# Homology Theory
## Notes & Exercises from my Independent Study

(Or: *If I could save Klein in a bottle ♪*)


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# Introduction

## What's this?

This document is a compendium of notes, exercises, and other miscellany from my independent study in Homology Theory. For this, I am working through the second half of *Topology Through Inquiry* by Michael Starbird and Francis Su (i.e., chapters 11-20), under supervision from Prof. Su himself. Rough topic coverage should be discernable from the table of contents, as I've tried to name each section identically to the corresponding title in the book.

## Notation

Most notation I use is fairly standard. Here's a (by no means exhaustive) list of some stuff I do.

- "WTS" stands for "want to show," s.t. for "such that." WLOG, as usual, is without loss of generality.
- End-of-proof things: $\blacksquare$ is QED for exercises and theorems. $\square$ is used in recursive proofs (e.g., proving a Lemma within a theorem proof). If doing a proof with casework, ✓ will be used to denote the end of each case.
- $(\Rightarrow\Leftarrow)$ means contradiction
- $\mathcal{T}(U)$ will denote the topology of a topological space $U$.
- $\mathcal{P}(A)$ is the powerset of $A$. I don't like using $2^A$.
- $\twoheadrightarrow$ denotes surjection.
- $\hookrightarrow$ denotes injection.
- Thus, $\hookrightarrow\!\!\!\!\!\twoheadrightarrow$ denotes bijection.
- **Important:** I use $f^{\rightarrow}(A)$ for the image of $A$ under $f$, and $f^{\leftarrow}(B)$ for the inverse image of $B$ under $f$.
- $\sim$ and $\equiv$ are used for equivalence relations. $\cong$ is used to denote homeomorphism. $\simeq$ is for Homotopy equivalence.
- $\epsilon$ is for trivial elements (e.g., the trivial path), while $\varepsilon$ is for small positive quantities.
- $\overline{U}$ denotes the closure of $U$, $U^{\circ}$ is the interior of $U$.
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# 1. Chapter 12: Classification of 2-Manifolds

## 1.1 Examples of 2-Manifolds

**12.1.** Show that the torus $\mathbb{T}^2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$

*Proof.* Let $\sim$ be an equivalence relation on $\mathbb{R}^2$ defined by $\forall (x, y) \in \mathbb{R}^2$,

$$\begin{cases} (x,y) \sim (x, y+1), & \text{and} \\ (x,y) \sim (x+1, y) \end{cases}$$

then $\mathbb{T}^2 \cong (\mathbb{R}^2/\sim) \cong \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{R}^2/$. Note that $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, which, by similar reasoning is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$. ■

**12.2.** For a given number of holes, demonstrate that the $n$-holed torus where the holes are lined up is homeomorphic to an $n$-holed torus where the holes are arranged in a circle.
*Note: for exercises like this that ask you to demonstrate a geometric homeomorphism, we are not askign you to define a formal homeomorphism — no equations are expected. Rather, it suffices to describe a process by which you would systematically distort one figure to look like the other.*

*Description:* First, twist the lined-up-holes torus (stretching as needed) such that the holes rest on the vertices of a regular $n$-gon. Then, simply stretch the body outwards until a disk shape is achieved. □

**Definition 1.1.1**

Define the *projective plane* (also called the *real projective 2-space*), denoted $\mathbb{R}\mathrm{P}^2$, to be the space of all lines in $\mathbb{R}^3$ that pass through the origin. The basis for the topology is the collection of open cones with the cone point at the origin.

**12.3.**
(a) Show that $\mathbb{R}\mathrm{P}^2 \cong \mathbb{S}^2/\langle x \sim -x \rangle$, that is, the projective plane is homeomorphic to the 2-sphere with diametrically opposite points identified.
(b) Show that $\mathbb{R}\mathrm{P}^2$ is also homeomorphic to a disk with two edges on its boundary (called a *bigon*) identified.
(c) Show that the klein bottle can be realized as a square with certain edges identified.

*Proof.* **COME BACK TO THIS AFTER ASKING PROF. SU ABOUT THE LEVEL OF RIGOR EXPECTED HERE!**

(a) **Claim:** Take some arbitrary point $\mathbf{r}(t)$, and give it a parameterization by $\mathbf{r}_0(t) = t\hat{\mathbf{r_0}}$, with the orientation chosen arbitrarily. Now, take a parameterization of each of the other lines in $\mathbb{R}\mathrm{P}^2$ by $\mathbf{r}(t) = t\hat{\mathbf{r}}$ chosen such that $\langle \hat{\mathbf{r}}, \hat{\mathbf{r}}_\mathbf{0} \rangle \geq 0$. Then $f$ given by $f(\mathbf{r}) = \hat{\mathbf{r}}$ is a homeomorphism.

**Proof of claim:** Let $U \in \mathcal{T}(\mathbb{R}\mathrm{P}^2)$, and let $\mathcal{B}$ be the basis described above. Since $U$ is open, $\exists B = \{B_i \mid i \in I\} \subseteq \mathcal{B}$ s.t.

$$U = \bigcup_{i \in I} B_i.$$

Suppose that $U$ contains no lines $\mathbf{r}(t)$ s.t. $\langle \hat{\mathbf{r}}, \hat{\mathbf{r}}_{\mathbf{0}} \rangle = 0$. Then

$$\begin{aligned} f^{\rightarrow}(U) &= f^{\rightarrow}\left(\bigcup_{i \in I} B_i\right) \\ &= \bigcup_{i \in I} f^{\rightarrow}(B_i) \end{aligned}$$

is a union of spherical caps.

(b) Take any two antipodal points, draw a great circle through them, and apply the result above.

(c)

■

**Recall:** A set $A \subset X$ is said to be *dense* iff $\overline{A} = X$. A topological space $X$ is said to be *separable* iff $X$ has a countable dense subset.

**Definition 1.1.2: 1-manifold**

A topological space is an *1-manifold* iff it is a separable metrizable space in which every point is in an open set homeomorphic to an open interval in $\mathbb{R}^1$.

**12.4.** Suppose $M$ is a compact, connected 1-manifold. Then $M$ is triangulable. That is, $M$ is homeomorphic to a subset $C$ of $\mathbb{R}^n$ consisting of a finite collection of straight line segments where any two segments of $C$ are either disjoint or meet at an endpoint of each.

*Proof.* Let $\forall x \in M$, let $C_x$ denote an open set containing $x$ such that Let $C = \{C_i \mid i \in I\}$ be an open cover of $M$. Then because $M$ is compact, there exists a finite subcover $C' = \{C_i \mid i \in I'\}$, where $I'$ is finite. (Unfinished) ■