Homology Theory
Notes & Exercises from my Independent Study

(Or: *If I could save Klein in a bottle ♪*)

Forest Kobayashi

Department of Mathematics
Harvey Mudd College

Supervised By
Francis Su[1]

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[1]Department of Mathematics, Harvey Mudd College

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Introduction

What's this?

This document is a compendium of notes, exercises, and other miscellany from my independent study in Homology Theory. For this, I am working through the second half of *Topology Through Inquiry* by Michael Starbird and Francis Su (i.e., chapters 11-20), under supervision from Prof. Su himself. Rough topic coverage should be discernable from the table of contents, as I've tried to name each section identically to the corresponding title in the book.

Notation

Most notation I use is fairly standard. Here's a (by no means exhaustive) list of some stuff I do.

- "WTS" stands for "want to show," s.t. for "such that." WLOG, as usual, is without loss of generality.
- End-of-proof things: $\blacksquare$ is QED for exercises and theorems. $\square$ is used in recursive proofs (e.g., proving a Lemma within a theorem proof). If doing a proof with casework, ✓ will be used to denote the end of each case.
- $(\Rightarrow\Leftarrow)$ means contradiction
- $\mathcal{T}(U)$ will denote the topology of a topological space U.
- $\mathcal{P}(A)$ is the powerset of A. I don't like using 2^A.
- $\twoheadrightarrow$ denotes surjection.
- $\hookrightarrow$ denotes injection.
- Thus, $\hookrightarrow\!\!\!\!\!\twoheadrightarrow$ denotes bijection.
- $\sim$ and $\equiv$ are used for equivalence relations. $\cong$ is used to denote homeomorphism. $\simeq$ is for Homotopy equivalence.
- ϵ is for trivial elements (e.g., the trivial path), while ε is for small positive quantities.
- $\overline{U}$ denotes the closure of U, U° is the interior of U.
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1. Chapter 12: Classification of 2-Manifolds

1.1 Examples of 2-Manifolds

12.1. Show that the torus $\mathbb{T}^2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$

Proof. Let $\sim$ be an equivalence relation on $\mathbb{R}^2$ defined by $\forall (x, y) \in \mathbb{R}^2$,

$$\begin{cases} (x, y) \sim (x, y+1), & \text{and} \\ (x, y) \sim (x+1, y) \end{cases}$$

then $\mathbb{T}^2 \cong (\mathbb{R}^2/\sim) \cong \mathbb{R}^2/\mathbb{Z}^2$. Now, let $\equiv$ be an equivalence relation on $\mathbb{R}$ defined by $\forall x \in \mathbb{R}, x \equiv x + 1$. Then $\mathbb{S}^1 \cong (\mathbb{R}/\equiv) \cong \mathbb{R}/\mathbb{Z}$. Then $\mathbb{S}^1 \times \mathbb{S}^1 \cong (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$. WTS $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}^2/\mathbb{Z}^2$.[1] ∎

[1] In general, $(X/\sim_X) \times (Y/\sim_Y) \cong (X \times Y)/(\sim_X \times \sim_Y)$ doesn't hold), but we'll see if we can make it happen here.