Homology Theory
Notes & Exercises from my Independent Study

(Or: *If I could save Klein in a bottle ♪*)

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Introduction

What's this?

This document is a compendium of notes, exercises, and other miscellany from my independent study in Homology Theory. For this, I am working through the second half of *Topology Through Inquiry* by Michael Starbird and Francis Su (i.e., chapters 11-20), under supervision from Prof. Su himself. Rough topic coverage should be discernable from the table of contents, as I've tried to name each section identically to the corresponding title in the book.

Notation

Most notation I use is fairly standard. Here's a (by no means exhaustive) list of some stuff I do.

- "WTS" stands for "want to show," s.t. for "such that." WLOG, as usual, is without loss of generality.
- End-of-proof things: $\blacksquare$ is QED for exercises and theorems. $\square$ is used in recursive proofs (e.g., proving a Lemma within a theorem proof). If doing a proof with casework, ✓ will be used to denote the end of each case.
- ($\Rightarrow\Leftarrow$) means contradiction
- $\mathcal{T}(U)$ will denote the topology of a topological space U.
- $\mathcal{P}(A)$ is the powerset of A. I don't like using 2^A.
- $\twoheadrightarrow$ denotes surjection.
- $\hookrightarrow$ denotes injection.
- Thus, $\hookrightarrow\!\!\!\!\rightarrow$ denotes bijection.
- **Important:** I use $f^{\rightarrow}(A)$ for the image of A under f, and $f^{\leftarrow}(B)$ for the inverse image of B under f.
- $\sim$ and $\equiv$ are used for equivalence relations. $\cong$ is used to denote homeomorphism. $\simeq$ is for Homotopy equivalence.
- ϵ is for trivial elements (e.g., the trivial path), while ε is for small positive quantities.
- $\overline{U}$ denotes the closure of U, U° is the interior of U.
-

Updates:

(01/29/2019)

Summary of previous week:

- It appears that *Topology Through Inquiry* is much more thorough than Kosniowski's *A First Course in Algebraic Topology* in its treatment of point-set topology. I suppose this should have been inferable from the title of the latter. Anyways, I think it'd be prudent to go back and do a quick survey of some selected topics from the first half of the book before going on to the second half. I've found that so far, even when I know most of the vocabulary involved in a problem statement in the second half, I'm just not quite comfortable with the process of putting all the pieces together. To me, this indicates a problem that could be fixed with maybe a short week of review.
- Speaking of the first half: so far, I've found all the exercises and concepts here to be very straightforward so far, partially owing to the fact that I've seen lots of the material already. I did most of

chapter 3 between Sunday (1/27/2019) and today (1/29/2019). I found most of the problems fairly straightforward and progress was generally fast. Those solutions I chose to typeset are tabulated in `first-half/solutions.pdf`. If difficulty is consistent throughout the book, then it would probably be feasible to get all the way through a selected subset of topics prior to next week, at which point I could attack the homology section with confidence.

- My current plan: do selected exercises from chapter 4 today and tomorrow. Thursday, do the same for chapter 5 (this chapter looks short). Friday, chapter 6 (also looks short, but might present some new material). Over the weekend, do 7.2, 7.4, 7.5, then all of chapter 8 (this shouldn't take *too* too long seeing as continuous functions were emphasized by Kosniowski, and I've done lots of the proofs of "is property X preserved by continuous functions" before), and parts of 9, 10. Start off the new week with a return to chapter 12, adjusting schedule if needed.

Takeway from meeting:

- Don't worry about chapter 5, 6, 7.4, 7.5.
- The course will probably do 5.1 and 5.2, maybe 6.1, 6.2, course will do 7.1 through 7.3, 8.1 through 8.5, 9.1 and 9.2, 10 mostly skipped
- Start in chapter 16! Then work way onwards

Hm

- Try and go through the theorems before the exercises. The theorems are more important. Try and do about 20 things per week, theorems are most important

1. Manifolds, Simplexes Complexes, and Triangulability: Building Blocks

1.1 Manifolds

We define some basic Euclidean sets for use in homeomorphisms.

