CSCI 5451 Fall 2015

Week 10 Notes

Professor Ellen Gethner

October 18, 2015

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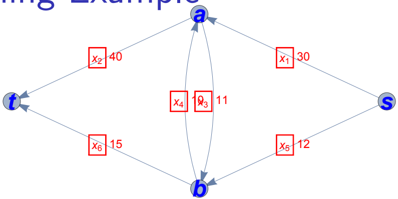
- For a long time, there was no efficient solution to solve the above problem, but there was (and is) a technique called Linear Programming that was used to find a solution.
- Let's see how linear programming works on network flows in an example.

Linear Programming Example



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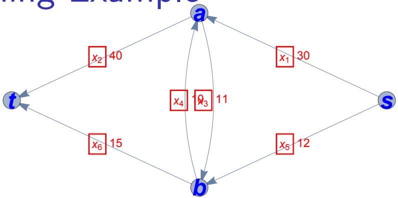

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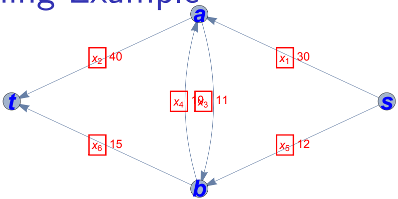

- The number in the $\square$ box is the flow and can change.
- The number in the non-box is the capacity and can't change.
- **Problem.** Maximize $x_1 + x_5$ subject to
  1. $0 \leq x_1 \leq 30$
  2. $0 \leq x_2 \leq 40$
  3. $0 \leq x_3 \leq 11$
  4. $0 \leq x_4 \leq 10$
  5. $0 \leq x_5 \leq 12$
  6. $0 \leq x_6 \leq 15$ together with
  7. $x_2 + x_3 - x_1 - x_4 = 0$, and
  8. $x_6 + x_4 - x_5 - x_3 = 0$.

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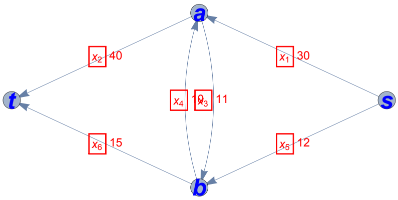

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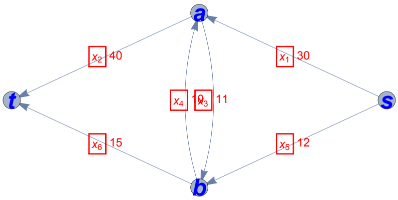

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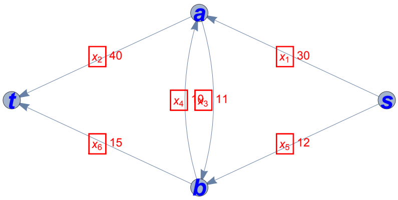

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- $x_1 + x_2 = 42$ by way of $x_1 = 30$, $x_2 = 40$, $x_3 = 0$, $x_4 = 10$, $x_5 = 12$, and $x_6 = 2$.

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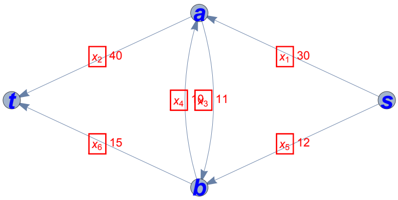

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- Why do we know that we've maximized the flow?

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- The **Network Flow Problem** happens to be an optimization problem that *can* be solved using linear programming, but there is a much better technique.

- **History.** Before 1956, it was not known whether or not for a given arbitrary (weighted, directed) graph G that a solution exists.

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- The **Network Flow Problem** happens to be an optimization problem that *can* be solved using linear programming, but there is a much better technique.
- **History.** Before 1956, it was not known whether or not for a given arbitrary (weighted, directed) graph G that a solution exists.
- In 1956, the first algorithm, a graph algorithm was proposed by Ford and Fulkerson, and over the years, their solution was improved.

