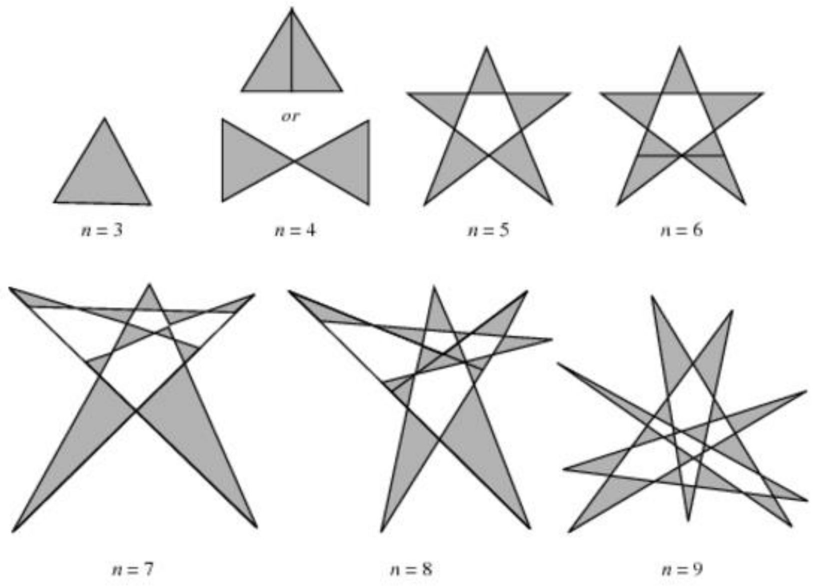
Kobon Triangles

The question is: What is the maximum number of non-overlapping triangles realizable by *n* straight lines in a plane?

- It was originally stated in the book “The Tokio Puzzle” by Kobon Fujimara, in 1978 [1-2].
- For up to six lines it’s easy to find K(n) and the corresponding optimal configurations, Figure 1.
- According to [3] up to now, this is an unsolved problem in combinatorial geometry.

- [1] K. Fujimura, “The Tokyo Puzzle”, Charles Scribner’s Sons, New York, 1978.
- [2] M. Gardner, “Wheels, Life, and Other Mathematical Amusements”. W. H. Freeman, New York, pp. 170-171 and 178, 1983.
- [3] G. Clément, J. Bader, “Tighter Upper Bound for the Number of Kobon Triangles”, Dec. 21, 2007



- Figure 1.

Saburo Tamura upper bound proof

- On Table I, the number of non-overlapping triangles *K(n)* that the best known configuration of *n* straight lines realizes.
- The bold numbers reach the bound of Tamura, the (*) indicates the configuration which reach the tighter bound (2), and therefore are maximum as well.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$(n \mod 6)$	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2
$K(n)$	**1**	**2**	**5**	7*	**11**	15*	**21**	25	32	38	**47**	53	**65**	72	**85**	93	104	115
Bound of Tamura	1	2	5	8	11	16	21	26	33	40	47	56	65	74	85	96	107	120
Authors' bound	1	2	5	7	11	15	21	26	33	39	47	55	65	74	85	95	107	119

Table I

- Saburo Tamura has proved an upper bound which is describe in (1)

$$K(n) \leq \binom{n(n-2)}{3} \quad \forall\, n \geq 3 \quad (1)$$

$$K(n) \leq \left\lfloor \frac{n(n-2)}{3} \right\rfloor - \mathbf{I}_{\{n|(n \mod 6) \in \{0,2\}\}}(n) \qquad (2)$$

- Definition 1.- A perfect configuration is an arrangement of *n* pairwise intersecting lines, where each segment is the side of exactly one non-overlapping triangle and *K(n)* meets the upper bound (1). For example n=3 or 5.

