Vibrational Resonance in a Duffing Oscillator. A Study with Word-Series Averaging

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Abstract

We have analyzed the synchronization of a small-world network of chaotic Rulkov neurons with an electrical coupling that contains a delay. We have developed an algorithm to compute a certain delay whose result is to improve the synchronization of the network when it was slightly synchronized, or to get synchronized when it was desynchronized. Our general approach has been to use tools from signal analysis, such as Fourier and wavelet transforms. With these tools, we have characterized the behavior of the neurons for different parameters in frequency and time-frequency domains. Finally, the robustness of the algorithm has been tested by using non-homogeneous neurons affected with a parametric noise.

I. INTRODUCTION

A. The Duffing Oscillator

A general equation for an oscillator is

$$\ddot{x} + \nabla V(x) + \delta\dot{x} = F(x,t) \quad x \in \mathbb{R}^n,$$

where $F(x,t)$ is the external force which acts over the system, V is the potential of the oscillator and $\delta\dot{x}$ is a dissipation force ($\delta > 0$).

If $n = 1$ and $V(x) = -x^2/2 + x^4/4$, the system is called Duffing Oscillator [1, Chapter 1]. It is a paradigmatic model of a nonlinear oscillator, and furthermore may have chaotic behaviour for some alternative choices of parameters. It may also be used to model a buckling beam. For the choice of parameters above, the quartic potential has two wells; other choices of parameters lead to a single well, a single unstable equilibrium or one wells with two unstable equilibria. Therefore the system may present monostability and bistability, see Figure 1. The double-well produces orbits around the minima of the potential ($x = -1$ and $x = 1$), or over both points [1, Chapter 3].

This work is focused on the case

$$\ddot{x} - x + x^3 + \delta\dot{x} = \omega B\cos(\omega t) + A\cos(\nu t), \quad \omega \gg \nu \tag{1}$$

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Later, we will discuss the implications of the presence of two periodic external forces with very different frequencies.

If the inertia term $\ddot{x}$ is small compared to the dissipation term $\delta\dot{x}$, the first can be omitted and the system reduces to the overdamped form

$$\dot{x} = x - x^3 + \omega B\cos(\omega t) + A\cos(\nu t), \quad \omega \gg \nu. \tag{2}$$

B. Vibrational Resonance

Vibrational Resonance (VR) is a phenomenon that occurs in dynamical systems with two different periodic forces; one of them with low frequency and the other with high frequency. The response of the system may grow significantly for some parameter values. This growth is measured by the response amplitude Q

$$B_s = \frac{2}{nT}\int_0^{nT} x(t)\sin\nu t dt, \quad B_c = \frac{2}{nT}\int_0^{nT} x(t)\cos\nu t dt, \tag{3}$$

$$Q = \frac{\sqrt{B_s^2 + B_c^2}}{B}, \tag{4}$$

where n is integer and $T = 2\pi/\nu$, ν and B are the same parameters of equation (1). The value of Q will be larger if the resonance effect is larger. The high frequency force can also induce a phase shift in the response to the low frequency force. This is measured by the function

$$\psi = -\arctan\left(\frac{B_s}{B_c}\right).$$

VR was described for the first time in [2] who worked with two bistable system, the underdamped and the overdamped Duffing Oscillator, (1) (2) respectively. In their numerical simulations, they observed that the Q factor strongly depends on the amplitude of the high frequency force in both cases (underdamped and overdamped). For the underdamped oscillator, the case of interest in this thesis, they obtained Figure 2, where the effect of the resonance is obvious.

The system (2) is also a paradigmatic model for the study of vibrational resonance and its possible applications, e.g. energy harvesting [3].

There are other types of resonances which may occur [1, Preface]:

1. Simple resonance: One oscillatory force with a frequency near the natural frequency of the system.

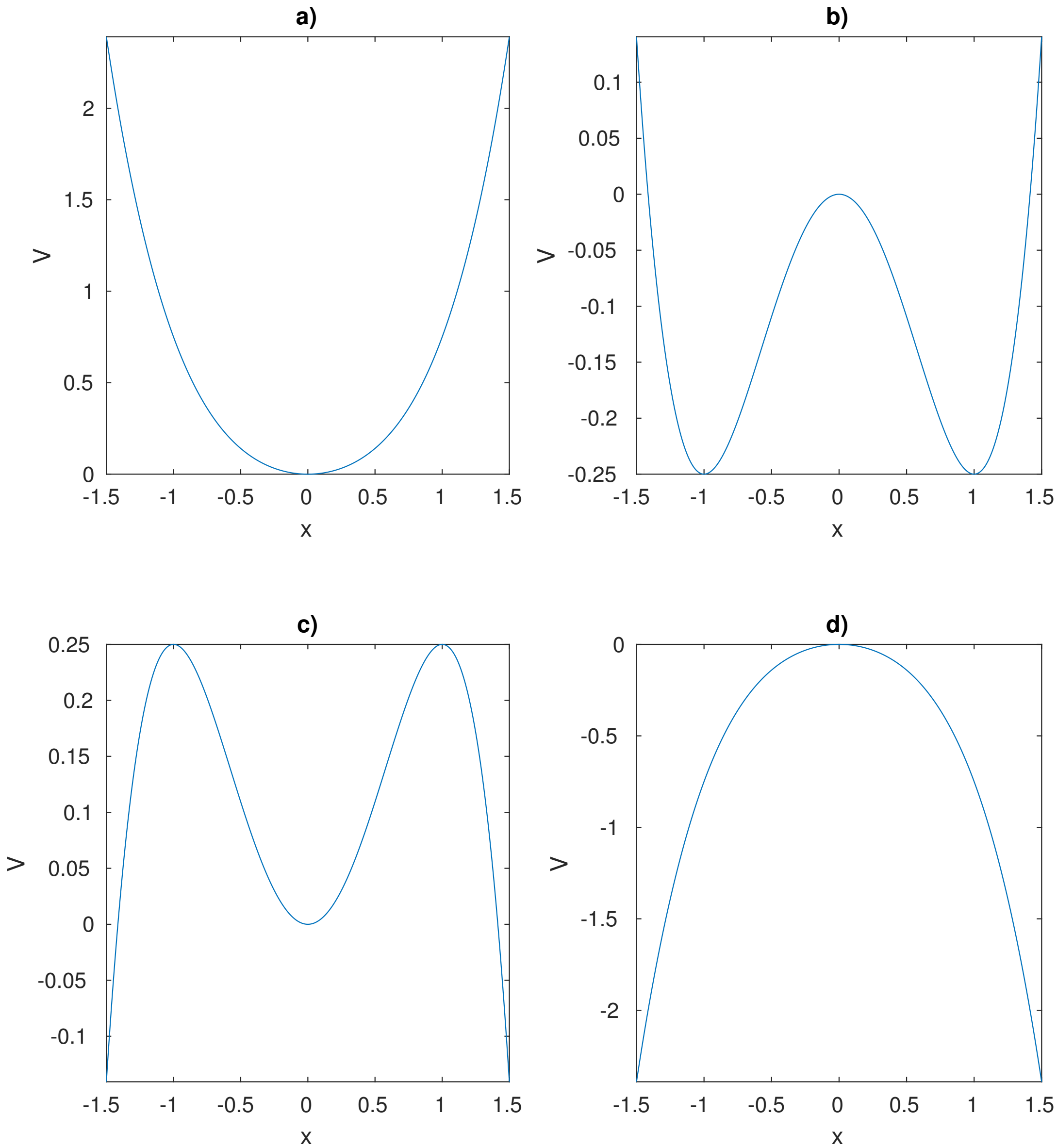

Figure 1: Duffing Potential $V = \alpha x^2/2 + \beta x^4/4$ for: a) $\alpha > 0$ and $\beta > 0$; b) $\alpha < 0$ and $\beta > 0$, c) $\alpha > 0$ and $\beta < 0$; d) $\alpha < 0$ and $\beta < 0$.

