CSCI 5451 Fall 2015

Week 6 Notes

Professor Ellen Gethner

September 20, 2015

Graph Theory and Graph Algorithms

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- Among any six people, there are always three who are all strangers or are all friends.
- Let's learn enough tools to model other problems by way of graph theory.

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- Alternatively, we say that v_i and v_j are **neighbors**.

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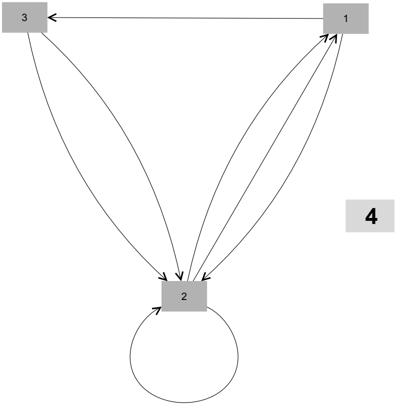

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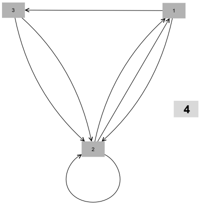

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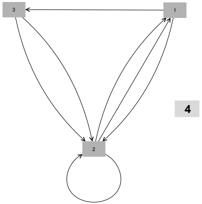
- 

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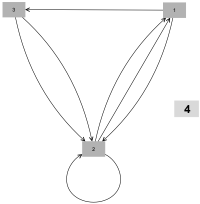

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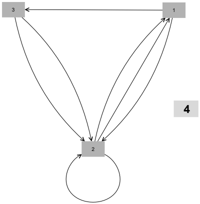

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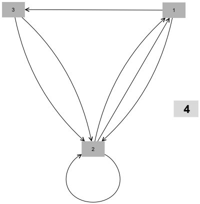

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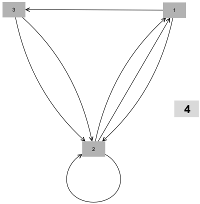

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- If edge $v_i v_j$ appears more than once in the set $E(G)$ then $v_i v_j$ is called a **multiple edge** (or multi-edge or parallel edge).
- Thus, for example, the edge (3,2) above is a multiple edge.

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- In particular, we consider an undirected edge to “allow” both directions.
- Our discussion so far has relied on pictures to understand the meaning of a graph.
- How do we represent a graph on a computer?

Representing a graph on a computer

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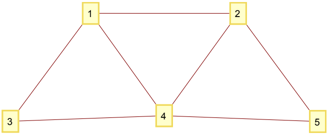
- First, denote the ijth entry of A by a_{ij}.

- For a *simple* graph G, define

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{if } v_i v_j \notin E(G) \end{cases}$$

Adjacency Matrix example

- Here's a simple undirected graph on five vertices.



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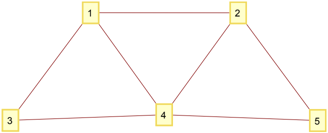

- And here is the associated adjacency matrix.

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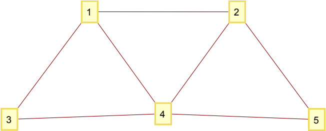

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- Note that a change in the numbering of the vertices will change the adjacency matrix.

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- the run time of any algorithm that must read the contents of the matrix will be $\Omega(n^2)$.
- Thus, when G is **sparse** (which means $|E(G)|$ is relatively small) we should look for another way of storing G.

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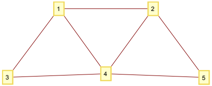
- 

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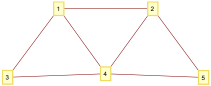
- 

- 1. 2,3,4
  2. 1,4,5
  3. 1,4
  4. 1,2,3,5
  5. 2,4

Basics, continued

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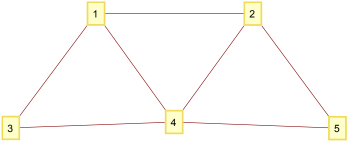
- 

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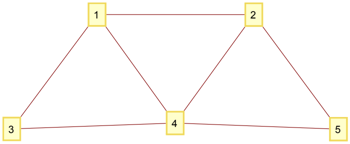
- 

- In this example, $deg(1){=}3$, $deg(2){=}3$, $deg(3){=}2$, $deg(4){=}4$, and $deg(5){=}2$.

