CSCI 5451 Fall 2015

Week 5 Notes

Professor Ellen Gethner

September 14, 2015

Greedy Algorithms: Second Big Example

- **Big Greedy Example 2: Data Compression by way of Huffman Encoding**

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- **Big Greedy Example 2: Data Compression by way of Huffman Encoding**

- **Problem.** Given a text file, find a compressed file, as small as possible, such that the original file can be reconstructed fully and efficiently.

- **Some Applications.**
  1. Storage savings

  2. Communications (when the cost of sending information is greater than the cost of reconstruction)

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- **Better Idea.** Choose smaller bit representations for characters that occur more frequently (like "e" in the English language).

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- Thus, in an encoding, we should as that all of the prefixes of the encoding of one character are not the same as a complete encoding of any other character.
- The above constraint is called the Prefix Constraint.

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- that minimizes the total number of bits needed to encode the text.

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- C_i will be represented by big string S_i of length s_i.
- Then the length of F encoded by E is

$$L(E,F) = \sum_{i=1}^{n} s_i f_i.$$

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- **Goal.** Find an encoding E that satisfies the prefix constraint, and
- that minimizes $L(E, F)$.

More Design Thoughts

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- Scan through the bit sequence left to right until we arrive at a character.
- Record the character and continued scanning until another character is found.
- Record and repeat.

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- If we build such a tree and somehow use the leaves of the tree in the encoding, then the encoding will satisfy the prefix constraint.
- That is, when the encoded file is scanned and we reach a leaf, we determine the encoding of the character.

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- We **won't** eliminate a character from the alphabet!

More Design Thoughts: Binary Tree

- **Another Problem Restatement.** Construct the binary tree that minimizes $L(E, F)$.
- **Goal.** Reduce the problem with n characters to one of $n - 1$ characters so that we can use inductive reasoning.
- We **won't** eliminate a character from the alphabet!
- Instead, we'll create a new artificial character made up of two low frequency characters.

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- In fact, since each node has either 0 or 2 children, we can always assume that C_i and C_j are siblings.
- The new character is called C_{ij} and will have frequency $f_i + f_j$, where recall that f_x is the frequency of C_x.

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- We'll do an example before writing down the algorithm.

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- Here comes the Huffman Tree...

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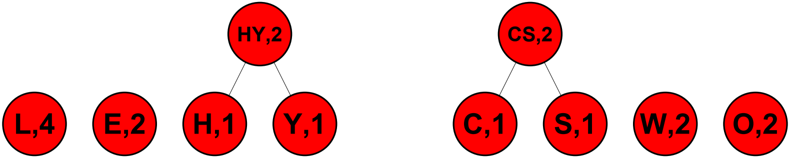

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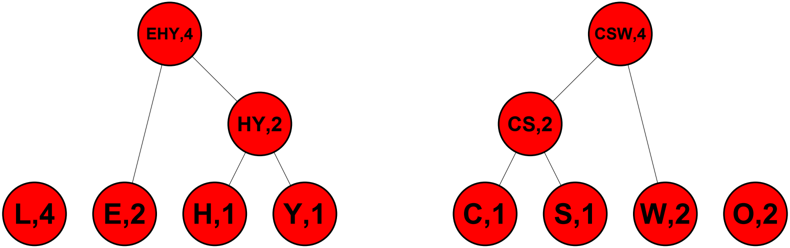

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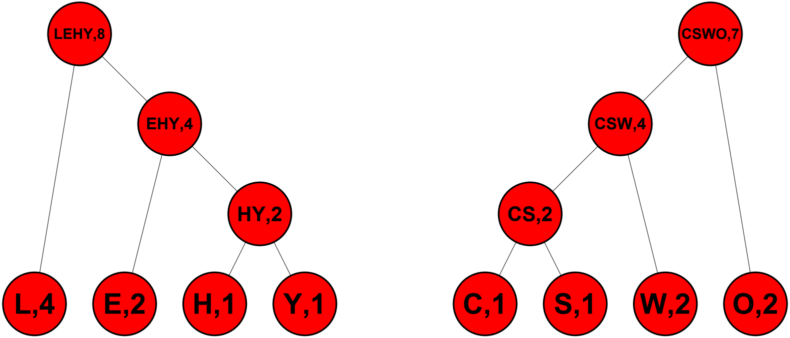

