CSCI 5451 Fall 2015

Week 14 Notes

Professor Ellen Gethner

November 15, 2015

Polygon Triangulation and Art Galleries

- **Problem.** Given a polygonal floor plan, what is the fewest number of guards[1] that must be posted

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Polygon Triangulation and Art Galleries

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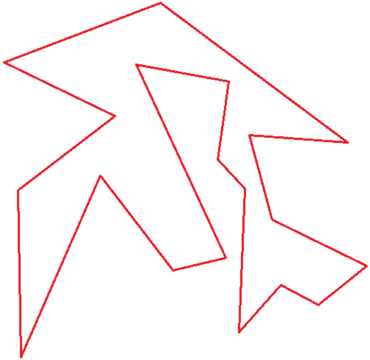
- in order that all points interior to the polygon are visible to at least one guard?

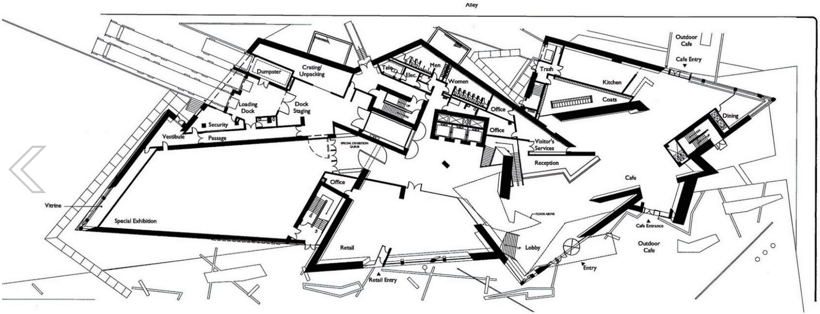


[1]guards have 360^o vision

Denver Art Museum (DAM)

- In case you were wondering about the existence of art galleries with odd floor plans, the Denver Art Museum downtown is the perfect example.

DAM Floorplan



Guarding Art Galleries

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- See, for example, `https://imaginary.org/film/point-guards-and-point-clouds-solving-general-art-gallery-problems`

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Preliminary Definitions[3], etc

- **Definition 1.** A polygon is a region in the plane bounded by a finite collection of line segments.

[3] Computational Geometry in C by J. O'Rourke

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  2. $e_i \cap e_j = \varnothing$ whenever $j \not\equiv i + 1 \pmod{n}$.
- That is, the union of the line segments form a closed non-self intersecting curve.

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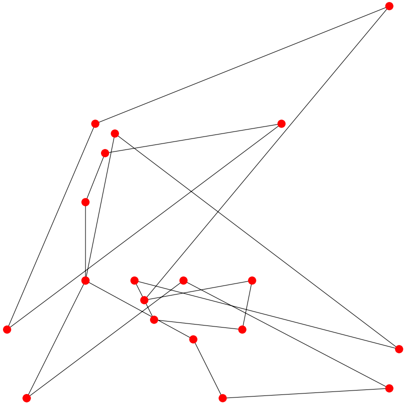
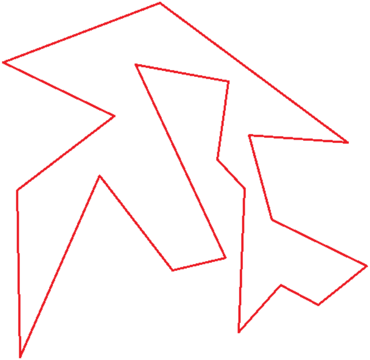
Non-example

- A non-simple polygon is shown below.



A Simple Polygon



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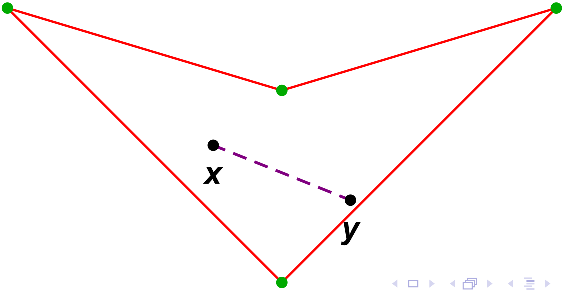