Definition 1.1.1. The *n-dimensional cube*, denoted $\mathbb{D}^n$, is defined as

$$\begin{aligned}\mathbb{D}^n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1 \text{ for } i = 1, \dots, n\} \\ &= \underbrace{\overbrace{[0,1] \times [0,1] \times \cdots \times [0,1]}}_{n \text{ times}} \subset \mathbb{R}^n.\end{aligned}$$

Definition 1.1.2. The *standard n-ball*, denoted B^n, is

$$B^n = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n \bigm| x_1^2 + \cdots + x_n^2 \le 1\big\}.$$

Definition 1.1.3. The *standard n-sphere*, denoted $\mathbf{s}^n$, is

$$\mathbf{s}^n = \big\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \bigm| x_0^2 + \cdots + x_n^2 = 1\big\}.$$

note that here, our indices start at 0.

Definition 1.1.4. An *n-dimensional manifold* or *n-manifold* is a separable metric space M such that $\forall p \in M$, $\exists U \in \mathfrak{T}(M)$ s.t. $p \in U$ and $U \cong V \subset \mathbb{R}^n$.

15.8. If M is an n-manifold and U is an open subset of M, then U is also an n-manifold.

15.9. If M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold.

15.10. Let M^n be an n-dimensional manifold with boundary. Then ∂M^n is an $(n-1)$-manifold.

1.2 Simplicial Complexes

Definition 1.2.1 (Affine Independence)**.** Let $X = \{v_0, \dots, v_k\} \subset \mathbb{R}^n$. We say X is *affinely independent* if $\{v_1 - v_i, \dots, v_k - v_i\}$ is linearly independent for all v_i.

Example 1.2.1. $X = \big\{(0,1), (-\sqrt{3}/2, -1/2), (\sqrt{3}/2, -1/2)\big\}$ is affinely independent.

Definition 1.2.2 (Convex combination)**.** A *convex combination* of $v_0, \dots, v_k$ is a linear combination of these points whose coefficients are nonnegative and sum to 1.

Definition 1.2.3. A *k-simplex* is the set of all convex combinations of $k+1$ affinely independent points in $\mathbb{R}^n$. For affinely independent points $v_0, \dots, v_k$ in $\mathbb{R}^n$, $\{v_0 \cdots v_k\}$ denotes the k-simplex

$$\left\{ \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k \;\middle|\; \forall i - 1, \dots, k;\ 0 \le \lambda_i \le 1 \text{ and } \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

each v_i is called a *vertex* of $\{v_0 \cdots v_k\}$. Any point x in the k-simplex is specified uniquely by the $k+1$ coefficients (λ_i); these coefficients are called the *barycentric coordinates of x*. The *barycentric coordinate of x with respect to vertex v_i* is the coefficient λ_i.

Definition 1.2.4. Any simplex τ whose vertices are a nonempty subset of the vertices of a k-simplex σ is called a *face* of σ. If the number of vertices is $i+1$, then τ has *dimension* i and is called an i-face of σ and τ has *codimension* $k-i$, the number of dimensions below the top dimension.

Notational Note: if $\sigma = \{v_0 \cdots v_k\}$, the $(k-1)$-dimensional face of σ obtained by deleting the vertex v_j from the list of vertices of σ is denoted by $\{v_0 \cdots \widehat{v}_i \cdots v_k\}$.

15.11. Show that if σ is a simplex and τ is one of its faces, then $\tau \subset \sigma$.

Solution. This is fairly trivial, so we offer just a sketch. Suppose $\mathbf{v} \in \tau$. Then write $\mathbf{v}$ as an element of σ by taking $\lambda_i = 0$ for all those $v_i \notin \tau$. ■

Definition 1.2.5. A *simplicial complex K* (in $\mathbb{R}^n$) is a collection of simplicies in $\mathbb{R}^n$ satisfying the following conditions.

1. If a simplex σ is in K, then each face of σ is also in K.
2. Any two simplices in K are either disjoint or their intersection is a face of each.

15.13. Exhibit a collection of simplices that satisfies condition (1) but not condition (2) in the definition of a simplicial complex.

Solution. Consider the following diagram, where the interior of each simplex is taken to be in the complex.

