CSCI 5451 Fall 2015

Week 1 Notes

Professor Ellen Gethner

August 19, 2015

Graduate Algorithms: Preliminaries

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- **Assignments.** There will be six homework assignments, none of which are handed in.

- **Quizzes.** Your mastery of each assignment will be assessed by way of a closed-book closed-notes quiz.

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- Quizzes count toward 30% of your final grade.

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- **Basic Algorithms** including but not limited to *Sequences and sets, graph algorithms, algebraic algorithms* and many more.
- **Advanced Topics** including but not limited to Approximation, NP-Completeness, Cryptography, Computational Geometry and more.

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- Recipe, process, method, technique, procedure, routine, ...
- Which do you think best describes what and algorithm is supposed to be?

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- **A5: Effectiveness.** Loosely speaking, an algorithm is effective if its operations are all basic enough that they can be done in a finite amount of time by someone using pencil and paper.

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- efficiency:
    1. How fast does your algorithm run, say, with respect to the size of the input? Can you do better?
    2. Alternatively, can you prove that you **cannot** do better?

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- **Analysis of Efficiency.** Use of mathematics and common sense.

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- Connections among the latter two items.

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- Generally speaking, our goal will be to prove that $P(n)$ is true for all integers $n \geq N$ for some fixed $N \in \mathbb{Z}$.

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5. **Conclusion.** Conclude that $P(n)$ is true $\forall n \geq N$.

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- Note that $f(n) = \sum_{i=1}^{n}(2i - 1) = 1 + 3 + 5 + \cdots + 2n - 1$.

Example One, continued

The first thing we should do is make a guess at a formula for $f(n)$ so let's experiment.

- $f(1) = \sum_{i=1}^{1}(2i - 1) = 2 \times 1 - 1 = 1.$

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- $f(3) = \sum_{i=1}^{3}(2i-1) = 1 + 3 + 5 = 9.$

Ready for a conjecture

Conjecture. $f(n) = n^2 \qquad \forall n \in \mathbb{Z}^+.$

Proof by induction on n

The statement $P(n)$ says that $f(n) = n^2$ and we wish to prove that $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

- **Base Cases.** We verified that $P(1), P(2)$, and $P(3)$ were all true on the previous slide.

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- **Induction Hypothesis.** Assume that $P(k)$ is true for some **fixed** integer $k \geq 3$.

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- $= k^2 + 2k + 2 - 1$ by the induction hypothesis and algebra
- $= k^2 + 2k + 1$
- $= (k+1)^2$, as desired.

Induction proof, completed

Conclusion. $P(n)$ is true $\forall n \in \mathbb{Z}^+$. That is, $f(n) = n^2 \forall n \in \mathbb{Z}^+$.

QED

Example Two: Counting Squares

Problem. How many squares are there in an $n \times n$ grid-square?

Notation. Let $f(n)$ denote the number of squares in an $n \times n$ grid square.

Accounting for small values of n: $n = 1$

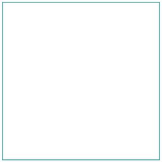

Figure: A 1×1 grid square

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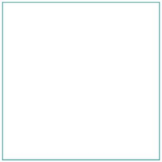

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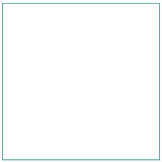

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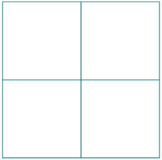

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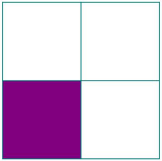

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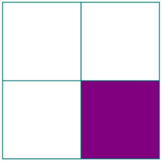

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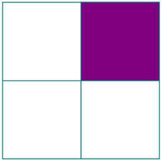

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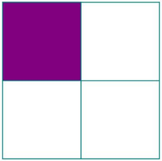

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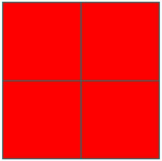

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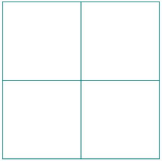

