure 2.3: Simplicial complex K with simple edges

Since $\mathsf{B}_0(K)$ is a subgroup of $\mathsf{C}_0(K)$, by closure under $+$, we see that any $v_i + v_j$ in K such that there exists a path from v_i to v_j (when K is considered a graph) is an element of $\mathsf{B}_0(K)$. In fact, we can say more:

Claim: Since K is connected as a graph, any even collection of vertices is in $\mathsf{B}_n(K)$.

Proof of Claim: Suppose we have $\sigma = \{v_{i_1}\} + \{v_{i_2}\} + \cdots + \{v_{i_{2k}}\}$, where $k \in \mathbb{N}$. Then for each $j = 1, \ldots, k$, let τ_j be a sum of edges representing a path from v_{i_j} to $v_{i_{j+1}}$. For example, if $v_{i_j} = v_6$ and $v_{i_{j+1}} = v_2$, we could take the following approaches:


v_1
v_2
v_3
v_4
v_5
v_6

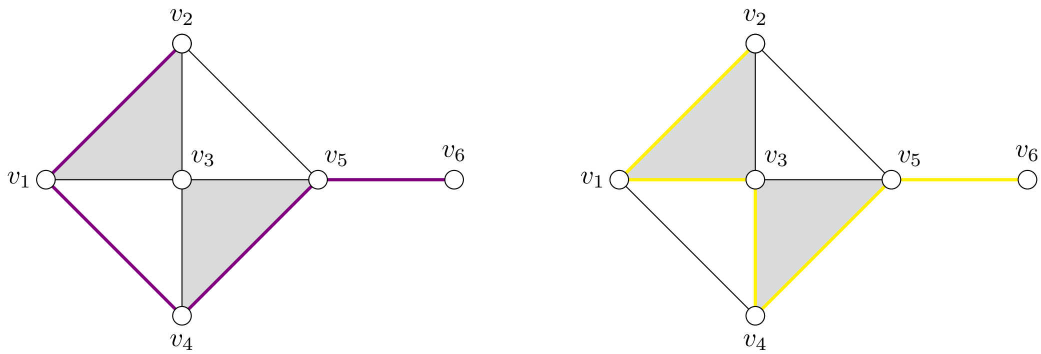


Figure 2.4: Some paths from v_6 to v_2

among others. Taking the sum of the constituent edges in each path yields a sum of 1-simplices with boundary v_6, v_2.[1]

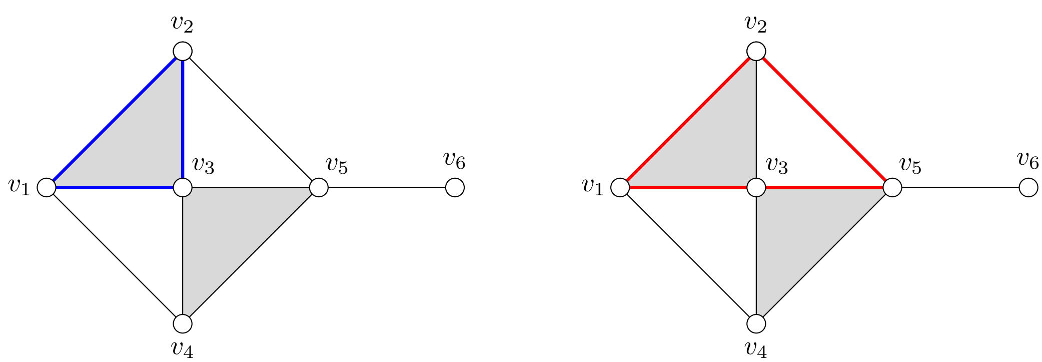
(d) Since $\mathsf{B}_n(K)$ is the group of all collections of even vertices in $\mathsf{C}_n(K)$, we have $\mathsf{H}_n(K) = \mathsf{C}_n(K)/\mathsf{B}_n(K) \cong \mathbb{Z}/2\mathbb{Z}$.

(ii) Now, we calculate the $k = 1$ case.

(a) Elements of $\mathsf{C}_1(K)$ are collections of linear combinations of the edges

$$\mathsf{C}_1(K) = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9 \rangle$$

(b) Elements of $\mathsf{Z}_1(K)$ are collections of edges such that each vertex contained in an edge in the collection has even degree. This corresponds to cyclic subgraphs of K (as well as the empty cycle), e.g.:



[1]Justification: note that the coefficient on any given vertex when we apply ∂ is the degree of the vertex in our path. Hence, only the initial and terminal vertex don't get mapped to 0.

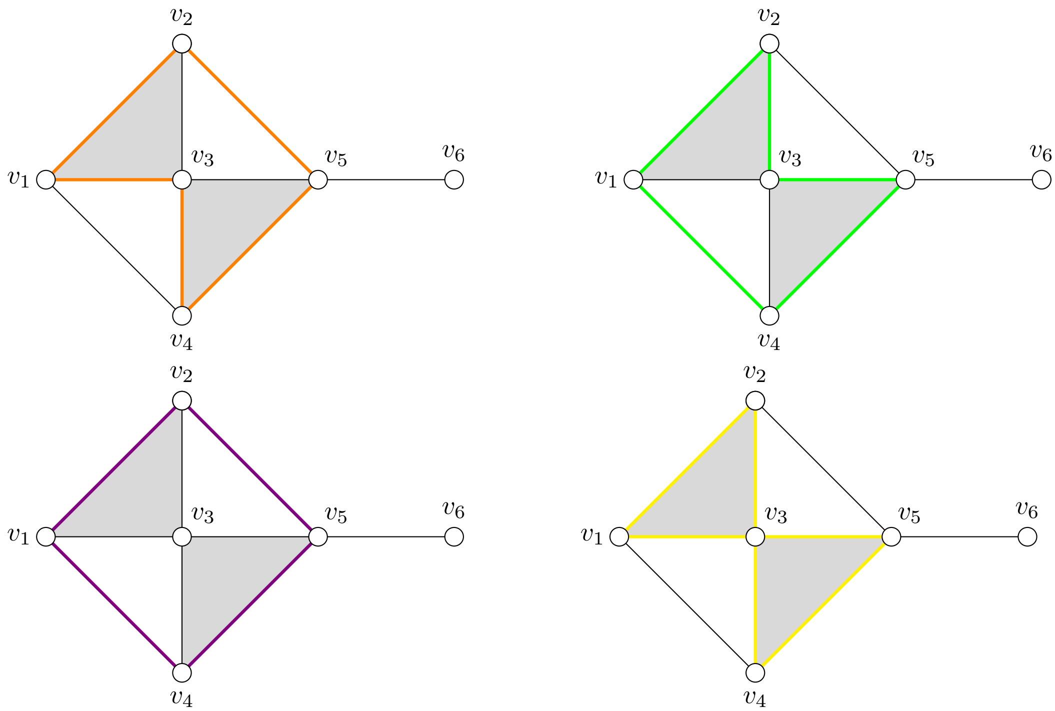


Figure 2.5: Some cycles in K

(c) First, consider the following diagram:

