

APPENDIX G
POLYNOMIAL ZONOTOPE ARITHMETIC

We introduce several useful operations on polynomial zonotopes: interval conversions, set addition and multiplication, slicing, computing bounds, and set-valued function evaluation.

1) *Interval Arithmetic*: The Minkowski sum and difference of $[x]$ and $[y]$ are

$$[x] \oplus [y] = [\underline{x} + \underline{y}, \bar{x} + \bar{y}], \quad (105)$$

$$[x] \ominus [y] = [\underline{x} - \bar{y}, \bar{x} - \underline{y}]. \quad (106)$$

The product of $[x]$ and $[y]$ is

$$[x][y] = [\min(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}), \max(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y})]. \quad (107)$$

The i^{th} and j^{th} entry of the product of an interval matrix $[Y]$ multiplied by an interval matrix $[X]$ is

$$([X][Y])_{ij} = \bigoplus_{k=1}^n ([X]_{ik}[Y]_{kj}), \quad (108)$$

where n is the number of columns of $[X]$ and number of rows of $[Y]$. Given interval vectors $[x], [y] \subset \mathbb{R}^3$, their cross product is

$$[x] \otimes [y] = [x]^\times [y], \quad (109)$$

where $[x]^\times$ is the skew-symmetric matrix representation of $[x]$ as in (1) (i.e., a matrix with interval entries).

2) *Intervals Conversion*: Intervals can also be written as polynomial zonotopes. For example, let $[z] = [\underline{z}, \bar{z}] \subset \mathbb{R}^n$, then one can convert $[z]$ to a polynomial zonotope \mathbf{z} using

$$\mathbf{z} = \frac{\bar{z} + \underline{z}}{2} + \sum_{i=1}^n \frac{\bar{z}_i - \underline{z}_i}{2} x_i, \quad (110)$$

where $x \in [-1, 1]^n$ is the indeterminate vector.

3) *Set Addition and Multiplication*: The Minkowski Sum of two polynomial zonotopes $\mathbf{P}_1 \subset \mathbb{R}^n = \mathcal{PZ}(g_i, \alpha_i, x)$ and $\mathbf{P}_2 \subset \mathbb{R}^n = \mathcal{PZ}(h_j, \beta_j, y)$ follows from polynomial addition:

$$\mathbf{P}_1 \oplus \mathbf{P}_2 = \left\{ z \in \mathbb{R}^n \mid z = \sum_{i=0}^{n_g} g_i x^{\alpha_i} + \sum_{j=0}^{n_h} h_j y^{\beta_j} \right\}. \quad (111)$$

Similarly, we may write the matrix product of two polynomial zonotopes \mathbf{P}_1 and \mathbf{P}_2 when the sizes are compatible. Letting $\mathbf{P}_1 \subset \mathbb{R}^{n \times m}$ and $\mathbf{P}_2 \subset \mathbb{R}^{m \times k}$, we obtain $\mathbf{P}_1 \mathbf{P}_2 \subset \mathbb{R}^{n \times k}$:

$$\mathbf{P}_1 \mathbf{P}_2 = \left\{ z \in \mathbb{R}^{n \times k} \mid z = \sum_{i=0}^{n_g} g_i \left(\sum_{j=0}^q h_j y^{\beta_j} \right) x^{\alpha_i} \right\}. \quad (112)$$

When $\mathbf{P}_1 \subset \mathbb{R}^{n \times n}$ is square, exponentiation \mathbf{P}_1^m may be performed by multiplying \mathbf{P}_1 by itself m times. Furthermore, if $\mathbf{P}_1 \subset \mathbb{R}^3$ and $\mathbf{P}_2 \subset \mathbb{R}^3$, we implement a set-based cross product as matrix multiplication. We create $\mathbf{P}_1^\times \subset \mathbb{R}^{3 \times 3}$ as

$$\mathbf{P}_1^\times = \left\{ A \in \mathbb{R}^{3 \times 3} \mid A = \sum_{i=0}^{n_g} \begin{bmatrix} 0 & -g_{i,3} & g_{i,2} \\ g_{i,3} & 0 & -g_{i,1} \\ -g_{i,2} & g_{i,1} & 0 \end{bmatrix} x^{\alpha_i} \right\} \quad (113)$$

where $g_{i,j}$ refers to the j^{th} element of g_i . Then, the set-based cross product $\mathbf{P}_1 \otimes \mathbf{P}_2 = \mathbf{P}_1^\times \mathbf{P}_2$ is well-defined.

