CSCI 5451 Fall 2015

Week 12 Notes

Professor Ellen Gethner

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- **FFT Application:** Polynomial multiplication (keyword= convolution for other applications).

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- That is, if $p(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $q(x) = b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ then the straightforward multiplication would require $O(n^2)$ multiplications.

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- The above method is more than adequate, but there are many applications that require real-time dynamic computations (such as interactive rendering of 3D graphics, for example)
- and thus any reduction we can make in the complexity will be quite useful.

A Different Approach to Polynomial Multiplication

- **Representing a Polynomial.** The typical representation of $p(x)$ would be as a $1 \times n$ array $[a_{n-1}, a_{n-2}, \ldots, a_1, a_0]$.

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- We can represent $\ell(x)$ by way of its coefficients (been there, done that) or
- instead we could represent $\ell(x)$ by any two distinct points on $\ell(x)$.
- Another way of making the above point (no pun intended) is to remember that **two points uniquely determine a line**."

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- **Example.** Suppose $p(x) = a_2x^2 + a_1x + a_0$ and that points (3,65), (1, 41), and (2, 57) are all on the curve determined by $p(x)$.

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- **Example.** Suppose $p(x) = a_2x^2 + a_1x + a_0$ and that points (3,65), (1, 41), and (2, 57) are all on the curve determined by $p(x)$.
- Find the coefficients of $p(x)$. That is, solve for a_2, a_1, and a_0.
- We have $67 = a_23^2 + a_13 + a_0$, $51 = a_21^2 + a_11 + a_0$, and $57 = a_22^2 + a_12 + a_0$, and thus we have three equations in three unknowns.

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- I used *Mathematica* and the command **LinearSolve** to solve for the coefficients of $p(x)$:

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```
In[25]:= Clear[matrixA]; matrixA = {{9, 3, 1}, {1, 1, 1}, {4, 2, 1}};
         matrixA // MatrixForm
```

Out[26]//MatrixForm=

$$\begin{pmatrix} 9 & 3 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$$

```
In[27]:=
         Clear[vectorOfAnswers]; vectorOfAnswers = {65, 51, 57};
         vectorOfAnswers // MatrixForm
```

Out[28]//MatrixForm=

$$\begin{pmatrix} 65 \\ 51 \\ 57 \end{pmatrix}$$

```
In[29]:= LinearSolve[matrixA, vectorOfAnswers]
Out[29]= {1, 3, 47}
```

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- Thus $p(x) = x^2 + 3x + 47$.

Another Example and the FFT Journey Continued

- **Example.** The polynomial $p(x) = x^2 + 3x + 1$ is uniquely determined by the points (1,5), (2, 11), and (3,19) as well as many other choices of three points.

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- **Quick Check:**