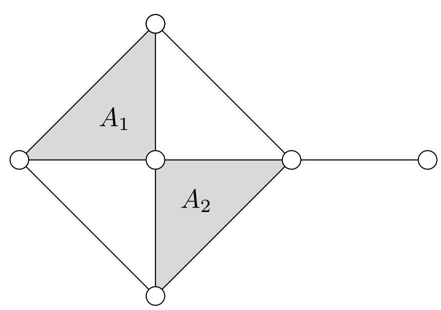


Figure 2.6: Two $n = 2$ simplices

$\mathbf{0}_1$ bounds $\mathbf{0}_2$. Since $\partial(A_1) \cap \partial(A_2) = \varnothing$, then the other two cycles in $\mathsf{B}_1(K)$ are just $\partial(A_1)$ and $\partial(A_2)$, respectively.

(d) $\mathsf{H}_1(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (equivalence classes have representative elements $\mathbf{0}, \partial(A_1), \partial(A_2), \partial(A_1) + \partial(A_2)$)

(iii) For $k = 2$, we have

(a) $\mathsf{C}_2(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(b) $\mathsf{Z}_2(K) \cong \mathbf{0}$

(c) $\mathsf{B}_2(K) \cong \mathbf{0}$

(d) And hence $\mathsf{H}_2(K) \cong \mathbf{0}$.

∎

16.7. If K is a one-point space, $\mathsf{H}_n(K) \cong 0$ for $n \geq 0$, and $\mathsf{H}_0(K) \cong \mathbb{Z}/2\mathbb{Z}$.

Solution. For $n > 0$, $\mathsf{C}_n(K)$ is the trivial group. Since $\mathsf{Z}_n(K) \leq \mathsf{C}_n(K)$, we thus have $\mathsf{Z}_n(K) \cong 0$, and so $\mathsf{H}_n(K) \cong 0$.

For the $n = 0$, note that $\mathsf{Z}_0(K) = \mathsf{C}_0(K) \cong \mathbb{Z}/2\mathbb{Z}$ (every point is definitionally a 0-cycle). Since K contains no 1-simplices, $\mathsf{B}_0(K) = \mathbf{0}$, hence $\mathsf{H}_0(K) \cong \mathbb{Z}/2\mathbb{Z}$. ∎

Definition 2.2.8 (Acyclic)**.** Any space with the homology groups of a point is called *acyclic.*

Definition 2.2.9 (Simplicially connected)**.** Let K be a simplicial complex. Then we call K *simplicially connected* iff for all pairs of 0-simplices $v_0, v_n \in K$, there exists a sequence of 0-simplices $\{v_i\}_{i\in[n]}$ such that for all $i \in [n]$ (with $i \neq n$), $\{v_i v_{i+1}\}$ is a 1-simplex in K. Note, this corresponds exactly to K being connected as a graph, where the 0-simplices represent vertices, and the 1-simplices represent edges.

16.8. If K is simplicially connected, then $\mathsf{H}_0(K) \cong \mathbb{Z}/2\mathbb{Z}$. If K has r simplicially connected components, then
$$\mathsf{H}_0(K) \cong \prod_{i=1}^{r} \mathbb{Z}/2\mathbb{Z}$$

Solution.

(a) Suppose K is simplicially connected. We want to show $\mathsf{H}_0(K) \cong \mathbb{Z}/2\mathbb{Z}$. First, observe that $\mathsf{Z}_0(K) \cong \mathsf{C}_0(K)$ (every 0 simplex has trivial boundary). By properties of module homomorphisms, for all $\sigma \in \mathsf{B}_0(K)$, σ is a basis element of $\mathsf{B}_0(K)$ iff $\exists \tau \in \mathsf{C}_1(K)$ such that τ is a basis element of $\mathsf{C}_1(K)$, and $\partial_1(\tau) = \sigma$. Thus, $\mathsf{B}_0(K)$ is spanned by $\{\{\{v_i\} + \{v_j\}\} \mid \{v_i v_j\} \in K\}$. It follows that $\mathsf{B}_0(K)$ contains exactly those elements of $\mathsf{C}_0(K)$ with an even number of vertices.[2]

It follows that $\mathsf{H}_0(K) = \mathsf{Z}_0(K)/\mathsf{B}_0(K) \cong \mathbb{Z}/2\mathbb{Z}$ (any 0-chain has either an even or odd number of vertices).

(b) This follows by applying the above argument to each of the connected components.

∎

16.9. Let K be a triangulation of a 3-dimensional ball that consists of a 3-simplex together with its faces. Compute $\mathsf{H}_n(K)$ for each n.

Solution.

[2] Since $\mathsf{B}_0(K)$ is generated by pairs.