4) *Slicing*: One can obtain subsets of polynomial zonotopes by plugging in values of known indeterminates. For instance, if a polynomial zonotope \mathbf{P} represented a set of possible positions of a robot arm operating near an obstacle. It may be beneficial to know whether a particular choice of \mathbf{P} 's indeterminates yields a subset of positions that could collide with the obstacle. To this end, "slicing" a polynomial zonotope $\mathbf{P} = \mathcal{PZ}(g_i, \alpha_i, x)$ corresponds to evaluating an element of the indeterminate x . Given the j^{th} indeterminate x_j and a value $\sigma \in [-1, 1]$, slicing yields a subset of \mathbf{P} by plugging σ into the specified element x_j :

$$\text{slice}(\mathbf{P}, x_j, \sigma) \subset \mathbf{P} = \left\{ z \in \mathbf{P} \mid z = \sum_{i=0}^{n_g} g_i x^{\alpha_i}, x_j = \sigma \right\}. \quad (114)$$

5) *Bounding a PZ*: Our algorithm requires a means to bound the elements of a polynomial zonotope. We define the \sup and \inf operations which return these upper and lower bounds, respectively, by taking the absolute values of generators. For $\mathbf{P} \subseteq \mathbb{R}^n$ with center g_0 and generators g_i ,

$$\sup(\mathbf{P}) = g_0 + \sum_{i=1}^{n_g} |g_i|, \quad (115)$$

$$\inf(\mathbf{P}) = g_0 - \sum_{i=1}^{n_g} |g_i|. \quad (116)$$

Note that for any $z \in \mathbf{P}$, $\sup(\mathbf{P}) \geq z$ and $\inf(\mathbf{P}) \leq z$, where the inequalities are taken element-wise. These bounds may not be tight, but they are quick to compute.

6) *Set-Valued Function Evaluation*: One can overapproximate any analytic function evaluated on a polynomial zonotope using a Taylor expansion, which itself can be represented as a polynomial zonotope [59, Sec 4.1], [2, Prop. 13]. Consider an analytic function $f: \mathbb{R} \rightarrow \mathbb{R}$ and $\mathbf{P}_1 = \mathcal{PZ}(g_i, \alpha_i, x)$, with each $g_i \in \mathbb{R}$, then $f(\mathbf{P}_1) = \{y \in \mathbb{R} \mid y = f(z), z \in \mathbf{P}_1\}$. We generate \mathbf{P}_2 such that $f(\mathbf{P}_1) \subseteq \mathbf{P}_2$ using a Taylor expansion of degree $d \in \mathbb{N}$, where the error incurred from the finite approximation is overapproximated using a Lagrange remainder. The method follows the Taylor expansion found in the reachability algorithm in [2], which builds on previous work on conservative polynomialization found in [59]. Recall that the Taylor expansion about a point $c \in \mathbb{R}$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (z - c)^n, \quad (117)$$

where $f^{(n)}$ is the n^{th} derivative of f . The error incurred by a finite Taylor expansion can be bounded using the Lagrange remainder r [60, p. 7.7]:

$$|f(z) - \sum_{n=0}^d \frac{f^{(n)}(c)}{n!} (z - c)^n| \leq r, \quad (118)$$

where

$$r = \max_{\delta \in [c, z]} \frac{(|f^{d+1}(\delta)|) |z - c|^{d+1}}{(d+1)!}. \quad (119)$$

For a polynomial zonotope, the infinite dimensional Taylor expansion is given by

$$f(\mathbf{P}_1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (\mathbf{P}_1 - c)^n. \quad (120)$$

In practice, only a finite Taylor expansion of degree $d \in \mathbb{N}$ can be computed. Letting $c = g_0$ (i.e., the center of \mathbf{P}_1), and noting that $(z - c) = \sum_{i=1}^{n_g} g_i x^{\alpha_i}$ for $z \in \mathbf{P}_1$, we write

$$\mathbf{P}_2 := \left\{ z \in \mathbb{R} \mid z \in \sum_{n=0}^d \left(\frac{f^{(n)}(g_0)}{n!} \left(\sum_{i=1}^{n_g} g_i x^{\alpha_i} \right)^n \right) \oplus [r] \right\}, \quad (121)$$

and the Lagrange remainder $[r]$ is computed using interval arithmetic as

$$[r] = \frac{[f^{(d+1)}(\mathbf{P}_1)][(\mathbf{P}_1 - c)^{d+1}]}{(d+1)!} \quad (122)$$

where $[(\mathbf{P}_1 - c)^{d+1}] = [\inf((\mathbf{P}_1 - c)^{d+1}), \sup((\mathbf{P}_1 - c)^{d+1})]$ is an overapproximation of $(\mathbf{P}_1 - c)^{d+1}$. \mathbf{P}_2 can be expressed as a polynomial zonotope because all terms in the summation are polynomials of x , and the interval $[r]$ can be expressed as a polynomial zonotope as in (110). We denote the polynomial zonotope overapproximation of a function evaluated on a zonotope using bold symbols (i.e., $\mathbf{f}(\mathbf{P}_1)$ is the polynomial zonotope overapproximation of f applied to \mathbf{P}).