Visibility, continued

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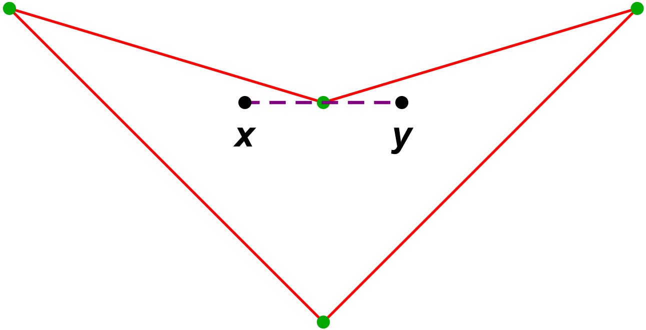

-

Figure: point x is still visible to point y

Visibility, continued

- And finally, here is an example in which the line segment $\overline{xy}$ has points that are exterior to P and hence x is not visible to y.

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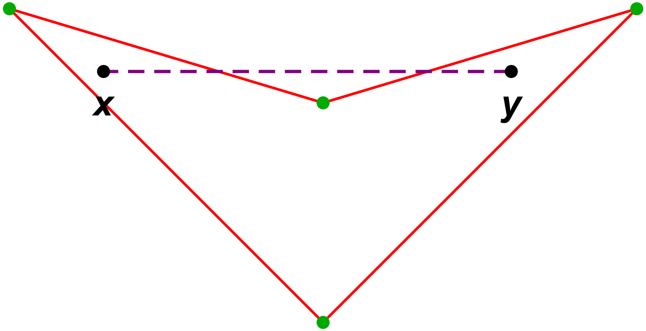
- 

Figure: point x is *not* visible to point y

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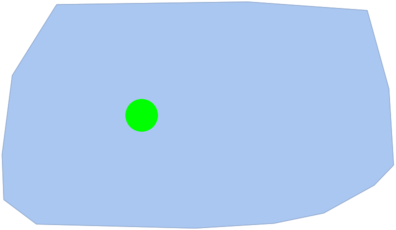
- 

Minimizing the number of guards

- **One Possible Approach.** Triangulate polygon P by connecting suitable pairs of vertices, and then place one guard in each triangle.

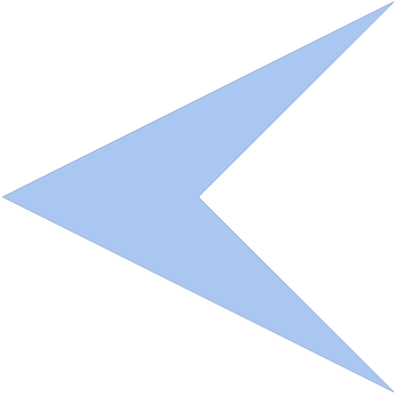


Figure: a simple polygon P

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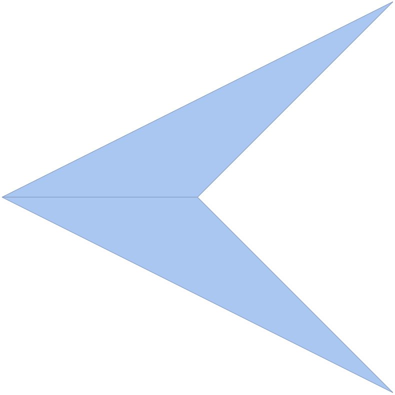

Figure: a simple polygon P triangulated

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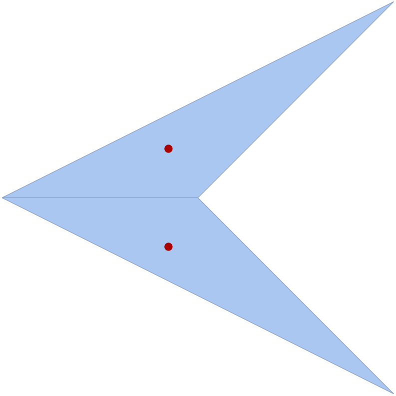

Figure: a simple polygon P triangulated and guarded

Motivation for a Theorem

- The only problem is that we have no guarantee that an arbitrary simple polygon actually **can** be triangulated.

A Helpful Theorem

- **Theorem 3.1[4].** Every simple polygon P with three or more vertices admits a triangulation. And the triangulation of a simple polygon with n vertices contains exactly $n-2$ triangles.