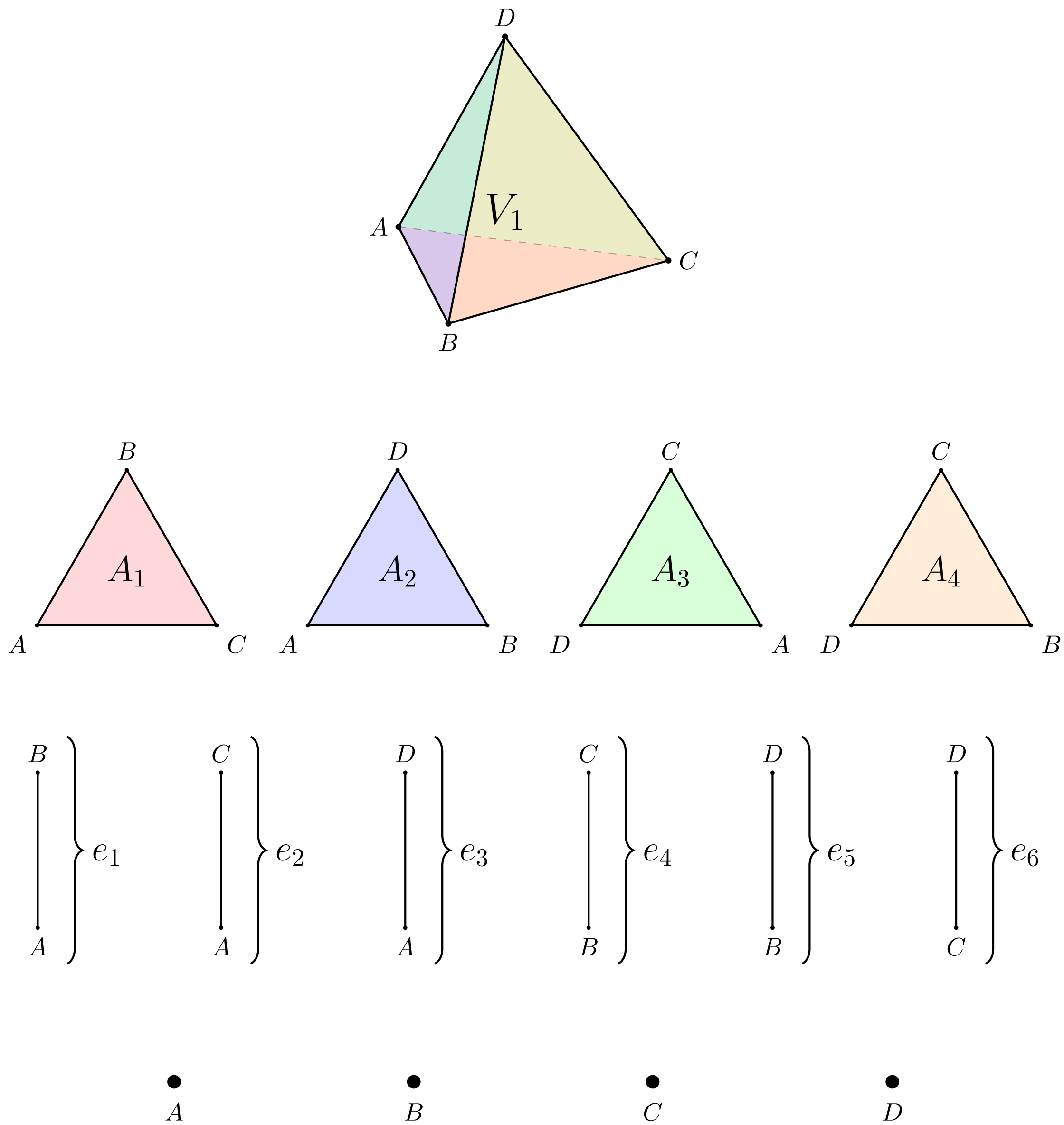


Figure 2.7: 3-simplex and its basis faces. Note the 1, 4, 6, 4 relationship. Gotta love Pascal's 2-simplex!

(0) K is connected, so $\mathsf{H}_0(K) \cong \mathbb{Z}/\mathbb{Z}2$.

(1) Elements of $\mathsf{Z}_1(K)$ are all just closed loops (linear combinations of the $\partial_2(A_i)$). But elements of $\mathsf{B}_1(K)$ are also just linear combinations of the $\partial_2(A_i)$. Hence $\mathsf{H}_1(K) \cong \mathbf{0}$.

(2) $\mathsf{Z}_2(K) = \{\mathbf{0}, A_1 + A_2 + A_3 + A_4\} = B_2(K) \cong \mathbb{Z}/2\mathbb{Z}$, so $\mathsf{H}_2(K) \cong \mathbf{0}$.

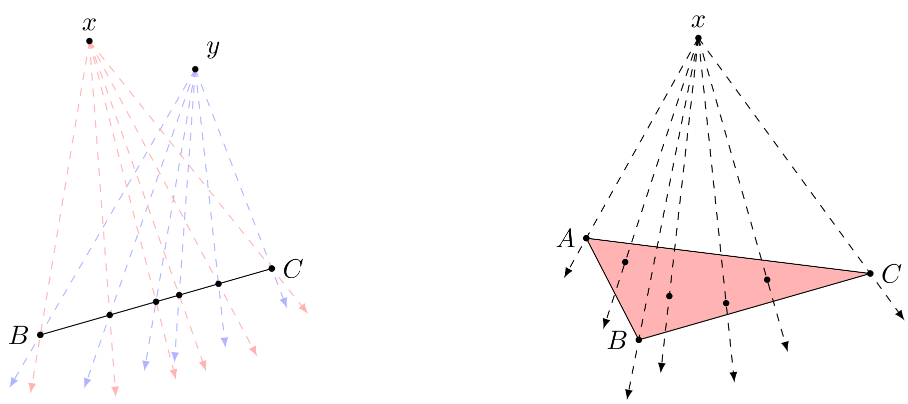
(3) $\mathsf{H}_3(K) \cong \mathbf{0}$.

It follows that the 3-simplex with all its faces is acyclic, which makes sense, since the underlying space is homeomorphic to the 3-ball, and the 3-ball is homeomorphic to a point. ■

16.10. Let K be a triangulation of a 2-sphere that consists of the proper faces of a 3-simplex. Compute $\mathsf{H}_n(K)$ for each n.

Solution. Proceed as before for $k = 0, 1$. For $k = 2$, note $\mathsf{B}_2(K) \cong \mathbf{0}$. Hence, $\mathsf{H}_2(K) \cong \mathbb{Z}/2\mathbb{Z}$. ■

Definition 2.2.10 (Seeing a simplex)**.** Let K be a simplicial complex with $|K| \subset \mathbb{R}^n$. A point $x \notin K$ can *see* K if any ray from x intersects $|K|$ at most once (as seen in the following diagram).



(a) x and y both see a 1-simplex σ_1

(b) x sees a 2-simplex σ_2

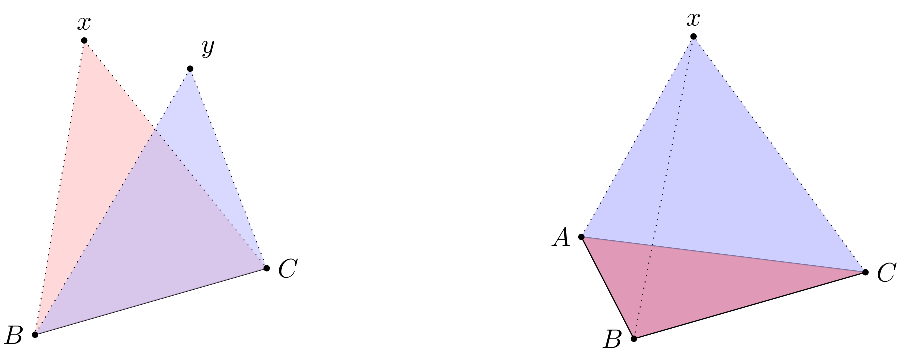
Figure 2.8: Simplices being seen

Remark. Note, that when there are multiple k-simplices in K, the picture might not be quite as simple.

