CSCI 5451 Fall 2015

Week 13 Notes

Professor Ellen Gethner

November 9, 2015

NP-Completeness

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  5. Operation nd-choice is associated with a fixed number of choices such that at each choice, the algorithm follows a different computation path.

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- you walk from vertex to vertex along directed edges;
- at some vertices, namely the nd-choice ones, you will have to decide which way to go.

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  - the outcome of the algorithm will be to accept x if and only if $x \in L$.

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- Otherwise the algorithm outputs **DO NOT ACCEPT**.

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- **Definition.** The class of problems for which there exists a nondeterministic algorithm whose run time is polynomial in the size of the input is called **NP**.

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- **Challenge.** Is $P = NP$?
- Nobody knows!

Skip the final? Get a PhD? Get an A?

- If you show either that $P = NP$ or $P \neq NP$ then you win the following prizes: $\{$A, PhD, skip the final, \$1,000,000 from the Clay Institute[3]$\}$.

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- **Giant Implication:** if any single problem in P is proved to be in NP, then P=NP !!!
- **Moral of the story.** Researchers continue collecting NP complete and NP hard problems. The list continues to grow...

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- His theory would have no substance if there were no NP complete problems;
- his great breakthrough came in 1971 when he proved that the **Satisfiability Problem** (or SAT for short) was NP complete and that was the first such problem so proved.

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- That is, we do arithmetic $\pmod 2$ to determine the value of S, which is, a priori, 0 or 1.

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- $(1 + 1 + \overline{0})(\overline{1} + 1 + 0)(\overline{1} + \overline{1} + \overline{0}) = 1 \times 1 \times 1 = 1.$

Overview and Summary

- In 1972, Richard Karp proved that 21 important problems[4] were NP complete and the list continues to grow.

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- The idea of the proof that SAT is NP complete is, briefly, that

    1. we can guess at a truth assignment and check in polynomial time to see of the outcome of S is 1; if so, then SAT $\in$NP.
    2. Then it remains to show that SAT is NP hard, the proof of which is nontrivial and uses the idea of a Turing Machine[5].

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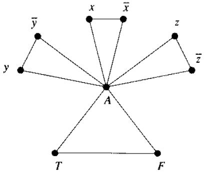
- **Tool.** 3SAT is NP complete (3SAT is a CNF expression in which all clauses have exactly three variables).
- **Theorem.** Graph 3-colorability is NP complete.
- **Idea of Proof.** We will reduce the 3SAT problem to the 3-colorability problem, but first note that if we guess at a 3-coloring of a graph G we can check it's validity in $O(E) = O(V^2)$-time. Why?

Reduction to 3SAT

- From a CNF expression, we construct a graph as follows. The three colors are T, F, and A.



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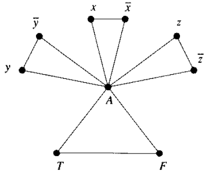
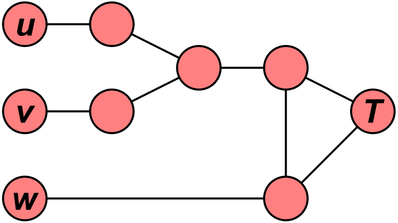


- In general, for a CNF expression S that has k variables, there will be $k+1$ triangles at this stage.

Attach a widget to each clause

- Next, attach the following widget to each clause $U + V + W$ in expression S.



