Solutions to Toplogy Through Inquiry

(*My very own home-cooked grading key!*)

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1. Introduction

2. Sets and Cardinality

2.8. Prove that every subset of $\mathbb{N}$ is either finite or has the same cardinality as $\mathbb{N}$.

Proof. We first prove a small Lemma.

Lemma Let X be an infinite set. Then $\forall x \in X$, $X - \{x\}$ is infinite.

Proof of Lemma: Suppose, to obtain a contradiction, that $\exists x_0 \in X$ s.t. $X - \{x_0\}$ is finite. Then $\exists n \in \mathbb{N}$ s.t. $\exists f : X - \{x_0\} \hookrightarrow\!\!\!\!\rightarrow \{1, \ldots, n\}$. Then construct $f' : X \hookrightarrow\!\!\!\!\rightarrow \{1, \ldots, n\}$ by taking

$$f'(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ n+1 & \text{if } x = x_0 \end{cases}$$

and note that f' is a bijection. It follows that X is finite, a contradiction. Thus, $\forall x \in X$, $X - \{x\}$ is infinite.

Main Proof: Let $X \subseteq \mathbb{N}$. Suppose that X is finite. Then we're done. Now, suppose X is infinite. WTS $|X| = |\mathbb{N}|$. We will apply the Well-Ordering Principle to construct $f : X \hookrightarrow\!\!\!\!\rightarrow \mathbb{N}$ as follows: let $X_1 = X$. Since X_1 is a nonempty set of natural numbers, there exists a least element $x_1 = \inf(X_1)$. Define $f(1) = x_1$. Now, take $X_2 = X_1 - \{x_1\}$. By the Lemma, X_2 is infinite as well. Applying a similar process, define $f(x_2) = \inf(X_2)$, and take $X_3 = X_2 - \{x_2\}$. In general $f(n+1) = \inf(X_{n+1}) = \inf(X_n - \{x_n\})$. By the lemma, we can continue this process indefinitely, yielding a bijection $f : X \hookrightarrow\!\!\!\!\rightarrow \mathbb{N}$. It follows that every subset of $\mathbb{N}$ is either finite or has the same cardinality as $\mathbb{N}$. ■

2.9. Every infinite set has a countably infinite subset.

Proof. Let X be an infinite set. Suppose X is countable. Then $X \subset X$ is a countably infinite subset, as desired. Now suppose X is uncountable. We apply the axiom of choice. Let $g : \mathcal{P}(X) - \{\varnothing\} \to X$ be a choice function. That is, $\forall S \subseteq X$ s.t. $S \neq \varnothing$, $g(S) \in S$. Then define $f : \mathbb{N} \hookrightarrow X$ as follows:

$$f(n) = \begin{cases} g(X) & \text{if } n = 1 \\ g\left(X - \bigcup_{i=1}^{n-1} \{f(i)\}\right) & \text{if } n > 1 \end{cases}$$

We claim f is injective. To see this, suppose, to obtain a contradiction, that f is not injective. Then $\exists n, m \in \mathbb{N}$ s.t. $f(n) = f(m)$, and $n \neq m$. WLOG, suppose $n > m$. Then by definition,

$$f(m) = f(n) \in X - \bigcup_{i=1}^{n-1} \{f(i)\}.$$

but $m \in \{1, \ldots, n-1\}$, hence $f(m) \in \bigcup_{i=1}^{n=1} \{f(i)\}$ and $f(m) = f(n) \notin X - \bigcup_{i=1}^{n-1} \{f(i)\}$, a contradiction. Thus f is injective. It follows that $f(\mathbb{N})$ is a countable subset of X, as desired. ■

2.10. A set is infinite if and only if there is an injection from the set to a proper subset of itself.

Proof.

($\Rightarrow$) : Let X be an infinite set. Then there exists a countable subset $S = \{s_1, \ldots, s_n, \ldots\}$. Define $f : X \hookrightarrow X - \{s_1\}$ by

$$f(x) = \begin{cases} x & \text{if } x \notin S \\ \operatorname{succ}(x) & \text{if } x \in S. \end{cases}$$

and note that this is an injection to a proper subset of X, as desired.

($\Leftarrow$) : Let X be a set, and S a proper subset of X. Suppose there exists an injection $f : X \hookrightarrow S$. WTS X is infinite. Suppose, to obtain a contradiction, that X were finite. Then $\exists n, m \in \mathbb{N}$ s.t. $\exists g : X \hookrightarrow\!\!\!\!\rightarrow \{1, \ldots, n\}$, and $h : S \hookrightarrow\!\!\!\!\rightarrow \{1, \ldots, m\}$, where $m < n$. Apply the pigeonhole principle to $g \circ f = h$ to obtain a contradiction.