Remark. As far as I can tell, a point x sees K iff x is in orthogonal complement of the k-hyperplane containing K. Not sure if this is actually correct though?

Definition 2.2.11 (Cone of x over σ)**.** Let K be a finite complex and x a point that sees K. If $\sigma = \{v_0 \cdots v_k\}$ is a simplex of K, define the *cone* of x over σ to be the simplex

$$\mathsf{Cone}_x(\sigma) = \{x v_0 \cdots v_k\}.$$



(a) $\mathsf{Cone}_x(\sigma_1)$, $\mathsf{Cone}_y(\sigma_1)$ (b) $\mathsf{Cone}_x(\sigma_2)$

Figure 2.9: Some cones

Definition 2.2.12 (Cone over K)**.** Define $x * K$, the *cone over* K to be the simplicial complex comprising all simplices $\mathsf{Cone}_x(\sigma)$ for $\sigma \in K$, and all faces of such simplices.

Remark. Essentially, this just includes the base point and edges in fig. 2.9a and the base point and edges *and* faces in fig. 2.9b.

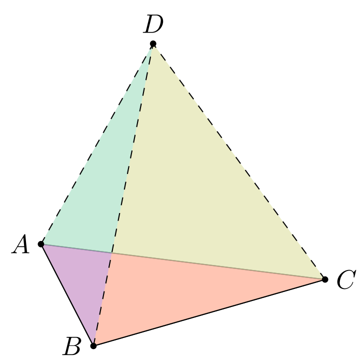


Figure 2.10: $x * \sigma_2$. Note, each of the faces is colored to indicate inclusion in the complex.

Definition 2.2.13 (Simplicial cone operator)**.** Define the *simplicial cone operator* $\mathsf{Cone}_x : \mathsf{C}_n(K) \to \mathsf{C}_{n+1}(x * K)$ by extending the definition of $\mathsf{Cone}_x(\sigma)$ linearly to chains.

16.11. For x seeing K, and σ a simplex of K,

$$\partial\, \mathsf{Cone}_x(\sigma) + \mathsf{Cone}_x(\partial\sigma) = \sigma.$$

Solution. Let $\sigma = \{v_0 \cdots v_k\}$. Then

$$\partial\, \mathsf{Cone}_x(\sigma) + \mathsf{Cone}_x(\partial\sigma) = \left(\{\widehat{x} v_0 \cdots v_k\} + \sum_{i\in[k]} \{v_0 \cdots \widehat{v_i} \cdots v_k\}\right) + \mathsf{Cone}_x\left(\sum_{i\in[k]} \{v_0 \cdots \widehat{v_i} \cdots v_k\}\right)$$

$$\begin{aligned}
&= \sigma + \sum_{i \in [k]} \{v_0 \cdots \widehat{v_i} \cdots v_k\} + \{v_0 \cdots \widehat{v_i} \cdots v_k\} \\
&= \sigma + \sum_{i \in [k]} \mathbf{0} \\
&= \sigma
\end{aligned}$$

as desired. ■

16.12. For any complex K and x seeing K, the complex $x * K$ is acyclic.

Solution.

■

16.13.

Solution.

■

2.3 Induced Homomorphisms and Invariance

Fix two simplicial complexes K and L.

16.14. Let $f : K \to L$ be a simplicial map. Carefully write out the definition of the natural induced map from n-chains of K to n-chains of L:

$$f_{\#n} : \mathsf{C}_n(K) \to \mathsf{C}_n(L)$$

and show that it is a homomorphism.

Solution. We define $f_{\#}$ by its action on basis elements, then apply linear extension. Let $\sigma = \{v_0 \cdots v_n\} \in \mathsf{C}_n(K)$ be a basis element. Then define

$$f_{\#n}(\sigma) = \begin{cases} \mathbf{0} & \text{if } f(\sigma) \text{ is not a } n\text{-simplex, and} \\ f(\sigma) & \text{otherwise} \end{cases}$$

We now apply linear extension. That is, for all $\tau = \sum_{i \in I} \left\{v_0^{(i)} \cdots v_n^{(i)}\right\} \in \mathsf{C}_n(K)$, define

$$\begin{aligned}
f_{\#n}(\tau) &= f_{\#n}\left(\sum_{i \in I} \sigma_i\right) \\
&= \sum_{i \in I} f_{\#n}(\sigma_i)
\end{aligned}$$

we want to show this is a homomorphism.

(1) Let $\sigma_1, \sigma_2 \in \mathsf{C}_n(K)$ be arbitrary. Then

$$\begin{aligned}
f_{\#n}(\sigma_1 + \sigma_2) &= f_{\#n}\left(\sum_{k \in I \cup J} \left\{v_0^{(k)} \cdots v_n^{(k)}\right\}\right) \\
&= f_{\#n}\left(\sum_{i \in I} \left\{v_0^{(i)} \cdots v_n^{(i)}\right\} + \sum_{j \in J} \left\{v_0^{(j)} \cdots v_n^{(j)}\right\}\right)
\end{aligned}$$

$$\begin{aligned}
&= f_{\#n}\left(\sum_{i\in I}\left\{v_0^{(i)}\cdots v_n^{(i)}\right\}\right) + f_{\#n}\left(\sum_{j\in J}\left\{v_0^{(j)}\cdots v_n^{(j)}\right\}\right) \\
&= f_{\#n}(\sigma_1) + f_{\#n}(\sigma_2)
\end{aligned}$$

(2) We want to show $f_{\#n}(\mathbf{0}) = \mathbf{0}$. Note, for any $\sigma \in \mathsf{C}_n(K)$, $\mathbf{0} = \sigma + \sigma$. By linearity, $f_{\#n}(\mathbf{0}) = f_{\#n}(\sigma + \sigma) = f_{\#n}(\sigma) + f_{\#n}(\sigma) = \mathbf{0}$ as well, as desired.

thus $f_{\#n}$ is a homomorphism. ■

The map $f_{\#n}$ is called the *induced chain map*. The next exercise contains an important technicality about the induced chain map in the case where the image of an n-simplex is an $(n-1)$-simplex.