[4] from Computational Geometry, Algorithms and Applications by M. de Berg

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- **Inductive Step.** And now let P be a simple polygon with n vertices.
- Our goal is to prove that P admits a triangulation.

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Proof of triangulation theorem, continued

- **Claim.** There exist two vertices v_i and v_j in P so that the line segment[5] $\overline{v_i v_j} \subset P$.

[5] Such a line segment is called a diagonal of polygon P.

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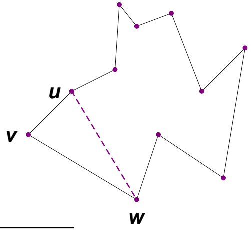
- If $\overline{uw} \subset P$, then $\overline{uw}$ is a diagonal of P, and we are done.



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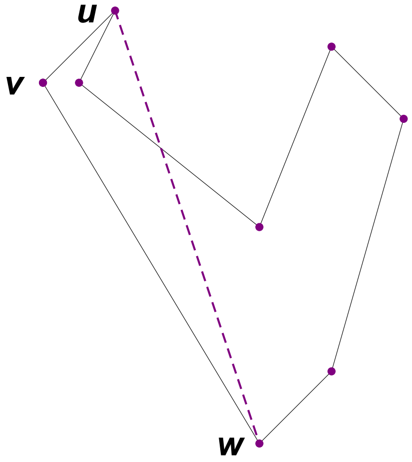
- **Else** there is at least one vertex of P contained in triangle uvw. Call it v'.



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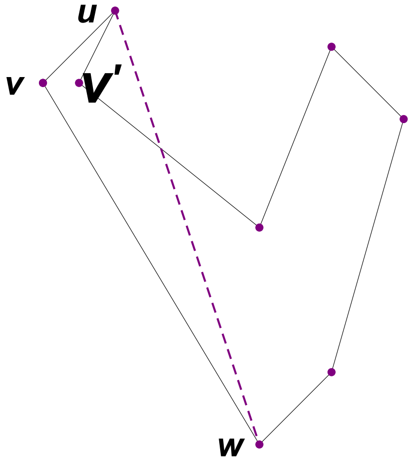

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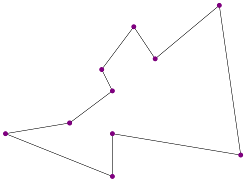
- In fact choose v' to be the vertex in triangle uvw that is farthest from the line through u and w.



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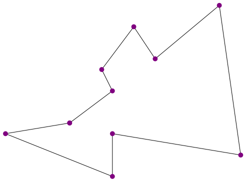

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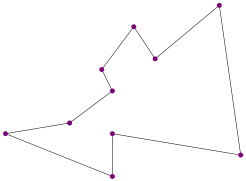

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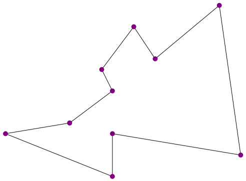

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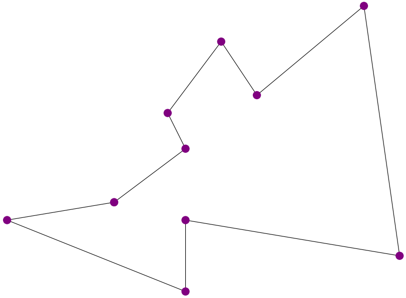
- Thus $\overline{vv'}$ is a diagonal of P, which completes the proof of the claim.
- **Back to the induction argument.**

Induction, continued

- Any diagonal of a simple polygon P splits P into two simple polygons P_1 and P_2.



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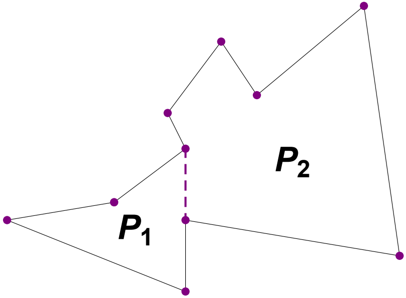
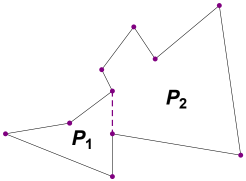


Induction, continued

- If P has n vertices, and P_1 has m vertices, then P_2 has $n - m + 2$ vertices. Why?