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- Hence the size of the adjacency list is $2|E(G)|$.

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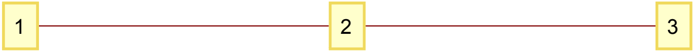
- the adjacency matrix has special, sometimes magical, properties that make it preferable to use.
- Let's illustrate the latter by way of an example.

Magical property of adjacency matrix

- Start with graph $G =$



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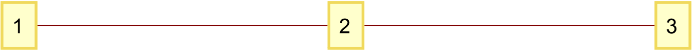

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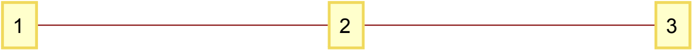

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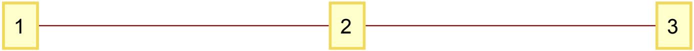

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- Consider the matrix
$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

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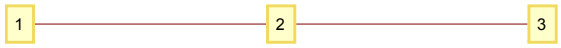

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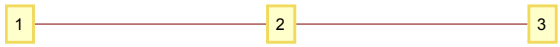

- **Observations.**
  1. $A^2(1,1) = 1$ and there is exactly one path of length 2 from vertex 1 to vertex 1, namely $1 \to 2 \to 1$.
  2. $A^2(2,2) = 2$ and there are exactly two paths of length 2 from vertex 2 to vertex 2, namely $2 \to 3 \to 2$ and $2 \to 1 \to 2$.
  3. **ETC**

Magical property of adjacency matrix A

- Theorem
  Let G be any simple graph on n vertices with $(n \times n)$ adjacency matrix A. Then $A^k(i,j)$ is the number of paths of length k from vertex i to vertex j for all $1 \leq i, j \leq n$.

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  - **Proof.** Take CSCI 5408 in the spring...

One more thing about the idea of the adjacency matrix

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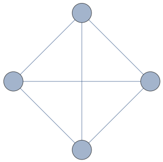

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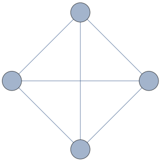

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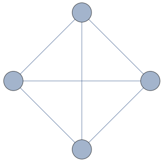

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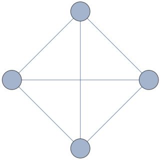

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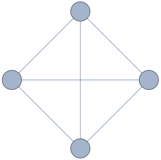

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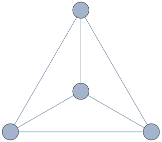

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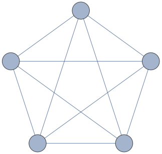

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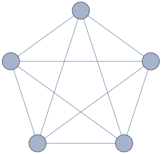

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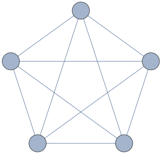

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- **Exercise.**

A Tool for Planar Graphs

- **Euler's Theorem.** Let G be a connected planar graph that has been embedded in the plane with no edge crossings (except, possibly, at endpoints). Suppose $V = |V(G)|$, $E = |E(G)|$, and $F = |F(G)|$, where $F(G)$ is the number of faces of G in the embedding. Then $V - E + F = 2$.

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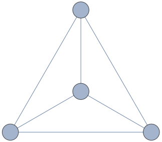
- 

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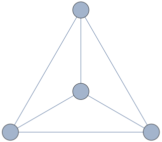
- 
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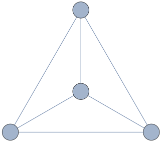
- 

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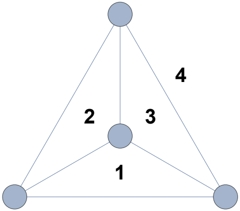
- 

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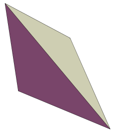
- 

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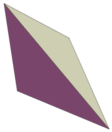
- 
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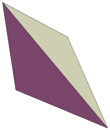
- 

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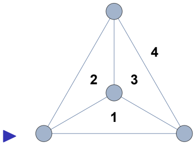
- So don't forget to count that infinite outer face when using Euler's formula.

