**CSC 5451, Professor E. Gethner**
**Assignment 1**
**3 September 2015**
**Quiz related to this homework is in class on Thursday, 17 September 2015**

Please feel free to collaborate with one another on this assignment. Consider writing up the solutions on your own for quiz practice. **Important information regarding the quizzes: Be neat, write complete sentences, and show all of your work. The way you communicate the solution to your answer is as important as the answer itself.** Good luck!

1. **(Another Number Theory Problem)** Prove by induction that $2^{2n} - 1$ is divisible by 3 $\forall n \in \mathbb{Z}^+$.

2. **(Size of $n$th Fibonacci Number)** CLR problem **4-4** (p. 109) part **d**. You may call upon parts **a, b** and **c** since we verified those in class.

3. **(Elegant Fibonacci)** In class we saw that the $2 \times 2$ matrix $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is a generator of sorts for the Fibonacci numbers. That is, the lower left entry of $F^n$ is exactly $F_n$, the $n$th Fibonacci number. The following elegant algorithm, fibel($n$), computes the $n$th Fibonacci number using the matrix $F$. In particular, if $n$ is a power of 2, the $n$th power of $F$ is computed by squaring $F$ lg $n$ times. Otherwise, if $n$ is not a power of 2, and say that $n = b_k b_{k-1} \ldots, b_2 b_1$ (in binary), then $F^n$ is the product of $F^{2^i}$ over all $i$ such that $b_i = 1$.

   **fibel($n$)**

   compute $n$ in binary, say $n =$ b[k]b[k-1]... b[1], where each b[i]$\in \{0, 1\}$.
   **if** b[1]=1 then total:= F
   **else** total:=$I_2$ ($2 \times 2$ identity matrix)
   **for** $i := 2$ to $k$ **do**
   $F := F * F$
   **if** b[i]=1 **then** total:=total$*F$
   **return** lower left entry of total

   You may assume that the time it takes to compute $n$ in binary is $\Theta(lg\ n)$. What is the runtime of fibel($n$)?

   **Please turn over**

4. **(Serendipity)** A positive integer is said to be **serendipitous** if it survives the following ordeal:
**Step 1:** $S_1 = \mathbb{Z}^+$
**Step 2:** Remove every 2nd number from $S_1$, and call this set $S_2$.
**Step 3:** Remove every 3rd number from $S_2$, and call this set $S_3$,

$$\vdots$$

**Step i:** Remove every $i$th number from $S_{i-1}$, and call this set $S_i$,

$$\vdots$$

Say that $S_\infty$ is the set of positive integers that remain after the ordeal is over. Then $S_\infty$ is the set of all serendipitious numbers; in fact, $S_\infty = \{1, 3, 7, 13, 19, \ldots\}$.

Your mission, should you choose to accept it, is to find all serendipitous numbers smaller than $n$. Consider the following two (ideas for) algorithms.

- **A1:** Use a boolean array of length $n$ and make repeated passes over the array, removing numbers until it completes a pass without any removals.
- **A2:** Maintain a linked list of numbers still thought to be serendipitious; at each pass shrink the list as needed, halting when a pass is completed with no removals.

Analyze the runtime of **A1** and **A2**. Suggestion: first write pseudocode (in English and readable by a human) for each algorithm.

5. **(Colorful Checkerboards)** A colorful skinny checkerboard of length $n$ is a $1 \times n$ board, where each square is colored either red or blue. Find a closed form formula for the number of colorful skinny checkerboards of length $n$ in which no two adjacent squares are both colored blue. Verify that your formula is correct with a proof. See Figure 1 for examples of three skinny uncolored checkerboards. Figure 2 contains examples of valid and invalid colored checkerboards.

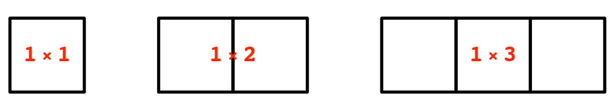


Figure 1: Three examples of $1 \times n$ skinny checkerboards: $n = 1, 2$ and 3

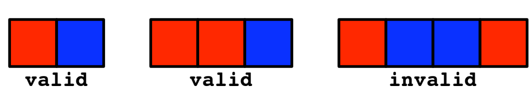


Figure 2: Examples of colorful skinny checkerboards