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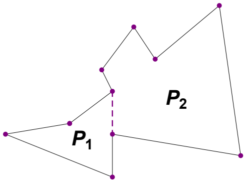

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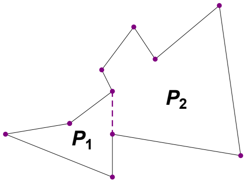

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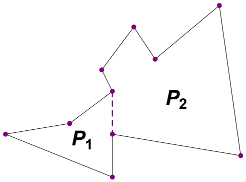

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- Moreover, by the induction hypothesis, P_1 contains $n - m$ triangles and P_2 contains $m - 2$ triangles,

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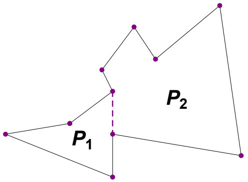

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- Moreover, by the induction hypothesis, P_1 contains $n - m$ triangles and P_2 contains $m - 2$ triangles,
- which means P contains $n - m + m - 2 = n - 2$ triangles, as desired.

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- **Can we do better??**

Some Slight Tweaks for further Improvement

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- Thus by Euler's formula, we have $n - (n + d) + n - 1 = 2 \Rightarrow d = n - 3$.

Some Slight Tweaks for further Improvement

- Start with simple polygon P as before, and triangulate it, which we know can be done by Theorem 3.1.
- **Question.** How many diagonals are there in any triangulation of P?
- **Answer:** $n - 3$. Why?
- Use Euler's formula for embedded planar graphs: our triangulated P has $n - 2 + 1$ faces, $n + d$ edges (where d is the number of diagonals) and n vertices.
- Thus by Euler's formula, we have $n - (n + d) + n - 1 = 2 \Rightarrow d = n - 3$.
- Slightly Better Result: Put a camera on each diagonal, thus reducing the number of guards from $n - 2$ to $n - 3$.

Can we do better??

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- **A new idea.** What about placing cameras on selected vertices instead of selected edges?
- **Why should this help?** Each triangle sees *at least* two triangles, maybe more.
- The following lemma from graph theory is key to our next step towards success.

Triangulated simple polygon graph

- **Lemma 1.** A triangulated simple polygon P has a vertex of degree 2.

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- You can check the cases $n = 3, 4, 5, 6$ by hand by producing all possible triangulations and show that in all cases, there exists a vertex of degree 2.
- **Conclusion.** Every simple triangulated polygon has a vertex of degree 2. **QED**

Help from graph coloring

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- **Induction Hypothesis.** For some fixed integer $k \geq 3$, the graph G of any triangulated simple polygon on k vertices satisfies $\chi(G) = 3$.
- **Inductive Step.** We must now prove that the graph G representing any triangulated simple polygon on $k + 1$ vertices satisfies $\chi(G) = 3$.

Proof of Theorem 5, continued

- By Lemma 1, G contains a vertex of degree 2; call it v. Consider the graph $G \setminus v$, which is a simple triangulated polygon on k vertices, thus satisfying our induction hypothesis.

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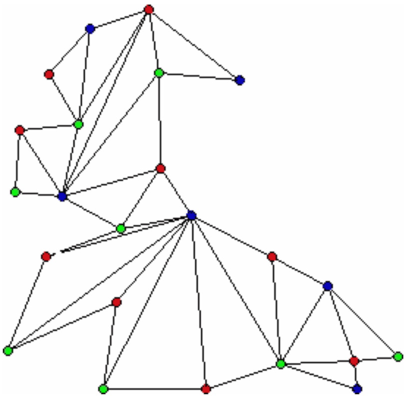
- **Conclusion.** If G is the graph of a simple triangulated polygon, then $\chi(G) = 3$.

3-coloring a triangulated polygon



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- Let C_R be the set of vertices colored red, C_G be the set of vertices colored green, and C_B be the set of vertices colored blue.

- **Fact.** $|C_R| + |C_G| + |C_B| = n$. Why?

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- By algebra, if n objects are distributed among three sets, at least one of those sets contains no more than $\lfloor \frac{n}{3} \rfloor$ objects.

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- Place a guard on each red vertex; since P is triangulated, every triangle is 3-colored, and thus P is covered.
- **QED**

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- Construct a polygon on n vertices for any $n \geq 3$ that requires a cover of $\lfloor \frac{n}{3} \rfloor$ vertices, thus proving that Theorem 6 is best possible.

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- **Next.** Speaking of Art, we'll look at a project inspired by M.C. Escher's Artwork.