CSCI 5451 Fall 2015

Week 9 Notes

Professor Ellen Gethner

October 14, 2015

Shortest Path Problems, continued from last week

How to Grow a Tree

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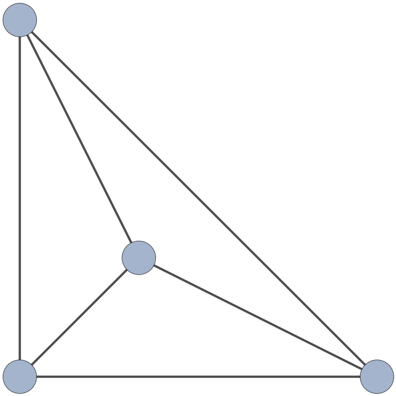
- A **frontier edge** of T is an edge $uv \in E(G)$ such that $u \in V(T)$ and $v \notin V(T)$.

- **Example.** $G = K_4$

Example: frontier edges



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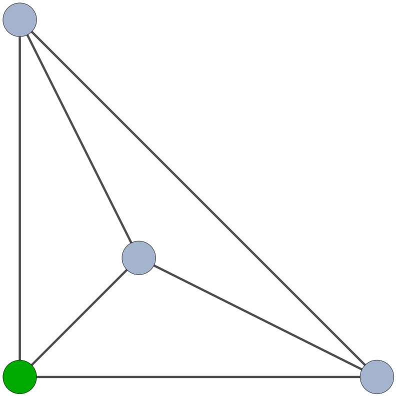
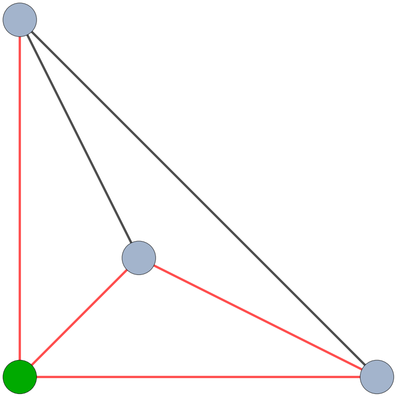


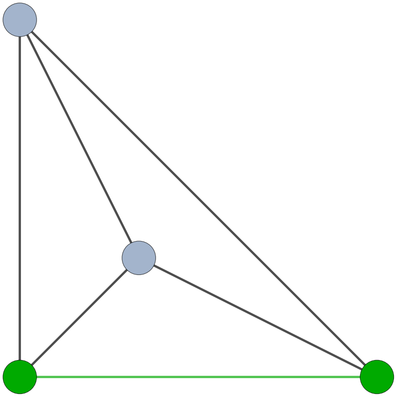
Tree T, so far, is the single green vertex.

Example: frontier edges



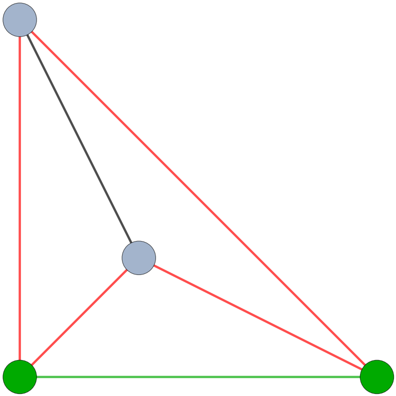
The current frontier edges are red.

Example: frontier edges



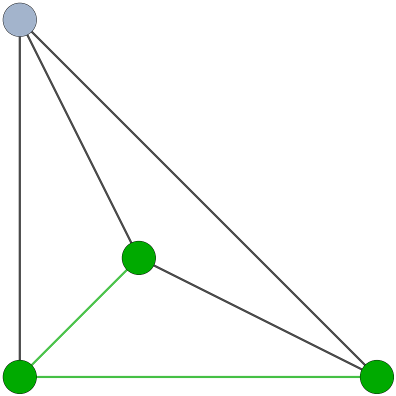
Choose a frontier edge and update the tree.