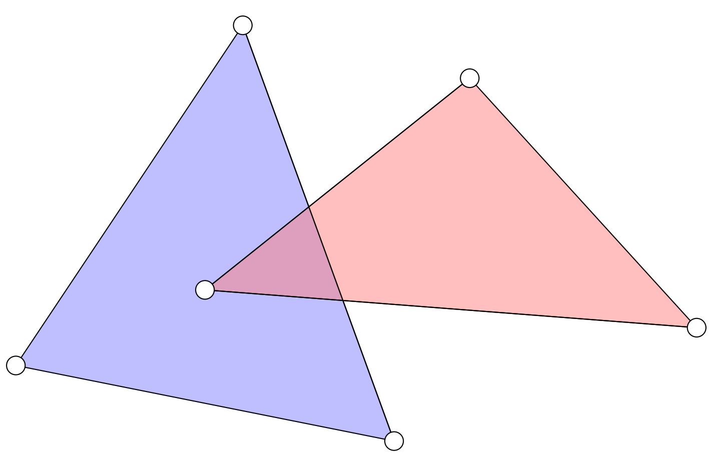

Figure 1.1: An unfortunate collision

Note that to fix this sorry situation, we can't just add two vertices at the points of intersections of the lines above (then the intersection of the resulting simplex with the two shone above would be non-trivial, but still not a face of the larger ones). We'd actually need something much more complicated.

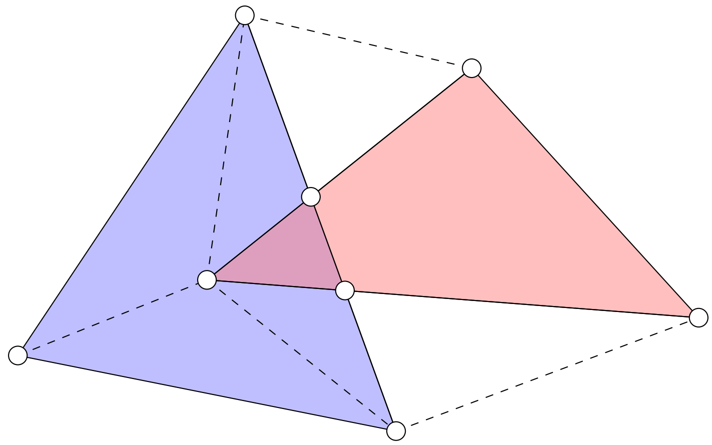

Figure 1.2: Constructing a resolution

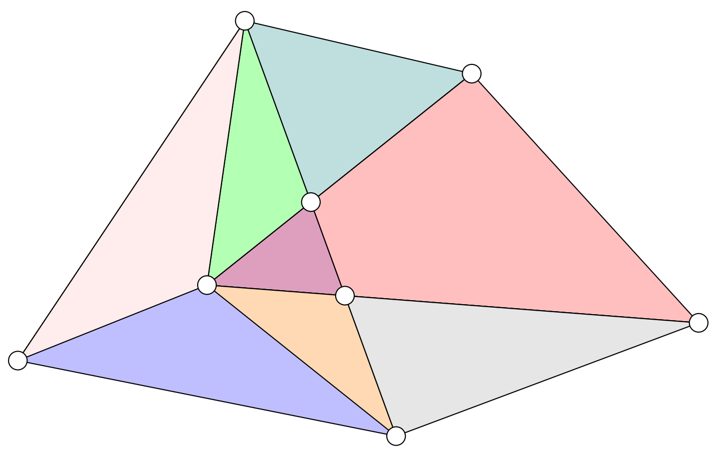

Figure 1.3: The completed resolution

■

Definition 1.2.6. The *underlying space* $|K|$ of a simplicial complex K is the set

$$|K| = \bigcup_{\sigma \in K} \sigma,$$

the union of all simplices in K, with a topology consisting of sets whose intersection with each simplex $\sigma \in K$ is open in σ. For finite simplicial complexes, this topology is the topology inherited as a subspace of $\mathbb{R}^n$.

15.14. Let K be the following simplicial complex:

(Omitted because it takes a long time to TeX out)

draw K and its underlying space.

Solution.

Figure 1.4: K (left) and its underlying space (right).

■

Definition 1.2.7. A topological space X is said to be *triangulable* if it is homeomorphic to the underlying space of a simplicial complex K. In that case, we say K is a *triangulation* of X.

15.15. Show that the space shown in Figure 15.2 (not included here) is triangulable by exhibiting a simplicial complex whose underlying space it is homeomorphic to.

Solution.

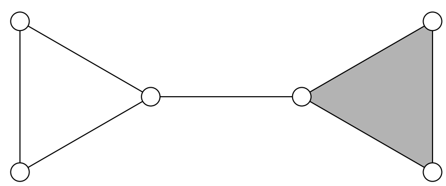

Figure 1.5: Such a simplicial complex. Note, the left triangle is unfilled.

■

15.6. For each $n \in \mathbb{N}$, $\mathbf{s}^n$ is triangulable.

Proof. We proceed by induction.

Base Case: Note that S^0 is trivially triangulable by taking $K = \{\{v_0\}, \{v_2\}\}$.