Saburo Tamura upper bound proof

- **Lemma 1**. If (n mod 3) ∈ $\{0,2\}$ then all configurations that meet the upper bound (1) are perfect configurations.
- In these cases *n(n-2)*=0 mod 3; hence *K(n)*=*n*(*n*-2)/3.
- **Proof:** it is necessary to show BWOC that no configuration with common side triangles and (n mod 3) ∈ $\{0,2\}$ exist with $\mathrm{K(n)} = \mathrm{n(n-2)/3}$.
- This is equal to show that the number of line segments needed is larger than 3K(n).
- We all know that a triangle needs three line segments if they don't side another triangle.

- From Figure 2 We can see:
- 1) A line segment can be the side of one or more triangles
- 2) If a line segment is the side of two triangles then the corresponding line intersects at one of the two endpoints an existing point which belongs to both triangles.
- 3) Every intersection point with more than two corresponding lines is part of at most two pairs of triangles that share a common side.

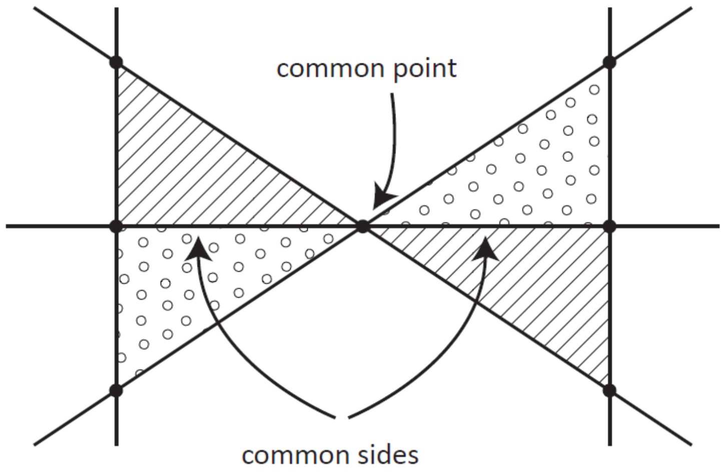


Figure 2. Two pairs of triangles share one common side each and a common point which is the intersection of (at least) three lines.

Saburo Tamura upper bound proof

- By looking at Figure 3 we see that if a line intersects an existing point then the number of points decreases by 2 and the number of segments by 3. Hence we ”save” at most two segments (the common ones) but we loose three due to the intersection of more than two lines in one point.
- Hence the number of line segments needed to build triangles increases if one side belongs to two triangles.
- Clearly, if a side does not belong to any triangle the number increases as well.
- Therefore the number of line segments needed is minimized, if each line segment belongs to exactly one triangle and cannot be reached in all other cases.

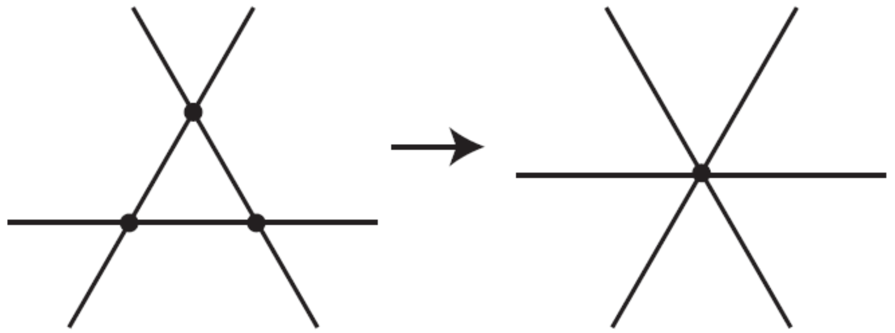


Figure 3. The number of line segments and points decreases by 3 and 2 respectively if a line intersects an existing point.

Biplanar Crossing Number

On the Biplanar Crossing Number [1]

- When designing printed circuit boards (PCBs) or integrated circuits (ICs) it is desirable to minimize the number of jumpers and via-holes.
- Replacing the components' pins with vertices and the paths between them with edges yields the linear graph G.
- We want to split G into two subgraphs G_1 and G_2 that correspond to the two sides of a PCB or to two layers of metallization on an IC.
- Our problem becomes minimize $v_2(G)$.