2. Stochastic resonance: An external noise and an oscillatory force.

3. Coherence resonance: An external noise without external forces.

4. Parametric resonance: Periodic variations of the system parameters.

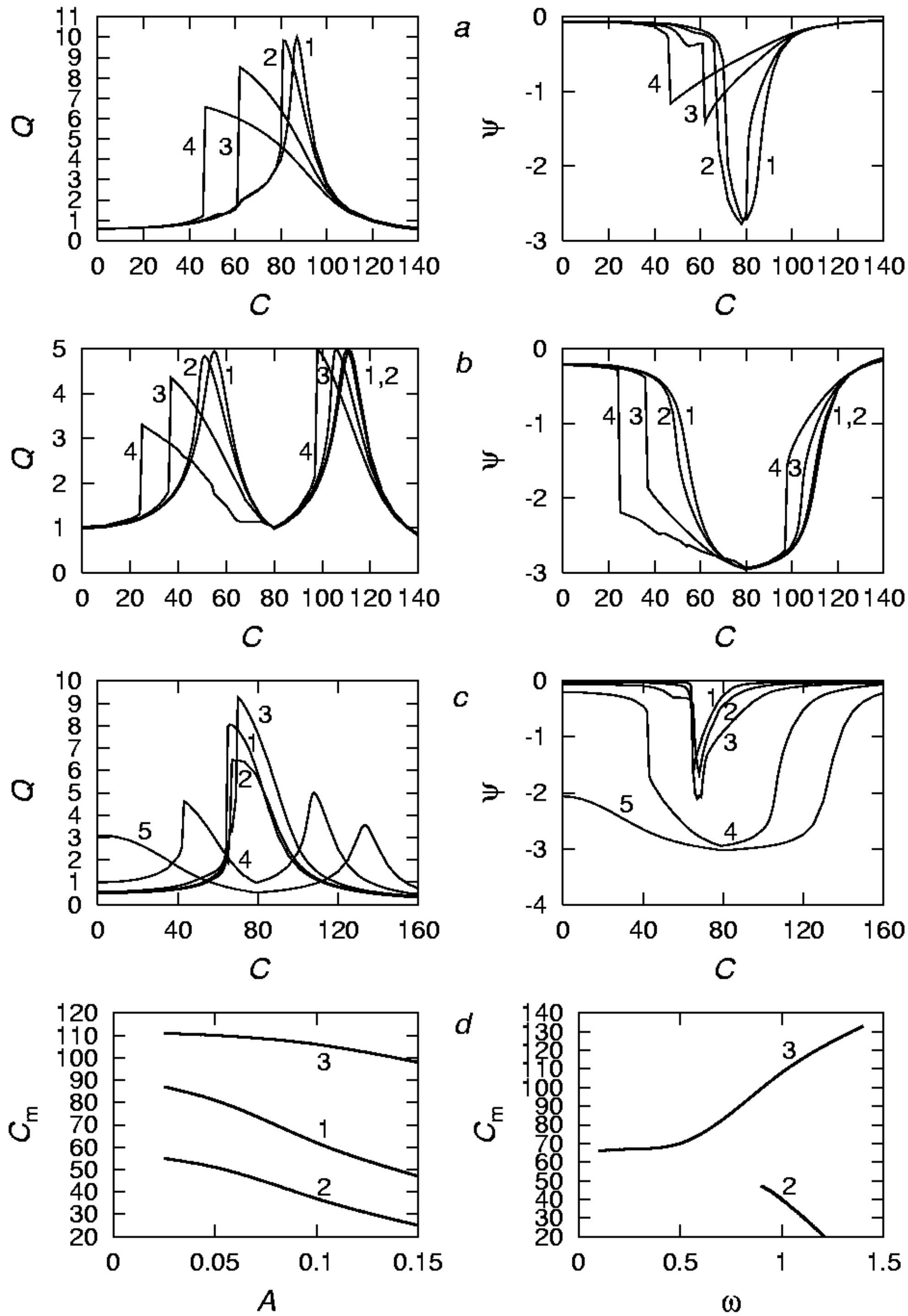

Figure 2: Response of the underdamped system (5) with $\delta = 0.1$ to the weak periodic signal $A\cos\omega t$, as influenced by the high-frequency vibrational force $C\cos t$ with $\Omega = 9.842$, for different conditions: (a) the response amplitude Q and corresponding phase shift ψ are plotted as functions of C with $\omega = 0.5$ for $A = 0.025$, $A = 0.05$, $A = 0.1$ and $A = 0.15$ (curves 1, 2, 3 and 4, respectively); (b) as in (a) but for $\omega = 1$; (c) as in (a) but for $A = 0.08$, ω=0.1, $\omega = 0.25$, $\omega = 0.5$, $\omega = 1$ and $\omega = 1.4$ (curves 1, 2, 3, 4 and 5, respectively); (d) plots of the characteristic vibration amplitudes C_m corresponding to maxima in $Q(C)$ as functions of signal amplitude A (left) for $\omega = 0.5$ (curve 1) and $\omega = 1$

5. Autoresonance: An external force that has a time-dependent frequency.

6. Chaotic resonance: A chaotic perturbation acts upon the system.

C. Integrating the Duffing Oscillator

System (1) has no closed-form solution and so it has to be solved numerically. Using ode45 of MATLAB, which is a standard ODE solver, Figure 3 is obtained.

High frequency oscillatory components are a source of difficulties for most common numerical ODE solvers as they typically need to operate with step sizes smaller than the forcing period. For this reason, in this simulation the default values of the absolute and relative tolerance of ode45 function had to be modified to 10^{-12} and 10^{-9} respectively to ensure accuracy; this makes the method slower. The numerical solution may be facilitated by using averaging techniques, in particular, our problem is susceptible to be treated by word-series averaging.

Table I will be used to compare computation times and error with those of word-series averaging method to be presented later. We did not use greater values of ω because the method is even more imprecise and we had to reduce even more the tolerances, as we said before.

If the system is solved numerically after the change of variable $\dot{x} = y + B\sin\omega t$,

$$\begin{aligned} \dot{y} &= x - x^3 + A\cos\nu t - \delta(y + B\sin\omega t), \\ \dot{x} &= y + B\sin\omega t, \end{aligned} \tag{5}$$

Proof.

$$\begin{aligned} \ddot{x} &= \dot{y} + \omega B\cos\omega t = x - x^3 + A\cos\nu t - \delta(y + B\sin\omega t) + \omega B\cos\omega t \\ \ddot{x} &= x - x^3 + A\cos\nu t - \delta\dot{x} + \omega B\cos\omega t \end{aligned} \tag{6}$$

□

we obtained Figure 4 and the computational costs are given in Table II.

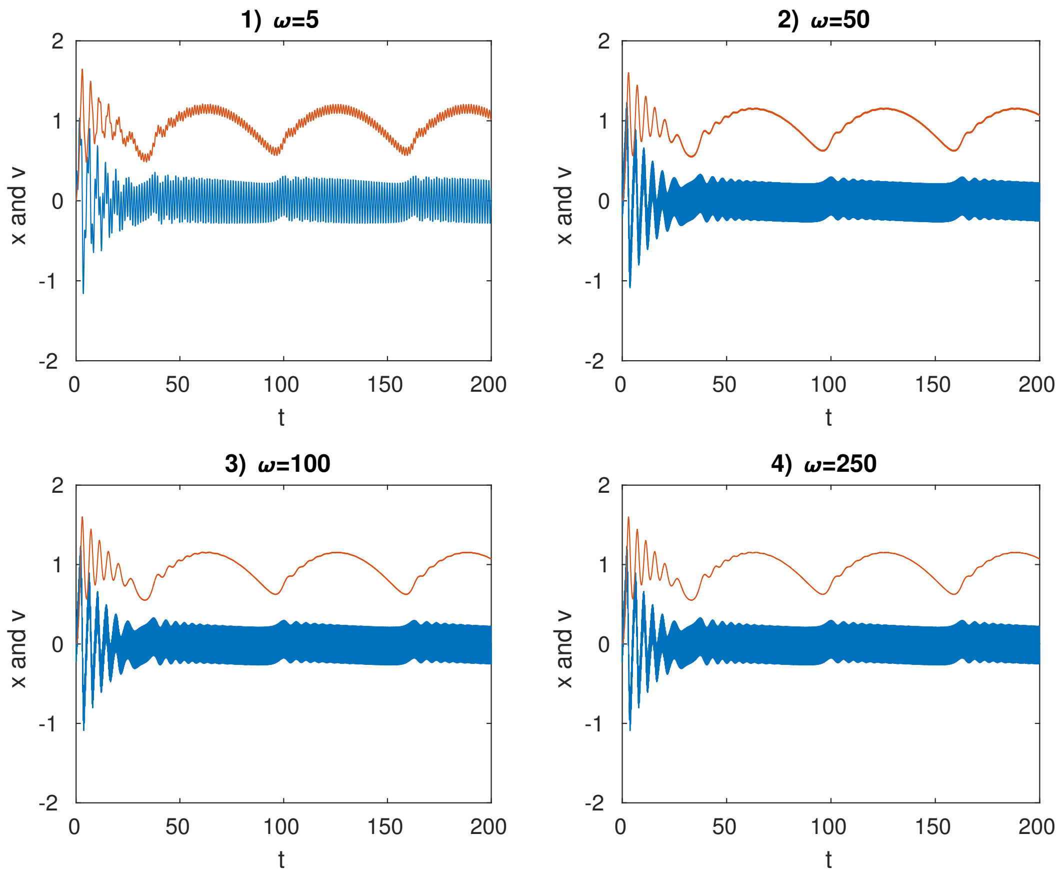

Figure 3: (Position in orange and velocity in blue.) Solutions of (2) for different ω. 1) $\omega = 5$, 2) $\omega = 50$, 3) $\omega = 100$, 4) $\omega = 250$ and $B = 0.25$, $A = 0.375$, $\nu = 0.1$ and $\delta = 0.1$ for 200 units of time.

D. Classical Averaging Methods

Averaging methods are useful to simplify the equations of dynamical systems prior to their study.

For systems of the form

$$\dot{x} = \varepsilon f(x, t, \varepsilon), \tag{7}$$

where $x \in U \subseteq \mathbb{R}^n$, $0 < \varepsilon \ll 1$ and $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n$ is T-periodic in t, bounded in a bounded set and C^r with $r \geq 2$, the solution does not vary too much over a period and it makes sense to consider the averaged system

$$\dot{y} = \varepsilon \frac{1}{T} \int_0^T f(y, t, 0) dt := \varepsilon \tilde{f}(y), \tag{8}$$

ω	Time (s)
5	0.948764
50	7.618521
100	15.325027
250	38.223647

Table I: Integration times for Figure 3.

where the explicit dependence of t is eliminated by taking the average value of f over one period.

The following formulation of the averaging theorem may be found in Guckenheimer and Holmes [**?** , Chapter 4]:

Theorem I.1. *There is a C^r change of coordinates $x = y + \varepsilon w(y, t, \varepsilon)$ under which* (7) *becomes:*

$$\dot{y} = \varepsilon \tilde{f}(y) + \varepsilon^2 f_1(y, t, \varepsilon),$$

where f_1 is T-periodic in t.