16.15. If the simplicial map $f : K \to L$ maps an n-simplex σ to an $(n-1)$-simplex τ, what is $f_{\#n}(\sigma)$?

Solution. By the definition given above, $f_{\#n}(\sigma) = \mathbf{0}$. ■

16.16. Let $f : K \to L$ be a simplicial map, and let $f_\#$ be the induced map $f_\# : \mathsf{C}_n(K) \to \mathsf{C}_n(L)$. Then for any chain $c \in \mathsf{C}_n(K)$,

$$\partial(f_\#(c)) = f_\#(\partial(c))$$

In other "words," we have the following commutative diagram

$$\begin{tikzcd}
\mathsf{C}_n(K) \arrow[r, "f_\#"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowright"] & \mathsf{C}_n(K) \arrow[d, "\partial"] \\
\mathsf{C}_{n-1}(k) \arrow[r, "f_\#"'] & \mathsf{C}_{n-1}(k)
\end{tikzcd}$$

Figure 2.11: Commutative diagram

Solution. Let $c \in \mathsf{C}_n(K)$. Express c as a sum of basis elements $\{\sigma\}_{i\in I}$. Let $\{\sigma_i\}_{i\in I'}$ be those σ_i for which $f_\#(\sigma_i) \neq \mathbf{0}$. Then

$$\begin{aligned}
\partial_n(f_{\#n}(c)) &= \partial_n\left(\sum_{i\in I'} f_{\#n}(\sigma_i)\right) \\
&= \partial_n\left(\sum_{i\in I'} f(\sigma_i)\right) \\
&= \partial_n\left(\sum_{i\in I'}\left\{f\left(v_0^{(i)}\right)\cdots f\left(v_n^{(i)}\right)\right\}\right) \\
&= \sum_{i\in I'}\sum_{j=1}^{n}\left\{f\left(v_0^{(i)}\right)\cdots \widehat{f\left(v_j^{(i)}\right)}\cdots f\left(v_n^{(i)}\right)\right\} \\
&= \sum_{i\in I'}\sum_{j=1}^{n} f_{\#n-1}\left(\left\{v_0^{(i)}\cdots \widehat{v_j^{(i)}}\cdots v_n^{(i)}\right\}\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in I'} f_{\#n-1}(\partial(\sigma_i)) \\
&= f_{\#n-1}\left(\partial_n\left(\sum_{i \in I'} \sigma_i\right)\right) \\
&= f_{\#n-1}(\partial_n(c))
\end{aligned}$$

as desired. ∎

Definition 2.3.1 (Induced Homomorphism)**.** Let $f : K \to L$ be a simplicial map. The *induced homomorphism* $f_* : \mathsf{H}_n(K) \to \mathsf{H}_n(L)$ is defined by $f_*([z]) = [f_\#(z)]$ (where the square brackets indicate an equivalence class).

16.17. Let $f : K \to L$ be a simplicial map. Then the induced homomorphism $f_* : \mathsf{H}_n(K) \to \mathsf{H}_n(L)$ is a well-defined homomorphism.

Solution. That f_* is a homomorphism follows directly from the definition.

(1) That $f_*([\mathbf{0}]) = [\mathbf{0}]$ follows by the definition of $f_\#$.

(2) Similarly for $f_*(\sigma + \tau) = f_*(\sigma) + f_*(\tau)$.

We now show that f_* is well-defined. Let $[\sigma] \in \mathsf{H}_n(K)$ and $[\tau] \in \mathsf{H}_n(K)$ with $[\sigma] = [\tau]$. Then $\exists \rho \in \mathsf{B}_n(K)$ s.t. $\sigma = \tau + \rho$. Observe that

$$\begin{aligned}
f_*([\sigma]) &= [f_\#(\sigma)] \\
&= [f_\#(\tau + \rho)] \\
&= [f_\#(\tau) + f_\#(\rho)] \\
&= [f_\#(\tau)] + [f_\#(\rho)] \\
&= [f_\#(\tau)] + [\mathbf{0}] \\
&= [f_\#(\tau)],
\end{aligned}$$

as desired. ∎

16.18. Let K be a complex comprising the proper faces of a hexagon: six edges and six vertices $v_0, \ldots, v_5$. Let L be the complex comprising the proper faces of a triangle: three edges and three vertices w_0, w_1, w_2. Let f be a simplicial map that sends v_i to $w_{i \bmod 3}$. Compute the homology groups of K and L and describe the simplicial map f and the induced homomorphism f_*.

Solution.

(1) We compute the homology groups of K. Observe, $\mathsf{H}_2(K) \cong \{\mathbf{0}\}$ (since $\mathsf{Z}_n(K)$ is trivial). $\mathsf{H}_1(K) \cong \mathbb{Z}/2\mathbb{Z}$ (since $\mathsf{Z}_1(K) \cong \mathbb{Z}/2\mathbb{Z}$, as we either have the whole hexagon or we don't, and $\mathsf{B}_1(K) \cong \{\mathbf{0}\}$). Finally, by theorem 16.8, we have $\mathsf{H}_0(K) \cong \mathbb{Z}/2\mathbb{Z}$.

(2) The homology groups of L are the same.

(3) The map f folds the circle $|K|$ onto itself

(4) f_* is an isomorphism.

∎

Definition 2.3.2 (λ-map)**.** Let K be a simplicial complex. Let $\lambda : \mathsf{sd}\, K \to K$ be defined as follows: for any vertex $v \in \mathsf{sd}\, K$, there exists $\sigma \in K$ such that v is the barycenter of σ. Then let

$$\lambda(v) = v_\sigma$$

where v_σ is a vertex in σ.

Definition 2.3.3 (λ_*)**.** Let $\lambda_* : \mathsf{H}_n(\mathsf{sd}\, K) \to \mathsf{H}_n(K)$ be defined by linear extension of λ to simplices. Since λ is a well-defined simplicial map, λ_* is a well-defined homomorphism (theorem 16.17).

16.19. Suggest a homomorphism from $\mathsf{C}_n(K) \to \mathsf{C}_n(\mathsf{sd}\, K)$ that commutes with ∂. Could its induced homomorphism on homology be an inverse for λ_*?

Solution. Consdier $f : \mathsf{C}_n(K) \to \mathsf{C}_n(\mathsf{sd}\, K)$ defined by

$$f(\sigma) = \text{sum of maximal } n\text{-simplices in } \mathsf{sd}\,\sigma$$

■

We give this a name.