[1] A. Owens, "On the biplanar crossing number," in *IEEE Transactions on Circuit Theory,* vol.18, no.2, pp.277-280, Mar 1971

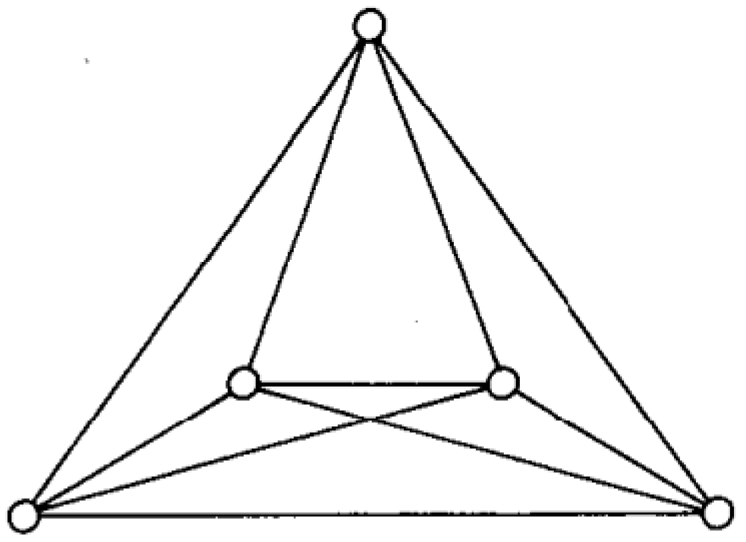
On the Biplanar Crossing Number [1]

- The first step is to determine if G is planar. Which would imply $v_2(G) = 0$.
- The second step is to determine the thickness t of G. Given t and the t subgraphs we could construct a PCB with no crossings in t layers.
- The third step is to draw G in one plane so that the edges have the fewest number $v_1(G)$ of crossings.
- The final step is to determine the fewest number of edge crossings when the graph is drawn on two planes.

[1] A. Owens, "On the biplanar crossing number," in *IEEE Transactions on Circuit Theory,* vol.18, no.2, pp.277-280, Mar 1971

On the Biplanar Crossing Number [1]

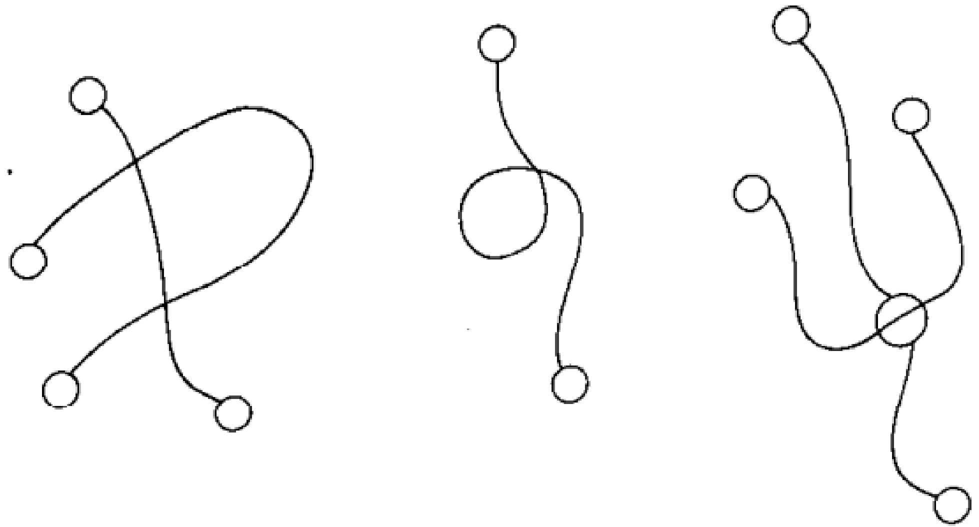
- Side note…
- K_5 drawn with one edge crossing, so
- $v_1(K_5) = 1$.
- Since we know $v_1(K_5) > 0$ and with the figure on the right we conclude $v_1(K_5) = 1$.