Inductive Hypothesis Suppose that for $k \in \mathbb{N} \cup \{0\}$, $\mathbf{s}^k$ is triangulable by a simplicial complex K.

Inductive Step: Take $v_{k+1} \in \mathbb{R}^{k+1}$ such that $v_{k+1} \in (\text{span}(K))^\perp$. Then

This proof is unfinished. Hey, future Forest — you should return to this later! ■

1.3 Simplicial Maps and PL Homeomorphisms

We now define structure-preserving maps bewteen simplicial concepts.

Definition 1.3.1. Let X, Y be topological spaces. A function $f : X \to Y$ is called a *simplicial map* iff there exist simplicial complexes K and L such that $|K| = X$, $|L| = Y$, and f maps each simplex of K linearly onto a (possibly lower-dimensional) simplex in L.

Definition 1.3.2. A simplicial map f is a simplicial homomorphism iff it's a bijection; in that case, the two complexes are *simplicially homeomorphic*

15.17. A simplicial map from K to L is determined by the images of the vertices of K.

Solution. Apply linearity and show the analog of images of liner combinations being uniquely determined by the action on the basis. ■

15.18. A composition of simplicial maps is a simplicial map.

Solution. Simply plug in arbitrary points and verify the properties hold. ■

Definition 1.3.3. Let K be a simplicial complex. Then a simplicial complex K' is a *subdivision* of K iff each simplex of K' is a subset of a simplex of K and each simplex of K is the union of finitely many simplices of K'.

Definition 1.3.4. If K and L are complexes, a continuous map $f : |K| \to |L|$ is called *piecewise linear* or *PL* if and only if there are subdivisions K' of K and L' of L such that f is a simplicial map from K' to L'. If there exist subdivisions such that f is a simplicial homomorphism, then f is a *PL homomorphism* and the spaces are *PL homeomorphic.*

15.21. A composition of PL maps is PL. A PL homeomorphism is an equivalence relation.

15.22. PL homeomorphic complexes are homeomorphic as topological spaces.

2. Simplicial $\mathbb{Z}_2$-Homology: Physical Algebra

2.1 Intro

This chapter, we'll talk about *homology*, which captures holes in a much more satisfying way than higher homotopy groups do.

Remark. Although not exactly accurate, a good way to start to understand homology for a space X is to view an n-manifold in X that is not the boundary of an $(n+1)$-manifold-with-boundary as capturing some geometry of X while an n-manifold that is the boundary of an $(n+1)$-dimensional manifold-with-boundary is not detecting any hole or structure.

2.2 Chains, Cycles, Boundaries, and the Homology Groups

Definition 2.2.1. An *n-chain* of K is a finite formal sum

$$\sum_{i=1}^{k} \sigma_i$$

of distinct n-simplices in K. Note that the dimensions of the simplices must be the same. So *chain* will mean n-chain whenever the dimension is either unimportant or understood.

Definition 2.2.2. The *n-chain group of K* (with coefficients in $\mathbb{Z}/2\mathbb{Z}$), denoted $\mathsf{C}_n(K)$, is the collection of n-chains in K under formal addition modulo 2. If there are no n-simplices in K, the n-chain group of K is defined to be trivial (containing the "empty" chain).

16.1. Check that $\mathsf{C}_n(K)$ is an abelian group.

Solution.

(1) $\epsilon = \sum_{i\in\varnothing} \sigma_i$.

(2) Associativity inherited from $\cup$.

(3) Closure inherited from $\cup$ over the domain given.

(4) Existence of inverses — since we're taking formal linear combinations over $\mathbb{Z}/2\mathbb{Z}$, then every element is its own inverse.

Finally, to see that $\mathsf{C}_n(K)$ is abelian, observe that $+$ in $\mathsf{C}_n(K)$ inherits commutativity from $\cup$. ■

Definition 2.2.3. The *$\mathbb{Z}/2\mathbb{Z}$-boundary of an n-simplex* $\sigma = \{v_0 \cdots v_n\}$ is defined by

$$\partial\sigma = \sum_{i=0}^{n} \{v_0 \cdots \widehat{v}_i \cdots v_n\}$$

the formal sum of the $(n-1)$-faces of σ.

For a 0-simplex, the $\mathbb{Z}/2\mathbb{Z}$ boundary is defined to be $0 \in \mathsf{C}_{-1}(K)$.

