CSCI 5451 Fall 2015

Week 3 Notes

Professor Ellen Gethner

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- we'll use this as a tool to measure the run time (or possibly memory consumption) with respect to the input.
- What follows is a very brief review of a subset of Chapter 3 in our textbook.

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- We write $f(n) = O(g(n))$.
- Note that $g(n)$ may not provide the best possible upper bound for $f(n)$.

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- $2^{lg(lg(n))} \geq 2^{c lg(n)}$
- $\Rightarrow 2^{lg(lg(n))} \geq 2^{lg(n^c)}$
- $\Rightarrow lg(n) \geq n^c$, a contradiction.

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- The best possible asymptotic bound is Big Θ.
- In particular, Θ will give both an upper and lower bound at the same time,
- which means you've found a function that mimics the exact run time of your algorithm.
- That is, $\Theta(g(n)) =$

  $\{f(n) : \exists c_1, c_2, n_0 > 0 \text{ such that } 0 \leq c_1 g(n) \leq f(n) \leq c_2(g(n) \ \forall n \geq n_0\}$.

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- $\forall a, b, C \in \mathbb{R}$ with $a \neq 0$ we have that $an^2 + bn + C = \Theta(n^2)$.
- Unravel the definition of Θ and come up with values for n_0, c_1, and c_2.

Artist's Interpretation of O, Ω, and Θ

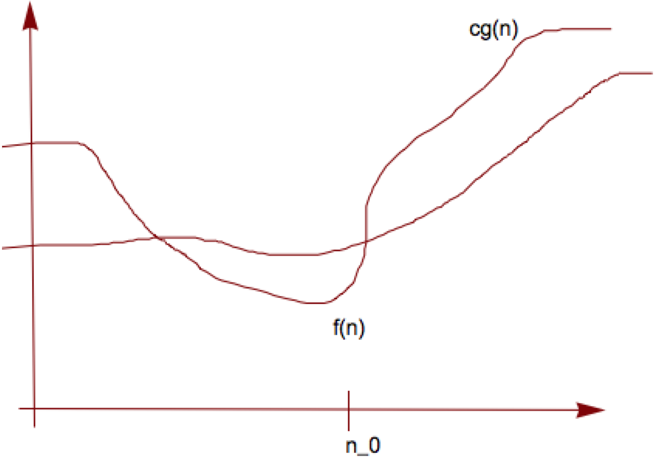

Figure: $f(n) = O(g(n))$

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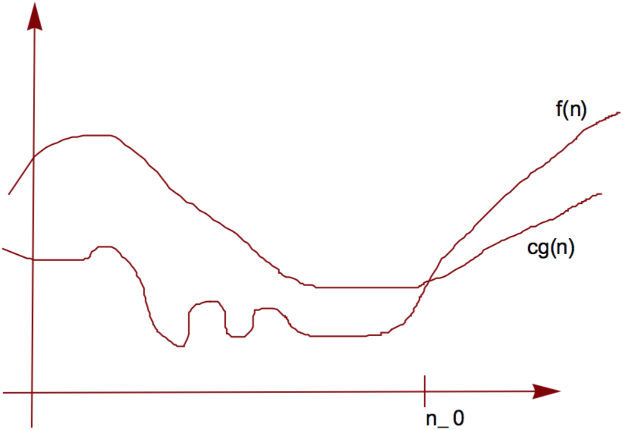

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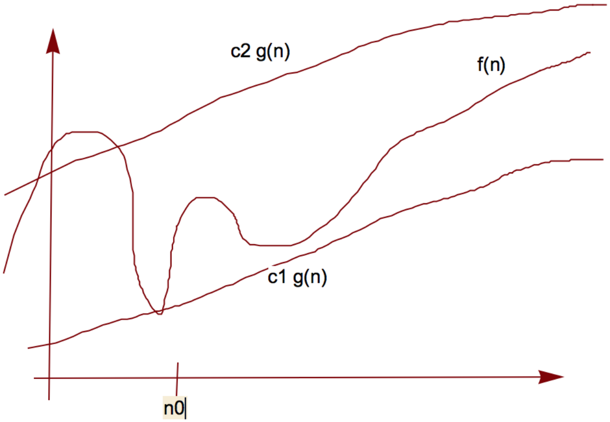

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- For a familiar example of this scenario, read **2-3** in our text on Mergesort.
- **Goal.** We want a closed-form asymptotic formula for $T(n)$.
- The final answer will depend on the nature of $f(n)$.