■

2.14. Prove that the set of all finite subsets of a countable set is countable.

Proof. Let X be a countable set, and consider the poset of finite subsets of X ordered by inclusion. One can show by induction that $\forall k \in \mathbb{N}$, the st of k-element subsets of $\mathbb{N}$ is countable. Apply this to obtain a bijection from the set of all finite subsets of X to $\mathbb{N} \times \mathbb{N}$, which can be shown to be countable in the usual way.

A much nicer proof is to first create a bijection from X to the set of all primes, and then multiply things together. ■

2.15. Suppose a submarine is moving in the plane along a straight line at a constant speed such that at each hour, the submarine is at a lattice point, that is, a point whose two coordinates are both integers. Suppose at each hour you can explode one depth charge at a lattice point that will hit the submarine if it is there. You do not know the submarine's direction, speed, or its current position. Prove that you can explode one depth charge each hour in such a way that you will be guaranteed to eventually hit the submarine.

Proof. Let the initial position of the submarine be given by $(x_0, y_0) \in \mathbb{Z} \times \mathbb{Z}$. Then $\exists v_x, v_y \in \mathbb{Z}$ s.t. the position of the submarine after t hours is given by

$$\mathbf{p}(t) = (v_x t + x_0, v_y t + y_0).$$

Note, $\mathbf{p}(t)$ is constrained by the parameters (v_x, x_0, v_y, y_0), hence our configuration space is simply $\mathbb{Z}^4$, which is countable. Hence we can simply try each possible configuration of $\mathbf{p}(t)$ sequentially, thus guaranteeing we'll eventually hit the submarine. ■

2.20. Let A be a set, and let P be the set of all functions $f : A \to \{0, 1\}$. Then $|P| = |\mathcal{P}(A)|$.

Proof. Trivial. Define a bijection by including things in a subset iff the function maps it to 1. ■

2.23. Consider $A = [0, 1]$ and $B = [0, 1)$, and consider injections $f : A \hookrightarrow B$, $g : B \hookrightarrow A$ defined by

$$f(x) = \frac{x}{3} \qquad g(x) = x$$

constuct a bijection $h : A \hookrightarrow\!\!\!\!\rightarrow B$ such that on some points of A, $h(x) = f(x)$, and for others, $h(x) = g^{-1}(x)$.

Proof. Note $g^{-1}(x) = g(x)$. Then let

$$h(x) = \begin{cases} f(x) & x \in \mathbb{Q} \\ g^{-1}(x) & x \notin \mathbb{Q} \end{cases}$$

and note that it works. ■

2.24. Consider A, B, and f as before, and let $g(x) = x/2$. Repeat the exercise above with this new definition of g.

Proof. Let h be given by

$$h(x) = \begin{cases} g^{-1}(x) & \text{if } x \in \bigcup_{n\in\mathbb{N}} \left(\frac{1}{6^n}, \frac{1}{2\cdot 6^{n-1}}\right] \\ f(x) & \text{if } x \in \bigcup_{n\in\mathbb{N}} \left(\frac{1}{2\cdot 6^{n-1}}, \frac{1}{6^{n-1}}\right] \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly, the union of the sets given in the three cases above covers $[0, 1]$. It remains to show that their images do as well. Note that

$$g^{-1}\left(\bigcup_{n\in\mathbb{N}} \left(\frac{1}{6^n}, \frac{1}{2\cdot 6^{n-1}}\right]\right) = \bigcup_{n\in\mathbb{N}} \left(\frac{1}{3\cdot 6^n}, \frac{1}{6^{n-1}}\right] \qquad f\left(\bigcup_{n\in\mathbb{N}} \left(\frac{1}{2\cdot 6^n}, \frac{1}{6}\right]\right) = \bigcup_{n\in\mathbb{N}} \left(\frac{1}{6^n}, \frac{1}{3\cdot 6^n}\right]$$

and that the union of the right-hand-sides together with $\{0\}$ yields $[0, 1]$. ■

2.25. Prove the Schroeder-Bernstein Theorem. That is, let A and B be sets. Suppose there exists injections $f : A \hookrightarrow B$, and $g : B \hookrightarrow A$. Then there exists a bijection $h : A \hookrightarrow\!\!\!\!\rightarrow B$.

3. Topological Spaces: Fundamentals

3.1. Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in a topological space $(X, \mathcal{T})$. Then

$$\bigcap_{i=1}^n U_i$$

is open.