Definition 2.2.4. The $\mathbb{Z}/2\mathbb{Z}$ *boundary of an n-chain* is the sum of the boundaries of the simplices. That is, $\partial_n : \mathsf{C}_n(K) \to \mathsf{C}_{n-1}(K)$ is given by

$$\partial\left(\sum_{i=1}^{k} \sigma_i\right) = \sum_{i=1}^{k} \partial(\sigma_i)$$

> **16.2.** Verify that ∂ is a homomorphism, and use the definition to compute the $\mathbb{Z}/2\mathbb{Z}$ boundary of $\sigma_1 + \sigma_2$ in Figure 16.1

Solution. We want to show ∂ is a homomorphism.

(a) Let $\epsilon_n \in \mathsf{C}_n(K)$ be identity. We want to show $\partial(\epsilon_n) = \epsilon_{n-1}$. Taking the empty sum to be identity, we see

$$\begin{aligned}\partial(\epsilon_n) &= \partial\left(\sum_{i\in\varnothing} \sigma_i\right) \\ &= \sum_{i\in\varnothing} \partial(\sigma_i) \\ &= \epsilon_{n-1}\end{aligned}$$

as desired.

(b) That ∂ respects addition is definitional.

We have $\partial(\sigma_1 + \sigma_2) = e_1 + e_2 + e_4 + e_5$. ■

Definition 2.2.5. An *n-cycle* is an n-chain of K whose boundary is zero. The set of all n-cycles on K is denoted $\mathsf{Z}_n(K)$. An *n-boundary* is an n-chain that is the boundary of an $(n+1)$-chain of K. The set of all n-boundaries is denoted $\mathsf{B}_n(K)$.

> **16.4.** Both $\mathsf{Z}_n(K)$ and $\mathsf{B}_n(K)$ are subgroups of $\mathsf{C}_n(K)$. Moreover,
>
> $$\partial \circ \partial = 0.$$
>
> In other words, $\partial_n \circ \partial_{n+1} = 0$ for each index $n \geq 0$. Hence, $\mathsf{B}_n(K) \subset \mathsf{Z}_n(K)$.

Solution. Let $\sigma_1, \sigma_2 \in \mathsf{Z}_n(K)$. Then by linearity of ∂_n, we have

$$\begin{aligned}\partial_n(\sigma_1 + \sigma_2) &= \partial_n(\sigma_1) + \partial_n(\sigma_2) \\ &= 0\end{aligned}$$

and hence $\mathsf{Z}_n(K) < \mathsf{C}_n(K)$.

Now, let $\sigma_1, \sigma_2 \in \mathsf{B}_n(K)$. Then $\exists \tau_1, \tau_2 \in \mathsf{Z}_{n+1}(K)$ such that $\partial_{n+1}(\tau_1) = \sigma_1, \partial_{n+1}(\tau_2) = \sigma_2$. Since $\mathsf{Z}_{n+1}(K) < \mathsf{C}_{n+1}(K)$, then $\tau_1 + \tau_2 \in \mathsf{Z}_{n+1}(K)$. Now, by linearity of ∂, we have

$$\begin{aligned}\partial_{n+1}(\tau_1 + \tau_2) &= \partial_{n+1}(\tau_1) + \partial_{n+1}(\tau_2) \\ &= \sigma_1 + \sigma_2\end{aligned}$$

hence $\mathsf{B}_n(K)$ is a subset closed under the operation, so we have $\mathsf{B}_n(K) < \mathsf{C}_n(K)$. ■

Definition 2.2.6. Two n-cycles α and β in K are *equivalent* or *homologous* iff $\alpha - \beta = \partial(\gamma)$ for some $(n+1)$-chain γ. In other words, α and β are homologous iff they differ by an element of the subgroup $\mathsf{B}_n(K)$, denoted by

$$\alpha \sim_{\mathbb{Z}/2\mathbb{Z}} \beta.$$

The equivalence class of α is denoted by enclosing it in brackets thusly: $[\alpha]$. For $\mathbb{Z}/2\mathbb{Z}$ n-chains, observe that $\alpha - \beta = \alpha + \beta$. So we see that two n-cycles are equivalent if together they bound an $(n+1)$-chain.