Moreover:

1. *If $x(t)$ and $y(t)$ are solutions of* (7) *and* (8) *based at x_0 and y_0 respectively, at $t = 0$ and $|x_0 - y_0| = \mathcal{O}(\varepsilon)$, then $|x(t) - y(t)| = \mathcal{O}(\varepsilon)$ on a time scale $t \sim 1/\varepsilon$.*

2. *If p_0 is a hyperbolic fixed point of* (8) *then exist $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_0$,* (7) *possesses a unique hyperbolic periodic orbit $\gamma_\varepsilon(t) = p_0 + \mathcal{O}(\varepsilon)$ of the same stability type as p_0.*

3. *If $x^s(t) \in W^s(\gamma_\varepsilon)$ is a solution of* (7) *lying in the stable manifold of the hyperbolic periodic orbit $\gamma_\varepsilon = p_0 + \varepsilon$, $y^s(t) \in W^s(p_0)$ is a solution of* (8) *lying in the stable manifold of the hyperbolic fixed point p_0 and $|x^s(0) - y^s(0)| = \mathcal{O}(\varepsilon)$, then $|x^s(t) - y^s(t)| = \mathcal{O}(\varepsilon)$ for $t \in [0, \infty)$. Similar results apply to solutions lying in the unstable manifolds on the time interval $t \in (-\infty, 0]$.*

Applying this method to the high frequency force of Duffing oscillators (2)(1), the corre-

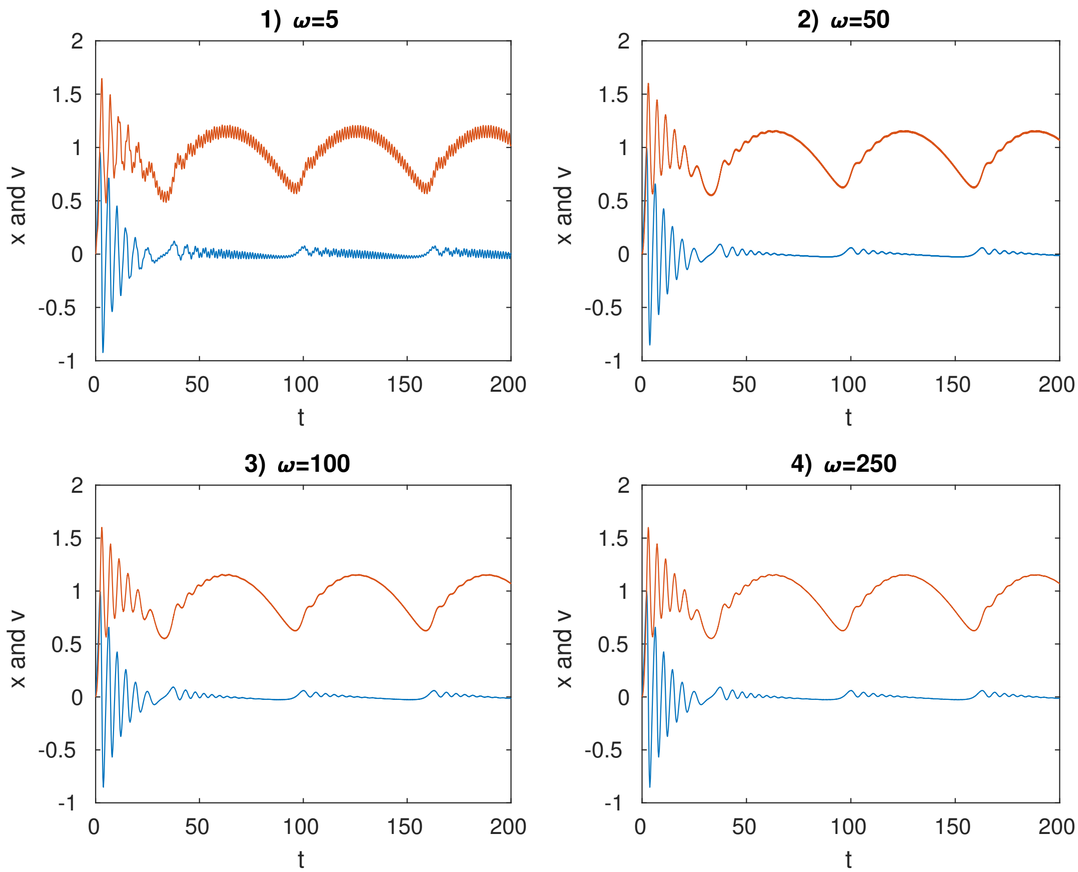

Figure 4: (Position in orange and velocity in blue.) Solutions of (5) for different ω. 1) $\omega = 5$, 2) $\omega = 50$, 3) $\omega = 100$, 4) $\omega = 250$ and $B = 0.25$, $A = 0.375$, $\nu = 0.1$ and $\delta = 0.1$ for 200 units of time.

spondent systems with one low frequency force are obtained as it was expected:

$$\begin{aligned} \ddot{x} &= x - x^3 \delta \dot{x} + A \cos \nu t, \\ \dot{x} &= x - x^3 + A \cos \nu t. \end{aligned} \tag{9}$$

This is a first order averaging, simpler but less accurate than high order methods. For higher order, the error $|x(t) - y(t)|$ will be of order $\mathcal{O}(\varepsilon^r)$, where r is the order of the method. There are many alternative ways to obtain higher order averaged systems. Typically one has to keep performing changes of variables to successively eliminate oscillations forcing terms of size $\mathcal{O}(\varepsilon)$, $\mathcal{O}(\varepsilon^2)$,... The algebra required is usually cumbersome and has to be carried out with the help of a symbolic manipulation package.

In this work, the word-series averaging method, introduced by Murua and Sanz-Serna [4]

ω	Time (s)
5	1.218775
50	4.882155
100	8.123174
250	16.807128

Table II: Integration times for Figure 4.

will be explained and applied to the underdamped Duffing oscillator (1). This method has te advantage of being based on expansions that use scalar coefficients hat are universal and may be computed easily by recurrence.

II. WORD-SERIES AVERAGING

A. Word-Series

Letters l are elements of an alphabet A, that can be finite or infinite. Sequences of letter form words, so $l_1 l_2 ... l_n = w \in \mathscr{W}$ (we denote by $\mathscr{W}$ the set of all word); the empty word is included. We assume that for each letter l, a function $f_l : \mathbb{C}^d \to \mathbb{C}^d$ is given. We then define for each word $w \in \mathscr{W}$ a *word basis function* f_w, which is defined recursively by

$$f_{l_1 \ldots l_n}(y) = f'_{l_2 \ldots l_n}(y) f_{l_1}(y), \quad y \in \mathbb{C}^d, \quad n > 1,$$

where $f'_{l_2 \ldots l_n}(y)$ is the Jacobian matrix of $f_{l_2 \ldots l_n}(y)$, and for the empty word is the identity map, $f_\emptyset(y) = y$.

Furthermore, we may define the complex vector space $\mathbb{C}^{\mathscr{W}}$ of the functions $\delta : \mathscr{W} \to \mathbb{C}$. For each δ there exist a *word series* relative to the word basis function f_w. By definition this is expressed

$$W_\delta(y) = \sum_{w \in \mathscr{W}} \delta_w f_w(y),$$

where δ_w are called the coefficients of the word series [4].

III. DYNAMICAL SYSTEMS AND WORD-SERIES

If we can write a dynamical system with the form

$$\dot{x} = \sum_{a\in A} \lambda(t) f_a(x), \tag{10}$$

where $\lambda : \mathbb{R} \to \mathbb{C}$ and $f_a : \mathbb{C}^d \to \mathbb{C}^d$, for each $a \in A$ we may define a first-order linear differential operator E_a by the formula

$$E_a g(x) = \sum_{j=1}^{d} f_a^j \frac{d}{dx^j} g(x),$$

where g is a scalar-valued function and superscripts denote vector components. By the chain rule

$$\dot{g}(x(t)) = \sum_{a\in A} \lambda(t)(E_a g)(x(t)),$$

and integrating in both sides we obtain

$$g(x(t)) = g(x(t_0)) + \sum_{a\in A} \int_0^t \lambda(t_1)(E_a g)(x(t_1))dt_1.$$

Performing now a Picard iteration we have

$$\begin{aligned} g(x(t)) =& g(x(t_0)) + \sum_{a\in A} \int_{t_0}^t \lambda(t_1)(E_a g)(x(0))dt_1 + \\ &+ \sum_{a\in A}\sum_{b\in A} \int_{t_0}^t \lambda(t_1)dt_1 \int_{t_0}^t \lambda(t_2)(E_b E_a g)(x(t_2)). \end{aligned}$$

Iterating this procedure and setting $g(x) = x$ leads to

$$x(t) = x_0 + \sum_{n=1}^{\infty} \sum_{a_1,\ldots,a_n\in A} \alpha_{a_1\ldots a_n}(t) f_{a_1\ldots a_n}(x_0), \tag{11}$$

where

$$\begin{aligned} \alpha_{a_1\ldots a_n}(t) &= \int_{t_0}^t \lambda_{a_n}(t_n)\alpha_{a_1\ldots a_{n-1}}(t_n)dt_n = \\ &= \int\cdots\int_{\mathcal{S}_n(t)} \lambda_{a_1}(t_1)\cdots\lambda_{a_n}(t_n)dt_1\cdots dt_n, \end{aligned} \tag{12}$$

with $\mathcal{S}_n(t) = \{(t_1,\ldots t_n) \in \mathbb{R}^n : t_0 \le t_1 \le \cdots \le t_n \le t\}$. Formula (11) can be written compactly using the previous notation [5]

$$x(t) = \sum_{w\in\mathscr{W}} \alpha_w(t) f_w(x_0) = W_{\alpha(t;t_0)}(x_0). \tag{13}$$

IV. THE CONVOLUTION PRODUCT AND UNIVERSAL FORMULATIONS

The convolution product $\delta \star \delta' \in \mathbb{C}^{\mathscr{W}}$ of $\delta, \delta' \in \mathbb{C}^{\mathscr{W}}$ is defined by

$$(\delta \star \delta')_{l_1 \cdots l_n} = \delta_\emptyset \delta'_{l_1 \cdots l_n} + \sum_{j=1}^{n-1} \delta_{l_1 \cdots l_j} \delta'_{l_{j+1} \cdots l_n} + \delta_{l_1 \cdots l_n} \delta'_\emptyset, \quad n \geq 1, \quad (\delta \star \delta')_\emptyset = \delta_\emptyset \delta'_\emptyset.$$

This operation is not commutative, but it is associative and has a unit $\mathbb{1}$ defined by

$$\mathbb{1}_w = \begin{cases} 1 \text{ if } w = \emptyset \\ 0 \text{ if } w \neq \emptyset \end{cases}.$$

The shuffle product $w ⧢ w'$ of two words is the formal sum of all words that may be formed by interleaving the letters of w with those of w' without altering the order in which those letters appear within w or w', (e.g. $lm ⧢ n = lmn + lnm + nlm$).