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- $T(n) = \Theta(f(n))$.
- We will prove Case 1 for the special case that n is an exact power of b.

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- For notational convenience, define $n_i = \frac{n}{b^i}$.

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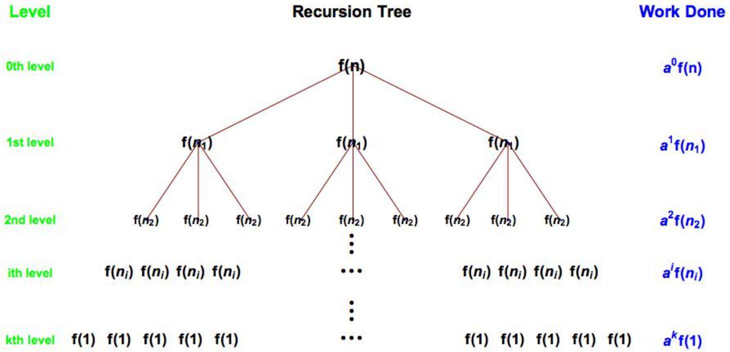

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- How much work is done at the remaining $k - 1$ levels?

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- **QED**

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- How can we recognize such problems?

Recognizing a dynamic programming problem: two properties

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2. **Overlapping Subproblems.** Solving the Big Problem recusively involves solving the same little problem over and over.

Big Example 1: The Knapsack Problem

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- **Set-up and Notation.**
  - Let $P(n, K)$ denote the original problem (n items and knapsack of size K)
  - Assume that the n items long with their sizes $k_1, k_2, \ldots k_n$ are given as input.
  - Let $P(i, k)$ denote the problem with the first i items and a knapsack of size k.

Knapsack problem, continued

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- later on, if a solution exists, work on the particulars.

Algorithmic Solution to the Knapsack Problem

- **Algorithm Sadsack**
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- and we simply choose not to pack the nth item, and we are done.

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- Again ☹

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- the (i, k)th entry contains information about the solution of $P(i, k)$.

- If we want to know the actual subset of items to be packed, we can add a field to the (i, k)th entry that indicates whether or not item i is to be packed.

Algorithm Knapsack(S, K)

Input. S, a $1 \times n$ array storing item sizes $k_1, \ldots, k_n$ and K, the size of the knapsack. So $S[i] = k_i$.

Output. P, a 2-dimensional array such that

$$\begin{cases} P[i,k].\mathit{exist} = \text{true} & \text{if } \exists \text{a solution to } P(i,k) \\ P[i,k].\mathit{belong} = \text{true} & \text{if the ith item is to be packed} \end{cases}$$

Algorithm Knapsack, continued

- **begin**
- $P[0,0].exist = \text{true}$
- **for** $k = 1$ **to** K **do**
- $P[0,k].belong = \text{false}$
- **for** $i = 1$ **to** n **do**
- for $k = 0$ to K do
- $P[i,k].exist = \text{false}$ *(default value)*
- **if** $P[i-1,k].exist = \text{true}$ **then**
- $P[i,k].exist = \text{true}$
- **else if** $k - S[i] \geq 0$ **then**
- **if** $P[i-1, k - S[i].exist = \text{true}$ **then**
- $P[i,k].exist = \text{true}$
- $P[i,k].belong = \text{true}$
- **end**

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- each of which is computed in constant time from two other entries in the array
- $\Rightarrow$ the total run time for the decision problem is $O(nK)$, which is better than exponential in general.
- Tracing back through the array $P[n, K]$ to find the valid "belong" elements takes time $O(n)$.

Next

We'll do another dynamic programming example next week, and then move on to greedy algorithms.