CSCI 5451 Fall 2015

Week 8 Notes

Professor Ellen Gethner

October 4, 2015

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- The problem was reduced to one of checking a finite number of graphs (about 2000) for special properties.

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- The number of configurations to check was reduced to about 500, and of course computers had become much faster in the intervening two decades.
- There is still no known computer-free proof of the Four Color Theorem.

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- When Heawood found and could not repair the flaw in Kempe's four color argument,
- he started thinking about and inventing graph color problems that he *could* solve.
- Here is one such problem.

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- **Proof.** Recall that G is a planar graph, and let G^* be derived from G by
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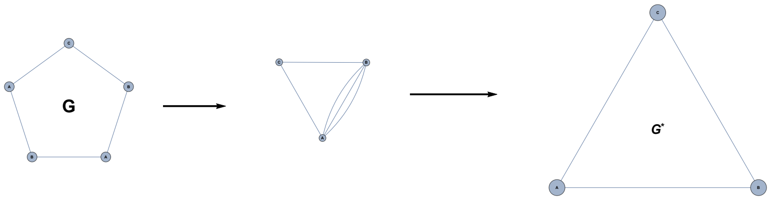
  2. maintaining the original adjacencies, but removing multiple edges.
  3. There won't be any loops in G^*. Why?

Example of deriving G^* from G

Below G is an example of a 3-pire map; the names of the 3-pires are A, B, and C.



Back to $\chi(G)$ for G an M-pire graph

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- Then $deg(n_1^*) + deg(n_2^*) + \cdots + deg(n_{v^*}^*) = 2e^*$

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- $\leq 2e$ (by Fact 2)
- $\leq 6v - 12$ (follows from the proof of Lemma A)
- $\leq 6Mv^* - 12$ (by Fact 1).
- In that case, the average vertex degree in graph G^* is

$$\frac{\sum_{i=1}^{v^*} n_i^*}{v^*} = \frac{2e^*}{v^*} \leq \frac{2e}{v^*} \leq \frac{6v - 12}{v^*} \leq \frac{6Mv^*}{v^*} - \frac{12}{v^*}$$

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- Thus if G is an M-pire graph, then $\chi(G) \leq 6M$.
- We know that the bound is not best possible when $M = 1$ (why?), but it is best possible for all $M \geq 2$ as is shown by the next theorem.

Theorem M give best result for all $M \geq 2$

- **Theorem M' (G. RIngel and B. Jackson 1983).** For every $M > 1$ there is an M-pire graph whose chromatic number is exactly $6M$.

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- What exactly *is* Ringel's Earth-Moon question?

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- **Definition.** If such an Earth-Moon graph G can be decomposed into two, and no fewer, planar graphs, then G is said to have thickness two.

Generalizing thickness

- **Definition T.** Let G be an undirected graph such that the edges of G can be partitioned into t sets, each of which induces a planar graph, and for which t is smallest possible. Then G is said to have thickness t, denoted by $\theta(G) = t$.

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- The notion of the thickness of a graph helps to measure the *near planarity* of a graph, and is a tool used in VLSI design, for example.

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- ... what is $\theta(K_8)$, $\theta(K_9)$?
- What is $\theta(P)$, where P is the Petersen graph?

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- Can anybody reduce the list at all?

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- **QED**

Sulanke's 9-chromatic thickness-2 graph

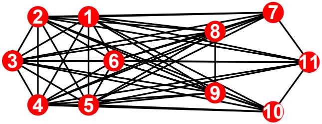

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Figure: K_6 joined with C_5

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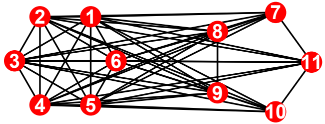
- 

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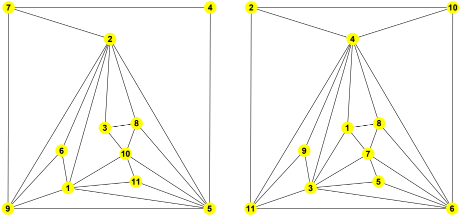
- 

Figure: thickness-2 decomposition

Thickness-2 application to chip testing

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- This problem was posed by Garey, Johnson, and So of AT&T labs in 1976.

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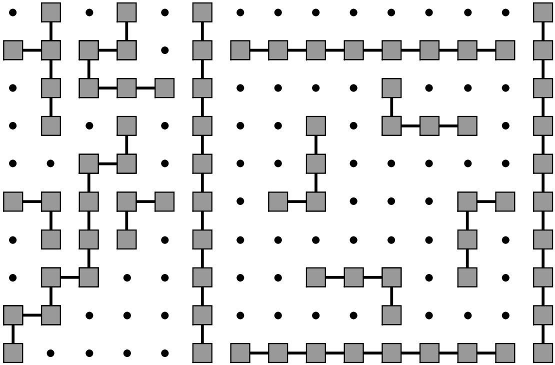
- 

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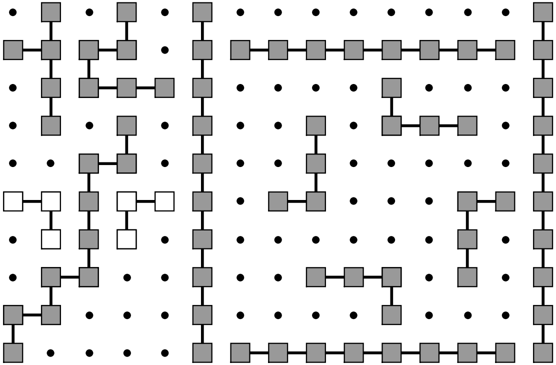

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- **Question.** How can we efficiently check for these types of errors?