[1] A. Owens, "On the biplanar crossing number," in *IEEE Transactions on Circuit Theory,* vol.18, no.2, pp.277-280, Mar 1971

On the Biplanar Crossing Number [1]

- Forbidden crossings:
- No crossings at vertices.
- No two edges may cross more than once.



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On the Biplanar Crossing Number [1]

- Assuming that we have to subgraphs of G (G_1, G_2) with $G = G_1 \cup G_2$ we can define the biplanar crossing number $v_2(G)$ as

$$v_2(G) = \min(v_1(G_1) + v_1(G_2))$$

- So the problem is to obtain these two graphs.
- The problem becomes more constrained given that some sets of vertices must have specific relative positions.

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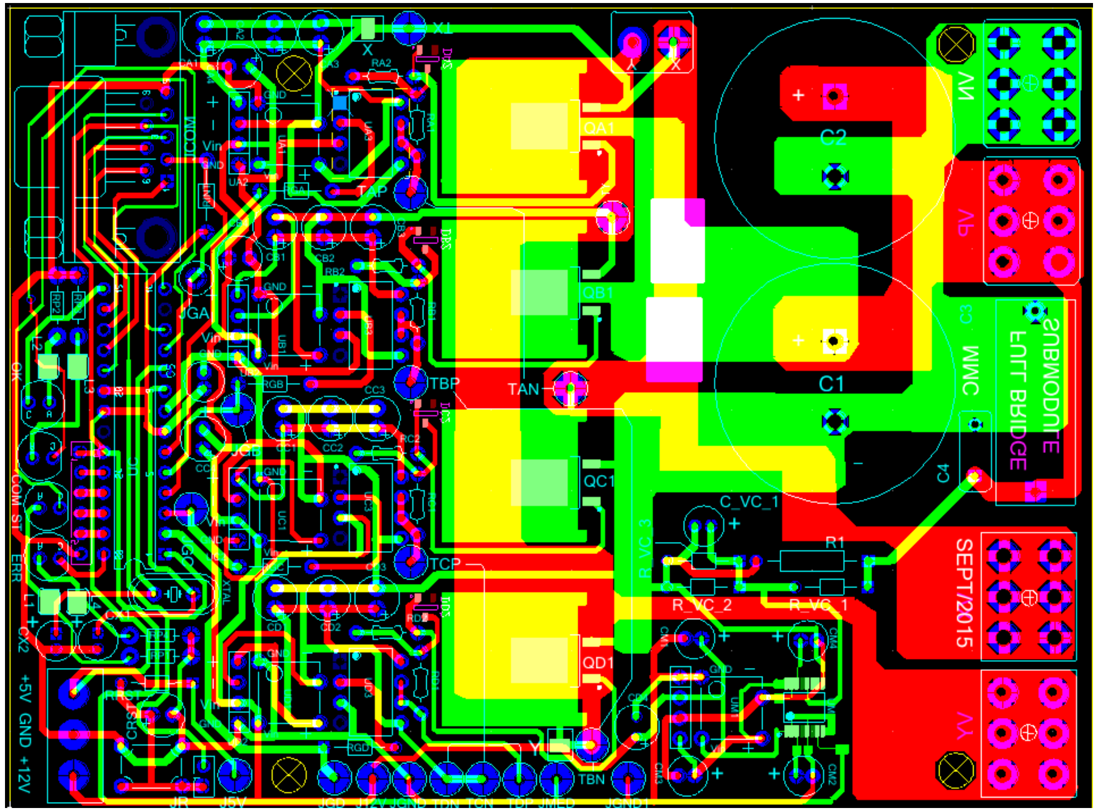
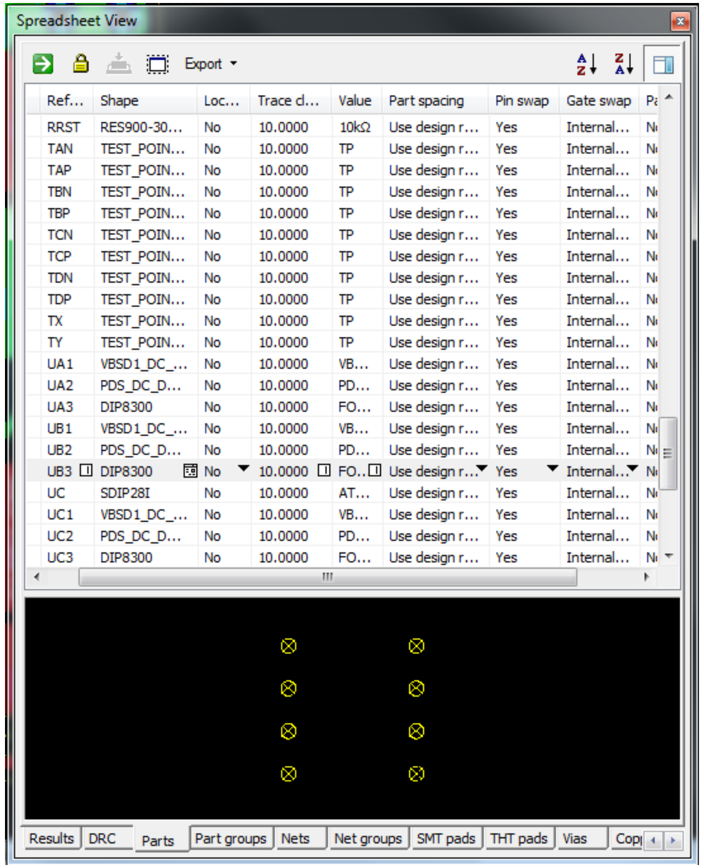
On the Biplanar Crossing Number [1]