Definition 2.3.4 (Subdivision operator)**.** Define the *subdivision operator* $\mathsf{SD} : \mathsf{C}_n(K) \to \mathsf{C}_n(\mathsf{sd}\, K)$ by first defining SD on a simplex:

$$\mathsf{SD}(\{v_0 \cdots v_n\}) = \sum_{\pi \in \mathcal{S}_{n+1}} \{b_0^\pi \cdots b_n^\pi\}$$

where $\mathcal{S}_{n+1}$ is the symmetric group, and b_k^π is the barycenter of the face $\left\{v_{\pi(0)} \cdots v_{\pi(k)}\right\}$.

Note. I'm pretty sure this gets us all the maximal simplices, as we want. To verify it, I think we proceed as follows: let $\sigma \in \mathsf{sd}\, K$ be an arbitrary n-simplex. Then one of the vertices in σ is a vertex in K (def. of maximal simplex). Take π such that $v_{\pi(0)}$ is this maximal vertex. Also restrict π such that $v_{\pi(1)}$ is the vertex such that the barycenter of $\left\{v_{\pi(0)} v_{\pi(1)}\right\}$ is in σ (again, since σ is maximal, I think this works). Continue this.

I think also that one can show this necessitates the resulting simplices be disjoint?

16.20. The subdivision operator commutes with the boundary operator, that is, if c is a chain in K, then $\mathsf{SD}(\partial c) = \partial\, \mathsf{SD}(c)$.

Solution. We show the result for a simplex. For intuition, observe the following diagram in the case c is a 2-simplex:

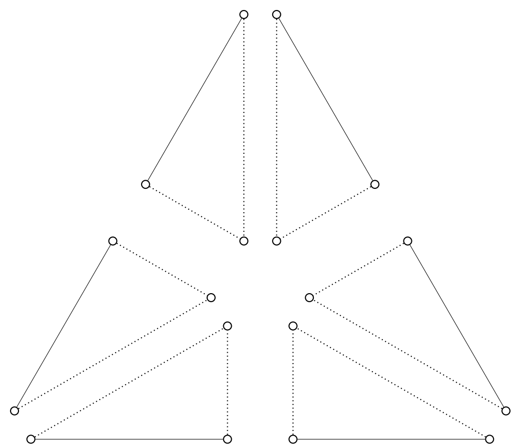


Figure 2.12: $\mathsf{SD}(c)$

under ∂, only the solid lines are not annihilated. In generality,

$$\begin{aligned}
\partial\,\mathsf{SD}(c) &= \partial \sum_{\pi \in \mathcal{S}_{n+1}} \{b_0^\pi \cdots b_n^\pi\} \\
&= \sum_{\pi \in \mathcal{S}_{n+1}} \sum_{j=0}^{n} \left\{b_0^\pi \cdots \widehat{b_j^\pi} \cdots b_n^\pi\right\} \\
&= \sum_{j=0}^{n} \sum_{\pi \in \mathcal{S}_{n+1}} \left\{b_0^\pi \cdots \widehat{b_j^\pi} \cdots b_n^\pi\right\} \\
&= \sum_{j=0}^{n-1} \sum_{\pi' \in \mathcal{S}_{n}} \left\{b_0^{\pi'} \cdots b_{n-1}^{\pi'}\right\} \\
&= \mathsf{SD}(\partial c)
\end{aligned}$$

as desired.[3] ■

Note. I think that it's getting a little tricky here to see which concepts are the "important parts." Maybe let's shift to trying `http://www.indiana.edu/ lniat/m621notessecondedition.pdf`

2.4 The Mayer-Vietoris Theorem

Definition 2.4.1 (Subcomplex)**.** If K is a simplicial complex, a *subcomplex* is a simplicial complex L such that $L \subset K$.

Note. The thing to note here is that if we choose some simplex to be in our subcomplex, we must bring all its faces with us as well.

16.31. If K is a finite simplicial complex, verify that the intersection of two subcomplexes of K is a subcomplex.

[3] π' is the permutation given by $\pi^{-1}(j_0) = \pi(j_0)$, where j_0 is the deleted vertex.

Solution. Let L, M be subcomplexes of K. Then for all $\sigma \in L \cap M$, $\sigma \in L$, $\sigma \in M$, hence for all faces $\sigma' \subset \sigma$, we have $\sigma' \in L$, $\sigma' \in M$, and thus $\sigma' \in L \cap M$. Hence $L \cap M$ is a simplicial complex.

The disjointness condition follows similarly. ■

We'll now examine cases where we have two subcomplexes A, B of a simplicial complex K. We want to look at relationships between cycles in $A, B, A \cap B$, and K.

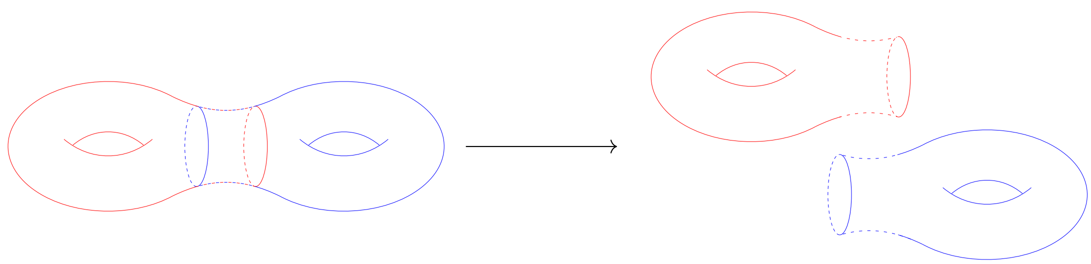


Figure 2.13: An example of such A, B

16.32. Note that a cycle in $A \cap B$ is still a cycle in A, B, and K. Then answer:
(a) Can a trivial cycle in $A \cap B$ be non-trivial in A?
(b) Can a non-trivial cycle in $A \cap B$ be trivial in A?
(c) Can a non-trivial cycle in $A \cap B$ that's also non-trivial in A and in B be trivial in K?

Solution. Let $\sigma \in A \cap B$. I'll assume this is asking us to just consider just the inclusion map applied to σ

(a) Nope. Including into A won't change σ at all.

(b) No?

(c) No?