The set $\mathscr{G}$ consists of those $\gamma \in \mathbb{C}^{\mathscr{W}}$ that satisfy

$$\gamma_w \gamma_{w'} = \sum_{j=1}^{N} \gamma_{w_j} \quad \text{if} \quad w ⧢ w' = \sum_{j=1}^{N} w_j,$$

for each pair of words and it is a group for the operation $\star$. The element α in (12) belongs to this group [4].

For $\gamma \in \mathscr{G}$, $\delta \in \mathbb{C}^{\mathscr{W}}$

$$W_\delta(W_\gamma(x)) = W_{\gamma \star \delta}(x).$$

Another useful property is that $W_\gamma(x)$ is equivariant with respect to changes of variables $x = C(\bar{x})$. If $\bar{f}_l(\bar{y}) = C'(\bar{y})^{-1} f_l(C(\bar{y}))$, i.e. $\bar{f}_l(\bar{y})$ is the result of the change on the field, then

$$C(\bar{W}_\gamma(\bar{x})) = W_\gamma(C(\bar{x})),$$

where $\bar{W}_\gamma(\bar{y})$ is the word series with coefficient γ_w and basis $\bar{f}_w(\bar{y})$.

Furthermore, associated with the group $\mathscr{G}$ there is a Lie algebra $\mathfrak{g}$ which is formed by the elements $\beta \in \mathbb{C}^{\mathscr{W}}$ that satisfy

$$\beta_\emptyset = 0 \quad \text{and} \quad \sum_{j=1}^{N} \beta_{w_j} = 0 \text{ if } w ⧢ w' = \sum_{j=1}^{N} w_j.$$

For each fixed time t we define $\beta(t) \in \mathfrak{g}$ by $\beta_l(t) = \lambda_l(t)$ and $\beta_w(t) = 0$ when the word w is empty or has more than one letter. Then (10) may be rewritten as

$$\dot{x} = W_{\beta(t)}(x),$$

If the solution is sought in the form (13)

$$\frac{\partial}{\partial t}W_{\alpha(t;t_0)}(x_0) = W_{\beta(t)}(W_{\alpha(t;t_0)}(x_0)) = W_{\alpha(t;t_0)\star\beta(t)}(x_0), \quad W_{\alpha(t_0;t_0)} = x_0,$$

and this will hold if

$$\frac{\partial}{\partial t}\alpha(t;t_0) = \alpha(t;t_0)\star\beta(t), \quad \alpha(t_0;t_0) = \mathbb{1}. \tag{14}$$

The element α may be obtained by recursion from these relations and the result is (12). Note that this transforms the problem any problem of the form (10) into (14), which is in $\mathscr{G}$ but is linear and therefore easy to solve.

Since α is independent from the basis f_l and the dimension of the problem (10), the formulation (14) is called *universal*.

A. Word-Series Averaging for Oscillatory Problems

Oscillatory problems of the form

$$\dot{x}(t) = \varepsilon f(x,t) = \sum_{k\in\mathbb{Z}} \exp(ik\omega t)\hat{f}_k(x) \tag{15}$$

are a particular case of (10) with alphabet $A = \mathbb{Z}$, $f_k(x) = \varepsilon\hat{f}_k(x)$ and $\lambda_k(t) = \exp(ik\omega t)$.

Proposition 1. *The coefficient α from* (12) *for* (15) *can be calculated by the recursive formulas*

$$\begin{aligned}
\alpha_k(t;t_0) &= \frac{i(\exp(ik\omega t_0) - \exp(ik\omega t))}{k\omega},\\
\alpha_{0^r}(t,t_0) &= \frac{(t-t_0)^r}{r!},\\
\alpha_{0^r k}(t;t_0) &= \frac{i}{kw}(\alpha_{0^{r-1}k}(t;t_0) - \alpha_{0^r}(t;t_0)\exp(ik\omega t)),\\
\alpha_{kl_1\cdots l_s}(t;t_0) &= \frac{i}{k\omega}(\exp(ik\omega t_0) - \alpha_{(k+l_1)l_2\cdots l_s}(t;t_0)),\\
\alpha_{0^r kl_1\cdots l_s}(t;t_0) &= \frac{i}{k\omega}(\alpha_{0^{r-1}kl_1\cdots l_s}(t;t_0) - \alpha_{0^r(k+l_1)l_2\cdots l_s}(t;t_0)),
\end{aligned} \tag{16}$$

where $r \geq 1$, $k \in \mathbb{Z}\setminus\{0\}$, and $l_1, ..., l_s \in \mathbb{Z}$.

Proof. Fulfilling (12) is equivalent to fulfilling

$$\dot{\alpha}_{k_1\cdots k_n}(t;t_0) = \exp(ik_n\omega t)\alpha_{k_1\cdots k_{n-1}}(t;t_0), \tag{17}$$

where (12) have been derived in both sides.

It easy to prove that (16) satisfy (17) for $n = 1$ and by induction the result can be proved for all n. $\square$

We may define another scalar-valued function related to α

$$\Gamma_w(\tau;\theta;\theta_0) = \alpha_w(t;t_0), \tag{18}$$

where $\tau = t - t_0$ and $\theta = \omega t$. The values of Γ may also be computed recursively.

Theorem IV.1. *For each $w \in \mathscr{W}$, $r \geq 1$, $k \in \mathbb{Z} \setminus \{0\}$, and $l_1, ..., l_s \in \mathbb{Z}$ and with $\Gamma_\emptyset(\tau, \theta, \theta_0) = 1$,*

$$\begin{aligned}
\Gamma_k(\tau,\theta;\theta_0) &= \frac{i}{k\omega}(\exp(ik\theta_0) - \exp(ik\theta)), \\
\Gamma_{0^r}(\tau,\theta;\theta_0) &= \frac{\tau^r}{r!}, \\
\Gamma_{0^r k}(\tau,\theta;\theta_0) &= \frac{i}{kw}(\Gamma_{0^{r-1}k}(\tau,\theta;\theta_0) - \Gamma_{0^r}(\tau,\theta;\theta_0)\exp(ik\omega t)), \\
\Gamma_{kl_1\cdots l_s}(\tau,\theta;\theta_0) &= \frac{i}{k\omega}(\exp(ik\omega t_0) - \Gamma_{(k+l_1)l_2\cdots l_s}(\tau,\theta;\theta_0)), \\
\Gamma_{0^r kl_1\cdots l_s}(\tau,\theta;\theta_0) &= \frac{i}{k\omega}(\Gamma_{0^{r-1}kl_1\cdots l_s}(\tau,\theta;\theta_0) - \Gamma_{0^r(k+l_1)l_2\cdots l_s}(t;t_0)),
\end{aligned} \tag{19}$$

then $\Gamma_w(\tau,\theta;\theta_0)$ is a polynomial in τ and a trigonometric polynomial in θ and satisfy (18).

Theorem IV.2. *For each $\tau \in \mathbb{R}$, $\theta, \theta_0 \in \mathbb{T}$, $\Gamma(\tau,\theta;\theta_0) \in \mathbb{C}^{\mathscr{W}}$ belongs to $\mathscr{G}$.*

Theorem IV.3. *For arbitrary $\tau_1, \tau_2 \in \mathbb{R}$ and $\theta_0, \theta_1, \theta_2 \in \mathbb{T}$,*

$$\Gamma(\tau_1,\theta_1;\theta_0) \star \Gamma(\tau_2,\theta_2;\theta_0) = \Gamma(\tau_1+\tau_2,\theta_2;\theta_0).$$

Proofs of the previous theorems are given in [4].

