Homology Theory
Notes & Exercises from my Independent Study

(Or: *If I could save Klein in a bottle ♪*)

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Last Updated: January 29, 2019

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Introduction

What's this?

This document is a compendium of notes, exercises, and other miscellany from my independent study in Homology Theory. For this, I am working through the second half of *Topology Through Inquiry* by Michael Starbird and Francis Su (i.e., chapters 11-20), under supervision from Prof. Su himself. Rough topic coverage should be discernable from the table of contents, as I've tried to name each section identically to the corresponding title in the book.

Notation

Most notation I use is fairly standard. Here's a (by no means exhaustive) list of some stuff I do.

- "WTS" stands for "want to show," s.t. for "such that." WLOG, as usual, is without loss of generality.
- End-of-proof things: ■ is QED for exercises and theorems. □ is used in recursive proofs (e.g., proving a Lemma within a theorem proof). If doing a proof with casework, ✓ will be used to denote the end of each case.
- ($\Rightarrow\Leftarrow$) means contradiction
- $\mathcal{T}(U)$ will denote the topology of a topological space U.
- $\mathcal{P}(A)$ is the powerset of A. I don't like using 2^A.
- $\twoheadrightarrow$ denotes surjection.
- $\hookrightarrow$ denotes injection.
- Thus, $\hookrightarrow\!\!\!\!\!\twoheadrightarrow$ denotes bijection.
- **Important:** I use $f^{\rightarrow}(A)$ for the image of A under f, and $f^{\leftarrow}(B)$ for the inverse image of B under f.
- $\sim$ and $\equiv$ are used for equivalence relations. $\cong$ is used to denote homeomorphism. $\simeq$ is for Homotopy equivalence.
- ϵ is for trivial elements (e.g., the trivial path), while ε is for small positive quantities.
- $\overline{U}$ denotes the closure of U, U° is the interior of U.
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Updates:

The current lay of the land (01/29/2019)

- It appears that *Topology Through Inquiry* is much more thorough than Kosniowski's *A First Course in Algebraic Topology* in its treatment of point-set topology. I suppose this should have been inferable from the title of the latter. Anyways, I think it'd be prudent to go back and do a quick survey of some selected topics from the first half of the book before going on to the second half. I've found that so far, even when I know most of the vocabulary involved in a problem statement in the second half, I'm just not quite comfortable with the process of putting all the pieces together. To me, this indicates a problem that could be fixed with maybe a short week of review.
- Speaking of the first half: so far, I've found all the exercises and concepts here to be very straightforward so far, partially owing to the fact that I've seen lots of the material already. I did most of chapter 3 between Sunday (1/27/2019) and today (1/29/2019). I found most of the problems fairly straightforward and progress was generally fast. Those solutions I chose to typeset are tabulated in

`first-half/solutions.pdf`. If difficulty is consistent throughout the book, then it would probably be feasible to get all the way through a selected subset of topics prior to next week, at which point I could attack the homology section with confidence.

- My current plan: do selected exercises from chapter 4 today and tomorrow. Thursday, do the same for chapter 5 (this chapter looks short). Friday, chapter 6 (also looks short, but might present some new material). Over the weekend, do 7.2, 7.4, 7.5, then all of chapter 8 (this shouldn't take *too* too long seeing as continuous functions were emphasized by Kosniowski, and I've done lots of the proofs of "is property X preserved by continuous functions" before), and parts of 9, 10. Start off the new week with a return to chapter 12, adjusting schedule if needed.

1. Chapter 12: Classification of 2-Manifolds

1.1 Examples of 2-Manifolds

12.1. Show that the torus $\mathbb{T}^2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$

Proof. Let $\sim$ be an equivalence relation on $\mathbb{R}^2$ defined by $\forall (x, y) \in \mathbb{R}^2$,

$$\begin{cases} (x, y) \sim (x, y+1), & \text{and} \\ (x, y) \sim (x+1, y) \end{cases}$$

then $\mathbb{T}^2 \cong (\mathbb{R}^2/\sim) \cong \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{R}^2/$. Note that $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, which, by similar reasoning is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$. ■

12.2. For a given number of holes, demonstrate that the n-holed torus where the holes are lined up is homeomorphic to an n-holed torus where the holes are arranged in a circle.
Note: for exercises like this that ask you to demonstrate a geometric homeomorphism, we are not askign you to define a formal homeomorphism — no equations are expected. Rather, it suffices to describe a process by which you would systematically distort one figure to look like the other.

