CSC 5451, Professor E. Gethner
Assignment 3
5 October 2015
Quiz 3 in class on Thursday, 22 October 2015

Please feel free to collaborate with one another on this assignment. Consider writing up the solutions on your own for quiz practice. **Important information regarding the quizzes: Be neat, write complete sentences, and show all of your work. The way you communicate the solution to your answer is as important as the answer itself.** Good luck!

1. **(Four Color Fallacy):** Given the validity of the Four Color Theorem, comment in detail on Figure 1.

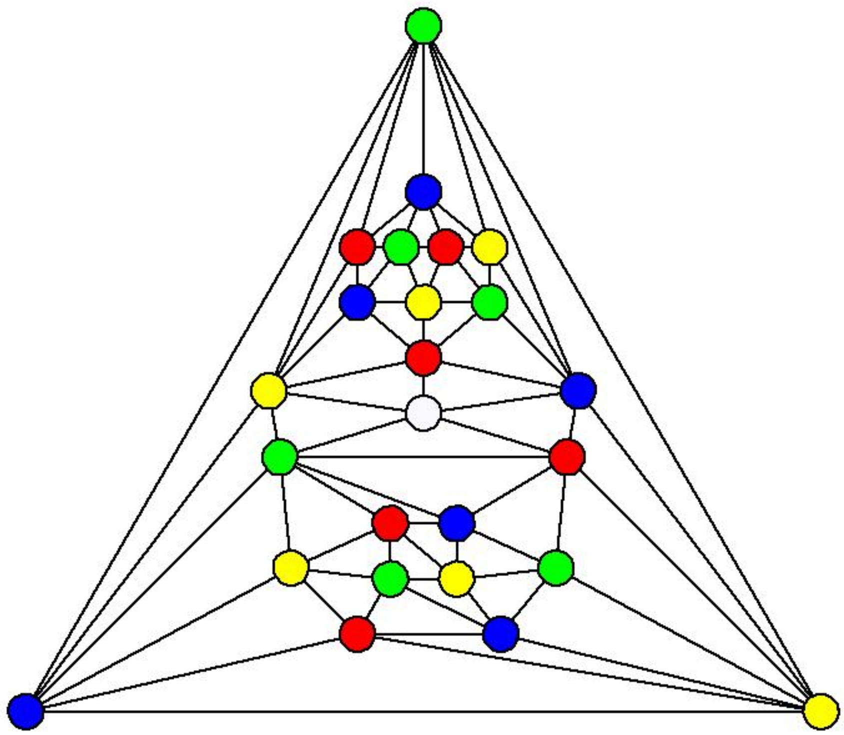

Figure 1: Is the Four Color Theorem false?

2. **(Compression)** Prove that there is no single compression* scheme that compresses all n-bit files.

   *A compression scheme must permit the reconstruction of the original n-bit file from the compressed version, and for a compression scheme to compress a file, the compressed version must be smaller than the original.

3. **(Party On)** Suppose that in the world every pair of people either

   (a) likes one another,

   (b) dislikes one another, or

   (c) is indifferent toward one another.

   Prove that in any gathering of 17 people, there is a group of three people all of whom satisfy *one* of conditions (a), (b) or (c).

4. **(Greedy Coloring Algorithm)** The following psuedo-code that (allegedly) colors the vertices of a graph so that no two adjacent vertices receive the same color. Such a coloring is called a **vertex-coloring** of G.

   Algorithm GCA

   Input: A simple undirected graph G with vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$. (The list of colors to be drawn from will be $\{1, 2, \ldots, n\}$; GCA does not necessarily use all of these colors.
   Output: A vertex-coloring of G.

   For $i = 1$ to n **do**
   $L_i = \{1, \ldots, i\}$ (L_i is the list of colors that may be assigned to v_i)
   For $i = 1$ to n **do**
   Set $c_i =$ the first color in L_i (c_i is the color assigned to v_i)
   For each j with $i < j$ and $(v_i, v_j) \in E(G)$ **do**
   Set $L_j := L_j \setminus c_i$
   Return each vertex, the color it was assigned, and the total number of colors used.

   (a) Prove that algorithm GCA is correct.

   (b) Show that the number of colors used by GCA on the vertices of G could be strictly greater than $\chi(G)$ with an example and an explanation.

   (c) Under what general circumstances will the number of colors used by GCA be equal to $\chi(G)$? Illustrate your claim with an example and an explanation. Hint: the algorithm depends on the way the vertices are numbered to begin with.

   (d) What is the runtime of GCA (in terms of n)?

   (e) How can GCA be utilized to construct an algorithm to find the exact value of $\chi(G)$ using part (c)? What is the runtime of your algorithm?

   The next three problems are both thinking and coding projects. In groups of two or (at most) three, do implementations of the three problems given below. The due date for your implementations is Thursday, 29 October; you will present your algorithms and findings in class. For a better understanding of presentation expectations, see the appendix at the end of this assignment.