We can see that the function Γ still depends on ωt, so we can define the averaged function

$$\bar{\alpha}(t;t_0) = \Gamma(t - t_0, \omega t_0; \omega t_0). \tag{20}$$

where the second argument has been frozen. This satisfies

$$\bar{\alpha}(t_0,t_0) = \Gamma(0,\omega t_0,\omega t_0) = \alpha(t_0;t_0) = \mathbb{1},$$

and from the trigonometric dependence on θ of Γ, for times of the form $t_s = t_0 + 2k\pi/\omega$, $k \in \mathbb{Z}$,

$$\begin{aligned}\alpha(t_s; t_0) &= \Gamma(t_s - t_0, t\omega; t_0\omega) = \Gamma\left(\frac{2\pi}{\omega}, \omega t_0 + 2k\pi, t_0\omega\right) = \Gamma\left(\frac{2\pi}{\omega}, \omega t_0, t_0\omega\right) \\ &= \bar{\alpha}(t; t_0).\end{aligned}$$

Then the averaged initial value problem is

$$\dot{\bar{\alpha}}(t; t_0) = \bar{\alpha}(t; t_0) \star \bar{\beta}(t_0), \quad \bar{\alpha}(t_0, t_0) = \mathbb{1}, \tag{21}$$

with

$$\bar{\beta}(t_0) = \frac{d}{dt}\bar{\alpha}(t; t_0)\Big|_{t=t_0} \in \mathfrak{g}. \tag{22}$$

To see the relation between α and $\bar{\alpha}$, Theorem IV.3 is used,

$$\begin{aligned}\alpha(t; t_0) &= \Gamma(t - t_0, t\omega; t_0\omega) = \Gamma(t - t_0, t_0\omega; t_0\omega) \star \Gamma(0, t\omega; t_0\omega) \\ &= \bar{\alpha}(t; t_0) \star \Gamma(0, t\omega; t_0\omega).\end{aligned}$$

If we define

$$\kappa(t\omega; t_0) = \Gamma(0, t\omega; t_0\omega), \tag{23}$$

we can write α

$$\alpha(t; t_0) = \bar{\alpha}(t; t_0) \star \kappa(t\omega; t_0). \tag{24}$$

Using formulas (18), (22) and (23), we can prove the following proposition

Proposition 2. *For each $w \in \mathscr{W}$, $r \geq 1$, $k \in \mathbb{Z}\backslash\{0\}$, and $l_1, ..., l_s \in \mathbb{Z}$ and with $\bar{\alpha}_\emptyset(t; t_0) = 1$, $\bar{\alpha}_w$ can be calculated recursively with the formulas*

$$\begin{aligned}\bar{\alpha}_k(t; t_0) &= 0, \\ \bar{\alpha}_{0^r}(t; t_0) &= \frac{(t - t_0)^r}{r!}, \\ \bar{\alpha}_{0^r k}(t; t_0) &= \frac{i}{k\omega}(\bar{\alpha}_{0^{r-1}k}(t; t_0) - \bar{\alpha}_{0^r}(t; t_0)e^{ik\omega t_0}), \\ \bar{\alpha}_{kl_1\cdots l_s}(t; t_0) &= \frac{i}{k\omega}(e^{ik\omega t_0}\bar{\alpha}_{l_1\cdots l_s}(t; t_0) - \bar{\alpha}_{(k+l_1)l_2\cdots l_s}(t; t_0)), \\ \bar{\alpha}_{0^r kl_1\cdots l_s}(t; t_0) &= \frac{i}{k\omega}(\bar{\alpha}_{0^{r-1}kl_1\cdots l_s}(t; t_0) - \bar{\alpha}_{0^r(k+l_1)l_2\cdots l_s}(t; t_0)).\end{aligned}$$

Proposition 3. *The coefficients $\bar{\beta}_w$ can be computed by*

$$\begin{aligned}
\bar{\beta}_k(t_0) &= 0,\\
\bar{\beta}_0(t_0) &= 1,\\
\bar{\beta}_{0^{r+1}}(t_0) &= 0,\\
\bar{\beta}_{0^r k}(t_0) &= \frac{i}{k\omega}(\bar{\beta}_{0^{r-1}k}(t_0) - \bar{\beta}_{0^r}(t_0)),\\
\bar{\beta}_{kl_1\cdots l_s}(t_0) &= \frac{i}{k\omega}(e^{ik\omega t_0}\bar{\beta}_{l_1\cdots l_s}(t_0) - \bar{\beta}_{(k+l_1)l_2\cdots l_s}(t_0)),\\
\bar{\beta}_{0^r kl_1\cdots l_s}(t_0) &= \frac{i}{k\omega}(\bar{\beta}_{0^{r-1}kl_1\cdots l_s}(t_0) - \bar{\beta}_{0^r(k+l_1)l_2\cdots l_s}(t_0)).
\end{aligned} \tag{25}$$

Proposition 4. *The coefficients κ_w may be computed recursively by*

$$\begin{aligned}
\kappa_k(t;t_0) &= \frac{i}{k\omega}(e^{ik\omega t_0} - e^i k\omega t),\\
\kappa_{0^r}(t;t_0) &= 0,\\
\kappa_{0^r k}(t;t_0) &= \frac{i}{k\omega}\kappa_{0^{r-1}k}(t;t_0),\\
\kappa_{kl_1\cdots l_s}(t;t_0) &= \frac{i}{k\omega}(e^{ik\omega t_0}\kappa_{l_1\cdots l_s}(t;t_0) - \kappa_{(k+l_1)l_2\cdots l_s}(t;t_0)),\\
\kappa_{0^r kl_1\cdots l_s}(t;t_0) &= \frac{i}{k\omega}(\kappa_{0^{r-1}kl_1\cdots l_s}(t;t_0) - \kappa_{0^r(k+l_1)l_2\cdots l_s}(t;t_0)).
\end{aligned}$$

Compiling this information, we obtain the following theorems [4]:

Theorem IV.4. *The solution of the problem* (14) *has the representation* (24)*, where $\bar{\alpha}(t;t_0)$ satisfies the autonomous averaged problem* (21)*. The coefficients can be calculated using* (20) *and* (19)*.*

And finally:

Theorem IV.5. *The solution of* (15) *can be represented as*

$$x(t) = W_{\kappa(t;t_0)}(X(t)),$$

where $X(t) = W_{\bar{\alpha}(t;t_0)}(x_0)$ solves the autonomous averaged initial value problem

$$\dot{X}(t) = W_{\bar{\beta}(t_0)}(x_0), \quad X(t_0) = x_0.$$

Then, $\bar{\beta}_w$ are the coefficients that will be used for the calculation of the averaged version of (1) and κ_w the coefficients that will be used to undo the change of variables in the averaging procedure.

V. RESULTS

A. Averaged Equations

We now proceed as in [6]. We will work with (1) after the change of variable in (5). The problem is rewritten in the following autonomous form

$$\dot{S}(t) = \begin{pmatrix} \dot{y} \\ \dot{x} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} x - x^3 + A\cos\varphi - 2\delta(y + B\sin\omega t) \\ y + B\sin\omega t \\ \nu \end{pmatrix}, \quad S(t_0) = S_0,$$

and we will look for averaged solutions for the averaged system $\dot{\bar{S}}(t) = W_{\bar{\beta}(t_0)}(S_0)$ and after that we obtain the solution of the original system of the form $S(t) = W_{\kappa(t;t_0)}(X(t))$, see Theorems IV.5 and IV.4.

After Fourier expansion we obtain the coefficients

$$f_{-1}(S) = \begin{pmatrix} -i\delta B \\ \frac{i}{2}B \\ 0 \end{pmatrix},$$

$$f_1(S) = \bar{f}_{-1}(S) = \begin{pmatrix} i\delta B \\ -\frac{i}{2}B \\ 0 \end{pmatrix},$$

$$f_0(S) = \begin{pmatrix} x - x^3 + A\cos\varphi - 2\delta y \\ y \\ \nu \end{pmatrix},$$

where $\bar{f}_{-1}$ means the complex conjugate of f_{-1} and $f_l = 0$ for others indices (i.e. for $l \neq -1, 0, 1$).

With words of one letter the averaged system is

$$\bar{S}_1(t) = \bar{\beta}_{-1}(t_0)f_{-1}(S) + \bar{\beta}_0(t_0)f_0(S) + \bar{\beta}_1(t_0)f_1(S).$$

The next step is to calculate the coefficients $\bar{\beta}_w$ by means of (25). Then the previous equation becomes

$$\bar{S}_1(t) = f_0(S) = \begin{pmatrix} x - x^3 + A\cos\varphi - 2\delta y \\ y \\ \nu \end{pmatrix}. \tag{26}$$

We can see this is the same system we obtained in (9). So, with words of one letter, we obtain a first order averaging method. Higher order can be obtained by using words with more letters, i.e. using a space $\mathcal{W} = \sum_{n=1}^{r} \mathscr{W}_n$ we obtain a method of order r, where $\mathscr{W}_n$ is the set of word with n letters.

To simplify the notation we are going to omit the variables of all the functions in subsequent equations. To simplify the calculations we have set $t_0 = 0$.

For words with two and one letter we obtain

$$\begin{aligned}\bar{S}_2 = \bar{S}_1 &+ \bar{\beta}_{-1-1} f_{-1-1} + \bar{\beta}_{-10} f_{-10} + \bar{\beta}_{-11} f_{-11} \\ &+ \bar{\beta}_{0-1} f_{0-1} + \bar{\beta}_{00} f_{00} + \bar{\beta}_{01} f_{01} \\ &+ \bar{\beta}_{1-1} f_{1-1} + \bar{\beta}_{10} f_{10} + \bar{\beta}_{11} f_{11},\end{aligned}$$

where

$$\begin{aligned}
f_{-1-1} &= f'_{-1}(S) f_{-1}(S) = 0 \begin{pmatrix} -i\delta B \\ \frac{i}{2} B \\ 0 \end{pmatrix} = 0, \\
f_{0-1} &= 0, \\
f_{1-1} &= 0, \\
f_{-10} &= \begin{pmatrix} -2\delta & 1-3x^2 & -A\sin\varphi \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -i\delta B \\ \frac{i}{2} B \\ 0 \end{pmatrix} = \begin{pmatrix} 2i\delta^2 B + \frac{i}{2} B - \frac{3}{2} i x^2 B \\ -i\delta B \\ 0 \end{pmatrix}, \\
f_{10} &= \bar{f}_{-10} = \begin{pmatrix} -2i\delta^2 B - \frac{i}{2} B + \frac{3}{2} i x^2 B \\ +i\delta B \\ 0 \end{pmatrix}, \\
f_{-11} &= 0, \\
f_{11} &= 0,
\end{aligned}$$

f_{0-1}, f_{1-1}... are 0 for the same reason as f_{-1-1}. We did not calculate f_{00} because $\bar{\beta}_{00} = 0$ by definition and therefore $f_{00}\bar{\beta}_{00}$ does not contribute anything.