Proof. Trivial proof by induction. ■

3.2. Why does your proof not prove the false statement that the infinite intersection of open sets is necessarily open?

Solution. Induction only proves that a claim holds for any natural number $n \in \mathbb{N}$. It does not show that the claim holds for $\aleph_0$. ■

3.3. Prove that a set U is open in a topological space $(X, \mathcal{T})$ if and only if for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$.

Proof. Let $U \subseteq X$.

($\Rightarrow$) : Suppose U is open. Let $x \in U$ be arbitrary. Then take $U_x = U$ to obtain the desired result.

($\Leftarrow$) : Suppose that $\forall x \in U$, there exists an open set U_x s.t. $x \in U_x \subset U$. Then let

$$V = \bigcup_{x \in U} U_x.$$

WTS $V = U$. Observe that $\forall x \in U$, $\exists U_x \subset V$ s.t. $U_x \ni x$ (by definition of V), hence $x \in U \implies x \in V$. Thus $U \subset V$. Now, since each $U_x \subset U$, we obtain $V \subset U$. Thus $V = U$.

This proves the claim. ■

3.4. Verify that $\mathcal{T}_{\text{std}}$ is a topology on $\mathbb{R}^n$; in other words, it satisfies the four conditions of the definition of a topology.

Proof.

1. It is vacuously true that $\forall x \in \varnothing$, $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \in \varnothing$, hence we have $\varnothing \in \mathcal{T}_{\text{std}}$.
2. Let $x \in \mathbb{R}^n$. Then $\forall \epsilon > 0$, we have $B_\epsilon(x) \subset \mathbb{R}^n$, thus $\mathbb{R}^n \in \mathcal{T}_{\text{std}}$.
3. Let $U, V \in \mathcal{T}_{\text{std}}$ be arbitrary, and let $W = U \cap V$. Let $x \in W$ be arbitrary. Then $x \in U$ and $x \in V$, hence $\exists \epsilon_U, \epsilon_V > 0$ s.t. $B_{\epsilon_U}(x) \subset U$ and $B_{\epsilon_V}(x) \subset V$. Let $\epsilon_x = \min\{\epsilon_U, \epsilon_V\}$. Then $B_{\epsilon_x}(x) \subset U$ and $B_\epsilon(x) \subset V$, and thus $B_{\epsilon_x}(x) \subset W$. Thus $W \in \mathcal{T}_{\text{std}}$, as desired.
4. Let $\{U_\alpha\}_{\alpha \in \lambda}$ be an arbitrary collection of open sets, and let $V = \bigcup_{\alpha \in \lambda} U_\alpha$. Let $x \in V$ be arbitrary. Then $\exists U_x \in \{U_\alpha\}_{\alpha \in \lambda}$ s.t. $x \in U_x$. Since $U_x \in \mathcal{T}_{\text{std}}$, $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subset U_x$. But $U_x \subset V$, hence $B_\epsilon(x) \subset V$. Since x was chosen to be arbitrary, this shows $V \in \mathcal{T}_{\text{std}}$.

Since $\mathcal{T}_{\text{std}}$ satisfies the topological axioms, we see that it indeed is a topology on $\mathbb{R}^n$. ■

3.5. Verify that the discrete, indiscrete, finite complement, and countable complement topologies are indeed topologies for any set X.

Proof.

(a) Discrete topology — trivial

(b) Indiscrete topology — trivial

(c) Finite complement topology (denoted here by $\mathcal{T}_{\text{fc}}$). Let $\{U_\alpha\}_{\alpha\in\lambda} \subset \mathcal{T}_{\text{fc}}$, and let $U, V \in \{U_\alpha\}_{\alpha\in\lambda}$. Then

1. Suppose $U = \varnothing$. By definition, we have $\varnothing \in \mathcal{T}_{\text{fc}}$, so axiom 1 holds.
2. Let $U = X$. Then $X - U = X - X = \varnothing$, which is finite. Thus $X \in \mathcal{T}_{\text{fc}}$.
3. Let $W = U \cap V$. Then by DeMorgan's Laws,

$$\begin{aligned} X - W &= X - U \cap V \\ &= (X - U) \cup (X - V). \end{aligned}$$

   But $U, V \in \mathcal{T}_{\text{fc}} \implies X - U, X - V$ are finite. Since the union of two finite sets is finite, we have $U \cap V \in T_{\text{fc}}$, as desired.

4. Let $W = \bigcup_{\alpha\in\lambda} U_\alpha$. Again, by DeMorgan's Laws, we have

$$X - \bigcup_{\alpha\in\lambda} U_\alpha = \bigcap_{\alpha\in\lambda} X - U_\alpha.$$

   Arbitrary intersections of finite sets are finite, hence we have $X - W$ is finite, whence $W \in \mathcal{T}_{\text{fc}}$.