```
In[34]:= p[x_] := x^2 + 3 x + 1

In[35]:= p[1] == 5

Out[35]= True

In[36]:= p[2] == 11

Out[36]= True

In[37]:= p[3] == 19

Out[37]= True
```

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- But $p(x)q(x) = 2x^4 + 5x^3 + 2x^2 + 8x + 3$ and so is not uniquely determined by the three points above.
- In particular, we must represent the above polynomial of degree four by five points.

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- We'll add (0,1) and (-1,-1) to the set of points representing $p(x)$.
- And we'll add (0,3) and (-1,6) to the set of points representing $q(x)$.
- Thus the five points that we'll use to represent $p(x)q(x)$ are (1,20), (2,99), (3,342), (0,3), and (-1,-6).

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- **Our insight thus far:** We need an efficient method both for converting from points on a curve representing a polynomial
- **and** of evaluating polynomials at those points.
- The ideas above are the foundations of the FFT: it accomplishes both tasks efficiently.

Restatement of the Problem

- **Problem (again).** How can we evaluate two polynomials $p(x)$ and $q(x)$ of degree $n-1$, each at $2n-1$ distinct x-values so that the coefficients of the product polynomial $p(x)q(x)$ can be determined?

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- In fact, by the same reasoning, our polynomials of degree $n-1$ can be viewed as polynomials of degree $2n-2$.

- **One more restatement.** Evaluate an arbitrary polynomial $p(x) = \sum_{i=1}^{n-1} a_i x^i$ of degree $n-1$ at n distinct points.

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- The matrix equation that represents the evaluation is:

- $$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & & & \ddots & \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} p(x_0) \\ p(x_1) \\ \vdots \\ p(x_{n-1}) \end{bmatrix} \quad (*)$$

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- Then $x_0 = -x_{\frac{n}{2}}$, $x_1 = -x_{\frac{n}{2}+1}, \ldots, x_{\frac{n}{2}-1} = -x_{n-1}$.

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- Then $x_0 = -x_{\frac{n}{2}}$, $x_1 = -x_{\frac{n}{2}+1}, \ldots, x_{\frac{n}{2}-1} = -x_{n-1}$.
- We will use the above set of functional equations to reduce the problem to two subproblems.

Reduction to two subproblems

- Rewrite matrix equation (*) on the previous slide using the functional equations as shown next:

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- (**)

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Similarities in the red and black parts of (**)

- $$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & & & \ddots & \\ 1 & x_{\frac{n}{2}-1} & x_{\frac{n}{2}-1}^2 & \cdots & x_{\frac{n}{2}-1}^{n-1} \\ & & & & \\ 1 & -x_0 & (-x_0^2) & \cdots & (-x_0^{n-1}) \\ 1 & -x_1 & (-x_1^2) & \cdots & (-x_1^{n-1}) \\ \vdots & & & \ddots & \\ 1 & (-x_{\frac{n}{2}-1}) & (-x_{\frac{n}{2}-1}^2) & \cdots & (-x_{\frac{n}{2}-1}^{n-1}) \end{bmatrix}$$

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- **Observation 1.** The coefficients of the even powers of x are the same in both the red and the black submatrices.

Similarities in the red and black parts of (**)

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- **Observation 1.** The coefficients of the even powers of x are the same in both the red and the black submatrices.
- **Observation 2.** The coefficients of the odd powers of x in the red submatrix are the negatives of the corresponding odd powers of x in the black submatrix.

Red versus black; even powers versus odd powers of x

- **Notation.** Let $P(x) =$

$$\sum_{i=0}^{\frac{n}{2}-1} a_{2i}x^{2i} + \sum_{i=0}^{\frac{n}{2}-1} a_{2i+1}x^{2i+1}$$

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- The purple summation contains the even powers of x, and
- the orange summation contains the odd powers of x.

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- The purple summation contains the even powers of x, and
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- The cost is $\frac{n}{2}$ additions, $\frac{n}{2}$ subtractions, and n multiplcations.
- **So Far.** We now have two subproblems of size $\frac{n}{2}$ and $O(n)$ additional computations. Sound familiar?

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- Review Mergesort for more explanation.
- **The Point.** We've significantly reduced the complexity from the naive $O(n^2)$-time algorithm!!

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- Geometrically, the complex numbers $\omega^0, \omega^1, \omega^2, \ldots, \omega^{n-1}$ are all vectors of length one spaced evenly around the unit circle centered at the origin of the complex plane.
- The vector ω^1 has polar coordinates $(1, \frac{2\pi}{n})$ and to move on to the next vector on the list, we simply add $\frac{2\pi}{n}$ to the current angle: ω^2 has polar coordinates $(1, \frac{4\pi}{n})$, and so on.

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  2. $\omega^0, \omega^1, \ldots, \omega^{n-1}$ are all distinct.
- **Observation.** Every primitive nth root of unity has a multiplicative inverse since $\omega^k \omega^{n-k} = 1$.

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- Altogether, we have $1 + \omega^{\frac{n}{2}} = 0 \Rightarrow \omega^{\frac{n}{2}} = -1$. **QED**

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- That is, substitute ω^2 for ω and repeat the procedure.

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- evaluate an arbitrary polynomial of degree $n-1$ at n points
- FFT accomplishes this in time $O(n \log n)$
- recover coefficients of product polynomial with inverse FFT[1] in time $O(n \log n)$.

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Next Week

NP-Completeness