The $\bar{\beta}_w$ with $f_w \neq 0$ are

$$\begin{aligned}\bar{\beta}_{-10} &= \frac{-i}{\omega}(\bar{\beta}_0 - \bar{\beta}_{-1}) = -\frac{i}{\omega}, \\ \bar{\beta}_{10} &= \frac{i}{\omega}(\bar{\beta}_0 - \bar{\beta}_1) = \frac{i}{\omega}.\end{aligned}$$

Then

$$\bar{S}_2 = \bar{S}_1 + \bar{\beta}_{-10} f_{-10} + \bar{\beta}_{10} f_{10} = \begin{pmatrix} x - x^3 + A\cos\varphi - 2\delta y + \dfrac{4\delta^2 B}{\omega} + \dfrac{B}{\omega} - \dfrac{3x^2 B}{\omega} \\ y - \dfrac{2\delta B}{\omega} \\ \nu \end{pmatrix}. \tag{27}$$

We can see that the difference between this equation and (26) is that there are corrections of order $1/\omega$.

If we continue for words with three or less letters, we obtain

$$\bar{S}_3 = \begin{pmatrix} x - x^3 + A\cos\varphi - 2\delta y + \dfrac{4\delta^2 B}{\omega} + \dfrac{B}{\omega} - \dfrac{3x^2 B}{\omega} - \dfrac{9xB^2}{2\omega^2} \\ y - \dfrac{2\delta B}{\omega} \\ \nu \end{pmatrix}. \tag{28}$$

Now the new correction is of the order $1/\omega^2$. This corrections give us the order of the error and the order of the method r, in this notation $|S - \bar{S}_r| = \mathcal{O}(1/w^r)$ if corrections are $1/w^{r-1}$.

B. Change of Variables

The systems (26), (27) and (28) will be integrated numerically and in this way we have the solutions of the averaged problems. If we wish to have the solutions without averaging, we must use the coefficients κ to undo the change of variable and also to undo the preliminary change in (5).

For the first order averaged system we have

$$S_1 = \bar{S}_1 + \begin{pmatrix} B\sin(\omega t) \\ 0 \\ 0 \end{pmatrix}.$$

For the second order

$$S_2 = \bar{S}_2 + \kappa_{-1} f_{-1} + \kappa_0 f_0 + \kappa_1 f_1 + \begin{pmatrix} B\sin(\omega t) \\ 0 \\ 0 \end{pmatrix},$$

where

$$\kappa_{-1} = -\frac{i}{\omega}(1 - e^{-i\omega t}),$$

$$\kappa_0 = 0,$$

$$\kappa_1 = \bar{\kappa}_{-1} = \frac{i}{\omega}(1 - e^{i\omega t}),$$

then

$$\begin{aligned}
\kappa_{-1} f_{-1} + \kappa_1 f_1 &= -\frac{i}{\omega}(1 - e^{-i\omega t}) \begin{pmatrix} -i\delta B \\ \frac{i}{2}B \\ 0 \end{pmatrix} + \frac{i}{\omega}(1 - e^{i\omega t}) \begin{pmatrix} i\delta B \\ -\frac{i}{2}B \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\delta B}{\omega} \\ \frac{1}{2\omega}B \\ 0 \end{pmatrix} (1 - e^{-i\omega t} + 1 - e^{i\omega t}) \\
&= \frac{1}{\omega}(1 - \cos\omega t) \begin{pmatrix} -2\delta B \\ B \\ 0 \end{pmatrix},
\end{aligned}$$

and finally

$$S_2 = \bar{S}_2 + \frac{1}{\omega}(1 - \cos\omega t) \begin{pmatrix} -2\delta B \\ B \\ 0 \end{pmatrix} + \begin{pmatrix} B\sin(\omega t) \\ 0 \\ 0 \end{pmatrix}.$$

For third order

$$S_3 = \bar{S}_3 + \frac{1}{\omega}(1 - \cos\omega t) \begin{pmatrix} -2\delta B \\ B \\ 0 \end{pmatrix} + \kappa_{-10} f_{-10} + \kappa_{10} f_{10} + \begin{pmatrix} B\sin(\omega t) \\ 0 \\ 0 \end{pmatrix},$$

$$\kappa_{-10} = -\frac{i}{\omega}(\kappa_0 - \kappa_{-1}) = \frac{1}{\omega^2}(1 - e^{-i\omega t}),$$

$$\kappa_{10} = \bar{\kappa}_{-10} = \frac{1}{\omega^2}(1 - e^{i\omega t}).$$

So

$$S_3 = \bar{S}_3 + \frac{B}{\omega}(1 - \cos\omega t) \begin{pmatrix} -2\delta \\ 1 \\ 0 \end{pmatrix} + \frac{B}{\omega^2}\sin\omega t \begin{pmatrix} 4\delta^2 + 1 - 3x^2 \\ -2\delta \\ 0 \end{pmatrix} + \begin{pmatrix} B\sin(\omega t) \\ 0 \\ 0 \end{pmatrix}. \tag{29}$$

We only used word with $n-1$ or less letters, where n is the maximum number of letter of the averaged system, because the order of the correction in κ is $1/\omega^{n-1}$ for words of $n-1$

letters. This means that adding words with more letters in the change of variable will not improve the result because the errors of the method are larger than the corrections we would add.

VI. COMPUTATIONAL RESULTS

1. Under Ordinary Conditions

In this section we integrate (26), (27) and (28), corresponding to one, two and three letter word expansions, plot the results and compare with ode45 solutions to measure the error. The times were computed from the beginning of the integration of the dynamical system to the conclusion of the computation of the change of variable. The errors were calculated with the supremum norm of the difference of the solution of Figure 3 and the solutions of the word-series averaging.

For the same values of the parameters as in Figure 3 and integrating the system $\bar{S}_3$ we obtained the Figures 5 and 6. And for all the three system we obtained the Tables III, IV and V.

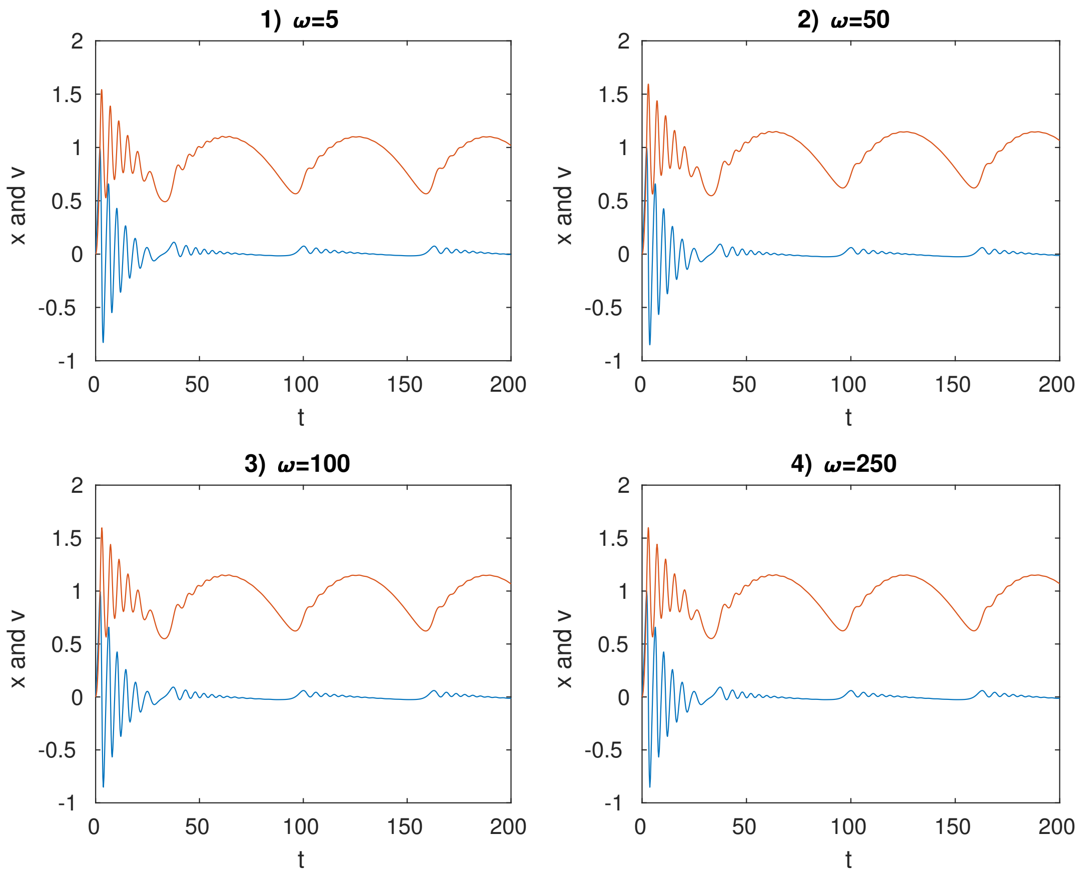

Figure 5: (Position in orange, velocity in blue.) Averaged solutions of (28) for different ω. 1) $\omega = 5$, 2) $\omega = 50$, 3) $\omega = 100$, 4) $\omega = 250$ and $B = 0.25$, $A = 0.375$, $\nu = 0.1$ and $\delta = 0.1$ for 200 units of time.