Thus, $\mathcal{T}_{\text{fc}}$ is a topology, as desired.

(d) Countable complement (denoted here by $\mathcal{T}_{\text{cc}}$). Quantify all variables as above, replacing $\mathcal{T}_{\text{fc}}$ with $\mathcal{T}_{\text{cc}}$. The proofs are identical to those above, replacing "finite" with "countable."

■

3.6. Describe some of the open sets you get if $\mathbb{R}$ is endowed with the topologies described above (standard, discrete, indiscrete, finite complement, and countable complement). Specifically, identify sets that demonstrate the differences among these topologies, that is, find sets that are open in some topologies but not in others. For each of the topologies, determine if the interval $(0, 1)$ is an open set in that topology.

Proof. (a) In the discrete topology, $\{x\}$ is open for all x, while $(0, 1)$ is not. $\{x\}$ is not open in the other topologies listed.

(b) In the indiscrete topology, our only open sets are $\varnothing and X$.

(c) In the finite complement topology, $X - \{x\}$ is open, but it is not in the indiscrete topology. $(0, 1)$ is not open here.

(d) In the countable complement topology, $\mathbb{R} - \mathbb{Q}$ is open, but $(0, 1)$ is not.

■

3.7. Give an example of a topological space and a collection of open sets in that topological space that show that the infite intersection of open sets need not be open.

Proof. Endow $\mathbb{R}$ with the standard topology. Then

$$\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{2^n}, \frac{1}{2^n}\right) = \{0\}$$

is an infinite intersection of open sets yielding a set that is not open. ∎

3.3 — Limit Points and Closed Sets

Definition 3.0.1

Let $(X, \mathfrak{T})$ be a topological space, A a subset of X, and p a point in X. Then p is a *limit point* of A iff $\forall U \in \mathfrak{T}$ s.t. $p \in U$, $(U - \{p\}) \cap A \neq \varnothing$. Notice p need not be in A

3.8. Let $X = \mathbb{R}$ and $A = (1, 2)$. Verify that 0 is a limit point of A in the indiscrete topology and the finite complement topology, but not in the standard topology nor the discrete topology on $\mathbb{R}$.

Proof. In the indiscrete topology, the only open set containing 0 is $\mathbb{R}$, and $\mathbb{R} \cap (0, 1) \neq \varnothing$. Hence, 0 is a limit point. In the case of the finite complement topology, let U be an arbitrary open set containing 0. Then $U = \mathbb{R} - X$, where X is finite. Take $X' = X \cup \{0\}$, and $U' = \mathbb{R} - X'$, and observe that X' is still finite, hence U' is open. ∎

3.9. Suppose $p \notin A$ in a topological space $(X, \mathfrak{T})$. Then p is not a limit point of A if and only if there exists a neighborhood U of p such that $U \cap A = \varnothing$.

Proof. p is not a limit point of A iff $\exists U \in \mathfrak{T}$ s.t. $p \in U, (U - \{p\}) \cap A = \varnothing \iff U \cap A = \varnothing$ (since $p \notin A$). ∎

Definition 3.0.2

Let $(X, \mathfrak{T})$ be a topological space, A be a subset of X, and p be a point in X. If $p \in A$ but p is not a limit point of A, then p is an *isolated point* of A.

3.10. If p is an isolated point of a set A in a topological space X, then there exists an open set U such that $U \cap A = \{p\}$.

Proof. Let $p \in A$, and suppose that p is not a limit point of A. Then $\exists U \in \mathfrak{T}$ s.t. $p \in U, (U - \{p\}) \cap A = \varnothing$. Then $U \cap A = \{p\}$. ∎

3.11. Give an example of sets A in various topological spaces $(X, \mathfrak{T})$ with

1. A limit point of A that is an element of A;
2. A limit point of A that is not an element of A;
3. An isolated point of A;
4. A point not in A that is not a limit point of A.

Proof.

1.

∎

3.12. Which sets are closed in a set X with
(a) The discrete topology?
(b) The indiscrete topology?
(c) The finite complement topology?
(d) The countable complement topology?

Proof. ∎

3.13. For any topological space $(X, \mathfrak{T})$ and $A \subset X$, $\overline{A}$ is closed. That is, for any set A in a topological space, $\overline{\overline{A}} = \overline{A}$.

Proof. Let ∎

3.14. Let $(X, \mathfrak{T})$ be a topological space. Then the set A is closed iff $X - A$ is open.

Proof.