Network Flow Hall of Fame (see Herb Wilf's book, ch. 3)

Author(s)	Year	Complexity
Ford, Fulkerson	1956	not known
Edmonds, Karp	1969	$O(E^2 V)$
Dinic	1970	$O(EV^2)$
Karzonov	1973	$O(V^3)$
Cherkassky	1976	$O(V^2\sqrt{E})$
Malhotra, et al	1978	$O(V^3)$
Galil	1978	$O(V^{\frac{5}{3}} E^{\frac{2}{3}})$
Galil, Naamed	1979	$O(EV log^2(V))$
Skeater, Tarjan	1980	$O(EV log(V))$
Goldberg, Tarjan	1985	$O(EV log(\frac{V^2}{E}))$
King, Rao, Tarjan	1994	$O(EV log(\frac{E}{V}) log(V^V))$
Goldberg, Rao	1998	$O(E min(V^{\frac{2}{3}}, \sqrt{E}) log(\frac{V^2}{E}) log(U)$[1]
Orlin, King, Rao, Tarjan	2013	$O(VE)$ with extra constraints[2]

[1] U is the maximum capacity of the network

[2] Orlin, James B. (2013). "Max flows in O(nm) time, or better". STOC '13 Proceedings of the forty-fifth annual ACM symposium on Theory of computing: 765-774.

Connection Between the Ford/Fulkerson Algorithm and a Famous Theorem

- The proof of correctness of the Ford/Fulkerson[3] algorithm to solve the network flow problem is related to the **Max Flow/Min Cut Theorem**.

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  - let $\overline{A} = A \setminus V(G)$ (ie, $\overline{A}$ is the complement of A)
  - Then a **cut** is the set of edges $\{vw \in E(G) : v \in A,\ w \in \overline{A}\}$

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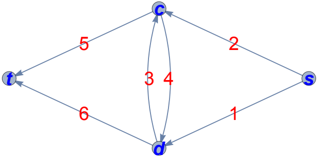
Example of a weighted graph G and all possible cuts



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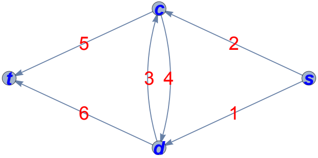

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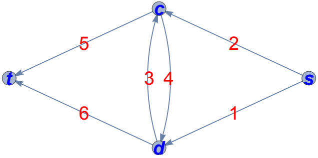

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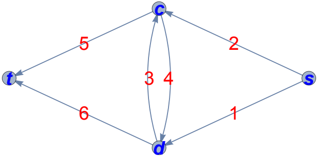

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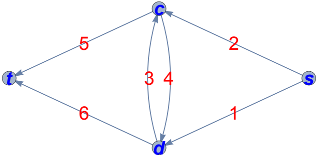

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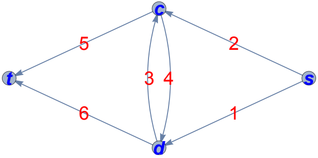

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- **Definition.** The **capacity** of a cut is the sum of the capacities of its edges.
- Thus, for example, the capacity of **cut 1** is 3, and the capacity of **cut 4** is 11.

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- **Theorem.** The value of a the maximum flow in a flow network G is the minimum capacity over all cuts in G.

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- The Greek word cryptos means secret or hidden.

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  2. And the legitimate recipient must know some procedure to translate the cryptogram back to the original plaintext message. This process is called decryption and the recipient must have the encryption key.

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- For example, linguistics and the use of frequency analysis may be enough to crack an encryption such as the Ceasar Cipher.
- In a Ceasar Cipher, one simply shifts the entire alphabet by a given fixed integer to scramble the message.
- Then since, for example, in the English language, the most frequently used symbol is the letter e, in an encrypted message, one simply looks for the most frequently used letter and from that, one can then determine the value of the shift.

Pubic Key Cryptography: Current Applications

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- **Exercise.** Surf the web for more applications of cryptography.

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- primality testing, which is a key component in most cryptographic tools.