Example: frontier edges



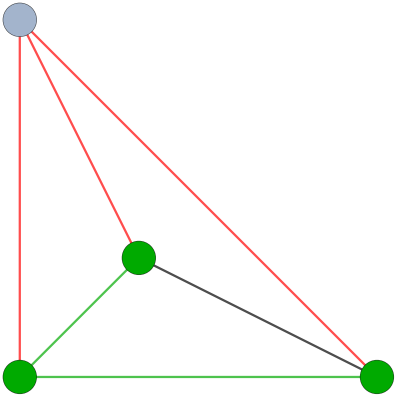
Now update the frontier edges.

Example: frontier edges



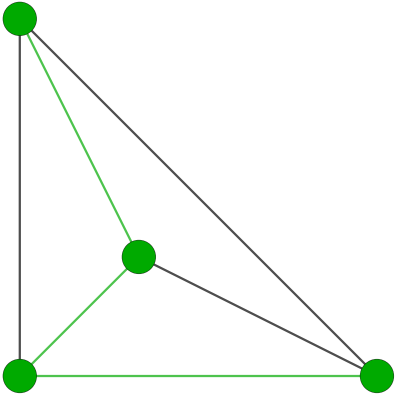
Choose any frontier edge and update the tree.

Example: frontier edges



And update the frontier edges again.

Example: frontier edges



And finally, choose one last frontier edge and update the tree.
Why are we done?

Tree Growing Algorithm

- **Input:** A connected graph G and starting vertex $v \in V(G)$
- **Output:** A spanning tree T of G and a vertex labeling of $V(G)$.
- **initialize** T as vertex v
- **initialize** set of frontier edges for T as **empty**
- **write** label 0 on vertex v
- **initialize** label counter $i = 1$
- **while** T does not yet span G
- **update** the set of frontier edges for T
- **let** e be the frontier edge for T of highest priority
- **let** w be the unlabeled endpoint of e
- **add** edge e (and vertex w) to tree T
- **write** label i on vertex w
- $i = i + 1$
- **return** T with its vertex labeling

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  2. a way to assign priorities to the frontier edges.

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- Two things that are missing so far from our tree growing algorithm are

  1. the fact that the edges are weighted, and

  2. a way to assign priorities to the frontier edges.

- So next let's use the weights of the edges to prioritize frontier edges in a way that makes sense for an eventual shortest path algorithm.

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- recall that $d(s, x)$ is the distance from vertex s to vertex x, and $\omega(e)$ is the weight of edge e (see week 8's notes).

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- recall that $d(s,x)$ is the distance from vertex s to vertex x, and $\omega(e)$ is the weight of edge e (see week 8's notes).
- Now that we know how to prioritize the frontier edges, we have all of the tools needed to describe Dijkstra's Single Source Shortest Path Algorithm.

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  - $dist[y] = P(e)$
  - **write** label $dist(y)$ on vertex y.
- **return** Dijkstra tree T and its vertex labels.

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- simply delete frontier edges that include the endpoint y.
- Then recompute the priorities of the remaining edges.
- **Question.** Is Djikstra's algorithm correct?

Proof of Correctness of Djikstra's Algorithm

- **Theorem.** Let T_j be the Dijkstra tree after j iterations of Algorithm Dijkstra on connect graph G for $j = 0, 1, 2, \ldots, |V(G)| - 1$.Then for each $v \in V(T_j)$, the unique path from s to v in T_j is a shortest path from s to v in G.

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- **Proof by induction.**
- **Base Cases.**
  - The claim is true for T_0. Why?
  - What must you check to know that the claim is true for T_1?

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- In particular, e is a frontier edge, x is labeled, and y is unlabeled.

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- Suppose e is the frontier edge added to T_j in the $(j+1)$st iteration of Algorithm Dijkstra.