16.5. List all the equivalence classes of 0-cycles, 1-cycles, and 2-cycles in the complex in Figure 16.1.

Solution. maybe the next exercise will be more informative. ■

Definition 2.2.7. The n^{th}*-homology group* (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) of a finite simplicial complex K, denoted $\mathsf{H}_n(K)$, is the additive group whose elements are equivalence classes of cycles under the $\mathbb{Z}/2\mathbb{Z}$-equivalence defined above, with $[\alpha] + [\beta] = [\alpha + \beta]$. I.e.,

$$\mathsf{H}_n(K) = \mathsf{Z}_n(K)/\mathsf{B}_n(K)$$

F1. Consider the simplicial complex given below in Figure (2.1). Then for $n = 0, 1, 2$,
(a) describe elements of $\mathsf{C}_n(K)$,
(b) compute $\mathsf{Z}_n(K)$,
(c) compute $\mathsf{B}_n(K)$, and
(d) compute $\mathsf{H}_n(K)$.

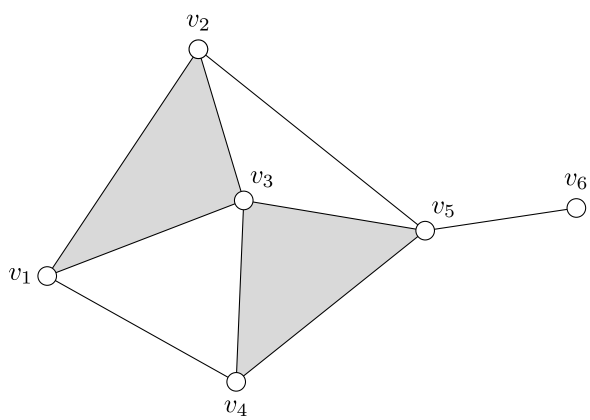

Figure 2.1: Simplicial complex K

Solution. First, we redraw the simplicial complex as follows:

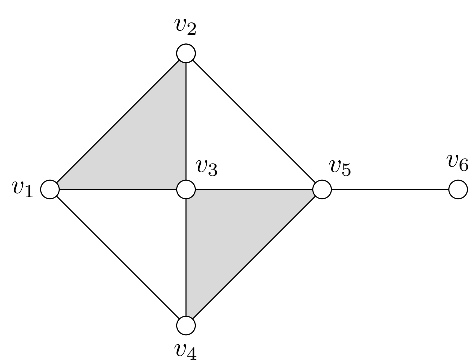

Figure 2.2: Simplicial complex K, straightened out

For the purposes of this problem, take angled brackets indicate span. We have

(i) We calculate the $k = 0$ case.

(a) Elements of $\mathsf{C}_0(K)$ are formal linear combinations over the set $\{v_1, v_2, \ldots, v_6\}$. Then

$$\mathsf{C}_0(K) = \left\langle v_1, v_2, v_3, v_4, v_5, v_6 \right\rangle$$

that is, collections of points in $\mathsf{C}_0(K)$.

(b) Let $\sigma_1, \ldots, \sigma_k \in \mathsf{C}_0(K)$. Then by definition,

$$\begin{aligned}\partial\left(\sum_{i=1}^{k} \sigma_i\right) &= \sum_{i=1}^{k} \partial(\sigma_i) \\ &= \sum_{i=1}^{k} 0 \\ &= 0\end{aligned}$$

hence $\mathsf{Z}_n(K) = \mathsf{C}_n(K)$.

(c) A $\sigma \in \mathsf{C}_0(K)$ is an n-boundary if $\exists \tau \in \mathsf{C}_1(K)$ with $\partial(\tau) = \sigma$. Note, for any 1-dimensional face $\{v_i v_j\} \in K$,

$$\begin{aligned}\partial(\{v_i v_j\}) &= \{v_i \widehat{v_j}\} + \{\widehat{v_i} v_j\} \\ &= \{v_i\} + \{v_j\} \\ &= \delta_{ij}.\end{aligned}$$

Hence, any edge formed of a pair of two distinct vertices yields a nonempty boundary. We first count all edges:

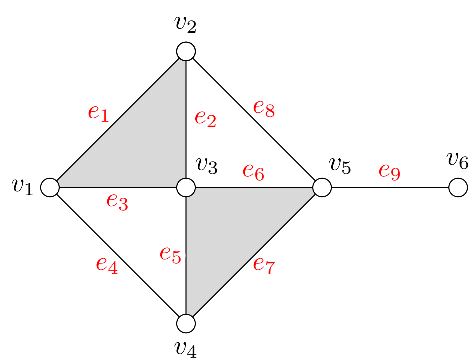

Figure 2.3: Simplicial complex K with simple edges

Since $\mathsf{B}_0(K)$ is a subgroup of $\mathsf{C}_0(K)$, by closure under $+$, we see that any $v_i + v_j$ in K such that there exists a path from v_i to v_j (when K is considered a graph) is an element of $\mathsf{B}_0(K)$. In fact, we can say more:

Claim: Since K is connected as a graph, any even collection of vertices is in $\mathsf{B}_n(K)$.