■

16.33. Let K be a finite simplicial complex and A and B be subcomplexes such that $K = A \cup B$. If α, β are k-cycles in A and B respectively, and if $\alpha \sim_{\mathbb{Z}/2\mathbb{Z}} \beta$ in K, then there is a k-cycle c in $A \cap B$ such that $\alpha \sim_{\mathbb{Z}/2\mathbb{Z}} c$ in A and $\beta \sim_{\mathbb{Z}/2\mathbb{Z}} c$ in B.

Solution. The question can be rephrased as

> Let $\alpha \in \mathsf{Z}_k(A)$, and $\beta \in \mathsf{Z}_k(B)$. Suppose that
>
> $$[\alpha]_K = [\beta]_K.$$
>
> Then there exists $c \in \mathsf{Z}_k(A \cap B)$ such that
>
> $$[\alpha]_A = [c]_A \qquad [\beta]_B = [c]_B$$

> Or, show that if $\alpha - \beta = 0 \in \mathsf{H}_k(K)$, then $\exists c \in \mathsf{Z}_k(A \cap B)$ such that $(\alpha, \beta) = (c, c) \in \mathsf{H}_k(A) \oplus \mathsf{H}_k(B)$. This gives us maps
>
> $$\mathsf{H}_n(K) \xrightarrow{\delta^k} \mathsf{H}_k(A \cap B) \xrightarrow{\phi^k} \mathsf{H}_k(A) \oplus \mathsf{H}_k(B)$$
>
> by
>
> $$[\alpha] = [\beta] \xmapsto{\delta^k} [c] \xmapsto{\phi^k} [(c, c)]$$

Define the homomorphisms $\pi_A : \mathsf{Z}_k(K) \to \mathsf{Z}_k(A)$, $\pi_B : \mathsf{Z}_k(K) \to \mathsf{Z}_k(B)$ as follows:

$$\pi_A(\sigma) = \begin{cases} \sigma & \text{if } \sigma \in \mathsf{Z}_k(A) \\ \mathbf{0} & \text{otherwise} \end{cases} \qquad \pi_B(\sigma) = \begin{cases} \sigma & \text{if } \sigma \in \mathsf{Z}_k(B) \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Observe that the following diagram commutes:[4]

$$\begin{tikzcd} \mathsf{Z}_k(K) \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] \arrow[dr, phantom, "\circlearrowright"] & \mathsf{Z}_k(A) \arrow[d, "\pi_B"] \\ \mathsf{Z}_k(B) \arrow[r, "\pi_A"'] & \mathsf{Z}_k(A \cap B) \end{tikzcd}$$

Figure 2.14: $\pi_A \circ \pi_B = \pi_B \circ \pi_A$

And also note that π_A, π_B are idempotent. Now, since $[\alpha]_K = [\beta]_K$, there exists $c \in \mathsf{B}_k(K)$ such that $\alpha + \beta = c$. Note that

$$\begin{aligned} \partial((\pi_A \circ \pi_B)(c)) &= (\pi_A \circ \pi_B)(\partial c) \\ &= (\pi_A \circ \pi_B)(\partial(\alpha + \beta)) \\ &= (\pi_A \circ \pi_B) \end{aligned}$$

Hence $c \in \mathsf{Z}_k(A \cap B)$. Now,

$$\pi_A(\alpha - c) = \pi_A(\pi_A(\alpha + \beta) - \pi_A(\beta) - c)$$

Hence, there exists $c_A \in \mathsf{B}_k(A)$, $c_B \in \mathsf{B}_k(B)$ such that

$$\alpha =$$

■

16.34. Let K be a finite simplicial complex and A and B be subcomplexes such that $K = A \cup B$. Let z be a k-cycle in K. Then there exist k-chains α and β in A and B respectively such that:

(1) $z = \alpha + \beta$ and
(2) $\partial\alpha = \partial\beta$ is a $(n-1)$-cycle c in $A \cap B$.
(3) If $z = \alpha' + \beta'$, a sum of n-chains in A and B respectively, and $c' = \partial\alpha' = \partial\beta'$ is a $(n-1)$-cycle, then c' is homologous to c in $A \cap B$.

[4] Ok, technically $\operatorname{dom}(\pi)_A = \mathsf{Z}_k(K) \neq \mathsf{Z}_k(B)$, but you could throw an inclusion map in there if you so pleased.

Solution.

(1) Let $\alpha = \pi_A(z)$. Then $\alpha \in \mathsf{C}_k(A)$. Now, taking $\beta = z - \alpha$, we see

$$\begin{aligned}\pi_A(\beta) &= \pi_A(z-\alpha)\\ &= \pi_A(z - \pi_A(\alpha))\\ &= \pi_A(z) - \pi_A(\pi_A(z))\\ &= \pi_A(z) - \pi_A(z)\\ &= \mathbf{0}\end{aligned}$$

hence $\beta \in \mathsf{C}_k(B)$.

(2) WTS $\partial\alpha = \partial\beta \in \mathsf{Z}_{n-1}(A \cap B)$. Note,

$$\begin{aligned}\partial(\beta) &= \partial(z-\alpha)\\ &= \partial(z) - \partial(\alpha)\\ &= \mathbf{0} - \partial(\alpha) = \partial(\alpha),\end{aligned}$$

which are $(n-1)$-boundaries (and hence $(n-1)$-cycles). Projection onto $A \cap B$ yields the desired result.

(3) In $A \cap B$, c and c' are both $(n-1)$ boundaries. Hence, they are trivially homologous.

■

16.36. Let K be a simplicial complex and A and B be subcomplexes such that $K = A \cup B$. Construct natural homomorphisms ϕ, ψ, δ among the groups below and show that $\psi \circ \phi = 0$ and $\delta \circ \psi = 0$.
(a) $\phi : \mathsf{H}_n(A \cap B) \to \mathsf{H}_n(A) \oplus \mathsf{H}_n(B)$.
(b) $\psi : \mathsf{H}_n(A) \oplus \mathsf{H}_n(B) \to \mathsf{H}_n(K)$.
(c) $\delta : \mathsf{H}_n(K) \to \mathsf{H}_{n-1}(A \cap B)$.

Solution. Let

(1) $\phi : \mathsf{H}_n(A \cap B) \to \mathsf{H}_n(A) \oplus \mathsf{H}_n(B)$ be given by

$$\phi(\rho) = (\rho, \rho).$$

That this is a well-defined homomorphism follows immediately.

(2) Let $\psi(\sigma, \tau)$ be given by

$$\psi(\sigma, \tau) = \sigma - \tau.$$

It is straightforward to verify this is a well-defined homomorphism.