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- Since y is the only new vertex in T_{j+1}, it suffices to show that the path from s to y in T_{j+1} is a shortest path from s to y in G.

Proof of correctness of Dijkstra, continued

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- Thus $\mathit{length}(Q) = P(e)$. Why?

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- Thus $length(Q) = P(e)$. Why?
- Now let R be *any* path in G from s to y.
- To finish the proof, it suffices to show that $length(R) \geq length(Q)$. Why?

Proof of correctness of Dijkstra, continued

- WLOG assume R and Q are different and let edge $f = vz$ be the first edge in path R that is not in T_j.

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- WLOG assume R and Q are different and let edge $f = vz$ be the first edge in path R that is not in T_j.
- Assume also that $v \in V(T_j)$ (and hence $z \notin T_j$).

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- Let K be the subpath of R from z to y.
- Since e was a frontier edge used in the $(j+1)$st iteration of Algorithm Dijkstra, we have $P(e) \leq P(f)$. Why?

Dijkstra proof, continued: $P(e) \leq P(f)$

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- $= P(f) + length(K)$
- $\geq P(e) \geq length(Q)$.

- In total, we have $length(R) \geq length(Q)$, as desired.

- **Conclusion.** For any $j \in \{0, 1, \ldots, |V(G)| - 1\}$ we have that T_j is a shortest path tree for G. **QED**

A Question

Where did we use the fact that the edge weights of G are non-negative?

Final Remarks about Dijkstra

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1. **Implementation.** Use a minimum priority queue by way of a binary heap: see section **24.3** for details.

2. **BFS tree.** Algorithm Dijkstra gives us a method with which we can construct a **B**readth **F**irst **S**earch (BFS) tree.

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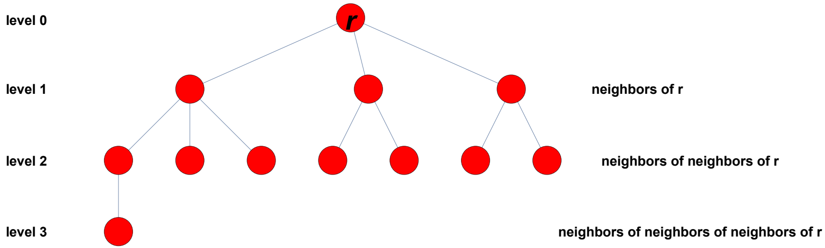
  3. ETC

BFS tree, continued



Generalization of Dijkstra: Bellman-Ford Algorithm

- Recall that the input to Dijkstra's algorithms required that the edges are undirected and have non-negative weight.

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- and tries to improve (ie, decrease) the value of the distance label $d[u]$.

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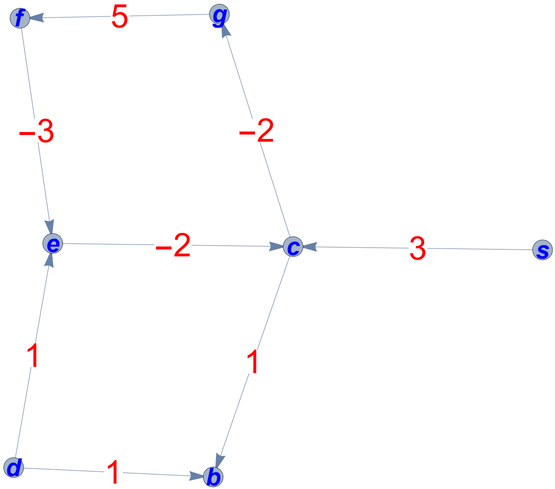

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Negative-weight cycles are bad

- **The Point.** If there exists a negative-weight cycle reachable from source vertex s then there is no shortest path from s to any other vertex reachable from s.