Proof of Claim: Suppose we have $\sigma = \{v_{i_1}\} + \{v_{i_2}\} + \cdots + \{v_{i_{2k}}\}$, where $k \in \mathbb{N}$. Then for each $j = 1, \ldots, k$, let τ_j be a sum of edges representing a path from v_{i_j} to $v_{i_{j+1}}$. For example, if $v_{i_j} = v_6$ and $v_{i_{j+1}} = v_2$, we could take the following approaches:

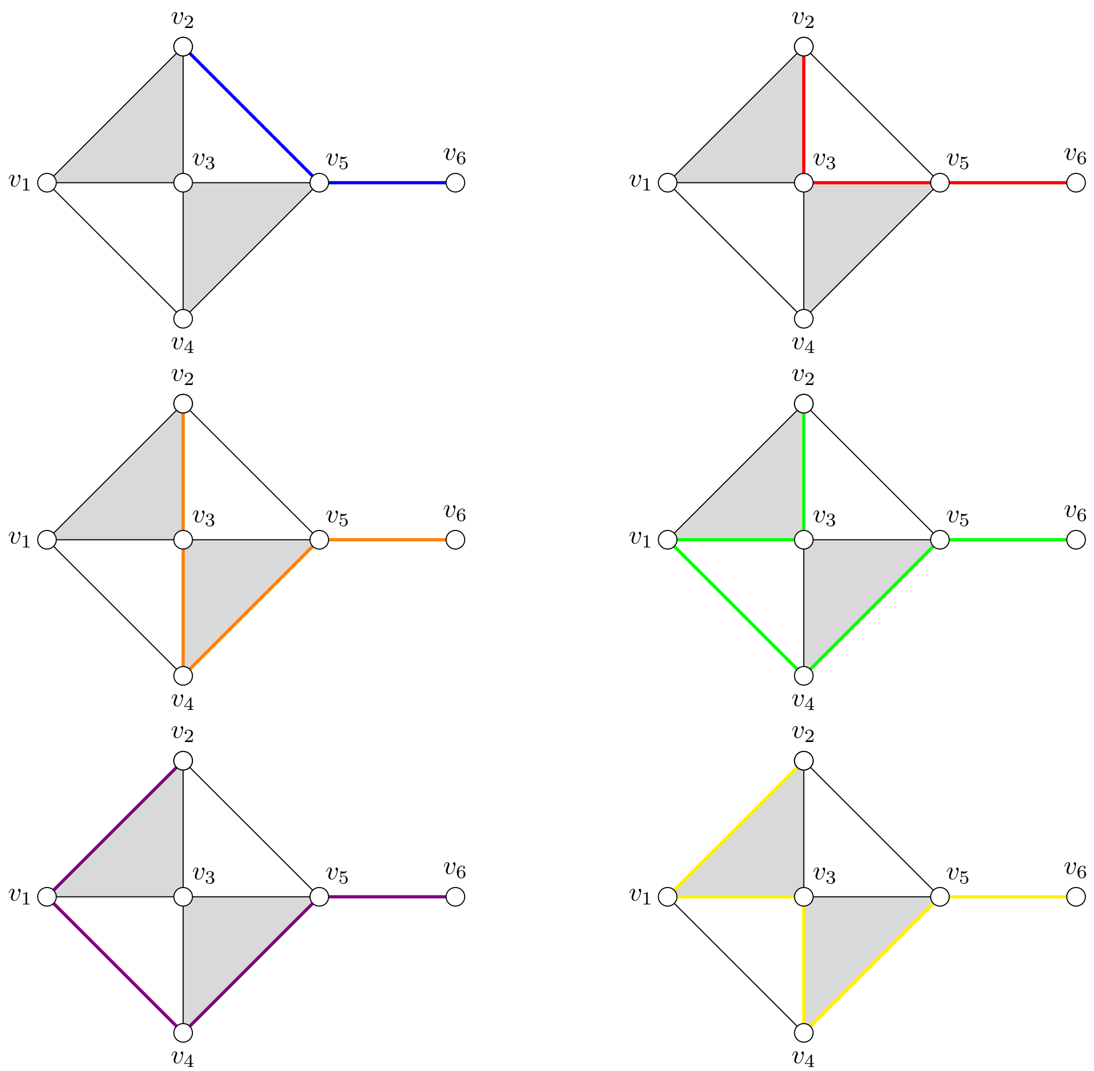

Figure 2.4: Some paths from v_6 to v_2

among others. Taking the sum of the constituent edges in each path yields a sum of 1-simplices with boundary v_6, v_2.[1]