($\Rightarrow$) : Suppose A is closed. Then A contains all of its limit points. Hence, if $p \notin A$, p is not a limit point of A. Thus, $\forall p \in X - A,\ \exists U_p \in \mathfrak{T}$ s.t. $p \in U_p,\ U_p \cap A = \varnothing$. Hence, we have

$$X - A = \bigcup_{p \in X - A} U_p.$$

Since this is a union of open sets, it follows that $X - A$ is open.

($\Leftarrow$) : Suppose $(X - A)$ is open. Let p be a limit point of A, and suppose, to obtain a contradiction, that $p \notin A$. Then $p \in (X - A)$. Since $(X - A)$ is open and contains p, then by the definition of a limit point, $((X - A) - \{p\}) \cap A \neq \varnothing$. But $(X - A) - \{p\} \subset (X - A)$, and $(X - A) \cap A = \varnothing$, a contradiction. Thus $p \in A$. Since p was an arbitrary limit point of A, we thus have A contains all its limit points, so $\overline{A}$.

∎

3.15. Let $(X, \mathfrak{T})$ be a topological space, and let U be an open set and A be a closed subset of X. Then the set $U - A$ is open and the set $A - U$ is closed.

Proof.

(a) WTS $(U - A)$ is open. Let $p \in U$, and suppose that $p \notin A$. Then because A closed, it follows that p cannot be a limit point of A. Hence $\exists V_p \in \mathfrak{T}$ s.t. $p \in V_p$, $V_p \cap A = \varnothing$. V is open implies $U_p = V \cap U$ is open (finite intersection of open sets is open). Note that $U_p \subset U$, and $U_p \cap A = \varnothing$. Hence

(b)

∎

4. Bases, Subspaces, Products: Creating New Spaces

4.1 Bases

Definition 4.1.1

Let $\mathcal{T}$ be a topology, and $\mathcal{B} \subset \mathcal{T}$. Then call $\mathcal{B}$ a *basis* for $\mathcal{T}$ iff $\forall U \in \mathcal{T}$, $\exists \mathcal{B}_u = \{B_\alpha\}_{\alpha \in \lambda}$ such that

$$\bigcup_{\alpha \in \lambda} B_\alpha = U.$$

If $B \in \mathcal{B}$, we say B is a *basis element* or *basic open set.*

4.1. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ iff

1. $\mathcal{B} \subset \mathcal{T}$, and
2. $\forall U \in \mathcal{T}$ and $p \in U$, $\exists V \in \mathcal{B}$ s.t. $p \in V \subset U$.

Proof. Actually, both directions here are pretty trivial, so we'll omit proof. ■

4.2.

(a) Show that $\mathcal{B}_1 = \{(a, b) \subset \mathbb{R} \mid a, b \in \mathbb{Q}\}$ is a basis for the standard topology on $\mathbb{R}$.

(b) Let $\mathcal{B}_2 = \{(a, b) \cup (c, d) \subset \mathbb{R} \mid a < b < c < d$ are distinct irrational numbers$\}$. Show that this is also a basis for $\mathcal{T}_{\text{std}}$.

Proof. We offer a sketch.

(a) We show that $\mathcal{B}_1$ generates the standard basis on $\mathcal{T}_{\text{std}}$ as follows: first, we show that $\mathcal{B}_1$ generates at least the set of all (a, b) such that $a, b \in \mathbb{R} - \mathbb{Q}$. To do so, select any such (a, b), and take the infinite union of (a_k, b_k) where $(a_k)_{i=1}^{\infty}$ is a rational sequence converging to a, and similar for b. Once this is done, use closure under union again to get all (a, b) where only one of $\{a, b\}$ is rational. From this, we obtain the standard basis.

(b) Proceed by a similar idea to the above, this time using irrational sequences, and performing a union with some other open set at the end to "plug up" the hole in the middle.

■

4.3. Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a topology on X iff

(a) Each point of X is in some element of $\mathcal{B}$, and

(b) If U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.

Proof.

$(\Rightarrow)$: Suppose $\mathcal{B}$ is a basis for some topology on X. Then the first point follows trivially from the fact that $\mathcal{B}$ covers X. Now, let $U, V \in \mathcal{B}$, and $p \in U \cap V$. Observe that every element of $\mathcal{B}$ is open, hence U, V are open as well. Then $U \cap V$ is open, hence there must exist a subset $\mathcal{B}' \subset \mathcal{B}$ the union of whose elements is $(U \cap V)$. The claim follows by simply selecting an element of $\mathcal{B}'$ containing p.

($\Leftarrow$) : For the reverse direction, take an arbitrary open set Y. Build up a superset of Y by unioning elements of $\mathcal{B}$ together.

$\blacksquare$