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- **Input.** G, a directed graph and $s \in V(G)$, a source vertex.
- **Output.** Either **TRUE** if there is no reachable-from-s negative weight cycle or **FALSE**. The former will also keep track of the shortest path tree.

B-F(G, s), continued

- $d[v] = \infty$ for all $v \in V(G)$
- $d[s] = 0$
- **For** $i = 1$ **to** $|V(G)| - 1$
-     **For** each $uv \in E(G)$
-         **if** $d[u] + \omega(uv) < d[v]$
-         **then** $d[v] = d[u] + \omega(uv)$
-         $parent[v] = u$
-     **For** each edge uv
-         **if** $d[u] + \omega(uv) < d[v]$
-         **then return** "False"
- **return** "True"

The red code is the edge-relaxation part of the algorithm.

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- **Claim.** If the input graph G contains a negative weight cycle reachable form s, then $\mathbf{B\text{-}F}(G, s)$ returns "False."

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- **QED**

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- Finally, a good alternative source for both Dijkstra and Bellman-Ford is Gross and Yellen's textbook **Graph Theory and Applications**, especially for the proof of Dijkstra and the use of frontier edges in tree growing.

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- Finally, the **value of flow** f is $\sum_{v \in V(G) \setminus s} f(sv)$. That is, the flow is the amount of water leaving the source vertex s.

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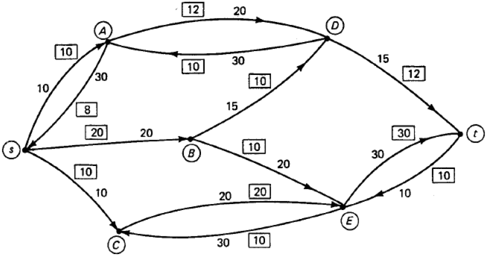
- 

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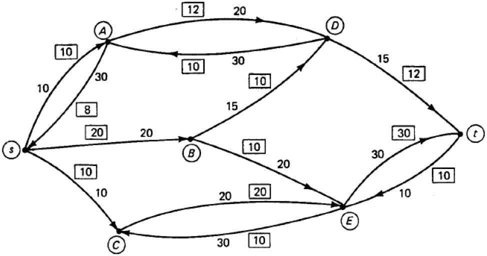
- 

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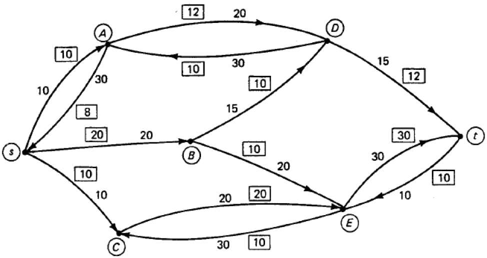
- 

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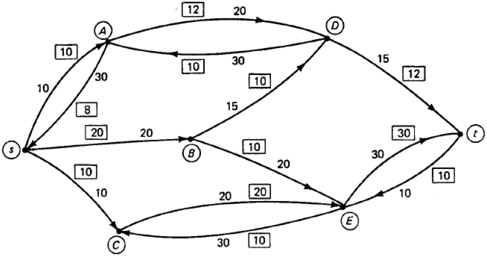

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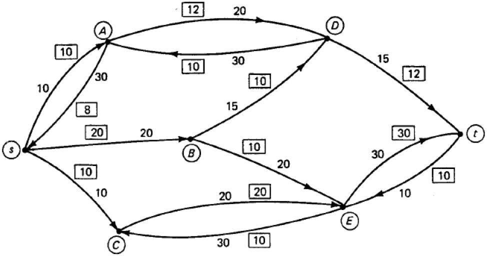
- 

- **Exercise.** What is the flow of the above network?

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- **The Network Flow Problem.** Given a flow network G with capacity constraint function c, find the flow of maximum value.

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- **Next Week.** More on network flows by way of a side-trip to linear programming, the max-flow/min cut theorem, and the Ford-Fulkerson hall of fame.