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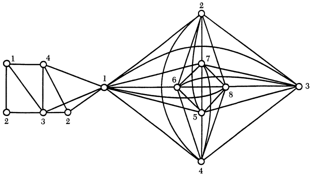
- 

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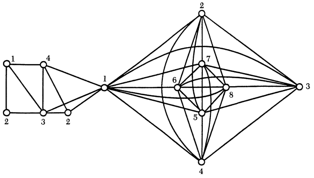
- 

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- In our example, $\chi(G) = 8$ (why?) and in general we can properly color G with 12 or fewer colors (why?).

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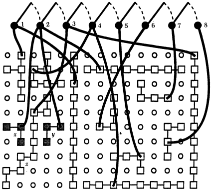
- 

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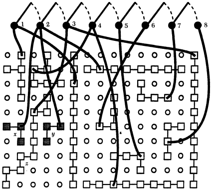
- 

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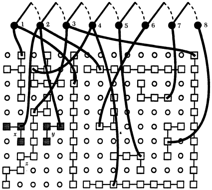
- 

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- Otherwise engage Net 1 and Net 2. If a circuit is completed then there is an error.

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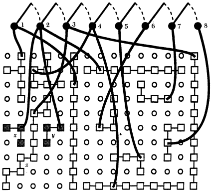
- 

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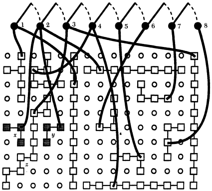
  - 

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- $\vdots$
- Engage all possible pairs of Nets 1-8. There are $\frac{8\times 7}{2} = 28$ possibilities.

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- Total number of checks: 11. Whew.

See the student webpage for a copy of the journal article that contains the full treatment of the M-pire problem, *Coloring Ordinary Maps, Maps of Empires, and Maps of the Moon*.

Next Topic: Graph Algorithms

Single Source Shortest Path Algorithms

A Scenario

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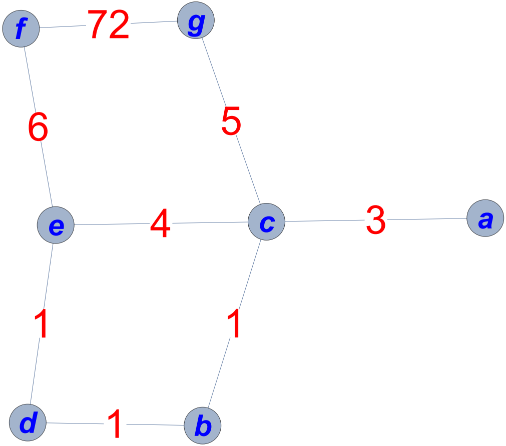

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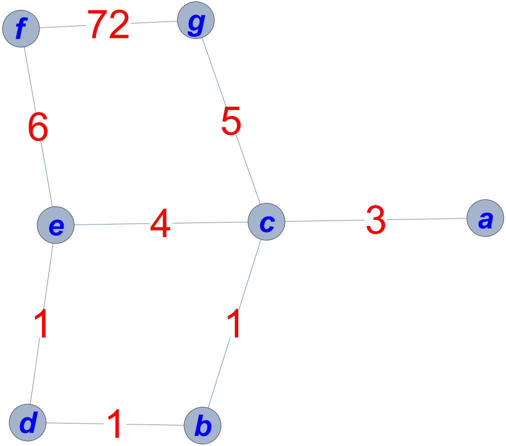

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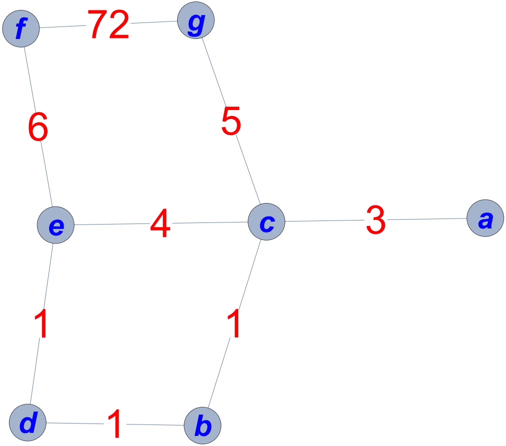

- **Answer.** $a \to c \to b \to d \to e \to f$.
- **Total Weight** $= 12$.

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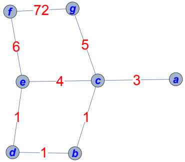
- 

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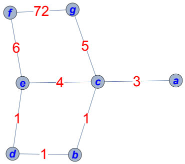
- 

- In our example, we have $\omega(ac) = 3$, $\omega(bd) = 1$, $\omega(gf) = 72$, etc.
- Importantly, if $xy \notin E(G)$ then we define $\omega(xy) = \infty$ so that there is no chance of a non-edge being chosen in a shortest path algorithm.

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- If there is no path from v_i to v_j define $d(v_i, v_j) = \infty$.

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- For a proof of the above theorem, take CSCI 5408 (Applied Graph Theory) in the spring.

Next: A Tree Growing Algorithm