Appendix G
Polynomial Zonotope Arithmetic

We introduce several useful operations on polynomial zonotopes: interval conversions, set addition and multiplication, slicing, computing bounds, and set-valued function evaluation.

1) Interval Arithmetic: Let $\mathbb{IR}^n$ be the set of all real-valued n-dimensional interval vectors. The Minkowski sum and difference of $[x]$ and $[y]$ are

$$[x] \oplus [y] = [\underline{x}+\underline{y}, \overline{x}+\overline{y}], \tag{104}$$

$$[x] \ominus [y] = [\underline{x}-\overline{y}, \overline{x}-\underline{y}]. \tag{105}$$

The product of $[x]$ and $[y]$ is

$$[x][y] = \left[\min\left(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right), \max\left(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right)\right]. \tag{106}$$

The i^{th} and j^{th} entry of the product of an interval matrix $[Y]$ multplied by an interval matrix $[X]$ is

$$([X][Y])_{ij} = \bigoplus_{k=1}^{n}([X]_{ik}[Y]_{kj}), \tag{107}$$

where n is the number of columns of $[X]$ and number of rows of $[Y]$. Given interval vectors $[x],[y] \subset \mathbb{R}^3$, their cross product is

$$[x] \otimes [y] = [x]^{\times}[y], \tag{108}$$

where $[x]^{\times}$ is the skew-symmetric matrix representation of $[x]$.

2) Interval Conversion: Intervals can also be written as polynomial zonotopes. For example, let $[z] = [\underline{z}, \overline{z}] \subset \mathbb{R}^n$, then one can convert $[z]$ to a polynomial zonotope $\mathbf{z}$ using

$$\mathbf{z} = \frac{\overline{z}+\underline{z}}{2} + \sum_{i=1}^{n} \frac{\overline{z}_i - \underline{z}_i}{2} x_i, \tag{109}$$

where $x \in [-1,1]^n$ is the indeterminate vector.

3) Set Addition and Multiplication: The Minkowski Sum of two polynomial zonotopes $\mathbf{P}_1 \subset \mathbb{R}^n = \mathcal{PZ}(g_i, \alpha_i, x)$ and $\mathbf{P}_2 \subset \mathbb{R}^n = \mathcal{PZ}(h_j, \beta_j, y)$ follows from polynomial addition:

$$\mathbf{P}_1 \oplus \mathbf{P}_2 = \left\{ z \in \mathbb{R}^n \mid z = \sum_{i=0}^{n_g} g_i x^{\alpha_i} + \sum_{j=0}^{n_h} h_j y^{\beta_j} \right\}. \tag{110}$$

Similarly, we may write the matrix product of two polynomial zonotopes $\mathbf{P}_1$ and $\mathbf{P}_2$ when the sizes are compatible. Letting $\mathbf{P}_1 \subset \mathbb{R}^{n \times m}$ and $\mathbf{P}_2 \subset \mathbb{R}^{m \times k}$, we obtain $\mathbf{P}_1\mathbf{P}_2 \subset \mathbb{R}^{n \times k}$:

$$\mathbf{P}_1\mathbf{P}_2 = \left\{ z \in \mathbb{R}^{n \times k} \mid z = \sum_{i=0}^{n_g} g_i (\sum_{j=0}^{q} h_j y^{\beta_j}) x^{\alpha_i} \right\}. \tag{111}$$

When $\mathbf{P}_1 \subset \mathbb{R}^{n \times n}$ is square, exponentiation $\mathbf{P}_1^m$ may be performed by multiplying $\mathbf{P}_1$ by itself m times. Furthermore, if $\mathbf{P}_1 \subset \mathbb{R}^3$ and $\mathbf{P}_2 \subset \mathbb{R}^3$, we implement a set-based cross product as matrix multiplication. We create $\mathbf{P}_1^{\times} \subset \mathbb{R}^{3 \times 3}$ as

$$\mathbf{P}_1^{\times} = \left\{ A \in \mathbb{R}^{3 \times 3} \mid A = \sum_{i=0}^{n_g} \begin{bmatrix} 0 & -g_{i,3} & g_{i,2} \\ g_{i,3} & 0 & -g_{i,1} \\ -g_{i,2} & g_{i,1} & 0 \end{bmatrix} x^{\alpha_i} \right\} \tag{112}$$

where $g_{i,j}$ refers to the j^{th} element of g_i. Then, the set-based cross product $\mathbf{P}_1 \otimes \mathbf{P}_2 = \mathbf{P}_1^{\times}\mathbf{P}_2$ is well-defined.