(d) Since $\mathsf{B}_n(K)$ is the group of all collections of even vertices in $\mathsf{C}_n(K)$, we have $\mathsf{H}_n(K) = \mathsf{C}_n(K)/\mathsf{B}_n(K) \cong \mathbb{Z}/2\mathbb{Z}$.

(ii) Now, we calculate the $k = 1$ case.

(a) Elements of $\mathsf{C}_1(K)$ are collections of linear combinations of the edges

$$\mathsf{C}_1(K) = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9 \rangle$$

(b) Elements of $\mathsf{Z}_1(K)$ are collections of edges such that each vertex contained in an edge in the collection has even degree. This corresponds to cyclic subgraphs of K (as well as the empty cycle), e.g.:

[1] Justification: note that the coefficient on any given vertex when we apply ∂ is the degree of the vertex in our path. Hence, only the initial and terminal vertex don't get mapped to 0.

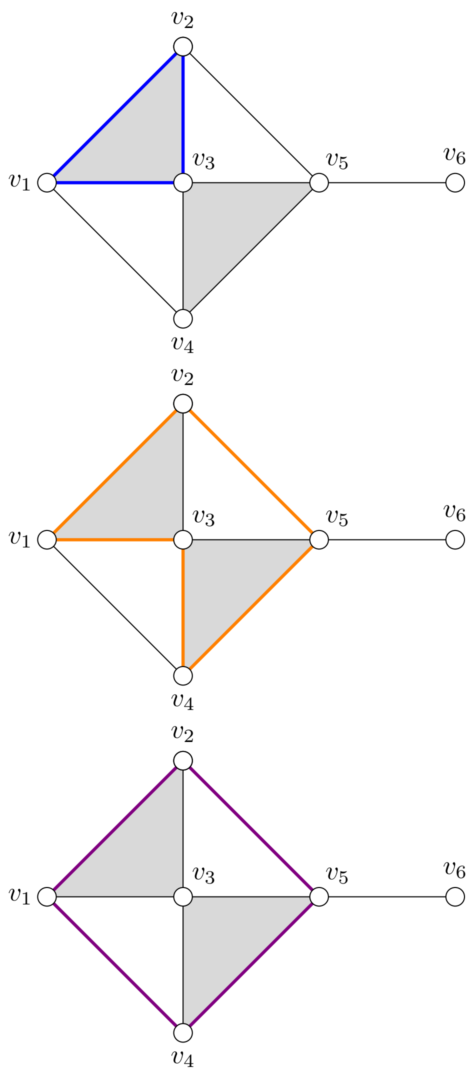

Figure 2.5: Some cycles in K

(c) First, consider the following diagram:

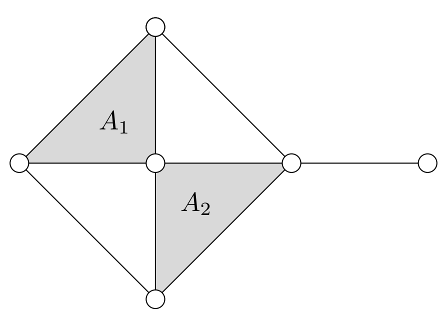

Figure 2.6: Two $n = 2$ simplices

$\mathbf{0}_1$ bounds $\mathbf{0}_2$. Since $\partial(A_1) \cap \partial(A_2) = \varnothing$, then the other two cycles in $\mathsf{B}_1(K)$ are just

$\partial(A_1)$ and $\partial(A_2)$, respectively.

(d) $\mathsf{H}_1(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (equivalence classes have representative elements $\mathbf{0}, \partial(A_1), \partial(A_2), \partial(A_1) + \partial(A_2)$)

■

16.7. If K is a one-point space, $\mathsf{H}_n(K) \cong 0$ for $n \geq 0$, and $\mathsf{H}_0(K) \cong \mathbb{Z}$.

Solution. For $n > 0$, $\mathsf{C}_n(K)$ is the trivial group, and so $\mathsf{H}_n(K) \cong 0$. ■