Back to the example



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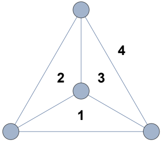
- 

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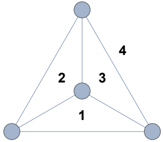
- 

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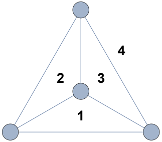
- 

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- For a proof of Euler's Theorem, take CSCI 5408 in the spring.

More Examples on Whiteboard

Another Planarity Tool

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- **2.** Suppose $d_1, d_2, \ldots, d_V$ are the degrees of the V vertices of G.
- Then $\sum_{i=1}^{n} d_i = 2E$ (why?)
- **3.** Now $3F \leq 2E$ because each face of G has three or more edges.

More explanation about $3F \leq 2E$

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Proof of Lemma A, continued

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- graph G must have a vertex of degree 5 or less. **QED**

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- **Every simple planar graph has a vertex of degree 5 or less.**

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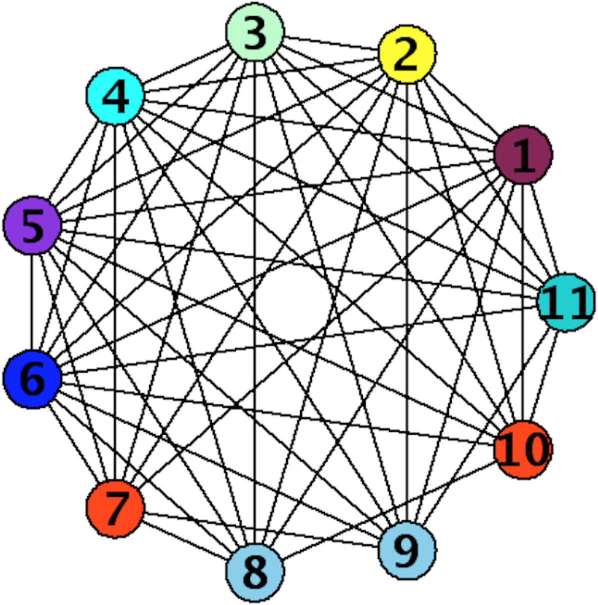
- **Exercise:** why must we assume that G is simple in the hypotheses of Lemma A?

Next Topic: Graph Coloring



Graph Coloring Basics

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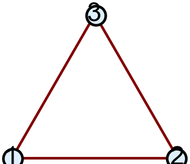
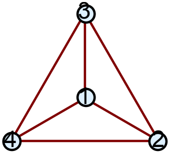
- **Examples.**

χ [] =

Chromatic Number $\chi(G)$ Examples, continued

χ [] = ?

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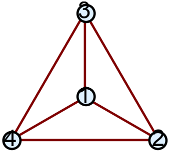
- $\chi\Big[$ 

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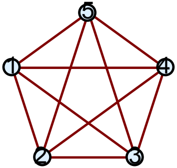
- $\chi\Big[$ 

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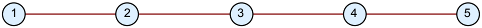
- x[] = ?

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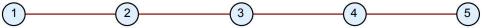
- x[] = ?

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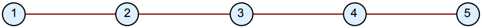
- χ[] = ?