4) Slicing: One can obtain subsets of polynomial zonotopes by plugging in values of known indeterminates. For instance, if a polynomial zonotope $\mathbf{P}$ represented a set of possible positions of a robot arm operating near an obstacle. It may be beneficial to know whether a particular choice of $\mathbf{P}$'s indeterminates yields a subset of positions that could collide with the obstacle. To this end, "slicing" a polynomial zonotope $\mathbf{P} = \mathcal{PZ}(g_i, \alpha_i, x)$ corresponds to evaluating an element of the indeterminate x. Given the j^{th} indeterminate x_j and a value $\sigma \in [-1,1]$, slicing yields a subset of $\mathbf{P}$ by plugging σ into the specified element x_j:

$$\texttt{slice}(\mathbf{P}, x_j, \sigma) \subset \mathbf{P} = \left\{ z \in \mathbf{P} \mid z = \sum_{i=0}^{n_g} g_i x^{\alpha_i}, x_j = \sigma \right\}. \tag{113}$$

5) Bounding a PZ: Our algorithm requires a means to bound the elements of a polynomial zonotope. We define the `sup` and `inf` operations which return these upper and lower bounds, respectively, by taking the absolute values of generators. For $\mathbf{P} \subseteq \mathbb{R}^n$ with center g_0 and generators g_i,

$$\texttt{sup}(\mathbf{P}) = g_0 + \sum_{i=1}^{n_g} |g_i|, \tag{114}$$

$$\texttt{inf}(\mathbf{P}) = g_0 - \sum_{i=1}^{n_g} |g_i|. \tag{115}$$

Note that for any $z \in \mathbf{P}$, $\texttt{sup}(\mathbf{P}) \geq z$ and $\texttt{inf}(\mathbf{P}) \leq z$, where the inequalities are taken element-wise. These bounds may not be tight, but they are quick to compute.

6) Set-Valued Function Evaluation: One can overapproximate any analytic function evaluated on a polynomial zonotope using a Taylor expansion, which itself can be represented as a polynomial zonotope [60, Sec 4.1],[2, Prop. 13]. Consider an analytic function $f : \mathbb{R} \to \mathbb{R}$ and $\mathbf{P}_1 = \mathcal{PZ}(g_i, \alpha_i, x)$, with each $g_i \in \mathbb{R}$, then $f(\mathbf{P}_1) = \{y \in \mathbb{R} \mid y = f(z), z \in \mathbf{P}_1\}$. We generate $\mathbf{P}_2$ such that $f(\mathbf{P}_1) \subseteq \mathbf{P}_2$ using a Taylor expansion of degree $d \in \mathbb{N}$, where the error incurred from the finite approximation is overapproximated using a Lagrange remainder. The method follows the Taylor expansion found in the reachability algorithm in [2], which builds on previous work on conservative polynomialization found in [60]. Recall that the Taylor expansion about a point $c \in \mathbb{R}$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(z-c)^n, \tag{116}$$

where $f^{(n)}$ is the n^{th} derivative of f. The error incurred by a finite Taylor expansion can be bounded using the Lagrange remainder r [61, p. 7.7]:

$$|f(z) - \sum_{n=0}^{d} \frac{f^{(n)}(c)}{n!}(z-c)^n| \leq r, \tag{117}$$

where

$$r = \max_{\delta \in [c,z]} \frac{(|f^{d+1}(\delta)|)|z-c|^{d+1}}{(d+1)!}. \tag{118}$$

For a polynomial zonotope, the infinite dimensional Taylor expansion is given by

$$f(\mathbf{P}_1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(\mathbf{P}_1 - c)^n. \tag{119}$$

In practice, only a finite Taylor expansion of degree $d \in \mathbb{N}$ can be computed. Letting $c = g_0$ (i.e., the center of $\mathbf{P}_1$), and noting that $(z-c) = \sum_{i=1}^{n_g} g_i x^{\alpha_i}$ for $z \in \mathbf{P}_1$, we write

$$\mathbf{P}_2 \coloneqq \left\{ z \in \mathbb{R} \,|\, z \in \sum_{n=0}^{d} \left(\frac{f^{(n)}(g_0)}{n!} (\sum_{i=1}^{n_g} g_i x^{\alpha_i})^n \right) \oplus [r] \right\}, \quad (120)$$

and the Lagrange remainder $[r]$ is computed using interval arithmetic as

$$[r] = \frac{[f^{(d+1)}([\mathbf{P}_1])][(\mathbf{P}_1 - c)^{d+1}]}{(d+1)!} \quad (121)$$

where $[(\mathbf{P}_1 - c)^{d+1}] = [\mathtt{inf}((\mathbf{P}_1 - c)^{d+1}), \mathtt{sup}((\mathbf{P}_1 - c)^{d+1})]$ is an overapproximation of $(\mathbf{P}_1 - c)^{d+1}$. $\mathbf{P}_2$ can be expressed as a polynomial zonotope because all terms in the summation are polynomials of x, and the interval $[r]$ can be expressed as a polynomial zonotope as in (109). We denote the polynomial zonotope overapproximation of a function evaluated on a zonotope using bold symbols (i.e., $\mathbf{f}(\mathbf{P}_1)$ is the polynomial zonotope overapproximation of f applied to $\mathbf{P}$).