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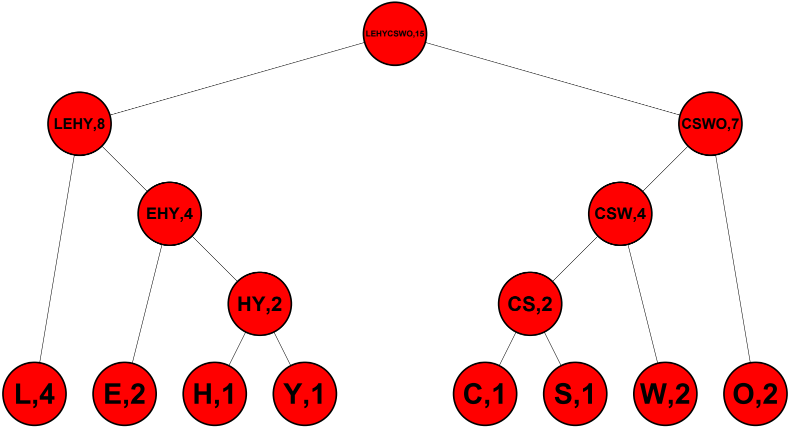
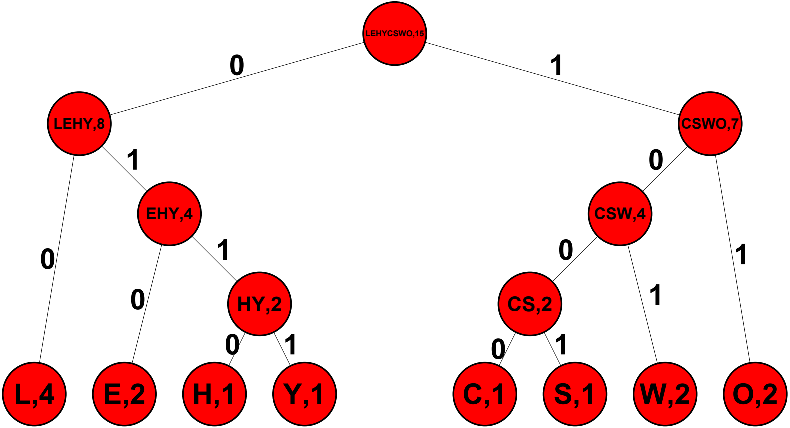


LEHYCSWO,15
LEHY,8
CSWO,7
EHY,4
CSW,4
HY,2
CS,2
L,4
E,2
H,1
Y,1
C,1
S,1
W,2
O,2

Example, continued

- Each internal node has two children; assign the left child a 0 and the right child a 1 (on edges to avoid clutter).

Example, continued



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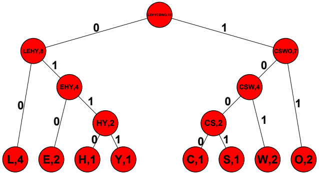

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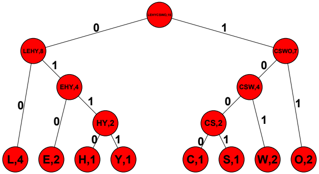

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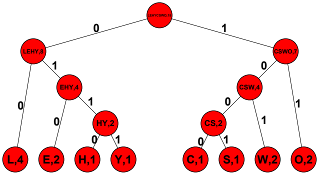

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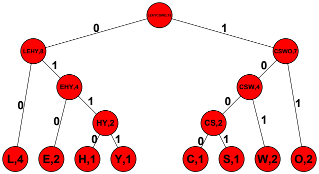

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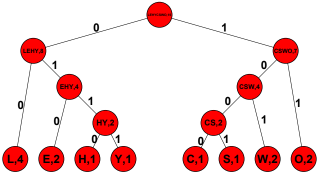

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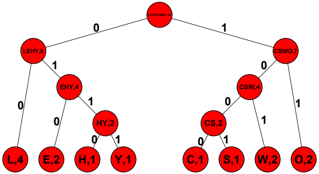

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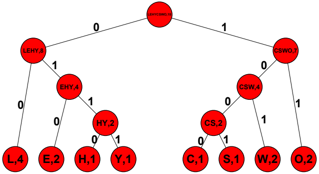

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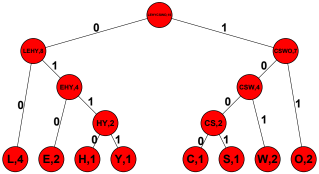

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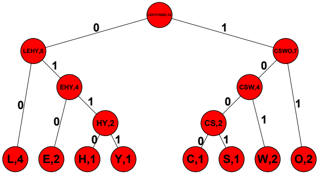

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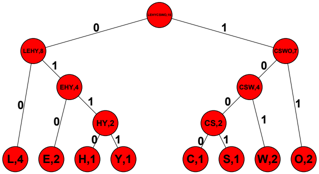

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- Thus, tallying the results, we see that $L(E, \mathit{helloyellowcows}) = 42$.

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- runs in time $O(lg(n)$ in the worst case.

Algorithm HuffmanEncoding(S, f)

- **Input** S (a string of characters) and f (the array of frequencies of S)
- **Output** T (the Huffman Tree for S)
- **begin**
- **insert** characters into **heap** H according to their
- frequencies (lowest to highest)
- **while** H is not empty **do**
- **if** H contains only one character X **then**
- make X the **root** of tree T
- **else** pick two characters X and Y with lowest
- frequencies f_x and f_y and **delete** them from H;
- **replace** X and Y with a new character Z whose
- frequency is $f_X + f_Y$;
- **insert** Z to H.
- make X and Y **children** of Z in T.
- **end**

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- with the frequency of new character $\underline{ab}$ given by $f_a + f_b$.