(3) Let $\delta(\sigma)$ be given by

$$\delta(\upsilon) = \partial(\pi_A \circ \pi_B(\upsilon)).$$

All the maps in this definition are linear on simplices. Homomorphism follows.

Now, we show the composition things.

(a) $\psi \circ \phi = 0$ is trivial

(b) Note

$$\begin{aligned}\delta(\psi(\alpha,\beta)) &= \partial(\pi_A \circ \pi_B(\alpha - \beta)) \\ &= \end{aligned}$$

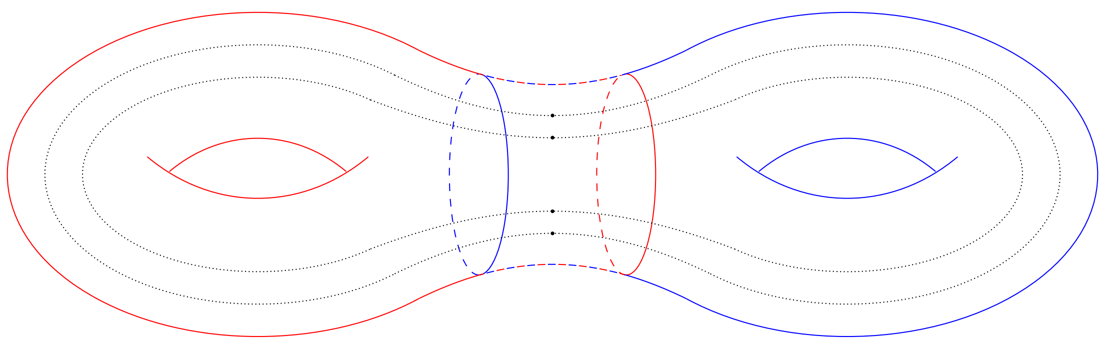


Figure 2.15: Example of A, B with α, β.

■

Definition 2.4.2 (Exact sequence)**.** Given a (finite or infinite) sequence of groups and homomorphisms:

$$\cdots \to G_{i-1} \xrightarrow{\phi_{i-1}} G_i \xrightarrow{\phi_i} G_{i+1} \to \cdots$$

the sequence is **exact at** G_i if and only if $\operatorname{im}\phi_{i-1} = \ker\phi_i$. The sequence is called an **exact sequence** if and only if it is exact at each group (except at the first and last groups if they exist).

16.37. Let K be a finite simplicial complex and A and B be subcomplexes such that $K = A \cup B$. The sequence

$$\cdots \to \mathsf{H}_n(A \cap B) \to \mathsf{H}_n(A) \oplus \mathsf{H}_n(B) \to \mathsf{H}_n(K) \to \mathsf{H}_{n-1}(A \cap B) \to \ldots$$

using the homomorphisms ϕ, ψ, δ above, is exact.

3. Some Homological Algebra

The big idea: algebraic topology assigns discrete algebraic invariants to topological spaces and continuous maps. Book for this section: James May's *A Concise Course in Algebraic Topology*

3.1 Chain complexes

Definition 3.1.1 (Chain/Cochain Complexes)**.** Let R be a commutative ring. A *chain complex* over R is a sequence of maps of R-modules

$$\cdots \to X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \to \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all i. We generally abbreviate $d = d_i$. A *cochain complex* over R is an analogous sequence

$$\cdots \to Y^{i-1} \xrightarrow{d^{i-1}} Y^i \xrightarrow{d^i} Y^{i-1} \to \cdots$$

with $d^i \circ d^{i-1}$.

We usually require chain complexes to satisfy $X_i = 0$ for $i < 0$, and cochain complexes to satisfy $Y^i = 0$ for $i < 0$. Without this distinction, the definitions are equivalent.

Definition 3.1.2 (Some definitions)**.** Elements of $\ker d_i$ are called cycles. Elements of $\operatorname{im} d_{i+1}$ are called boundaries. Write $B_i(X) \subset Z_i(X) \subset X_i$ for the submodules of boundaries and dycles, and define the i^{th} homology group $H_i(X)$ by

$$H_i(X) = Z_i(X)/B_i(X).$$

We write $H_*(X)$ for the sequence of R-modules $H_i(X)$. We understand

4. Rotman

The big idea: algebraic topology assigns discrete algebraic invariants to topological spaces and continuous maps. Book for this section: Joseph Rotman's *A First Course in Algebraic Topology*

4.1 A sketch of the Brouwer Fixed Point Theorem

R 0.1. Every continuous function $f : D^1 \to D^1$ has a fixed point.

Solution. We'll prove this without the techniques of analysis, so as to make the connection to the general argument slightly more obvious. Let $f(-1) = a$ and $f(1) = b$.

(1) Suppose $a = -1$ or $b = 1$, then we're done.

(2) Else, $a > -1$ and $b < 1$. Consider the graph of f:

$$G = \left\{(x, f(x)) \;\middle|\; x \in D^1\right\}$$

since f is continuous and D^1 is connected, G is connected as well. Let

$$A = \{(x, f(x)) \mid f(x) > x\} \qquad \text{and} \qquad B = \{(x, f(x)) \mid f(x) < x\}.$$

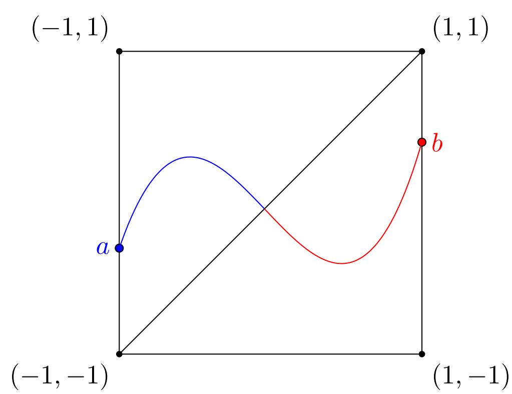


Figure 4.1: G

And let $\Delta = \{(x, x) \mid x \in [0, 1]\}$. Note $a \in A$, and $b \in B$, so $A \neq \varnothing \neq B$.

Suppose $G \cap \Delta = \varnothing$. Then $G = A \sqcup B$. Note A, B are open in G, hence G is not connected, a contradiction.

■

Definition 4.1.1 (retract)**.** A subspace X of a topological space Y is a *retract* of Y if there is a continuous map $r : Y \to Y$ with $r(x) = x$ for all $x \in X$. Such a map is called a *retraction*.

Problem.

5. Appendix

5.1 List of Definitions