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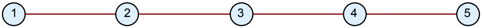
- $x[$  $] = ?$

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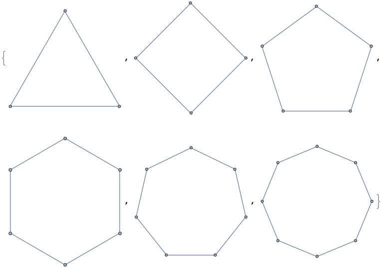
Out[35]= 

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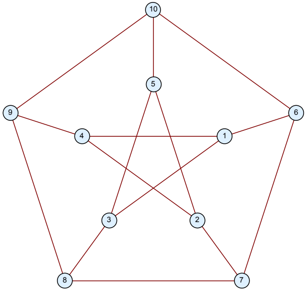
Out[29]=



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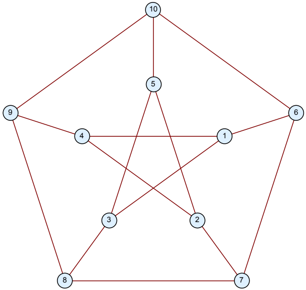

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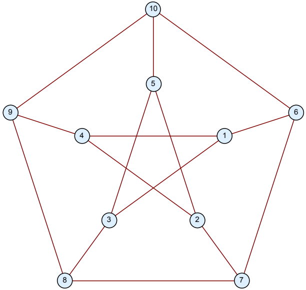

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Scheduling and Graph Coloring

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- What follows is an example of how to avoid scheduling conflicts, and finding an optimal schedule.

- **Problem.** How many days must be scheduled for committee meetings of Pariliament if every committee meets all day, and some members serve on more than one committee?

- **Set-up.** Each committee is represented by a vertex; two vertices (committees) are adjacent iff the vertices have (at least) one committee member in common.

Parliamentary Scheduling, best case

- The best possible case, in terms of scheduling, would be a graph with no edges.

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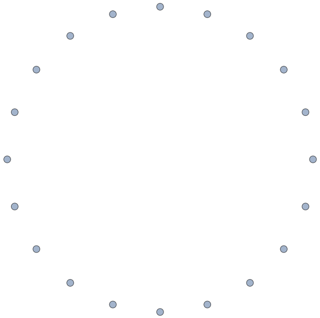

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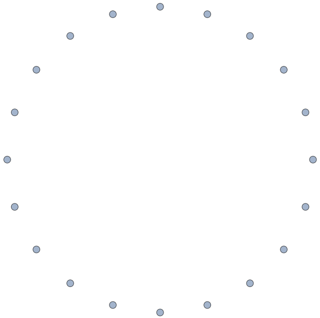
- 

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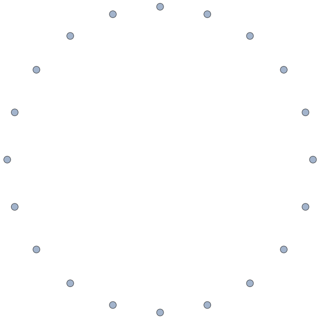
- 

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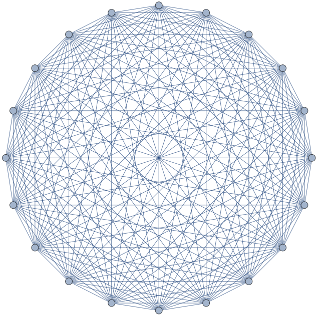

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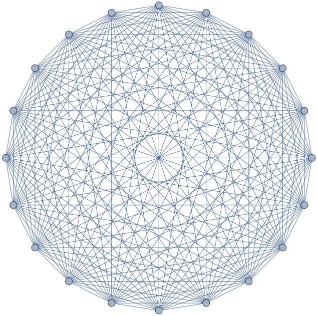
- 
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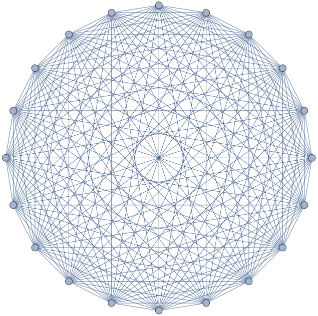
- 
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Parliamentary Scheduling, random case

- What about the random graph (on 20 vertices) below?