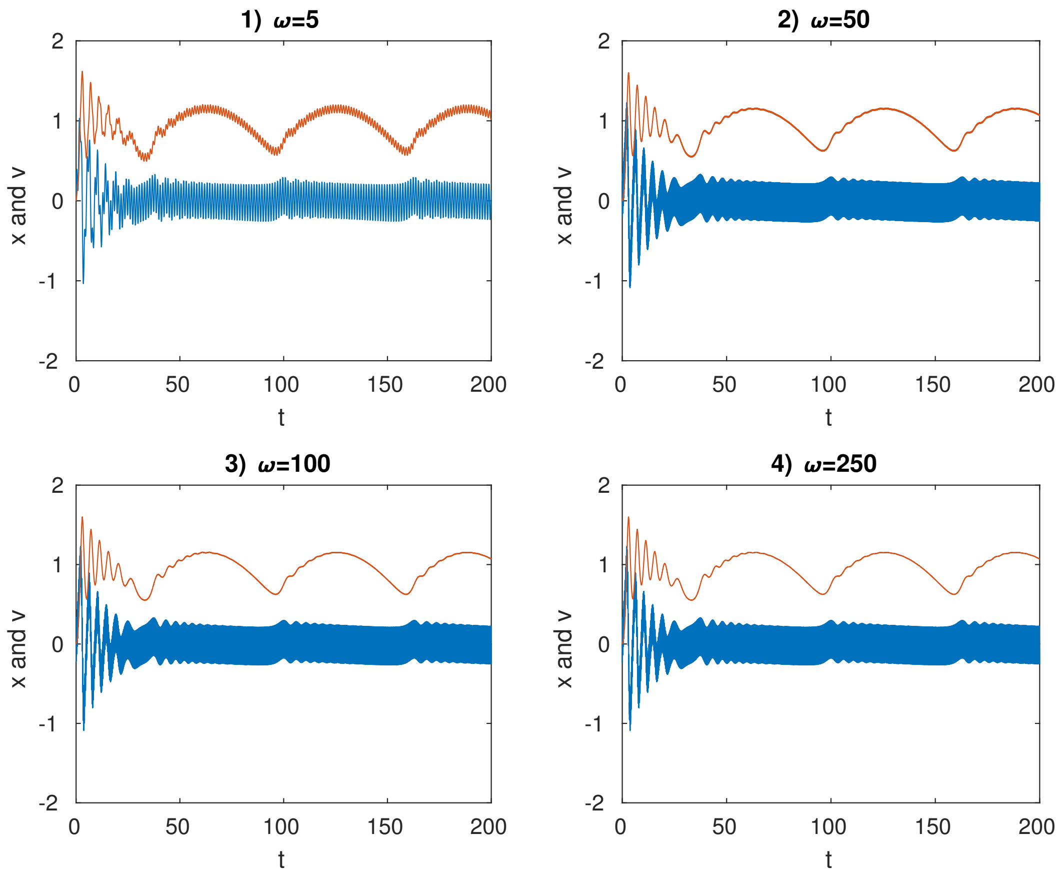

Figure 6: (Position in orange, velocity in blue.) Solutions of (28) after the change of variable for different ω. 1) $\omega = 5$, 2) $\omega = 50$, 3) $\omega = 100$, 4) $\omega = 250$ and $B = 0.25$, $A = 0.375$, $\nu = 0.1$ and $\delta = 0.1$ for 200 units of time.

ω	Time (s)	Error
5	0.459532	0.933967914915814
50	0.850018	0.090997796650348
100	1.191527	0.045455223508130
250	2.335278	0.018140110616174

Table III: Integration times and error for (26).

ω	Time (s)	Error
5	0.472348	0.529023082945981
50	0.859552	0.007610810804546
100	1.197445	0.003223083177153
250	2.385031	0.001123027911783

Table IV: Integration times and error for (27).

ω	Time (s)	Error
5	0.767218	0.135643159694801
50	0.955357	0.000123941932990
100	1.271647	0.000020042606107
250	2.379124	0.000002021668552

Table V: Integration times and error for (28).

2. Vibrational Resonance

In the previous section we used a set of parameters that does not produce vibrational resonance to study word-series averaging. Now we will modify the values of the parameters to provoke this phenomenon. We will use words with three letters or less.

We obtained Figure 7 omitting the high frequency force; Figure 8 for an amplitude of this force near to the maximum of the vibrational resonance; Figure 9 for an amplitude that provoke bad performance of the word-series method; and Figure 10 for an amplitude that maximizes the vibrational resonance.

We also obtained the Table VI with the comparison of times of computation for word-series averaging and ode45 and the differences between the solutions.

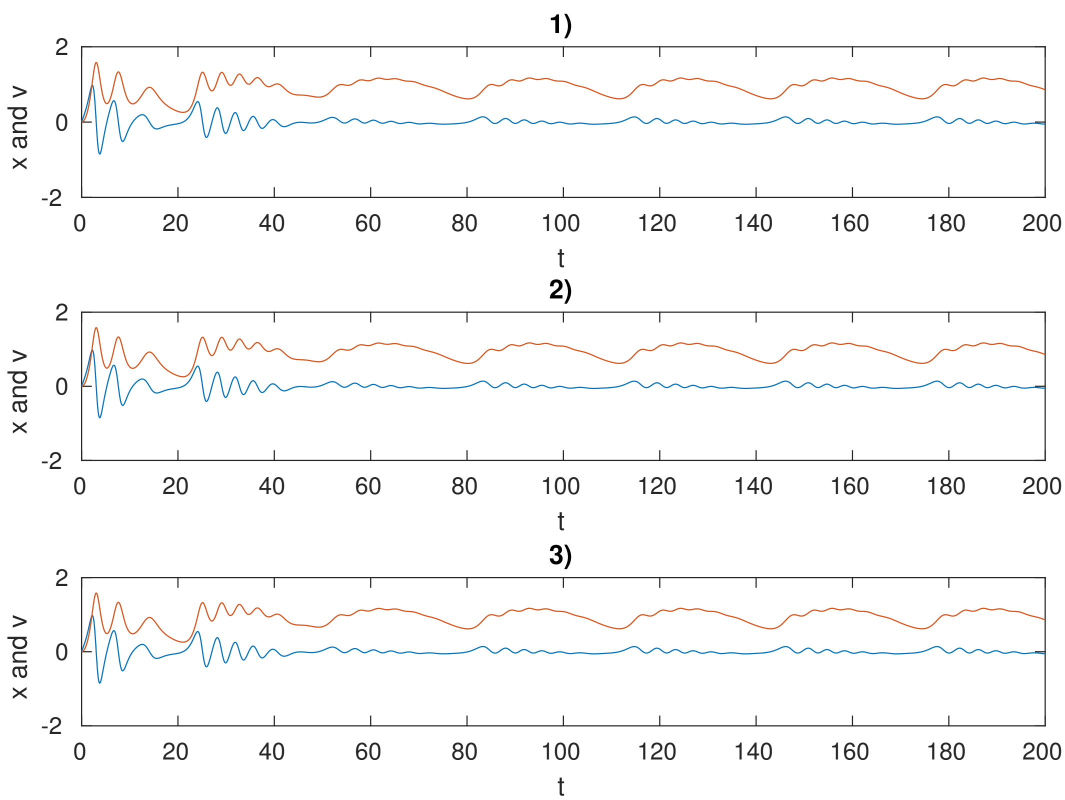

Figure 7: (Position in orange and velocity in blue.) 1) Solution of (1), 2) Solution of (28) and 3) Solution of (28) with change of variable. Computed with the parameters: $B = 0$, $A = 0.37$, $\nu = 0.2$, $\delta = 0.1$, for 200 units of time.

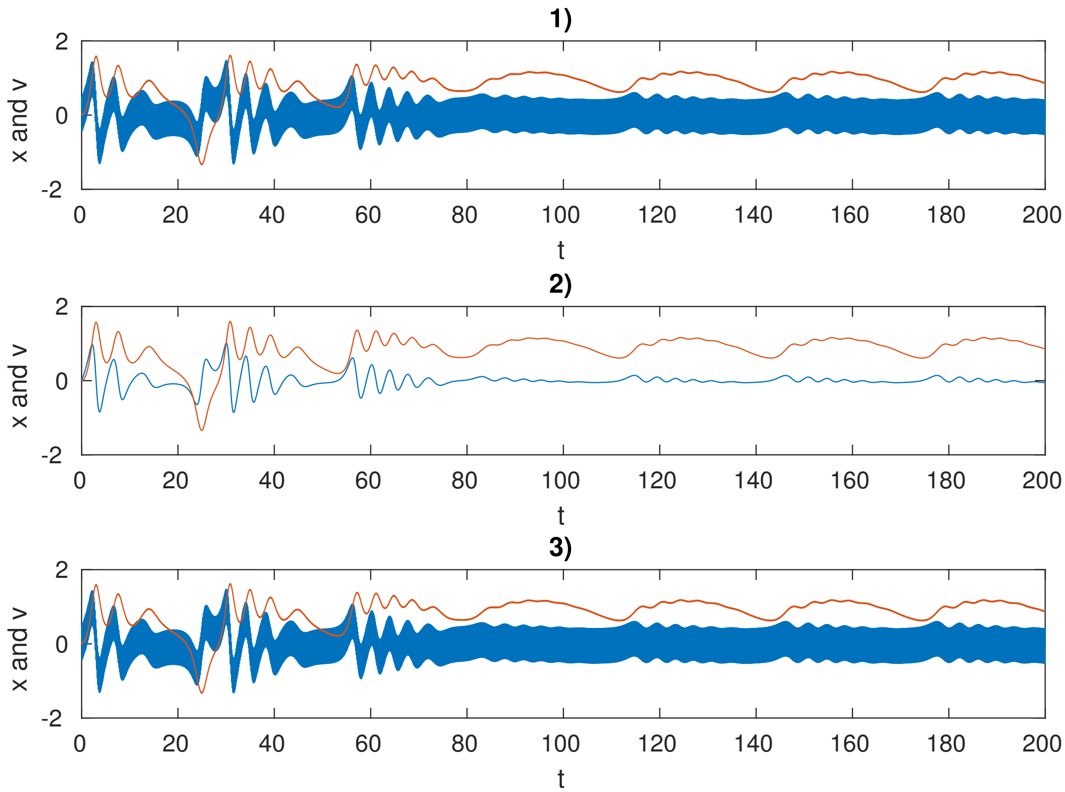

Figure 8: (Position in orange and velocity in blue.) 1) Solution of (1), 2) Solution of (28) and 3) Solution of (28) with change of variable. Computed with the parameters: $B = 0.49$, $\omega = 50$, $A = 0.37$, $\nu = 0.2$, $\delta = 0.1$, for 200 units of time.