5. **(A Problem out of the Blue)** Find a formula for the largest number of non-overlapping triangles formed with n line segments for all $n \geq 3$. Suppose $T(n)$ represents the sought after function. Then $T(3) = 1$, $T(4) = 2$, $T(5) = 5$, and $T(6) = 7$; see Figure 2. Prove that your formula is correct.

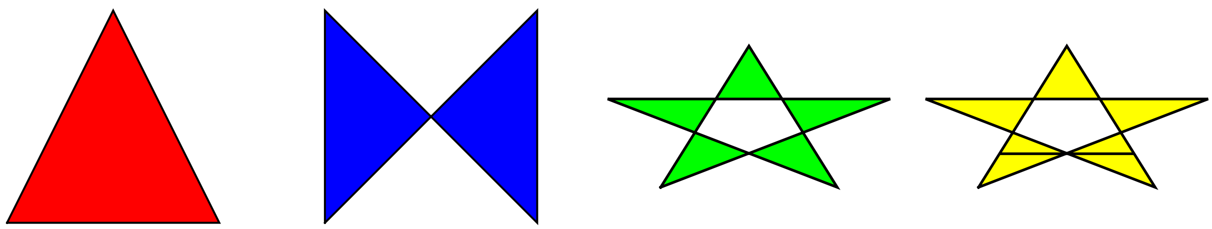

Figure 2: non-overlapping triangles formed from line segments

6. **(Biplanar Crossing Number)** A graph G drawn in the plane may contain edge crossings, namely intersections among pairs of edges that are not vertices. One constraint in crossing number problems is that no three edges may intersect in a single point (with the exception of vertices). The *crossing number* of a graph G, written $\nu(G)$, is the fewest possible number of (non-vertex) edge crossings over all drawings of G. For example if G is planar then $\nu(G) = 0$. And $\nu(K_5) = 1$ (verify this claim). The *biplanar crossing number* of a graph G is the smallest number of crossings of G over all drawings of G, where you are now allowed to use two (2) layers to draw the graph. In other words, replicate the set of vertices and distribute the edges among the two layers. The biplanar crossing number of a graph G is written $\nu_2(G)$.

   Problem. Determine $\nu_2(K_n)$ for $n = 5, 6, 7, 8, 9, 10, 11$, and 12. Prove that your answers are correct.

   Problem. You define the t-planar crossing number and investigate the above problem for "many" values of t. Note that 2-planar and biplanar mean the same thing.

7. **(Great Circle Problem)** A *great circle* is any circle on a sphere whose radius is the same as the radius of the sphere (so it is largest possible). A circle that goes through both the North and South poles is an example of a great circle on the Earth. Given n ($n \geq 3$) great circles on a sphere, no three of which pass through a single point, form a *great circle graph* by making

points of intersection into vertices, and connect two vertices by an edge if and only if there is an arc between them. See Figures 3 and 4.

Problem: What is the largest chromatic number of any great circle graph? Generate 50 examples of great circle graphs and find the chromatic number of each one. What is your conjecture?

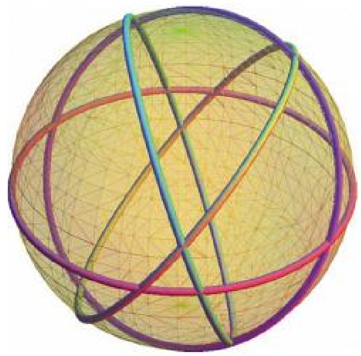

Figure 3: A great circle graph

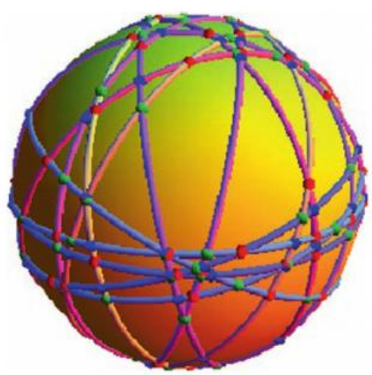

Figure 4: Another great circle graph

Appendix: Presentation Expectations

At the very least, I will be watching for the following during your presentations. Creativity and thinking ”outside of the box” is always appreciated.

1. Timing: is your presentation too short, too, long, or just right? Was the presentation given at a good pace or was it rushed? (I suggest practicing at least once before giving your talk)
2. Have you motivated your subject to an audience of non-experts? You may assume, of course, that we all understand the lecture material presented in class.
3. Do you have both a depth and breadth of understanding of your subject and did you convey that to the audience?
4. Have you made clear who is responsible for the work that you are presenting by referencing the appropriate sources (name(s) + year published) within and at the end of the presentation? A complete list of ****appropriately formatted references**** should be given at the end. As usual, be careful to avoid sources that are not peer-reviewed.
5. Have you given enough examples (concocted by you) to make difficult concepts more palatable? Have you defined new terms before using them?
6. If you do a demonstration, was it done well? Was the code well-explained? Were there any technical problems during the talk– avoid this at all costs!
7. Did you formulate sound conclusions at the end of the talk?
8. Were there questions at the end of your talk? If so, how well did you handle them?
9. Was the discussion following the talk interesting?
10. Finally, *don’t overload your slides with small text.* Keep your slides sparse.