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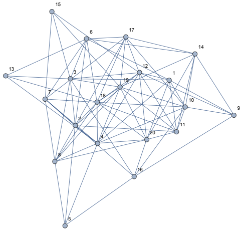
- What about the random graph (on 20 vertices) below?

```
RandomGraph[{20, 79}, VertexLabels → "Name", ImageSize → 500]
```

Out[50]=



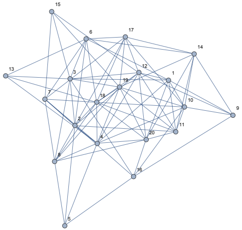

```
In[51]:= EdgeList[%50]

Out[51]= {1 ⟷ 3, 1 ⟷ 4, 1 ⟷ 6, 1 ⟷ 9, 1 ⟷ 10, 1 ⟷ 11, 1 ⟷ 12, 1 ⟷ 20, 2 ⟷ 3, 2 ⟷ 4, 2 ⟷ 5, 2 ⟷ 6, 2 ⟷ 7, 2 ⟷ 8, 2 ⟷ 11, 2 ⟷ 12, 2 ⟷ 17, 2 ⟷ 18,
  3 ⟷ 6, 3 ⟷ 8, 3 ⟷ 12, 3 ⟷ 13, 3 ⟷ 15, 3 ⟷ 16, 3 ⟷ 19, 4 ⟷ 5, 4 ⟷ 7, 4 ⟷ 8, 4 ⟷ 10, 4 ⟷ 12, 4 ⟷ 13, 4 ⟷ 18, 4 ⟷ 19, 4 ⟷ 20, 5 ⟷ 8,
  5 ⟷ 16, 6 ⟷ 7, 6 ⟷ 13, 6 ⟷ 14, 6 ⟷ 17, 6 ⟷ 18, 7 ⟷ 8, 7 ⟷ 15, 7 ⟷ 16, 7 ⟷ 17, 7 ⟷ 18, 8 ⟷ 18, 8 ⟷ 19, 8 ⟷ 20, 9 ⟷ 10, 9 ⟷ 12,
  9 ⟷ 14, 9 ⟷ 16, 10 ⟷ 11, 10 ⟷ 14, 10 ⟷ 16, 10 ⟷ 17, 10 ⟷ 18, 10 ⟷ 19, 11 ⟷ 12, 11 ⟷ 14, 11 ⟷ 18, 11 ⟷ 19, 11 ⟷ 20, 12 ⟷ 15,
  12 ⟷ 17, 12 ⟷ 18, 12 ⟷ 20, 13 ⟷ 18, 14 ⟷ 17, 14 ⟷ 19, 14 ⟷ 20, 15 ⟷ 19, 16 ⟷ 20, 17 ⟷ 18, 17 ⟷ 19, 18 ⟷ 19, 18 ⟷ 20, 19 ⟷ 20}
```

Parliamentary Scheduling, random case

- What about the random graph (on 20 vertices) below?

```
RandomGraph[{20, 79}, VertexLabels → "Name", ImageSize → 500]
```