Description: First, twist the lined-up-holes torus (stretching as needed) such that the holes rest on the vertices of a regular n-gon. Then, simply stretch the body outwards until a disk shape is achieved. □

Definition 1.1.1

Define the *projective plane* (also called the *real projective 2-space*), denoted $\mathbb{R}\mathrm{P}^2$, to be the space of all lines in $\mathbb{R}^3$ that pass through the origin. The basis for the topology is the collection of open cones with the cone point at the origin.

12.3.
(a) Show that $\mathbb{R}\mathrm{P}^2 \cong \mathbb{S}^2/\langle x \sim -x \rangle$, that is, the projective plane is homeomorphic to the 2-sphere with diametrically opposite points identified.
(b) Show that $\mathbb{R}\mathrm{P}^2$ is also homeomorphic to a disk with two edges on its boundary (called a *bigon*) identified.
(c) Show that the klein bottle can be realized as a square with certain edges identified.

Proof. **COME BACK TO THIS AFTER ASKING PROF. SU ABOUT THE LEVEL OF RIGOR EXPECTED HERE!**

(a) **Claim:** Take some arbitrary point $\mathbf{r}(t)$, and give it a parameterization by $\mathbf{r}_0(t) = t\hat{\mathbf{r_0}}$, with the orientation chosen arbitrarily. Now, take a parameterization of each of the other lines in $\mathbb{R}\mathrm{P}^2$ by $\mathbf{r}(t) = t\hat{\mathbf{r}}$ chosen such that $\langle \hat{\mathbf{r}}, \hat{\mathbf{r}}_\mathbf{0} \rangle \geq 0$. Then f given by $f(\mathbf{r}) = \hat{\mathbf{r}}$ is a homeomorphism.

Proof of claim: Let $U \in \mathcal{T}(\mathbb{R}\mathrm{P}^2)$, and let $\mathcal{B}$ be the basis described above. Since U is open, $\exists B = \{B_i \mid i \in I\} \subseteq \mathcal{B}$ s.t.

$$U = \bigcup_{i \in I} B_i.$$

Suppose that U contains no lines $\mathbf{r}(t)$ s.t. $\langle \hat{\mathbf{r}}, \hat{\mathbf{r}}_{\mathbf{0}} \rangle = 0$. Then

$$\begin{aligned} f^{\rightarrow}(U) &= f^{\rightarrow}\left(\bigcup_{i \in I} B_i \right) \\ &= \bigcup_{i \in I} f^{\rightarrow}(B_i) \end{aligned}$$

is a union of spherical caps.

(b) Take any two antipodal points, draw a great circle through them, and apply the result above.

(c)

■

Recall: A set $A \subset X$ is said to be *dense* iff $\overline{A} = X$. A topological space X is said to be *separable* iff X has a countable dense subset.

Definition 1.1.2: 1-manifold

A topological space is an *1-manifold* iff it is a separable metrizable space in which every point is in an open set homeomorphic to an open interval in $\mathbb{R}^1$.

12.4. Suppose M is a compact, connected 1-manifold. Then M is triangulable. That is, M is homeomorphic to a subset C of $\mathbb{R}^n$ consisting of a finite collection of straight line segments where any two segments of C are either disjoint or meet at an endpoint of each.

Proof. Let $\forall x \in M$, let C_x denote an open set containing x such that Let $C = \{C_i \mid i \in I\}$ be an open cover of M. Then because M is compact, there exists a finite subcover $C' = \{C_i \mid i \in I'\}$, where I' is finite. (Unfinished) ■