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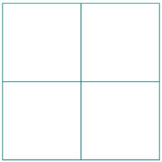

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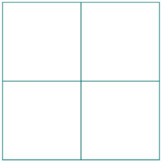

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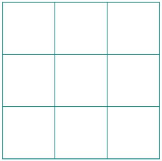

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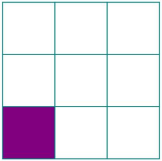

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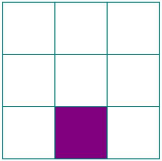

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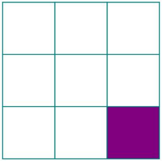

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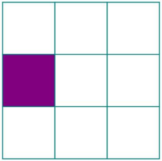

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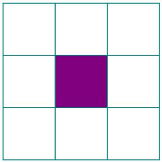

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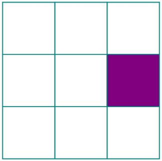

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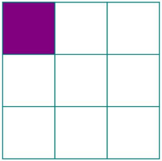

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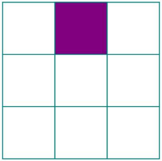

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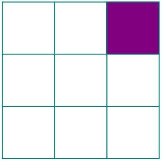

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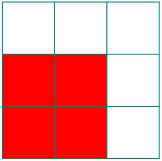

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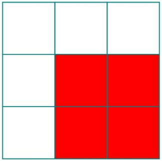

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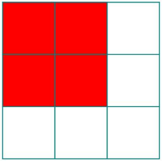

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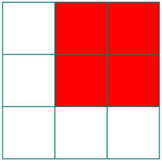

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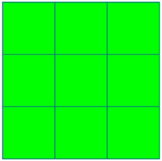

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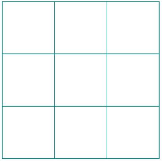

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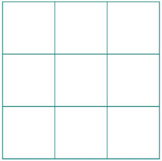

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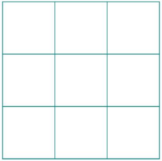

Figure: A 3×3 grid square

- Looks like $f(3) = 9 + 4 + 1$.
- Curiouser and curiouser...
- Time for a conjecture...

Conjecture

The statement $P(n)$, namely our conjecture, is as follows:

If $f(n)$ is the number of squares in an $n \times n$ grid square then

$$f(n) = \sum_{i=1}^{n} i^2 = 1^1 + 2^2 + \cdots + n^2$$

Induction Argument to verify that $P(n)$ is true $\forall n \in \mathbb{Z}^+$

- **Base Cases.** We have verified that $P(1)$, $P(2)$, and $P(3)$ are all true.

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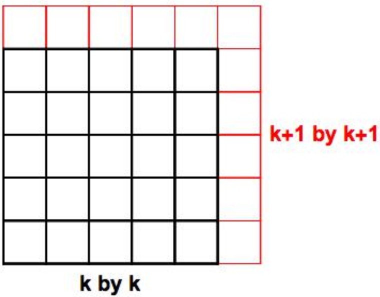
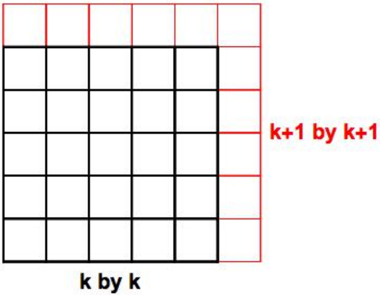
- Consider a $(k+1) \times (k+1)$ grid square.



Inductive Step

- Since the lower left $k \times k$ (black) square has been accounted for by the induction hypothesis,



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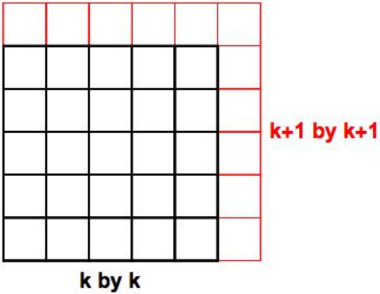
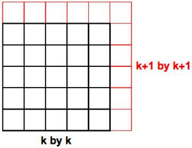


- it remains to count all possible new squares, namely only the ones that include the top row and right column of red squares.