Out[50]=




```
In[51]:= EdgeList[%50]
Out[51]= {1 ⟷ 3, 1 ⟷ 4, 1 ⟷ 6, 1 ⟷ 9, 1 ⟷ 10, 1 ⟷ 11, 1 ⟷ 12, 1 ⟷ 20, 2 ⟷ 3, 2 ⟷ 4, 2 ⟷ 5, 2 ⟷ 6, 2 ⟷ 7, 2 ⟷ 8, 2 ⟷ 11, 2 ⟷ 12, 2 ⟷ 17, 2 ⟷ 18,
  3 ⟷ 6, 3 ⟷ 8, 3 ⟷ 12, 3 ⟷ 13, 3 ⟷ 15, 3 ⟷ 16, 3 ⟷ 19, 4 ⟷ 5, 4 ⟷ 7, 4 ⟷ 8, 4 ⟷ 10, 4 ⟷ 12, 4 ⟷ 13, 4 ⟷ 18, 4 ⟷ 19, 4 ⟷ 20, 5 ⟷ 8,
  5 ⟷ 16, 6 ⟷ 7, 6 ⟷ 13, 6 ⟷ 14, 6 ⟷ 17, 6 ⟷ 18, 7 ⟷ 8, 7 ⟷ 15, 7 ⟷ 16, 7 ⟷ 17, 7 ⟷ 18, 8 ⟷ 18, 8 ⟷ 19, 8 ⟷ 20, 9 ⟷ 10, 9 ⟷ 12,
  9 ⟷ 14, 9 ⟷ 16, 10 ⟷ 11, 10 ⟷ 14, 10 ⟷ 16, 10 ⟷ 17, 10 ⟷ 18, 10 ⟷ 19, 11 ⟷ 12, 11 ⟷ 14, 11 ⟷ 18, 11 ⟷ 19, 11 ⟷ 20, 12 ⟷ 15,
  12 ⟷ 17, 12 ⟷ 18, 12 ⟷ 20, 13 ⟷ 18, 14 ⟷ 17, 14 ⟷ 19, 14 ⟷ 20, 15 ⟷ 19, 16 ⟷ 20, 17 ⟷ 18, 17 ⟷ 19, 18 ⟷ 19, 18 ⟷ 20, 19 ⟷ 20}
```

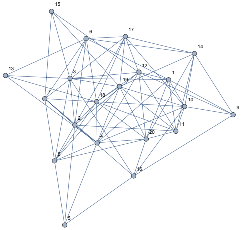
- What is the chromatic number of the above graph (20 vertices)?

Parliamentary Scheduling, random case

- What about the random graph (on 20 vertices) below?

```
RandomGraph[{20, 79}, VertexLabels → "Name", ImageSize → 500]
```

Out[50]=




```
In[51]:= EdgeList[%50]
Out[51]= {1 ↔ 3, 1 ↔ 4, 1 ↔ 6, 1 ↔ 9, 1 ↔ 10, 1 ↔ 11, 1 ↔ 12, 1 ↔ 20, 2 ↔ 3, 2 ↔ 4, 2 ↔ 5, 2 ↔ 6, 2 ↔ 7, 2 ↔ 8, 2 ↔ 11, 2 ↔ 12, 2 ↔ 17, 2 ↔ 18,
  3 ↔ 6, 3 ↔ 8, 3 ↔ 12, 3 ↔ 13, 3 ↔ 15, 3 ↔ 16, 3 ↔ 19, 4 ↔ 5, 4 ↔ 7, 4 ↔ 8, 4 ↔ 10, 4 ↔ 12, 4 ↔ 13, 4 ↔ 18, 4 ↔ 19, 4 ↔ 20, 5 ↔ 8,
  5 ↔ 16, 6 ↔ 7, 6 ↔ 13, 6 ↔ 14, 6 ↔ 17, 6 ↔ 18, 7 ↔ 8, 7 ↔ 15, 7 ↔ 16, 7 ↔ 17, 7 ↔ 18, 8 ↔ 18, 8 ↔ 19, 8 ↔ 20, 9 ↔ 10, 9 ↔ 12,
  9 ↔ 14, 9 ↔ 16, 10 ↔ 11, 10 ↔ 14, 10 ↔ 16, 10 ↔ 17, 10 ↔ 18, 10 ↔ 19, 11 ↔ 12, 11 ↔ 14, 11 ↔ 18, 11 ↔ 19, 11 ↔ 20, 12 ↔ 15,
  12 ↔ 17, 12 ↔ 18, 12 ↔ 20, 13 ↔ 18, 14 ↔ 17, 14 ↔ 19, 14 ↔ 20, 15 ↔ 19, 16 ↔ 20, 17 ↔ 18, 17 ↔ 19, 18 ↔ 19, 18 ↔ 20, 19 ↔ 20}
```

- What is the chromatic number of the above graph (20 vertices)?
- How many days of committee meetings must be scheduled?

Scheduling

In general, if $\chi(G) = k$, then k days of meetings must be scheduled.

Next

- **Map Coloring**

Next

- **Map Coloring**
-