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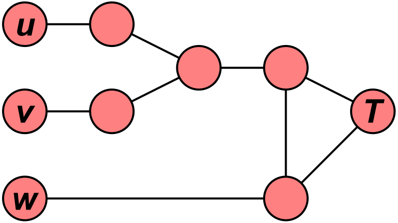

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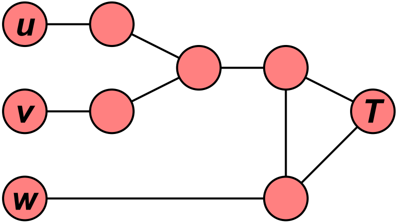
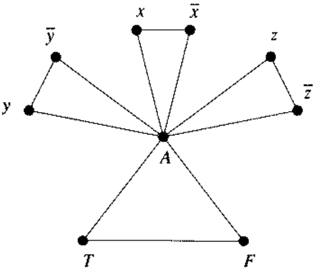


- **Facts.** The widget graph above is locally 3-colorable and forces one of u, v, or w to be assigned T.
- The above fact is important since otherwise S would never be satisfiable.

Widgets and 3-colorability, continued



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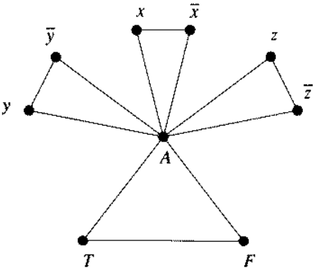

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- Each of x, $\overline{x}$, y, $\overline{y}$, z, and $\overline{z}$ will never be colored A because all are adjacent to a vertex colored A.

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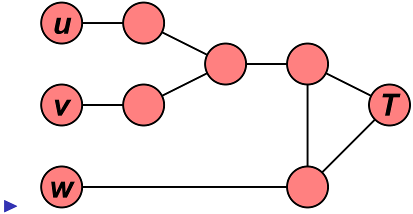

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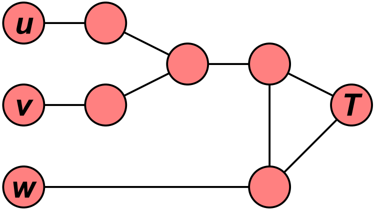
- 

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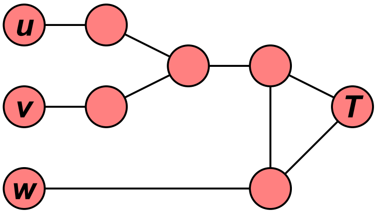
- 

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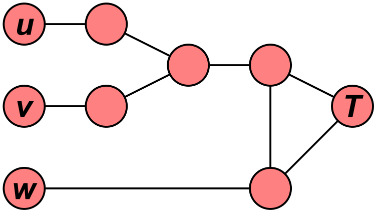
- 

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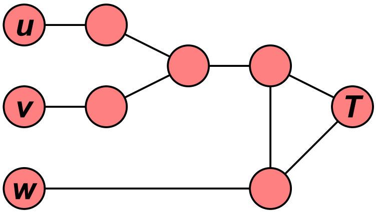
- 

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- That is, vertex 3-colorability is NP-complete.
- **Exercise.** Create the entire graph associated with the example $S = (x + y + \overline{z})(\overline{x} + y + z)(\overline{x} + \overline{y} + \overline{z})$.

Next Week

The Art Gallery Problem

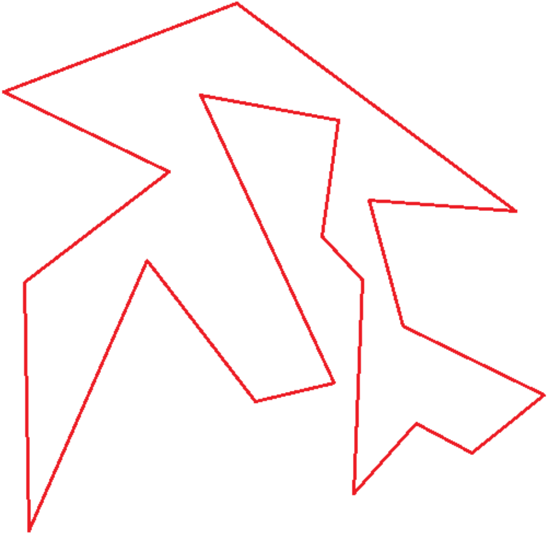

Figure: example caption