Counting the remaining squares



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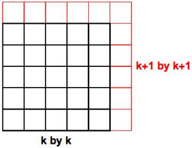

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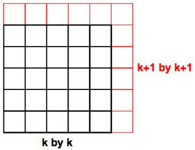

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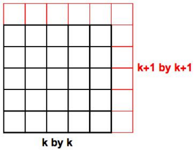

- 1×1's: $(k+1)+k=2k+1$
- 2×2's: $k+(k-1)=2k-1$
- 3×3's: $(k-1)+(k-2)=2k-3$
- $\vdots$

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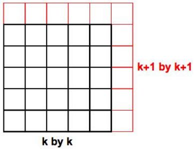

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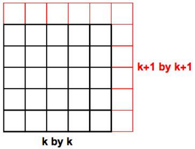

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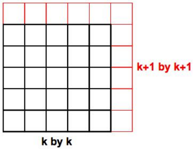

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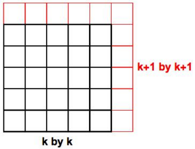

- $\vdots$
- $k \times k$'s: $(k-(k-1))+(k-(k-2))=3$
- $k+1 \times k+1$'s: $(k-(k))+(k-(k-1))=1$
- Thus the total number of **new** squares is $\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$ by Example 1.

Total number of squares in a $k+1 \times k+1$ grid square

- **Conclusion.** Thus, by induction, the total number of squares in a $k+1 \times k+1$ grid square is

$$(\sum_{i=1}^{k} i^2) + (k+1)^2 = \sum_{i=1}^{k+1} i^2$$

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- **QED**

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- $F_0 = 0$, $F_1 = 1$, and for all integers $n > 1$ we define $F_i = F_{i-1} + F_{i-2}$.
- For example, the first 12 numbers in the *Fibonacci sequence* are 1,1,2,3,5,8,13,21,34,55,89,144

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- Generally speaking, we will study the growth of functions, namely *asymptotics*, with the intention of measuring the efficiency of our algorithms.

Example 1

Problem. Show by induction that $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$ for all integers $n \geq 1$.

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  $F_1 + F_3 + F_5 = 1 + 2 + 5 = 8 = F_6 = F_{2\times 3}$. ✓

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- But by the definition of the Fibonacci numbers we have that $F_{2k} + F_{2k+1} = F_{2k+2}$, as desired.

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- **Base Cases.** $n = 1$ (test $P(1)$):
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- That is, assume $F_{k+1}F_{k-1} - {F_k}^2 = (-1)^k$ for some fixed integer $k \geq 1$.

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- $= (-1)^{k+1}$, as desired.

Make a conclusion

Conclusion. By induction $F_{n+1}F_{n-1} - F_n^2 = (-1)^n \quad \forall n \in \mathbb{Z}^+$.
QED

A generating function for the Fibonacci numbers

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- The above is called a *formal power series*, which you've learned about in your calculus classes.

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- To see why claim 1 is true, let's deconstruct each of $zg(z)$ and $z^2 g(z)$ from the original definition of $g(z)$ as a formal power series.

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- Now let's carry out the same procedure of $z^2 g(z)$.

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- Next, we'll massage (1) and (2) to make the final algebra easier to digest when we finally compute $z + zg(z) + z^2 g(z)$.

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- $= F_0 z^0 + F_1 z + \sum_{i=2}^{\infty} F_i z^i = \sum_{i=0}^{\infty} F_i z^i$
- $= g(z)$, as claimed. This completes the proof of Claim 1.

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- Then formally speaking (all issues of convergence aside), we have $S = \frac{a}{1-r}$.

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- Let's experiment with *Mathematica*.