[1] A. Owens, "On the biplanar crossing number," in *IEEE Transactions on Circuit Theory,* vol.18, no.2, pp.277-280, Mar 1971

On the Biplanar Crossing Number [1]

- The authors propose to obtain a simple biplanar model for K_n by decomposing it into three subgraphs:

$$K_{[n/2]}; \qquad K_{\{n/2\}}; \qquad K_{[n/2],\{n/2\}}.$$

[x] implies the ceiling function {x} implies the floor function

- The crossing numbers of the three graphs added would yield the crossing number of K_n.

$$v_2(K_n) = v_1\left(K_{[n/2]}\right) + v_1\left(K_{\{n/2\}}\right) + v_1\left(K_{[n/2],\{n/2\}}\right)$$

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On the Biplanar Crossing Number [1]

- Using previous conjectures from other papers, the authors give an upper bound for this biplanar crossing number with n even as:

$$v_2(K_n) \leq \frac{1}{2}\left[\frac{r}{2}\right]\left[\frac{r-1}{2}\right]\left[\frac{r-2}{2}\right]\left[\frac{r-3}{2}\right] + \left[\frac{r}{2}\right]^2\left[\frac{r-1}{2}\right]^2,$$

- where $n = 2r$.

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On the Biplanar Crossing Number [1]

- Furthermore, in the text the authors mention that K_9 **is the smallest complete graph which cannot be drawn on two planes without crossings**.
- From this statement we imply $v_2(K_{1\dots 8}) = 0$.
- Applying the formula to K_{10} and K_{12} yields

$$v_2(K_{10}) \leq 34 \text{ and } v_2(K_{12}) \leq 81.$$

[1] A. Owens, "On the biplanar crossing number," in *IEEE Transactions on Circuit Theory,* vol.18, no.2, pp.277-280, Mar 1971

Questions?