A Side-Trip: Algorithms for Primality Testing

- **Algorithm Trial(n)**
- **Input.** $n \in \mathbb{Z}$; $n \geq 2$
- **Output.**

$$\begin{cases} 1 & \text{if } n \text{ is composite} \\ 0 & \text{if } n \text{ is prime} \end{cases}$$

- $i = 2$
- **while** $i^2 \leq n$ **repeat** (a *divisor of* n *must be* $\leq \sqrt{n}$)
- **if** $i | n$
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- **Analysis.** Is the algorithm correct? If so, what is the runtime?

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- then how long would the computation take in this example?

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- **Exercise.** Generalize the argument above to determine a big Oh runtime for Algorithm Trial for an arbitrary positive integer n.

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- Examples at white board.

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- Consider the set $\{aa_1, aa_2, \ldots, aa_{\phi(n)}\}$.

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- **Claim.** The two sets $L_1 = \{a_1, a_2, \ldots, a_{\phi(n)}\}$ and $L_2 = \{aa_1, aa_2, \ldots, aa_{\phi(n)}\} \pmod{n}$ are the same.

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- Consider the set $\{aa_1, aa_2, \ldots, aa_{\phi(n)}\}$.
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- **Claim.** The two sets $L_1 = \{a_1, a_2, \ldots, a_{\phi(n)}\}$ and $L_2 = \{aa_1, aa_2, \ldots, aa_{\phi(n)}\} \pmod{n}$ are the same.
- **Proof of claim.** It suffices to show that the set L_2 has no duplicate elements. Why?

Proof of Euler's Theorem, continued

- BWOC suppose $\exists i, j$ with $i \neq j$ and $1 \leq i, j \leq \phi(n)$ such that $aa_i \equiv aa_j \pmod{n}$.

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- This completes the proof of the claim.
- Now on to the remainder of the proof of Euler's Theorem

Proof of Euler's Theorem, continued

- The claim has shown us that the sets $L_1 = \{a_1, a_2, \ldots, a_{\phi(n)}\}$ and $L_2 = \{aa_1, aa_2, \ldots, aa_{\phi(n)}\} \pmod{n}$ are identical sets of elements.

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- **QED**

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- It follows from Euler's Theorem that for any prime p and any $a \in \mathbb{Z}$ such that $gcd(a, p) = 1$ we have $a^{p-1} \equiv 1 \pmod{p}$.
- This special case of Euler's Theorem is due to Fermat and is called Fermat's Little Theorem, not to be confused with Fermat's *Last* Theorem.

Back to primality testing algorithms...

A Better Method: Randomization

- **Algorithm Lehann's Primality Test (1982)**
- **Input.** $n \geq 3$; n odd and $\ell \geq 2$ with $b[1..\ell]$ a $1 \times \ell$ array.
- **Output.** *Is n prime? Can we tell?*
- **for** $i = 1$ **to** ℓ **do**
- a is a randomly chosen element from $\{1, 2, \ldots, n\}$
- $c = a^{\frac{n-1}{2}} \pmod{n}$
- **if** $c \notin \{1, n-1\}$
- **then return** 1
- **else** $b[i] = c$
- **if** $b[1] = b[2] = \cdots = b[\ell] = 1$ (*)
- **then return** 1
- **else return** 0

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- By Fermat's Little Theorem, we have that $c \in \{1, -1\} \equiv \{1, n-1\} \pmod{n}$.
- Thus if c turns out to be anything other than 1 or -1 $\pmod{n}$, then n cannot be prime.
- In line (*) we will have some entry $\notin \{1, n-1\}$
- and can conclude that n is not prime since Fermat's Little Theorem is not satisfied.

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- **Case 1.** The integer n is actually a prime number and thus we hope that the output is "0".
- **Fact.** When n is an odd prime, exactly half of $\{1, 2, \ldots, n\}$ satisfy $a^{\frac{n-1}{2}} \equiv 1 \pmod{n}$ and the other half satisfy $a^{\frac{n-1}{2}} \equiv n-1 \pmod{n}$ (*take number theory in the spring to see why...*)

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- Using fast exponentiation (ie, repeated squaring) leads to at most $2 log n$ multiplications and divisions of integers that are $\leq n^2$.

Next week...

We'll study the RSA public encryption algorithm.