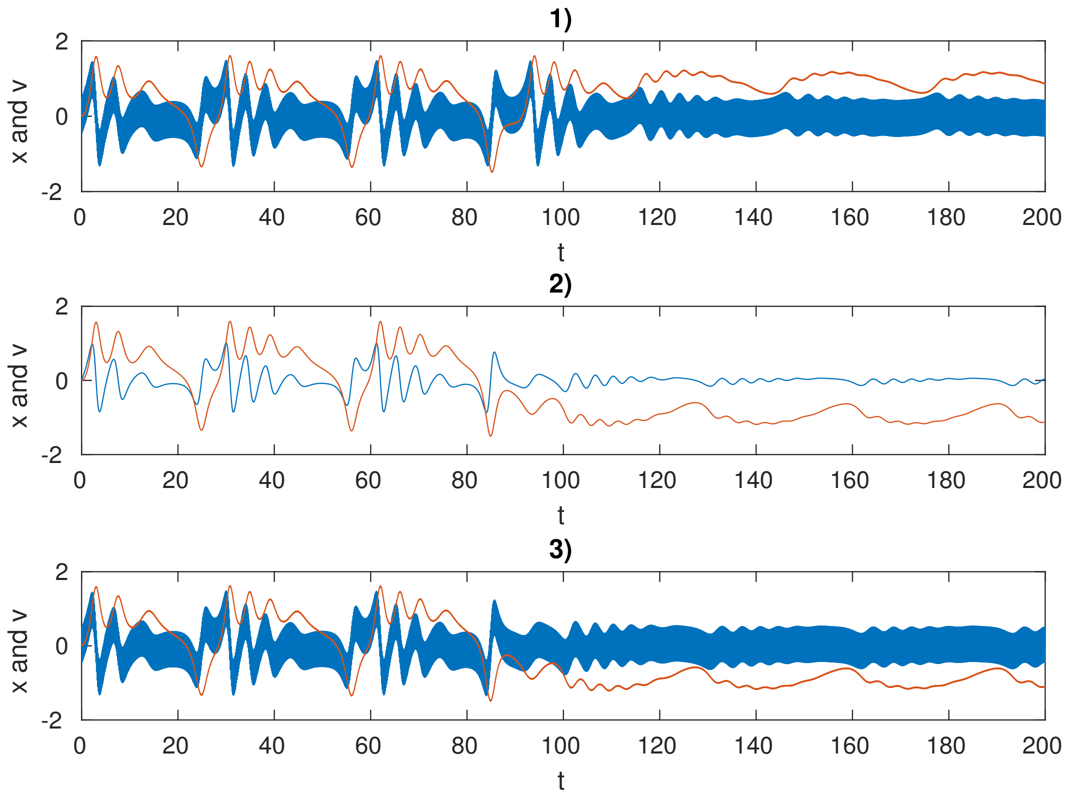

Figure 9: (Position in orange and velocity in blue.) 1) Solution of (1), 2) Solution of (28) and 3) Solution of (28) with change of variable. Computed with the parameters: $B = 0.50$, $\omega = 50$, $A = 0.37$, $\nu = 0.2$, $\delta = 0.1$, for 200 units of time.

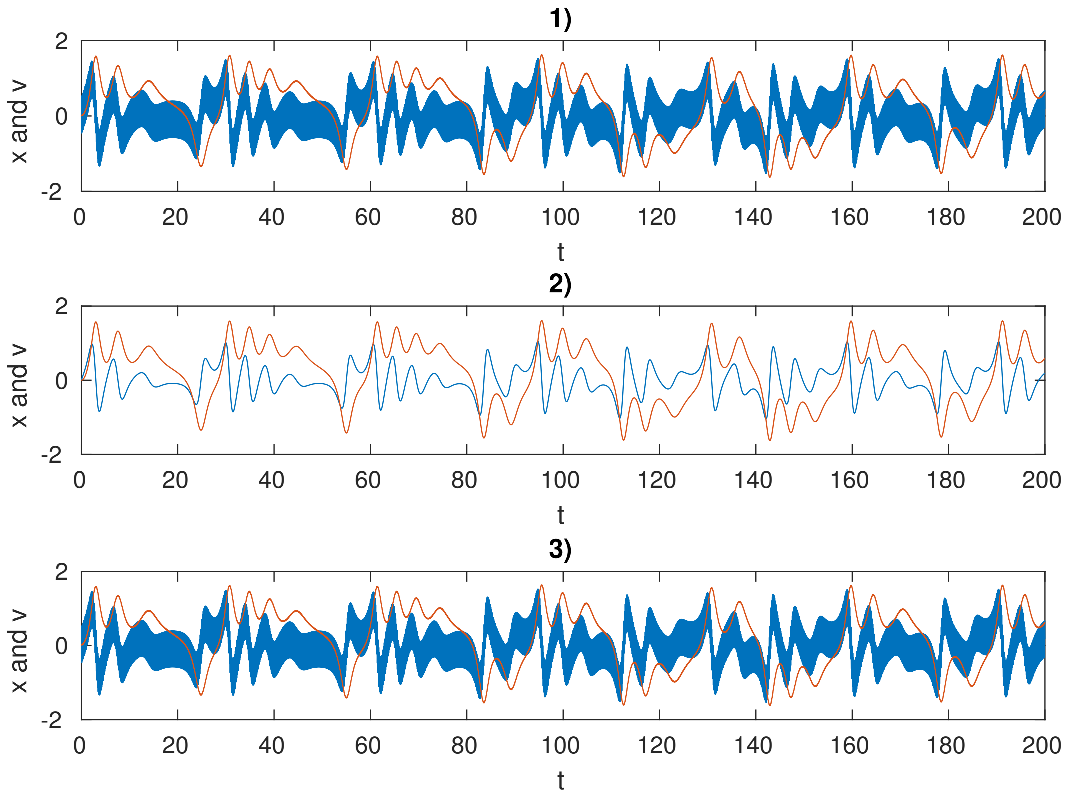

Figure 10: (Position in orange and velocity in blue.) 1) Solution of (1), 2) Solution of (28) and 3) Solution of (28) with change of variable. Computed with the parameters: $B = 0.51$, $\omega = 50$, $A = 0.37$, $\nu = 0.2$, $\delta = 0.1$, for 200 units of time.

B	No Averaging Time (s)	Averaging Time (s)	Maximum of Difference
0	1.006566	0.862332	0.000000000000131
0.49	7.683202	0.841099	0.066689388063071
0.50	7.690391	0.963427	2.480010274851534
0.51	7.580135	0.849172	0.022594421213023

Table VI: Comparison of methods for VR.

VII. DISCUSSION

A. Analysis of Word-Series Averaging

If we compare the times of word-series averaging of Table I with Tables III, IV and V, we can see there is a significant difference between them. Larger values of ω larger difference in times because the costs of integrating a more oscillatory system grows quicker than that of integrating the correspondent averaged system.

Errors behave as predicted, smaller errors for higher orders. Furthermore, for each method, the errors decrease as ω increases and they do following the order of the method, i.e. proportional to the inverse of the order power of ω.

B. Vibrational Resonance discussion

For vibrational resonance we obtain the same except for some values of the amplitude of the high frequency force (we only show one of them in results above). For these values, the word-series method of fourth and less order was unable to average the real motion, the method was in the wrong basin, as we can see in Figure 9.

We argue that this problem is provoked by the extreme sensibility around the unstable fixed point (0,0) in the phase space, i.e. $x = 0$ and $\dot{x} = v = 0$, for this choices of B. A small error committed near this point can bring the averaged solution to the wrong basin. In fact, in (28), the term $-9/2\, B^2/\omega^2\, x$ 'softens' the force x and therefore lowers the energy barrier at $x = 0$. This make it easier for solutions to transition between the two basin.

VIII. CONCLUSIONS AND FUTURE WORK

The word-series averaging method is useful to solve high frequency oscillatory problems because it improves the time of computation and without paying a high price in the error. In fact, the errors of the method will be smaller the higher oscillatory behaviour of the problem.

One example of this kind of problem are systems with vibrational resonance. In previous works, Murua and Sanz-Serna used this method for (2) and they did not find any issue, but we discover in this work a strange behaviour for the method for (1). For some choices of parameters, the method is accurate and stable but for a small perturbation of a parameter, it may not identify correctly the behaviour of the solution.

In the future we may apply the method to more complex dynamical systems and also increase the order of the method by considering longer words. In this thesis all algebraic computations have been carried out by hand. Future work will benefit of the use of symbolic manipulating programs.

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