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- Let $\ell_{T'}(x)$ be the length of encoded character x in tree T'.

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Proof of Huffman Encoding correctness, continued

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- $= (f_c - f_a)(\ell_T(c) - \ell_T(a)) + (f_d - f_b)(\ell_T(d) - \ell_T(b))$

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- ≥ 0 because
  - $f_d \geq f_b$ and $f_c \geq f_a$, and
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- This completes the verification of (1) (the Greedy Choice Property).

Proof of Huffman Encoding correctness, continued

- **Subconclusion.** We may assume WLOG that any optimal code tree has low frequency characters at maximum distance from the root of T.

Proof of Huffman Encoding correctness, continued

- **Subconclusion.** We may assume WLOG that any optimal code tree has low frequency characters at maximum distance from the root of T.
- That is, low frequency characters are leaves of T.

Proof of Huffman Encoding correctness, continued

- **Verification of (2), The Optimal Substructure Property.**

Proof of Huffman Encoding correctness, continued

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- Let T be any optimal code tree for file F with alphabet C.

Proof of Huffman Encoding correctness, continued

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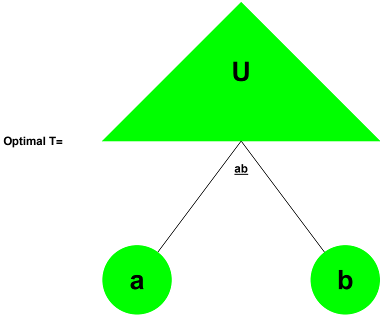
- 

Proof of Huffman Encoding correctness, continued

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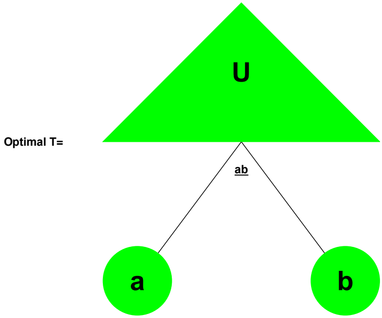

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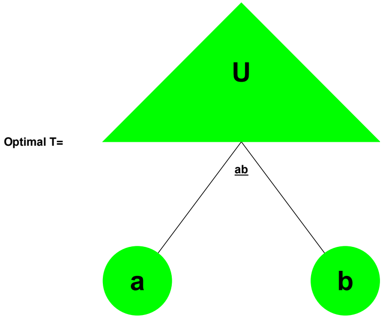
- 

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Proof of Huffman Encoding correctness, continued

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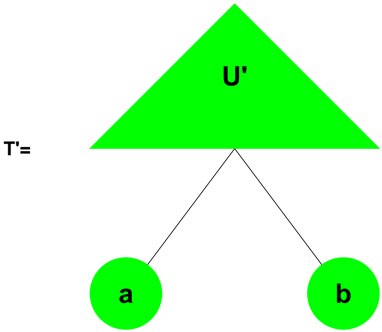

Proof of Huffman Encoding correctness, continued

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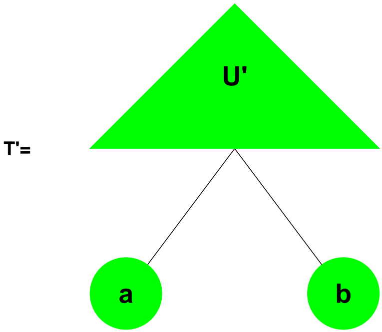

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Proof of Huffman Encoding correctness, continued

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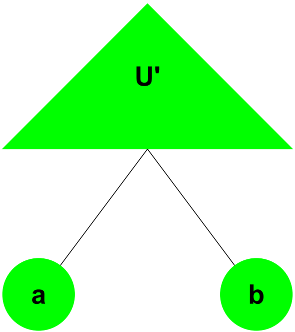

- By earlier work we know a and b are leaves of T', and
- call the remainder of the tree (minus a and b) U'.

Proof of Huffman Encoding correctness, continued

- Then $COST(T) = \sum_{x \in C \setminus \{a,b\}} f_x \ell_T(x) \quad + f_a \ell_T(a) + f_b \ell_T(b)$

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- $= \sum_{x \in C \setminus \{a,b\}} f_x \ell_T(x) + (f_a + f_b)(\ell_U(\underline{ab}) + 1)$
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- $= \sum_{x \in C \setminus \{a,b\}} f_x \ell_T(x) + f_{\underline{ab}} \ell_U(\underline{ab}) + f_{\underline{ab}}$

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- So far we have
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- $COST(U') + f_{\underline{ab}} < COST(U) + f_{\underline{ab}}$
- $\Rightarrow COST(T') < COST(T)$, a contradiction (since T is optimal).

Conclusion

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- **QED**
- **Exercise:** What is the run time of Algorithm HuffmanEncoding?

Next

Graph Theory and Graph Algorithms